AN ABSTRACT OF THE THESIS OF

Antje Höring for the degree of Master of Science in Physics

presented on August 31, 1988.

Title: Application of the Schematic Model to Four-Quasiparticle States

Abstract approved: Redacted for Privacy

V. A. Madsen

The Tamm-Dancoff equation of motion for four BCS quasiparticles is derived. It is solved in the limit of degenerate energy levels using a seperable interaction. The result is then applied to the coupling between collective states with two and four quasiparticles, which corresponds to a coupling between collective states with one and two phonons.

A case of special interest involves the coupling of the collective two-quasiparticle state to the collective four-quasiparticle state. This interaction represents the leading-order anharmonic effect in the vibrational motion. Its effect on the energy level diagram and the strength of the crossover transition $2_2^+ \rightarrow 0_1^+$ are investigated for ^{106}Pd, ^{108}Pd, ^{110}Pd, ^{74}Se, ^{76}Se and ^{78}Se.

Application of the Schematic Model to Four-Quasiparticle States

by

Antje Höring

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

Completed August 31, 1988

Commencement June 1989

Approved:

Redacted for Privacy

Professor of Physics in charge of major

Redacted for Privacy

Chairman of Department of Physics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented August, 31, 1988

Typed by Antje Höring for Antje Höring

ACKNOWLEDGEMENT

This thesis is dedicated to my fiance, Scott Henderson, who died last Febuary. I wish, he could have seen the completion of this work.

First and foremost I wish to thank my adviser, Dr. Victor A. Madsen; for his never-ending patience with all my questions; for his guidance, councel, support and help throughout the two years; for being an excellent teacher and the best major professor I could have wished for.

Special thanks to Larry Winiarski for his help with the computer programming. Thanks also to Dr. Philip J. Siemens for enlightening discussions and to Dr. Donald Henderson for helping me with my English.

I am very grateful to Dr. Henry J. F. Jansen, Dr. Rubin H. Landau and to Dr. Kenneth S. Krane for reading my thesis and making helpful suggestions.

Last, but far from least I want to thank my parents for whom five thousand miles are nothing when they are needed. This work would not have been possible without their encouragement, their moral and financial support.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

APPLICATION OF THE SCHEMATIC MODEL TO FOUR-QUASIPARTICLE STATES

I. INTRODUCTION

In the few decades that nuclear physics has existed large progress has been made in both explaining nuclear structure and nuclear reactions. Nuclear structure theory can be subdivided into macroscopic models, such as the liquid drop model, and microscopic models, such as the nuclear shell model.

The liquid drop model of the nucleus was historically the first model to be proposed as an explanation of the different properties of the nucleus. To describe the qualitative features of the nucleus, especially collective motions, it is still generally accepted and above all it gives transparent results. It is therefore a good first order check for any microscopic model that describes collective phenomena.

Microscopic models work best for one body excitation, which can be described using an average potential and assuming that the nucleons move independently obeying the Pauli principle. However, it turns out that collective excitations can be explained only if we suppose that coherent participation by many nucleons takes places in the nucleus. The independent shell model fails to describe such states. If the shell-model states are to be obtained as basis states, a residual interaction has to be used. In general, collective phenomena cannot be solved exactly, so approximations have to be made. Using realistic

interactions, including a large number of possible particle configurations, requires large numerical calculations and the transparency of the original problem and its results is usually lost. However, there are several problems that can be solved analytically to a reasonably good approximation.

In this first chapter we will discuss the background needed to understand the calculations to follow.

1.1 Macroscopic Description of Collective Phenomena

Depending on the number of valence nucleons there are different kinds of nuclear excitation spectra. One obtains the spectrum of a one-particle excitation for closed shell nuclei, if there is only one nucleon outside the major shell, or if only one nucleon is missing in the major shell. As the number of valence nucleons increases toward the middel of the shell the effect of the long-range nuclear force increases and a collective motion of all nucleons arises. An important collective motion for a spherically symmetric nucleus is the surface vibration, which will be discussed in this study. In the nuclear drop model, surface vibrations correspond to small elastic vibrations in the shape of the drop about its spherical equilibrium. If the number of valence nuleons is increased slightly away from a closed shell, the equilibrium state of the nucleus is still a sphere (due to the pairing force) but deformations are possible as vibrations about its spherical shape. The frequency and therefore the energy of

the surface vibration then decreases and the vibrational states are observable in a spectrum. The collective quadrupole motion appears to be best developed and well separated from single-particle motion for nuclei at least four protons and four neutrons (or four proton holes and four neutron holes) away from closed proton and neutron shells.[1]

If the number of nucleons increases even more the tendency of deformation increases. Finally, the spherical shape of the nucleus becomes unstable and an elliptical equilibrium arises. The deformed nucleus can now rotate as a whole, which yields a rotational spectrum.

The collective states discussed so far are part of the low energy spectrum. At considerably higher energies of 10 - 20 MeV strong, broad collective excitations appear, which are referred to as giant resonances. The structure of the giant resonances varies smoothly through magic numbers and shows no difference between nuclei with even and odd numbers of nucleons. The excitation thus involves the entire nucleus, and not just a few nucleons in the outermost shell. Although our formalism can be applied to the giant resonance states it will not be a subject of this study.

In order to describe the vibrational states in the liquid drop model we will first parametrize the surface of the vibrating drop.[2]

$$R(\vartheta,\varphi) = R_0\left(1+\sum_{\lambda,\mu} a_{\lambda\mu}\, Y^{*}_{\lambda\mu}(\vartheta,\varphi)\right) \tag{1.1.1}$$

Since λ=0 gives the spherical shape and λ=1 describes a translation of the sphere, λ=2 is the lowest interesting order in the expansion,

which is a quadrupole vibration. For $\lambda=2$ μ can have five different values (2,1,0,-1,-2) corresponding to five independent vibrations. If we limit ourselves to small deformations each amplitude $a_{\lambda\mu}$ oscillates. Each oscillation corresponds to a vibrational energy

$$H_{vib} = \frac{1}{2} B_\lambda \dot{a}^2_{\lambda\mu} + \frac{1}{2} C_\lambda q^2_{\lambda\mu} \qquad (1.1.2)$$

where B_λ are the inertia parameters and C_λ are the stiffness parameters. They characterize the inertia of the collective motion and the stiffness of the collective potential energy, respectively. Since the Hamiltonian has to be a rotational invariant, the constants are independent of μ. The frequency of each vibration is given by

$$\omega_\lambda = \sqrt{\frac{C_\lambda}{B_\lambda}} \qquad (1.1.3)$$

and their energy is given by

$$E_\lambda = \hbar \omega_\lambda \qquad (1.1.4)$$

The quantum mechanical vibrational spectrum of the individual oscillators is given by a series of states with equal distances $\hbar\omega$. Each phonon of a quadrupole vibration with $\lambda=2$ has an angular momentum quantum number of 2. Since phonons are bosons, only symmetric total wavefunctions are possible. The degeneracy of the energy levels vanishes for anharmonic vibrations.

1.2 Collective Phenomena and the Tamm-Dancoff Approximation

In the last two decades considerable research has been done in the "microscopic" description of collective excitations of a many-body system. In a quantal system like a nucleus, density variations occur due to transitions of one or more particles between different states. When a particle is excited (a particle-hole state created out of the ground state), the corresponding fluctuations in the nuclear field affect the motion of the other particles and tend to generate other particle-hole excitations. Thus, because of the interaction between the particles through the field, the randomly distributed fluctuations from different single-particle excitations come in phase, and a more or less collective movement of the particles, a vibration, arises. We shall study the quadrupole vibrations which are known from experiment to be the most collective among all vibrations.

Spherical even-even nuclei will now be considered. The particles are supposed to move in a one-body potential, interacting by a short-range and a long-range force. Thus the Hamiltonian is

$$H = H(\text{shell mod.}) + H(\text{short range}) + H(\text{long range}) \qquad (1.2.1)$$

The advantage of the Tamm-Dancoff approximation (TDA) is that the fermion character of the nucleons is preserved, whereas in most other microscopic collective models, such as the interacting boson model[3] or the various boson expansion models as used for example by S. G. Lie and G. Holzwarth[1], one deals with nucleon pairs, which are approximately treated as bosons.

The Tamm-Dancoff equation diagonalizes the shell model and the long-range part of the Hamiltonian. The ansatz of the Tamm-Dancoff secular equation is the following. If the shell model potential is filled with A nucleons up to a certain Fermi level, then all zero, one, two, three, four,...,N particle shell model excitations form a complete orthogonal set which can be used to expand the true many-nucleon wave functions of the ground state $|0\rangle$ or the excited states $|n\rangle$. The exact diagonalization of H within the full shell model space is a task which cannot be solved. In ordinary TDA it is assumed that the ground state is a Hartree-Fock ground state: that is, ground state correlations are not taken into account, which is the main drawback of this procedure. Since experimental operators are one-body operators (that is they excite one particle-hole pair at a time) the excited state is usually approximated by a linear combination of one-particle, one-hole excitations.

$$|0\rangle = |HF\rangle$$

$$|n\rangle = \sum_{m,i} C_{mi}\ a_m^+\ a_i\ |0\rangle \qquad (1.2.2)$$

The Hamiltonian in second quantiziation is given by

$$H = \sum_{m,i} \varepsilon_{mi}\ a_m^+\ a_i + \frac{1}{4} \sum_{m,n,i,j} a_m^+\ a_n^+\ a_i\ a_j\ V_{mnji} \qquad (1.2.3)$$

Minimizing the energy and contracting all the operators, we obtain the so-called Tamm-Dancoff equation of motion:

$$\sum_{n,j} \left(\varepsilon_{mi}\, \delta_{m,n}\, \delta_{i,j} + v_{mjin} \right) C_{nj} = E\, C_{mi} \tag{1.2.4}$$

where v stands for the residual interaction and indices m,n (i,j) refer to states above (below) the Fermi level. The interaction matrix element is given by

$$v_{mjin} = \int \varphi_m^*(\xi_1)\, \varphi_i^*(\xi_2)\, v(\xi_1, \xi_2) \times \left(\varphi_i(\xi_1)\, \varphi_n(\xi_2) - \varphi_n(\xi_1)\, \varphi_i(\xi_2) \right) d\xi_1\, d\xi_2 \tag{1.2.5}$$

It is very useful to represent the interaction graphically by a Feynman diagram.[4]

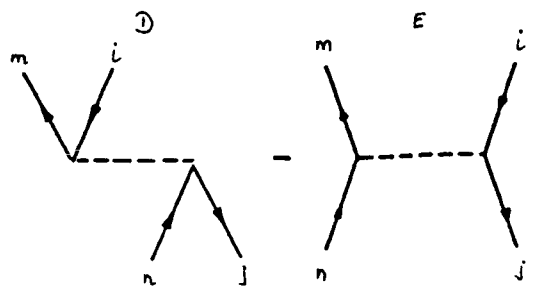

Figure 1.2.1 Graphical representation of the matrix element v_{mjin} and v_{mjni}

A dashed line stands for v and lines with arrows for the single-particle functions (arrow up for particles; arrow down for holes), which according to their coordinates, are linked to the interaction points 1 and 2 of the dashed line. If we imagine a time scale perpendicular to the dotted line, a particle-hole pair (nj) is annihilated and another one created in the direct term, whereas in the exchange term a pair (nj) is scattered into a new one (mi).

The solution of the Tamm-Dancoff equation is, in general, not obtained in the full particle-hole space, which is an infinite numerical task. In order to understand the qualitative features of this solution and what is meant by "collective" states we will solve the equation in the schematic model using a seperable interaction and degenerate energies. The interaction is approximated by

$$V_{mjin} = \chi D_{mi} D_{nj} \qquad (1.2.6)$$

The justification for this ansatz is the following:[5] v is the residual interaction, i. e. it describes the change of the mean field due to the nucleons being no longer in a self-consistent state. The mean field can change the particle density by either absorbing or emitting a gamma ray. This process is described formally by[6]

$$\delta H = Ce \sum_{k_1, k_2} \langle k_1 | r^\lambda Y_\lambda | k_2 \rangle a^+_{k_1} a_{k_2} \qquad (1.2.7)$$

The multipolarity λ agrees, of course, with the angular momentum to which the particle-hole pair (mi) is coupled.

Now define

$$D_{mi} = \langle mi | \delta H | 0 \rangle \qquad (1.2.8)$$

graphically this can be represented by Figure 1.2.2.

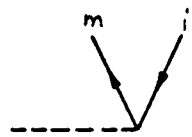

Figure 1.2.2 The absorption of a gamma ray

The direct term of the residual interaction looks exactly like the product of two such matrix elements. The physical process then is described by the direct term rather than the exchange term. Neglecting the exchange term is only a fair approximation.

