A REVIEW OF SEVERAL CLASSICAL MEASURE FUNCTIONS

by

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CHAPTER I

INTRODUCTION

In elementary geometry the notions of length, area, and volume are familiar concepts and easily defined for a restricted number of geometrical figures. The problem of extending these ideas of length, area, and volume to more complicated figures and sets in spaces of several dimensions gave rise to the theory of measure. The first attempt to solve the problem was made by Hankel in a discussion of integration. He originated a theory of what he called content of a linear set of points. This theory was further developed by Harnack. Cantor generalized the concept of content to sets of points in a space of any number of dimensions. These early theories have little value today other than historical, but a brief account of them will be given in order to point out certain defects which are remedied in later theories of measure.

Let A be any given set of points in the interval (a,b) . There corresponds to A a definite number S , which is such that all the points of A are interior points of a finite number of intervals the sum of whose lengths exceeds S by less than an arbitrary positive number ϵ , the number of intervals depending upon ϵ . This number S is called the <u>content</u> of the set A .

The preceding definition of content was used by Hankel and Harnack (4, pp. 161 - 162). An equivalent one was given by Cantor.

Instead of enclosing the points of the set A in a finite number of intervals, he enclosed each point x in an interval of length $2r$ where x is the midpoint of the interval. Any parts of these covering intervals which lie outside of (a,b) are disregarded. If this infinite set of overlapping intervals is replaced by the set of non-overlapping intervals with the same interior points, a finite set of intervals of length $\geq r$ is obtained. With the sum of the lengths of these intervals denoted by S_r, the <u>content</u> of the set A is defined to be the greatest lower bound of the numbers S_r. Cantor extended this definition to a set A of points in n-dimensions by enclosing each point x in an n-dimensional sphere of radius r and center x and in a similar manner defined content of A as the greatest lower bound of the volumes made up of the points of A contained within the spheres (4, pp. 163 - 164).

This theory of content has certain defects when applied to non-closed sets, however. In order to be completely satisfactory, content should be a generalization of the notion of length of a linear set of points in a space of one dimension and the content of a continuous interval should be equivalent to the length of the interval. Thus, if two sets A_1 and A_2 are two complementary sets in the interval $(0,1)$, then the sum of their contents should be 1 in order to correspond to the length of the interval. However, this is not necessarily so if A_1 and A_2 are not closed. For example, let A_1 be the set of rational numbers and A_2 the set of irrationals.

The content of each of the sets A_1 and A_2 is 1, as is the content of (0,1) itself.

In order to remedy this defect in the definition of content, Peano and Jordan introduced what they called outer content and inner content and from it defined a measure which is applicable to sets in a space of n-dimensions (12, pp. 58 - 61). The definition may be stated as follows:

Let A be a set contained in a bounded interval. Let A be covered by sets $\{I_j\}$ of intervals. Let $|I_j|$ represent the volume of each interval I_j. That is, $|I_j| = \prod_{i=1}^{n} s_{ji}$ where s_{ji}, $i = 1,2,\ldots,n$, represent the edge lengths of I_j. The <u>outer</u> <u>content</u> of A, $\bar{c}(A)$, is defined to be the greatest lower bound of the sums $\sum |I_j|$ for all possible finite coverings of A. The <u>inner</u> <u>content</u> of A, $\underline{c}(A)$, is defined to be the least upper bound of the sums of the volumes of intervals I_j where the $\{I_j\}$ are such that $A \supset \bigcup_{j=1}^{k} I_j$. If $\bar{c}(A) = \underline{c}(A)$, then the set A is said to be <u>Jordan</u> <u>measurable</u> and the common value of $\bar{c}(A)$ and $\underline{c}(A)$ is called the <u>Jordan</u> <u>content</u> $c(A)$ of A. The outer content as defined here is equivalent to content as defined by Hankel and Cantor.

Although the theory of content as developed by Peano and Jordan has been largely replaced by other theories of measure, it has played a part in the theory of integration. The connection between Jordan content and Riemann integration has been discussed by Kestelman in his book <u>Modern</u> <u>Theories</u> <u>of</u> <u>Integration</u> in which the following statement is proved (6, pp. 64 - 65).

Let $f(x)$ be a bounded, non-negative function defined for the closed interval $[a,b]$ and let G be the plane set of points (x,y) consisting of all the points $a \leq x \leq b$, $0 \leq y \leq f(x)$. Then

$$\underline{c}\,(G) = \underline{\int_a^b} f(x)dx \quad \text{and} \quad \bar{c}\,(G) = \overline{\int_a^b} f(x)dx$$

and each of the following three statements implies the other two:

i) $f(x)$ is Riemann integrable over $[a,b]$.

ii) $f(x)$ is continuous almost everywhere in $[a,b]$.

iii) The set G is Jordan measurable.

One of the deficiencies of the theory of Jordan content is that the sum of infinitely many Jordan measurable sets need not be Jordan measurable. The theory of Riemann integration reflects the same deficiency in that a bounded function, defined as the limit of a sequence of integrable functions, may itself be non-integrable in the Riemann sense.

The theory of measure developed by Lebesgue overcomes this deficiency and the theory of integration based on Lebesgue measure is free from many of the limitations of the Riemann theory. The fundamental difference between Lebesgue measure and the outer content as defined by Jordan and Peano is that in the latter the covering intervals, in terms of which the content was defined, had to be finite in number, whereas Lebesgue's idea was to replace the finite systems of intervals by countable infinite ones. Thus any set which is Jordan measurable is also Lebesgue measurable and the two measures are the same. The converse, however, is not true.

Lebesgue's theory of measure has led to further generalizations. If, instead of starting with the notions of length, area, and volume in attempting to develop a general concept of measure of a set, we imagine a mass distribution in the space under consideration and assign to each set as its measure the amount of mass distributed on the set, we are led to a generalization of Lebesgue measure which is called Lebesgue - Stieltjes measure. This measure has important applications to physical problems and problems in probability and statistics and gives rise to a generalization of Lebesgue integration. Another theory of measure which is closely related to that of Lebesgue was developed by Hausdorff. It has special properties that enable it to be used to define dimensionality of a set. Although in general, the Hausdorff and Lebesgue measures of a set are not the same, they are identical for sets in E_1.

These special theories of measure, together with a theory developed by Haar, will be discussed in detail in the following chapters. Also, in Chapter 3, an abstract and general form of all these special theories which is due to Carathéodory will be given.

CHAPTER II

A GENERAL DEFINITION OF MEASURE

In order to present a unified account of the various theories of measure to be discussed, we shall present in this chapter a general, axiomatic definition of a measure function. Our procedure will be to determine a particular class of sets and then define a measure for sets belonging to this class. Our procedure in dealing with some special theories of measure in later chapters will be reversed. That is, a measure will be defined for all sets of a particular space under consideration, and then by restricting this measure to a smaller class of sets, a measure will be determined which satisfies the definition to be given in this chapter.

Definition 2.1. A class (A) of sets in an abstract space S will be called a completely additive class of sets if it satisfies the following postulates:

A-1. The empty set belongs to (A).

A-2. If a set A belongs to (A), then the complement of A belongs to (A).

A-3. If $\{A_n\}$ is any sequence of sets from (A), then $\bigcup_{n=1}^{\infty} A_n$ also belongs to (A).

The class (A) is called a finitely additive class of sets if A-3 is replaced by

A-3'. If A and B belong to (A), then $A \cup B$ belongs to (A) .

Two fundamental properties of completely additive classes are given in the following theorem.

Theorem 2.1. If $\{A_n\}$ is a sequence of sets belonging to (A) , then $\bigcap_{n=1}^{\infty} A_n$ belongs to (A) and $\overline{\lim}\, A_n$ and $\underline{\lim}\, A_n$ belong to (A) (8, p. 64) .

Note that because of postulates A-1 and A-2 the space S itself belongs to every additive class of sets in S since S is the complement of the empty set.

Two immediate examples of completely additive classes in S are the class consisting of the two sets S and the empty set and also the class consisting of S and all subsets of S . Another example which plays a useful role in the development of some of the special theories of measure and the applications of measure theory is the class of Borel sets. Before defining the Borel sets, however, some preliminary definitions and theorems are needed.

Definition 2.2. Let (M) be any class of subsets of S . A completely additive class (A) is called the minimal completely additive class containing (M) if $(A) \supset (M)$ and if for any completely additive class (B) such that $(B) \supset (M)$ it follows that $(B) \supset (A)$.

Theorem 2.2. Given any class (M) of subsets of S, there exists a minimal completely additive class of sets (A) containing (M) (10, p. 8).

Now let S be any space and (N) any class of subsets of S.

Define $(N)_\sigma$ to be the class of all sets which are countable unions of sets from (N) and $(N)_\delta$ to be the class of all sets which are countable intersections of sets from (N). Since the union of a countable class of countable sets is a countable set, it follows that $(N)_{\sigma\sigma} = (N)_\sigma$ and $(N)_{\delta\delta} = (N)_\delta$. In order to obtain new classes of sets it is sufficient to alternate the operators σ and δ. The usual notation is to write the subscripts in the order in which the operations are performed. Let F represent a closed set and (F) represent the class of all closed subsets of S. Similarly, G is an open set and (G) the class of all open sets of S. Also, to simplify notation, let

$$(F)^0 = (F);\ (F)^1 = (F)_\sigma;\ (F)^2 = (F)_{\sigma\delta};\ (F)^3 = (F)_{\sigma\delta\sigma};\ \ldots;$$

$$(G)^0 = (G);\ (G)^1 = (G)_\delta;\ (G)^2 = (G)_{\delta\sigma};\ (G)^3 = (G)_{\delta\sigma\delta};\ \ldots .$$

<u>Theorem 2.4</u>. For every positive integer n,

$(F)^{n-1} \subset (F)^n$; $(G)^{n-1} \subset (G)^n$; $(F)^{n-1} \subset (G)^n$; $(G)^{n-1} \subset (F)^n$ (8, p. 65).

