Shear waves, which are instabilities of the longshore current, are of interest to nearshore research because they can act as a cross-shore mixing mechanism for the current. The best method to study them is in a controlled environment, such as circular wave tank where continuous longshore currents can be generated.

Conservation of vorticity is found from the linearized equations of motion in cylindrical coordinates. The flow field is expressed as a small perturbation superimposed upon a mean longshore current. Using streamfunction definitions for the perturbation velocities, solutions to the vorticity equation are frequency (eigenvalues) and cross-shore structure (eigenvectors).

Solutions of a finite difference model, using cross-shore geometry suitable for laboratory testing, range from 0.0 to 0.8 Hz. The fastest growing wave occurs at 0.4 Hz. The u-v phase difference of the unstable modes are nonquadrature in the area of steep offshore shear, moving toward quadrature further offshore. The dispersion shows that shear waves are shorter than natural tank modes.

An experiment is designed to observe shear waves in a circular tank, using a cross-shore geometry similar to that of the model. Measurements are taken over
sequential runs at seven cross-shore positions using one current meter. A composite profile of the longshore current shows that the measurement positions fall within a region of steep offshore shear that includes an extremum in the potential vorticity profile.

The cross-shore power spectra of the demeaned velocity time series show an increase in energy below 0.4 Hz in both velocity components. The coherence spectra at the first three cross-shore positions show significant coherence below 0.4 Hz that correspond to nonquadrature phases over the same band. The positions fall over the steepest part of the shear.

Longshore wavenumbers are found by regressing phase onto spacial lags for eleven sensor pairs. Time series are taken over sequential runs, with different sensor spacings for each run. Twelve wavenumber slopes are found for frequencies between .15 Hz and .39 Hz. The dispersion of the measured signals compares favorably with shear wave solutions and indicates that the waves are too short to be natural tank modes.
Shear Waves in a Circular Wave Basin

by

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Instabilities of longshore currents, known as shear waves, have been studied and modeled in the coastal environment over the last decade. Their existence was discovered through the analysis of field data (Oltman-Shay et al., 1991). Bowen and Holman (1989) developed the basic theory for shear waves along a straight beach, and applied their analytical analysis to an idealized longshore current over a beach with a flat cross-shore profile. They found that the range of frequencies for the unstable modes is dependent upon the magnitude of the offshore gradient of the longshore current. They determined that the wavelengths are much shorter than those for infragravity waves of the same frequency and the wavenumbers determined from field measurements fell within the range predicted by the model. They speculated that shear instabilities may be important factors in longshore current mixing.

Since the work of Bowen and Holman the theory has evolved. Dodd and Thornton (1990) derived the energetics for the growth of the instability, showing that the growth must be accompanied by mixing between the perturbation and longshore current. They also studied the effects of the beach profile on the growth characteristics of the waves by applying the theory to a sloped bed and comparing it to results from a flat-bed model, concluding that the beach slope made little difference in the range of wavenumbers and
frequencies over which shear waves can grow. Other numerical studies have been completed by Putrevu and Svenden (1992), who tested the effects of various beach topographies on the shear wave solutions; Dodd, Oltman-Shay and Thornton (1991) who included friction in their numerical model and compared the results with test data; and Dodd (1994) who numerically investigated the destabilizing effects of the bottom shear stress on the current. In most of these studies the numerical solutions were checked against field data.

A problem that arises in the use of field data comparisons with shear wave model solutions is that the conditions under which they are generated are not always predictable and cannot be controlled. Field studies can also be expensive as the conditions which generate the strongest shear waves require a strong surf which can be damaging to the instruments. To study shear waves most effectively, it is necessary to test the effect of various parameters and geometries on the generation of these waves. The easiest and most economical way to do this is under the controlled conditions of a laboratory. Since rectangular tanks have short beaches and ends which make it difficult to sustain or even generate longshore currents, for this study a circular wave tank was used to generate a 'continuous' longshore current profile, driven by incident wave forcing. The purpose is to show that the waves can be generated and measured in the wave tank environment, and to understand how the wave parameters of the circular model relate to the case of a straight beach.
The major steps involved are first to develop shear wave theory for cylindrical coordinates and examine the effects of the curvature on the energetics and wave parameters; then develop a unique set of diagnostic criteria for shear wave identification; set up and implement an experiment which would generate data which could be compared with the criteria and model for both shear waves and the natural gravity modes of the tank.
Longshore Currents

The interest in longshore current instabilities expanded when the standard mixing and dissipation theories often failed to explain the longshore current profile over barred beaches during storms. According to present theories, storm waves breaking over a bar would force a sharp rise in the current velocities with a maximum over the bar; however, studies often show the current profiles to be smoother than predicted, with the maximum current velocities over the trough shoreward of the bars (Bowen and Holman, 1989).

The prevailing theory used to predict and model longshore currents was developed separately by Bowen (1969) and Longuet-Higgins (1970). The depth averaged and time averaged longshore force balance in the presence of an incident wave field, where the dynamic pressures are assumed small enough to be ignored (Mei, 1989) is:

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -g \frac{\partial \bar{\eta}}{\partial y} + \frac{1}{\rho(\bar{\eta} + h)} \left[ \frac{\partial (s_{sy})}{\partial x} + \frac{\partial (s_{xy})}{\partial y} \right] - \bar{B}_y
\]

where \( U \) and \( V \) are the depth averaged velocity components, \( \bar{\eta} \) is the variation in sea surface height, \( s_{sy} \) and \( s_{xy} \) are the longshore and cross-shore fluxes of longshore momentum (radiation stress tensors), and \( \bar{B}_y \) is the longshore component of friction directed against the longshore flow. The cross-shore and longshore radiation stress gradients arise from the fluctuating components of the nonlinear terms in the momentum equations and can provide a longshore directed stress. In shallow water these gradients are directly proportional to the
energy flux dissipation. In the shoaling region outside the breakpoint the waves approach at an angle so that the wave velocities have a cross-shore and long-shore component. Even as the waves shoal, however, energy flux is conserved until breaking, so that seaward of the breakpoint the radiation stress gradients are nonexistent. With wave breaking the energy flux is dissipated linearly between the breakpoint and the shoreline, so that the momentum flux gradient is non-zero. The cross-shore gradient of the longshore directed momentum flux sets up a current by exerting a 'push' on the water column. Bottom friction acts against the current and is proportional to the current velocity. The current reaches steady state when the longshore friction balances the radiation stress gradient.

When the system is at steady state, both $V$ and $\eta$ are non-time dependent, and if it is also assumed there are no longshore gradients in the mean components of the force balance then the continuity equation

$$\frac{\partial U(\eta + h)}{\partial x} = 0 ;$$

(2)

shows that $U$ is zero. At steady state the longshore momentum balance is:

$$0 = \frac{1}{\rho(\eta + h)} \frac{\partial (S_{\eta})}{\partial x} - \vec{B},$$

(3)

Under the condition where the incident waves are monochromatic (one frequency) the force balance expressed in equation 3 would result in a velocity profile that vanishes offshore of the breakpoint, has a sharp discontinuity at the breakpoint and decreases linearly from the breakpoint to the shoreline. This
is true because the bottom friction is a linear function of the time averaged velocity,

\[ \overline{B_y} = \left( \frac{2}{\pi} \right) \rho u_{\text{max}} \overline{v_y}, \]  

(4)

In nature momentum mixing would smooth such sharp velocity discontinuities. In the surf zone momentum mixing from breaking waves provides a mechanism which smoothes the longshore current profile. Longuet-Higgins (1970) parameterized this process by introducing a function which represents the exchange of momentum due to turbulent eddies. A realistic mixing coefficient was then determined experimentally. The time averaged momentum balance for monochromatic incident waves can be expressed as:

\[ 0 = -\int \frac{1}{\rho(\eta + h)} \frac{\partial S_{xy}}{\partial x} + \frac{\partial}{\partial x} \left( \mu_r h \frac{\partial v}{\partial x} \right) - \overline{B_y}, \]  

(5)

when the current reaches a steady state. The mixing term smoothes the velocity profile so that it is spread slightly offshore of the breakpoint.

Usually random waves are incident along the shoreline. Breaking wave height is determined by a height to depth ratio, and the wave height in shallow water is a function of the deep water wavelength and wave height (Komar, 1976). Therefore breaking depth in shallow water will vary for individual waves in a random wave field, resulting in a spatial range over which breaking takes place. In most applications that involve random waves equation 2 will provide a sufficient model for the longshore current profile that compares reasonably well to observations. However, in some situations where there is intense wave energy breaking over a very small region, such as storm waves breaking over a
bar, measured cross-shore profiles of longshore velocity are much more smoothed than would be expected from solutions to this force balance; artificial eddy mixing does not substantially contribute to the momentum balance in these cases. It has been hypothesized that the excursion of the current due to the shear wave cross-shore velocity components may account for mixing in some cases. Bowen and Holman (1989) showed by their linear analysis, using a typical (piece wise) geometry, that the cross-shore excursion can be as much as 50 meters and this discovery offered a new mixing mechanism that could potentially explain the inconsistencies between the theory and observations.
Theory

Vorticity equation

Bowen and Holman used the linearized equations for a perturbed flow along a straight beach to derive the stability equation (conservation of vorticity) for the system. The same method is used to find the vorticity equation for a curved flow. The derivation of the shallow water equations in cylindrical coordinates are given in appendix 1.

The shallow water equations for the radial and azimuthal flow are:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -g \frac{\partial \eta}{\partial r}
\]
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \theta} + \frac{uv}{r} = -g \frac{\partial \eta}{r\theta}
\]

where \( r \) is the radius from the center of the circular flow to any arbitrary point and theta (\( \theta \)) denotes displacements in the azimuthal direction. A perturbation to a longshore steady state flow can be described by

\[
U(r, \theta) = u(r, \theta)e_r + (v(r, \theta) + V(r))e_\theta
\]

where \( e_r \) and \( e_\theta \) represent the radial (cross-shore) and azimuthal (longshore) directions. \( V \) represents the mean background current and \( u \) and \( v \) are the cross-shore and longshore velocity components of the perturbation. The linearized momentum equations become:

\[
\frac{\partial u}{\partial t} + \frac{V}{r} \frac{\partial u}{\partial \theta} - \frac{V^2}{r} \frac{V}{r} - 2v \frac{V}{r} = -g \frac{\partial (\eta + \eta')}{\partial r}
\]
\[ \frac{\partial v}{\partial t} + u \frac{\partial V}{\partial r} + V \frac{\partial v}{\partial \theta} + u \frac{V}{r} = -g \frac{\partial (\bar{\eta} + \eta')}{r \partial \theta} \]  

(8b)

The surface fluctuation \( \eta \) has been decomposed into mean and fluctuating components. In the absence of a perturbation the balance is

\[ \frac{V^2}{r} = -g \frac{\partial \bar{\eta}}{\partial r}, \]  

(9)

which is the cross-shore pressure gradient forced by the longshore steady flow.