In general, the residual interaction is attractive for T=0 states and repulsive for T=1 states. Therefore, now choose

$$\chi < 0 \quad \text{for T=0}$$
$$\chi > 0 \quad \text{for T=1} \qquad (1.2.9)$$

With the ansatz (1.2.6), the secular equation (1.2.4) has the following form

$$\left(E - \varepsilon_m - \varepsilon_i\right) C_{mi} = \chi D_{mi} \sum_{nj} D^*_{nj} C_{nj} \qquad (1.2.10)$$

and is solved by

$$C_{mi} = N \frac{\mathcal{D}_{mi}}{E - \varepsilon_m + \varepsilon_i}$$

$$N^{-2} = \sum_{mi} \frac{|\mathcal{D}_{mi}|^2}{(E - \varepsilon_m + \varepsilon_i)^2} \qquad (1.2.11)$$

The eigenvalue equation for the excitation energies E is

$$\frac{1}{\chi} = \sum_{mi} \frac{|\mathcal{D}_{mi}|^2}{E - \varepsilon_m + \varepsilon_i} \qquad (1.2.12)$$

In the degenerate case we set all $\epsilon_{mi}=\epsilon_m-\epsilon_i$ equal to ϵ. This case is realized, for example, if only one major shell is taken into account for particles and another major shell is taken into account for holes in a spherical oscillator potential without a spin-orbit term. The solution for this case is

$$C_{mi} = \left(\sum_{mi} |\mathcal{D}_{mi}|^2 \right)^{-\frac{1}{2}} \mathcal{D}_{mi}$$

$$E = \varepsilon + \lambda \sum_{mi} |\mathcal{D}_{mi}|^2 \qquad (1.2.13)$$

The eigenvalue equation (1.2.12) can be solved graphically by plotting the r.h.s. as a function of E. The eigenvalues are obtained from the intersection of this function with the straight line $1/\chi$. All solutions are sandwiched between the original shell model excitation , with only one solution being pushed up (if $1/\chi < 0$) or down (if $1/\chi > 0$) in energy. The one excitation which is pushed down (T=0) or up (T=1) is a state which has a constructive superposition of

all terms of equation (1.2.12). This state will be refered to as the collective state. By "constructive" we mean that all particle-hole terms contribute to the multipole matrix element, $\Sigma\ C_{mi}\ D_{mi}$, with the same sign. Examples for such states are the first octupole states (T=0) appearing in ^{16}O, ^{40}Ca, and ^{208}Pb at low excitation energies and the giant dipole resonance (T=1).

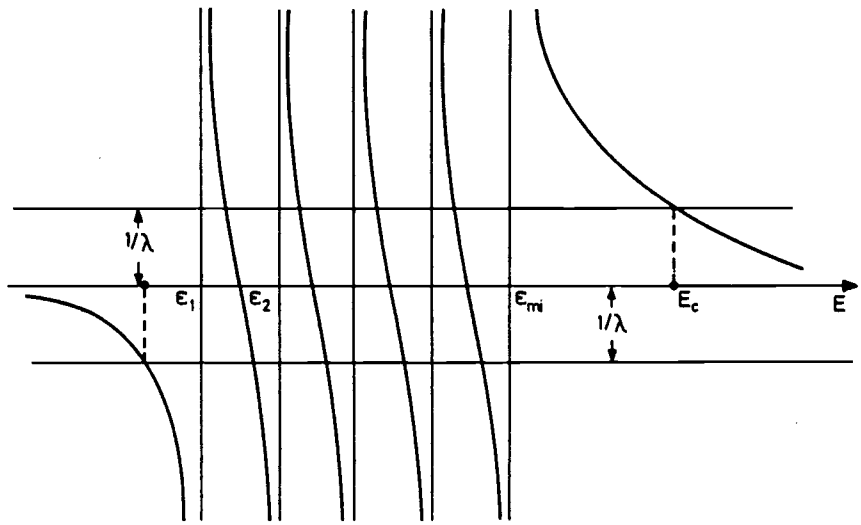

Figure 1.2.3 Graphical solution of equation (1.2.12)[4]
$1\backslash\lambda$ in the graph corresponds to $1/\chi$ in the secular equation

1.3 Quasiparticles and the BCS Theory Applied to Nuclei

For low lying states in open shell nuclei it is a crude approximation to use uncorrelated Hartree-Fock ground states. Even though the Fermi level is defined, it is rather artificial to seperate particle states from hole states. Furthermore pair correlations become so important that the "feedback" of these correlations on the single-particle motion cannot be neglected any more. As was already mentioned in the liquid drop model, the existence of low-lying 2^+ levels for open shell even-even nuclei is closely connected. Nuclei in the neighborhood of closed shells that are still spherical can easily be excited to shape vibrations around their spherical equilibrium position, since the restoring force, which is the difference between pairing and deformation effects, is rather small. The nucleus will therefore become deformed into an ellipsoid and vibrate about its spherical shape with a low frequency (quadrupole oscillations, 2^+ levels). Exactly these states will be investigated in this study.

Pairing correlations are due to the short-range part of the nucleon-nucleon interaction, and this interaction is most effective between (I=0)-coupled pairs. In our formulation we will use ground-state wave functions that diagonalize the shell model part and the short range part of the Hamiltonian. There are basically two formalisms that describe the pairing of the nucleons.

The Seniority Scheme is a very elegant method. It uses the SU(2) algebra, which is the same algebra as the spin algebra. The Seniority

Scheme is therefore very often called Quasispin. The main drawback of this procedure is that it can be used only for degenerate shells.

The most commonly used method is the BCS-Theory, which was invented by Bardeen, Cooper and Schrieffer who used a variational principle to determine the ground state energy of a superconductor. This method no longer provides an exact solution of the eigenvalue problem (like the Seniority Scheme does), but rather like the Hartree-Fock method, it can be derived from a variational principle. The exact ground state cannot be used for this purpose because the method makes use of a Hamiltonian that does not commute with the particle number operator. The main disadvantage of this method is that the particle number is not conserved any longer. For superconductors where one deals with thousands of electrons this does not have any effect. For nuclei, however, where there are only a few nucleons outside the major closed shell this effect cannot always be neglected. For the pairing energy in the nucleus by means of the BCS method the folowing assumptions are made:

- Its two-nucleon spectrum is approximately the same as the spectrum of a Dirac delta force
- Because of the Pauli exclusion principle pair correlations are expected only in the neighborhood of the Fermi limit.
- The pairing energy is significant only for the energy spectra of intermediate and heavy nuclei because the separations of the single-particle energies are then sufficiently small. In these nuclei, however, the protons and neutrons in unfilled shells occupy

different states. The pair correlations of the two kinds of nucleons may therefore be treated seperately.

The BCS method makes use of the Bogoliubov transformation, which makes it possible to account for ground-state correlations. This transformation changes from a particle-hole representation to a quasiparticle representation, where a quasiparticle is defined to be a linear combination of a particle and a hole. The coefficients in front of the particle creation and annihilation operators give the probability of the quasiparticle to be a particle or a hole. They are called the occupation number (v_k) and the non-occupation number (u_k) respectively.

$$\alpha^{+}_{jm} = u_j \; a^{+}_{jm} + (-1)^{j-m} \; v_j \; a_{j-m} \qquad (1.3.1)$$

where a^{+}_{jm} (a_{jm}) is the creation (destruction) operator of a particle and α^{+}_{jm} (α_{jm}) is the creation (destruction) operator of a quasiparticle with quantum number j and projection quantum number of the total angular momentum m. We require the quasiparticle operators to be fermion operators and therefore to satisfy fermion anti-commutation relations

$$\{ \alpha_{jm} , \alpha^{+}_{j'm'} \} = \delta_{jj'} \; \delta_{mm'} \qquad (1.3.2)$$

If ϵ_j is much above the Fermi sea of nucleons α^{+}_{jm} should be a particle and $u_j=1$, $v_j=0$; if ϵ_j is much below the Fermi limit then $u_j=0$, $v_j=1$.

The procedure for deriving the pairing equations starting from the assumptions above is the following[7]: The pairing force Hamiltonian is given by

$$H = \sum_{jm} \varepsilon_j \, a^{\dagger}_{jm} \, a_{jm} - \frac{1}{2} G \sum_{jm\, j'm'} a^{\dagger}_{j'm'} \, a^{\dagger}_{j'-m'} \, (-1)^{j'-m'} \, a_{jm} \, a_{j-m} \, (-1)^{j-m} \tag{1.3.3}$$

where the first part of the second term describes the creation of a pair coupled to angular momentum zero and the second part describes the destruction of a particle pair coupled to zero. A Boguliubov transformation of the Hamiltonian is made. Then $\langle 0| H-\lambda N |0\rangle$ is minimized with respect to u_j and v_j subject to the condition that the average particle number be the actual particle number and that

$$u_j^2 + v_j^2 = 1 \tag{1.3.4}$$

This procedure yields the pairing equations

$$1 = \frac{G}{4} \sum_{jm} \frac{1}{\sqrt{\Delta^2 + (\varepsilon_j - \lambda)}} \tag{1.3.5}$$

$$N = \sum_{j} (2j+1) \, v_j^2 \tag{1.3.6}$$

where Δ is the gap parameter of the spectrum.

In the degenerate case the pairing equations can be solved in closed form. The pairing constant G, the quasiparticle energy E_Q and the occupation and nonoccupation numbers u_j and v_j for this case are given by

$$G = \frac{23}{A} \tag{1.3.7}$$

$$E_Q = \frac{G N}{4} \tag{1.3.8}$$

$$v^2 = \frac{N}{N_0} \tag{1.3.9}$$

$$u^2 = 1 - \frac{N}{N_0} \tag{1.3.10}$$

where A is the total number of nucleons, No is the degeneracy of the major shell and N is the number of nucleons in the major shell.

Making use of the Bogoliubov transformation the TDA particle hole secular equation can be reformulated in quasiparticles. The particle hole states become two quasiparticle states. For detailed discussion see Chapter 2.

1.4 Four-Quasiparticle States

The anharmonicity of the vibrational states will now be investigated. The Hamiltonian includes higher order terms, which provide a coupling between the different phonon states. Whereas the TDA and RPA calculations usually take only one-phonon states into account, we will determine the mixing of the two-phonon (four-

quasiparticle) and the one-phonon (two-quasiparticle) states and determine its effect on the energy-level diagram.

The following states are used as basis states for a four-quasiparticle wavefuncion:

$$\alpha^{+}_{j_1 m_1}\ \alpha^{+}_{j_2 m_2}\ \alpha^{+}_{j_3 m_3}\ \alpha^{+}_{j_4 m_4}\ |\,0\rangle \tag{1.4.1}$$

where $|0\rangle$ is the quasiparticle vacuum defined by

$$\alpha_{jm}\,|\,0\rangle = 0 \tag{1.4.2}$$

Then each four-quasiparticle state can be written as a linear combination of these basis-states. The amplitude X_{1234} is then given by the overlap of its basis-state and the wavefunction $|\psi\rangle$.

$$X_{1234} = \langle 0|\ \alpha_{j_4 m_4}\ \alpha_{j_3 m_3}\ \alpha_{j_2 m_2}\ \alpha_{j_1 m_1}\,|\psi\rangle \tag{1.4.3}$$

Rather than using a product representation a transformation to the coupled representation is done.

$$\begin{aligned}
|\psi\rangle &= \sum_{j's,m's} X_{1234} \sum_{\lambda's,\mu's,J,M} \langle j_1 j_2 m_1 m_2 \,|\, \lambda_1 \mu_1\rangle \\
&\quad \langle j_3 j_4 m_3 m_4 \,|\, \lambda_2 \mu_2\rangle \langle \lambda_1 \lambda_2 \mu_1 \mu_2 \,|\, J M\rangle \\
&\quad \left[\left[\alpha^{+}_{j_1}\ \alpha^{+}_{j_2}\right]_{\lambda_1}\left[\alpha_{j_3}\ \alpha_{j_4}\right]_{\lambda_2}\right]_{JM}|\,0\rangle \\
&= \sum_{\lambda's,j's,J,M} X_{1234}\ (J\ (\lambda_1,\lambda_2)\ M) \\
&\quad \left[\left[\alpha^{+}_{j_1}\ \alpha^{+}_{j_2}\right]_{\lambda_1}\left[\alpha_{j_3}\ \alpha_{j_4}\right]_{\lambda_2}\right]_{JM}|\,0\rangle
\end{aligned} \tag{1.4.4}$$

The last equality of equation (1.4.4) defines the coupled amplitude of a four-quasiparticle state.

$$X_{1234}\left(J(\lambda_1,\lambda_2)M\right) = \langle 0 | \left[\left[\alpha_{j_4}\alpha_{j_3}\right]_{\lambda_2}\left[\alpha_{j_2}\alpha_{j_1}\right]_{\lambda_1}\right]_{JM} | \psi\rangle \tag{1.4.5}$$

The transformation from the product representation to the coupled representation and vice versa is given by

$$X_{1234}\left(J(\lambda_1,\lambda_2)M\right) = \sum_{m's,\mu's} \langle j_1 j_2 m_1 m_2 | \lambda_1\mu_1\rangle \langle j_3 j_4 m_3 m_4 | \lambda_2\mu_2\rangle \langle \lambda_1\lambda_2\mu_1\mu_2 | JM\rangle \; X_{1234} \tag{1.4.6}$$

and by

$$X_{1234} = \sum_{\lambda's,\mu's,J,M} \langle j_1 j_2 m_1 m_2 | \lambda_1\mu_1\rangle \langle j_3 j_4 m_3 m_4 | \lambda_2\mu_2\rangle \langle \lambda_1\lambda_2\mu\mu_2 | JM\rangle \; x_{1234}\left(J(\lambda_1,\lambda_2)M\right) \tag{1.4.7}$$

Since the creation and the destruction operators of fermions fulfill fermion commutation relations X_{1234} is completely antisymmetric, that is an even permutation of (1,2,3,4) gives the same amplitude back, whereas an odd permutation yields the negative amplitude.

