AN ABSTRACT OF THE THESIS OF

NANCY MANG-ZE HUANG for the MASTER OF SCIENCE
(Name) (Degree)

in MATHEMATICS presented on April 26, 1968
(Major) (Date)

Title: A CONSTRUCTION OF THE REAL NUMBERS USING NESTED CLOSED INTERVALS

Abstract approved: [redacted]
Dr. B. H. Arnold

It is well-know that a real number can be defined as an equivalence class of fundamental rational sequences. In fact, it is also possible to define a real number as an equivalence class of sequences of nested closed rational intervals. This paper is devoted to the latter case.

A Construction of the Real Numbers Using Nested Closed Intervals

by

Nancy Mang-ze Huang

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

June 1968

APPROVED:

[redacted]

Professor of Mathematics

in charge of major

[redacted]

Head of Department of Mathematics

[redacted]

Dean of Graduate School

Date thesis is presented April 26, 1968

Typed by Clover Redfern for Nancy Mang-ze Huang

ACKNOWLEDGMENT

Although it is difficult to find words to completely express my gratefulness to Dr. B.H. Arnold for his fatherly care and patient help during the past two years, I must at least try to let him know how much I appreciate his efforts to guide and counsel me in a manner which has helped me to fit into this American society. Thank you so much, Dr. Arnold.

Sincerely,

Nancy M. Huang

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A CONSTRUCTION OF THE REAL NUMBERS USING NESTED CLOSED INTERVALS

I. INTRODUCTION

Cohen and Ehrlich [1] have given a development of the real number system from Peano's postulates. The step from the rational numbers to the real numbers is made by equivalence classes of Cauchy sequences of rational numbers. In this paper, we assume the development of the rational numbers as given in [1] and we use their notation, then we define a real number as an equivalence class of sequences of nested closed rational intervals and prove that our development is equivalent to that of Cohen and Ehrlich.

In [1], an integer is defined as an equivalence class of ordered pairs of natural numbers, and a rational number is defined as an equivalence class of ordered pairs of integers. In our construction of the real numbers we again begin with an equivalence relation. In this paper, an equivalence relation will be defined in the set of all sequences of nested closed rational intervals. A real number will be defined to be an equivalence class under this equivalence relation. By suitable addition, multiplication, and order, the set R of all real numbers will be made into an ordered field which will be an extension of the ordered field Q of all rational numbers. The order in R will have no gaps in the sense of Definition 3.9 of [1].

Equivalently, every fundamental sequence of real numbers will have a limit in R, i. e. the converse of Theorem 3. 22 of [1] (which is false in Q by Theorem 3. 24 of [1]) will hold in R.

II. THE EQUIVALENCE RELATION IN F

Before we define our equivalence relation in the rational numbers, we need the concept of a sequence of nested closed rational intervals.

Definition 1: We define $\{[a_n, b_n]\}_{n=1}^{\infty}$ as a sequence of nested closed rational intervals if

(1) a_n and b_n are rational numbers.

(2) $a_n \leq b_n$ for all n in N.

(3) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n in N.

(4) $L(a_n - b_n) = 0$, where L stands for $\lim\limits_{n\to\infty}$

We shall write $\{[a_n, b_n]\}$ for $\{[a_n, b_n]\}_{n=1}^{\infty}$, and use F to denote the set of all sequences of nested closed rational intervals. Actually, from the definition of $\{[a_n, b_n]\}$, we can prove that (a_n) and (b_n) are fundamental rational sequences.

Theorem 2: If $\{[a_n, b_n]\} \in F$, then (a_n) and (b_n) are fundamental rational sequences.

Proof:

$\{[a_n, b_n]\} \in F$ means that $L(a_n - b_n) = 0$. Hence, for any given $\varepsilon > 0$ in Q, there exists $n(\varepsilon)$ in N such that $|a_n - b_n| < \varepsilon$ for all $n > n(\varepsilon)$ in N.

But $[a_m, b_m] \subset [a_n, b_n]$ for all $m > n$ in N,

i.e. $a_n \leq a_m \leq b_m \leq b_n$ for all $m > n$ in N.

Therefore

$$|a_m - a_n| \leq |b_n - a_n| < \varepsilon \quad \text{and}$$

$$|b_n - b_m| \leq |b_n - a_n| < \varepsilon$$

for all $n, m > n(\varepsilon)$ in N.

Hence, (a_n) and (b_n) are fundamental rational sequences.

We are now ready to define our equivalence relation in F.

Theorem 3: The relation $\sim$ in F defined by

$\{[a_n, b_n]\} \sim \{[c_n, d_n]\} \Longleftrightarrow L(a_n - c_n) = 0$ is an equivalence relation.

Proof:

i) $\{[a_n, b_n]\} \sim \{[a_n, b_n]\}$ since $L(a_n - a_n) = 0$.

ii) $\{[a_n, b_n]\} \sim \{[c_n, d_n]\}$ implies $\{[c_n, d_n]\} \sim \{[a_n, b_n]\}$ since $L(a_n - c_n) = 0$ implies $L(c_n - a_n) = 0$.

iii) $\{[a_n, b_n]\} \sim \{[c_n, d_n]\}$ and $\{[c_n, d_n]\} \sim \{[e_n, f_n]\}$ imply $\{[a_n, b_n]\} \sim \{[e_n, f_n]\}$ since $L(a_n - c_n) = 0$ and $L(c_n - e_n) = 0$ imply $L(a_n - e_n) = 0$.