We are now able to define the class of Borel sets.

<u>Definition 2.3</u>. The <u>class</u> (B) <u>of Borel sets</u> is the minimal completely additive class of sets containing (F).

<u>Theorem 2.5</u>. For each positive integer n, $(F)^n \subset (B)$ and $(G)^n \subset (B)$.

Proof: Using induction, we shall first prove that $(F)^n \subset (B)$. By Definition 2.3, $(F)^0 \subset (B)$. Assume $(F)^{n-1} \subset (B)$. If n is odd and $H \in (F)^n$, then $H = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in (F)^{n-1} \subset (B)$. Using postulate A-3, since (B) is completely additive, the union of sets from (B) also belongs to (B). Therefore $H \in (B)$ and $(F)^n \subset (B)$. If n is even and $H \in (F)^n$, then $H = \bigcap_{n=1}^{\infty} A_n$, where $A_n \in (F)^{n-1} \subset (B)$. Using A-2 and A-3, $-A_n \in (B)$, $[\bigcup_{n=1}^{\infty}(-A_n)] \in (B)$, $[-\bigcup_{n=1}^{\infty}(-A_n)] \in (B)$. But $-\bigcup_{n=1}^{\infty}(-A_n) = \bigcap_{n=1}^{\infty} A_n = H$. Therefore $H \in (B)$ and $(F)^n \subset (B)$.

The proof that $(G)^n \subset (B)$ follows from the above and Theorem 2.4.

Definition 2.4. A set from either $(F)^n$ or $(G)^n$ is called a Borel set of order n.

We have defined a completely additive class (A) of sets for any space and shown some examples of such classes. Our final step is to define a measure for sets which are elements of (A). Such a measure is defined in terms of a completely additive set function.

Definition 2.5. Let (A) be a completely additive class of sets in a space S . A set function ω will be called a completely additive set function on (A) provided it satisfies the following postulates:

F-1. The function $\omega(X)$ is defined in the extended real number system for each X in (A).

F-2. If $\{X_n\}$ is a sequence of disjoint sets from (A), then $\sum_{n=1}^{\infty} \omega(X_n)$ is defined in the extended real number system and $\omega(\bigcup_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} \omega(X_n)$.

F-3. If A is the empty set, then $\omega(A) = 0$.

Definition 2.6. A completely additive set function μ on (A) will be called a measure for sets belonging to the class (A) if it is non-negative for every set in (A) . A set X is said to be μ measurable if X belongs to (A) .

The specific examples of measures to be discussed in the following chapters will be shown to satisfy Definition 2.5. Two rather trivial examples of measures which can be mentioned now are the following:

1. Let (X) be the class consisting of a space S and all of its subsets. Let $\mu(X) \equiv 0$ for every set X in (X) . Then μ is obviously a measure as defined by Definition 2.5.
2. Let (X) be defined as in the preceding example and choose a point p which is an element of S . Define $\mu(X) = 1$ if p is in X and $\mu(X) = 0$ if p is not in X . Then μ is a measure as defined.

CHAPTER III

CARATHÉODORY MEASURE

In the preceding chapter we defined a measure function by starting with a particular class of sets and then defined a measure for sets belonging to this class. Our procedure in dealing with some special measures will be reversed. That is, we will begin by postulating a non-negative function of a set called an outer measure which is defined for all sets of the space under consideration. This is not necessarily a measure as defined in the last chapter, but we will show that a suitable restriction of an outer measure to a smaller class of sets will determine a measure as previously defined. A general theory of such special measures is due to C. Carathéodory (10, pp. 43 - 47), so in this chapter we shall define and discuss some of the properties of Carathéodory measure. The special measures discussed in the following chapter will then be shown to be Carathéodory measures.

Definition 3.1. Let (S,ρ) be a metric space. A function μ^+ of a set, defined and non-negative for all sets in (S,ρ) will be called a Carathéodory outer measure if it satisfies the following postulates:

C-1. $\mu^+(A) \leq \mu^+(B)$ if $A \subset B$.

C-2. $\mu^+(\bigcup_{n=1}^{\infty} X_n) \leq \sum_{n=1}^{\infty} \mu^+(X_n)$ for any sequence $\{X_n\}$ of sets of (S,ρ) .

C-3. $\mu^+(A + B) = \mu^+(A) + \mu^+(B)$ whenever $\rho(A,B) > 0$.

C-4. If A is the empty set, $\mu^+(A) = 0$.

Note that only C-3 in these four conditions is metrical in character. Therefore, any of the results which are obtained in this chapter without use of C-3 are valid in any abstract space.

Definition 3.2. A set E is said to be measurable with respect to μ^+ if for every $X \subset (S,\rho)$.

$$\mu^+(X) = \mu^+(X \cap E) + \mu^+(X - E) .$$

Theorem 3.1. A necessary and sufficient condition that a set E be measurable with respect to μ^+ is that for every $X \subset (S,\rho)$, $\mu^+(X) \geq \mu^+(X \cap E) + \mu^+(X - E)$.

Proof: Since $(X \cap E) \cup (X - E) = X$, it follows from C-3 that $\mu^+(X) \leq \mu^+(X \cap E) + \mu^+(X - E)$. This, together with the above definition, establishes the proof of the theorem.

Denote the class of all sets that are measurable with respect to μ^+ by (M). We want to establish that (M) is a completely additive class so that the restriction of μ^+ to (M) will be a measure as defined in Definition 2.5. In order to do this we need to establish some fundamental properties of μ^+ .

Theorem 3.2. If $\mu^+(A) = 0$, then A is measurable (8, p. 88).

Theorem 3.3. If A is measurable, then the complement of A is measurable (11, p. 136) .

Theorem 3.4. Any finite union or intersection of measurable sets is measurable (11, p. 137) .

Theorem 3.5. Any countable union of disjoint measurable sets is measurable (8, p. 89).

Theorem 3.6. Any countable union of measurable sets is measurable (8, p. 90).

We are now able to prove the following theorem.

Theorem 3.7. (M) is a completely additive class of sets in the space (S,ρ).

> Proof: Referring to the definition of a completely additive class given in Definition 2.1, postulate A-1 is satisfied by C-4 and Theorem 3.2; A-2 is satisfied by Theorem 3.3; A-3 by Theorem 3.6.

Thus the restriction of μ^+ to the class of sets (M) is a measure as defined in Chapter 2.

We would like to show now that there does exist a class of sets which is μ^+ measurable. In particular, we shall show that the Borel sets are measurable with respect to μ^+. In order to do this, we need to make use of a lemma which is due to Carathéodory (10, pp. 51 - 52).

Carathéodory's Lemma. If G is an open set, A is any set contained in G, and A_n denotes the set of points $\{x\}$, x in A, such that $\rho(x, -G) \geq 1/n$ for each positive integer n, then $\lim_n \mu^+ (A_n) = \mu^+ (A)$.

Theorem 3.10. The class of all Borel sets is contained in (M).

> Proof: The proof will be established if we show that every closed set F is μ^+ measurable, hence $(F) \subset (M)$. For, by Definition 2.3, the class of Borel sets (B) is the

minimal completely additive class containing (F) and by Definition 2.2, it will follow that $(B) \subset (M)$. To do this let X be any set. Then $X - F \subset -F$ and $-F$ is an open set. By the preceding lemma, there is a sequence of sets $\{A_n\}$ such that $A_n \subset X - F$, $\rho(A_n, F) \geq \frac{1}{n}$ for $n = 1, 2, \ldots$, and $\lim\limits_{n} \mu^+(A) = \mu^+(X - F)$.

Then we have $X = [(X \cap F) \cup (X - F)] \supset [(X \cap F) \cup A_n]$.

Using C-3, $\mu^+(X) \geq \mu^+(X \cap F) + \mu^+(A_n)$. Letting $n \to \infty$, we have $\mu^+(X) \geq \mu^+(X \cap F) + \mu^+(X - F)$. Therefore, by Theorem 3.1, F is μ^+ measurable.

CHAPTER IV

LEBESGUE MEASURE

The theory of measure developed by H. Lebesgue was the starting point of further extensions of ideas of measure and integration. The method of developing Lebesgue measure to be given in this chapter will be to define a measure for open intervals in the Euclidean space E_n which is a generalization of length, area, and volume. Using this we will define an outer measure for sets in E_n which was used by Lebesgue to define a class of measurable sets.

Definition 4.1. Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ be real numbers such that $a_k < b_k$ for $k = 1, 2, \ldots, n$. The set $I = \{(x_1, x_2, \ldots, x_n)\}$ in E_n, where $a_k < x_k < b_k$, is defined to be an n-dimensional open interval. If $a_k \leq x_k \leq b_k$, then I is defined to be an n-dimensional closed interval.

Definition 4.2. The measure $m(I)$ of an open interval I in E_n is defined to be the non-negative number $m(I) = \prod_{k=1}^{n} (b_k - a_k)$.

Thus the measure of a one-dimensional open interval is its length, of a two-dimensional one its area, and of a three-dimensional one its volume.

Definition 4.3. The Lebesgue exterior (outer) measure of a set A in E_n is the greatest lower bound of the sums $\sum_{n=1}^{\infty} m(I_n)$, where $\{I_n\}$ is any sequence of open intervals which cover A. Denote the Lebesgue exterior measure of A by $m_e(A)$.