From this property, symmetry conditions for the coupled amplitude $x_{1234}(JM(\lambda_1\lambda_2))$ can be derived, which will be very useful for further calculations and which can be found in the appendix. One important result which evolves from these symmetry properties should be mentioned here. In the case of (j_1, j_2) being equal to (j_3, j_4) and λ_1 being equal to λ_2

$$x_{1212}(J(\lambda\lambda)M) = (-1)^J x_{1212}(J(\lambda\lambda)M) \qquad (1.4.8)$$

i. e. for odd J values the amplitudes are exactly equal to their negatives and the wavefunction is identically zero. This result is found experimentally and accounted for by the liquid drop model.

II. THE TAMM-DANCOFF APPROXIMATION FOR QUASIPARTICLES

In this chapter the Tamm-Dancoff secular equations for two and for four quasiparticles will be derived. These two equations will then be solved in the limit of degenerate energy levels using a separable interaction. Again, doing more realistic calculations is an infinite numerical task. The intent, however, is merely to study the qualitative features.

2.1 The Two Body Interaction in Quasiparticle Representation

Doing a Bogoliubov transformation of the two body residual interaction yields

$$V = \sum_{j's\ m's} \mathcal{X} \left(u_{j_a} \alpha^{+}_{j_a m_a} + (-1)^{j_a - m_a} v_{j_a} \alpha_{j_a - m_a} \right)$$
$$\left(u_{j_b} \alpha^{+}_{j_b m_b} + (-1)^{j_b - m_b} v_{j_b} \alpha_{j_b - m_b} \right)$$
$$\left(u_{j_c} \alpha_{j_c m_c} + (-1)^{j_c - m_c} v_{j_c} \alpha^{+}_{j_c - m_c} \right)$$
$$\left(u_{j_d} \alpha_{j_d m_d} + (-1)^{j_d - m_d} v_{j_d} \alpha^{+}_{j_d - m_d} \right)$$
$$V_{abdc} \qquad (2.1.1)$$

The part of the interaction, that doesn't change the quasi-particle number is of the following operators

$$\alpha^{+} \alpha^{+} \alpha \alpha \qquad (2.1.2)$$

This part of the interaction is given by

$$
\begin{aligned}
V = \sum_{j's\,m's} \Big[& u_{ja} u_{jb} u_{jc} u_{jd} \langle \varphi_{ja m_a} \varphi_{jc m_c} | V | \varphi_{jd m_d} \varphi_{jb m_b} \rangle \\
& + v_{ja} v_{jb} v_{jc} v_{jd} \langle \tilde{\varphi}_{ja m_a} \tilde{\varphi}_{jc m_c} | V | \tilde{\varphi}_{jd m_d} \tilde{\varphi}_{jb m_b} \rangle \\
& + \tfrac{1}{2} \Big(u_{ja} u_{jd} v_{jb} v_{jc} \langle \tilde{\varphi}_{ja m_a} \tilde{\varphi}_{jb m_b} | V | \tilde{\varphi}_{jc m_c} \varphi_{jd m_d} \rangle_A \\
& - u_{jc} u_{jd} v_{ja} v_{jb} \langle \varphi_{jc m_c} \tilde{\varphi}_{jb m_b} | V | \tilde{\varphi}_{ja m_a} \varphi_{jd m_d} \rangle_A \\
& - u_{ja} u_{jb} v_{jc} v_{jd} \langle \varphi_{ja m_a} \tilde{\varphi}_{jd m_d} | V | \tilde{\varphi}_{jc m_c} \varphi_{jb m_b} \rangle_A \\
& + u_{jb} u_{jc} v_{ja} v_{jd} \langle \varphi_{jc m_c} \tilde{\varphi}_{jd m_d} | V | \tilde{\varphi}_{ja m_a} \varphi_{jb m_b} \rangle_A \Big) \Big] \\
& \alpha^{+}_{ja m_a} \alpha^{+}_{jc m_c} \alpha_{jb m_b} \alpha_{jd m_d}
\end{aligned}
\tag{2.1.3}
$$

where A means that the matrix element is antisymmetrized and the wavy line has the following meaning

$$
| \tilde{\varphi}_{jm} \rangle = (-1)^{j-m} | \varphi_{j-m} \rangle \tag{2.1.4}
$$

It is often very convenient to consider the closed shell limit, where the state weighted by a u_j corresponds to a pure particle state and the one weighted by a v_j corresponds to a pure hole state (see also chapter 1.3). In this limit the first two terms of equation (2.1.3) describe transitions from a two-particle state to a two-particle state and from a two-hole state to a two-hole state, respectively, which stand for two-particle pick-up and stripping reactions. The other four terms correspond in the closed shell limit to the particle-hole residual interaction matrix element discussed in

section 1.2 with the only difference that no distinctions are made between particles and holes and therefore all possible index permutations have to be considered, which corresponds to the four different terms. In analogy to the particle-hole case the last four terms of equation (2.1.3) will be considered.

Using a seperable interaction gives a result

$$V = -\frac{1}{2}\chi \sum_{j's\, m's} u_{a,c}\, \mathcal{D}_{ac}(\lambda)\; u_{b,d}\, \mathcal{D}_{db}(\lambda) \; \alpha^{+}_{j_a m_a}\, \alpha^{+}_{j_c m_c}\, \alpha_{j_b m_b}\, \alpha_{j_d m_d} \tag{2.1.5}$$

or in coupled representation

$$V = -\frac{1}{2}\chi \sum_{j's} u_{a,c}\, \mathcal{D}_{ac}(\lambda)\; u_{b,d}\, \mathcal{D}_{db}(\lambda) \; \left[\alpha^{+}_{j_a}\, \alpha^{+}_{j_c}\right]_\lambda \left[\alpha_{j_b}\, \alpha_{j_d}\right]_\lambda \tag{2.1.6}$$

where the brackets denote angular momentum coupling. For the reduced matrix elements the phase convention of Yoshida[8] is used.

$$q_{ac}(\lambda) = \langle j_a \| i^\lambda r^\lambda Y_\lambda \| j_c \rangle \tag{2.1.7}$$

and for convenience we have defined

$$u_{a,c} = u_{j_a} v_{j_c} + u_{j_c} v_{j_a} \tag{2.1.8}$$

$$\mathcal{D}_{ac}(\lambda) = \frac{q_{ac}(\lambda)}{\hat{\lambda}} \tag{2.1.9}$$

$$\hat{\lambda} = \sqrt{2\lambda+1} \tag{2.1.10}$$

Except for the weighting of the q's with the occupation and nonoccupation numbers the interaction stands in direct analogy to the particle hole case with the difference that quasiparticle-operators are used instead of particle and hole operators.

2.2 Derivation and Solution of the TDA Equation for Two Quasiparticles

The derivation and solution of the Tamm-Dancoff equation for two quasiparticles is in direct analogy to that in the particle hole case, discussed in section 1.2. Therefore the details will be bypassed at this time, stating only the results. The two-quasiparticle amplitudes in product and coupled representation are defined in direct analogy to the four-quasiparticle case (see section 1.4).

The equation of motion for a seperable interaction for this case is

$$\begin{aligned}&(\varepsilon_m + \varepsilon_n - \lambda)\, x_{mn}(JM) \\ &= -\sum_{j_i j_j} \chi\, x_{ij}(JM)\, u_{i,j}\, \mathcal{D}_{ij}(\lambda)\, u_{m,n}\, \mathcal{D}_{mn}(\lambda)\end{aligned} \tag{2.2.1}$$

In the case of degenerate energy levels this is solved by

$$x_{mn}(J\,M) = \frac{N}{\sqrt{2}}\; u_{m,n}\; \mathcal{D}_{mn}(J)$$

$$N^{-\frac{1}{2}} = \sum_{m,n} u_{m,n}^{2}\; \mathcal{D}_{mn}^{2}(J) \qquad (2.2.2)$$

Different from the particle-hole case is again the weighting by the u's and v's. Furthermore since quasiparticles are indistiguishable particles and because we have used redundent sums the amplitudes have to be normalized to 1/2, whereas in the particle-hole case they were normalized to 1.

The two-quasiparticle energy is given by

$$E_{2QP} = 2\epsilon + \chi\, Y(2) \qquad (2.2.3)$$

where ϵ: degenerate single quasiparticle energy

χ: interaction strength parameter

$Y(2) = \sum_{m,n} u_{m,n}^{2}\, D_{mn}^{2}(2)$

2.3 Derivation and Solution of the TDA Equation for Four Quasiparticles

The procedure is the same as for two quasiparticles (see 2.2). First the matrix element between two four-quasiparticle collective states is calculated, then the variation is done which yields the equation of motion, finally the secular equation will be solved in the degenerate limit.

For convenience the one- and the two-body part of the Hamiltonian is calculated seperately. From the BCS-theory the one-body Hamiltonian is given by

$$H_1 = \sum_k E_k \alpha_k^+ \alpha_k \qquad (2.3.1)$$

where E_k is the quasiparticle energy

$$E_k = \sqrt{(\epsilon_k - \lambda)^2 + \Delta} \qquad (2.3.2)$$

with Δ : pairing gap

ϵ : shell energy

The matrix element of the one-body operator in coupled representation then is

$$\langle \psi'_{4QP} | H_1 | \psi_{4Q} \rangle$$

$$= \sum_{j's\,\lambda's\,J\,M\,J'\,M'} E_k \; x_{1234}\left(J'(\lambda_1\lambda_2)M'\right) x_{5678}\left(J(\lambda_3\lambda_4)M\right)$$

$$\langle 0 | [[[\alpha_{j_4}\alpha_{j_3}]_{\lambda_2}[\alpha_{j_2}\alpha_{j_1}]_{\lambda_1}]_{J'M'} | \alpha^+_{j_k m_k}\alpha_{j_k m_k}$$

$$| [[[\alpha^+_{j_5}\alpha^+_{j_6}]_{\lambda_3}[\alpha^+_{j_7}\alpha^+_{j_8}]_{\lambda_4}]]_{JM} | 0\rangle \qquad (2.3.3)$$

In order to do the contractions of the operators they have to be decoupled first. There are 2x4! = 48 different possible contractions. With the symmetry conditions of the coupled four-quasiparticle amplitude (see appendix A) it is straightforward to show that all 48 terms are equal. Thus the matrix element is given by

$$H_1 = 48 \sum_{j's\,\lambda's} E_1 \left(x_{1234}\left(J(\lambda_1\lambda_2)M\right)\right)^2 \qquad (2.3.4)$$

The two-body term is derived similarily

$$\langle \psi'_{4QP} | V | \psi_{4QP} \rangle$$

$$= \sum_{j's,\lambda's,J,M,J'M'} x_{1234}\left(J'(\lambda_1\lambda_2)M'\right) x_{5678}\left(J(\lambda_3\lambda_4)M\right)$$

$$\langle 0 | [[[\alpha_{j_4}\alpha_{j_3}]_{\lambda_2}[\alpha_{j_2}\alpha_{j_1}]_{\lambda_1}]_{J'M'} | V$$

$$| [[[\alpha^+_{j_5}\alpha^+_{j_6}]_{\lambda_3}[\alpha^+_{j_7}\alpha^+_{j_8}]_{\lambda_4}]_{JM} \qquad (2.3.5)$$

Again the operators have to be decoupled first in order to do the contractions. Using Wick's theorem to calculate the contractions there are $(4x3)^2x2 = 288$ terms. Using the symmetry conditions of the $x_{1234}(jm(\lambda_1\lambda_2))$ (see appendix A) all terms turn out to be the same and the matrix element is given by

$$\langle \Psi'_{4QP} | V | \Psi_{4QP} \rangle$$
$$= 144 \chi \sum_{j's\,\lambda's} X_{1234}\left(J(\lambda_1,\lambda_2)\,M\right) X_{5678}\left(J(\lambda,\lambda_2)\,M\right)$$
$$u_{1,2}\, D_{12}(\lambda_1)\, u_{5,6}\, D_{56}(\lambda_1)\, \delta_{j_3 j_7}\, \delta_{j_4 j_8} \qquad (2.3.6)$$

Graphically equation (2.3.6) can be represented by Figure 2.3.1.