Hence, $\sim$ is an equivalence relation in F.

Theorem 4: If $\{[a_n, b_n]\}, \{[c_n, d_n]\} \in F$, then $L(a_n - c_n) = 0$ if and only if $L(b_n - d_n) = 0$.

Proof:

If $L(a_n - c_n) = 0$, for any given $\varepsilon > 0$ in Q, there exist n_1, n_2, n_3 in N such that

$$|a_n - c_n| < \frac{\varepsilon}{3} \quad \text{for all} \quad n > n_1 \quad \text{in} \quad N,$$

$$|a_n - b_n| < \frac{\varepsilon}{3} \quad \text{for all} \quad n > n_2 \quad \text{in} \quad N,$$

$$|c_n - d_n| < \frac{\varepsilon}{3} \quad \text{for all} \quad n > n_3 \quad \text{in} \quad N.$$

Let $n(\varepsilon) = \max\{n_1, n_2, n_3\}$, then

$$|b_n - d_n| \leq |b_n - a_n| + |a_n - c_n| + |c_n - d_n|$$

$$< \varepsilon \quad \text{for all} \quad n > n(\varepsilon) \quad \text{in} \quad N.$$

Hence $L(b_n - d_n) = 0$. Likewise, we can prove the other implication.

Corollary 1: For $\{[a_n, b_n]\} \epsilon F$, $L(a_n) = a$ if and only if $L(b_n) = \alpha$.

Proof:

$L(a_n) = a$ i.e. $L(a_n - a) = 0$. But $L(b_n - a_n) = 0$. Then $L(b_n - a) = L(b_n - a_n + a_n - a) = L(b_n - a_n) + L(a_n - a) = 0$. Therefore $L(b_n) = a$.

Conclusion: If two sequences of nested closed rational intervals are equivalent, then their left endpoints and right endpoints form two fundamental rational sequences; furthermore, if either one of them converges, then both of them do so, and they have the same limit.

Definition 5: A real number is an equivalence class of the set F under the equivalence relation $\sim$. We use $C_{\{[a_n, b_n]\}}$ to denote the equivalence class containing the element $\{[a_n, b_n]\} \in F$. We denote the set of all real numbers by R and use $\xi, \eta, \ldots$ for real numbers.

III. ARITHMETIC OPERATIONS AND ORDER IN R

Before we define our addition and multiplication in the real numbers R, we shall show that the expressions we shall use for sum and product of $\xi = C_{\{[a_n, b_n\}]}$ and $\eta = C_{\{[c_n, d_n\}]}$ are independent of the choice of $\{[a_n, b_n\}] \in \xi$ and $\{[c_n, d_n\}] \in \eta$.

<u>Theorem 6</u>: If $\{[a_n, b_n]\} \sim \{[a'_n, b'_n]\}$ and $\{[c_n, d_n]\} \sim \{[c'_n, d'_n]\}$, then

(1) $\{[a_n+c_n, b_n+d_n]\} \sim \{[a'_n+c'_n, b'_n+d'_n|\}$.

(2) $\{[a_nc_n, b_nd_n]\} \sim \{[a'_nc'_n, b'_nd'_n]\}$.

<u>Proof:</u>

(1) By the definition of $\sim$, we know $L(a_n-a'_n) = 0$ and $L(c_n-c'_n) = 0$. But $L(a_n+c_n-(a'_n+c'_n)) = 0$ in Q. Hence

$\{[a_n+c_n, b_n+d_n]\} \sim \{[a'_nc'_n, b'_n+d'_n]\}$.

(2) Since (a_n), (a'_n), (c_n), and (c'_n) are fundamental rational sequences, by Theorem 3. 19 in [1], there exist a and c' in Q such that $|a_n| < a$ and $|c'_n| < c'$ for all n in N; by Theorem 3. 20 in [1], $(a_na'_n)$ and $(c_nc'_n)$ are fundamental rational sequence in Q. Since $L(a_n-a'_n) = 0$ and $L(c_n-c'_n) = 0$, there are, for each positive e in Q, $n_1(e)$ and $n_2(e)$ in N such that

$|a_n - a'_n| < \frac{e}{2c'}$ in Q for all $n \geq n_1(e)$ in N and

$|c_n - c'_n| < \frac{e}{2a}$ in Q for all $n \geq n_2(e)$ in N.

Hence

$$|a_n c_n - a'_n c'_n| \leq |a_n||c_n - c'_n| + |a_n - a'_n||c'_n| < a\frac{e}{2a} + \frac{e}{2c'}c' = e$$

for all $n \geq \max\{n_1(e), n_2(e)\}$ in N. Therefore $L(a_n c_n - a'_n c'_n) = 0$ in Q and $\{[a_n c_n, b_n d_n]\} \sim \{[a'_n c'_n, b'_n d'_n]\}$.

Theorem 7: There are binary operations f and g on R such that if $\{[a_n, b_n]\} \epsilon\, \xi$ and $\{[c_n, d_n]\} \epsilon\, \eta$, then

(1) $f(\xi, \eta) = C_{\{[a_n + c_n, b_n + d_n]\}}$.

