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This thesis gives a procedure for generalizing the concept of conditional probability applied to the case of the bivariate probability density function. The conditional probability density function is desired where the two variables vary according to some prescribed path ϕ which lies in the plane of the two variables. Instead of the usual restriction of one or the other variable being a constant, we can vary both variables simultaneously, so long as the relationship between them, expressed by the function ϕ, remains a constant. Then we have the conditional probability of the bivariate probability density function along the path ϕ.

Conditional Probability Along a Path

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CONDITIONAL PROBABILITY ALONG A PATH

EXPLANATION OF THE PROBLEM

The problem is to find the density function of a random variable along a known path within a known bivariate density function. We know the density $f(x_1, x_2; \theta)$ of a two-dimensional random variable (x_1, x_2), where both components, x_1 and x_2, are statistically independent random variables. The assumption of independence allows us to write the simplest case of the bivariate density function, where the joint density is just the product of the two marginal densities for x_1 and x_2. Symbolically, $f(x_1, x_2) = f(x_1)f(x_2)$. Thus we can simplify the illustrations which follow, though independence of x_1 and x_2 does not seem to be a necessary condition for the process to work. θ denotes the parameter and may be multi-dimensional to accomodate densities with more than one parameter.

We also have a path in the x_1x_2-plane denoted by $\phi(x_1, x_2) = a$ of any general nature--conic section, polynomial, etc.--along which we want to find the density function of another variable, which assumes values on the path ϕ as it is described under the bivariate density f. We describe the distance along the path by the random variable z, a function of x_1 and x_2, starting from some prescribed point of origin on ϕ. The maximum value of z will depend on the domain of definition of f; that is, upon the limitations of x_1 and

x_2 prescribed by the given bivariate density, and by our arbitrary restriction to consider the density and the path only in the first quadrant for the sake of simplicity. Thus, z is sometimes finite and sometimes infinite.

Our final object is to determine $g(z|a)$, the conditional probability density function as a function of distance travelled from a point of origin along a specified path. $g(z|a)$ may or may not involve θ, the parameter of the given bivariate density function f.

We arrive at the notation $g(z|a)$ by noticing that $g(z|a)$ is the density function giving the probability of being z distance from the point of origin given that we satisfy the condition suggested by a; that is, that we are on the path $\phi(x_1, x_2) = a$. Thus, we have a type of conditional density function for z, given that z lies on ϕ under the bivariate density f, and we say $g(z|a)$ is the conditional density of z given that we are on a.

The usual interpretation of conditional probability involving a bivariate density function gives the density function of one of the variables while the other is held constant. For example, if we are given $f(x_1, x_2)$, a bivariate density, we may have the density of x_1 where x_2 is a constant value over the whole domain of x_1. Now we want to extend the concept to let us vary both x_1 and x_2 in some specific manner, namely in such a manner that x_1 and x_2 are related to each other by the equation $\phi(x_1, x_2) = a$. Now our example becomes the density of z, the distance from some point of origin, when x_1 and x_2 are combined in the manner of ϕ. The usual concept of a conditional density is a special case of the extended concept, where ϕ is a straight line parallel to either the x_1- or x_2-axis. We are therefore generalizing the usual concept to allow both x_1 and x_2 to vary, and further, to vary in a nonlinear relationship.

The idea for expressing the conditional probability in the form using $\phi(x_1, x_2)$ as a general curve rather than merely a line parallel to either the x_1- or x_2-axis came about as a result of a study of the sufficient statistic $\bar{x}$. Consider a population with density $f(x;\mu)$ from which we draw a sample of $n = 2$, consisting of x' and x''. Let us estimate the mean by $\hat{\mu} = \bar{x} = \frac{1}{2}(x'+x'')$. In studying the subject of estimators, we learn that $\hat{\mu}$ will be a sufficient statistic if the conditional distribution of x' and x'', given $\hat{\mu}$ is a certain value, does not depend on μ. This means that the joint (bivariate) density of the sample may be put in the form

$$f(x';\mu)f(x'';\mu) = g(x', x'' \,|\, \hat{\mu})h(\hat{\mu};\mu) \ ,$$

where the function g does not involve μ.

The factor $g(x', x'' \,|\, \hat{\mu})$ can be considered a bivariate conditional density function of the two variables x' and x'', given that we have the relationship $\hat{\mu} = \bar{x} = \frac{1}{2}(x'+x'')$; that is, given that x' and

x'' are linearly related in such a way that each may vary while their sum remains constant. Thus we have an example which we can put into a more general notation, and from the more general description consisting of conditional probability along a general curve, we will deduce additional examples corresponding to the original one using $\overline{x}$. We will try different combinations of common bivariate densities with specific curves for $\phi(x_1, x_2) = a$. In the original example concerning $\hat{\mu}$, we see that, letting "$\sim$" mean "corresponds to," $x' \sim x_1$, $x'' \sim x_2$, $\hat{\mu} \sim a$, where the symbol $\hat{\mu}$ is an abbreviation for the fact that $x' + x'' = 2\overline{x}$, just as a is an abbreviation for $\phi(x_1, x_2) = a$. The notation $g(x', x'' | \hat{\mu})$ means the conditional probability in terms of x' and x'' given that $\hat{\mu} = \frac{1}{2}(x' + x'')$; later we will write the two variables in terms of one, z, being the distance travelled along the curve, and express the conditional probability in terms of z.

PROCEDURE

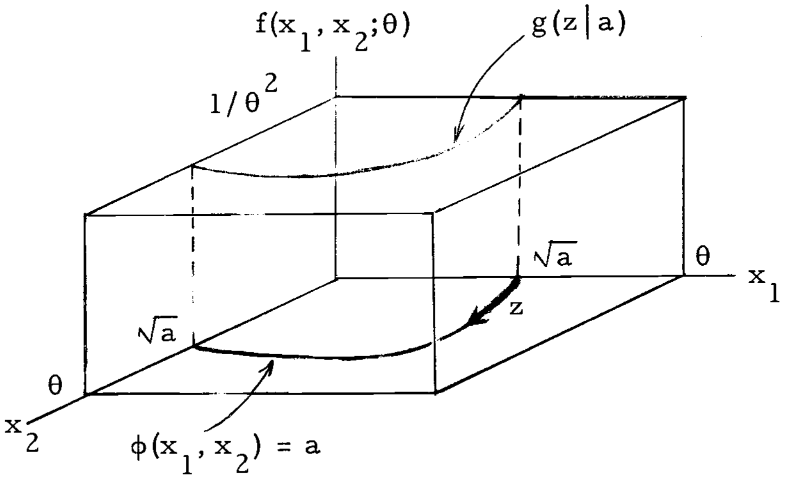
We begin with any bivariate density function $f(x_1, x_2;\theta)$ and its domain of definition $0 \leq x_1 \leq c_1$, $0 \leq x_2 \leq c_2$, where c_1 and c_2 may be finite or not. From this density we want to write a conditional density function involving x_1 and x_2 indirectly, given that x_1 and x_2 vary according to the equation $\phi(x_1, x_2) = a$. The conditional density will be a function of z and a, which are in turn functions of x_1 and x_2.

Geometrically, our problem is to find the equation of the curve described by the upward projection of the curve $\phi(x_1, x_2) = a$ onto the surface defined by $f(x_1, x_2;\theta)$. This intersection will be a curve directed through 3-dimensional space, and not in general lying in just one plane.

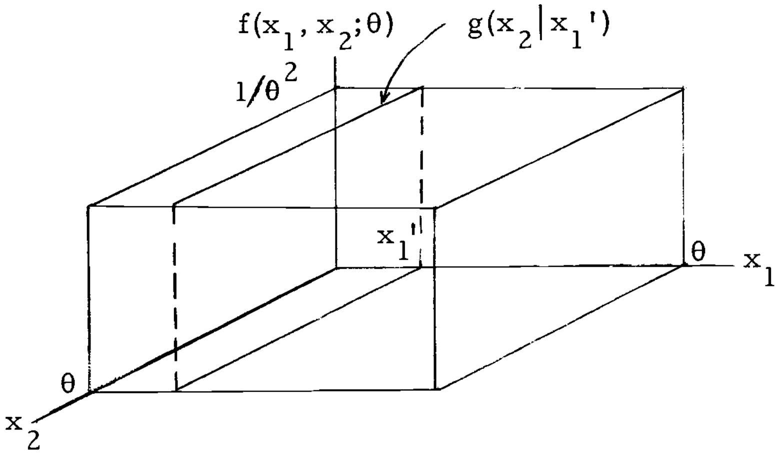
In usual conditional problems the projection of the planar curve onto the surface consisting of the density function would define a curve lying at a constant distance from a vertical plane through either the x_1- or x_2-axis. But in the new problem, the curve which we obtain as the projection of $\phi(x_1, x_2) = a$ upward onto $f(x_1, x_2;\theta)$ is not generally parallel to either of the two coordinate planes. This difference, which is clarified by the following pair of illustrations, is evidence of the extension of the usual concept to a more general one. Here both x_1 and x_2 are free to vary in the plane so long as the

ϕ relationship between them is equal to the constant a.

The following drawings compare the new and the usual types of conditional probability density function, using $f(x_1, x_2; \theta) = \frac{1}{\theta^2}$, and the conditions $0 \leq x_1 \leq \theta$, $0 \leq x_2 \leq \theta$.



New problem



Usual problem

We use $\phi(x_1, x_2) = x_1^2 + x_2^2 = a$ for the new problem and $x_1 = x_1'$ for the usual problem. This example, involving the uniform distribution and the circle centered about the origin, is fully described later on.

In order to arrive at $g(z|a)$, we go through a succession of steps where we transform the variables (x_1, x_2) into (z, a) by means of the relationship $\phi(x_1, x_2) = a$, so that we can transform $f(x_1, x_2; \theta)$ into a bivariate density using z and a and perhaps θ; this we call $g(z, a; \theta)$. We then find the marginal density of a and divide the joint density function of z and a by it to obtain the conditional density of z given a.

In order to apply the process, we break it down into four main steps. In the first step we make the transformation of variables of the bivariate density from x_1 and x_2 to z and a, where we treat the constant a as a variable, due to its parametric nature in designating a particular function $\phi(x_1, x_2)$. So that we will know the nature of z, we sketch $\phi(x_1, x_2) = a$, define a convenient point of origin for z on the curve, and describe the direction in which z is measured along the path ϕ. We will use this illustration of ϕ later to help us find the limits of integration during the computation of the marginal density of a.

Next we need to express x_1 and x_2 in terms of z and a. This step can be accomplished by solving a pair of equations for x_1

and x_2 in terms of z and a. We already have one equation of the pair needed, $\phi(x_1, x_2) = a$. For the other equation we must produce an equation $z = z(x_1, x_2)$ relating z and x_1 or x_2, or both, which we can solve for either x_1 or x_2. It is not always possible to obtain $z = z(x_1, x_2)$, and even if $z(x_1, x_2)$ can be found, not always possible to solve for x_1 or x_2. However, numerical methods could be used in applied problems using specific constants, so that the integration and algebraic difficulties could be bypassed.