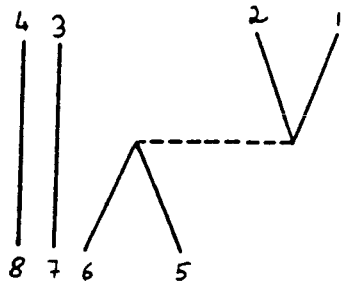

Figure 2.3.1 graphical description of equation (2.3.6)

Since a quasiparticle is partly a particle and partly a hole, there are no arrrows drawn on the lines in figure 3.2.1. A two-quasiparticle pair is annihalated at position 1, another one is created at position 2, and the other two-quasiparticle pair is a spectator.

This corresponds to the direct matrix element in the particle-hole excitation with one particle-hole pair being a spectator. In the closed shell limit this can be represented by Figure 2.3.2.

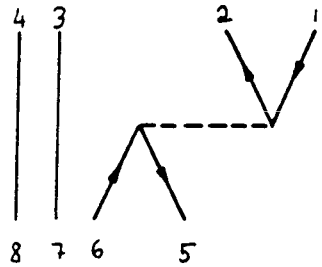

Figure 2.3.2 graphical description of equation (2.3.6) in the closed shell limit

The equation of motion is derived by a variational principle

$$\frac{2}{\partial x_{mnij}(J(\lambda_1'\lambda_2')M)}\Big[6\sum_{j's\,\lambda's}\chi\; x_{1234}(J(\lambda_1\lambda_2)M)$$

$$x_{5634}(J(\lambda_1\lambda_2)M)\,u_{12}\mathcal{D}_{12}(\lambda_1)\,u_{5\ell}\mathcal{D}_{5\ell}(\lambda_2)$$

$$+(4E_1 - E_{4QP})\left(x_{1234}(J(\lambda_1\lambda_2)M)\right)^2\Big] = 0 \qquad (2.3.7)$$

Since we are using an overcomplete basis, there are 24 different permutations of $x_{mnij}(jm(\lambda_1\lambda_2))$, that contribute to this differentiation. In the one-body term all 24 terms turn out to be equal, because of the product of the two antisymmetric amplitudes. The two-body term consists of six different types of terms.

The equation of motion can be written as

$$
\begin{aligned}
&\left(E_m + E_n + E_i + E_j - E_{HQP} \right) X_{mnij} \left(J(\lambda_1 \lambda_2) M \right) \\
&= \chi \sum_{j_5 j_6} \Big\{ -X_{56ij} \left(J(\lambda_1 \lambda_2) M \right) u_{5,6} D_{56}(\lambda_1) u_{m,n} D_{mn}(\lambda_1) \\
&\quad - (-1)^{J-\lambda_1-\lambda_2} X_{56mn} \left(J(\lambda_2 \lambda_1) M \right) u_{56} D_{56}(\lambda_2) u_{i,j} D_{ij}(\lambda_2) \\
&\quad + \sum_{\lambda_1' \lambda_2'} \Big[U \begin{pmatrix} j_m & j_n & \lambda_1 \\ j_i & j_j & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{pmatrix} X_{56nj} \left(J(\lambda_1' \lambda_2') M \right) \\
&\qquad u_{m,i} D_{mi}(\lambda_1') u_{5,6} D_{56}(\lambda') \\
&\qquad + U \begin{pmatrix} j_n & j_m & \lambda_1 \\ j_j & j_i & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{pmatrix} X_{56mi} \left(J(\lambda_1' \lambda_2') M \right) \\
&\qquad \times (-1)^{\lambda_1 + \lambda_2 + j_m + j_n + j_i + j_j} \\
&\qquad \times u_{5,6} D_{56}(\lambda_1') u_{n,j} D_{nj}(\lambda_1') \\
&\qquad - U \begin{pmatrix} j_m & j_n & \lambda_1 \\ j_j & j_i & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{pmatrix} X_{56ni} \left(J(\lambda_1' \lambda_2') M \right) \\
&\qquad \times (-1)^{\lambda_2 - j_i - j_j} u_{5,6} D_{56}(\lambda_1') u_{mj} D_{mj}(\lambda_1') \\
&\qquad - U \begin{pmatrix} j_n & j_m & \lambda_1 \\ j_i & j_j & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{pmatrix} X_{5i mj} \left(J(\lambda_1' \lambda_2') M \right) \\
&\qquad \times (-1)^{\lambda_1' - j_n - j_m} u_{5,6} D_{56}(\lambda_1') u_{ni} D_{ni}(\lambda_1') \Big] \Big\}
\end{aligned}
\tag{2.3.8}
$$

where, in terms of the Wigner 9-j coefficient (written with braces) the unitary 9-j recoupling coefficient is

$$U\begin{pmatrix} j_m & j_n & \lambda_1 \\ j_i & j_j & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{pmatrix} = \hat{\lambda}_1\,\hat{\lambda}_2\,\hat{\lambda}_1'\,\hat{\lambda}_2' \begin{Bmatrix} j_m & j_n & \lambda_1 \\ j_i & j_j & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{Bmatrix} \qquad (2.3.9)$$

The secular equation (equation 2.3.8) can be interpreted as follows: In the first term the pair (ij) is the spectator pair, its angular momentum coupling doesn't change. The quasiparticle pair (mn) is destroyed and the pair (56) is created. It is summed over all possible final states (56). In order for the interaction to be a scalar it is necessary that the two quasiparticle pairs, which take part in the interaction ((56) and (mn)), are both coupled to the same angular momentum.

All the other terms are exchange terms of the first term: that is, each possible pair is in one of the terms the spectator and in another term it takes part in the interaction. The 9j-recouplings evolve from using an overcomplete basis for the four-quasiparticle basis states.

For further calculations the first two terms will be referred to as the direct terms (in that there is no recoupling necessary), even though the second term is actually a boson exchange term. The other four terms are fermion exchange terms.

The equation of motion for the degenerate case, i.e. $E_m = E_n = E_i = E_j$, will now be solved. Equation (2.3.8) is multiplied through by $u_{1,2}\, q_{12}$, then it is summed over j_m and j_n. The fermion exchange

terms will be neglected, which will be justified with some numerical examples in the appendix.

$$k_{ij}(\lambda, T(\lambda,\lambda_2)M) = -\frac{\chi}{4E - E_{4QP}}\Big(k_{ij}(\lambda, T(\lambda,\lambda_2)M)\,Y(\lambda_1) + k_{mn}(\lambda_2\, T(\lambda_2\lambda_1)M)\,u_{i,j}\,\mathcal{D}_{ij}(\lambda_2)\,u_{mn}\,\mathcal{D}_{mn}(\lambda_1)\Big) \tag{2.3.10}$$

where the following convention has been used

$$k_{ij}(\lambda_1, T(\lambda_1,\lambda_2)M) = \sum_{j_m j_n} u_{m,n}\,\mathcal{D}_{mn}(\lambda_1)\,z_{mnij}(T(\lambda\lambda_2)M) \tag{2.3.11}$$

Multiplying equation (2.3.10) by $u_{i,j}\,D_{ij}(\lambda)$, then summing over j_i, j_j and solving for the four-quasiparticle energy yields

$$E_{4QP} = 4E + \chi\left(Y(\lambda_1) + Y(\lambda_2)\right) \tag{2.3.12}$$

Substituting all this back into equation 2.3.7 yields the four-quasiparticle amplitude in coupled representation

$$
\begin{aligned}
&x_{mnij}\ (J(\lambda_1,\lambda_2)M) \\
&= \frac{k(J(\lambda_1,\lambda_2)M)}{\chi\, Y(\lambda_1)\, Y(\lambda_2)}\ u_{ij}\, D_{ij}(\lambda_2)\ u_{mn}\, D_{mn}(\lambda_1) \\
&+ \sum_{\lambda_1'\lambda_2'} \frac{Y(\lambda_1')+Y(\lambda_2')}{Y(\lambda_1)+Y(\lambda_2)}\ \frac{k(J(\lambda_1'\lambda_2')M)}{Y(\lambda_1')\,Y(\lambda_2')} \\
&\quad \times\Bigg(-U\begin{pmatrix} j_m & j_n & \lambda_1 \\ j_i & j_j & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{pmatrix} u_{mi}\, D_{mi}(\lambda_1')\, u_{nj}\, D_{nj}(\lambda_2') \\
&\quad + U\begin{pmatrix} j_m & j_n & \lambda_1 \\ j_j & j_i & \lambda_2 \\ \lambda_1' & \lambda_2' & J \end{pmatrix} u_{mj}\, D_{mj}(\lambda_1')\, u_{ni}\, D_{ni}(\lambda_2')\,(-1)^{\lambda_2-j_i-j_j}\Bigg)
\end{aligned}
\tag{2.3.13}
$$

Since m, n, i, j label quantum numbers of indistinguishable particles the four quasiparticle amplitudes have to be normalized to 4!=24.

$$
\sum_{m,n,i,j} \left(x_{mnij}(J(\lambda_1,\lambda_2)M)\right)^2 = 24 \tag{2.3.14}
$$

Unlike in the boson-expansion methods as used for example by Lie and Holzwarth[1] or in the interacting boson model as used by F. Iachello and I. Talmi[3] in our result the fermion character of the wavefunction is still present, that is all possible exchanges of quasiparticles (fermions) and of quasiparticle pairs (bosons) are

taken into account. Therefore the Pauli princple is obeyed and the blocking effect is contained.

III. FOUR QUASIPARTICLE - TWO QUASIPARTICLE COUPLING MATRIX ELEMENT FOR COLLECTIVE STATES

As an example of the effects of four-quasiparticle states, the coupling between states with two and four quasiparticles will be considered, which corresponds to a coupling between states with one and two phonons. A case of special interest involves the coupling of the one-phonon (two-quasiparticle) state to the state of two phonons (four-quasiparticles) of the same type. This interaction represents the leading-order anharmonic effect in the vibrational motion, and its consequences on the energy level diagram will be investigated in the following chapters.

In this chapter we shall calculate the matrix element that describes the transition from the four-quasiparticle collective state to the two-quasiparticle collective state. The operator and its Hermitian conjugate that contribute to this transition has the form

$$\alpha^+ \alpha \alpha \alpha \tag{3.1}$$

that is three quasiparticles are destroyed and one is created.

This part of the operator (equation 2.1.1) is given by

$$
\begin{aligned}
V_{24} = \frac{1}{2} \sum_{j's\ m's} \Big\{ & u_{b,d}\, u_{ja}\, u_{jc} \\
& \big(\langle \tilde{\varphi}_{j_b m_b}\, \varphi_{j_a m_a} | V | \varphi_{j_d m_d}\, \varphi_{j_c m_c} \rangle \\
& - \langle \varphi_{j_a m_a}\, \tilde{\varphi}_{j_b m_b} | V | \varphi_{j_d m_d}\, \varphi_{j_c m_c} \rangle \big) \\
& + u_{c,d}\, v_{ja}\, v_{jb} \\
& \big(\langle \tilde{\varphi}_{j_b m_b}\, \tilde{\varphi}_{j_c m_c} | V | \tilde{\varphi}_{j_a m_a}\, \varphi_{j_d m_d} \rangle \\
& - \langle \tilde{\varphi}_{j_b m_b}\, \tilde{\varphi}_{j_c m_c} | V | \varphi_{j_d m_d}\, \tilde{\varphi}_{j_a m_a} \rangle \big) \Big\} \\
& \alpha^{+}_{j_a m_a}\, \alpha_{j_b m_b}\, \alpha_{j_c m_c}\, \alpha_{j_d m_d}
\end{aligned}
$$

(3.2)

The collective matrix element of this interaction has the following form

$$
\begin{aligned}
V_{24}^{C} &= \langle \Psi'_{2QP} \mid V_{24} \mid \Psi_{4QP} \rangle \\
&= \frac{1}{2} \sum_{j's} X_{12}(J'M')\, X_{3456}(J(\lambda_1,\lambda_2)M) \\
&\quad \langle 0 \mid [\alpha_{j_2}\alpha_{j_1}]_{J'M'}\, \alpha^{+}_{j_a m_a} \alpha_{j_b m_b} \alpha_{j_c m_c} \alpha_{j_d m_d} \\
&\quad [[\alpha^{+}_{j_3}\alpha^{+}_{j_4}]_{\lambda_1}[\alpha^{+}_{j_5}\alpha^{+}_{j_6}]_{\lambda_2}]_{JM} \mid 0\rangle \\
&\quad \{ u_{b,d}\, u_{j_a}\, u_{j_c} \\
&\quad (\langle \tilde{\varphi}_{j_b m_b}\, \varphi_{j_a m_a} \mid V \mid \varphi_{j_d m_d}\, \varphi_{j_c m_c}\rangle \\
&\quad - \langle \varphi_{j_a m_a}\, \tilde{\varphi}_{j_b m_b} \mid V \mid \varphi_{j_d m_d}\, \varphi_{j_c m_c}\rangle) \\
&\quad + u_{c,d}\, v_{j_a}\, v_{j_b} \\
&\quad (\langle \tilde{\varphi}_{j_b m_b}\, \tilde{\varphi}_{j_c m_c} \mid V \mid \tilde{\varphi}_{j_a m_a}\, \varphi_{j_d m_d}\rangle \\
&\quad - \langle \tilde{\varphi}_{j_b m_b}\, \tilde{\varphi}_{j_c m_c} \mid V \mid \varphi_{j_d m_d}\, \tilde{\varphi}_{j_a m_a}\rangle)\}
\end{aligned}
\tag{3.3}
$$