(2) $g(\xi, \eta) = C_{\{[a_n c_n, b_n d_n]\}}$.

Proof:

The sets

$$f = \left\{((\xi, \eta), C_{\{[a_n + c_n, b_n + d_n]\}}) \mid \{[a_n, b_n]\} \epsilon\, \xi, \{[c_n, d_n]\} \epsilon\, \eta, \xi, \eta \,\epsilon\, R\right\}$$

and

$$g = \left\{((\xi, \eta), C_{\{[a_n c_n, b_n d_n]\}}) \mid \{[a_n, b_n]\} \epsilon\, \xi, \{[c_n, d_n]\} \epsilon\, \eta, \xi, \eta \,\epsilon\, R\right\}$$

are subsets of $(R \times R) \times R$.

If $(\xi, \eta) \in R \times R$, then $\xi = C_{\{[a_n, b_n]\}}$, $\eta = C_{\{[c_n, d_n]\}}$ for some $\{[a_n, b_n]\}$, $\{[c_n, d_n]\}$ in F. Since $\{[a_n+c_n, b_n+d_n]\} \in F$, the pair $((\xi, \eta), \zeta) \in f$ where $\zeta = C_{\{[a_n+c_n, b_n+d_n]\}}$. If $((\xi, \eta), \zeta') \in f$, then $\zeta' = C_{\{[a'_n+c'_n, b'_n+d'_n]\}}$ where $\{[a_n, b_n]\} \sim \{[a'_n, b'_n]\}$ and $\{[c_n, d_n]\} \sim \{[c'_n, d'_n]\}$.
By the previous Theorem 6, it follows that

$$\{[a_n+c_n, b_n+d_n]\} \sim \{[a'_n+c'_n, b'_n+d'_n]\} \quad \text{and} \quad \zeta = \zeta'.$$

Thus f is a mapping of $R \times R$ into R, and hence a binary operation on R.

If $(\xi, \eta) \in R \times R$, then $\{[a_n, b_n]\} \in \xi$ and $\{[c_n, d_n]\} \in \eta$ for some $\{[a_n, b_n]\}$, $\{[c_n, d_n]\}$ in F. Since $\{[a_n c_n, b_n d_n]\} \in F$, hence the pair $((\xi, \eta), \zeta) \in g$, where $\zeta = C_{\{[a_n c_n, b_n d_n]\}}$. If $((\xi, \eta), \zeta') \in g$, then $\zeta' = C_{\{[a'_n c'_n, b'_n d'_n]\}}$ where $\{[a'_n, b'_n]\} \in \xi$, $\{[c'_n, d'_n]\} \in \eta$. By the previous theorem again, it follows that $\{[a_n c_n, b_n d_n]\} \sim \{[a'_n c'_n, b'_n d'_n]\}$ and $\zeta = \zeta'$. Thus g is a mapping of $R \times R$ into R, and hence a binary operation on R.

We are now ready to define our addition and multiplication in the real numbers R.

Definition 8: We call the binary operations f and g of

Theorem 7 addition and multiplication in R, respectively, and write "$\xi +_R \eta$" and "$\xi \cdot_R \eta$" for $f(\xi, \eta)$ and $g(\xi, \eta)$. As usual, we shall feel free to omit the subscript "R".

Now that we have defined addition and multiplication in the real numbers R, we can prove that $(R, +, \cdot)$ is a field.

Theorem 9: $(R, +, \cdot)$ is a field.

Proof:

i) Associative Laws: If

$$\{[a_n, b_n]\} \epsilon\ \xi, \quad \{[c_n, d_n]\} \epsilon\ \eta, \quad \{[e_n, f_n]\} \epsilon\ \zeta,$$

then

$$(\xi+\eta) + \zeta = C_{\{[a_n+c_n, b_n+d_n]\}} + C_{\{[e_n, f_n]\}}$$

$$= C_{\{[(a_n+c_n)+e_n,(b_n+d_n)+f_n]\}} = C_{\{[a_n+(c_n+e_n),b_n+(d_n+f_n)]\}}$$

$$= C_{\{[a_n, b_n]\}} + C_{\{[c_n+e_n,d_n+f_n]\}} = \xi + (\eta+\zeta) ,$$

and

$$(\xi\cdot\eta) \cdot \zeta = C_{\{[a_nc_n, b_nd_n]\}} \cdot C_{\{[e_n, f_n]\}} = C_{\{[(a_nc_n)e_n, (b_nd_n)f_n]\}}$$

$$= C_{\{[a_n(c_ne_n), b_n(d_nf_n)]\}} = C_{\{[a_n, b_n]\}} \cdot C_{\{[c_ne_n, d_nf_n]\}}$$

$$= \xi \cdot (\eta\cdot\zeta).$$

ii) Commutative Laws: If

$$\{[a_n, b_n]\} \epsilon \xi, \quad \{[c_n, d_n]\} \epsilon \eta,$$

then

$$\xi + \eta = C_{\{[a_n, b_n]\}} + C_{\{[c_n, d_n]\}} = C_{\{[a_n+c_n, b_n+d_n]\}}$$

$$= C_{\{[c_n+a_n, d_n+b_n]\}} = C_{\{[c_n, d_n]\}} + C_{\{[a_n, b_n]\}} = \eta + \xi,$$