The linearized momentum equations for the perturbation can be found by dropping the basic flow balance:

\[ \frac{\partial u}{\partial t} + V \frac{\partial u}{\partial r} - \frac{2vV}{r} = -g \frac{\partial \eta}{\partial r} \]  

(10a)

\[ \frac{\partial v}{\partial t} + u \left( \frac{V}{r} + \frac{\partial V}{\partial r} \right) + \frac{V}{r} \frac{\partial v}{\partial \theta} + \frac{uV}{r} = -\frac{g}{r} \frac{\partial \eta}{\partial \theta} \]  

(10b)

where the primes have been dropped from the surface deformation, \( \eta \). The linearized continuity equation is derived in appendix 2. It is shown in that it is reasonable to apply the rigid approximation for geometries where \( V_0^2/gh_0 << 1 \)

Then the relationship can be is expressed as

\[ \frac{1}{r} \frac{\partial rh u}{\partial r} + \frac{1}{r} \frac{\partial h v}{\partial \theta} = 0 \]  

(11)

The vorticity relationships between the inertial and background components of the flow are derived by cross differentiating the momentum equations, as in the cartesian case (appendix 3). Using the continuity relationship the terms can be arranged to show that the linearized conservation of vorticity is analogous to the case of straight flow:
\[
\left( \frac{\partial}{\partial t} + \frac{V}{r} \frac{\partial}{\partial \theta} \right) \left( \frac{1}{r} \frac{\partial v}{\partial r} - \frac{\partial u}{\partial \theta} \right) = -uh \frac{\partial}{\partial r} \left[ \frac{V + \partial V}{r} / h \right] \tag{12}
\]

where \( h \) is the mean depth measured from the surface. Changes to the local relative vorticity (i.e. the vorticity at any local cross-shore position) will arise as a response to cross-shore perturbations to the local background vorticity. The local curvature and shear of the current contribute to the local potential vorticity. An extension of Raleigh's inflection point criteria can show that if the RHS derivative of equation 10 crosses 0 somewhere in the cross-shore unstable solutions are possible. An extremum in the potential vorticity will guarantee this condition is met.

The linearized continuity relationship with the rigid lid approximation applied allows stream function definitions to be introduced for \( u \) and \( v \), since the flow is approximated as two dimensional.

If a right handed frame of reference is chosen such that positive \( u \) is onshore, then positive \( v \) is counterclockwise. The stream function can then be defined as follows:

\[
u = -\frac{1}{h} \frac{\partial \Psi}{\partial \theta}, \quad v = \frac{1}{h} \frac{\partial \Psi}{\partial r} \tag{13}
\]

Expressed in terms of a stream function the vorticity equation becomes:

\[
\left[ \frac{\partial}{\partial t} + \frac{V}{r} \frac{\partial}{\partial \theta} \right] \left( \frac{\partial v}{\partial r} \frac{\partial \Psi}{\partial \theta} + \frac{\partial u}{\partial \theta} \frac{\partial \Psi}{\partial r} + \frac{\partial \Psi}{\partial \theta} \frac{\partial \Psi}{\partial r} \right) = \frac{\partial \Psi}{\partial \theta} \frac{\partial}{\partial r} \left[ \frac{(V + \partial V)}{r} / h \right] \tag{14}
\]

Wave like solutions for the stream function can be expressed as

\[
\Psi(r, \theta) = \text{Re}[A \psi(r)e^{i(N\theta - \sigma)}] \tag{15}
\]

where \( \psi(r) \) and \( \sigma \) can be complex, and \( A \) is an arbitrary constant; \( N \) is a
dimensionless longshore wavenumber equivalent to $2\pi$ divided by a radian wavelength $\theta_L$. This is similar to a straight flow stream function except

$$k = k(r) = \frac{2\pi}{\theta_L} r$$

(where $\theta_L r$ equals the wavelength at an arbitrary radius)

and arbitrary longshore distances are arc lengths, $S = \theta r$. Then

$$k(r) \cdot S = (2\pi/\theta_L) \theta = N \theta.$$ This relationship shows that integer values of $N$ represent the number of whole wavelengths that fit within the circumference of the circular flow.

Using this form of the stream function the vorticity equation can be expressed in terms of $\psi(r)$, $\sigma$ and $N$:

$$\left[ \frac{V}{r} - \frac{\sigma}{N} \right] \left[ \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial \psi}{\partial r} \left( \frac{1}{h} \frac{\partial h}{\partial r} \right) - N^2 \frac{\psi}{r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right] = \frac{h}{r} \psi(r) \frac{\partial}{\partial r} \left[ \frac{V}{r} - \frac{\partial V}{\partial r} \right]$$

(16)

Solutions for the range of wavenumbers over which instabilities can exist and their respective celerities can be found numerically since the vorticity equation is an eigensystem.

The dimensionless wavenumber, $N$, along with the cross-shore mean velocity and beach profiles, can be used as input parameters to the model. The unstable solutions are complex eigenvalues and eigenvectors, yielding celerities ($\sigma/N$) and a cross-shore structure $\psi(r)$ for each wavenumber where instabilities can exist.

**Energetics**

The energetics of the perturbation along a straight beach were derived by Dodd and Thornton (1990). By integrating the longshore average energy
equations in the cross-shore they found that the time rate of change in the kinetic energy of the perturbation velocity components is inversely related to the cross-shore shear in the longshore current, i.e.,

\[
\frac{\partial KE}{\partial t} = -\int_0^\infty \frac{\partial V}{\partial x} dx
\]

Their right handed coordinate system defined \( u \) as positive in the offshore direction. The direction of positive \( v \) when facing offshore flows to the left. In that frame of reference the longshore current is defined as moving in the positive longshore direction.

The offshore shear is negative, because \( V \) is decreasing with increasing offshore distance. Net energy is extracted from the mean flow in the area of negative shear in the current profile when \( \bar{uv} \) is positive.

In cylindrical coordinates (see appendix 4) the temporal change in kinetic energy of the perturbation is

\[
\frac{\partial KE}{\partial t} = - \int_{k_0}^r \bar{uv} \left( \frac{\partial V}{\partial r} - \frac{V}{r} \right) r dr - \int_{k_0}^r \bar{u} \frac{\partial \eta}{\partial r} + \bar{v} \frac{\partial \eta}{\partial \theta} \right) r dr
\]

The coordinate system here is chosen so that positive \( u \) is onshore (positive in the positive radial direction) and positive \( v \) is in the clockwise direction. The longshore current is moving in the same direction with respect to its shoreline as in the previous case of a straight beach, but that direction is negative in this system; the offshore shear is still negative because it represents a positive cross-shore change in the velocity of the longshore current with increasing negative cross shore distance. In this case as well as that of
the straight beach growth takes place when the shear in the current is negative when \( \bar{uv} \) is positive (on average instantaneous values of \( u \) and \( v \) are either both in positive or negative quadrants as defined by the coordinate system). If equations (14) and (15) are compared with respect to the growth rate of kinetic energy due to the shear production, then (15) implies that the angularity of the flow also affects this rate. However, the local shear in a jet, except in a few regions (where the current is small or the gradients equal zero), can be an order of magnitude or more higher than the angular velocity over most of the profile. It is unlikely that the interaction between the Reynolds stresses and angular velocity would have a significant effect on the KE growth rate since profiles that maximize the angular velocity, i.e. small radius with high velocities, would also create large shears.

The work term on the far right represents the work done by the surface gradients on the perturbation velocities and it can be shown from the continuity relationship that this term is equal to the work performed by the velocity gradients to deform the surface. This term is non-zero when integrated over the entire area because the horizontal velocities and surface expression are in complex phase with one another. Using the volume integral of the continuity relationship to derive the time rate of change in potential energy and combining it with the volume integral of the kinetic energy (see appendix 4) it can be seen that the source of energy for both the potential and kinetic energy of the perturbation comes from momentum transfer between the perturbation and the mean flow:
The problem defines the perturbation velocities as \( u \sim v \). In terms of the stream function \( \partial \Psi / \partial \theta \sim \partial \Psi / \partial r \). The stream function definitions for \( u \) and \( v \) can be used to scale the wavelengths and periods. Using \( R_0 \) as a standard radius (such as the maximum radius along the outer perimeter of the flow), \( \theta_i \) as the radian wavelength of an unstable wave, and \( \Delta R \) as the maximum width of the surf zone, then

\[
R_0 \theta_i = S_0
\]

where \( S_0 \) is a wavelength. Choosing \( \Psi \) as the corresponding stream function value and \( H_0 \) the depth, then \( u \sim v \) scale as

\[
\frac{\Psi}{H_0 S_0} \sim \frac{\Psi}{H_0 \Delta R}
\]

so that \( S_0 \sim \Delta R \). for any stream function and \( K_0 \sim 1 / \Delta R_0 \). The length of the waves in the system remain on the order of the width of the surf zone; they are independent of the radius, so that the magnitudes of the wavenumbers remain constant as the radius of curvature increases.

The cross-shore momentum equation can be used to scale the periods in a similar fashion as scaling the longshore balance to scale \( \eta \) in appendix 3:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + (V + v) \frac{\partial v}{\partial \theta} - \left( \frac{(V + v)^2}{r} \right) = -g \frac{\partial (\eta + \tilde{\eta})}{\partial r}
\]

Subtracting the basic flow balance and using a similar approach to Dodd and
Thornton (1990) to scaling the variables:

\[ r = R_0 r' \quad (23a) \]
\[ V(r) = V_0 V' \quad (23b) \]
\[ u(r, \theta), v(r, \theta) = u_0(u', v') \quad (23c) \]
\[ \eta(z) = \eta_0 \eta' \quad (23d) \]
\[ t = T_0 t' \quad (23e) \]
\[ g = g_0(1) \quad (23f) \]

where the subscripts indicate characteristic values.