In simple cases $z = z(x_1, x_2)$ can be found by inspection, using trigonometric definitions and formulas. In general, $z(x_1, x_2)$ cannot be found by inspection but must be expressed as an arc length by the formula derived in calculus:

$$z = \int_{x_1'}^{t} \sqrt{1+\left(\frac{dx_2}{dx_1}\right)^2}\, dx_1 .$$

Provided that the derivative can be found and then the integral evaluated, provisions which are by no means guaranteed for arbitrary $\phi(x_1, x_2)$, we will have an expression $z(x_1, x_2)$ for z. Again, we are not assured of being able to solve $z = z(x_1, x_2)$ for either x_1 or x_2, and the procedure stops if we cannot. But if we can, we will have an equation in z and a, which we use with $\phi(x_1, x_2) = a$ to find expressions for x_1 and x_2. Symbolically, from $a = \phi(x_1, x_2)$ and $z = z(x_1, x_2)$ we arrive at $x_1 = x_1(z, a)$, $x_2 = x_2(z, a)$ by solving two simultaneous equations in two unknowns, the unknowns being x_1 and x_2.

Now that we have expressions for x_1 and x_2 in terms of z and a, we can proceed to the second step, which consists of transforming the bivariate density $f(x_1, x_2; \theta)$ into a density involving z and a and perhaps θ again, which we call $g(z, a; \theta)$. The transformation is accomplished by substituting the expressions $x_1(z, a)$ and $x_2(z, a)$ for x_1 and x_2, respectively, into $f(x_1, x_2; \theta)$ and multiplying $f[x_1(z, a), x_2(z, a)]$ by the Jacobian consisting of the absolute value of the determinant of all four combinations of partial derivatives of $x_1(z, a)$ and $x_2(z, a)$ with respect to z and a. Here as before, a is considered a variable, though strictly speaking it is a parameter to be determined by any specific example. The transformation is written as follows:

$$g(z, a) = f[x_1 = x_1(z, a), x_2 = x_2(z, a)] \left| \begin{matrix} \dfrac{\partial x_1(z, a)}{\partial z} & \dfrac{\partial x_1(z, a)}{\partial a} \\ \dfrac{\partial x_2(z, a)}{\partial z} & \dfrac{\partial x_2(z, a)}{\partial a} \end{matrix} \right|_+ .$$

The equation above will express the original bivariate density function in terms of z and a instead of x_1 and x_2. Thus we have made the step of writing the density in terms of our curve $\phi(x_1, x_2) = a$, which may be nonlinear, instead of the usual rectangular coordinates x_1 and x_2.

We now progress to the third step, the computation of the marginal density function $g_2(a)$. The notation signifies that we have the marginal density of the second variable a, and that we call the function g_2. The probability function from which we derive $g_2(a)$ is $g(z, a)$. We determine $g_2(a)$ by the usual process of evaluating the integral of $g(z, a)$ with respect to z, the limits being chosen to go from the minimum value of z to the maximum value of z.

In several of the examples, the integration must be done for two or more cases, due to the different algebraic values of minimum and maximum z depending on the location of $\phi(x_1, x_2) = a$ relative to the region defined by $f(x_1, x_2)$. In other words, z is sometimes represented by one function of x_1 and x_2 for a certain a and a different function of x_1 and x_2 for a different a. This complication arises, for example, when varying a moves the starting point of z from one axis to the other, or, in general, from one boundary of the region defined by $f(x_1, x_2; \theta)$ to another boundary of the region having a different algebraic representation. Also, the ending point (maximum) of z may shift when a changes relative to the parameter θ, as in the example using the uniform density and the circular path: we have different cases depending on whether the radius $\sqrt{a}$ is less than or greater than θ. If it is necessary to use several cases in the integration, we will arrive at as many expressions for $g_2(a)$ as we have cases, though they may not all be distinct.

The step in the procedure which follows finding $g_2(a)$ is to find the conditional probability of z, given a, called $g(z|a)$, by dividing $g(z, a)$ by $g_2(a)$. Since we may have several expressions for $g_2(a)$, we will have as many expressions for $g(z|a)$, and which expression for $g(z|a)$ is to be considered depends on the value of a in any specific problem.

We interpret the notation $g(z|a)$ as the probability of having a particular value for z, given that we are on the path $\phi(x_1,x_2) = a$. Thus we have our desired function, giving the probability density function of z as it is defined along the path $\phi(x_1, x_2) = a$.

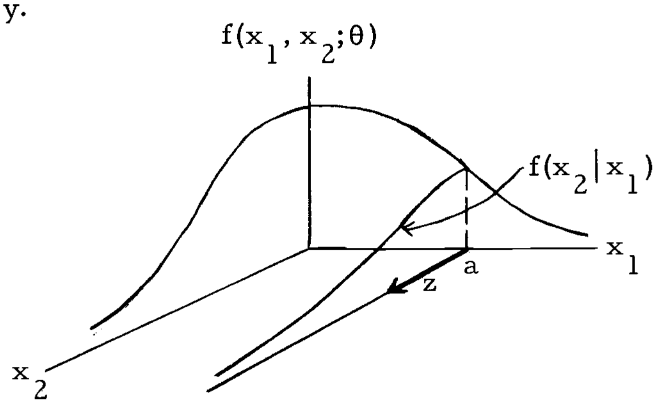
DISCUSSION

The examples where $\phi(x_1, x_2) = a$ is a straight line parallel to either axis give us the common cases of conditional probability, the traditional approach of textbooks. Let us illustrate the four step procedure by using such a path and showing that the new procedure, when applied to the path, gives as a final result for $g(z|a)$, the usual conditional probability density function.

We use $\phi(x_1, x_2) = x_1 = a$ and the general bivariate density $f(x_1, x_2; \theta)$. Step 1 consists of changing variables from (x_1, x_2) to (z, a), solving for x_1 and x_2 in terms of (z, a). We have $\phi(x_1, x_2) = x_1 = a$ as one equation, and we simply define $z = x_2$. The point of origin is $x_1 = a$, $x_2 = 0$. The following illustration uses the bivariate normal density, but the restriction to the normal is not necessary in the theory.

$$x_1 = x_1(z, a) = a$$

$$x_2 = x_2(z, a) = z$$



Now we go to step 2 and transform the bivariate density from a function of (x_1, x_2) to one of (z, a).

$$g(z, a) = f[x_1(z, a), x_2(z, a)]\, J = f(a, z) J \ .$$

$$J = \begin{vmatrix} \frac{\partial x_1(z, a)}{\partial z} & \frac{\partial x_1(z, a)}{\partial a} \\ \frac{\partial x_2(z, a)}{\partial z} & \frac{\partial x_2(z, a)}{\partial a} \end{vmatrix}_+ = \begin{vmatrix} \frac{\partial a}{\partial z} & \frac{\partial a}{\partial a} \\ \frac{\partial z}{\partial z} & \frac{\partial z}{\partial a} \end{vmatrix}_+ = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}_+ = |-1| = 1 \ .$$

Therefore $g(z, a) = f(a, z)$.

In step 3 we compute the marginal density function of a, $g_2(a)$:

$$g_2(a) = \int_{k_1}^{k_2} g(z, a) dz = \int_{k_1}^{k_2} f(a, z) dz \ .$$

We note that the k's may be finite or not, and that $g_2(a)$ must be finite in order that the following step will have meaning.

In step 4 we write the conditional density as the quotient of the previously evaluated expressions, according to the definition of conditional probability as the quotient whose numerator is the bivariate density function and denominator is the marginal density of the variable held constant in the conditional density function. Thus

$$g(z|a) = \frac{g(z, a)}{g_2(a)} \ ,$$

and by substitution and subsequent use of the definition of conditional density,

$$g(z \mid a) = \frac{f(a,z)}{\int_{k_1}^{k_2} f(a,z)dz} = f(z \mid a)\ .$$

We conclude that the conditional density function of z given that we are on the line $\phi(x_1, x_2) = a = x_1$ is $f(z \mid a)$; that is, it is the original f density with z substituted for x_2, and a for x_1.

The four step process has thus been illustrated for the ordinary case of one variable held constant, the other free to vary throughout the domain of definition of $f(x_1, x_2; \theta)$. However, when both variables are allowed to vary simultaneously, according to the specified relationship $\phi(x_1, x_2) = a$, the results are not so transparent as they are in the preceding example, and the four step process gives us a means of finding the required conditional density.

We have discussed the case for three dimensions, where $\phi(x_1, x_2) = a$ is no more than two-dimensional and $f(x_1, x_2; \theta)$ is three-dimensional. Let us increase the number of dimensions of f to n. Then we are dealing with the joint probability density function $f(x_1, x_2, \ldots, x_n; \theta)$ and the curve $\phi(x_1, x_2, \ldots, x_n) = a$, both defined on the same n-dimensional space. Can we apply the previously discussed process to this n-dimensional case, and if so, what does the result mean?

In the usual bivariate case with two independent random variables, the conditional probability density function gives us the probability of one variable, knowing the other is a constant. In the new problem, we again have the conditional density function giving the probability of one variable, z, knowing that the other, a, is a constant. But here z is a function of x_1 and x_2, and a is always a constant.

In the case involving n variables, we are looking for a function to give us the probability associated with those n variables, knowing that they are related according to the function $\phi(x_1, x_2, \ldots, x_n) = a$. We define the variable z similarly as in the bivariate case to be the distance along the curve ϕ through n-dimensional space, starting at some point of origin. The distance can be expressed, theoretically at least, by the formula for arc length in n variables, after introducing the parameter t and expressing each x_i in terms of t. Then

$$z = z(x_1, x_2, \ldots, x_n) = z[x_1(t), x_2(t), \ldots, x_n(t)]$$

$$= \int_{\xi_1}^{\xi_2} \left[\sum_{i=1}^{n} \left(\frac{dx_i(t)}{dt}\right)^2 \right]^{1/2} dt\ .$$

We define the curve ϕ by n parametric equations, each involving the parameter a, which we again treat as a variable. We assign a subscript to each a, so that we have the necessary number of variables. Then ϕ is defined by

$$x_1 = x_1(t) = a_1$$

$$x_2 = x_2(t) = a_2$$

$$\vdots$$

$$x_n = x_n(t) = a_n .$$

We use the expression for z together with $n-1$ of the parametric equations defining ϕ for the substitution:

$$f(x_1, x_2, \cdots, x_n; \theta) \rightarrow g(z, a_1, a_2, \cdots, a_{n-1}; \theta).$$

The n-th equation is used in elimination of t. The Jacobian is an $n \times n$ determinant, constructed analogously to the three-dimensional case.

The marginal density becomes the joint density

$$g_{n-1}(a_1, a_2, \cdots, a_{n-1}) = \int_{k_1}^{k_2} g(z, a_1, a_2, \cdots, a_{n-1}; \theta) dz ,$$

where k_1 and k_2 are again the minimum and maximum values of z. The resulting conditional density is the quotient

$$g(z | a_1, a_2, \cdots, a_n) = \frac{g(z, a_1, a_2, \cdots, a_{n-1}; \theta)}{g_{n-1}(a_1, a_2, \cdots, a_{n-1})} .$$

No examples will be worked out for the n-dimensional case.