and

$$\xi \cdot \eta = C_{\{[a_n c_n, b_n d_n]\}} = C_{\{[c_n a_n, d_n b_n]\}} = \eta \cdot \xi .$$

iii) Distributive Laws: If

$$\{[a_n, b_n]\} \epsilon \xi, \quad \{[c_n, d_n]\} \epsilon \eta, \quad \{[e_n, f_n]\} \epsilon \zeta,$$

then

$$(\xi+\eta) \cdot \zeta = C_{\{[a_n+c_n, b_n+d_n]\}} \cdot C_{\{[e_n, f_n]\}} = C_{\{[(a_n+c_n)e_n, (b_n+d_n)f_n]\}}$$

$$= C_{\{[a_n e_n + c_n e_n, b_n f_n + d_n f_n]\}}$$

$$= C_{\{[a_n e_n, b_n f_n]\}} + C_{\{[c_n e_n, d_n f_n]\}} = \xi \cdot \zeta + \eta \cdot \zeta .$$

iv) Identity Elements: We note that $C_{\{[0_Q, 0_Q]\}}$ serves as the additive identity, 0_R; $C_{\{[1_Q, 1_Q]\}}$ as the multiplicative identity, 1_R; where 0_Q and 1_Q are the additive and multiplicative identity of Q, respectively.

v) Additive Inverse Elements: For any $C_{\{[a_n, b_n]\}} \neq 0_R$ in R, $\{[a_n, b_n]\} \in F$ implies $\{[-b_n, -a_n]\} \in F$ and $\{[a_n - b_n, b_n - a_n]\} \sim \{[0_Q, 0_Q]\}$.

$$C_{\{[a_n, b_n]\}} + C_{\{[-b_n, -a_n]\}} = C_{\{[a_n - b_n, b_n - a_n]\}} = C_{\{[0_Q, 0_Q]\}} = 0_R .$$

Hence $C_{\{[-b_n, -a_n]\}}$ serves as the additive inverse $-C_{\{[a_n, b_n]\}}$ of $C_{\{[a_n, b_n]\}}$.

vi) Multiplicative Inverse Elements: For any $C_{\{[a_n, b_n]\}} \neq 0_R$, we know $L(a_n) \neq 0$. Since (a_n) does not have limit zero in Q, there exists a positive element e_1 in Q such that for every n in N, $|a_k| > e_1$ for some $k > n$ in N. Since (a_n) is fundamental rational sequence, there exists n_1 in N such that $|a_n - a_m| < \frac{e_1}{2}$ for all $n, m \geq n_1$ in N. If, for $k_1 > n_1$, $|a_{k_1}| > e_1$, then

$$|a_n| = |a_{k_1} - (a_{k_1} - a_n)| \geq |a_{k_1}| - |a_{k_1} - a_n| > e_1 - \frac{e_1}{2} = \frac{e_1}{2}$$

for all $n \geq n_1$ in N. Hence $a_n \neq 0$ for all $n \geq n_1$ in N. By the same process we can prove $b_n \neq 0$ for all $n \geq n_2$ in N. Let $\overline{e} = \min(e_1, e_2)$, $\overline{n} = \max(n_1, n_2)$, then

$$|a_n| > \frac{\bar{e}}{2} \quad \text{and} \quad |b_n| > \frac{\bar{e}}{2} \quad \text{for all } n \geq \bar{n} \text{ in } N.$$

Take

$$a'_n = 1 \quad \text{for} \quad n < \bar{n}, \quad a'_n = \frac{1}{b_n} \quad \text{for} \quad n \geq \bar{n};$$

$$b'_n = 1 \quad \text{for} \quad n < \bar{n}, \quad b'_n = \frac{1}{a_n} \quad \text{for} \quad n \geq \bar{n}.$$

But for every $e > 0$ in Q, there exist $n(e)$ and $n'(e)$ in N such that

$$|b_n - b_m| < \frac{e\bar{e}^2}{4} \quad \text{for all} \quad n, m \geq n(e) \quad \text{in} \quad N$$

and

$$|a_n - a_m| < \frac{e\bar{e}^2}{4} \quad \text{for all} \quad n, m \geq n'(e) \quad \text{in} \quad N.$$

Then

$$|a'_n - a'_m| = \frac{|b_n - b_m|}{|b_n||b_m|} < \frac{\frac{e\bar{e}^2}{4}}{\frac{\bar{e}}{2}\frac{\bar{e}}{2}} = e \quad \text{for} \quad n, m \geq \max(n(e), \bar{n})$$

and

$$|b'_n - b'_m| = \frac{|a_n - a_m|}{|a_n||a_m|} < \frac{\frac{e\bar{e}^2}{4}}{\frac{\bar{e}}{2}\frac{\bar{e}}{2}} = e \quad \text{for} \quad n, m \geq \max(n'(e), \bar{n}).$$

Therefore (a'_n) and (b'_n) are fundamental rational sequences. Since $\{[a_n, b_n]\} \in F$, for any $e > 0$ in Q, there exists $\bar{n}(e)$ in N such that $|a_n - b_n| < \frac{e\bar{e}^2}{4}$ for all $n > \bar{n}(e)$ in N. Then

$$|a'_n - b'_n| = \frac{|a_n - b_n|}{|a_n||b_n|} < \frac{\frac{e\bar{e}^2}{4}}{\frac{\bar{e}}{2}\frac{\bar{e}}{2}} = e \quad \text{for all} \quad n > \max(\bar{n}(e), \bar{n}) \ .$$