Using the identity \( S_0 \sim \Delta R \):

\[ \frac{u_0}{T_0} \frac{\partial u'}{\partial r'} + \frac{u_0^2}{S_0} \frac{\partial u'}{\partial r} + \frac{V_0 u_0}{S_0} V' \frac{\partial V'}{\partial \theta} + \frac{u_0^2}{S_0} \frac{u'}{r'} \frac{\partial u'}{\partial \theta} - \frac{u_0 V_0}{R_0} u' V' \frac{1}{r'} - \frac{u_0^2}{R_0} \frac{v^2}{r'} = - \frac{S_0}{S_0} \eta_0 \frac{\partial \eta'}{\partial r'} \quad (24) \]

Multiplying through by \( S_0 \) dividing by \( V_0 \) and \( u_0 \) nondimensionalizes the equation:

\[ \frac{S_0}{T_0 V_0} \frac{\partial u'}{\partial r'} + \frac{u_0}{V_0} \frac{u'}{r'} \frac{\partial u'}{\partial r} + \frac{V'}{r'} \frac{\partial V'}{\partial \theta} + \frac{u_0}{V_0} \frac{u'}{r'} \frac{\partial u'}{\partial \theta} - \frac{S_0}{R_0} u' \frac{V'}{r'} - \frac{S_0 u_0}{R_0} \frac{v^2}{r'} = - \frac{S_0}{V_0 u_0} \frac{\eta_0}{r'} \frac{\partial \eta'}{\partial r'} \quad (25) \]

Dropping lower order terms yields the linearized balance:

\[ \frac{S_0}{T_0 V_0} \frac{\partial u'}{\partial r'} + \frac{V'}{r'} \frac{\partial V'}{\partial \theta} - \frac{S_0}{R_0} \frac{v^2}{r'} = - \frac{S_0}{V_0 u_0} \frac{\eta_0}{r'} \frac{\partial \eta'}{\partial r'} \quad (26) \]

The curvature term is included only in the centripetal force term and does not effect the order of the coefficients, since the width of the surf zone (noting the relationship between it and the wavelength) is approximately the lower limit for the smallest possible maximum radius of curvature. The coefficient of the nondimensional centripetal force term is never greater than one and diminishes as the radius of curvature of the surf zone increases; the highest
order of the coefficients in the force balance is one, as in the cartesian case, and $T_0 - S_0/V_0$. This shows that the real celerities are on the order of the longshore current.
Velocity Model

The cross-shore profile of longshore mean flow was generated by numerically integrating the Longuet-Higgins longshore balance for a steady flow (see eq. 2 in Longshore Currents) using a finite difference scheme. The input parameters for the numerical integration were a constant angle of wave incidence, a constant wave height at the break point, a cross-shore beach profile, bottom friction coefficient and an estimated 'mixing coefficient'. The breakpoint, wave heights and angle of wave incidence were estimated from experimental observations, as described in the following section. The angle of incidence was visually estimated to be 5 degrees over the wave runs, and the wave heights to be approximately 3.8 cm (1.5 in) at the breakpoint.

A depth profile was generated using the parabolic equation

\[ h_b = \alpha x^2 \]  \hspace{1cm} (24)

where \( h_b \) is the height of the beach measured from the bottom of the tank; \( x \) is the cross-shore distance from the shoreline toward the center of the tank, and \( \alpha \) is a fixed ratio between \( h_b \) and \( x^2 \) (see Experiment section). The breakpoint was estimated to be 14.5 in (36.8 cm) from the mean shoreline. The solution for the mean longshore current from the longshore force balance yields a current profile that contains a strong offshore shear. The current profile and the beach geometry, figure 1a, shows the extremum in the
offshore shear region of the potential vorticity profile \((V/r + \partial V/\partial r)/h\) (see fig (1b)). The scaled geometry ratio, \(V_0^2/gH_0\), is 35/980 \(~ 0.04\). The ratio of the variables depend on the cross-shore location and can get larger (order 1 or more) close to the shoreline. It shows, however, that it is not unreasonable to assume that the rigid lid condition applies for this geometry.

Figure 1(a,b). Cross-shore profiles; a) mean longshore current velocity and depth
b) cross-shore potential vorticity profile
Vorticity Model Results:

The parameters applied to the vorticity model are the longshore current and beach geometry profiles and the integer wavenumber $N$. The numerical solutions for the dispersion relationship and range of integer wavenumbers over which instabilities can exist are shown in figures 2a and 2b. As was
found with the straight beach the dispersion relationship is nearly linear. The fastest growing waves occur at $N=15$ with a corresponding wave period of 2.47 seconds and a wavelength of 0.42 radians. The e-folding time of the fastest growing integer wave mode is 5.36 sec. The solutions can be compared with solutions for a straight beach (see table 1) of the same cross-shore geometry. The units of length were computed at the maximum radius of 187.96 cm.

<table>
<thead>
<tr>
<th></th>
<th>straight beach</th>
<th>circular beach</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$ (cm)</td>
<td>74.40</td>
<td>78.90</td>
</tr>
<tr>
<td>$T$ (sec)</td>
<td>2.90</td>
<td>2.47</td>
</tr>
<tr>
<td>$GT$ (sec)</td>
<td>4.70</td>
<td>5.36</td>
</tr>
<tr>
<td>$C$ (cm/sec)</td>
<td>25.96</td>
<td>31.90</td>
</tr>
</tbody>
</table>

Table 1. Straight and circular beach comparisons of wave parameters

The e-folding time is 14% higher for the circular system with the specified radius. At this radius of curvature the wavelengths close to the shoreline compare well with the straight beach solution while those at the breakpoint and beyond are significantly shorter; however the celerities at the breakpoint and in the area of high offshore shear are closer to the straight beach solution than at the shoreline. Table 2 compares the wavelengths at the shoreline with those at the breakpoint (25 cm from the shoreline) for two radius of curvatures: the shoreline radius of 187.96 cm and 687.96 cm.
Table 2. Shoreline and breakpoint comparisons of wavelength and celerities

<table>
<thead>
<tr>
<th></th>
<th>max R=187.96</th>
<th>max R=187.96</th>
<th>max R=687.96</th>
<th>max R=687.96</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>shoreline</td>
<td>breakpoint</td>
<td>shoreline</td>
<td>breakpoint</td>
</tr>
<tr>
<td>L (cm)</td>
<td>78.90</td>
<td>68.26</td>
<td>75.83</td>
<td>73.08</td>
</tr>
<tr>
<td>C (cm/sec)</td>
<td>31.90</td>
<td>27.64</td>
<td>27.38</td>
<td>26.38</td>
</tr>
</tbody>
</table>

A comparison of the range of N (over which instabilities exist) for two radius of curvatures are shown in curves in figure 3. Solutions for the growth rate of the fastest growing wave are closer to that of the straight beach as the maximum radius doubles while holding the width of the surf zone constant. The behavior of K can also be examined from the two curves; K (ratio of N to the radius) for the fastest growing wave, remains essentially fixed at slightly greater than 0.08 rad/cm as the radius of curvature increases.

Figure 3 Real and imaginary frequency vs. N at two radius of curvatures: maximum radius of 332.74 cm and maximum radius of 622.30 cm
Figure 4 shows the slightly dispersive nature of the waves at longer wavelengths (values of $N$ lower than 13) for the wavetank geometry.

Solutions for the cross shore phase structure are shown in figure 5. As with the case of a straight beach the solutions show $u$ and $v$ in nonquadrature relationship in the area of steep off shore shear containing the extremum in the potential vorticity profile. The area of nonquadrature phase extends...
further offshore for longer waves (low values of N). Figures 6a and 6b show that for the instability waves with an N value of 24 the positive quadrature region begins at approximately 42 cm offshore, while for waves associated with an N value of 3 have a phase relationships at positive quadrature starting at about 68 cm offshore.

Figure 6(a,b). Phase relationships for short and long waves; a) u-v phase for N=24, b) u-v phase for N=3

This relationship between the radian wavelength and point in the cross-
shore where the phase relationship level off toward positive quadrature is plotted in figure 7 for N values of 3, 7, 10, 15, 20, 24 and 27. The frequency range of nonquadrature phase narrows toward the lower frequencies for positions offshore.

![Figure 7. Cross-shore position of phase asymptote to positive quadrature](image)

**Comparisons with Natural Sloshing Modes of the Tank**

Natural tangential and radial sloshing modes can occur over the same frequency range as the instability waves in a circular tank. Solutions are sought for the Laplace equation

\[ \nabla^2 \phi = 0. \]  \hspace{1cm} (25)

with a rigid boundary at \( r = r_0 \), a bottom at \( z = -h \), and no singularity at \( r = 0 \) (Paterson, 1989). As with other plane wave solutions the combined kinematic and dynamic free surface condition is
\[ \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = 0. \] (26)

Solution are sought of the form

\[ \phi = f(r)g(\theta) \cosh(k(z+h))e^{i\omega} \] (27)

For a flat bottom the solutions to the Laplace equation are Bessel functions.

A cross shore dependency on h cannot be solved analytically. A numerical solution for an arbitrary bottom profile was found and modeled by W. F. Elsasser (Elsasser, 1989). The cross-shore beach geometry is approximated with a polynomial and the wave period and wavelengths can be found for all combinations of longshore and radial modes (given a specified range of modes). Figures 8a. and 8b show the dispersion relationship for gravity wave modes (modes 0 through 5 for both radial and tangential components) and instability waves (for N values 2 through 27). Gravity waves with wavelengths of 0.4 radians will have a periods of about 0.4 seconds, whereas shear waves of the same length will have periods of around 2.4 seconds.
Figure 8(a,b). Period vs. radian wavelength for natural sloshing modes and shear wave modes; a) natural tank modes, b) shear wave solutions
Figures 9a and 9b compare the dispersive qualities of the shear wave solutions and the natural tank modes.

Figure 9(a,b). Radian celerity vs. radian wavelength for natural sloshing mode solutions and shear wave solutions; a) natural modes, b) shear waves.
The shear wave celerities fall between 0.165 and 0.205 radians. The shorter waves, including the fastest growing mode, are nondispersive with a celerity of approximately 0.17 cm per second, while natural gravity waves (modes 0-5 are shown here) are nondispersive with a celerity slightly greater than 1 rad per second.

The analytical solutions can be used as diagnostic features that distinguish shear waves from other waves that are generated in the tank. Features to look for are: 1) a significant rise in coherent energy at low frequencies in both u and v, within the range of frequencies predicted by the model, 2) non-quadrature u - v phase structure in the offshore shear region of the longshore current profile moving toward positive quadrature further offshore of this region, 3) dispersion relationships and celerities resembling those found in the shear wave solutions.
**Experiment**

Laboratory tests of shear wave dynamics were based in the small spiral wavemaker at the O. H. Hindsdale Wave Research Laboratory of Oregon State University. The experiments took place in three stages. Studies were first carried out to ensure that a steady longshore current could be repeatedly generated using the same incident wave field. Secondly measurements of the low frequency velocity components $u$ and $v$ were taken at cross-shore positions in order to observe their power spectra and relative phase. In the final stage time series were collected at longshore spacial lags in order to find the wavelengths and periods of the waves that are present in the low frequency range.

**Circular Wave Tank and Wavemaker**

The wavemaker is at the center of a circular tank of 4.57 meters (15 feet) diameter. It consists of a drum mounted on two orthogonal tracks. The track mounts move back and forth along the tracks, 90 degrees out of phase with one another. The sum of these motions results in the drum's circular orbit about the center of its resting position so that waves spiral off in a clockwise (in this case) direction. These waves break at an angle to the shoreline, forcing a longshore current. For all runs in this study the incident waves were one second monochromatic waves which provided a wave height of about 2.5 cm at the breakpoint.
A cement beach with a parabolic shape, \( h = \alpha x^2 \), was designed and installed by Dr. William McDougal and Dave Katzev of Oregon State University Ocean Engineering department (see figure 10).

Figure 10. Cross-shore geometry of wave tank bathymetry and sensor positions

The parabolic slope was chosen to minimize reflections from the shoreline. The depth at the wavemaker was 35.6 cm (14 in) and the beach extended approximately 1.8 meters (6 ft.) from the edge of the tank. The distance between the center and the wavemaker's maximum displacement is 45.7 cm. (18 in.)