Theorem 4.1. The Lebesgue exterior measure is a Caratheodory outer measure.

Proof: We must show that the postulates given in Definition 3.1 are satisfied. Let A and B be any two sets in E_n such that $A \subset B$. Any sequence of open intervals which covers B will also cover A so that it follows directly from Definition 4.3 that $m_e(A) \leq m_e(B)$. This satisfies C-1.

Let $\{X_n\}$ be any sequence of sets in E_n and let $S = \bigcup_{n=1}^{\infty} X_n$. We want to show that $m_e(S) \leq \sum_{n=1}^{\infty} m_e(X_n)$.

If the series on the right diverges the inequality is true. Assume that the series is convergent and let ϵ be any positive number. For each positive integer n, there exists a sequence of open intervals such that

$$\bigcup_{k=1}^{\infty} I_{nk} \supset X_n \quad \text{and} \quad \sum_{k=1}^{\infty} m(I_{nk}) \leq m_e(X_n) + \frac{\epsilon}{2^n} .$$

Then $S = \bigcup_{n=1}^{\infty} X_n \subset \bigcup_{n=1}^{\infty}\left(\bigcup_{n=1}^{\infty} I_{nk}\right)$ so that

$$m_e(S) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m(I_{nk}) \leq \sum_{n=1}^{\infty} \left[m_e(X_n) + \frac{\epsilon}{2^n}\right]$$

$$= \sum_{n=1}^{\infty} m_e(X_n) + \epsilon .$$

Since this is true for all ϵ, C-2 is satisfied.

Let A and B be two sets in E_n such that $\rho(A,B) = d > 0$. From the preceding paragraph, $m_e(A \cup B) \leq m_e(A) + m_e(B)$. Given $\epsilon > 0$, there exists a sequence of open intervals of edge lengths less than $\frac{d}{\sqrt{n}}$ where n is the dimensionality of the space E_n such that $\bigcup_{n=1}^{\infty} I_n \supset A \cup B$ and $\sum_{n=1}^{\infty} m(I_n) \leq m_e(A \cup B) + \epsilon$.

Then no I_n contains points of both A and B. Therefore part of the sequence covers A and the rest covers B. Thus $m_e(A) + m_e(B) \leq \sum_{n=1}^{\infty} m(I_n) \leq m_e(A \cup B) + \epsilon$. Since this is true for every ϵ, $m_e(A) + m_e(B) \leq m_e(A \cup B)$. This result combined with the reverse inequality above fulfils condition C-3. The satisfaction of C-4 follows directly from Definition 4.3.

Definition 4.4. A set which is measurable (cf. Definition 3.2) with respect to Lebesgue exterior measure is said to be Lebesgue measurable.

Denote the class of sets which are Lebesgue measurable by (L). It follows from Theorem 3.9 that (L) is a completely additive class of sets in E_n and all of the theorems concerning sets which were proved in Chapter 3 hold for sets which are elements of (L). The measure of a set A which is in (L) is denoted by m(A).

Theorem 4.2. The Lebesgue exterior measure of a set A is the greatest lower bound of the measure of all open sets which contain A (11, pp. 152 - 153).

Definition 4.5. The Lebesgue inner measure of a set A, denoted by $m_i(A)$, is the least upper bound of the measures of all closed sets which are contained in A .

Theorem 4.3. $m_i(A) \leq m_e(A)$.

Proof: Let $\{F_n\}$ be the set of all closed sets such that $F_n \subset A$ for each n . Then $m(F_n) \leq m_e(A)$ for all n and $m_e(A)$ is an upper bound of the set $\{m(F_n)\}$. But $m_i(A)$ is the least upper bound of this set. Therefore $m_i(A) \leq m_e(A)$.

In some discussions of Lebesgue measure a bounded set A is called Lebesgue measurable if the condition $m_e(A) = m_i(A)$. The following theorem shows that this condition is equivalent to the condition for measurability given in Definition 4.4.

Theorem 4.4. A necessary and sufficient condition for a bounded set A to be Lebesgue measurable is that $m_i(A) = m_e(A) = m(A)$ (11, p. 153).

Since Lebesgue measure is a Carathéodory outer measure, it follows from Theorem 3.10 that the class (B) of Borel sets is Lebesgue measurable. The restriction of Lebesgue measure to the class (B) is sometimes called Borel measure. Although it is true that every Borel set is Lebesgue measurable, not every Lebesgue measurable set is a Borel set. The following theorem indicates a relationship that exists between the classes (L) and (B) .

Theorem 4.5. Each of the following conditions is necessary and sufficient for a set E to be Lebesgue measurable (10, pp. 69 - 70):

(i) given $\epsilon > 0$, there exists an open set $G \supset E$ such that $m(G - E) < \epsilon$;

(ii) there exists a set $H \epsilon (G)_\delta$ such that $H \supset E$ and $m(H - E) = 0$;

(iii) given $\epsilon > 0$, there exists a closed set $F \subset E$ such that $m(E - F) < \epsilon$.

(iv) there exists a set $K \epsilon (F)_\sigma$ such that $E \supset K$ and $m(E - K) = 0$.

Many investigations in measure theory would be simplified if it were true that all sets in E_n are Lebesgue measurable. This, however, is not the case. In order to show this we shall construct a set which is non-measurable in the Lebesgue sense. It is interesting to note, however, that a Lebesgue non-measurable set is such a strange thing that to date none has been constructed without using the axiom of choice.

Before constructing such a set, we need to establish the property of invariance of Lebesgue measure of sets in E_n under linear transformations. The following theorem which is proved by P. Halmos in his book Measure Theory does this.

Theorem 4.6. Let T be the one to one transformation of the entire real line onto itself, defined by $T(x) = \alpha x + \beta$, where α and β are real numbers and $\alpha \neq 0$. If, for every subset E of the real

line E_1, $T(E)$ denotes the set of all points of the form $T(x)$ with x in E, i.e.,
$T(E) = \{\alpha x + \beta : x \in E\}$, then $m_e(T(E)) = |\alpha|\, m_e(E)$ and $m_i(T(E)) = |\alpha|\, m_i(E)$. The set $T(E)$ is a Lebesgue measurable set if and only if E is a Lebesgue measurable set, (3, pp. 64-65).

Let A be a set in E_1 such that $A \subset [0,1)$, and let $a \in [0,1]$. Express A as the union of two disjoint sets A_1 and A_2 where $A_1 = \{x : x \in A,\ x + a < 1\}$ and $A_2 = \{x : x \in A,\ x + a \geq 1\}$. We say that $T = A + a$ [mod. 1] provided $T = T_1 \cup T_2$ where $T_1 = \{x + a : x \in A_1\}$ and $T_2 = \{x + a - 1 : x \in A_2\}$.

Theorem 4.7. If $A \subset [0,1)$, if $a \in [0,1)$, if $T = A + a$ [mod. 1], then $m_e(T) = m_e(A)$ and $m_i(T) = m_i(A)$.

Proof: From Theorem 4.6, $m_e(A_1) = m_e(T_1)$ and $m_e(A_2) = m_e(T_2)$. Now $T = T_1 \cup T_2$ and $A = A_1 \cup A_2$ where A_1 and A_2 are disjoint sets as are T_1 and T_2. Therefore, from Theorem 3.5,

$$m_e(T) = m_e(T_1) + m_e(T_2) = m_e(A_1) + m_e(A_2) = m_e(A) .$$

Similarly, $m_i(T) = m_i(A)$.

Theorem 4.8. If $T = A + a$ [mod. 1], then H is Lebesgue measurable if and only if A is Lebesgue measurable.

Proof: This follows directly from the preceding theorem and from Theorem 4.4.

Let R be the set of rational numbers in $[0,1)$. For each $x \in [0,1)$ let $A_x = R + x$ [mod.1]. Any two of these sets, say A_{x_1} and A_{x_2}, are either identical or disjoint. For let $y \in (A_{x_1} \cap A_{x_2})$ and let $z \in A_{x_1}$. We have $z - x_2 = (z - x_1) + (x_1 - y) + (y - x_2)$ where each term on the right side of the equation is rational. Then $z - x_2$ is rational. Therefore $z \in A_{x_2}$ and $A_{x_1} \subset A_{x_2}$. Similarly, $A_{x_2} \subset A_{x_1}$. Therefore, if $A_{x_1} \cap A_{x_2}$ is non-empty, it follows that A_{x_1} and A_{x_2} are identical. Let (C) be the class of disjoint sets of the form A_x. Using the axiom of choice there is a set P_0 consisting of one point from each of the sets belonging to (C). We now enumerate all rational numbers in $(0,1)$, obtaining a sequence of rationals $r_1, r_2, \ldots, r_k, \ldots$. Let $P_k = P_0 + r_k$ [mod. 1]. The sets P_k are disjoint. For assume $P_{k_1} \cap P_{k_2}$ is non-empty and let y be an element of this intersection. Then either $(y - r_{k_1})$ and $(y - r_{k_2})$ are elements of P_0 or $(y - r_{k_1} + 1)$ and $(y - r_{k_2} + 1)$ are elements of P_0. Thus P_0 contains two points whose difference is a rational number. But this cannot be, since these two points must belong to the same set A_x and P_0 contains only one point from each set of the form A_x. Therefore, the sets P_k are disjoint.

Now for each $k = 0,1,\ldots,$ $P_k \subset [0,1)$. Therefore, $\bigcup_{k=0}^{\infty} P_k \subset [0,1)$. Also, if $x \in [0,1)$, then for some r_k, $x - r_k$ is the point of A_x (or differs by 1 from the point of A_x) which belongs to P_0. Therefore, $x \in P_k$ and $[0,1) \subset \bigcup_{k=1}^{\infty} P_k$.