Therefore $L(a'_n - b'_n) = 0$. On the other hand $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ implies $a'_n \leq a'_{n+1} \leq b'_{n+1} \leq b'_n$ for all n in N, hence $\{[a'_n, b'_n]\} \in F$. We find

$$|a_n a'_n - 1| = |\frac{a_n}{b_n} - 1| = \frac{|a_n - b_n|}{|b_n|} < \frac{\frac{e\bar{e}^2}{4}}{\frac{\bar{e}}{2}} = \frac{e\bar{e}}{2}$$

for all $n > \max(\bar{n}(e), \bar{n})$. This means $(a_n a'_n)$ has limit 1, i.e. $L(a_n a'_n) = 1$. Now we have proved: for any $C_{\{[a_n, b_n]\}} \neq 0_R$, there exist fundamental rational sequences (a'_n) and (b'_n) such that $\{[a'_n, b'_n]\} \in F$ and $L(a_n a'_n) = 1_Q$. Hence $C_{\{[a'_n, b'_n]\}}$ serves as the multiplicative inverse,

$$\frac{1}{C_{\{[a_n, b_n]\}}} ,$$

of $C_{\{[a_n, b_n]\}}$. This completes the proof that $(R, +, \cdot)$ is a field.

We shall now define an order relation in R. First, we define positive elements of R as the equivalence classes containing positive sequences in F. Before we do that, we prove the following theorem.

<u>Theorem 10:</u> If $\{[a_n, b_n]\} \in F$, then (a_n) is positive if and only if (b_n) is positive.

<u>Proof:</u>

(a_n) is positive, i.e. for some positive e in Q, there exists $n(e)$ in N such that $a_n \geq e$ for all $n \geq n(e)$ in N. Since $\{[a_n, b_n]\} \in F$, we know for any $e > 0$ in Q, there exists $n'(e)$ in N such that $-\frac{e}{2} < b_n - a_n < \frac{e}{2}$ for all $n \geq n'(e)$ in N. But then $b_n = b_n - a_n + a_n > \frac{-e}{2} + e = \frac{e}{2}$ for all $n \geq \max(n(e), n'(e))$ in N. Hence (b_n) is positive. By the same process, we can prove the other implication.

<u>Definition 11:</u> $\{[a_n, b_n]\}$ is positive in F if and only if (a_n) is positive. Notation: We let

$$R^+ = \left\{\xi \in R \mid \text{ for some } \{[a_n, b_n]\} \in \xi, \{[a_n, b_n]\} \text{ is positive in } F\right\}$$
$$= \left\{\xi \in R \mid \text{ for some } \{[a_n, b_n]\} \in \xi, (a_n) \text{ is positive.}\right\}$$

Next, we will prove that if $\{[a_n, b_n]\} \in \xi$ is positive in F, then all $\{[a'_n, b'_n]\} \in \xi$ are positive in F.

<u>Theorem 12:</u> If $\{[a_n, b_n]\} \sim \{[a'_n, b'_n]\}$ and $\{[a_n, b_n]\}$ is positive, then $\{[a'_n, b'_n]\}$ is positive.

Proof:

By definition, we know $\{[a_n, b_n]\}$ is positive if and only if (a_n) is positive. (a_n) is positive means for any $e > 0$ in Q, there exists $n(e)$ in N such that $a_n \geq e$ in Q for all $n \geq n(e)$ in N. On the other hand, $\{[a_n, b_n]\} \sim \{[a'_n, b'_n]\}$ if and only if $L(a_n - a'_n) = 0$. That means for any $e > 0$ in Q, there exists $n'(e)$ in N such that $-\frac{e}{2} < a_n - a'_n < \frac{e}{2}$ for all $n \geq n'(e)$ in N. Then $a'_n = a'_n - a_n + a_n > -\frac{e}{2} + e = \frac{e}{2} > 0$ for all $n \geq \max\{n(e), n'(e)\}$ in N. Hence (a'_n) is positive and $\{[a'_n, b'_n]\}$ is positive in F.

Now we can say

Corollary 1: $R^+ = \{\xi \in R \mid \{[a_n, b_n]\}$ is positive for all $\{[a_n, b_n]\} \in \xi\}$.

Theorem 13: R^+ is a set of positive elements for R.

Proof:

We shall show that

(1) $\xi + \eta \in R^+$ for all $\xi, \eta \in R^+$.

(2) $\xi \cdot \eta \in R^+$ for all $\xi, \eta \in R^+$.