The still water shoreline was approximately 40.6 cm (16.0 in.) to 43.2 cm (17.0 in.) from the edge of the tank so that the distance between the mean shoreline and the breakpoint was 36.8 cm (14.5 in.). At the breakpoint the depth was measured to be about 3.8 cm (1.5 in.). Plots of the calculated and actual measured profile are shown in figures 11a and 11b.
Figure 11 (a,b). Comparison of depth profiles; a) depth profile applied to the model, b) measured depth profile
The measured profile was applied to the vorticity model with results that were identical to the solutions obtained by using the calculated beach profile and the same longshore current profile.

**Mean Longshore Current.**

To measure the longshore current profile, and later shear wave characteristics, observations were obtained over sequential runs. This required the same longshore current profile to be repeatedly generated. It was also important that the currents remain at steady state, i.e. no mean increase due to tank spin up. Preliminary testing by Dr. William McDougal and Dave Ketzev showed that increasingly large currents and wave amplitudes developed during short runs of 232 seconds. They installed six iron 'fins' at the base of the wavemaker (fig. 10), each of 38.1 cm (15 in.) length. Tests following this emplacement showed that the current and wave amplitudes remained steady. Tests were carried out to see if the waves would induce a steady longshore current profile. Time series of 1024 seconds were repeatedly generated at seven cross-shore positions using one acoustic velocity sensor attached from a beam which extended across the wave tank in such a way that the sensor aligned with the diameter of the tank. The cross-shore positions are shown in figure 10, with the distance from the tank's center, as well as offshore of the still water line, listed in table 3:
Table 3. Cross-shore positions of the acoustic velocity sensor

<table>
<thead>
<tr>
<th>Position number</th>
<th>Distance from center (in.)</th>
<th>Distance from center (cm)</th>
<th>Distance from shoreline (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>56</td>
<td>142.24</td>
<td>44.45</td>
</tr>
<tr>
<td>2</td>
<td>54</td>
<td>137.16</td>
<td>49.53</td>
</tr>
<tr>
<td>3</td>
<td>52</td>
<td>132.08</td>
<td>54.61</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>127.00</td>
<td>59.69</td>
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<tr>
<td>5</td>
<td>46</td>
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</tr>
<tr>
<td>6</td>
<td>42</td>
<td>106.68</td>
<td>80.01</td>
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<tr>
<td>7</td>
<td>38</td>
<td>96.52</td>
<td>90.17</td>
</tr>
</tbody>
</table>

Table 4 shows the mean longshore velocity for six separate runs at position two. Each run lasted 256 seconds. The mean variance is listed for each position. Over the repeated runs the means varied within 6% of the average mean of the runs with a standard deviation of 3.4% of the average mean. The variance of each of the runs fell within 14.5% of the mean variance with a standard deviation of 1.1% of this mean.

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 1</td>
<td>17.66</td>
<td>9.87</td>
</tr>
<tr>
<td>Run 2</td>
<td>19.64</td>
<td>7.95</td>
</tr>
<tr>
<td>Run 3</td>
<td>18.60</td>
<td>8.61</td>
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<tr>
<td>Run 4</td>
<td>18.39</td>
<td>7.56</td>
</tr>
<tr>
<td>Run 5</td>
<td>18.46</td>
<td>8.44</td>
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<tr>
<td>Run 6</td>
<td>18.41</td>
<td>8.89</td>
</tr>
<tr>
<td>mean</td>
<td>18.53</td>
<td>8.56</td>
</tr>
<tr>
<td>S D</td>
<td>0.638</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Table 4. Repeatability: runs at pos. 2

To establish that the currents were stationary a time series of 1024 seconds
was taken at the first cross-shore position (table 2) using one second incident waves. The series was broken up into 250 second intervals after subtracting 24 seconds from the time series to allow for maximum spin up of the longshore current. The interval means are listed in table 5. Over the segments the mean longshore flow velocity remained constant within 2.4% of the average of the segments. The maximum variance fell within 6.7% of the mean of the variances.

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>seg1</td>
<td>22.35</td>
<td>0.2081</td>
</tr>
<tr>
<td>seg2</td>
<td>23.47</td>
<td>0.1840</td>
</tr>
<tr>
<td>seg3</td>
<td>23.57</td>
<td>0.1798</td>
</tr>
<tr>
<td>seg4</td>
<td>23.20</td>
<td>0.1849</td>
</tr>
<tr>
<td>mean</td>
<td>23.15</td>
<td>0.1892</td>
</tr>
<tr>
<td>S.D.</td>
<td>0.055</td>
<td>0.0126</td>
</tr>
</tbody>
</table>

Table 5. Stationarity: segments at pos. 1

The small variations in the averages and variances of the segments and repeated runs indicate that essentially the currents are, for the purposes of these tests, stationary and reproducible. A composite cross-shore profile of the longshore current was assembled with longshore velocity means taken over sequential runs at the seven cross-shore positions. The composite profiles are shown in are shown in figures 12a-c.

From the velocity magnitudes at the first few cross-shore positions (1-3) in Figure 12a, it appears that there is a substantial offshore shear in the mean current that tapers at positions further offshore. Figure 12b shows the current velocity normalized by the radius to distinguish the effects of solid body rotation.
on the current profile. It indicates that the angular velocity consists of a solid body rotation of about 0.06 rad/s added to a steeply sheared wave driven current in the surf zone.

Figures 12(a-c). Measured cross-shore profiles; a) cross-shore profile of longshore current, b) angular velocity, c) potential vorticity
The potential vorticity, shown in figure 12c, indicates that position 1 is on the offshore side of an extremum located in the area of high shear in the current profile and moves toward zero further offshore. Figure 13 provides a visual comparison between the longshore current applied to the model and the profile composed from the runs. Surf zone distances are with respect to the center of the tank.

![Figure 13. Longshore current profile from data and model](image)

**Cross-Shore Measurements of U and V**

The second set of experiments were directed at measuring the cross-shore characteristics of the perturbation velocities of the time series. The method of sequential runs was used to build a set of velocity time series over the seven cross-shore positions. At each position 20 minute time series were obtained; 176 seconds were subtracted from the beginning of each of the time series, to allow for the settling of transients, leaving a series of 1024 seconds (16384 data points...
sampled at 16Hz) for each location. The power spectral density was found for the radial and tangential (u and v) demeaned time series, as well as the coherence and relative phase between them. Spectral values were bandaveraged using 22 degrees of freedom. The power spectra for u and v are shown in figures 14a,b for position 1 through 4 and 15a,b for positions 5 through 7. The coherence and u-v phase spectra for the first four positions closest to the shoreline are shown in figure 16a,b and 17a,b for the last 3 positions off shore.
Figure 14(a,b). Log power spectra for u and v component at positions 1 - 4; a) tangential component (v), b) radial component (u)
Figure 15(a,b). Log power spectra for u and v at positions 5 - 7; a) tangential (v), b) radial (u)
Figure 16(a,b). Coherence and u-v phase spectra for positions 1-4, a) coherence, b) phase
A rise in energy below 0.4 HZ (about the lower half of the frequency range predicted by the model) is seen at most positions in figures 14 and 15 for both velocity components. Figures 16a and 17a show that over this low frequency range u and v are significantly coherent at the first 5 positions while the coherence disappears at positions 6 and 7. The u-v phase difference is shown in figures 16b.1-4 and 17b.1-3 for frequencies below 0.5 HZ.
Figure 16a shows that the range of frequencies over which \(u\) and \(v\) are significantly coherent falls below 0.4 HZ over the first four positions. Comparisons over the first 5 positions show a progressive decrease in the range of these frequencies offshore from position 1. At position 1 the upper limit for this range of frequencies is about 0.375 HZ whereas at positions 4 and 5, this region extends only up to about 2 HZ or less. In figure 16b over the same range of high \(u\)-\(v\) coherence, the relative phases follow a pattern similar to what would be expected to be seen for shear waves. The plots show a region for which phases are closely dispersed about a mean at the coherent frequencies, while above this range, where \(u\) and \(v\) are not significantly coherent, the phases show no particular structure. The narrowing of the coherent frequency range with offshore position is evident in both the coherent and phase spectra.

The gradual increase of \(u\)-\(v\) phase from nonquadrature to quadrature as distance increases from the shoreline is consistent with the model's solutions of the phase structure (figs. 6a and 6b) over the range of wavenumbers for which solutions exist (small wavenumbers correspond to low frequency). At position 1 the range of \(u\)-\(v\) relative phase falls between negative and positive quadrature, with the lower phases occurring at the lower end of the frequency range. This supports the observation made from the solutions for cross-shore phase relationships taken at a similar cross-shore location (figs. 6a-b, 7); the lowest phases occur at the lowest frequencies are negative at this position, becoming positive and nonquadrature at the higher frequencies with an average phase of less than 50 degrees. Position 2 shows a similar pattern, except the phases at all
frequencies have shifted upward from those found for position 1, with an average phase above 50 degrees. Some of the phases appear to be close to 90 degrees here. At position 3, the band of coherent frequencies has narrowed in both spectra with an upper limit at about .25 HZ. While the average phase is at 66 degrees, the variance of the phase over the whole coherent frequency range has increase. At position 4 most of the phases are between 100 and 120 degrees, with an average of 112 degrees. This position is close to 60 cm offshore which is indicated by the theory as a point at which the longer wave phases near positive quadrature. So the absence of any phases below quadrature is consistent with the model, although the values are higher than expected. Again, the range of u-v coherence has narrowed, with an upper limit slightly below 0.2 HZ, and there is a wider distribution of phases about the mean. At position 5 many of the phases are spread out between 100 and 200 degrees, with an average of 120 degrees. This trend extends to higher frequencies than those for positions 1 through 4 and the phase does not show a unique pattern over the range of uv coherence. At position 6 and 7 the frequency range of significant u-v coherence has disappeared, and the phases are scattered, showing no discernible pattern.

Although in theory the phase solutions extend across the surf zone to the wavemaker, the trends beyond position 4 or do not resemble the theoretical u-v phase solutions. Since shear wave amplitudes diminish past the region of sharp offshore shear, its possible that natural tank modes dominate the signals in this region and the shear waves are undetectable.
Longshore Measurements

The final set of tests were directed at measuring the longshore characteristics of the waves; wavelength, period and dispersion, and to compare the measured values with shear wave and gravity wave solutions. In the field the direction of wave propagation, as well as its wavenumber, are often calculated by examining the relationship of the phase difference between pairs of sensors and their distance of separation. These parameters are measured by an array of sensors spatially lagged in the longshore. A similar method was used in this study, except that instead of a longshore array of instruments, longshore velocity time series were collected simultaneously with two sensors at various lag spacings over sequential runs; for each run one sensor remained fixed in position while the location of the other was moved in the longshore.