Thus $\bigcup_{k=0}^{\infty} P_k = [0,1)$.

From Theorem 4.8 it follows that the sets P_k are either all Lebesgue measurable or all non-measurable. Assume they are measurable. Then $m_e(\bigcup_{k=0}^{\infty} P_k) = \sum_{k=0}^{\infty} m_e(P_k) = m_e\,[0,1) = 1$.

From Theorem 4.7 we have that $m_e(P_0) = m_e(P_1) = \cdots m_e(P_k) = \cdots$.

If this common value of the measures is 0, then we have $\sum_{k=0}^{\infty} m_e(P_k) = 0 = m_e\,[0,1] = 1$, a contradiction. If the common value is a positive number, then $\sum_{k=0}^{\infty} m_e(P_k)$ is infinite, again a contradiction. Therefore, the sets P_k are not Lebesgue measurable.

CHAPTER V

HAUSDORFF MEASURE

A measure which is closely related to Lebesgue measure was defined by F. Hausdorff for any metric space. It is of particular interest in that it can be used to define the dimension of a set. While Hausdorff measure is a metrical concept and a dimension a topological one, it can be shown that there is a connection between the two concepts.

<u>Definition 5.1.</u> For a separable metric space (S,ρ) and $\epsilon > 0$ let p be a positive real number and E a subset of (S,ρ). Define $\mu_p^\epsilon(E) = \text{g.l.b.}\ \{\sum_{i=1}^{\infty} [\delta(E_i)]^p\}$ for all decompositions $E = \bigcup_{i=1}^{\infty} E_i$ such that $\delta(E_i) < \epsilon$ where $\delta(E)$ denotes the diameter of a set E. Let $\mu_p^+(E) = \lim_{\epsilon \to 0} \mu_p^\epsilon(E)$. Then $\mu_p^+(E)$ is called the <u>Hausdorff p-dimensional outer measure of E</u>.

<u>Theorem 5.1.</u> μ_p^+ is a Carathéodory outer measure.

Proof: First note that $\mu_p^+(E)$ is non-negative and defined for all sets in (S,ρ).

If $A \subset B$, then $\mu_p^+(A) \leq \mu_p^+(B)$. For let $\sum_{i=1}^{\infty} B_i$ be any decomposition of B such that $\delta(B_i) < \epsilon$.

Let $A_i = A \cap B_i$ so that $A_i \subset B_i$, $i = 1,2,\cdots$.

Then $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (A \cap B_i) = A \cap (\bigcup_{i=1}^{\infty} B_i) = A \cap B = A$.

Hence $\bigcup_{i=1}^{\infty} A_i$ is a decomposition of A such that

$\delta(A_i) \leq \delta(B_i) < \epsilon$. Then $\mu_p^{\epsilon}(A) \leq \sum_{i=1}^{\infty} [\delta(A_i)]^p \leq \sum_{i=1}^{\infty} [\delta(B_i)]^p$.

Therefore $\mu_p^{\epsilon}(A)$ is a lower bound of $\{\sum_{i=1}^{\infty} [\delta(B_i)]^p\}$ so

that $\mu_p^{\epsilon}(A) \leq \mu_p^{\epsilon}(B)$. Taking the limit as $\epsilon \to 0$,

$\mu_p^{+}(A) \leq \mu_p^{+}(A)$. Thus postulate C-1 in Definition 3.1

is satisfied.

If $E = \bigcup_{n=1}^{\infty} E_n$, then $\mu_p^{+}(E) \leq \sum_{n=1}^{\infty} \mu_p^{+}(E_n)$.

If $\sum_{n=1}^{\infty} \mu_p^{+}(E_n) = +\infty$, the inequality is obviously true.

Assume $\sum_{n=1}^{\infty} \mu_p^{+}(E_n)$ is finite. Let σ be a positive

number. There exists a sequence of sets $\{E_{nk}\}$ such that

for each n

$$\bigcup_{k=1}^{\infty} E_{nk} = E_n \quad , \qquad \delta(E_{nk}) < \epsilon$$

and

$$\sum_{k=1}^{\infty} [\delta(E_{nk})]^p \leq \mu_p^{\epsilon}(E_n) + \sigma/2^n$$

We have $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{nk} = E$ so that

$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{nk}$ is a decomposition of E such that

$\delta(E_{nk}) < G$.

Then

$$\mu_p^\epsilon(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} [\delta(E_{nk})]^p \leq \sum_{n=1}^{\infty} [\mu_p^\epsilon (E_n) + \epsilon/2^n]$$

$$= \sum_{n=1}^{\infty} \mu_p^\epsilon (E_n) + \sigma \quad .$$

Since this is true for every σ ,

$$\mu_p^\epsilon (E) \leq \sum_{n=1}^{\infty} \mu_p^\epsilon (E_n)$$

Letting $\epsilon \to 0$, $\mu_p^+(E) \leq \sum_{n=1}^{\infty} \mu_p^+ (E_n)$ so that

postulate C-2 is satisfied.

If $p(A,B) = 0$, then $\mu_p^+ (A + B) = \mu_p^+(A) + \mu_p^+(B)$.

Given an $\sigma > 0$ there exists a sequence $\{E_n\}$ such that

$\bigcup_{n=1}^{\infty} E_n = A \cup B$, where $\delta(E_n) < \epsilon < \rho(A,B)$ and

$\mu_p^\epsilon (A \cup B) + \sigma \geq \sum_{n=1}^{\infty} [\delta(E_n)]^p$. Then no E_n contains

points of both A and B . Thus $\bigcup_{n=1}^{\infty} E_n$ is a combination of a decomposition of A and a decomposition of B

so that $\mu_p^\epsilon(A) + \mu_p^\epsilon (B) \leq \sum_{M=1}^{\infty} [\delta(E_n)]^p \leq \mu_p^\epsilon(A \cup B) + \sigma$.

Since this is true for every σ, $\mu_p^\epsilon(A) + \mu_p^\epsilon(B) \leq \mu_p^\epsilon(A \cup B)$.

Then taking the limit as $\epsilon \to 0$, $\mu_p^+(A) + \mu_p^+(B) \leq \mu_p^+(A \cup B)$.

But from C-2, we have $\mu_p^+(A \cup B) \leq \mu_p^+(A) + \mu_p^+(B)$.

Therefore, $\mu_p^+(A) + \mu_p^+(B) = \mu^+(A \cup B)$ and C-3 is satisfied.

C-4 follows from the fact that if A is empty, then $\delta(A) = 0$.

Thus μ_p^+ determines in (S,ρ) a class of sets which are measurable, i.e., a class of sets which satisfy Definition 3.2. Let (H) denote the class of sets which are measurable with respect to Hausdorff p-dimensional outer measure and denote the measure of a set A which is in (H) by $\mu_p(H)$. By Theorem 3.10, the Borel sets (B) are contained in (H), hence the Borel sets are μ_p-measurable.

The next two theorems show a relationship between Lebesgue and Hausdorff measures.

<u>Theorem 5.2.</u> If A is a set in E_n such that $\mu_n(A) = 0$, then $m_e(A) = 0$.

Proof: Since $\mu_n^+(A) = 0$, $\mu_n^1(A) = 0$. Hence, given a $\sigma > 0$ there exists a sequence of sets $\{A_j\}$ such that $A = \bigcup_{j=1}^{\infty} A_j$, $\delta(A_j) < 1$, and $\mu_n^1(A) + \sigma = \sigma \geq \sum_{j=1}^{\infty} [\delta(A_j)]^n$.

Define the projection of each set A_j onto the k-th axis by

$$P_k(A_j) = \{x_k : (x_1, \ldots, x_k, \ldots, x_n) \in A_j$$

$$\text{for some } x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\} .$$

Each A_j is bounded since $\delta(A_j) < 1$. Let

$a_{jk} = \text{g.l.b.}\{P_k(A_j)\}$, $b_{jk} = \text{l.u.b.}\{P_k(A_j)\}$,

$d_{jk} = b_{jk} - a_{jk}$ for $k = 1, 2, \ldots, n$. Let I_j be the open interval defined by

$I_j=\{(x_1,\ldots,x_n):a_{ji} < x_i < b_{ji},\ i=1,2,\ldots,n\}$.

For each k, $d_{jk} < 1$, since $\delta(A_j) = \delta(\bar{A}_j) < 1$ and there exist points y_j and z_j in $\bar{A}_j$ such that

$y_j = a_{jk}$ and $z_j = b_{jk}$.

Then $d_{jk} = b_{jk}-a_{jk} \leq \sqrt{\ldots+(b_{jk}-a_{jk})^2+\ldots} = \rho(y_j,z_j) \leq \delta(\bar{\bar{E}}_j) < 1.$

Thus, $\delta(A_j) \geq \max_k d_{jk}$ so that

$[\delta(A_j)]^n \geq n(\max_k d_{jk}) \geq \prod_{k=1}^{n} d_{jk} = m(I_j)$.

But $\bar{I}_j$ A_j for $j = 1,2,\ldots,n$. We have then

$\sigma \geq \sum_{j=1}^{\infty}[\delta(A_j)]^n \geq \sum_{j=1}^{\infty} m(I_j) = \sum_{j=1}^{\infty} m(\bar{I}_j) \geq \sum_{j=1}^{\infty} m_e(A_j) \geq m_e(A)$.

But this is true for every $\sigma > 0$ so that $m_e(A) = 0$. Then by Theorem 3.2 , A is Lebesgue measurable and $m(A) = 0$.