(3) For $\xi \in R$, exaclty one of the following holds:

$$\xi \in R^+, \quad \xi = 0, \quad -\xi \in R^+$$

If $\xi, \eta \in R^+$, then $\xi = C_{\{[a_n, b_n]\}}$, $\eta = C_{\{[c_n, d_n]\}}$ where $\{[a_n, b_n]\}$ and $\{[c_n, d_n]\}$ are positive in F. Therefore (a_n) and (c_n) are positive sequences in Q. By the exercise 3.24 in [1], we know (a_n+c_n) and $(a_n c_n)$ are positive in Q. Hence $\xi + \eta = C_{\{[a_n+c_n, b_n+d_n]\}} \in R^+$ and $\xi \cdot \eta = C_{\{[a_n c_n, b_n d_n]\}} \in R^+$, so that (1) and (2) are fulfilled. If $\xi = C_{\{[a_n, b_n]\}}$, by the Corollary of Theorem 12, $\xi \in R^+$ if an only if $\{[a_n, b_n]\}$ is positive in F if and only if (a_n) is positive in Q. $\xi = 0_R = C_{\{[0_Q, 0_Q]\}}$ if and only if $L(a_n) = 0_Q$. $-\xi = C_{\{[-b_n, -a_n]\}} \in R^+$ if and only if $(-b_n)$ is positive if and only if $(-a_n)$ is positive. Hence, by Theorem 3.26 in [1], exactly one of $\xi \in R^+$, $\xi = 0_R$, $-\xi \in R^+$ must hold. Thus (3) is fulfilled, and R^+ is a set of positive elements for R.

By Theorem 2.19 and Definition 3.5 in [1], we have the following theorems.

<u>Theorem 14:</u> The set $T = \{(\xi, \eta) | \eta - \xi \in R^+\}$ is an order relation in R.

<u>Notation:</u> We write "$\xi <_R \eta$" ("$\eta >_R \xi$") if $(\xi, \eta) \in T$. Usually, we omit the subscript R.

<u>Theorem 15:</u> $(R, +, \cdot, <)$ is an ordered field.

So far we have defined addition, multiplication and an order relation in the real numbers R, and we have shown that $(R,+, \ ,<)$ is an ordered field. Now, we are going to prove the ordered field R of real numbers which we made is an extension of the ordered field Q of rational numbers.

<u>Theorem 16</u>: The mapping E of Q into R such that $E(a) = C_{\{[a,a]\}}$ is an isomorphism of Q into R preserving addition, multiplication and order.

<u>Proof:</u>

E is a one to one mapping of Q into R. For $C_{\{[a,a]\}} = C_{\{[b,b]\}}$ if and only if $\{[a,a]\} \sim \{[b,b]\}$ in F if and only if $L(a-b) = 0$ if and only if $a = b$.

If $a, b \in Q$, then

$$E(a+b) = C_{\{[a+b,\, a+b]\}} = C_{\{[a,a]\}} + C_{\{[b,b]\}} = E(a) + E(b),$$

and

$$E(ab) = C_{\{[ab,\, ab]\}} = C_{\{[a,b]\}} \cdot C_{\{[a,b]\}} = E(a) \quad E(b) .$$

Thus, E preserves addition and multiplication. Also, $a < b$ in Q if and only if $b - a > 0$ in Q, so that $(b-a)$ is a positive sequence in Q. On the other hand, $C_{\{[a,a]\}} < C_{\{[b,b]\}}$ in R if and only if $C_{\{[b,b]\}} - C_{\{[a,a]\}} > 0$ in R, so that $C_{\{[b-a,\, b-a]\}}$

is a positive element in R, i.e. $(b-a)$ is a positive sequence in Q. Thus $a < b$ in Q if and only if $C_{\{[a,a]\}} < C_{\{[b,b]\}}$ in R, and E preserves order.

We have just seen that the ordered field Q of rational numbers is isomorphic to a subfield of the real numbers R. As usual, we will identify Q with its image in R and use interchangeably the symbols a and $C_{\{[a,a]\}}$.

IV. COMPLETENESS OF R

In this paper, first we have defined real numbers as equivalence classes of sequences of nested closed rational intervals; second we have proved that the set R of all real numbers is an ordered field which is an extension of the ordered field Q of rational numbers. Now we are going to prove that R is complete, i.e. every Cauchy sequence in R is convergent. Before proving this completeness, we need some preliminary results.

<u>Theorem 17:</u> For every real number $\varepsilon > 0$ in R, there is a rational number e such that $0 < e < \varepsilon$ in R.

<u>Proof:</u>

Suppose $\varepsilon = C_{\{[a_n, b_n]\}}$ in R. Since $\varepsilon > 0$ in R, by definition, we know (a_n) is positive in Q. Then, for some positive e' in Q, there exists n(e') in N such that $a_n \geq e'$ in R for all $n \geq n(e')$ in N. Therefore, $a_n - \frac{e'}{2} > 0$ in R for all $n \geq n(e')$ in N, and $a_n - \frac{e'}{2} \leq b_n - \frac{e'}{2}$ for all n in N. Hence $C_{\{[a_n - \frac{e'}{2}, b_n - \frac{e'}{2}]\}}$ is a positive element in R. It follows that

$$\varepsilon = C_{\{[a_n, b_n]\}} > C_{\{[\frac{e'}{2}, \frac{e'}{2}]\}} = \frac{e'}{2} = e > 0 .$$

<u>Theorem 18:</u> (a_n) is a fundamental rational sequence in Q

if and only if (a_n) is a fundamental rational sequence in R.

Proof:

By the previous theorem, we know, for any real number $\varepsilon > 0$, there exists $e > 0$ in Q such that $0 < e < \varepsilon$. But for this $e > 0$, there exists $n(e)$ in N such that $|a_n - a_m| < e$ in Q for all $n, m \geq n(e)$ in N. It follows $|a_n - a_m| < \varepsilon$ for all $n, m \geq n(e)$ in N. Therefore (a_n) is fundamental in R. On the other hand, for any $e > 0$ in $Q \subset R$, there exists $n(e)$ in N such that $|a_n - a_m| < e$ in Q whenever $n, m \geq n(e)$ in N. Hence (a_n) is fundamental in Q.