Runs of 20 minutes were collected at offshore position 2 (see table 3) for spatial lags ranging between 10 cm and 41 cm. These spacings fall within one wavelength of the fastest growing wave as predicted by the model at this radius; at a radius of 137.16 cm this wavelength is 57.6 cm. (see Vorticity Model Results). The lag spacings are listed in table 6. The time series were collected at was 16HZ; 176 seconds were removed from beginning of each run. As in the tests for cross-shore spectral characteristics of the waves, the cross spectrum was band averaged, allowing 22 degrees of freedom for the spectral parameters. Slopes depicting the phase vs. spacial lag relationships were analyzed for frequencies between 0.12 HZ (8.45 sec) and 0.387 HZ (2.58 sec), the later being the high end of the frequency range over which significant coherent energy is found in the
<table>
<thead>
<tr>
<th>lag 1</th>
<th>10.67 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>lag 2</td>
<td>13.63 cm</td>
</tr>
<tr>
<td>lag 3</td>
<td>16.58 cm</td>
</tr>
<tr>
<td>lag 4</td>
<td>19.52 cm</td>
</tr>
<tr>
<td>lag 5</td>
<td>22.48 cm</td>
</tr>
<tr>
<td>lag 6</td>
<td>25.44 cm</td>
</tr>
<tr>
<td>lag 7</td>
<td>28.39 cm</td>
</tr>
<tr>
<td>lag 8</td>
<td>31.35 cm</td>
</tr>
<tr>
<td>lag 9</td>
<td>34.29 cm</td>
</tr>
<tr>
<td>lag 10</td>
<td>37.25 cm</td>
</tr>
<tr>
<td>lag 11</td>
<td>40.20 cm</td>
</tr>
</tbody>
</table>

Table 6. Longshore sensor lags

spectrum, and is also approximately the period of the fastest growing wave of the numerical solution. Figures 18(a-d) and 19(a-d) show power and coherence spectra at the first lag and at half the wavelength predicted for the fastest growing wave.
Figure 18(a-d). Log power spectra for lagged sensors at 10.67 cm and 28.39 cm; a-b) longshore sensor, c-d) fixed sensor
Although the power spectral density is similar for both sensors at both lags, significant coherence between them is stronger at shorter lags and diminishes for increasing spatial separation. This may be attributed frictional damping of the waves over distance. The phase-lag relationships for the 11 lags were used to find the wavenumber slopes for nineteen frequencies over the band where the u-v phase structure was between negative and positive quadrature in the
cross-shore phase spectra. The slopes were found by regressing v-v phase lag against the longshore spacial lag for periods between 2.58 and 6.64 seconds. The periods, slopes (beta) and intercepts are listed in table 7. The slopes,

<table>
<thead>
<tr>
<th>Period</th>
<th>Slope</th>
<th>Intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.58</td>
<td>6.47</td>
<td>23</td>
</tr>
<tr>
<td>2.73</td>
<td>4.43</td>
<td>64</td>
</tr>
<tr>
<td>2.82</td>
<td>2.66</td>
<td>82</td>
</tr>
<tr>
<td>2.90</td>
<td>3.64</td>
<td>64</td>
</tr>
<tr>
<td>3.00</td>
<td>5.16</td>
<td>45</td>
</tr>
<tr>
<td>3.10</td>
<td>3.11</td>
<td>60</td>
</tr>
<tr>
<td>3.32</td>
<td>0.24</td>
<td>170</td>
</tr>
<tr>
<td>3.44</td>
<td>3.15</td>
<td>27</td>
</tr>
<tr>
<td>3.58</td>
<td>5.23</td>
<td>5</td>
</tr>
<tr>
<td>3.88</td>
<td>3.35</td>
<td>65</td>
</tr>
<tr>
<td>4.04</td>
<td>4.64</td>
<td>27</td>
</tr>
<tr>
<td>4.22</td>
<td>4.10</td>
<td>25</td>
</tr>
<tr>
<td>4.43</td>
<td>4.89</td>
<td>31</td>
</tr>
<tr>
<td>5.49</td>
<td>3.06</td>
<td>33</td>
</tr>
<tr>
<td>5.81</td>
<td>2.08</td>
<td>28</td>
</tr>
<tr>
<td>6.20</td>
<td>2.50</td>
<td>35</td>
</tr>
<tr>
<td>6.64</td>
<td>1.65</td>
<td>53</td>
</tr>
</tbody>
</table>

Table 7. Phase vs. lag: slopes and intercepts

plotted as a function of period in figure 20, show a general decrease with wave period, particularly beyond wave periods of 4 seconds.

The non-zero intercepts may indicate that this method of measurement has inherent weaknesses in a system where waves with different dispersion relationships can exist at the same frequencies. This will be addressed in more detail in the following discussion. However if relatively low intercepts can reasonably be recalibrated with a zero phase intercept an estimate of the wave-
lengths and celerities of the dominate waves generated over the runs is possible. Twelve of the stronger trends were recalibrated with a zero intercept. Null hypothesis testing of the recalibrated slope and zero intercept, using a 0.1% level of significance (two tailed upper CI limits) and 8 degrees of freedom, showed that the recalibration was reasonable. Table 8 lists the results:

<table>
<thead>
<tr>
<th>Period (sec)</th>
<th>Freq. (cps)</th>
<th>Celerity (rps)</th>
<th>Slope (dg/cm)</th>
<th>L (cm)</th>
<th>Ltheta (radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.64</td>
<td>0.15</td>
<td>0.112</td>
<td>7.282</td>
<td>109.11</td>
<td>0.796</td>
</tr>
<tr>
<td>6.20</td>
<td>0.16</td>
<td>0.135</td>
<td>6.751</td>
<td>114.99</td>
<td>0.838</td>
</tr>
<tr>
<td>5.81</td>
<td>0.17</td>
<td>0.147</td>
<td>5.212</td>
<td>117.13</td>
<td>0.854</td>
</tr>
<tr>
<td>4.89</td>
<td>0.20</td>
<td>0.112</td>
<td>5.630</td>
<td>75.36</td>
<td>0.549</td>
</tr>
<tr>
<td>4.23</td>
<td>0.24</td>
<td>0.113</td>
<td>5.430</td>
<td>65.65</td>
<td>0.479</td>
</tr>
<tr>
<td>4.04</td>
<td>0.25</td>
<td>0.116</td>
<td>5.600</td>
<td>64.45</td>
<td>0.470</td>
</tr>
<tr>
<td>3.87</td>
<td>0.26</td>
<td>0.121</td>
<td>5.586</td>
<td>64.33</td>
<td>0.469</td>
</tr>
<tr>
<td>3.58</td>
<td>0.28</td>
<td>0.135</td>
<td>5.484</td>
<td>66.30</td>
<td>0.483</td>
</tr>
<tr>
<td>3.44</td>
<td>0.29</td>
<td>0.135</td>
<td>4.777</td>
<td>63.94</td>
<td>0.466</td>
</tr>
<tr>
<td>3.10</td>
<td>0.32</td>
<td>0.162</td>
<td>3.074</td>
<td>69.08</td>
<td>0.504</td>
</tr>
<tr>
<td>3.00</td>
<td>0.33</td>
<td>0.130</td>
<td>3.130</td>
<td>53.32</td>
<td>0.389</td>
</tr>
<tr>
<td>2.58</td>
<td>0.39</td>
<td>0.140</td>
<td>3.300</td>
<td>49.44</td>
<td>0.360</td>
</tr>
</tbody>
</table>

Table 8. Wave characteristics from recalibrated slopes
Figure 21(a,b) shows the dispersion and celerities of the natural modes, shear wave solutions and those from measurements, derived from wavelengths in table 8.

Figure 21(a,b). Comparisons between natural mode solutions (1), shear wave solutions (2) and measurements (3); a) wave period vs radian wavelength, b) radian celerity vs. wave period
The dispersion trend for the data, fig. 21a.3, has a slope of 7.7 (sec/cm), slightly greater than the slope of about 6 (sec/cm) of the shear wave model and substantially greater than the 1 (sec/cm) slope predicted for the natural mode dispersion. Similarly the range of the celerities in figure 21b.3 is on the order of the shear wave solutions, although the values are lower. The linear trend and the dispersiveness at small wavenumbers found in the model solutions are not apparent for the measured waves (figure 22).

![Figure 22 (a,b). Celerity comparisons between measurements and shear wave solutions; a) radian celerity vs. radian wavelength of the measurements, b) radian celerity vs. wavelength of the shear wave solutions](image)

Most of the celerities from wave tank measurements fall between 0.1 and 0.15 rads/sec, which is lower than the predicted range of 0.15 to 0.2 rads/sec.
However, natural tank modes are predicted to be an order of magnitude higher than these values; a result indicating that tank modes probably do not strongly affect the measurements.
Discussion

The power, coherence and phase spectra show that the measured waves fall within frequency range predicted by the model. However, they fall over the lower half of this range (below 0.4 Hz) and the highest energies fall at the lower end of the predicted band (below 0.2 Hz). The dispersion is similar to that of the model. Although the time series means and amplitudes remain essentially constant over the lengths of the runs the power spectra do show a rise in coherent energy, with phase structure similar to the model solutions. Since the purpose in generating these waves is to exam their effect on the longshore current in controlled experiments, it is important to know what forcings influence their growth and behavior. Factors which may effect these waves and their measurements will now be examined.

Figure 13 shows that while the magnitudes of the offshore shear and velocities of the model solutions are comparable to the measured points, the shear of the measured data is considerably further offshore, implying that the shape of the generated profile differs from that of the modeled profile. It isn't clear how or to what extent this effects the solutions. Although the depth limitations prevent the measurement of a complete current profile, from the measured shear and the known position of the breakpoint and shoreline, it may be possible to generate several reasonable profiles fitting these constraints. Using them to generate shear wave solutions and comparing them with the present solutions may provide an estimate of the extent and
manner in which the profile differences could effect the solutions.

The shallow water force balance used to derive the vorticity equation not account for the effects of friction. Noting the differences between the solutions to the basic force balance and the measured values can provide insight into the extent to which friction may be effecting the waves. H. J. de Vriend (1989) extended the Bowen and Holman analytical solution to include frictional forcing. He found that friction decreases the growth rates, and in the case where the friction is non-linear, its effects are wavenumber dependent, and will also reduce the wavenumbers of modes effected; the shorter waves are significantly more effected by friction than longer waves. The results, qualitatively, suggest a possible reason for the absence of the higher frequency modes in the measurements, as well the high spectral energies at lower frequencies (implying that the growth rates, and wavenumbers of the fastest growing waves have been reduced) Since his analysis showed that the real frequencies are also shifted, it may be possible that the short modes are operating at lower frequencies and contributing to the energy in that region, or that their growth rates are weakened such that the growth of the longer periods waves are more energetic.