EXAMPLES

The four step process for finding the conditional probability of z will now be illustrated by means of several examples. Some of these examples can be worked through to completion to give $g(z|a)$, while some cannot for various reasons. However, the obstacles that prevent finding the final expression for $g(z|a)$ are such that a numerical solution could probably be found in any practical problem, where constants or their approximations could be used.

The examples which follow consist of the combinations of five possibilities for $\phi(x_1, x_2) = a$ taken with three possibilities for $f(x_1, x_2; \theta)$. All of the 15 combinations are not feasible, but it is of some interest to note why the process breaks down in the particular instances where it does. The process of changing variables for any given ϕ need be done only once for each ϕ, so the results of $x_1 = x_1(z, a)$ and $x_2 = x_2(z, a)$ are used in examples for any f. Likewise, the evaluation of the Jacobian is specific only to the particular ϕ, and its value can be used in all the examples involving the same ϕ.

The examples chosen for ϕ are two straight lines with slopes of 45° and -45°, respectively; a circle; a parabola; and a curve resembling a parabola defined in terms of a parameter. The examples for f are the bivariate uniform density, bivariate exponential density, and bivariate normal density, where the two variables x_1 and x_2 are assumed to be independent, so that the densities can be written as the product of the marginal densities of x_1 and of x_2. This restriction to independence is made only to simplify computation. The following table gives the functions and indicates which examples were successful in producing an expression for $g(z|a)$ and which were unsuccessful. The reasons for the unsuccessful ones will be given in the discussion of each example.

Table of Examples

		Probability density function		
No.	Path	Uniform	Exponential	Normal
1	$x_1 + x_2 = a$	S	S	U
2	$x_2 - x_1 = a$	S	S	S
3	$x_1^2 + x_2^2 = a$	S	U	S
4	$x_1^2 - x_2 = a$	U	U	U
5	$x_1 = 1/3t^3 + a$ $x_2 = 1/2t^2$	S	U	U

The bivariate probability density functions are expressed as follows:

Uniform:
$$f(x_1, x_2) = \frac{1}{\theta^2} \quad \text{for} \quad \begin{array}{l} 0 \le x_1 \le \theta \\ 0 \le x_2 \le \theta \end{array}$$
$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

Exponential: $f(x_1, x_2) = \theta^2 e^{-\theta(x_1+x_2)}$ for $0 \le x_1 < \infty$, $0 \le x_2 < \infty$

$= 0$ for x_1, x_2 otherwise

Normal: $f(x_1, x_2) = \frac{1}{8\pi\theta^2} e^{-\frac{1}{2\theta^2}(x_1^2+x_2^2)}$ where θ^2 is the variance, for $0 \le x_1 < \infty$, $0 \le x_2 < \infty$

$= 0$ for x_1, x_2 otherwise

Outline of Procedure

The following scheme was used in the examples to arrive at $g(z|a)$:

1. $(x_1, x_2) \rightarrow (z, a)$.

   Given $a = \phi(x_1, x_2)$.

   Find $z = z(x_1, x_2)$ by inspection or by

$$z = \int_{x_1'}^{t} \sqrt{1+\left(\frac{dx_2}{dx_1}\right)^2}\, dx_1.$$

   Solve $a = \phi(x_1, x_2)$, $z = z(x_1, x_2)$ for $x_1 = x_1(z, a)$, $x_2 = x_2(z, a)$.

2. $f(x_1, x_2;\theta) \rightarrow g(z, a;\theta)$.

   $g(z, a;\theta) = f[x_1(z, a), x_2(z, a)]\, J$, where

$$J = \left| \begin{matrix} \frac{\partial x_1(z, a)}{\partial z} & \frac{\partial x_1(z, a)}{\partial a} \\ \frac{\partial x_2(z, a)}{\partial z} & \frac{\partial x_2(z, a)}{\partial a} \end{matrix} \right|_+ .$$

3. Find $g_2(a) = \int_{k_1}^{k_2} g(z, a; \theta)dz.$

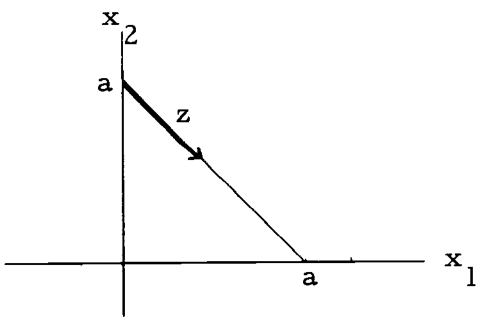
4. $g(z|a) = \frac{g(z, a; \theta)}{g_2(a)}$.

Example 1--Uniform

$$f(x_1, x_2) = \frac{1}{\theta^2} \quad \text{for} \quad \begin{matrix} 0 \leq x_1 \leq \theta \\ 0 \leq x_2 \leq \theta \end{matrix} \qquad \phi(x_1, x_2) = x_1 + x_2 = a$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

1. $(x_1, x_2) \rightarrow (z, a)$.



$\cos 45° = \frac{x_1}{z} = \frac{1}{\sqrt{2}}$

therefore $x_1 = \frac{z}{\sqrt{2}}$

and $x_2 = a - x_1 = a - \frac{z}{\sqrt{2}}$.

2. $f(x_1, x_2; \theta) \rightarrow g(z, a; \theta)$.

$$g(z, a; \theta) = f\left(\frac{z}{\sqrt{2}}, a - \frac{z}{\sqrt{2}}\right)J = \frac{1}{\theta^2}J$$

$$J = \begin{vmatrix} \frac{\partial}{\partial z}(\frac{z}{\sqrt{2}}) & \frac{\partial}{\partial a}(\frac{z}{\sqrt{2}}) \\ \frac{\partial}{\partial z}(a - \frac{z}{\sqrt{2}}) & \frac{\partial}{\partial a}(a - \frac{z}{\sqrt{2}}) \end{vmatrix}_+ = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 \end{vmatrix}_+ = \frac{1}{\sqrt{2}}$$

$$g(z, a;\theta) = \frac{1}{\sqrt{2}\,\theta^2}$$

3. $$g_2(a) = \int_{k_1}^{k_2} g(z, a;\theta)dz = \int_{k_1}^{k_2} \frac{1}{\sqrt{2}\,\theta^2} dz = \left[\frac{z}{\sqrt{2}\,\theta^2} \right]_{k_1}^{k_2}$$

From the diagram of step 1, we see that the minimum value of z is 0 and the maximum value of z is $\sqrt{a^2+a^2} = \sqrt{2}\,a$.

Thus $k_1 = 0$, $k_2 = \sqrt{2}\,a$.

$$g_2(a) = \frac{\sqrt{2}\,a}{\sqrt{2}\,\theta^2} - 0 = \frac{a}{\theta^2}.$$

4. $$g(z|a) = \frac{g(z, a;\theta)}{g_2(a)} = \frac{1}{\sqrt{2}\,\theta^2} \cdot \frac{\theta^2}{a} = \frac{1}{\sqrt{2}\,a}.$$

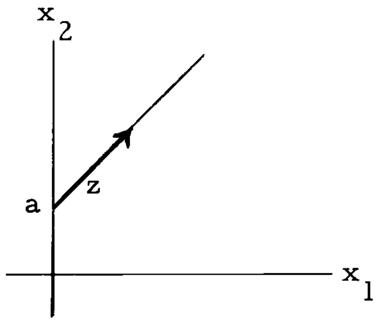
We again have a uniform density for z.

Example 2--Uniform

$$f(x_1, x_2) = \frac{1}{\theta^2} \quad \text{for} \quad \begin{matrix} 0 \le x_1 \le \theta \\ 0 \le x_2 \le \theta \end{matrix} \qquad \phi(x_1, x_2) = x_2 - x_1 = a$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

1. $(x_1, x_2) \rightarrow (z, a)$.



$$\cos 45^\circ = \frac{x_1}{z} = \frac{1}{\sqrt{2}}$$

$$x_1 = \frac{z}{\sqrt{2}}$$

and $x_2 = a + x_1 = a + \dfrac{z}{\sqrt{2}}$

2. $f(x_1, x_2; \theta) \rightarrow g(z, a; \theta)$.

$$g(z, a; \theta) = f\left(\frac{z}{\sqrt{2}}, a - \frac{z}{\sqrt{2}}\right)J = \frac{1}{\theta^2}\,J$$

$$J = \left|\begin{matrix} \dfrac{\partial}{\partial z}\left(\dfrac{z}{\sqrt{2}}\right) & \dfrac{\partial}{\partial a}\left(\dfrac{z}{\sqrt{2}}\right) \\ \\ \dfrac{\partial}{\partial z}\left(a + \dfrac{z}{\sqrt{2}}\right) & \dfrac{\partial}{\partial a}\left(a + \dfrac{z}{\sqrt{2}}\right) \end{matrix}\right|_+ = \left|\begin{matrix} \dfrac{1}{\sqrt{2}} & 0 \\ \\ \dfrac{1}{\sqrt{2}} & 1 \end{matrix}\right|_+ = \frac{1}{\sqrt{2}}$$

$$g(z, a; \theta) = \frac{1}{\sqrt{2}\,\theta^2} .$$

3. $$g_2(a) = \int_{k_1}^{k_2} g(z, a; \theta)\,dz = \int_{k_1}^{k_2} \frac{1}{\sqrt{2}\,\theta^2}\,dz = \left[\frac{z}{\sqrt{2}\,\theta^2}\right]_{k_1}^{k_2}$$

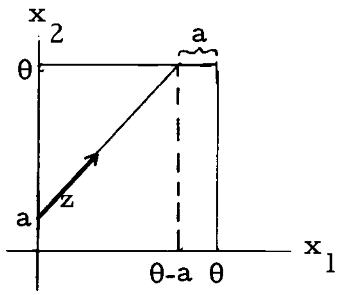
We see that a must be less than θ, since if $a = \theta$ we would have $g_2(a) = 0$, and $g(z|a)$ would be undefined; similarly, if $a > \theta$, a would be out of the domain of definition of f. The minimum value of z is obviously zero. To find k_2, the maximum value of z, we use the Phythagorean theorem, where

$x_1 = \theta - a$

$x_2 = \theta - a$

$z(\max) = \sqrt{2(\theta-a)^2} = \sqrt{2}\,(\theta-a)$

$$g_2(a) = \left[\frac{z}{\sqrt{2}\,\theta^2}\right]_0^{\sqrt{2}\,(\theta-a)} = \frac{\sqrt{2}\,(\theta-a)}{\sqrt{2}\,\theta^2} = \frac{\theta-a}{\theta^2}.$$



4. $$g(z\,|\,a) = \frac{g(z,a;\theta)}{g_2(a)} = \frac{1}{\sqrt{2}\,\theta^2} \cdot \frac{\theta^2}{\theta-a} = \frac{1}{\sqrt{2}\,(\theta-a)}.$$

As in the previous example, we have a uniform density for $g(z\,|\,a)$, but it happens to involve the parameter θ, where the previous density did not.