<u>Theorem 5.3.</u> If A is a set in E_1, then $\mu_1^+(A) = m_e(A)$.

Proof: For every $\epsilon > 0$, $\mu_1^\epsilon(A) \leq \sum_{n=1}^{\infty} \delta(A_n)$ where $A = \bigcup_{n=1}^{\infty} A_n$ and $\delta(A_n) < \epsilon$. Given a $\sigma > 0$ there exists a sequence of sets $\{A_n\}$ such that $A = \bigcup_{n=1}^{\infty} A_n$, $\delta(A_n) < \epsilon$ and $\mu_1^+(A) + \frac{\sigma}{2} \geq \sum_{n=1}^{\infty} \delta(E\ \ )$. Now $\delta(A_n) < \epsilon$ so that A_n is bounded. If any of the A_n's are empty, eliminate them from the sequence. Let $a_n = \text{g.l.b. } \{A_n\}$ and $b_n = \text{l.u.b. } \{A_n\}$ and define I_n to be the open interval $(a_n - \frac{\sigma}{2^{n+2}}, b_i + \frac{\sigma}{2^{n+2}})$. Note that $\delta(A_n) = b_n - a_n$ so that $A_n \subset I_n$. Then

$$\sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} (b_n + \frac{\sigma}{2^{n+2}} - a_n + \frac{\sigma}{2^{n+2}}) = \sum_{n=1}^{\infty} \delta(A_n) + \sum_{n=1}^{\infty} \frac{\sigma}{2^{n+1}}.$$

Hence
$$\mu_1^{\epsilon}(A) + \frac{\sigma}{2} \geq \sum_{n=1}^{\infty} \delta(A_n) = \sum_{n=1}^{\infty} m(I_n) - \frac{\sigma}{2}$$
$$\geq m_e(A) - \frac{\sigma}{2}.$$

But $\mu_1^{+}(A) \geq \mu_1^{\epsilon}(A)$. Therefore,

$\mu_1^{+}(A) + \frac{\sigma}{2} \geq \mu_1^{\epsilon}(A) + \frac{\sigma}{2} \geq m_e(A) - \frac{\sigma}{2}$, or

$\mu_1^{+}(A) + \sigma \geq m_e(A)$. Since this is true for every $\sigma > 0$, $\mu_1^{+}(A) \geq m_e(A)$.

Given an $\sigma > 0$, there exists an $\epsilon > 0$, such that $\mu_1^{+}(A) - \sigma \leq \mu_1^{\epsilon}(A)$. Let $\{I_n\}$ be any sequence of open intervals such that $A \subset \bigcup_{n=1}^{\infty} I_n$ and $\delta(I_n) < \epsilon$. Then

$\mu_1^{+}(A) - \sigma \leq \mu_1^{\epsilon}(A) \leq \sum_{n=1}^{\infty} \delta(I_n) = \sum_{n=1}^{\infty} m(I_n)$ so that

$\mu_1^{+}(A) - \sigma$ is a lower bound of the sum of the measures of all countable sequences of open intervals of diameter less than ϵ. But $m_e(A)$ is the greatest lower bound of this sum. Therefore, $m_e(A) + \sigma \geq \mu_1^{+}(A)$ and since this is true for every $\sigma > 0$, $m_e(A) \geq \mu_1^+ (A)$. This combined with the reverse inequality obtained in the preceding paragraph gives $\mu_1^+ (A) = m_e (A)$.

The property of Hausdorff measure given in the following theorem makes it possible to use Hausdorff measure to define dimensionality of a set.

<u>Theorem 5.4.</u> If $\mu_p^+ (A) < \infty$ and if $q > p$, then $\mu_q^+ (A) = 0$.

Proof: Let n be any positive integer and $\{A_k\}$ a sequence of sets such that $\bigcup_{k=1}^{\infty} A_k = A$ and $\delta(A_k) < \frac{1}{n}$ for $k = 1,2,3,\ldots$. Then

$$\frac{[\delta(A_k)]^q}{[\delta(A_k)]^p} = [\delta(A_k)]^{q-p} < \left(\frac{1}{n}\right)^{q-p} \quad \text{so that}$$

$$\mu_q^{\frac{1}{n}} (A) \leq \sum_{k=1}^{\infty} [\delta(A_k)]^q < \left(\frac{1}{n}\right)^{q-p} \sum_{k=1}^{\infty} [\delta(A_k)]^p .$$

Thus $\mu_q^{\frac{1}{n}}$ is a lower bound for $(\frac{1}{n})^{q-p} \sum_{k=1}^{\infty} [\delta(A_k)]^p$ and therefore $\mu_q^{\frac{1}{n}} (A) \leq \left(\frac{1}{n}\right)^{q-p} \mu_p^{\frac{1}{n}} (A)$.

But since $q - p > 0$, $\lim_{n \to \infty} \mu_p^{\frac{1}{n}} (A) = \mu_p^+ (A) < \infty$, we have, taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} \mu_q^{\frac{1}{n}}(A) = \mu_q^+(A) \le \lim_{n\to\infty} \left(\frac{1}{n}\right)^{q-p} \mu_p^{\frac{1}{n}}(A) = 0, \text{ and}$$

since $\mu_q^+(A)$ is non-negative, $\mu_q^+(A) = 0$.

Corollary: A set A can have finite, non-zero μ_p^+ - measure for at most one value of p.

Definition 5.2. Let E be a set in (S,ρ). The Hausdorff dimension of E, denoted Hausdorff dim (E), is defined to be the least upper bound of all positive real numbers p such that $\mu_p^+(E) > 0$.

It follows then from Theorem 5.4 that if a set E has finite μ_p^+ - measure for some $p = p'$, then Hausdorff dim $(E) = p'$.

The definition of dimension of a set given in Definition 5.2 differs from the topological definition as given by Hurewicz and Wallman in their book Dimension Theory (5, p. 24). The method of defining dimension which they use is an inductive one and is given in the following definition.

Definition 5.3. The empty set and only the empty set has dimension -1.

A space S has dimension $\le n (n \ge 0)$ at a point p if p has arbitrarily small neighborhoods whose boundaries have dimension $\le n - 1$.

S has dimension $\le n$, dim $S \le n$, if S has dimension $\le n$ at each of its points.

S has dimension n at a point p if it is true that S has dimension $\le n$ at p and it is false that S has dimension $\le n-1$ at p.

S has <u>dimension</u> n if dim $S \leq n$ is true and dim $S \leq n-1$ is false.

S has <u>dimension</u> ∞ if dim $S \leq n$ is false for each n.

The following two theorems which are proved by Hurewicz and Wallman (5, p. 107) show the relationship between dimension as defined above and the definition given in Definition 5.2.

<u>Theorem 5.4.</u> For an arbitrary metric space X,
Hausdorff dim $(X) \geq$ dim (X).

<u>Theorem 5.6.</u> If X' is allowed to range over all the spaces homeomorphic to a given space X, then
g.l.b. $\{$Hausdorff dim $(X')\}$ = dim (X).

CHAPTER VI

LEBESGUE - STIELTJES MEASURE

In this chapter we shall define an outer measure which constitutes a generalization of Lebesgue outer measure and which plays a fundamental role in applications to theories of probability and statistics.

Lebesgue outer measure might be thought of as being constructed by weighting each of the open intervals in E_n according to their volumes. The idea behind the construction of a Lebesgue-Stieltjes outer measure is to obtain from a point function a more general weighting of the intervals.

The definition of a Lebesgue - Stieltjes measure in E_n is rather cumbersome, so we shall first define it for sets in E_1. Then the generalization to sets in E_n will contain the same basic ideas.

Definition 6.1. Let f be a monotone increasing, everywhere finite, real valued function whose domain is the real line E_1 which is continuous on the right at every point. For every half open interval $(a,b]$, define $P_0(\ (a,b]\ ) = f(b) - f(a)$. Let $(I) = (\ (a,b)\ )$ be the class of open bounded intervals. For each I define the function $P(I)$ to be $P_0(\ (a,b]\ )$. For any set A in E_1 the Lebesgue-Stieltjes outer measure of A induced by f is defined to be

$\mu_f^+(A) = \text{g.l.b.} \{ \sum_{M=1}^{\infty} P(I_n) \}$ for all sequences $\{I_n\}$ of open intervals in E_1 such that $A \subset \bigcup_{M=1}^{\infty} I_n$. The function f is called a <u>distribution function</u> for μ_f^+.

Note that the function $P(I_n)$ is always non-negative so that the sequence of numbers $\{ \sum_{n=1}^{\infty} P(I_n) \}$ is bounded below. Thus, the greatest lower bound of this sequence does exist.

<u>Theorem 6.1.</u> μ_f^+ is a Caratheodory outer measure.