The tests for stationarity allowed the assumption of a steady state current, which is appropriate for the linear instability analysis. However, the both the mean current and the variance remain constant over the runs, some of which were 20 minutes. If instabilities were generated, then the growth was not detectable (statistically), which implies that on average the
system remains linear. One step toward understanding the problem is to
determine how long it would take the system to go unstable in the absence
of friction. One way to determine this would be to observe the growth in the
stream function using the e-folding time from the model. Initial values of
the stream function can be derived by using the estimated velocities from the
power spectra and wavenumbers from the measurements. These values could
also be used as the initial inputs into a non-linear model which could then be
stepped in time to determine when the system becomes non-linear.

The consistently positive non-zero phase intercepts discussed in the
previous section indicate that there may be a small systematic error in the
longshore measurements. This may be a result of both shear waves tank
modes affecting the measurements differently at longer lag spacings than at
shorter lags spacings, due to frictional effects. Figure 19 shows that the
coherence decreases at longer lags. If friction effects shorter waves more
strongly than longer waves, the coherence signals at longer lags may be
dominated by the tank modes, which should reduce the phase value. This
could flatten the phase-lag slopes and induce a positive intercept.
Conclusions

A vorticity model was developed for an unstable circular longshore current. A cross-shore beach geometry and longshore current profile were used to compare the solutions from the circular model to that of a straight beach (Bowen and Holman, 1989). The comparisons show similar results for wavelengths, growth rates and celerities. The growth time is slightly longer for the circular beach at a small radius than the straight beach. The growth time becomes smaller, approaching that of the straight beach when the radius is increased while holding the surf zone geometry fixed. It was shown through scaling arguments that the longshore length and time scales are dependent upon the cross-shore width of the surf zone and longshore current velocity for any radius.

An experiment was carried out at the Hindsdale Wave Research Laboratory in order to determine if the instability could be generated and measured under controlled conditions, and to evaluate how well the measurements compared with the developed theory. A steady state circular jet was generated by wave forcing in a circular tank using beach geometry similar to that applied to the model. The depth was too shallow to place instruments shoreward of the break zone; measurement positions closest to the shoreline were in the area of offshore shear. Runs were carried out sequentially to test the longshore current using a single acoustic sensor. The currents were found to be repeatable and steady state.
Cross-shore positions were also used to develop relative phase, coherence and power spectra for \( u \) and \( v \). The power spectrum shows a rise in low frequency energy for both \( u \) and \( v \) which corresponds to a range of significant \( u-v \) coherence; all within the band over which instabilities are predicted to occur by the model.

The composite profiles for the cross-shore positions show a non-quadrature phase between \( u \) and \( v \) in the strong offshore shear of the current, while leveling off and moving toward positive quadrature for positions further off shore of this area. This is similar phase structure to that of the model.

Longshore measurements were also obtained sequentially by changing the spacial separation between two sensors in the longshore. One sensor was fixed while the upstream sensor was moved along shore. Eleven lags were used to derive phase vs. lag slopes over the frequency range of low coherent \( u-v \) energy. The positive offset of the phase intercept was assumed to be an effect induced by the presence of tank wave modes at the same frequencies. The slopes were recalibrated for a zero intercept so that the wavelengths of the prevalent wave signals could be estimated for each frequency. It was found that the wavelengths and celerities of the measured signals resembled the model solutions much more closely than the natural mode signals.

There is evidence that instabilities have been generated in this laboratory study. All the tests performed to determine if waves with shear wave characteristics were present yielded positive results, but with some
ambiguities. The measurements may offer insight as to what extent other factors, such as friction, are affecting the system and help to modify the model to account for their effects.
Bibliography


Appendices
Appendix 1

Outline of appendix I

Conversion of cartesian system to cylindrical coordinates and the shallow water approximation:

1) A vector (or point) \( \vec{P} \) can be expressed as \( \vec{P} = ix + jy + zk \).

   a) an expression for \( \vec{P} \) can also be found in cylindrical coordinates by transforming \( x, y, z \) and \( i, j, k \) and then substitute the transformation back into \( \vec{P} = ix + jy + zk \).

   b) the result is \( \vec{P} = re_r + zk \), where scalars are now \( r, \theta, z \) and unit vectors \( e_r, e_\theta, e_z \). (SEE SECTION 1)

2) Velocity = \( \frac{d\vec{P}}{dt} = \frac{d(re_r + ze_z)}{dt} \).

   a) the time derivatives are found by using the identities established in section 1 for \( e \) (unit vectors) in terms of \( i, j, k \).

   b) \( e_r = \cos \theta + j \sin \theta, \quad e_\theta = -i \sin \theta + j \cos \theta \) (SEE SECTION 2)

   c) the result is \( \vec{V} = u e_r + r \dot{e}_\theta + w e_z \). Since \( r \dot{\theta} = \nu \) this is \( \vec{V} = u e_r + v e_\theta + w e_z \).

3) \( \frac{d^2\vec{P}}{dt^2} = \frac{d\vec{V}}{dt} = \left( \frac{du}{dt} - \frac{\nu^2}{r} \right) e_r + \left( \frac{dv}{dt} + \frac{u \nu}{r} \right) e_\theta + \frac{dw}{dt} = 0 \)

   a) the centripetal acceleration terms arise from the fact that there is a non-zero time derivative of the unit vectors in the \( r, \theta \) plane.

   b) \( \frac{du}{dt} = \frac{\partial u}{\partial t} + \vec{V} \cdot \vec{V} u, \quad \frac{dv}{dt} = \frac{\partial v}{\partial t} + \vec{V} \cdot \vec{V} v, \quad \frac{dw}{dt} = \frac{\partial w}{\partial t} + \vec{V} \cdot \vec{V} w \)
c) \( \vec{V} \) must be found in cylindrical coordinates (SEE SECTION 3)

4) Form of the Navier Stokes equations in cylindrical coordinates:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} + \frac{uv}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial \theta} \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + v \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} + \frac{uv}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g
\end{align*}
\]

\( \vdots \) (SEE SECTION 4)

5) Integrated Shallow water equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \theta} - \frac{v^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \theta} + \frac{uv}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial \theta}
\end{align*}
\]

\textbf{Section 1}

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

A. \( \vdots \) The unit vectors in the rotated coordinate system are:

\[
\begin{align*}
e_1 &= \cos \theta i + \sin \theta j = e_r \\
e_2 &= -\sin \theta i + \cos \theta j = e_\theta \\
e_3 &= k = e_z
\end{align*}
\]

B. Multiplying the first vector through by \( \sin \theta \) and the second by \( \cos \theta \) yields (a) and (b):
(a) \( e_r \sin \theta = \cos \theta \sin \theta i + \sin^2 \theta j \)

(b) \( e_\theta \cos \theta = -\cos \theta \sin \theta i + \cos^2 \theta \)

\[ \therefore j = e_r \sin \theta + e_\theta \cos \theta \]

C) Multiplying (a) through by \( \cos \theta \) and (b) through by \( -\sin \theta \):

\[ i = e_r \cos \theta - e_\theta \sin \theta \]

\[ \therefore \vec{P} = xi + yj + zk = (r \cos \theta)(e_r \cos \theta - e_\theta \sin \theta) + (r \sin \theta)(e_\theta \sin \theta + e_\theta \cos \theta) + ze \]

D) Multiplying this out:

\[ \vec{P} = re_r \cos^2 \theta + re_\theta \sin^2 \theta - re_\theta \cos \theta \sin \theta + re_\theta \cos \theta \sin \theta + ze_k \]

\[ \therefore \vec{P} = re_r + ze_k \]

Section 2

A) \[ \frac{d\vec{P}}{dt} = e_r \frac{dr}{dt} + r \frac{de_r}{dt} + e_k \frac{dz}{dt} \] (since \( \frac{de_r}{dt} = 0 \))

B) Finding the time derivatives of the unit vectors:

\[ e_r = \cos \theta i + \sin \theta j \quad \therefore \frac{de_r}{dt} = \frac{d(\cos \theta i + \sin \theta j)}{dt} = (-\sin \theta i + \cos \theta j) \frac{d\theta}{dt} = \dot{\theta} e_\theta \]

\[ e_\theta = \cos \theta j - \sin \theta i \quad \therefore \frac{de_\theta}{dt} = \frac{d(-\sin \theta i + \cos \theta j)}{dt} = (-\cos \theta i - \sin \theta j) \frac{d\theta}{dt} = -\dot{\theta} e_r \]

\[ e_k = k \quad \therefore \frac{de_k}{dt} = \dot{\theta} e_\theta \]

\[ \frac{d\vec{P}}{dt} = \left( e_r \frac{\partial}{\partial t} + e_\theta r \frac{\partial \theta}{\partial t} \right) + e_k \frac{dz}{dt} = \ddot{U} \]

\[ \ddot{V} = u e_r + r \dot{\theta} e_\theta + w e_k \quad \therefore \ddot{V} = u(r, \theta, z)e_r + v(r, \theta, z)e_\theta + w(r, \theta, z)e_k \]
Section 3

A) \[
\frac{d^2 \bar{P}}{dt^2} = \frac{d\bar{V}}{dt} = \frac{d u e_r}{dt} + \frac{d v e_\theta}{dt} + \frac{d w e_k}{dt}
\]

\[
\therefore \frac{d\bar{V}}{dt} = \left( \frac{du}{dt} e_r + u \frac{d\theta}{dt} e_\theta \right) + \left( \frac{dv}{dt} e_\theta + v \frac{d\theta}{dt} (-e_r) \right) + \frac{dw}{dt} e_k
\]

B) Using the identities for time derivatives from the last section:

\[
\frac{d\bar{V}}{dt} = \left( \frac{du}{dt} e_r + u \frac{d\theta}{dt} e_\theta \right) + \left( \frac{dv}{dt} e_\theta + v \frac{d\theta}{dt} (-e_r) \right) + \frac{dw}{dt} e_k
\]

C) Collecting the scalars for each unit vector yields:

\[
\frac{d\bar{V}}{dt} = \left( \frac{du}{dt} - v \frac{d\theta}{dt} \right) e_r + \left( \frac{dv}{dt} + u \frac{d\theta}{dt} \right) e_\theta + \frac{dw}{dt} e_k, \text{ where } \frac{d\theta}{dt} = \frac{v}{r}.
\]

\[
\frac{d\bar{V}}{dt} = \left( \frac{du}{dt} - \frac{v^2}{r} \right) e_r + \left( \frac{dv}{dt} + \frac{uv}{r} \right) e_\theta + \frac{dw}{dt} e_k = \frac{dV}{dt}
\]

Section 4

A) If \( P \) represents the pressure, than the full momentum becomes:

\[
\rho \frac{d\bar{V}}{dt} = -\bar{V} \bar{P} + \rho \vec{g} \quad \text{or} \quad \left( \frac{du}{dt} - \frac{v^2}{r} \right) e_r + \left( \frac{dv}{dt} + \frac{uv}{r} \right) e_\theta + \frac{dw}{dt} e_k = -\frac{1}{\rho} \bar{V} \bar{P} + \vec{g}
\]

where

\[
\frac{du}{dt} = \frac{\partial u}{\partial t} + \bar{V} \cdot \nabla u, \quad \frac{dv}{dt} = \frac{\partial v}{\partial t} + \bar{V} \cdot \nabla v, \quad \frac{dw}{dt} = \frac{\partial w}{\partial t} + \bar{V} \cdot \nabla w.
\]

It is necessary to derive the differential operator, \( \bar{V} \), for the cylindrical system in order to express the momentum balance in each direction.