Example 3--Uniform

$$f(x_1, x_2) = \frac{1}{\theta^2} \quad \text{for} \quad 0 \le x_1 \le \theta, \; 0 \le x_2 \le \theta \qquad \phi(x_1, x_2) = x_1^2 + x_2^2 = a$$

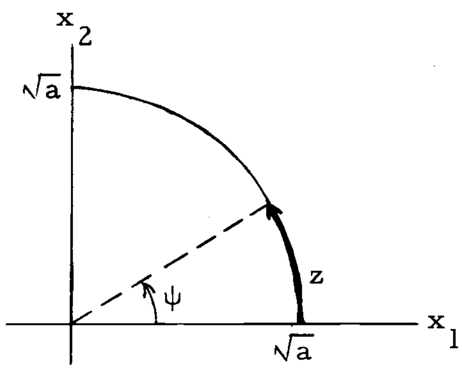
$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

1. $(x_1, x_2) \rightarrow (z, a)$.

$z = \sqrt{a}\,\psi$, if ψ is in radians

$$= \sqrt{a}\cos^{-1}\frac{x_1}{\sqrt{a}}$$

$$x_1 = \sqrt{a}\cos\frac{z}{\sqrt{a}}.$$



Since $x_1^2 + x_2^2 = a$,

$$x_2 = \sqrt{a-x_1^2} = [a-(\sqrt{a}\cos\frac{z}{\sqrt{a}})^2]^{1/2} = [a-a\cos^2\frac{z}{\sqrt{a}}]^{1/2}$$

$$= \sqrt{a}(1-\cos^2\frac{z}{\sqrt{a}})^{1/2} = \sqrt{a}\sin\frac{z}{\sqrt{a}}.$$

2. $f(x_1, x_2;\theta) \rightarrow g(x_1, x_2;\theta)$.

$$g(z, a;\theta) = f(\sqrt{a}\cos\frac{z}{\sqrt{a}}, \sqrt{a}\sin\frac{z}{\sqrt{a}})J = \frac{1}{\theta^2}J$$

$$J = \left|\begin{matrix} \frac{\partial}{\partial z}(\sqrt{a}\cos\frac{z}{\sqrt{a}}) & \frac{\partial}{\partial a}(\sqrt{a}\cos\frac{z}{\sqrt{a}}) \\ \frac{\partial}{\partial z}(\sqrt{a}\sin\frac{z}{\sqrt{a}}) & \frac{\partial}{\partial a}(\sqrt{a}\sin\frac{z}{\sqrt{a}}) \end{matrix}\right|_+$$

$$J = \left|\begin{matrix} -\sin\frac{z}{\sqrt{a}} & \frac{z}{2a}\sin\frac{z}{\sqrt{a}} + \frac{1}{2}a^{-1/2}\cos\frac{z}{\sqrt{a}} \\ \cos\frac{z}{\sqrt{a}} & -\frac{z}{2a}\cos\frac{z}{\sqrt{a}} + \frac{1}{2}a^{-1/2}\sin\frac{z}{\sqrt{a}} \end{matrix}\right|_+$$

$$= \left| -\frac{1}{2}a^{-1/2}\right|_+ = \frac{1}{2}a^{-1/2}.$$

Then $g(z, a;\theta) = \dfrac{1}{2\sqrt{a}\,\theta^2}$.

3. $$g_2(a) = \int_{k_1}^{k_2} g(z, a;\theta)dz = \int_{k_1}^{k_2}\frac{1}{2\sqrt{a}\,\theta^2}dz = \left[\frac{z}{2\sqrt{a}\,\theta^2}\right]_{k_1}^{k_2}$$

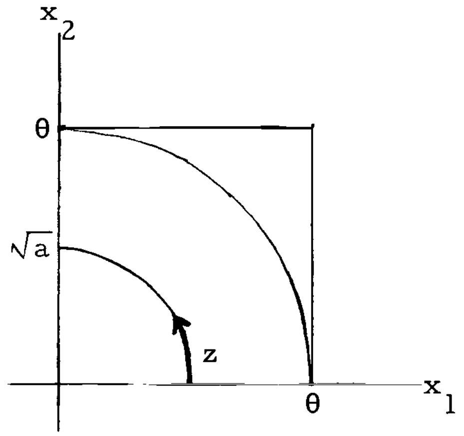
In order to evaluate the limits on the integral, we must divide the problem into two cases, according to the value of a relative to θ.

Case I: $0 < \sqrt{a} \leq \theta$

The minimum value of z is zero, the maximum is $\frac{1}{4}(2\pi\sqrt{a}) = \frac{\pi\sqrt{a}}{2}$.

Then $$g_2(a) = \left[\frac{z}{2\sqrt{a}\,\theta^2}\right]_0^{\frac{\pi\sqrt{a}}{2}}$$

$$= \frac{1}{2\sqrt{a}\,\theta^2}\left[\frac{\pi\sqrt{a}}{2}\right] = \frac{\pi}{4\theta^2}.$$

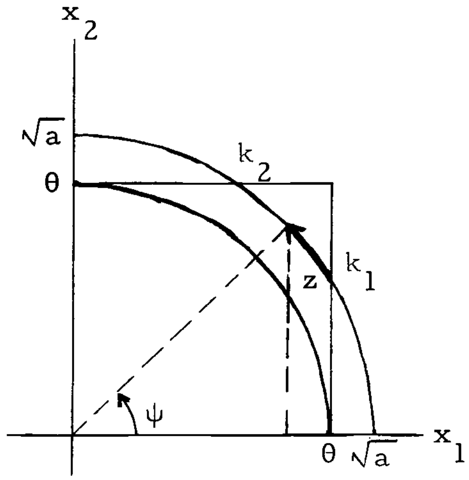


Case II: $\theta < \sqrt{a} < \sqrt{2}\theta$

We must find an expression for the length of z from the x_1-axis up to the line $x_1 = \theta$ in order to write the lower limit of the integral for $g_2(a)$.

When ψ is measured in radians, the arc length of $\phi(x_1, x_2) = a$ from the x_1-axis to k_1 is $\sqrt{a}\,\psi = \sqrt{a}\cos^{-1}\frac{\theta}{\sqrt{a}}$.

k_2 is located on the line $x_2 = \theta$. The arc length of ϕ from the x_1-axis to $x_2 = \theta$ is



$$\sqrt{a}\,\psi = \sqrt{a}\sin^{-1}\frac{\theta}{\sqrt{a}}.$$

$$g_2(a) = \left[\frac{z}{2\sqrt{a}\,\theta^2}\right]_{\sqrt{a}\cos^{-1}\frac{\theta}{\sqrt{a}}}^{\sqrt{a}\sin^{-1}\frac{\theta}{\sqrt{a}}}$$

$$= \frac{1}{2\sqrt{a}\,\theta^2}\left[\sqrt{a}\sin^{-1}\frac{\theta}{\sqrt{a}} - \sqrt{a}\cos^{-1}\frac{\theta}{\sqrt{a}}\right]$$

$$= \frac{1}{2\theta^2}\left[\sin^{-1}\frac{\theta}{\sqrt{a}} - \cos^{-1}\frac{\theta}{\sqrt{a}}\right].$$

4. $g(z\,|\,a) = \dfrac{g(z, a;\theta)}{g_2(a)}$.

Case I: $g(z\,|\,a) = \dfrac{1}{2\sqrt{a}\,\theta^2} \cdot \dfrac{4\theta^2}{\pi} = \dfrac{2}{\pi\sqrt{a}}$

Case II: $g(z\,|\,a) = \dfrac{1}{2\sqrt{a}\,\theta^2} \cdot \dfrac{2\theta^2}{\sin^{-1}\frac{\theta}{\sqrt{a}} - \cos^{-1}\frac{\theta}{\sqrt{a}}}$

$$= \frac{1}{\sqrt{a}\left(\sin^{-1}\frac{\theta}{\sqrt{a}} - \cos^{-1}\frac{\theta}{\sqrt{a}}\right)} .$$

In both cases we get a uniform density as a result of using the circle with the bivariate uniform density. In Case I the density is determined by choice of a only. However, in Case II the density of z will have to change with a change in θ, given the same radius $\sqrt{a}$, since the total length of z increases as θ increases, for any a. (See illustration under Case I.)

To get a uniform density for $g(z\,|\,a)$ is consistent with our intuitive expectation, since, considering the geometric interpretation of $g(z|a)$ as the projection of $\phi(x_1, x_2)$ upwards onto $f(x_1, x_2; \theta)$, we might expect to get another uniform density as the intersection of ϕ and f.

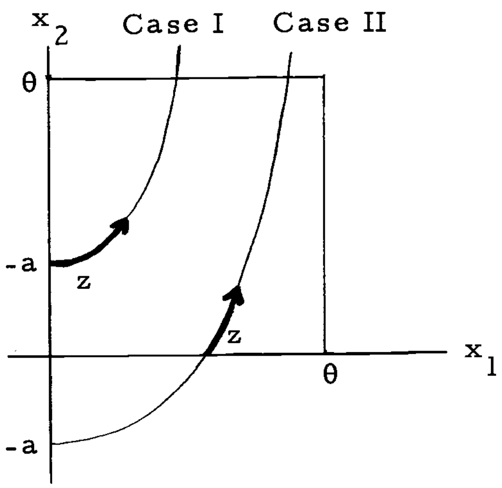
Example 4--Uniform

$$f(x_1, x_2) = \frac{1}{\theta^2} \quad \text{for} \quad \begin{array}{l} 0 \le x_1 \le \theta \\ 0 \le x_2 \le \theta \end{array} \qquad \phi(x_1, x_2) = x_1^2 - x_2 = a$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

1. $(x_1, x_2) \rightarrow (z, a)$.

First we find $z = z(x_1, x_2)$, where we already have $a = \phi(x_1, x_2)$.



$$z = \int_{x_1'}^{t} \sqrt{1 + \left(\frac{dx_2}{dx_1}\right)^2}\, dx_1 .$$

To find $\frac{dx_2}{dx_1}$, we use the relationship $\phi(x_1, x_2) = a$:

$$x_1^2 - x_2 = a$$

$$2x_1 \frac{dx_1}{dx_1} - \frac{dx_2}{dx_1} = 0, \quad \text{so} \quad \frac{dx_2}{dx_1} = 2x_1 .$$

$$z = \int_{x_1'}^{t} \sqrt{1+4x_1^2}\, dx_1 = \frac{1}{2}\left[\sqrt{1+4x_1^2} + \log\,(2x_1 + \sqrt{1+4x_1^2})\right]_{x_1'}^{t}$$

$$= \frac{1}{2}\,[\sqrt{1+4t^2} + \log\,(2t+\sqrt{1+4t^2})] - \frac{1}{2}\left[\sqrt{1+4(x_1')^2} + \log\,(2x_1'+\sqrt{1+4(x_1')^2})\right]$$

t is a dummy variable, which will be replaced later by x_1, and x_1' is a constant, the value of x_1 at the origin of z. We see that there will be two cases to consider in finding x_1'.