Proof: μ_f^+ is defined for all sets in E_1 and since the function f is monotone increasing, $P \geq 0$ and hence, $\mu_f^+ \geq 0$. If A and B are two sets such that $A \subset B$, then given an $\epsilon > 0$ there exists a sequence of open intervals $\{I_n\}$ such that $\bigcup_{n=1}^{\infty} I_n \supset B \supset A$ and $\mu_f^+(B) + \epsilon \geq \sum_{n=1}^{\infty} P(I_n)$. But $A \subset \bigcup_{n=1}^{\infty} I_n$, so that $\mu_f^+(A) \leq \sum_{n=1}^{\infty} P(I_n) \leq \mu^+(B) + \epsilon$. Since this is true for every $\epsilon > 0$, $\mu_f^+(A) \leq \mu_f^+(B)$ and C-1 of Definition 3.1 is satisfied. Let $\{A_n\}$ be any sequence of sets from E_1 and let $A = \bigcup_{n=1}^{\infty} A_n$. For each positive integer n and $G > 0$ there exists a sequence $\{I_n\}$ of open intervals such that $A_n \subset \bigcup_{k=1}^{\infty} I_{nk}$ and

$$\mu_f^+(A_n) + \epsilon/2^n \geq \sum_{k=1}^{\infty} P(I_{nk}) .$$

Now $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{nk} \supset \bigcup_{n=1}^{\infty} A_n \supset A$ so that $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{nk}$ is a countable sum of open intervals covering A . Then

$$\mu_f^+ (A) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} P(I_{nk})$$
$$\leq \sum_{n=1}^{\infty} [\mu_f^+ (A_n) + \epsilon/2^n] = \sum_{n=1}^{\infty} \mu_f^+ (A_n) + G .$$

But this is true for every ϵ so that

$\mu_f^+ (A) \leq \sum_{n=1}^{\infty} \mu_f^+ (A_n)$ and condition C-2 is satisfied.

Let A and B be two sets such that $\rho(A,B) = d > 0$. Given an $\epsilon > 0$, there exists a sequence of open intervals $\{I_n\}$ such that $\bigcup_{n=1}^{\infty} I_n \supset (A \cup B)$ and

1.) $\mu_f^+(A \quad B) + \epsilon/2 \geq \sum_{n=1}^{\infty} P(I_n)$. Now for each open interval $I_n = (a_n, b_n)$ form a partition of I_n by points x_{ni} such that $a_n = x_{n0} < x_{n1} < \ldots < x_{nm} = b_n$ and $x_{ni} - x_{n,i-1} < \frac{d}{n+1}$ for $i=1,2,\ldots, m$. Then $\bigcup_{i=1}^{m} (x_{n,i-1}, x_{ni}] = (a_n, b_n] \supset I_n$ and

2.) $\sum_{i=1}^{m} P_0((x_{n,i-1}, x_{ni}]) = f(x_m) - f(x_0) = P(I_n)$.

I_n is not covered by the open intervals $(x_{n,i-1}, x_{ni})$, but by extending each of them to the right to a point x'_{ni} obtaining an open covering $\bigcup_{i=1}^{m} (x_{n,i-1}, x'_{n,i})$ of I_n where

$x'_{ni}-x_{n,i-1} \leq \frac{d}{n+1}$, for $i = 1,2,\ldots,m$. Since f is right continuous we can choose the points x'_{ni} so that, given an $\epsilon > 0$, $f(x'_{ni}) - f(x_{n,i}) < \frac{\epsilon}{m2^{n+1}}$.

Then
$$P(\,(x_{n,i-1},x'_{ni})\,) = f(x'_{ni}) - f(x_{n,i-1})$$
$$= f(x'_{ni}) - f(x_{ni}) + f(x_{ni}) - f(x_{n,i-1})$$
$$< \frac{\epsilon}{m2^{n+1}} + f(x_{ni}) - f(x_{n,i-1})$$
$$= P_0((x_{n,i-1},x_{ni}) + \frac{\epsilon}{m2^{n+1}} \quad .$$

Using this last result together with 2.), we have

$$3.)\quad \sum_{i=1}^{m} P((x_{n,i-1},x'_{ni}))< \sum_{i=1}^{m} [P_0((x_{n,i-1},x_{ni}]) + \frac{\epsilon}{m2^{n+1}}]=P(I_n)+ \frac{\epsilon}{2^{n+1}} \quad .$$

Now we have

$$\bigcup_{i=1}^{m} (x_{n,i-1},x'_{ni}) \supset I_n$$ so that

$$\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{m} (x_{,i-1},x'_{n,i}) \supset \bigcup_{n=1}^{\infty} I_n \supset (A \cup B) .$$ But since

$x'_{ni} - x_{n,i-1} \leq \frac{d}{n+1} < d = \rho(A,B)$, no interval $(x_{n,i-1},x'_{n,i})$ contains points of both A and B. Therefore part of $\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{m} (x_{n,i-1},x'_{n,i})$ covers A and the rest covers B so that

$\mu_f^+(A) + \mu_f^+(B) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{m} P(x_{n,i-1}, x'_{n,1})$. Using this and

equations 3.) and 1.), we have

$\mu_f^+(A) + \mu_f^+(B) \leq \mu_f^+(A \quad B) + \epsilon$. Thus

$\mu_f^+(A) + \mu_f^+(B) \leq \mu_f^+(A \cup B)$. The reverse inequality

covers from C-2 so that $\mu_f^+(A) + \mu_f^+(B) = \mu_f^+(A \cup B)$, and

C-3 is satisfied. C-4 is obviously satisfied since

$P(A)$, where A is the empty set, is defined to be zero.

Before proceding to the generalization of μ_f^+ to sets in E_n, we might state two theorems which point out something of the significance of the distribution function f.

Theorem 6.2. If two distribution functions f_1 and f_2 yield the same Lebesgue-Stieltjes outer measure, then $f_1 - f_2$ is a constant (2, p.53).

Theorem 6.3. If f is a distribution function and μ_f^+ is the Lebesgue-Stieltjes measure induced by f, then for any half-open interval $(a,b]$, $\mu_f^+((a,b]) = f(b) - f(a)$, (8, p. 117).

The construction of a Lebesgue-Stieltjes measure in E_n from a distribution function which is a product of n real variables is similar to the one-dimensional case. A half-open interval $(a,b]$ in E_n is a set of the form $\{(x_1, x_2, \ldots, x_n)\}$ where $a_k < x_k \leq b_k$ for $k=1,2,\ldots,n$. In order to define a function $P_0((a,b])$, let $f(x) = f(x,\ldots,x_n)$ and define a set of difference operators $\Delta_1, \Delta_2, \ldots, \Delta_n$ such that

$\Delta_k(f) = f(x_1,\ldots,b_k,\ldots x_n) - f(x_1,\ldots,a_k,\ldots,x_n)$ for $k=1,2,\ldots,n$.

Define $P_0((a,b]) = \Delta_1(\Delta_2(\ldots(\Delta_n(f))\ldots))$. For each n-dimensional open interval $I = (a,b)$, define $P(I) = P_0((a,b])$.

Definition 6.2. If $f(x) = f(x_1,\ldots,x_n)$ is a function such that, for any half-open interval $(a,b]$, $P_0((a,b])$ is non-negative and f is continuous on the right in each variable $x_i, i=1=1,2,\ldots n$, separately and if A is any set in E_n, then the Lebesgue-Stieltjes outer measure of A induced by f is defined to be

$\mu_f^+(A) = \text{g.l.b.} \{ \sum_{n=1}^{\infty} P(I_n)\}$, where $\{I_n\}$ is a sequence of open intervals such that $A \subset \bigcup_{n=1}^{\infty} I_n$.

Theorem 6.4. μ_f^+ is a Carathéodory outer measure.

The proof of this theorem is similar to that of Theorem 6.1. It should be noted that in Theorem 6.1 the monotonicity of f insured that P_0 be non-negative and hence, $\mu_f^+ \geq 0$. However, in Definition 6.2, we had to require that f yield a non-negative function P_0 since monotonicity of f in each variable separately does not guarantee that P_0 be non-negative. For example let f be a function of two variables defined by

$$f(x,y) = \begin{cases} x + y, & \text{when } x + y < 0 \\ 0, & \text{when } x + y \geq 0 . \end{cases}$$

The function f is monotone in x and in y . For if $x_1 < x_2$,

then either $x_2 + y < 0$ or $x_2 + y \geq 0$. If $x_2 + y < 0$, then $x_1 + y < 0$ and $f(x_1,y) = x_1 + y < x_2 + y = f(x_2,y)$. If $x_2 + y \geq 0$, then either $x_1 + y < 0$ or $x_1 + y \geq 0$. If $x_1 + y < 0$, then $f(x_1,y) = x_1 + y < 0 = f(x_2,y)$. If $x_1 + y \geq 0$, then $f(x,y) = 0 = f(x_2,y)$. In all cases then, $x_1 < x_2$ implies that $f(x,y) \leq f(x_2,y)$. Similarly, $y_1 < y_2$ implies that $f(x,y_1) \leq f(x,y_2)$. However, if I is the half open interval defined by $I = \{(x,y)\}$ where $0 < x \leq 1$ and $-1 < y \leq 0$, $P_0(I)$ will be negative. For $P_0(I) = \Delta_x(\Delta_y(f))$
$= \Delta_x(f(x,0) - f(x,-1)) = f(1,0) - f(1,-1) - f(0,0) + f(0,-1)$
$= 0 - 0 - 0 + (0 -1) = -1$.

It follows from Theorem 6.4. that μ_f^+ determines a class of sets which are measurable in the sense of Definition 3.2 and that the Borel sets are a sub-class of this class of measurable sets.

That Lebesgue outer measure is a special case of Lebesgue-Stieltjes outer measure is shown by the following theorem.

<u>Theorem 6.5.</u> If $f(x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} x_i$, then the Lebesgue-Stieltjes measure induced by f is identically equal to Lebesgue outer measure.

Proof: For any half-open interval $(a,b]$,

$$P_0((a,b]) = \Delta_1(\Delta_2(\ldots(\Delta_n \prod_{i=1}^{n} x_i)\ldots)) = \prod_{i=1}^{n}(b_i - a_i) .$$

Then for any open interval $I = (a,b)$,

$$P(I) = P_0((a,b]) = \prod_{i=1}^{n} (b_i - a_i) = m_e(I) .$$

Thus for any set A in E_n, and any sequence of open intervals $\{I_n\}$ such that $A \subset \bigcup_{n=1}^{\infty} I_n$,

$$\mu_f^+(A) = \text{g.l.b.} \left\{ \sum_{n=1}^{\infty} P(I_n) \right\} = \text{g.l.b.} \left\{ \sum_{n=1}^{\infty} m_e(I_n) \right\} .$$ But this is precisely the definition of Lebesgue outer measure of A as given in Definition 4.3. Therefore, $\mu_f^+(A) = m_e(A)$.