Because of the symmetry-properties of the amplitudes (see appendix A) all possible 96 contractions give the same result.

$$
\begin{aligned}
V_{24}^{C} = 48 \sum_{j's, m's, \lambda's, \mu's} & X_{12}(J'M') \; X_{3456}(J(\lambda,\lambda_2)M) \\
& \langle j_1 j_2 m_1 m_2 | J'M' \rangle \langle j_3 j_4 m_3 m_4 | \lambda_1 \mu_1 \rangle \\
& \langle j_5 j_6 m_5 m_6 | \lambda_2 \mu_2 \rangle \langle \lambda_1 \lambda_2 \mu_1 \mu_2 | JM \rangle \\
& \{ u_{5,3} u_{j_1} u_{j_4} \\
& \big(\langle \tilde{\varphi}_{j_5 m_5} \varphi_{j_1 m_1} | v | \varphi_{j_3 m_3} \varphi_{j_4 m_4} \rangle \\
& - \langle \varphi_{j_1 m_1} \tilde{\varphi}_{j_5 m_5} | v | \varphi_{j_3 m_3} \varphi_{j_4 m_4} \rangle \big) \\
& + u_{3,4} v_{j_1} v_{j_5} \\
& \big(\langle \tilde{\varphi}_{j_5 m_5} \tilde{\varphi}_{j_4 m_4} | v | \tilde{\varphi}_{j_1 m_1} \varphi_{j_3 m_3} \rangle \\
& - \langle \tilde{\varphi}_{j_5 m_5} \tilde{\varphi}_{j_4 m_4} | v | \varphi_{j_3 m_3} \tilde{\varphi}_{j_1 m_1} \rangle \big) \}
\end{aligned}
\tag{3.4}
$$

Graphically the first and the third term of equation (3.4) have the form shown in Figure 3.1.

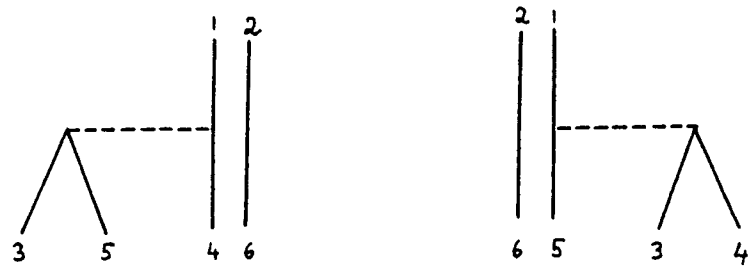

Figure 3.1 Graphical representation of the first and the third term of equation (3.4)

In the closed shell limit Figure 3.1 gets replaced by Figure 3.2.

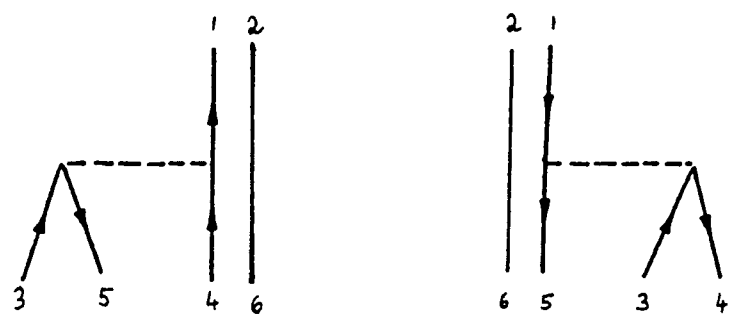

Figure 3.2 Graphical representation of the first and the third term of equation (3.4) in the closed shell limit

In the first term a particle-hole pair is destroyed at coordinate 1, a particle is scattered at coordinate 2 and the spectator can be either a particle or a hole. In the second term a particle-hole pair is destroyed at coordinate 2, a hole is scattered at coordinate 1 and the spectator can again be a particle or a hole. The second and fourth

term are exchange terms of the above and they can be rewritten in exactly the same form as the first and the third term.

Since the coupling of these matrix elements is very complex, the coupling of one of the terms of equation (3.4) will be discussed in detail. The third term has been chosen.

The operators shall first be rewritten, using

$$\sum_m \alpha^+_{jm} \, | \tilde{\varphi}_{jm} \rangle = -\sum_m \tilde{\alpha}^+_{jm} \, | \varphi_{jm} \rangle \tag{3.5}$$

The operator can be represented by

$$\begin{aligned} (V_{24})_3 = -\sum_{j's, m's} & u_{c,d} \, v_{ja} \, v_{jb} \\ & \langle \tilde{\varphi}_{j_b m_b} \, \tilde{\varphi}_{jc m_c} \, | V | \, \varphi_{ja m_a} \, \varphi_{jd m_d} \rangle \\ & \alpha^+_{ja m_a} \, \alpha_{jb m_b} \, \alpha_{jc m_c} \, \alpha_{jd m_d} \end{aligned} \tag{3.6}$$

In equation (3.6) the matrix element consists only of angular momentum eigenstates, whereas the operators are all complex conjugates of irreducible tensor operators (ITO).

An ITO is defined in the following way: An ITO , T^q_k, is one which transforms under rotations as an eigenstate of angular momentum k with projection q.

$$U\, T^q_k\, U^+ = \sum_{q'} \mathcal{D}^k_{q'q}(U)\, T^{q'}_k \tag{3.7}$$

where $D^k_{qq'}(U)$ are the matrix elements of the rotation, which are defined by the expansion of the rotated states in terms of eigenstates of $|jm\rangle$.[9] Therefore the same Clebsch-Gordan Coefficients can be used for both the coupling of the quasiparticle operators as well as the coupling of the matrix element.

$$\left(V_{24}\right)_3 = \sum_{j's,\lambda's,\mu's} (-1)^{\lambda_2 - j_c - j_d}\, u_{c,d}\, v_{j_a} v_{j_b} \langle \tilde{\varphi}_{j_b} \overbrace{\tilde{\varphi}_{j_c} \mid V \mid \varphi_{j_a}}^{\lambda_1 \mu_1} \underbrace{\varphi_{j_d}}_{\lambda_2 \mu_2} \rangle \left[\alpha^+_{j_a} \alpha_{j_b}\right]_{\lambda_1 \mu_1} \left[\alpha_{j_c} \alpha_{j_d}\right]_{\lambda_2 \mu_2}$$

(3.8)

The next step is to use the schematic model, that is use a separable interaction and degenerate energies. Let T_{LM} be an ITO. Then the interaction matrix element can be written as

$$\langle \overbrace{\tilde{\varphi}_{j_b}\ \tilde{\varphi}_{j_c}}^{\lambda_1 \mu_1} | V | \underbrace{\varphi_{j_a}\ \varphi_{j_d}}_{\lambda_2 \mu_2} \rangle$$

$$= \sum_{L,M} \langle \tilde{\varphi}_{j_b} | T_{LM} | \varphi_{j_a} \rangle \langle \tilde{\varphi}_{j_c} | T^{+}_{LM} | \varphi_{j_d} \rangle \tag{3.9}$$

Using the Wigner-Eckart theorem gives

$$\sum_{L,M,\,m's} \langle j_b\, j_a\, m_b\, m_a \mid \lambda_1 \mu_1 \rangle \langle j_c\, j_d\, m_c\, m_d \mid \lambda_2 \mu_2 \rangle$$
$$\langle j_a\, j_b\, m_a\, m_b \mid L\, -M \rangle \frac{1}{\hat{L}} (-1)^{j_a + j_b - m_a - m_b}$$
$$\langle j_c\, j_d\, -m_c\, -m_d \mid L\, -M \rangle \frac{1}{\hat{L}} (-1)^{2j_c}$$
$$\langle \varphi_{j_b} \| T_L \| \varphi_{j_a} \rangle \langle \varphi_{j_d} \| T_L \| \varphi_{j_c} \rangle^{*} \tag{3.10}$$

Using symmetry conditions of the Clebsch-Gordan coefficients and defining $D_{ba}(L)$ by

$$\langle \varphi_{j_b} \| T_L \| \varphi_{j_a} \rangle = \mathcal{D}_{ba}(L) \tag{3.11}$$

we can represent the interaction can finally by

$$(V_{2L})_3 = \frac{1}{2} \sum_{j's} u_{c,d}\, v_{j_a}\, v_{j_b}\, (-1)^{\lambda}\, \hat{\lambda}\, \mathcal{D}_{ba}(\lambda)\, \mathcal{D}_{dc}(\lambda)$$
$$\left[\left[\tilde{\alpha}^{+}_{j_a}\, \alpha_{j_b} \right]_{\lambda} \left[\alpha_{j_c}\, \alpha_{j_d} \right]_{\lambda} \right]_{00} \tag{3.12}$$

Similar calculations are done for the other three terms of this interaction. Substituting all the results back into equation 3.4 the complete collective matrix element is given by

$$V^{C}_{24} = 48 \sum_{j's} X_{12}(J'M') \; X_{345}(J(\lambda_1\lambda_2)M)$$
$$u_{34}\, \mathcal{D}(\lambda) \; (u_{j_1}u_{j_5} + v_{j_1}v_{j_5})\, \mathcal{D}_{15}(\lambda)\, (-1)^{\lambda}\hat{\lambda}$$
$$\langle J'(j_1 j_2) M' | [[\tilde{\alpha}^{+}_{j_1}\, \alpha_{j_5}]_{\lambda}\, [\alpha_{j_4}\alpha_{j_3}]_{\lambda}]_{00}$$
$$| J(\lambda_1(j_3 j_4)\, \lambda_2(j_5 j_6))M\rangle \qquad (3.13)$$

The part $[\alpha_{j_4}\alpha_{j_3}]_{\lambda}\, u_{3,4}\, q_{34}(\lambda)$ describes the destruction of a quasiparticle pair. The part $[\alpha^{+}_{j_1}\alpha_{j_5}]_{\lambda}\, (u_{j_1}u_{j_5} + v_{j_1}v_{j_1})$ describes the scattering of a quasiparticle, which in the closed shell limit either corresponds to a particle (uu) or to a hole (vv).

It is more "physical" to couple each of the two terms above than to couple the wavefunction and the operator seperately. Therefore a recoupling will be done.

$$[[\tilde{\alpha}^{+}_{j_1}\, \alpha_{j_5}]_{\lambda}\, [\alpha_{j_4}\alpha_{j_3}]_{\lambda}]_{00} \; | J(\lambda_1(j_3 j_4)\, \lambda_2(j_5 j_6))\rangle$$
$$= \sum_{L_1 L_2} U\begin{pmatrix} \lambda & \lambda & 0 \\ \lambda_2 & \lambda_1 & J \\ L_1 & L_2 & J \end{pmatrix} \; (-1)^{J-\lambda_1-\lambda_2}$$
$$[[[\widetilde{\tilde{\alpha}^{+}_{j_1}\, \alpha_{j_5}}]_{\lambda}\, |\lambda_2(j_5 j_6)\rangle]_{L_1}\, [[\widetilde{\alpha_{j_4}\alpha_{j_3}}]_{\lambda}|\lambda_1(j_3 j_4)\rangle]_{L_2}]_{JM}$$

(3.14)

where

$$[\widetilde{\alpha_{j_4}\alpha_{j_3}}]_{\lambda\mu} = (-1)^{\lambda-\mu}[\alpha_{j_4}\alpha_{j_3}]_{\lambda-\mu} \tag{3.15}$$

This convention is needed to transform the complex conjugate of an ITO to an ITO or vice versa. An angular momentum coupling is only possible of two ITO's or of two complex conjugate ITO's but never of an ITO and a complex conjugate ITO.