Case I: $-\theta < a \leq 0$

Here $x_1' = 0$, since the origin of z is the x_2-axis.

$$z = \frac{1}{2}\,[\sqrt{1+4t^2} + \log\,(2t+\sqrt{1+4t^2})] - \frac{1}{2}[1+\log 1]$$

$$= \frac{1}{2}\,[\sqrt{1+4t^2} + \log\,(2t+\sqrt{1+4t^2})].$$

Case II: $0 < a < \theta^2$

Here x_1' is the value of x_1 at $x_2 = 0$,

$$x_1' = \sqrt{a+x_2} = \sqrt{a}\,.$$

$$z = \frac{1}{2}[\sqrt{1+4t^2} + \log\,(2t+\sqrt{1+4t^2})] - \frac{1}{2}[\sqrt{1+4a} + \log\,(2\sqrt{a}+\sqrt{1+4a})]\,.$$

Ordinarily we would solve the above equations for t in terms of z, but it is not possible algebraically. However, any problem with specific, real values could be solved, and the subsequent steps of the process carried through. The difficulty arises as a result of the complicated nature of the parabola when we try to express it in terms of one variable, z.

Since we cannot even complete step 1 of the process for finding $g(z|a)$, we will abandon the parabola as a possibility for $\phi(x_1, x_2) = a$ with the last two density functions, the exponential and normal.

Example 5--Uniform

$$f(x_1, x_2) = \frac{1}{\theta^2} \quad \text{for} \quad 0 \leq x_1 \leq \theta$$

$$= 0 \quad \text{for} \quad x_1, x_2 \text{ otherwise}$$

$$\phi(x_1, x_2) = \phi[x_1(t), x_2(t)]$$

where

$$x_1 = x_1(t) = \frac{1}{3}t^3 + a$$

$$x_2 = x_2(t) = \frac{1}{2}t^2$$

1. $(x_1, x_2) \to (z, a)$.

First we find the arc length z as a function of the parameter t. Then, using the relations defining $\phi(x_1, x_2)$, we eliminate t to get z and a in terms of x_1 and x_2.

$$z = \int_{t_0}^{t'} \sqrt{\left(\frac{dx_1(t)}{dt}\right)^2 + \left(\frac{dx_2(t)}{dt}\right)^2}\, dt$$

$$\frac{dx_1(t)}{dt} = \frac{d}{dt}\left(\frac{1}{3}t^3 + a\right) = t^2$$

$$\frac{dx_2(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}t^2\right) = t$$

The limits consist of the value of t at the origin of the arc length z, called t_0, and the dummy variable t', which will be replaced by the parameter t used in defining ϕ. To know what t_0 is, we sketch $\phi(x_1, x_2)$.

Table of values

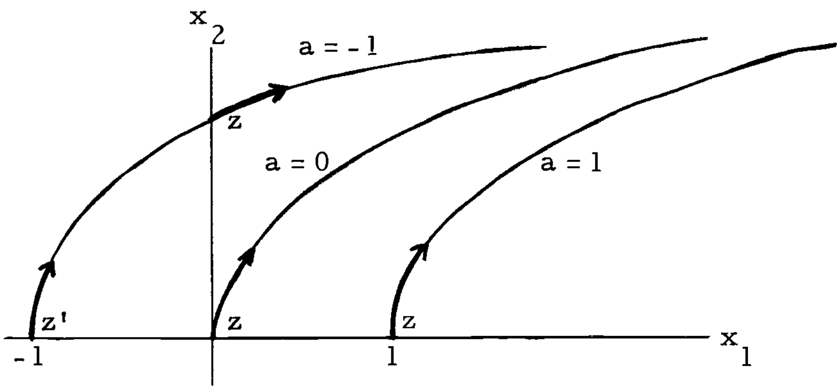
t	x_1	x_2
0	a	0
1	$\frac{1}{3} + a$	$\frac{1}{2}$
2	$\frac{8}{3} + a$	2
3	$9 + a$	$\frac{9}{2}$
-1	$-\frac{1}{3} + a$	$\frac{1}{2}$

$$x_1(t) = \frac{1}{3}t^3 + a$$
$$x_2(t) = \frac{1}{2}t^2$$

It is clear that negative values of t produce a branch of $\phi(x_1, x_2)$ which is identical to that for positive values of t but which is reflected in the x_2-axis. We will disregard the branch for negative t. The branch we will consider is a family of lines, depending on a for placement on the x_1-axis. We see that as t increases, so do both x_1 and x_2, but that x_1 increases faster than x_2 since it increases as the cube of t, where x_2 increases as the square of t. In fact the slope

$$\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{t^2}{t^3} = \frac{1}{t}$$

tells us that the curve is infinitely large on the x_1-intercept then levels out to become very small as t is large. Now we can draw $\phi(x_1, x_2)$ for various values of a.



We define z to originate on the x_1-axis for $0 \leq a \leq \theta$, and define it to originate on the x_2-axis for $-\frac{1}{3}(2\theta)^{3/2} \leq a < 0$. Thus it always originates at the left side of the domain of definition of f. It continues along ϕ as t increases to the boundary of the domain of f. However, in order to evaluate the integral for arc length, we consider the arc length as starting always on the x_1-axis and ending at an arbitrary point of ϕ, and call this revised arc length z'. z' is z for $a \geq 0$, and for $a < 0$ z' will be decreased later by the length of ϕ between the x_1-axis and the x_2-axis to give the z which is defined only in the domain of definition of f.

To continue the evaluation of z', we need t_0, the value of t at $\phi(x_1, 0)$. From the original equation for x_2 we have $x_2 = \frac{1}{2}t^2 = 0$, so that $t_0 = 0$. Substitution of the derivatives into the formula gives

$$z' = \int_0^{t'} \sqrt{(t^2)^2 + (t)^2}\, dt = \int_0^{t'} t\sqrt{t^2+1}\, dt$$

$$= \frac{1}{2}\int_0^{t'} \sqrt{t^2+1}\,(2t\,dt) = \frac{1}{2}\cdot\frac{2}{3}\left[(t^2+1)^{3/2}\right]_0^{t'}$$

$$= \frac{1}{3}[\,(t^2+1)^{3/2} - (1)] = \frac{1}{3}(t^2+1)^{3/2} - \frac{1}{3}.$$

We can eliminate the parameter by using the original relationship $t^2 = 2x_2$ to get

$$z = \frac{1}{3}(2x_2+1)^{3/2} - \frac{1}{3}.$$

Also going back to the original statement of ϕ gives

$$a = x_1 - \frac{1}{3}t^3 = x_1 - \frac{1}{3}[(2x_2)^{1/2}]^3 = x_1 - \frac{1}{3}(2x_2)^{3/2}.$$

We now have the pair of simultaneous equations

$$z = z(x_1, x_2)$$

$$a = a(x_1, x_2)$$

which we wish to solve for x_1 and x_2.

$$z = \frac{1}{3}(2x_2+1)^{3/2} - \frac{1}{3}$$

$$(2x_2+1)^{3/2} = 3z + 1$$

$$2x_2 + 1 = (3z+1)^{2/3}$$

$$x_2 = \frac{1}{2}[(3z+1)^{2/3} - 1]$$

$$a = x_1 - \frac{1}{3}(2x_2)^{3/2}$$

$$x_1 = a + \frac{1}{3}(2x_2)^{3/2}$$

$$= a + \frac{1}{3}[\,(3z+1)^{2/3} - 1]^{3/2}$$

2. $f(x_1, x_2; \theta) \rightarrow g(z, a; \theta)$.

$$g(z, a; \theta) = f\{a+\frac{1}{3}[(3z+1)^{2/3} - 1]^{3/2}, \quad \frac{1}{2}[(3z+1)^{2/3} - 1]\}\,J$$

$$= \frac{1}{\theta^2}\,J$$

Evaluation of J:

$$\frac{\partial}{\partial z} x_1(z,a) = \frac{\partial}{\partial z}\{a+\frac{1}{3}[(3z+1)^{2/3}-1]^{3/2}\}$$

$$= \frac{1}{3}\{\frac{3}{2}[(3z+1)^{2/3}-1]^{1/2}[\frac{2}{3}(3z+1)^{-1/3}(3)]\}$$

$$= [(3z+1)^{2/3}-1]^{1/2}(3z+1)^{-1/3}$$

$$\frac{\partial}{\partial z} x_2(z,a) = \frac{\partial}{\partial z}\{\frac{1}{2}[(3z+1)^{2/3}-1]\}$$

$$= \frac{1}{2}[\frac{2}{3}(3z+1)^{-1/3}(3)]$$

$$= (3z+1)^{-1/3}$$

$$\frac{\partial}{\partial a} x_1(z,a) = \frac{\partial}{\partial a}\{a+\frac{1}{3}[(3z+1)^{2/3}-1]^{3/2}\}$$

$$= 1$$

$$\frac{\partial}{\partial a} x_2(z,a) = \frac{\partial}{\partial a}\{\frac{1}{2}[(3z+1)^{2/3}-1]\}$$

$$= 0$$

$$J = \begin{vmatrix} [(3z+1)^{2/3}-1]^{1/2}(3z+1)^{-1/3} & 1 \\ (3z+1)^{-1/3} & 0 \end{vmatrix}_+$$

$$= (3z+1)^{-1/3}.$$

Therefore $g(z,a;\theta) = \dfrac{1}{\theta^2(3z+1)^{1/3}}$.

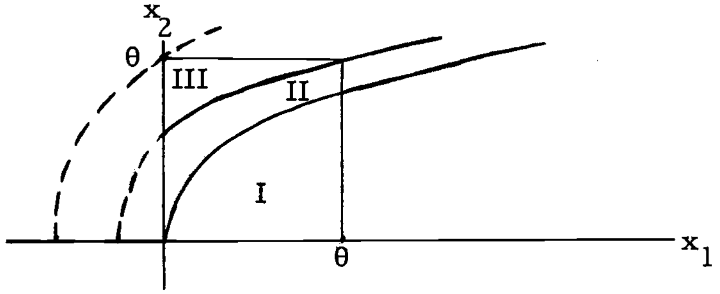
3. $$g_2(a) = \int_{k_1}^{k_2} g(z, a;\theta)dz = \int_{k_1}^{k_2} \frac{1}{\theta^2(3z+1)^{1/3}}\, dz$$

$$= \frac{1}{2\theta^2}\left[(3z+1)^{2/3}\right]_{k_1}^{k_2}$$

Reference to the illustration of ϕ will enable us to break up the problem of finding the limits on the integral into the necessary number of cases.

At first glance it appears we have three cases: the x_1-intercept a either less than zero, zero, or greater than zero. But there is a complication resulting from the character of ϕ, whose slope near the x_1-intercept is very much greater than its slope far from the x_1-intercept. Thus, if we have a small (to be defined later) value of θ, ϕ will leave the domain of definition of f through the line $x_2 = \theta$, while if θ is larger, ϕ will intercect the boundary line $x_1 = \theta$, given the same a for both. Therefore, we need to define the separate regions of a in terms of θ rather than in simple terms of positive, negative, zero.

We have the three different regions illustrated:



We note from the original equations that the x_1-intercept is a, so we need to find the x_1-intercept of ϕ for each boundary line of the three regions.

Case I: $\theta \geq a \geq 0$.

Region I is defined as the region where z originates on the x_1-axis and terminates on $x_1 = \theta$. z varies in length from zero to the arc length of ϕ where ϕ crosses $x_1 = \theta$. Thus $k_1 = 0$, k_2 is the arc length of ϕ from $\phi(0, 0)$ to $\phi(\theta, x_2)$.