CHAPTER VII

HAAR MEASURE

Alfred Haar developed a theory of measure which defines a measure in a locally compact separable metric space for which the notion of congruent sets is defined. This measure is a Caratheodory outer measure and has an important application in that it defines a measure in a locally compact separable metric space which has the algebraic structure of a group.

Definition 7.1. A set A is compact if every infinite subset of A has at least one limit point in A. A space S is locally compact if every x of S has neighborhood N_x whose closure is compact.

Definition 7.2. Let (S,ρ) be a locally compact separable metric space and let $(\vee)$ denote the class of open sets in (S,ρ) whose closures are compact. For sets in (S,ρ) the notion of congruence, denoted $\cong$, is defined so as to fulfil the following conditions:

S-1. $A \cong B$ implies $B \cong A$.

S-2. $A \cong B$ and $B \cong C$ imply $A \cong C$.

S-3. If $A \cong B$ and A is an element of $(\vee)$, then B is also an element of $(\vee)$.

S-4. If $A \cong B$ and $\{A_n\}$ is a sequence of sets belonging to $(\vee)$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$, then there exists a sequence of sets $\{B_n\}$ such that $B \subset \bigcup_{n=1}^{\infty} B_n$ and $A_n \cong B_n$ for $n = 1,2,\ldots$.

S-5. For any set A belonging to $(\vee)$ the class of sets which are congruent to A covers the space (S,ρ).

S-6. If $\{S_n\}$ is a sequence of concentric neighborhoods whose closures are compact and with radii approaching zero, and if $\{G_n\}$ is a sequence of sets such that $G_n \cong S_n$, $n = 1,2,\ldots$, then the relations $a = \lim_n a_n$ and $b = \lim_n b_n$, where a_n and b_n are elements of G_n, imply $a = b$.

Definition 7.3. A measure μ defined for all sets in the space (S,ρ) is called a Haar measure if it satisfies the following conditions:

H-1. μ is a Carathéodory outer measure.

H-2. If $A \cong B$, then $\mu(A) = \mu(B)$.

H-3. For every non-empty open set C whose closure is compact, $0 < \mu(C) < \infty$.

In order to construct a measure function which will be a Haar measure let A and B be two sets belonging to $(\vee)$. By S-5, the class of sets congruent to A covers the set $\bar{B}$ and by S-3, this class of sets consists of open sets. Since $\bar{B}$ is compact, by the Heine-Borel Theorem, there exists a finite collection of sets congruent to A which covers $\bar{B}$. Let $(B:A)$ denote the smallest number of sets which constitute such a collection. It follows then that for any sets A, B, C which belong to $(\vee)$, $(B:A) \leq (B:C)(C:A)$

Now let G be an element of (V) and $\{S_n\}$ a contracting sequence of concentric open spheres in G. Then the closure of each S_n is also compact. For any open set A whose closure is compact, define $l_n(A) = \dfrac{(A:S_n)}{(G:S_n)}$. Then since

$(A:S_n) \leq (A:G)\cdot(G\cdot S_n)$ and $(G:S_n) \leq (G:A)\cdot(A:S_n)$, it follows that

$l_n(A) = \dfrac{(A:S_n)}{(G:S_n)} \leq (A:G)$ and $l_n(A) = \dfrac{(A:S_n)}{(G:S_n)} \geq \dfrac{1}{(G:A)}$. Therefore,

$\dfrac{1}{(G:A)} \leq l_n(A) \leq (A:G)$ for $n = 1,2,\ldots$. Thus $\{l_n(A)\}$ is a bounded sequence of real numbers so that we can make use of the following theorem.

<u>Theorem 7.1.</u> To every bounded sequence $\{x_n\}$ of real numbers there corresponds a number $\underset{n}{\mathrm{Lim}}\, x_n$ (called the generalized limit of the sequence $\{x_n\}$, which has the following properties (8, pp. 58 - 59) and (4, p. 316):

7.1-1. $\underset{n}{\mathrm{Lim}}\,(ax_n + by_n) = a\,\underset{n}{\mathrm{Lim}}\, x_n + b\,\underset{n}{\mathrm{Lim}}\, y_n$.

7.1-2. If $x_n \geq 0$ for every n, then $\underset{n}{\mathrm{Lim}}\, x_n \geq 0$.

7.1-3. $\underset{n}{\mathrm{Lim}}\, x_{n+1} = \underset{n}{\mathrm{Lim}}\, x_n$.

7.1-4. If $x_n = 1$ for every n, then $\underset{n}{\mathrm{Lim}}\, x_n = 1$.

7.1-5. $\underset{n}{\underline{\lim}}\, x_n \leq \mathrm{Lim}\, x_n \leq \underset{n}{\overline{\lim}}\, x_n$.

Using this theorem then, for every set A belonging to (V) let $l(A) = \underset{n}{\mathrm{Lim}}\, l_n(A)$. For any arbitrary set X in (S,ρ), define $\mu(X) = \text{g.l.b.}\ \{\sum_{n=1}^{\infty} l(A_n)\}$ for all sequences $\{A_n\}$ of sets belonging to $(\mathbf{V})$ such that $X \subset \bigcup_{n=1}^{\infty} A_n$. If X is the empty set, define $\mu(X) = 0$ (1, pp. 314 - 316).

In order to establish that μ as defined in the preceding paragraph is a Carathéodory outer measure, we prove several properties of the function l.

Theorem 7.2. Let the function l be as defined above. Then for sets A and B which are elements of $(\mathbf{V})$ the following properties hold:

7.2-1. $0 < l(A)\ \infty$.

7.2-2. $A \subset B$ implies $l(A) \leq l(B)$.

7.2-3. $l(A \cup B) \leq l(A) + l(B)$.

7.2-4. $\rho(A,B) > 0$ implies $l(A \cup B) = l(A) + l(B)$.

Proof: Since each $l_n(A)$ is bounded below by a positive number, let $l_n(A) \geq \epsilon > 0$. Then for some non-negative number p_n, $l_n(A) = \epsilon + p_n$. Then $\frac{1}{\epsilon} l_n(A) - \frac{1}{\epsilon} p_n = 1$ and by Theorem 7.1-4, and 7.1-1,

$\operatorname{Lim}_n(\frac{1}{\epsilon} l_n(A) - \frac{1}{\epsilon} p_n) = \frac{1}{\epsilon} \operatorname{Lim}_n l_n(A) - \frac{1}{\epsilon} \operatorname{Lim}_n p_n = 1$. Thus

$\operatorname{Lim}_n l_n(A) = \epsilon + \operatorname{Lim}_n p_n$. But since $p_n \geq 0$, it follows from Theorem 7.1-2 that $\operatorname{Lim}_n p_n \geq 0$. Therefore,

$\operatorname{Lim}_n l_n(A) \geq \epsilon > 0$ and $l(A) > 0$. Also, from Theorem 7.1, $l(A) < \infty$.

If $A \subset B$, then $(A:S_n) \leq (B:S_n)$ so that

$$\ln(A) = \frac{(A:S_n)}{(G:S_n)} \leq \frac{(B:S_n)}{(G:S_n)} \leq l_n(B),$$ where S_n and G

are as defined on page 42, lines 1 and 2. Then, from Theorem 7.1 and the definition of l, $l(A) \leq l(B)$.

If A and B are sets from $(\vee)$, then

$(A \cup B:S_n) \leq (A:S_n) + (B:S_n)$ and

$$l_n(A \cup B) = \frac{(A \cup B:S_n)}{(G:S_n)} \leq \frac{(A:S_n)}{(G:S_n)} + \frac{(B:S_n)}{(G:S_n)} = l_n(A) + l_n(B).$$

It follows then, from Theorem 7.1, that

$l(A \cup B) \leq l(A) + l(B)$.

If $\rho(A,B) = d > 0$, then for $d > 1/n$ and $\{S_n\}$ a sequence of concentric spheres with radius $< 1/n$ it follows that $(A \cup B:S_n) = (A:S_n) + (B:S_n)$. Thus for $n > 1/d$,

$$l_n(A \cup B) = \frac{(A \cup B:S_n)}{(G:S_n)} = \frac{(A:S_n)}{(G:S_n)} + \frac{(B:S_n)}{(G:S_n)} = l_n(A) + l_n(B).$$

Using Theorem 7.1-3 and 7.1-1, it follows that

$l(A \cup B) = l(A) + l(B)$.

<u>Theorem 7.3.</u> $\mu(X)$ is a Haar measure.

Proof: Referring to Definition 7.3, we must show that μ is a Carathéodory outer measure in order to satisfy H-1. First, we have $\mu(X)$ defined for all X in (S,ρ) and also $\mu(X) \geq 0$ since for any sequence $\{A_n\}$ of open sets whose closures are compact $l_n(A_n) \geq 0$; hence by Theorem 7.1-2, $l(A_n) \geq 0$. If $A \subset B$, then $\mu(A) \leq \mu(B)$.