The rest of the calculation is straightforward and only the results will be stated.

$$\langle 0 | [[\alpha_{j_4}\alpha_{j_3}]_{\lambda} | \lambda_1 (j_3 j_4)\rangle]_{L_2M_2}$$

$$= \hat{\lambda}\ \delta_{L_2,0}\ \delta_{M_2,0}\ \delta_{\lambda,\lambda_1}\ \delta_{\mu_1,-\mu_1} \tag{3.16}$$

The 9-j coefficient can be calculated by

$$\mathcal{U}\begin{pmatrix} \lambda & \lambda & 0 \\ \lambda_2 & \lambda_1 & J \\ L_1 & 0 & J \end{pmatrix} = (-1)^{J+\lambda_1-\lambda_2}\ \frac{\delta_{\lambda,\lambda'}\ \delta_{J,L_1}}{\hat{\lambda}_1\ \hat{\lambda}} \tag{3.17}$$

With equation 6.25 of Rose[10]

$$\langle J'(j_1 j_2) M' | [[\widetilde{\alpha^{+}_{j_1}\alpha_{j_5}}]_{\lambda} | \lambda_2 (j_5 j_6)\rangle]_{JM}$$

$$= -(-1)^{\lambda_2+j_6-j_5}\ \frac{\hat{\lambda}_2}{\hat{\lambda}}\ \delta_{j_2,j_6}\ W(j_5\ \lambda_2\ j_1\ J;\ j_6\ \lambda)$$

$$\langle j_1 \| [\alpha^{+}_{j_1}\ \widetilde{\alpha}_{j_5}]_{\lambda} \| j_5\rangle \tag{3.18}$$

The reduced matrix element is given by

$$\langle j_1 \| [\alpha^+_{j_1} \tilde{\alpha}_{j_5}] \| j_5 \rangle = -\hat{\lambda} \tag{3.19}$$

Substituting equations (3.15) through (3.19) into (3.14) gives the final result for the one phonon to two phonon transition matrix element for collective states.

$$V_{24}^{c} = 48 \sum_{j_1 j_5} X_{12}(JM)\, X_{3452}(J(\lambda\lambda_2)M)\, \hat{\lambda}\,\hat{\lambda}_2\, (-1)^{\lambda_2}$$
$$u_{3,4}\, \mathcal{D}_{34}(\lambda)\, (u_{j_1} u_{j_5} + v_{j_1} v_{j_5})\, \mathcal{D}_{15}(\lambda)$$
$$W(j_5\, \lambda\, j_1\, J\, j_2\, \lambda) \tag{3.20}$$

Using the results for the amplitudes from the previous chapter yields

$$
\begin{aligned}
V_{24}^{C} = 48 \sum_{j'_1} & \left(2\,Y(\lambda)\right)^{-\frac{1}{2}} \; u_{1,2}\, \mathcal{D}_{12}(\lambda)\; u_{3,4}\, \mathcal{D}_{34}(\lambda)\; \hat{\lambda}\,\hat{\lambda}_2\, (-1)^{\lambda_2} \\
& \left(u_{j_1} u_{j_5} + v_{j_1} v_{j_5}\right) W\left(j_5\, \lambda\; j_1\, J\; j\; j_2\, \lambda\right) \\
& \Bigg[\frac{k(J(\lambda\,\lambda_2)M)}{\pi\, Y(\lambda)\, Y(\lambda_2)}\; u_{5,2}\, \mathcal{D}_{52}(\lambda_2)\; u_{3,4}\, \mathcal{D}_{34}(\lambda) \\
& + \sum_{\lambda'_1 \lambda'_2} \frac{Y(\lambda'_1) + Y(\lambda'_2)}{Y(\lambda) + Y(\lambda_2)}\; \frac{k(J(\lambda'_1 \lambda'_2)M)}{Y(\lambda'_1)\, Y(\lambda'_2)} \\
& \quad * \Bigg(-\mathcal{U}\begin{pmatrix} j_3 & j_4 & \lambda \\ j_5 & j_2 & \lambda_2 \\ \lambda'_1 & \lambda'_2 & J \end{pmatrix}\; u_{35}\, \mathcal{D}_{35}(\lambda'_1)\; u_{4,2}\, \mathcal{D}_{42}(\lambda'_2) \\
& \quad + \mathcal{U}\begin{pmatrix} j_3 & j_4 & \lambda \\ j_2 & j_5 & \lambda_2 \\ \lambda'_1 & \lambda'_2 & J \end{pmatrix}\; u_{3,2}\, \mathcal{D}_{32}(\lambda'_1)\; u_{4,5}\, \mathcal{D}_{45}(\lambda'_2)\; (-1)^{\lambda_2 - j_5 - j_1} \Bigg) \Bigg]
\end{aligned}
$$

(3.21)

IV. MIXING OF THE FOUR-QUASIPARTICLE COLLECTIVE STATE WITH THE TWO-QUASIPARTICLE COLLECTIVE STATE

The mixing of the four-quasiparticle collective state with the two-quasiparticle collective state is determined in diagonalizing the matrix

$$\begin{matrix} H_{22} & V_{24} \\ V_{24} & H_{44} \end{matrix} \qquad (4.1)$$

where H_{22} and H_{44} are the the two-quasiparticle energy and the four-quasiparticle energy, respectively. The eigenvalues of the matrix (4.1) give the shifted energy states. This calculation is done in a truncated space. Only quadrupole states are taken into account. As mentioned in the previous chapter, this interaction represents part of the leading-order anharmonic effect in the vibrational motion. As an example the isotopes of palladium, ^{106}Pd, ^{108}Pd and ^{110}Pd, and the isotopes of selenium, ^{74}Se, ^{76}Se and ^{78}Se, are chosen. The results of the calculations are listed in the tables 4.1 and 4.2.

The reduced matrix elements are given by

$$\begin{aligned} &\langle N' \ell' j' \| i^{\lambda} Y_{\lambda} r^{\lambda} \| N \ell j \rangle \\ &= (4\pi)^{-\frac{1}{2}} \; i^{\lambda + \ell - \ell'} (-1)^{j - \frac{1}{2}} \hat{j} \hat{j}' \langle j' \tfrac{1}{2} \; j - \tfrac{1}{2} \mid \lambda 0 \rangle \\ &\tfrac{1}{2} \left(1 + (-1)^{\ell' + \ell - \lambda} \right) \langle N' \ell' \mid r^{\lambda} \mid N \ell \rangle \left(\frac{m \omega_0}{\hbar} \right)^{\frac{1}{2}} \end{aligned} \qquad (4.2)$$

where the radial integrals were computed using Simpson's rule.

The pairing constant G, the quasiparticle energy E_Q and the occupation and nonoccupation numbers u_j and v_j are determined by solving the pairing equations for the degenerate case, as discussed in chapter 1.3. In the case of the palladium isotopes the pairing constants G_n and G_p for neutrons and protons were adjusted such that the quasiparticle energy is given by the weighted average of E_Q^n and E_Q^p.

The interaction strength χ is determined such that the first 2^+-state agrees with the experimental value. The experimental data are taken from the Nuclear Data Sheets.[11]

Table 4.1 Energies of the first and the second 2^+ state without the two-quasiparticle, four-quasiparticle coupling matrix element and including the two-quasiparticle, four-quasiparticle coupling matrix element compared to the experimental data[11]. The four-quasiparticle states are normalized including the exchange term.

Isotope	Energy in MeV					
	without coupling		including coupling		experimental values	
	2^+_1	2^+_2	2^+_1	2^+_2	2^+_1	2^+_2
^{106}Pd	0.633	1.265	0.512	1.385	0.512	1.128
^{108}Pd	0.568	1.136	0.433	1.271	0.433	0.931
^{110}Pd	0.519	1.038	0.374	1.183	0.374	0.813
^{74}Se	0.796	1.592	0.634	1.755	0.634	1.269
^{76}Se	0.733	1.465	0.559	1.639	0.559	1.216
^{78}Se	0.785	1.571	0.614	1.742	0.614	1.308

Table 4.2 Energies of the first and the second 2^+ state without the two-quasiparticle, four-quasiparticle coupling matrix element and including the two-quasiparticle, four-quasiparticle coupling matrix element compared to the experimental data[11]. The four - quasiparticle states are normalized using the direct term only.

Isotope	Energy in MeV					
	without coupling		including coupling		experimental values	
	2^+_1	2^+_2	2^+_1	2^+_2	2^+_1	2^+_2
^{106}Pd	0.594	1.188	0.512	1.271	0.512	1.128
^{108}Pd	0.528	1.056	0.433	1.152	0.433	0.931
^{110}Pd	0.478	0.957	0.374	1.061	0.374	0.813
^{74}Se	0.728	1.457	0.634	1.552	0.634	1.269
^{76}Se	0.662	1.323	0.559	1.426	0.559	1.216
^{78}Se	0.714	1.428	0.614	1.528	0.614	1.308

The strength of the crossover transition $2^+_2 \rightarrow 0^+_1$ can be determined by the ratio of the amplitudes of the two-quasiparticle state and the four-quasiparticle state. If the 2^+_2 state were a pure four - quasiparticle state, as the harmonic model predicts, no such transition would be possible. However, due to the two - quasiparticle four - quasiparticle mixing this state is actually a mixture of a two - quasiparticle and a four - quasiparticle state which makes such a transition possible. The ratio of the amplitudes of the two - quasiparticle state and the four - quasiparticle state gives the amount of this mixture. This is compared to ratio of the deformation parameters $\beta_{2^+_1 \rightarrow 0^+_1}$ and $\beta_{2^+_2 \rightarrow 0^+_1}$. That those two ratios are actually the same can be explained in the following way: Both the first and

the second 2^+ states ($|2_1^+\rangle$ and $|2_2^+\rangle$, respectively) are taken to be a linear combination of the harmonic pure one-phonon and pure two-phonon states 2^+ states ($|2_1^+\rangle$ and $|2_2^+\rangle$, respectively).

$$|2_1^+) = a\,|2_1^+\rangle + c\,|2_2^+\rangle \tag{4.3}$$

$$|2_2^+) = -c\,|2_1^+\rangle + a\,|2_2^+\rangle \tag{4.4}$$

The reduced matrix elements of the dressed and pure states are given by

$$\langle 2_1^+ \| Q(2) \| 0_1^+ \rangle = d\,\beta_H \tag{4.5}$$

$$(2_1^+ \| Q(2) \| 0_1^+) = d\,\beta_{2_1^+ \to 0_1^+} \tag{4.6}$$

$$(2_2^+ \| Q(2) \| 0_1^+) = d\,\beta_{2_2^+ \to 0_1^+} \tag{4.7}$$

where d is a constant of proportionality. With equations (4.3) through (4.7) the following identity is deduced

$$\frac{\beta_{2_2^+ \to 0_1^+}}{\beta_{2_1^+ \to 0_1^+}} = \frac{c}{a} \tag{4.8}$$

The deformation parameters β are directly proportional to the square root of the corresponding transiton strength.[12] Therefore

$$\frac{c}{a} = \sqrt{\frac{B(E2_2^+)}{B(E2_1^+)}} \tag{4.9}$$

The transition strength was taken from the Nuclear Data Sheets.[10]

Table 4.3 $2_2^+ \rightarrow 0_1^+$ crossover transition strength. The first theoretical value is calculated using the direct term only for the four quasiparticle normalization. In the third column the exchange term was included for calculating the four quasiparticle norm. The numbers in the fourth column give the experimental values.[11]

Isotope	theory c/a	c/a	experiment $\beta_{2_2^+ \rightarrow 0_1^+} / \beta_{2_1^+ \rightarrow 0_1^+}$
^{106}Pd	0.348	0.402	0.162
^{108}Pd	0.390	0.438	0.150
^{110}Pd	0.421	0.468	0.126
^{74}Se	0.337	0.410	0.172
^{76}Se	0.368	0.438	0.139
^{78}Se	0.340	0.423	0.183

V. DISCUSSION OF RESULTS

We have derived the Tamm-Dancoff secular equation for four-quasiparticle states and have solved this equation in the degenerate limit using a separable interaction. The resulting amplitudes for the four-quasiparticle states include a term that has boson character, that is, only quasiparticle pairs are interchanged. This term was called the direct term. It also contains a term with fermion character, where an odd number of quasiparticles is interchanged. This term has been referred to as the exchange term. The exchange term also couples a state which is made out of quadrupole phonons to all other possible angular momenta.

We then applied the results of equation (2.2.2) and equation (2.3.15) to calculate the four-quasiparticle, two-quasiparticle coupling matrix element for low lying collective states for even nuclei. This was done in using the part of the two-body interaction of the form: $\alpha^+ \alpha \alpha \alpha$. This coupling matrix element has two important physical applications:

- It is part of the leading order anharmonic effect in the vibrational motion.
- It also describes damping procsses, a property not investigated in this study. This is especially useful for the description of giant resonance states, where it contributes to the damping of giant resonances.

The first of the above properties can be understood best in going to one of the boson expansion models.