Expressing the differential in terms of cartesian coordinates:
\[ ds^2 = dx^2 + dy^2 + dz^2 \]

where \( ds \) is the differential arc length that is described by the cartesian differential components. The same arc length can be described by differential components of the cylindrical system. These can be found by first finding the expression for the cartesian system in terms of the cylindrical system:

\[
\begin{align*}
    dx &= \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial z} dz, \\
    dy &= \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial z} dz, \\
    dz &= \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial z} dz
\end{align*}
\]

\[ \therefore ds^2 = \left( \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta \right)^2 + \left( \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta \right)^2 + dz^2 \]

and therefore,

\[
\begin{align*}
    ds^2 &= \left[ \frac{\partial x}{\partial r} \right]^2 + \left( \frac{\partial y}{\partial r} \right)^2 dr^2 + \left[ \frac{\partial x}{\partial \theta} \right]^2 + \left( \frac{\partial y}{\partial \theta} \right)^2 d\theta^2 + 2drd\theta \left[ \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} \right] + dz^2.
\end{align*}
\]

Noting that \( x = r \cos \theta, \quad y = r \sin \theta \)

then \( \frac{\partial x}{\partial r} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial \theta} = 0 \) due to orthogonality,

\[
\begin{align*}
    \left( \frac{\partial x}{\partial r} \right)^2 + \left( \frac{\partial y}{\partial r} \right)^2 &= \cos^2 \theta + \sin^2 \theta = 1 \\
    \left( \frac{\partial x}{\partial \theta} \right)^2 + \left( \frac{\partial y}{\partial \theta} \right)^2 &= r^2 (\cos^2 \theta + \sin^2 \theta) = r^2
\end{align*}
\]

so that \( ds = \sqrt{(dr)^2 + (r d\theta)^2 + (dz)^2} \).

\[ \therefore \vec{V} = \left( e_r \frac{\partial}{\partial r}, e_\theta \frac{\partial}{r \partial \theta}, e_z \frac{\partial}{\partial z} \right) \]
Since the pressure is a scalar, the momentum in each direction can now be expressed along the $r, \theta, k$ respectively:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial \theta} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial \theta} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial \theta}$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + v \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g$$

Since $g = ge_k$.

Section 5

If it can be assumed that the net vertical accelerations are negligible compared to the horizontal accelerations (Kundu, 1990) then the third momentum equation becomes the hydrostatic approximation:

$$\frac{\partial p}{\partial z} = -\rho g.$$

When integrated in $z$ to find an expression for pressure ($p$), where $z$ is an arbitrary distance from the bottom:

$$p = -\rho g z + f(x,y,t).$$

Using the fact that the pressure vanishes at the surface, then

$$f(x,y,t) = \rho g (h + \eta)$$

and therefore the appropriate expression for pressure is

$$p = \rho g (h + \eta - z)$$

where $h$ is the distance from the bottom to the still water level and $\eta$ is the
displacement from the still water level, including the displace due to the basic state balance and surface fluctuations.

If an arbitrary depth, $D$, from mean water level is expressed as $D = h - z$ then horizontal pressure gradients are:

$$\frac{\partial p}{\partial r} = \rho g \frac{\partial (h + \eta - z)}{\partial r} = \rho g \frac{\partial \eta}{\partial r}$$

$$\frac{\partial p}{r \partial \theta} = \rho g \frac{\partial (h + \eta - z)}{r \partial \theta} = \rho g \frac{\partial \eta}{r \partial \theta}$$

since depth from the mean surface is not a function of $r$.

The pressure gradients are not $z$ dependent and so the horizontal accelerations are also independent of $z$ implying that the horizontal velocities are not $z$ dependent. Using the shallow water approximation the momentum equations become:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{r \partial \theta} - \frac{v^2}{r} = -g \frac{\partial \eta}{\partial r}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{r \partial \theta} + \frac{uv}{r} = -g \frac{\partial \eta}{r \partial \theta}$$
Appendix 2

The conservation of mass for an incompressible fluid:

\[ \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \]

Integrating from the bottom boundary to the surface yields:

\[ \int_{-h}^{h} \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \right) dz = \left[ \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + w \right]_{-h}^{h} \]

which can be written as:

\[ (\eta + h) \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) + w \eta - w_{-h} = 0 \]

Using the free surface kinematic boundary condition:

\[ \frac{D \eta}{Dt} = w_n \]

and the bottom boundary condition:

\[ w_{-h} = u \frac{\partial (-h)}{\partial r} \]

\[ \therefore -u \frac{\partial (-h)}{\partial r} = u \frac{\partial h}{\partial r} \]

the full integrated continuity equation for an incompressible fluid can be written as:

\[ \frac{1}{r} \frac{\partial (r(h + \eta)u)}{\partial r} + \frac{1}{r} \frac{\partial (h + \eta)v}{\partial \theta} + \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial t} = 0 \]

Since \( \eta \ll h \) the linearized continuity relationship is

\[ \frac{\partial \eta}{\partial t} + V \frac{\partial \eta}{\partial \theta} + \frac{1}{r} \frac{\partial hu}{\partial r} + \frac{\partial hv}{\partial r} = 0 \]
If the assumption can be made (rigid lid approximation)

\[ 0 < \frac{\partial \eta}{\partial t} + V \frac{\partial \eta}{r \partial \theta} - \frac{1}{r} \frac{\partial ru}{\partial r} + \frac{\partial hv}{\partial r} \]

a stream function can be used to define \( u \) and \( v \). (The exact identity

\[ \frac{V \cdot \partial \eta}{r \partial \theta} + \frac{\partial \eta}{\partial t} = 0 \]

is only possible for stable modes.) Scalings for these variables can be estimated using the momentum equations to find the conditions for which this approximation can be appropriate. Dodd and Thornton (1990) scaled the momentum equations using the width of the surf zone for the longshore and cross-shore length scales, and used the scaled equations to find the characteristic period. Using this approach to find an expression for \( \eta \), the components of the longshore momentum equation are:

\[
V = V_0 V', \\
u, v = u_0(u', v'), \\
\eta = \eta_0 \eta', \\
S = S_0 S', \\
r = R_0 r', \\
t = T_0 t',
\]

where \( u_0 << V_0 \) and \( S_0 - \Delta R \) (where \( \Delta R \) is the width of the surf zone).

The full longshore momentum equation:

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + V \frac{\partial v}{r \partial \theta} + \frac{\partial v}{r \partial \theta} + \frac{uv}{r} + \frac{uV}{r} = -g \frac{\partial \eta}{r \partial \theta}
\]

The terms can be expressed as nondimensional variables multiplied by their appropriate scalings:
Dividing by the velocity scalings and multiplying through by the surf zone length scale results in the nondimensional momentum balance:

\[
\frac{v'}{T_0 \frac{\partial v'}{\partial t}} + \frac{u_0^2 u' \frac{\partial v'}{\partial r}}{S_0 \frac{\partial r}{\partial r}} + \frac{u_0 V_0 u' \frac{\partial v'}{\partial r}}{S_0 \frac{\partial r}{\partial r}} + \frac{u_0^2 v' \frac{\partial v'}{\partial \theta}}{S_0 \theta \frac{\partial \theta}{\partial \theta}} + \frac{V_0 u_0 V' \frac{\partial v'}{\partial r}}{S_0 \frac{\partial r}{\partial r}} + \frac{u_0^2 u' v'}{R_0 \frac{\partial r}{\partial r}} + \frac{u_0 V_0 u' v'}{R_0 \frac{\partial r}{\partial r}} = g_0 \frac{\eta_0}{u_0 V_0 \frac{\partial \eta'}{r \frac{\partial \theta}{\partial \theta}}}
\]

Dropping terms of order \(u_0/V_0\) linearizes the balance:

\[
\frac{S_0}{T_0 V_0} \frac{\partial v'}{\partial t} + \frac{u_0}{V_0} \frac{\partial v'}{\partial r} + \frac{u'}{V_0} \frac{\partial v'}{\partial r} + \frac{V' \frac{\partial v'}{\partial \theta}}{R_0 \frac{\partial r}{\partial \theta}} + \frac{S_0 u' V'}{R_0 \frac{\partial r}{\partial \theta}} = -g_0 \frac{\eta_0}{u_0 V_0 \frac{\partial \eta'}{r \frac{\partial \theta}{\partial \theta}}}
\]

The highest order is one so that \(\eta_0 \sim \frac{u_0 V_0}{g}\).

Then the parameters of the continuity equation can be scaled as:

\[
\frac{\partial \eta}{\partial t} + \frac{u_0 V_0^2}{g S_0} \frac{1}{r \frac{\partial r}{\partial r}} \frac{\partial h u}{\partial \theta} + \frac{h_0 u}{S_0}
\]

Waves with longshore wavelengths that scale as the

The rigid lid is approximation (time and longshore gradients of \(\eta\) are negligible compared with the other terms in the linearized continuity equation) is reasonable for geometries where \(V_0^2/gh_0 \ll 1\) providing the scalings apply; i.e. wavelengths scale as the width of the surf zone.
Appendix 3

The linearized equations of motion in cylindrical coordinates:

\[
\frac{\partial u}{\partial t} + \frac{V}{r} \frac{\partial u}{\partial \theta} - 2v \frac{V}{r} = -g \frac{\partial \eta}{\partial r}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial V}{\partial r} + \frac{V}{r} \frac{\partial v}{\partial \theta} + u \frac{V}{r} = -g \frac{\partial \eta}{r \partial \theta}
\]

Multiplying through by \(r\) and cross differentiating:

\[
\frac{\partial}{\partial \theta} \left( \frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial \theta} \left( \frac{V}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\partial}{\partial \theta} \left( 2v \frac{V}{r} \right) = -g \frac{\partial}{\partial \theta} \left( \frac{\partial \eta}{\partial r} \right)
\]

\[
\frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial t} \right) + \frac{\partial}{\partial r} \left( ru \frac{\partial V}{\partial r} \right) + \frac{\partial}{\partial r} \left( V \frac{\partial v}{\partial \theta} \right) + \frac{\partial u V}{\partial r} = -g \frac{\partial}{\partial r} \frac{\partial \eta}{\partial \theta}
\]