For given an $\epsilon > 0$, there exists a sequence $\{B_n\}$ of sets belonging to $(\vee)$ such that $B \subset \bigcup_{n=1}^{\infty} B_n$ and $\mu(B) + \epsilon \geq \sum_{n=1}^{\infty} l(B_n)$. But since $\bigcup_{n=1}^{\infty} B_n \supset A$, $\mu(A) \leq \sum_{n=1}^{\infty} l(B_n)$. Therefore $\mu(A) \leq \mu(B) + \epsilon$. This is true for every $\epsilon > 0$ so that $\mu(A) \leq \mu(B)$. If $X = \bigcup_{n=1}^{\infty} X_n$, then $\mu(X) \leq \sum_{n=1}^{\infty} \mu(X_n)$. For let $\{X_{nk}\}$ be a sequence of sets belonging to $(\vee)$ such that, given $\epsilon > 0$, for each n, $\bigcup_{k=1}^{\infty} X_{nk} \supset X_n$ and $\sum_{k=1}^{\infty} l(X_{nk}) \leq \mu(X_n) + \epsilon/2^n$. Now $X = \bigcup_{n=1}^{\infty} X_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} X_{nk}$, a countable union of open sets. Therefore,

$$\mu(X) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} l(X_{nk}) \leq \sum_{n=1}^{\infty} [\mu(X_n) + \epsilon/2^n] \leq \sum_{n=1}^{\infty} \mu(X_n) + G.$$

But this is true for any $\epsilon > 0$, so that $\mu(X) \leq \sum_{n=1}^{\infty} \mu(X_n)$. If $\rho(A,B) > 0$, then $\mu(A \cup B) = \mu(A) + \mu(B)$. For let G_1 and G_2 be open sets such that $G_1 \supset A$ and $G_2 \supset B$, and $\rho(G_1, G_2) > 0$. Given an $\epsilon > 0$ there exists a sequence of sets $\{E_n\}$ belonging to $(\vee)$ such that $(A \cup B) \subset \bigcup_{n=1}^{\infty} E_n$ and $\mu(A \cup B) + \epsilon \geq \sum_{n=1}^{\infty} l(E_n)$. Let $E_{1n} = E_n \cap G_1$ and $E_{2n} = E_n \cap G_2$. Then $\rho(E_{1n}, E_{2n}) > 0$ so that, from from Theorem 7.2-4, $l(E_{1n} \cup E_{2n}) = l(E_{1n}) + l(E_{2n})$.

Also $(E_{1n} \cup E_{2n}) \subset E_n$ so that, from Theorem 7.2-2,

$l(E_{1n} \cup E_{2n}) = l(E_{1n}) + l(E_{2n}) \leq l(E_n)$. Now since
$A \subset \bigcup_{n=1}^{\infty} E_{1n}$ and $B \subset \bigcup_{n=1}^{\infty} E_{2n}$,
$\mu(A) \leq \sum_{n=1}^{\infty} l(E_{1n})$ and $\mu(B) \leq \sum_{n=1}^{\infty} l(E_{2n})$. Therefore,

$$\mu(A) + \mu(B) \leq \sum_{n=1}^{\infty} l(E_{1n}) + \sum_{n=1}^{\infty} l(E_{2n}) \leq \sum_{n=1}^{\infty} l(E_n) \leq \mu(A \cup B) + \epsilon.$$

But since this holds for every $\epsilon > 0$, $\mu(A) + \mu(B) \leq \mu(A \cup B)$. The reverse inequality was proved above. Thus, $\mu(A) + \mu(B) = \mu(A \cup B)$. If A is the empty set, then by definition, $\mu(A) = 0$. Thus postulate H-1 of Definition 7.3 has been satisfied, i.e., μ is a Caratheodory outer measure.

To prove H-2, let $A \cong B$ and let $\{B_n\}$ be any sequence of sets from (V) such that $B \subset \bigcup_{n=1}^{\infty} B_n$. Then from S-4, there exists a sequence of sets $\{A_n\}$ such that $A \subset \bigcup_{n=1}^{\infty} A_n$ and $A_n \cong B_n$ for $n=1,2,3,\ldots$. From S-3, the sets A_n also belong to (V). Then, since $l_n(B_n) = l_n(A_n)$, it follows that $l(B_n) = l(A_n)$ and $\mu(A) \leq \sum_{n=1}^{\infty} l(A_n) = \sum_{n=1}^{\infty} l(B_n)$. Thus $\mu(A)$ is a lower bound of $\{\sum_{n=1}^{\infty} l(B_n)\}$ so that $\mu(A) \leq \mu(B)$. Similarly, by letting $\{A_n\}$ be any sequence of sets from (V) such that $A \subset \bigcup_{n=1}^{\infty} A_n$ we obtain a sequence of sets $\{B_n\}$ from

$(\mathcal{V})$ such that $B \subset \bigcup_{n=1}^{\infty} B_n$ and $A_n \cong B_n$. From this we obtain $\mu(B) \leq \mu(A)$. Therefore, $\mu(A) = \mu(B)$.

To prove H-3, let C be any non-empty set belonging to $(\mathcal{V})$. From Theorem 7.2-1, $0 < l(C) < \infty$. Then $\mu(C) \leq l(C) < \infty$. Let x be an element of C and ϵ be any positive number such that $N(x,\epsilon) \subset C$. Let $H = N(x, \epsilon/2)$. Given $\sigma > 0$, there exists a sequence $\{A_n\}$ of sets belonging to $(\mathcal{V})$ such that $\bigcup_{n=1}^{\infty} A_n \supset C \supset \bar{H}$ and $\sum_{n=1}^{\infty} l(A_n) \leq \mu(C) + \sigma$. Since the space (S,ρ) is locally compact, $\bar{H}$ is compact. Therefore, there exists a finite collection of the sets A_n which covers $\bar{H}$, hence H. Thus we have

$$l(H) \leq l\left(\bigcup_{k=1}^{m} A_{n_k}\right) \leq \sum_{k=1}^{m} l(A_{n_k}) \leq \sum_{n=1}^{\infty} l(A_n) \leq \mu(C) + \sigma.$$

But since $l(H) > 0$ and $\mu(A) \geq l(A) > 0$, $\mu(X) > 0$. Therefore, H-3 is proved.

A set is called measurable with respect to the Haar measure μ if it satisfies Definition 3.2 with $\mu^{+} = \mu$. Thus the Haar measure μ determines an additive class of sets in (S,ρ) which are μ-measurable. Also since μ is a Carathéodory measure, Borel sets are μ-measurable.

To give an example of a space in which a Haar measure is defined, let (S,ρ) be a locally compact separable metric space which constitutes a group, i.e., for every pair x, y of elements in (S, ρ), there is also an element xy in (S, ρ) called product such that the following conditions are satisfied:

G-1. For every x, y, and z in (S, ρ), $(xy)z = x(yz)$.

G-2. There exists in (S, ρ) a unique identity element e such that $xe = ex = x$ for every x in (S, ρ).

G-3. To each element x in (S, ρ) there corresponds a unique inverse x^{-1} in (S, ρ) such that $x^{-1}x = xx^{-1} = e$.

Since we want the algebraic operation of multiplication to be continuous in (S, ρ), suppose that (S, ρ) fulfils the additional conditions:

G-4. If $\lim_n x_n = x$ and $\lim_n y_n = y$, then $\lim_n x_n y_n = xy$.

G-5. If $\lim_n x_n = x$, then $\lim x_n^{-1} = x^{-1}$.

If x is an element of (S, ρ) and A (S, ρ), define

$$xA = \{xz\} \text{ for all } z \text{ in } A$$

$$Ax = \{zx\} \text{ for all } z \text{ in } A.$$

The sets xA and Ax are called the left translation and right translation of A by x respectively. Also for A (S, ρ) and B (S, ρ), define

$$AB = \{xy\} \text{ for all } x \text{ in } A \text{ and } y \text{ in } B$$

$$A^{-1} = \{x^{-1}\} \text{ for all } x \text{ in } A.$$

<u>Definition 7.4.</u> A measure μ defined on the Borel sets of (S, ρ) is called a <u>left Haar measure</u> if it satisfies the following conditions:

LH-1. μ is invariant under left translations,

i.e., for every x in (S,ρ) and A an element of (B), $\mu(xA) = \mu(A)$.

LH-2. For every compact set C, $\mu(C) < \infty$.

LH-3. For every open non-empty set G, $\mu(G) > 0$.

A <u>right Haar measure</u> is one for which LH-1 postulates invariance under right translations.

Two sets A and B in (S,ρ) are congruent if there exists an element x such that $A = xB$. This satisfies the definition of congruence given in Definition 7.2.

<u>Theorem 7.4.</u> The restriction of the measure μ as defined on page 43 to the Borel sets is a left Haar measure.

Proof: By Theorem 7.3, μ is a Haar measure and hence satisfies Definition 7.3. Then by H-1, μ is a Carathéodory outer measure so that the Borel sets are μ-measurable. LH-1 is satisfied by H-2 and LH-2 follows from H-3. The proof of LH-3 is similar to the proof given in Theorem 7.3 to show that μ satisfied H-3.

In particular, when (S,ρ) is Euclidean space E_n and group multiplication is interpreted to be ordinary addition, the left Haar measure of Borel sets is invariant under translations. For let $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ be two points in E_n. Denote the sum $x + y$ by the point $(x_1 + y_1,\ldots,x_n + y_n)$. If X is any set in E_n and a any point from E_n, let $X^{(a)}$ denote the set of points of the form $a + x$ where x in an element of X. $X^{(a)}$ is called the translation of X by a. Let group multiplication be interpreted to be addition. This satisfies G-1 to G-5 where the identity element is taken to be $0 = (0,\ldots,0)$. Then for any set A which is a Borel set and any x in E_n, $xA = A^{(x)}$ and it follows from Theorem 7.4 that $\mu(xA) = \mu(A^{(x)})$.

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