5.1. Boson Expansion Methods

The basic idea of boson expansion methods is to represent the fermion Hamiltonian by pure boson operators B, B^+ and diagonalize it in a boson space.[4] Mathematically, this corresponds to a mapping of the Fock space of many fermion states into a space of boson states. There are many types of boson representations. All these methods start with a many-fermion Hilbert space H_f , which contains vectors and operators. In second quantization the space is completely specified by the vacuum $|0\rangle$ and a set of fermion operators α_k, $\alpha_k^\dagger$. The fermion space is then mapped onto a different Hilbert space, a space of bosons H_b. The boson space is given by a boson vacuum $|0)$ and boson creation and annihilation operators B_1, $B_1^\dagger$.

$$B_\mu |0) = 0 \tag{5.1.1}$$

These operators have to fulfill boson commutation relations

$$[B_\mu, B_{\mu'}^+] = \delta_{\mu\mu'} \tag{5.1.2}$$

It generally turns out that the boson space H_b is much larger than the fermion space H_f. This comes from the fact that in the boson expansion of a fermion operator there is usually an infinite number of boson operators needed. Therefore, the mapping is only unique in a certain subspace of the boson space. There are two basic procedures to introduce this mapping explicitly:

- Belyaev and Zelevinskii[13] propose to map the operators in such a way that the commutation relations are preserved. Usually all

important operators can be constructed from a set of basic operators whose commutation relations form an algebra. Vectors obtained in this way do not span the whole boson space but only a physical subspace.

- Marumori[14,15] proposed to map vectors in the Hilbert spaces H_f and H_b and to define the operators in such a way that the matrix elements are conserved by the mapping. The mapping of fermion states to boson states by this prescription, however, is not unique.

The advantage of such boson representations is that very collective modes - for instance, the quadrupole mode - can be approximated by rather simple wave functions, namely one-boson states $B^+ |0\rangle$. In all the applications, however, drastic approximations have to be used, which are justified only in special physical cases.

5.2 Anharmonicity

Harmonic oscillations are characterized by an equidistant spectrum. Experimental spectra of spherical nuclei in some regions of the periodic table qualitatively show this structure (see 1.1). In detail, however, some more or less dramatic deviations from this simple picture, as, for instance, splitting of the higher multiplets and a shift in the position of the energy centroid are found. Such anharmonicities are caused by two effects:[4]

- The collective fermion pairs are not exact bosons.

- The exact many-body Hamiltonian contains not only second-order terms (two-body term) in the bosons, which diagonalize the TDA approach, but also higher order terms. Such terms, neglected in the harmonic approximation, provide a coupling between the different harmonic modes.

The next higher order term in the Hamiltonian is the three-body operator. In the boson approximation it is represented by B^+ B B. $\alpha^+ \alpha \alpha \alpha$ is part of this boson operator not taken into account in a boson expansion. It can therefore be interpreted as part of the leading order anharmonic effect of the vibrational motion.

We have investigated this effect for low-lying 2^+ states. As expected, the 2^+ states push each other apart due to their mixing and a splitting of the higher multiplets can be observed. However, the $J=0^+$ and the $J=4^+$ are still degenerate.

In figure 5.2.1 and 5.2.2 our results for the energy spectrum are graphically compared with the experimental values for the case of ^{76}Se and ^{106}Pd. In the case of ^{76}Se they are, in addition, compared with the theoretical results of S. G. Lie and G. Holzwarth[1]. They start out with the fermion Hamiltonian as described in equation (1.2.3), then a Bogoliubov transformation is done. Finally they make use of a modified Marumori's expansion to map the fermion quasiparticle Hamiltonian into the boson space including boson operators up to fourth order. Marumori's expansion is modified in the sense that the boson expansion is formulated from the very beginning in terms of the collective operators or states. Their phenomenological Hamiltonian contains seven independent parameters which can be fitted to obtain

the eight triplet plus quintet members of the phonon spectrum of a given nucleus. Those parameters make it possible to fit the experimental data quite accurately as can be seen in figure 5.2.1 (f). It should be noted however, that dependent on their fitting parameters they can achieve any possible order for the triplet state and also a vast variety of energy shifts as shown in their paper for the case of ^{100}Ru.

In contrast to this we have only one free parameter, the interaction strength, which is fitted such that the first 2^+ state has the experimental value.

We have also compared the crossover transition strength for ^{76}Se and ^{78}Se to the results of Lie and Holzwarth. Those are listed in table 5.2.1.

Table 5.2.1 $2_2^+ \rightarrow 0_1^+$ crossover transition strength. The first two columns of numbers give our theoretical values, where the first column uses the direct term only for the four quasiparticle normalization. In the second column the exchange term was included for calculating the four quasiparticle norm. The third column of numbers gives the theoretical values of Lie and Holzwarth. The fourth column gives the experimental value.[11]

Isotope	theory c/a	theory c/a	c/a	experiment $\beta_{2_2^+ \rightarrow 0_1^+} / \beta_{2_2^+ \rightarrow 0_1^+}$
^{76}Se	0.368	0.438	0.247	0.139
^{78}Se	0.340	0.423	0.139	0.183

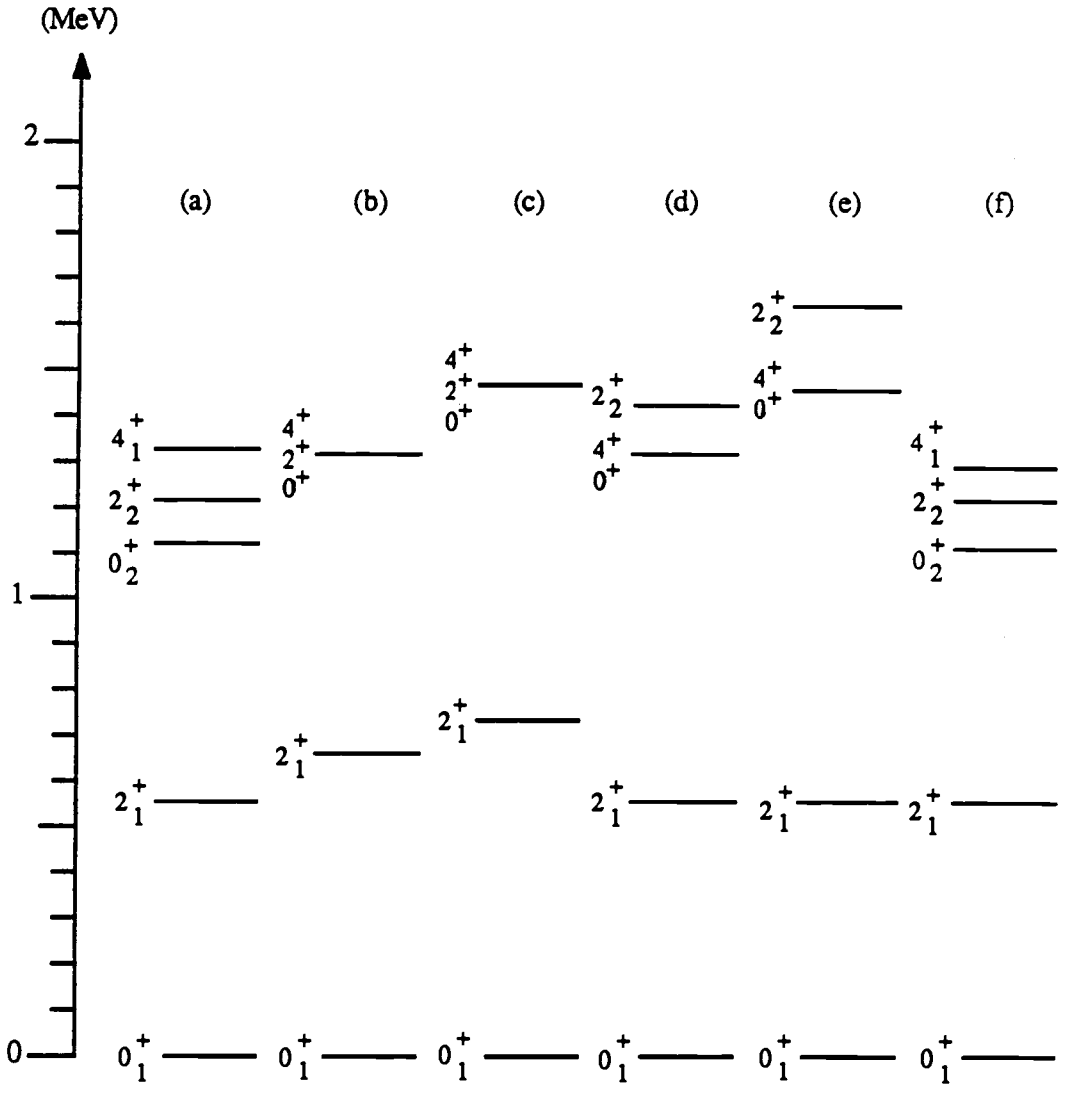

Figure 5.2.1 Energy level diagrams for ^{76}Se

(a) Experimental values.
(b) Harmonic spectrum using the direct term only for the four-quasiparticle normalization.
(c) Harmonic spectrum including the exchange term in the four-quasiparticle normalization.
(d) Results including the two-quasiparticle, four-quasiparticle coupling using the direct term only for the four quasiparticle normalization.
(e) Spectrum including including the two-quasiparticle, four-quasiparticle coupling. The exchange term is included for determining the normalization for the four quasiparticle state.
(f) Spectrum of Lie and Holzwarth.

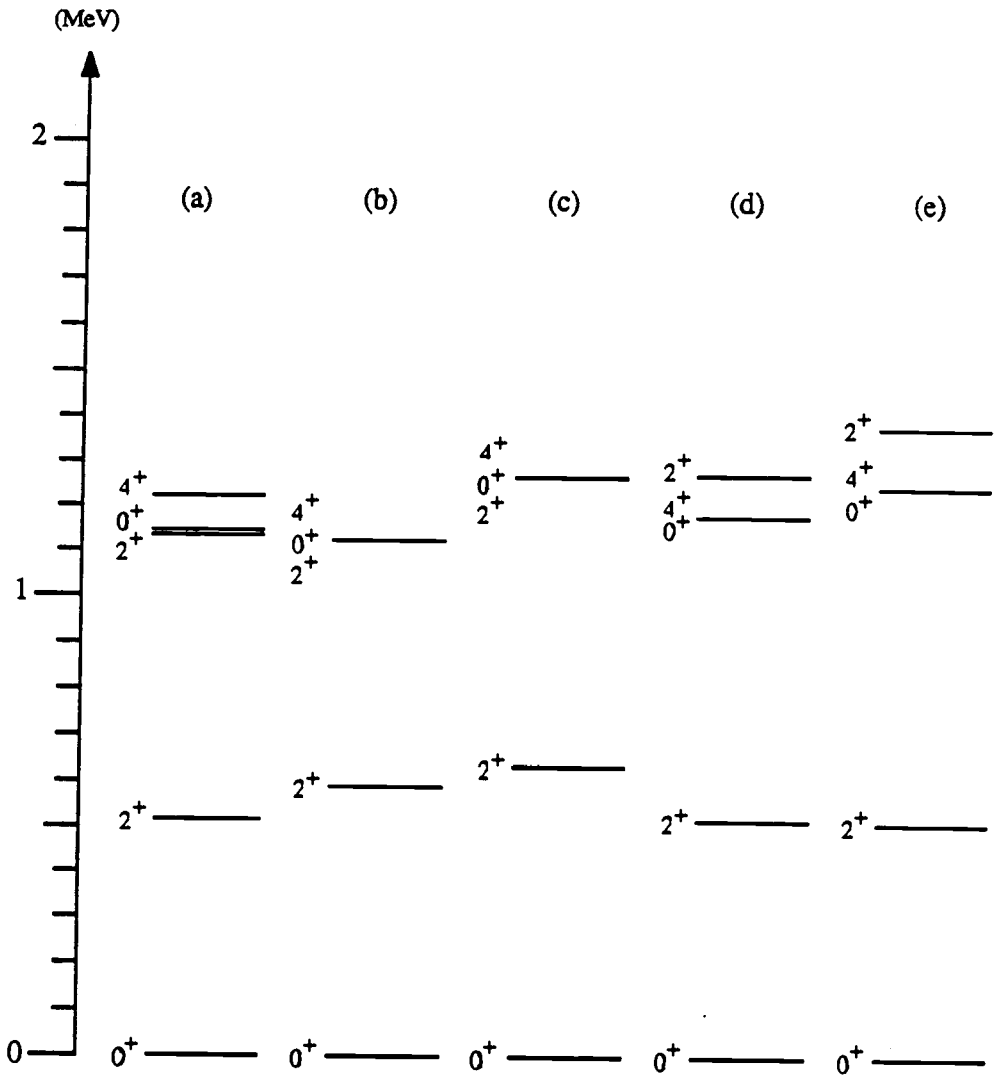

Figure 5.2.2 Energy level diagrams for ^{106}Pd

(a) Experimental values.

(b) Harmonic spectrum using the direct term only for the four-quasiparticle normalization.

(c) Harmonic spectrum including the exchange term in the four-quasiparticle normalization.

(d) Results including the two-quasiparticle, four-quasiparticle coupling using the direct term only for the four-quasiparticle normalization.