To find the arc length described above, we find the t corresponding to $x_1 = \theta$, then use the expression for z found previously in step 1, and eliminate the parameter t.

$$x_1 = \frac{1}{3}t^3 + a = \theta$$

$$t^3 = 3(\theta - a)$$

$$t = [3(\theta - a)]^{1/3}$$

$$z = \frac{1}{3}(t^2+1)^{3/2} - \frac{1}{3} = \frac{1}{3}\{[3(\theta-a)]^{2/3}+1\}^{3/2} - \frac{1}{3}$$

so that $k_2 = \frac{1}{3}\{[3(\theta-a)]^{2/3}+1\}^{3/2} - \frac{1}{3}$

Substitution of k_1 and k_2 into the integrated expression for $g_2(a)$ gives

$$g_2(a) = \frac{1}{2\theta^2}[(3k_2+1)^{2/3} - (3k_1+1)^{2/3}]$$

$$= \frac{1}{2\theta^2}\left[(\{[3(\theta-a)]^{2/3}+1\}^{3/2}-1+1)^{2/3} - (3\cdot 0+1)^{2/3}\right]$$

$$= \frac{1}{2\theta^2}\left[[3(\theta-a)]^{2/3}+1-1\right]$$

$$= \frac{1}{2\theta^2}[3(\theta-a)]^{2/3} .$$

Case II: $0 > a \geq \theta - \frac{1}{3}(2\theta)^{3/2}$.

Region II is defined as the region where ϕ enters the domain of definition of f on the left along the x_2-axis and leaves on the right along the line $x_1 = \theta$.

To explain the quantities used to define Region II, we note from the illustration that the x_1-intercept, which is equal to a, must be less than zero for z to originate on the x_2-axis.

The lower limit of a is found by getting the a corresponding to $x_1 = x_2 = \theta$.

$$x_1 = \frac{1}{3}t^3 + a = \theta$$

$$x_2 = \frac{1}{2}t^2 = \theta$$

$$t = \pm (2\theta)^{1/2}$$

$$a = \theta - \frac{1}{3}t^3 = \theta \pm \frac{1}{3}(2\theta)^{3/2} .$$

Since we have already established $a < 0$, we discard $\theta + \frac{1}{3}(2\theta)^{3/2}$ and have $a = \theta - \frac{1}{3}(2\theta)^{3/2}$.

We now return to finding the limits on the integral $g_2(a)$. The upper limit k_2 is the same expression as k_2 of Case I, since we want to find the length of ϕ up to the boundary line $x_1 = \theta$. The lower limit k_1 is found by evaluating z at $x_1 = 0$ by means of the arc length formula already obtained. First we find the t corresponding to $x_1 = 0$.

$$x_1 = \frac{1}{3}t^3 + a = 0$$

$$t^3 = -3a$$

$$t = (-3a)^{1/3}$$

$$z = \frac{1}{3}(t^2+1)^{3/2} - \frac{1}{3} = \frac{1}{3}[(-3a)^{2/3}+1]^{3/2} - \frac{1}{3}$$

$$= \frac{1}{3}[(3a)^{2/3}+1]^{3/2} - \frac{1}{3}.$$

Thus $k_1 = \frac{1}{3}[(3a)^{2/3}+1]^{3/2} - \frac{1}{3}$ and $k_2 = \frac{1}{3}\{[3(\theta-a)]^{2/3}+1\}^{3/2} - \frac{1}{3}$.

Substitution into the integrated expression for $g_2(a)$ gives

$$g_2(a) = \frac{1}{2\theta^2}[(3k_2+1)^{2/3} - (3k_1+1)^{2/3}]$$

$$= \frac{1}{2\theta^2}\left[(\{[3(\theta-a)]^{2/3}+1\}^{3/2}-1+1)^{2/3} - ([(3a)^{2/3}+1]^{3/2}-1+1)^{2/3}\right]$$

$$= \frac{1}{2\theta^2}\left[[3(\theta-a)]^{2/3}+1-[(3a)^{2/3}+1]\right]$$

$$= \frac{1}{2\theta^2}\left[[3(\theta-a)]^{2/3}-(3a)^{2/3}\right].$$

Case III: $\theta - \frac{1}{3}(2\theta)^{3/2} > a \geq -\frac{1}{3}(2\theta)^{3/2}$.

Region III is defined as the region where ϕ enters the domain of definition of f on the left along the x_2 - axis and leaves along the line $x_2 = \theta$.

To find the a corresponding to the upper limit on Region III, we need the value of a when $x_1 = 0$, $x_2 = \theta$. We solve the following two equations for a:

$$x_1 = \frac{1}{3}t^3 + a = 0$$

$$x_2 = \frac{1}{2}t^2 = \theta$$

$$t = \pm (2\theta)^{1/2}$$

$$a = -\frac{1}{3}t^3 = -\frac{1}{3}[\pm(2\theta)^{1/2}]^3 = \pm\frac{1}{3}(2\theta)^{3/2}$$

Since we have the restriction $a < 0$, we discard the positive a and have $a = -\frac{1}{3}(2\theta)^{3/2}$. This value of a is the least value that a can have and still give a curve $\phi(x_1, x_2) = a$ which goes through the domain of definition of f.

We now proceed to find k_1 and k_2. Since Region III begins on the x_2-axis as does Region II, we use the same value for k_1, so that

$$k_1 = \frac{1}{3}[(3a)^{2/3}+1]^{3/2} - \frac{1}{3} .$$

k_2 is found by the arc length formula $z = \frac{1}{3}(t^2+1)^{3/2} - \frac{1}{3}$, where the parameter is the t corresponding to $\phi(x_1, \theta)$.

$$x_2 = \frac{1}{2}t^2 = \theta$$

$$t^2 = 2\theta$$

$$z = \frac{1}{3}(2\theta+1)^{3/2} - \frac{1}{3} = k_2$$

Now we use k_1 and k_2 to evaluate $g_2(a)$.

$$g_2(a) = \frac{1}{2\theta^2}[(3k_2+1)^{2/3} - (3k_1+1)^{2/3}]$$

$$= \frac{1}{2\theta^2}\left[((2\theta+1)^{3/2}-1+1)^{2/3} - ([(3a)^{2/3}+1]^{3/2}-1+1)^{2/3}\right]$$

$$= \frac{1}{2\theta^2}\left[2\theta+1-[(3a)^{2/3}+1]\right]$$

$$= \frac{1}{2\theta^2}[2\theta-(3a)^{2/3}]$$

4. $g(z|a) = \dfrac{g(z, a;\theta)}{g_2(a)}$.

Case I:

$$g(z|a) = \frac{1}{\theta^2(3z+1)^{1/3}} \cdot \frac{2\theta^2}{[3(\theta-a)]^{2/3}}$$

$$= 2[3(\theta-a)]^{-2/3}(3z+1)^{-1/3}.$$

Case II:

$$g(z|a) = \frac{1}{\theta^2(3z+1)^{1/3}} \cdot 2\theta^2[[3(\theta-a)]^{2/3}-(3a)^{2/3}]^{-1}$$

$$= 2\{[3(\theta-a)]^{2/3}-(3a)^{2/3}\}^{-1}(3z+1)^{-1/3}.$$

Case III:

$$g(z|a) = \frac{1}{\theta^2(3z+1)^{1/3}} \cdot 2\theta^2[2\theta-(3a)^{2/3}]^{-1}$$

$$= 2[2\theta-(3a)^{2/3}]^{-1}(3z+1)^{-1/3}.$$

We conclude that the three expressions for $g(z|a)$ differ only by constant factors, which are functions of both θ and a.

<u>Example 1--Exponential</u>

$$f(x_1, x_2) = e^{-(x_1+x_2)} \quad \text{for} \quad \begin{array}{l} 0 \le x_1 < \infty \\ 0 \le x_2 < \infty \end{array} \qquad \phi(x_1, x_2) = x_1 + x_2 = a$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

1. $(x_1, x_2) \to (z, a)$.

We use the results from Example 1--Uniform as follows:

$$x_1 = \frac{z}{\sqrt{2}}$$

$$x_2 = a - \frac{z}{\sqrt{2}}$$

2. $f(x_1, x_2; \theta) \rightarrow g(z, a; \theta)$.

$$g(z, a; \theta) = f\left(\frac{z}{\sqrt{2}}, a - \frac{z}{\sqrt{2}}\right)J = \exp - \left\{\left(\frac{z}{\sqrt{2}} + a - \frac{z}{\sqrt{2}}\right)\right\}J = e^{-a}J.$$

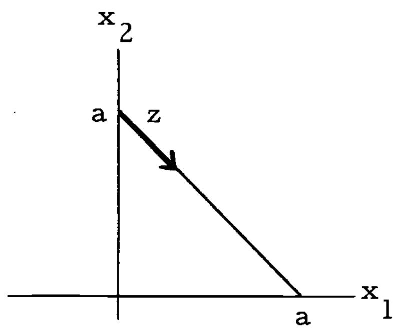
We have found the value of the Jacobian in Example 1--Uniform, so we have

$$J = \frac{1}{\sqrt{2}} \quad \text{and}$$

$$g(z, a; \theta) = \frac{1}{\sqrt{2}} e^{-a} .$$

3. $$g_2(a) = \int_{k_1}^{k_2} g(z, a; \theta)dz = \int_{k_1}^{k_2} \frac{1}{\sqrt{2}} e^{-a} dz = \frac{1}{\sqrt{2}} e^{-a} [z]_{k_1}^{k_2}$$

As in Example1--Uniform, we evaluate z along ϕ from 0 to its maximum in the first quadrant, which we find to be $\sqrt{2}\,a$ by the Pythagorean Theorem.



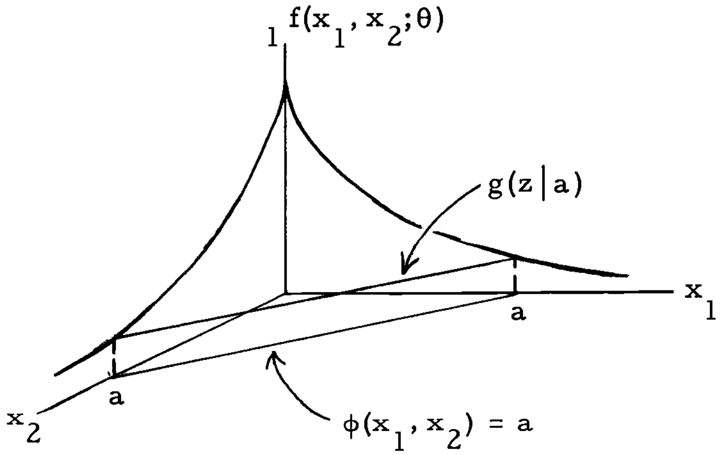
$$g_2(a) = \frac{1}{\sqrt{2}}e^{-a}[\sqrt{2}\,a - 0]$$

$$= ae^{-a}.$$

4. $$g(z|a) = \frac{g(z, a;\theta)}{g_2(a)} = \frac{1}{\sqrt{2}}\,e^{-a} \cdot \frac{1}{ae^{-a}}$$

$$= \frac{1}{\sqrt{2}\,a}$$

Thus, we have a uniform density when we consider the exponential bivariate density function along the line given by ϕ. We can see that this is plausible from the illustration of f and ϕ:



Example 2--Exponential

$$f(x_1, x_2;\theta) = e^{-(x_1+x_2)} \quad \text{for} \quad \begin{array}{l} 0 \le x_1 < \infty \\ 0 \le x_2 < \infty \end{array} \qquad \phi(x_1, x_2) = x_2 - x_1 = a$$

$= 0$ for x_1, x_2 otherwise

1. $(x_1, x_2) \to (z, a)$.

