Supplement to

BUCKLING OF FLAT PLYWOOD PLATES IN COMPRESSION, SHEAR, OR COMBINED COMPRESSION AND SHEAR

Buckling of Plates of any Symmetrical Construction. Edges Simply Supported. Buckling of Plates With Two Edges Clamped.

~~Information Reviewed and Reaffirmed~~

~~March 1956~~

INFORMATION REVIEWED
AND REAFFIRMED
1962

No. 1316-B

FOREST PRODUCTS LABORATORY
MADISON 5, WISCONSIN

UNITED STATES DEPARTMENT OF AGRICULTURE
FOREST SERVICE
In Cooperation with the University of Wisconsin

BUCKLING OF PLATES OF ANY SYMMETRICAL CONSTRUCTION. EDGES SIMPLY SUPPORTED BUCKLING OF PLATES WITH TWO EDGES CLAMPED.[1]

By

H. W. MARCH, Former Head Mathematician

Forest Products Laboratory,[2] Forest Service
U. S. Department of Agriculture

This report presents results supplementary to those in Forest Products Laboratory Report No. 1316, "Buckling of Flat Plywood Plates in Compression, Shear, or Combined Compression and Shear." In that publication the results of exact and approximate methods of analysis were given for certain types of flat plywood plates with simply supported edges. The formulas for the critical compressive stress and the critical shearing stress, respectively, were written in the form

$$p_{cr} = k_c \; E_L \; h^2/a^2 \tag{1}$$

$$q_{cr} = k_s \; E_L \; h^2/a^2 \tag{2}$$

The notation used here and elsewhere in this Supplement is that of Report No. 1316. In the case of combined compression and shear, the shearing stress was taken to be f times the compressive stress. Thus, $q = fp$.

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[1] This is one of a series of progress reports prepared by the Forest Products Laboratory relating to the use of wood in aircraft issued in cooperation with the Army-Navy-Civil Committee on Aircraft Design Criteria. Original report published 1942.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

In the report here presented the values of the constants of equations (1) and (2) are given for further types of plywood plates, viz., for 9-ply plates and for plates with an infinite number of plies, both with edges simply supported.

Approximate values of the constants k_c and k_s for types of plywood other than those for which calculations have been made can be obtained from curves showing k_c and k_s for various types of loading as functions of the ratio $E_1/(E_1 + E_2)$. Curves for the cases of uniform compression alone and uniform shear alone are given in figures 35 and 36.[3] They were constructed from the values of the constants given in this Supplement and in Report No. 1316. For combined compression (or tension) and shear, the corresponding curves can be drawn for various values of f, the ratio of the shearing stress to the compressive (or tensile) stress. The use of such curves is limited to plywood of symmetrical construction. For example, points corresponding to 2-ply plates will be found not to lie on the curves.

Further, methods were developed for calculating the constants k_c and k_s of equations (1) and (2) for plates in which the edges parallel to the direction of the applied loads are clamped. Calculations have been made of the constants for a few types of plates for the purpose of comparison with the constants for plates with simply supported edges. Actual edge conditions are usually intermediate between the simply supported condition and the clamped or fixed condition. By comparing the values of the constants for both clamped and simply supported edges for the types of plates for which calculations have been made, it is believed that a practical estimate of the effect of edge restraint can be made.

1. NINE-PLY AND INFINITE-PLY RECTANGULAR PLATES WITH SIMPLY SUPPORTED EDGES

The necessary formulas are to be found in Report No. 1316. The results are presented in table 7 and in the curves of figures 27 to 34. The values of E_1 and E_2 for infinite-ply plates are most readily obtained by noting that for a symmetrical construction

$$E_1 + E_2 = \left(1 + \frac{E_T}{E_L}\right) E_L$$

and that E_1 and E_2 approach equality as the number of plies is increased.

[3] The figures and tables in this Supplement are numbered consecutively with those of Report 1316.

2. PLATES WITH SIMPLY SUPPORTED EDGES AND OF SYMMETRICAL CONSTRUCTION BUT OF TYPES DIFFERENT FROM THOSE FOR WHICH CONSTANTS ARE GIVEN

Inspection of the formulas of Report No. 1316 reveals that the constants k_c and k_s depend upon the constants E_1, E_2, and A. The variation of the critical stress from one type of plywood to another will be determined chiefly by the variation of the ratio $E_1/(E_1 + E_2)$. As far as the constant A is concerned, it plays a less important role in the calculations of critical stresses than do the constants E_1 and E_2. The effect of the variation of the ratio of A to E_L in case the plies are made of wood of different species is neglected in the procedure about to be described for the determination of the constants k_c and k_s for types of plywood other than those for which these constants are given in Report No. 1316 and in this Supplement. Smoothed curves are constructed for the constants k_c and k_s as functions of the ratio $E_1/(E_1 + E_2)$ for each type of loading, using the values of these constants for the types of plywood for which calculations have been made. It is assumed that the plywood is of symmetrical construction. These curves are given in figures 35 and 36 for the cases of uniform compresssion alone or uniform shear alone. For combined compression and shear, corresponding curves can be drawn for various values of f. The curves cannot be used for unsymmetrical constructions. A reason for this can be found in the fact that for a symmetrical construction of plies of the same species:

$$E_1 + E_2 = E_L + E_T$$

and for any symmetrical construction the sum will be between the largest and the smallest of the sums $E_L + E_T$ formed for each ply. For an unsymmetrical construction -- for example, a 2-ply plate -- this is not the case.