(e) Spectrum including including the two-quasiparticle, four-quasiparticle coupling. The exchange term is included for determining the normalization for the four-quasiparticle state.

5.3 Methods for Further Improvements

Our results, including the energy level diagram and the crossover transition, indicate for all isotopes investigated that the two-quasiparticle, four-quasiparticle coupling matrix element came out too strong by a factor of approximately 2.5, i. e. our results are very close to the experimental values if a factor of 2.5 is included. Mechanisms that could account for this consitant behaviour are:

- Our calculations were done in a truncated space. Including the λ sum in the exchange term presumably lowers the value of the coupling matrix element.
- Since we have used TDA quasiparticles, only ground state correlations coming from the short range part of the nuclear force have been taken into account. In order to be more precise one should use RPA quasiparticle states, which include both ground state correlations due to the short range part and the long range part of the nuclear force. The ground state correlations due to the pairing force are larger by a factor of about three compared with the RPA particle-hole ground state correlations.[16]
- Ccore polarization, which is the effect on the interaction, if states outside the model space are eliminated.
- Coupling to noncollective states[17].

All of those effects push the first 2^+ level down in energy. If they are included, the interaction strength needed to fit the first 2^+

state is not as strong, giving therefore a smaller two-quasiparticle, four-quasiparticle mixing.

In the case of the Palladium isotopes the second 2+ state is experimentally lower than the two other states of the triplet, whereas our theory predicts the second 2^+ state to be higher than the 0^+ and the 4^+ state. This deviation can have its origin in two effects. In the schematic model the first 2^+ state gets pushed down in energy whereas the second 2^+ state gets pushed up in energy. Noncollective two-quasiparticle states, which have an energy of about 3 MeV and which are not accounted for in this study, tend to push the second 2^+ state down in energy. Another effect being worth investigating is the influence of the rotational spectrum. Although the nuclei we discussed are of primarily vibrational character, the coexistance of rotations and vibrations must not be neglected completely. Coexistance of the rotational 4^+ state and the vibrational 4^+ state is expected to raise the 4^+_1 state in energy.

We have used a degenerate model in order to solve the TDA equation of motion, since this is the only case in which the equation is soluble in closed form. Although this procedure is admittedly crude and is not meant to be a substitute for detailed nuclear-structure calculations, the assumption of degeneracy for the major filling shell is not as crude as one might think. The inclusion of pairing forces puts in a gap between the ground state and the two-quasiparticle excitations of about 3 MeV for medium-mass nuclei. With this gap as a base, the variation of two-quasiparticle energies is very much less than that of the independent particle levels.[18]

We have not made a distinction between protons and neutrons. However, because of the shell structure and the Pauli principle, differences in the freedom of neutrons and protons to vibrate are expected and actually observed. The ratio of neutron and proton multipole matrix elements for collective vibrations differs, in general, from N/Z. This is particularly true for single-closed-shell nuclei, where one type of nucleon is "frozen in" by the shell closure.[19] In our work we are dealing with open-shell nuclei since closed-shell nuclei don't have vibrational spectra as discussed here. For these nuclei the ratio of neutron and proton multipole matrix elements for collective vibrations may also depart from N/Z but the differences tend to be much smaller than for single closed shell nuclei. In addition to this, we are dealing with nearly isoscalar states, so the use of an isoscalar force, which does not distingush between neutrons and protons, is expected to be quite accurate.[20]

5.4 Suggested Applications for our Formalism

We have investigated the effect of the four-quasiparticle, two-quasiparticle coupling matrix element for low-lying 2^+ states, but it can also be applied to high-lying states ($2\hbar\omega$). Especially interesting in this context would be to investigate its effect on the broadening of giant resonances.

The effects of core polarization on the isospin has so far been investigated for $2\hbar\omega$ two-quasiparticle states.[21] Formally this is

done using the Feshbach formalism to eliminate the $2\hbar\omega$ space. This calculation shows that after the transformed Hamiltonian is projected into the $0\hbar\omega$ space it is no longer isospin conserving, i. e. the proton - proton force is unequal to the neutron - neutron force. It would be interesting to see how the inclusion of four-quasiparticle states affects this feature, i. e. using the Feshbach formalism to eliminate both, the $2\hbar\omega$ space and the four-quasiparticle $1\hbar\omega$ space, which corresponds to the same harmonic oscillator energy.

REFERENCES

1. S. G. Lie and G. Holzwarth, Phys. Rev. C12, 1035 (1975)

2. J. M. Eisenberg and W. Greiner, Nuclear Models, North Holland, Amsterdam (1970)

3. F. Iachello and I. Talmi, Rev. Mod. Phys., 59 (1987)

4. P. Ring and P. Schuck, The Nuclear Many-Body Problem, Springer Verlag (1980)

5. P. J. Siemens, private communication (June 1988)

6. C. E. Brown, Unified Theory of Nuclear Models and Forces, North Holland Publishing Company, third edition (1971)

7. V. Madsen, Seminar Talk (1987)

8. S. Yoshida, Nuc. Phys. 38, 380 (1962)

9. A. R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press (1974)

10. M. E. Rose, Elementary Theory of Angular Momentum, Wiley, New York (1957)

11. Nuclear Data Sheets, 53, 1 (1988), 37,2 (1982), 38,4 (1983), 51, 2 (1987), 42, 3 (1984), 33, 2 (1981)

12. A. Bohr and B. R. Mottelson, Nuclear Structure, volume I and II, W. A/ Benjamin, INC. (1969) and (1975)

13. T. Belyaev and V. G. Zelevinskii, Nucl. Phys 39, 582 (1962)

14. T. Marumori, M. Yamamura and A. Tokunaga, Progr. Theor. Phys. 31, 1007 (1964)

15. T. Marumori, M. Yamamura, A. Tokunaga and A. Takada, Progr. Theor. Phys. 32, 726 (1964)

16. J. Heisenberg, lecture in the Nuclear Physics Summer School 1988

17. V. A. Madsen and V. R. Brown, to be published in Phys. Rev. C (1988)

18. V. A. Madsen and V. R. Brown, Phys. Rev. Lett. 52, 176 (1984)

19. V. A. Madsen and V. R. Brown, AIP Conference Proceedings 124, 171 (1984)

20. C. J. Veje, Mat. Fys. Medd. Dan. Vid. Selsk. 35, 1 (1966)

21. V. A. Madsen and V. R. Brown, private communication (1988)

Appendices

A. SYMMETRY PROPERTIES OF THE COUPLED FOUR-QUASIPARTICLE AMPLITUDE

The fact that the four-quasiparticle amplitude in product representation (equation 1.4.3) is completely antisymmetric with respect to permutations of quasiparticles can mathematically expressed by

$$X_{1234} = (-1)^{P} P\left(X_{1234}\right) \tag{A.1}$$

where P stands for permutation and $(-1)^{P}$ is positive for any even permutation and negative for any odd permutation. Using equation (1.4.7) the following relation holds for the amplitude in coupled representation.

$$\begin{aligned}
&\sum_{\lambda's,\mu's,J,M} \langle j_1 j_2 m_1 m_2 \mid \lambda_1 \mu_1 \rangle \langle j_3 j_4 m_3 m_4 \mid \lambda_2 \mu_2 \rangle \\
&\qquad \langle \lambda_1 \lambda_2 \mu_1 \mu_2 \mid J M \rangle \, x_{1234}\left(J(\lambda_1 \lambda_2) M\right) \\
&= (-1)^{P} P\Big(\sum_{\lambda's,\mu's,J,M} \langle j_1 j_2 m_1 m_2 \mid \lambda_1 \mu_1 \rangle \langle j_3 j_4 m_3 m_4 \mid \lambda_2 \mu_2 \rangle \\
&\qquad \langle \lambda_1 \lambda_2 \mu_1 \mu_2 \mid J M \rangle \, x_{1234}\left(J(\lambda_1 \lambda_2) M\right)\Big)
\end{aligned} \tag{A.2}$$

Using angular momentum algebra, 24 symmetry properties of the coupled four-quasiparticle amplitude can be derived. Five different examples are listed below. The 9j-coefficients come from angular momentum recouplings.

$$x_{2134}\left(J(\lambda_1\lambda_2)M\right) = (-1)^{\lambda_1+j_1-j_2}\, x_{1234}\left(J(\lambda_1\lambda_2)M\right) \tag{A.3}$$

$$x_{3412}\left(J(\lambda_1\lambda_2)M\right) = (-1)^{J-\lambda_1-\lambda_2}\, x_{1234}\left(J(\lambda_2\lambda_1)M\right) \tag{A.4}$$

$$x_{1423}\left(J(\lambda_1\lambda_2)M\right) = \sum_{\lambda_1'\lambda_2'} u\begin{pmatrix} j_1 & j_2 & \lambda_1' \\ j_4 & j_3 & \lambda_2' \\ \lambda_1 & \lambda_2 & J \end{pmatrix} (-1)^{\lambda_2'-j_3-j_4}\, x_{1234}\left(J(\lambda_1'\lambda_2')M\right) \tag{A.5}$$

$$x_{2314}\left(J(\lambda_1\lambda_2)M\right) = \sum_{\lambda_1'\lambda_2'} u\begin{pmatrix} j_2 & j_1 & \lambda_1' \\ j_3 & j_4 & \lambda_2' \\ \lambda_1 & \lambda_2 & J \end{pmatrix} (-1)^{\lambda_1'-j_1-j_2}\, x_{1234}\left(J(\lambda_1'\lambda_2')M\right) \tag{A.6}$$

$$x_{2413}\left(J(\lambda_1\lambda_2)M\right) = -\sum_{\lambda_1'\lambda_2'} u\begin{pmatrix} j_2 & j_1 & \lambda_1' \\ j_4 & j_3 & \lambda_2' \\ \lambda_1 & \lambda_2 & J \end{pmatrix} (-1)^{\lambda_1+\lambda_2'-j_1-j_2-j_3-j_4}\, x_{1234}\left(J(\lambda_1'\lambda_2')M\right) \tag{A.7}$$

B. EXCHANGE TERMS

In equations (2.3.10) and (2.3.14) the exchange terms have been suppressed. That they are smaller can be understood qualitatively in that the direct terms comprise constructive interference ($Y(\lambda)$) whereas in the exchange terms the random phases of the individual summands tend to cancel each other. Furthermore, the exchange terms are all multiplied by a coefficient coming from the recoupling, which is always less than or equal to one - usually well below one. For the selenium and palladium isotopes discussed in this work the exchange terms turn out to be less than ten percent in magnitude of the direct terms and in most cases well below. As an example, the tables B.1, B.2, B.3 and B.4 give the ratio of the exchange term to the direct term (of equation 2.3.10) for ^{106}Pd and ^{76}Se, respectively.

Table B.1 Ratio of the exchange term to the direct term of equation (2.3.10) of ^{106}Pd for proton quadrupole transitions

initial state			final state			exchange term/direct term
n	l	j	n	l	j	
3	1	1.5	3	1	1.5	0.003
3	1	1.5	3	3	2.5	0.014
3	1	1.5	3	1	0.5	0.021
3	3	2.5	3	3	2.5	0.013
3	3	2.5	3	1	0.5	0.030
4	4	4.5	4	4	4.5	0.031

Table B.2 Ratio of the exchange term to the direct term of equation (2.3.10) of ^{106}Pd for neutron quadrupole transitions

initial state			final state			exchange term/direct term
n	l	j	n	l	j	
4	2	2.5	4	2	2.5	0.030
4	2	2.5	4	4	3.5	0.046
4	2	2.5	4	2	1.5	0.541
4	2	2.5	4	0	0.5	0.065
4	4	3.5	4	4	3.5	0.038
4	4	3.5	4	2	1.5	0.066
5	5	5.5	5	5	5.5	0.054
4	2	1.5	4	2	1.5	0.094
4	2	1.5	4	0	0.5	0.100

Table B.3 Ratio of the exchange term to the direct term of equation (2.3.10) of ^{76}Se for proton quadrupole transitions

initial state			final state			exchange term/direct term
n	l	j	n	l	j	
3	1	1.5	3	1	1.5	0.005
3	1	1.5	3	3	2.5	0.027
3	1	1.5	3	1	0.5	0.042
3	3	2.5	3	3	2.5	0.026
3	3	2.5	3	1	0.5	0.061
4	4	4.5	4	4	4.5	0.060

Table B.4 Ratio of the exchange term to the direct term of equation (2.3.10) of ^{76}Se for neutron quadrupole transitions

initial state			final state			exchange term/direct term
n	l	j	n	l	j	
3	1	1.5	3	1	1.5	0.006
3	1	1.5	3	3	2.5	0.030
3	1	1.5	3	1	0.5	0.046
3	3	2.5	3	3	2.5	0.028
3	3	2.5	3	1	0.5	0.066
4	4	4.5	4	4	4.5	0.066

The ratio of exchange to direct term of eqation (2.3.14) are 0.051 for ^{106}Pd and 0.048 for ^{76}Se.