We use the results from Example 2--Uniform as follows:

$$x_1 = \frac{z}{\sqrt{2}}$$

$$x_2 = a + \frac{z}{\sqrt{2}} .$$

2. $f(x_1, x_2; \theta) \to g(z, a; \theta)$.

$$g(z, a; \theta) = f(\frac{z}{\sqrt{2}}, a + \frac{z}{\sqrt{2}})J = \exp - \{\frac{z}{\sqrt{2}} + a + \frac{z}{\sqrt{2}}\}J = \exp - \{a + \sqrt{2}\, z\}J.$$

Using the value for J found in Example 2--Uniform gives

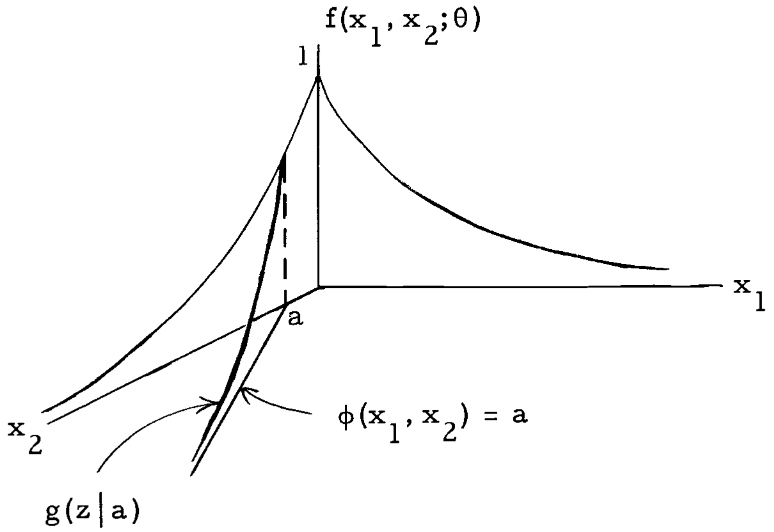
$$g(z, a; \theta) = \frac{1}{\sqrt{2}} e^{-(a+\sqrt{2}\, z)}$$

3. $g_2(a) = \int_{k_1}^{k_2} g(z, a; \theta) dz = \int_{k_1}^{k_2} \frac{1}{\sqrt{2}} e^{-(a+\sqrt{2}\, z)} dz$

$$= \frac{1}{\sqrt{2}} e^{-a} \int_{k_1}^{k_2} e^{-\sqrt{2}\, z} dz = \frac{1}{\sqrt{2}} e^{-a} [-\frac{1}{\sqrt{2}} e^{-\sqrt{2}\, z}]_{k_1}^{k_2}$$

$$g_2(a) = -\frac{1}{2} e^{-a} [e^{-\sqrt{2}\, z}]_{k_1}^{k_2} .$$

To find k_1 and k_2 it is helpful to have the illustration of f and ϕ.



We see that z goes from $k_1 = 0$ to $k_2 = \infty$.

$$g_2(a) = -\frac{1}{2}e^{-a}[e^{-\sqrt{2}\,z}]_0^\infty$$

$$= -\frac{1}{2}e^{-a}[\lim_{z\to\infty} e^{-\sqrt{2}\,z} - e^0]$$

$$= -\frac{1}{2}e^{-a}[\,0-1] = \frac{1}{2}e^{-a}\,.$$

4. $$g(z|a) = \frac{g(z,a;\theta)}{g_2(a)} = \frac{1}{\sqrt{2}}\,e^{-(a+\sqrt{2}\,z)}\cdot 2e^{a}$$

$$= \sqrt{2}\,e^{-a-\sqrt{2}\,z+a}$$

$$= \sqrt{2}\,e^{-\sqrt{2}\,z}.$$

Our final result is again an exponential density function.

Example 3--Exponential

$$f(x_1, x_2;\theta) = e^{-(x_1+x_2)} \quad \text{for} \quad \begin{array}{l} 0 \le x_1 < \infty \\ 0 \le x_2 < \infty \end{array} \quad \phi(x_1, x_2) = x_1^2 + x_2^2 = a$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

1. $(x_1, x_2) \rightarrow (z, a)$.

We use the results from step 1 of Example 3--Uniform, where

$$x_1 = \sqrt{a} \cos \frac{z}{\sqrt{a}}$$

$$x_2 = \sqrt{2} \sin \frac{z}{\sqrt{a}} .$$

2. $f(x_1, x_2;\theta) \rightarrow g(z, a;\theta)$.

$$g(z, a;\theta) = f(\sqrt{a} \cos \frac{z}{\sqrt{a}}, \sqrt{a} \sin \frac{z}{\sqrt{a}})J$$

$$= \exp -\{\sqrt{a} \cos \frac{z}{\sqrt{a}} + \sqrt{a} \sin \frac{z}{\sqrt{a}}\}J$$

Use of the Jacobian found in Example 3--Uniform gives

$$g(z, a;\theta) = [\exp -\{\sqrt{a} \cos \frac{z}{\sqrt{a}} + \sqrt{a} \sin \frac{z}{\sqrt{a}}\}][\frac{1}{2}a^{-1/2}]$$

$$= \frac{1}{2\sqrt{a}} \exp -\{\sqrt{a}(\cos \frac{z}{\sqrt{a}} + \sin \frac{z}{\sqrt{a}})\}$$

3. $g_2(a) = \int_{k_1}^{k_2} g(z, a;\theta)dz = \frac{1}{2\sqrt{a}} \int_{k_1}^{k_2} \exp -\{\sqrt{a}(\cos \frac{z}{\sqrt{a}} + \sin \frac{z}{\sqrt{a}})\}dz$

The integrand is not integrable by the usual formulas. It might be integrable by other methods; for instance, the exponential function could be expressed as a Maclaurin series. However, problems of convergence of the series are likely to arise. Numerical methods, where specific values are known for the z variable, can be resorted to in a given problem to arrive at an approximation for $g_2(a)$. Then $g(z|a)$ can be found as before.

Example 5--Exponential

$$f(x_1, x_2) = e^{-(x_1+x_2)} \quad \text{for} \quad 0 \leq x_1 < \infty, \quad 0 \leq x_2 < \infty$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

$$\phi(x_1, x_2) = \phi[x_1(t), x_2(t)]$$

where

$$x_1 = x_1(t) = \frac{1}{3}t^3 + a$$

$$x_2 = x_2(t) = \frac{1}{2}t^2$$

1. $(x_1, x_2) \rightarrow (z, a)$.

From Example 5--Uniform we have

$$x_1 = a + \frac{1}{3}[(3z+1)^{2/3} - 1]^{3/2}$$

$$x_2 = \frac{1}{2}[(3z+1)^{2/3} - 1].$$

2. $f(x_1, x_2; \theta) \rightarrow g(z, a; \theta)$.

$$g(z,a;\theta) = f\{a+\frac{1}{3}[(3z+1)^{2/3}-1]^{3/2},\ \frac{1}{2}[(3z+1)^{2/3}-1]\}J$$

$$= \exp -\ \{a+\frac{1}{3}[(3z+1)^{2/3}-1]^{3/2}+\frac{1}{2}[(3z+1)^{2/3}-1]\}J$$

We use the Jacobian found in Example 5--Uniform.

$$g(z,a;\theta) = (3z+1)^{-1/3}\ \exp -\ \{a+\frac{1}{3}[(3z+1)^{2/3}-1]^{3/2}+\frac{1}{2}[(3z+1)^{2/3}-1]\}.$$

3. $g_2(a) = \int_{k_1}^{k_2} g(z,a;\theta)dz.$

$g_2(a)$ obviously cannot be found by usual methods.

Example 1--Normal

$$f(x_1,x_2;\theta) = \frac{1}{8\pi\theta^2} e^{-\frac{1}{2\theta^2}(x_1^2+x_2^2)} \quad \text{for} \quad 0 \le x_1 < \infty$$

$$0 \le x_2 < \infty$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

$$\phi(x_1,x_2) = x_1 + x_2 = a$$

1. $(x_1,x_2) \rightarrow (z,a)$.

We use the results from Example 1--Uniform.

$$x_1 = \frac{z}{\sqrt{2}}$$

$$x_2 = a - \frac{z}{\sqrt{2}}.$$

2. $f(x_1, x_2;\theta) \rightarrow g(z, a;\theta)$.

$$g(z, a;\theta) = f(\frac{z}{\sqrt{2}}, a - \frac{z}{\sqrt{2}})J$$

$$= \frac{1}{8\pi\theta^2} \exp\{-\frac{1}{2\theta^2}[(\frac{z}{\sqrt{2}})^2 + (a - \frac{z}{\sqrt{2}})^2]\}J$$

$$= \frac{1}{8\pi\theta^2} \exp\{-\frac{1}{2\theta^2}[\frac{z^2}{2} + a^2 - \sqrt{2}\, az + \frac{z^2}{2}]\}J$$

$$= \frac{1}{8\pi\theta^2} \exp\{-\frac{1}{2\theta^2}[z^2 + a^2 - \sqrt{2}\, az]\}J$$

The Jacobian was found in Example 1--Uniform to be $\frac{1}{\sqrt{2}}$.

$$g(z, a;\theta) = \frac{1}{8\sqrt{2}\,\pi\theta^2} \exp\{-\frac{1}{2\theta^2}[z^2 + a^2 - \sqrt{2}\, az]\}$$

3. $g_2(a) = \int_{k_1}^{k_2} g(z, a;\theta)dz = \frac{1}{8\sqrt{2}\,\pi\theta^2} \int_{k_1}^{k_2} \exp\{-\frac{1}{2\theta^2}[z^2 + a^2 - \sqrt{2}\, az]\}dz.$

The integrand cannot be integrated by usual means.

<u>Example 2--Normal</u>

$$f(x_1, x_2;\theta) = \frac{1}{8\pi\theta^2} e^{-\frac{1}{2\theta^2}(x_1^2 + x_2^2)} \quad \text{for} \quad \begin{array}{l} 0 \leq x_1 < \infty \\ 0 \leq x_2 < \infty \end{array}$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

$$\phi(x_1, x_2) = x_2 - x_1 = a$$

1. $(x_1, x_2) \rightarrow (z, a)$.