After differentiating all the terms, adding the equations and dividing through by \(r\), the terms can be rearranged as:

\[
\frac{\partial}{\partial \theta} \left( \frac{\partial v}{\partial r} - \frac{\partial u}{\partial r \partial \theta} \right) + \left( \frac{V}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial v}{\partial \theta} \right) - \frac{V}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial u}{\partial r \partial \theta} \right) \right) + \left( \frac{1}{r} \frac{\partial v}{\partial t} + \frac{V}{r} \frac{\partial V}{\partial r} \right) + \left( \frac{u}{r \partial r} + \frac{\partial u}{\partial r} + \frac{u}{r \partial r} \frac{\partial}{\partial r} \frac{\partial V}{\partial r} \right) + \left( \frac{V}{r \partial r \partial \theta} + \frac{\partial v}{\partial \theta} + \frac{V}{r \partial \theta} + \frac{\partial V}{\partial r \partial \theta} + \frac{\partial u V}{\partial r} + \frac{u \partial V}{\partial r} \right) = 0
\]

Again collecting terms this expression becomes:

\[
\left( \frac{\partial}{\partial t} + \frac{V}{r} \frac{\partial}{\partial \theta} \right) \left( \frac{\partial v}{\partial r} - \frac{\partial u}{r \partial r \partial \theta} + \frac{u}{r} \right) + \left( \frac{u \partial V}{r \partial r \partial \theta} + \frac{\partial u}{\partial r} + \frac{u \partial V}{r \partial r} + \frac{u^2 \partial V}{r \partial r^2} \right) + \frac{\partial V}{\partial r} \frac{\partial v}{\partial r \partial \theta} + \frac{V}{r} \left( \frac{\partial v}{r \partial r \partial \theta} + \frac{\partial u}{\partial r} \right) + \frac{u \partial V}{r \partial r} = 0
\]

The first term is time and spatial derivative of the local relative vorticity. The continuity equation can be used to define the remaining terms, numbered 2,3, 4 and 5.
The continuity in cylindrical coordinates is

\[ \frac{1}{r} \frac{\partial r h u}{\partial r} + \frac{1}{r} \frac{\partial h v}{\partial \theta} = 0 \]

which is

\[ \frac{h u}{r} + u \frac{\partial h}{\partial r} + h \frac{\partial u}{\partial r} + \frac{h}{r} \frac{\partial v}{\partial \theta} = 0, \]

since \( \frac{\partial h}{\partial \theta} = 0. \)

By multiplying continuity through by \( r \) and dividing by \( h \),

\[ \frac{\partial v}{\partial \theta} = -u - ru \frac{1}{h} \frac{\partial h}{\partial r} - ru \frac{\partial u}{\partial r}. \]

Then 3rd term of the vorticity equation becomes,

\[ \frac{\partial V}{r \partial r} \left( -u - ru \frac{1}{h} \frac{\partial h}{\partial r} - ru \frac{\partial u}{\partial r} \right). \]

Dividing the continuity equation by \( h \) and rearranging yields,

\[ \frac{\partial u}{\partial r} + \frac{\partial v}{r \partial \theta} = - \frac{u}{h} \frac{\partial h}{\partial r}, \]

The 4th term of the vorticity equation becomes,

\[ - \frac{V}{r} \left( \frac{u}{r} + \frac{u}{h} \frac{\partial h}{\partial r} \right). \]

Adding the 3rd term to the 4th term reduces them to the expression on
the RHS of

\[ \left( \frac{u}{r} \frac{\partial V}{\partial r} + \frac{\partial u}{\partial r} \frac{\partial V}{\partial r} + u \frac{\partial^2 V}{\partial r^2} \right) + \frac{1}{r} \frac{\partial V}{\partial \theta} \left( -u - ru \frac{1}{h} \frac{\partial h}{\partial r} - r \frac{\partial u}{\partial r} \right) = u \frac{\partial^2 V}{\partial r^2} - u \frac{\partial V}{\partial r} \frac{1}{h} \frac{\partial h}{\partial r}. \]

Rearranging the sum expression of the 4th and 5th terms,

\[ - \frac{V}{r} \left( \frac{u}{r} + \frac{u}{h} \frac{\partial h}{\partial r} \right) + \frac{u}{r} \frac{\partial V}{\partial r} , \]
can be rewritten as

\[ u \left( \frac{\partial V}{r \partial r} - \frac{V}{r^2} \right) - \frac{u V}{h} \frac{\partial h}{\partial r}. \]

Using the identities

\[ u \left( \frac{\partial V}{r \partial r} - \frac{V}{r^2} \right) = u \frac{h}{h} \frac{\partial}{\partial r} \left( \frac{V}{r} \right) \]

and

\[ -u \frac{V}{r} \frac{1}{h} \frac{\partial h}{\partial r} = -uh \frac{1}{h^2} \frac{\partial h}{\partial r} \]

then the sum of the 4th and 5th terms can be expression as,

\[ uh \frac{\partial}{\partial r} \left( \frac{V}{h} \right). \]

The 2nd and 3rd term expression

\[ \frac{\partial^2 V}{\partial r^2} - u \frac{\partial V}{\partial r} \frac{1}{h} \frac{\partial h}{\partial r} \]

can be written as

\[ uh \frac{\partial}{\partial r} \left( \frac{\partial V}{h \partial r} \right), \]

so the vorticity equation can be expressed as

\[ \left( \frac{\partial}{\partial t} + \frac{V}{r} \frac{\partial}{\partial \theta} \right) \left( \frac{\partial rv}{\partial r} - \frac{\partial ru}{\partial \theta} \right) = -uh \frac{\partial}{\partial r} \left( \left( \frac{V + \partial V}{r} \right) / h \right). \]
Appendix 4

The shallow water radial and azimuthal momentum equations after subtracting the basic flow balance are:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + (V + v) \frac{\partial u}{r \partial \theta} - \frac{v^2}{r} - 2 \frac{vV}{r} = - g \frac{\partial \eta}{\partial r} \\
\frac{\partial v}{\partial t} + u \frac{\partial (V + v)}{\partial r} + \frac{(V + v) \partial v}{r \partial \theta} + \frac{uv}{r} + \frac{uV}{r} = - g \frac{\partial \eta}{r \partial \theta}
\]

Taking the dot product of the perturbations with the momentum equations yields the change in kinetic energy of the perturbation at every point:

\[
\frac{1}{2} \frac{\partial (u^2 + v^2)}{\partial t} + \frac{1}{2} \frac{V}{r} \frac{\partial (u^2 + v^2)}{\partial \theta} - u^2 \frac{\partial u}{\partial r} + uv \frac{\partial u}{r \partial \theta} + \frac{v^2}{r} + uv \frac{\partial v}{\partial r} +
\]

\[
u \left( \frac{\partial V}{\partial r} - \frac{V}{r} \right) + g \left( u \frac{\partial \eta}{\partial r} + v \frac{\partial \eta}{r \partial \theta} \right) = 0
\]

The dropping the second order terms and averaging in the longshore:

\[
\frac{1}{2} \frac{\partial (u^2 + v^2)}{\partial t} + \frac{1}{2} \frac{V}{r} \frac{\partial (u^2 + v^2)}{\partial \theta} + uv \left( \frac{\partial V}{\partial r} - \frac{V}{r} \right) + g \left( u \frac{\partial \eta}{\partial r} + v \frac{\partial \eta}{r \partial \theta} \right) = 0
\]

To find potential energy:

multiply the full continuity equation and KE equation by total depth:

\[
(\eta + h) \left( \frac{\partial \eta}{\partial t} + V \frac{\partial \eta}{r \partial \theta} + \eta \frac{\partial \eta}{r \partial \theta} + \nu \frac{\partial \eta}{r \partial \theta} + u \frac{\partial h}{r \partial \theta} + h \frac{\partial u}{r \partial \theta} + h \frac{\partial u}{r \partial \theta} + \eta \frac{\partial \eta}{r \partial \theta} + \eta \frac{\partial \eta}{r \partial \theta} + \eta \frac{\partial \eta}{r \partial \theta} \right) = 0
\]

Expanding the continuity equation:

\[
\frac{1}{2} \frac{\partial \eta^2}{\partial t} + h \frac{\partial \eta}{\partial t} + \nu \frac{\partial \eta}{r \partial \theta} + h \nu \frac{\partial \eta}{r \partial \theta} + \eta u \frac{\partial \eta}{r \partial \theta} + h u \frac{\partial \eta}{r \partial \theta} + \eta \frac{\partial \eta}{r \partial \theta} + \eta \frac{\partial \eta}{r \partial \theta} + h u \frac{\partial \eta}{r \partial \theta} + hu \frac{\partial \eta}{r \partial \theta} +
\]

\[
\frac{\eta h u \frac{\partial u}{r \partial r}}{r} + \frac{\eta h u \frac{\partial \eta}{r \partial \theta}}{r} + \frac{\eta h u \frac{\partial u}{r \partial \theta}}{r} + \frac{\eta h u \frac{\partial \eta}{r \partial \theta}}{r} + \frac{\eta h u \frac{\partial \eta}{r \partial \theta}}{r} + \frac{h^2 \frac{\partial u}{r \partial \theta}}{r} + \frac{h^2 \frac{\partial \eta}{r \partial \theta}}{r} + \frac{h^2 \frac{\partial \eta}{r \partial \theta}}{r} + \frac{h^2 \frac{\partial \eta}{r \partial \theta}}{r} = 0
\]
Averaging in the longshore and linearizing shows the first order relationship for the potential energy:

\[
g\left(\frac{1}{2}\frac{\partial \eta^2}{\partial t} + \frac{1}{r} \frac{\partial (\eta uh)}{\partial r} + \frac{\partial h^2 \eta}{r \partial \theta} + \frac{\partial^2 \eta}{r \partial \theta} + \frac{\eta \frac{\partial v}{r \partial r}}{} + \frac{\eta h \frac{\partial u}{r \partial r}}{}ight) = 0
\]

Multiplying the KE equation by the total depth \((\eta + h)\) and dropping the second order terms \((\eta \ll h)\);

\[
h\frac{1}{2} \frac{\partial \bar{u}^2}{\partial t} + huv \left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) + gh \frac{\partial \eta}{\partial r} + gh \frac{\partial \bar{u}}{r \partial \theta} = 0
\]

using the identity:

\[
hu \frac{\partial \eta}{\partial r} = ruh \frac{\partial \eta}{\partial r}
\]

Adding the longshore averaged PE and KE budgets:

\[
\frac{h}{2} \frac{\partial \bar{u}^2}{\partial t} + \frac{g}{2} \frac{\partial \eta^2}{\partial t} + huv \left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) + g\bar{V} \cdot h\bar{u} + gh\bar{V} \cdot \bar{u} = 0,
\]

Integrating in the cross-shore yields the volume energy budget

\[
\frac{\partial (KE + PE)}{\partial t} = -h \int_{r_1}^{r_2} \left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) \rho dr
\]

Since \(\frac{1}{r} \left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) = \frac{\partial (V/r)}{\partial r}\) the integral can be expressed as

\[
\frac{\partial (KE + PE)}{\partial t} = -h \int_{r_1}^{r_2} \left(\frac{\partial (V/r)}{\partial r} \right) \rho dr
\]