With the aid of two theorems above, we can prove the next.

Theorem 19: If $\{[a_n, b_n]\} \epsilon \xi$ in R, then $L(a_n) = L(b_n) = \xi$.

Proof:

The proof is by contraposition. Suppose $L(a_n) \neq \xi$, and $\xi = C_{\{[a'_n, b'_n]\}}$ in R; we shall prove that $L(a_n - a'_n) \neq 0$. Since $\{[a_n, b_n]\}$ is a sequence of nested closed rational intervals, by Theorem 2, (a_n) and (b_n) are fundamental in Q. From Theorem 18, we know (a_n) and (b_n) are fundamental in R too. But since $L(a_n) \neq \xi$, then there exists $\varepsilon > 0$ in R such that $|a_n - \xi| > \varepsilon$ in R for all large n in N. In other words

$$|a_n - C_{\{[a'_n, b'_n]\}}| > \varepsilon$$

or

$$|C_{\{[a_n - b'_n, a_n - a'_n]\}}| > \varepsilon > 0$$

for all large n in N. From Theorem 17, we can find $e > 0$ in Q such that $\varepsilon > e > 0$ for large n in N. But by the definition of absolute value (see page 68 of [1]), it follows that

$$|C_{\{[a_n - b'_n, a_n - a'_n]\}}| = C_{\{[a_n - b'_n, a_n - a'_n]\}}$$

or

$$|C_{\{[a_n - b'_n, a_n - a'_n]\}}| = - C_{\{[a_n - b'_n, a_n - a'_n]\}} .$$

If

$$|C_{\{[a_n - b'_n, a_n - a'_n]\}}| = C_{\{[a_n - b'_n, a_n - a'_n]\}} > e > 0$$

for large n in N, then $(a_n - a'_n)$ is positive, therefore $L(a_n - a'_n) \neq 0$ and $\{[a_n, b_n]\} \notin \xi$.

If

$$|C_{\{[a_n - b'_n, a_n - a'_n]\}}| = - C_{\{[a_n - b'_n, a_n - a'_n]\}} = C_{\{[a'_n - a_n, b'_n - a_n]\}} > e > 0$$

for large n in N, then $(a'_n - a_n)$ is positive; therefore $L(a'_n - a_n) \neq 0$ and $\{[a_n, b_n]\} \notin \xi$.

We now get a conclusion: for any $\{[a_n, b_n]\} \epsilon F$ if $L(a_n) \neq \xi$, then $\{[a_n, b_n]\} \notin \xi$; i.e. for any $\{[a_n, b_n]\} \epsilon F$, if $\{[a_n, b_n]\} \epsilon \xi$ in R, then $L(a_n) = \xi$. But from the Corollary of Theorem 4, we know $L(a_n) = L(b_n) = \xi$. This completes the proof.

Now we can say for any $\{[a_n, b_n]\} \epsilon F$, we are able to find a real number ξ such that $\{[a_n, b_n]\} \epsilon \xi$ and $L(a_n) = L(b_n) = \xi$. Since $\{[a_n, b_n]\}$ can only belong to one ξ, therefore we have a stronger statement that for any $\{[a_n, b_n]\} \epsilon F$, there exists a unique real number ξ such that $\{[a_n, b_n]\} \epsilon \xi$ and $L(a_n) = L(b_n) = \xi$.

The following statements are consequences of Theorem 19.

Corollary 1: If $\xi \epsilon R$, for any $\varepsilon > 0$ in R there exists a in Q such that $|\xi - a| < \varepsilon$ in R.

Proof:

If $\xi = C_{\{[a_n, b_n]\}}$ in R, from Theorem 19, we know $L(a_n) = \xi$. Then, for any $\varepsilon > 0$ in R there exists $n(\varepsilon)$ in N such that $|\xi - a_n| < \varepsilon$ whenever $n \geq n(\varepsilon)$ in N. In particular, if $a = a_{n(\varepsilon)}$, then $|\xi - a| < \varepsilon$ is as required.

Corollary 2: If $\xi < \eta$ in R, there exists a in Q such that $\xi < a < \eta$ in R.

Proof:

Since R is an ordered field, by Theorem 3.16 of [1], then there exists $\zeta \epsilon R$ such that $\xi < \zeta < \eta$ in R. Let $\varepsilon = \min\{\zeta - \xi, \eta - \zeta\} > 0$, then for this $\zeta \epsilon R$ and $\varepsilon > 0$ in R there exists a in Q such that $|\zeta - a| < \varepsilon$ in R, i.e. $-\varepsilon < \zeta - a < \varepsilon$ in R. Therefore $\xi \leq \zeta - \varepsilon < a < \zeta + \varepsilon \leq \eta$.

Corollary 3: R is Archimedean.

Proof:

For $0 < \xi < \eta$ in R, let a and b be two rational numbers such that $0 < a < \xi \leq b < \xi + \eta$ in R. Since Q is Archimedean (see page 69 of [1]) and the embedding isomorphism preserves addition and order, there is an interger n in N such that $na \geq b$. But $n\xi > na \geq b \geq \eta$ in R, therefore R is Archimedean.