We use the results from Example 2--Uniform.

$$x_1 = \frac{z}{\sqrt{2}}$$

$$x_2 = a + \frac{z}{\sqrt{2}}$$

2. $f(x_1, x_2; \theta) \rightarrow g(z, a; \theta)$.

$$g(z, a; \theta) = f\left(\frac{z}{\sqrt{2}}, a + \frac{z}{\sqrt{2}}\right)J = \frac{1}{8\pi\theta^2} \exp\left\{-\frac{1}{2\theta^2}\left[\left(\frac{z}{\sqrt{2}}\right)^2 + \left(a + \frac{z}{\sqrt{2}}\right)^2\right]\right\}J$$

$$= \frac{1}{8\pi\theta^2} \exp\left\{-\frac{1}{2\theta^2}[z^2 + a^2 + \sqrt{2}\,az]\right\}J$$

The Jacobian was found in Example 2--Uniform to be $\frac{1}{\sqrt{2}}$.

$$g(z, a; \theta) = \frac{1}{8\sqrt{2}\,\pi\theta^2} \exp\left\{-\frac{1}{2\theta^2}[z^2 + a^2 + \sqrt{2}\,az]\right\}.$$

3. $$g_2(a) = \int_{k_1}^{k_2} g(z, a; \theta)dz = \frac{1}{8\sqrt{2}\,\pi\theta^2} \int_{k_1}^{k_2} \exp\left\{-\frac{1}{2\theta^2}[z^2 + a^2 \sqrt{2}\,az]\right\}dz.$$

We can complete the square in the exponent as follows.

$$g_2(a) = \frac{1}{8\sqrt{2}\,\pi\theta^2}\int_{k_1}^{k_2} \exp\{-\frac{1}{2\theta^2}[(z^2+\frac{a^2}{2}+\sqrt{2}\,az)+\frac{a^2}{2}]\}dz$$

$$= \frac{1}{8\sqrt{2}\,\pi\theta^2}\exp\{-\frac{a^2}{4\theta^2}\}\int_{k_1}^{k_2} \exp\{-\frac{1}{2\theta^2}[z^2+\sqrt{2}\,az+\frac{a^2}{2}]\}dz$$

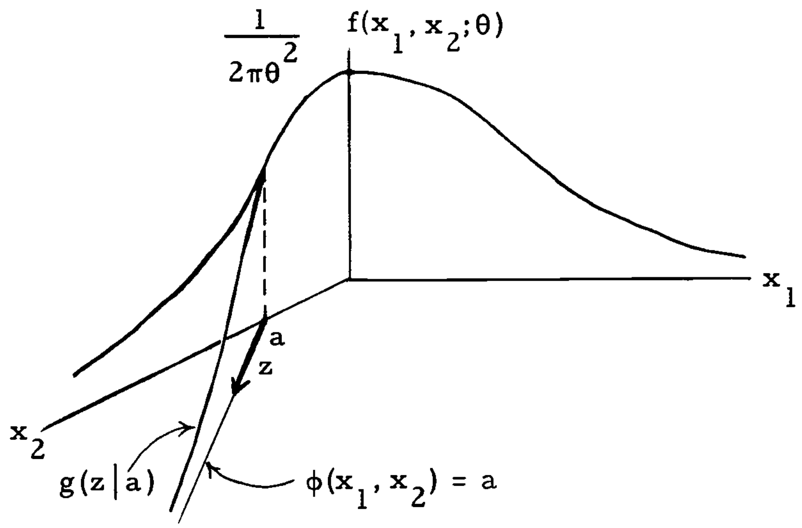
$$= \frac{1}{8\sqrt{2}\,\pi\theta^2}\exp\{-\frac{a^2}{4\theta^2}\}\int_{k_1}^{k_2} \exp\{-\frac{1}{2\theta^2}[(z+\frac{a}{\sqrt{2}})^2]\}dz$$

The integrand is of the form $e^{-a^2u^2}du$, where

$$a = \frac{1}{\sqrt{2}\,\theta}$$

$$u = z + \frac{a}{\sqrt{2}}\,.$$

The limits k_1 and k_2 are found by the following illustration to be 0 and ∞, respectively.



We can find the definite integral $\int_0^{\infty} e^{-a^2u^2} du$ in a table of integrals to have the value $\frac{\sqrt{\pi}}{2a}$, if $a > 0$. Thus we have

$$\int_0^{\infty} \exp\{-\frac{1}{2\theta^2}[(z+\frac{a}{\sqrt{2}})^2]\}dz = \frac{\sqrt{\pi}}{2}\cdot \sqrt{2}\,\theta = \frac{\sqrt{\pi}\,\theta}{\sqrt{2}}$$

$$g_2(a) = \left[\frac{1}{8\sqrt{2}\,\pi\theta^2}\exp\{-\frac{a^2}{4\theta^2}\}\right][\frac{\sqrt{\pi}\,\theta}{\sqrt{2}}]$$

$$= \frac{1}{16\sqrt{\pi}\,\theta}\exp\{-\frac{a^2}{4\theta^2}\}.$$

4. $g(z|a) = \frac{g(z, a;\theta)}{g_2(a)}$.

$$g(z|a) = \frac{1}{8\sqrt{2}\,\pi\theta^2}\exp\{-\frac{1}{2\theta^2}[z^2+a^2+\sqrt{2}\,az]\}4\sqrt{\pi}\,\theta\exp\{\frac{a^2}{4\theta^2}\}$$

$$= \frac{\sqrt{2}}{4\sqrt{\pi}\,\theta}\exp\{-\frac{1}{2\theta^2}[z^2+\frac{a^2}{2}+\sqrt{2}\,az]\}$$

$$= \frac{\sqrt{2}}{4\sqrt{\pi}\,\theta}\exp\{-\frac{1}{2\theta^2}[z+\frac{a}{\sqrt{2}}]^2\}.$$

The final density is therefore an exponential function.

Example 3--Normal

$$f(x_1, x_2;\theta) = \frac{1}{8\pi\theta^2} e^{-\frac{1}{2\theta^2}(x_1^2+x_2^2)} \quad \text{for} \quad 0 \le x_1 < \infty, \quad 0 \le x_2 < \infty$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

$$\phi(x_1, x_2) = x_1^2 + x_2^2 = a$$

1. $(x_1, x_2) \rightarrow (z, a)$.

We use the results from Example 3--Uniform:

$$x_1 = \sqrt{a}\cos\frac{z}{\sqrt{a}}$$

$$x_2 = \sqrt{a}\sin\frac{z}{\sqrt{a}}.$$

2. $f(x_1, x_2;\theta) \rightarrow g(z, a;\theta)$.

$$g(z, a;\theta) = f(\sqrt{a}\cos\frac{z}{\sqrt{a}}, \sqrt{a}\sin\frac{z}{\sqrt{a}})J$$

$$= \frac{1}{8\pi\theta^2}\exp\{-\frac{1}{2\theta^2}[(\sqrt{a}\cos\frac{z}{\sqrt{a}})^2 + (\sqrt{a}\sin\frac{z}{\sqrt{a}})^2]\}J$$

$$= \frac{1}{8\pi\theta^2}\exp\{-\frac{1}{2\theta^2}[a(\cos^2\frac{z}{\sqrt{a}} + \sin^2\frac{z}{\sqrt{a}})]\}J$$

$$= \frac{1}{8\pi\theta^2}\exp\{-\frac{a}{2\theta^2}\}J$$

From Example 3--Uniform $J = \frac{1}{2\sqrt{a}}$

$$g(z, a;\theta) = \frac{1}{16\pi\theta^2\sqrt{a}}\exp\{-\frac{a}{2\theta^2}\}.$$

3. $g_2(a) = \int_{k_1}^{k_2} g(z, a;\theta)dz$

$$= \int_{k_1}^{k_2}\frac{1}{16\pi\theta^2\sqrt{a}}\exp\{-\frac{a}{2\theta^2}\}dz = \frac{1}{16\pi\theta^2\sqrt{a}}\exp\{-\frac{a}{2\theta^2}\}[z]_{k_1}^{k_2}$$

As in Example 3--Uniform, z transcribes the part of the circle represented by ϕ which lies in the first quadrant. Thus $k_1 = 0$, $k_2 = \frac{1}{4}(2\pi\sqrt{a}) = \frac{\pi\sqrt{a}}{2}$.

$$g_2(a) = \frac{1}{16\pi\theta^2\sqrt{a}} \exp\{-\frac{a}{2\theta^2}\}[\frac{\pi\sqrt{a}}{2} - 0] = \frac{1}{32\theta^2} \exp\{-\frac{a}{2\theta^2}\}.$$

4. $g(z|a) = \dfrac{g(z, a;\theta)}{g_2(a)}$.

$$g(z|a) = \frac{1}{16\pi\theta^2\sqrt{a}} \exp\{-\frac{a}{2\theta^2}\} 8\theta^2 \exp\{\frac{a}{2\theta^2}\}$$

$$= \frac{1}{2\pi\sqrt{a}} \exp\{-\frac{a}{2\theta^2} + \frac{a}{2\theta^2}\} = \frac{1}{2\pi\sqrt{a}} .$$

Thus $g(z|a)$ is a uniform density function dependent only on the value of $\sqrt{a}$, the radius of ϕ.

<u>Example 5--Normal</u>

$$f(x_1, x_2;\theta) = \frac{1}{8\pi\theta^2} e^{-\frac{1}{2\theta^2}(x_1^2+x_2^2)} \quad \text{for} \quad \begin{array}{l} 0 \leq x_1 < \infty \\ 0 \leq x_2 < \infty \end{array}$$

$$= 0 \quad \text{for} \quad x_1, x_2 \quad \text{otherwise}$$

$$\phi = \phi[x_1(t), x_2(t)] \quad \text{where} \quad \begin{array}{l} x_1 = x_1(t) = \frac{1}{3}t^3 + a \\ x_2 = x_2(t) = \frac{1}{2}t^2 \end{array}$$

1. $(x_1, x_2) \rightarrow (z, a)$.

From Example 5--Uniform

$$x_1 = a + \frac{1}{3}[(3z+1)^{2/3} - 1]^{3/2}$$

$$x_2 = \frac{1}{2}[(3z+1)^{2/3} - 1].$$

2. $f(x_1, x_2; \theta) \rightarrow g(z, a; \theta)$.

$$g(z, a; \theta) = f(a + \frac{1}{3}[(3z+1)^{2/3} - 1]^{3/2}, \frac{1}{2}[(3z+1)^{2/3} - 1])J$$

$$= \frac{1}{8\pi\theta^2} \exp\{-\frac{1}{2\theta^2}(a + \frac{1}{3}[(3z+1)^{2/3} - 1]^{3/2})^2 + (\frac{1}{2}[(3z+1)^{2/3} - 1])^2\}J$$

In Example 5--Uniform J was found to be $(3z+1)^{-1/3}$.

$$g(z, a; \theta) = \frac{1}{8\pi\theta^2}(3z+1)^{-1/3} \exp\{-\frac{1}{2\theta^2}(a + \frac{1}{3}[(3z+1)^{2/3} - 1]^{3/2})^2 + (\frac{1}{2}[(3z+1)^{2/3} - 1])^2\}.$$

3. $g_2(a) = \int_{k_1}^{k_2} g(z, a; \theta)dz.$

$$= \int_{k_1}^{k_2} \frac{1}{8\pi\theta^2}(3z+1)^{-1/3} \exp\{-\frac{1}{2\theta^2}(a + \frac{1}{3}[(3z+1)^{2/3} - 1]^{3/2})^2 + (\frac{1}{2}[(3z+1)^{2/3} - 1])^2\}dz.$$

Due to the lack of proper differential, the integral of $g(z, a; \theta)$ cannot be found by the usual theoretical means.

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