Conclusion: From Theorem 2, we only know that for every sequence $\{[a_n, b_n]\}$ of nested closed rational intervals, the left endpoints (a_n) and the right endpoints (b_n) form fundamental sequences. But now we know more. That is, they are convergent, and converge to the real number which the sequence $\{[a_n, b_n]\}$ represents. We also know every real number is the limit of at least one fundamental rational sequence. Finally, we know that for any real number, we can find a rational number as near the real number as desired; no matter how close together two real numbers are, there are rational numbers in between. We are now ready to prove the completeness of R.

Theorem 20: R is complete.

Proof:

Suppose (ξ_p) is Cauchy in R, where $\xi_p = C_{\{[a_n^p, b_n^p]\}}$, then, for any k in N, there exists n_k in N such that

$$\xi_{n_k} - \frac{1}{k} < \xi_p < \xi_{n_k} + \frac{1}{k}$$

is true for all $p \geq n_k$ in N. But, on the other hand,

$$\xi_{n_k} = C_{\{[a_n^{n_k}, b_n^{n_k}]\}},$$

$$L(a_n^{n_k}) = \xi_{n_k} = L(b_n^{n_k}),$$

and

$$a_n^{n_k} \leq \xi_{n_k} \leq b_n^{n_k}$$

for all n in N. Therefore, for any k in N, there exist n(k) and n'(k) in N such that

$$|\xi_{n_k} - a_n^{n_k}| < \frac{1}{k} \quad \text{for} \quad n \geq n(k) \quad \text{in} \quad N$$

and

$$|\xi_{n_k} - a_n^{n_k}| < \frac{1}{k} \quad \text{for} \quad n \geq n'(k) \quad \text{in} \quad N, \quad \text{respectively;}$$

i. e.

$$\xi_{n_k} - \frac{1}{k} < a_{n(k)}^{n_k} \leq a_n^{n_k} \leq \xi_{n_k} \qquad \text{for} \quad n \geq n(k) \quad \text{in} \quad N$$

and

$$\xi_{n_k} \leq b_n^{n_k} \leq b_{n'(k)}^{n_k} < \xi_{n_k} + \frac{1}{k} \quad \text{for} \quad n \geq n'(k) \quad \text{in} \quad N .$$

Hence, for all m in N, there exist

$$p_m = \max \{n_k | k \leq m\} \qquad \text{in} \quad N,$$

$$a_m = \max \{a_{n(k)}^{n_k} - \frac{1}{k} | k \leq m \} \quad \text{in} \quad Q,$$

$$b_m = \min \{b_{n'(k)}^{n_k} + \frac{1}{k} | k \leq m\} \quad \text{in} \quad Q,$$

such that

$$a_{m-1} \leq a_m < \xi_p < b_m \leq b_{m-1} \quad \text{for all} \quad m \quad \text{in} \quad N \quad \text{and}$$
$$p \geq p_m \quad \text{in} \quad N.$$

But then we have $\{[a_m, b_m]\} \subset \{[a_{m-1}, b_{m-1}]\}$ for all m in N,

and

$$|b_m - a_m| \leq |b^{n_m}_{n'(m)} + \frac{1}{m} - (a^{n_m}_{n(m)} - \frac{1}{m})|$$

$$= |\frac{2}{m} + b^{n_m}_{n'(m)} - \xi_{n_m} - (a^{n_m}_{n(m)} - \xi_{n_m})|$$

$$< \frac{2}{m} + \frac{1}{m} + \frac{1}{m} = \frac{4}{m}$$

tends to 0 as m tends to infinity. Hence $\{[a_m, b_m]\}$ is a nested closed rational sequence. For this $\{[a_m, b_m]\}$, we know there exists some ξ in R such that $L(a_m) = L(b_m) = \xi$. Then

$$a^{n_m}_{n(m)} - \frac{1}{m} \leq a_m \leq \xi \leq b_m \leq b^{n_m}_{n'(m)} + \frac{1}{m} \qquad \text{for all } m \text{ in } N;$$

$$a^{n_m}_{n(m)} - \frac{1}{m} \leq a_m < \xi_p < b_m \leq b^{n_m}_{n'(m)} + \frac{1}{m}$$

for all m in N and $p \geq n_m$ in N. Hence

$$|\xi_p - \xi| = |\xi_p - b^{n_m}_{n'(m)} + b^{n_m}_{n'(m)} - \xi_{n_m} + \xi_{n_m} - a^{n_m}_{n(m)} + a^{n_m}_{n(m)} - \xi|$$

$$< \frac{1}{m} + \frac{1}{m} + \frac{1}{m} + \frac{1}{m} = \frac{4}{m}.$$

Since R is Archimedean, $\frac{4}{m} < \varepsilon$ for some m in N. Therefore $|\xi_p - \xi| < \varepsilon$ for large p in N. Hence R is complete.

We have just proved that the ordered field R which we have constructed by using nested closed rational intervals is complete and

Archimedean. Although the way we build up the system of real numbers from rational numbers is different from that of Cohen and Ehrlich, our end results (i.e. the field of real numbers) are isomorphic by Theorem 5.3 of [1].

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