3. RECTANGULAR PLYWOOD PLATES UNDER UNIFORM COMPRESSION. LOADED EDGES SIMPLY SUPPORTED. REMAINING EDGES CLAMPED.

Reference is made to figure 4 and the notation of Report No. 1316.

Case 1. Finite plate. $\theta = 0°$ or $90°$.

An exact method of solution could be employed. This would consist of extending to plywood plates the analysis used for the isotropic plate.[4]

[4] Timoshenko, S. Theory of Elastic Stability. p. 344.

As this method would require the solution of a transcendental equation for each case considered, the energy method is used to obtain an approximate formula for the critical stress.

The form of the buckled surface is assumed to be given by the equation:

$$w = H \sin^2\frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{3}$$

where b is either the length of the plate or the length of a half-wave of the buckled surface, if the plate buckles into more than one-half wave. The strain energy of bending due to buckling over a half-wave length b is obtained by substituting (3) in the integral.[5]

$$\frac{h^3}{24\lambda}\int_0^a \int_0^b \left[E_1\left(\frac{\delta^2 w}{\delta x^2}\right)^2 + E_2\left(\frac{\delta^2 w}{\delta y^2}\right)^2 \right. \tag{4}$$

$$\left. + 2\, E_L\, \sigma_{TL} \frac{\delta^2 w}{\delta x^2}\frac{\delta^2 w}{\delta y^2} + 4\lambda\, \mu_{LT}\left(\frac{\delta^2 w}{\delta x \delta y}\right)^2\right] dydx$$

By equating this strain energy of bending to the work done by the load in producing the corresponding shortening of the plate, it is readily found that the critical compressive stress is given by the formula:

$$p = \frac{4\pi^2}{9\lambda}\left[E_1\frac{b^2}{a^2} + \frac{3}{16}E_2\frac{a^2}{b^2} + \frac{A}{2}\right]\frac{h^2}{a^2} \tag{5}$$

When this equation is written in the form (1), the constant k_c has the value:

$$k_c = \frac{4\pi^2}{9\lambda E_L}\left[E_1\frac{b^2}{a^2} + \frac{3}{16}E_2\frac{a^2}{b^2} + \frac{A}{2}\right] \tag{6}$$

The least critical stress is found for a half-wave length b such that:

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[5] See Forest Products Laboratory Rept. No. 1312, equation (3.21), p. 46.

$$\frac{b}{a} = \frac{1}{2}\left(\frac{3E_2}{E_1}\right)^{1/4} \qquad (7)$$

In this case:

$$k_c = \frac{2\pi^2}{9\lambda}\,\frac{1}{E_L}\left(\sqrt{3\,E_1E_2} + A\right) \qquad (8)$$

The transition from n to n + 1 half-waves will occur when the ratio of the length of a plate to its breadth is equal to

$$\left[\frac{n\,(n+1)}{2}\right]^{1/2}\left(\frac{3E_2}{E_1}\right)^{1/4}$$

Case 2. Infinite strip. $\theta = 45°$.

Proceeding exactly as in the corresponding case[6] for a plate with simply supported edges, except that the form of the buckled surface is taken to be:

$$w = H \sin^2\frac{\pi x}{a}\,\sin\frac{\pi}{b}(y - \gamma x) \qquad (9)$$

it is found by the energy method that the constant k_c of equation (1) is given by the formula:

$$k_c = \frac{\pi^2}{9\lambda E_L}\left[8\,G\,z^2 + 6G\gamma^2 + R - 3\,M\,\gamma\right] \qquad (10)$$

where z and γ satisfy the simultaneous equations:

$$z^4 = \frac{3}{16}\left[1 + \gamma^4 + \frac{R\gamma^2 - M\,(\gamma^3 + \gamma)}{G}\right] \qquad (11)$$

$$z^2 = \frac{M\,(3\gamma^2 + 1) - 4G\gamma^3 - 2\,R\gamma}{16\,G\gamma - 4M} \qquad (12)$$

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[6] Rept. No. 1316.

The constants that appear in these equations are defined in the Appendix of Report No. 1316.

4. INFINITELY LONG PLATES (LONG NARROW PLATES) UNDER UNIFORM SHEAR. EDGES CLAMPED.

Case 1. $\theta = 0°$ or $\theta = 90°$.

Recourse is had to the energy method because a solution corresponding to that used for this case in Report No. 1316 is not available. Using the form of the buckled surface given by (9) and the integral (4) for the strain energy of bending, it is found that the constant k_s of equation (2) is given by the formula:

$$k_s = \frac{\pi^2}{9\lambda\gamma E_L}\left[4E_1 z^2 + 3E_1\gamma^2 + A\right] \qquad (13)$$

where z and γ satisfy the simultaneous equations:

$$z^4 = \frac{3(E_1\gamma^4 + E_2 + 2A\gamma^2)}{16E_1} \qquad (14)$$

$$z^2 = \frac{3E_2 - 3E_1\gamma^4}{12E_1\gamma^2 - 4A} \qquad (15)$$

Case 2. $\theta = 45°$.

Proceeding exactly as in the corresponding case[7] for a plate with simply supported edges, but using equation (9) for the buckled surface, the formula for the constant k_s of equation (2) is found to be:

$$k_s = \frac{\pi^2}{18\lambda\gamma E_L}\left[8\,Gz^2 + 6G\gamma^2 + R - 3M\gamma\right] \qquad (16)$$

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[7] Rept. No. 1316.

where z and γ satisfy the simultaneous equations:

$$z^4 = \frac{3}{16}\left[1 + \gamma^4 + \frac{R\gamma^2 - M(\gamma^3 + \gamma)}{G}\right] \qquad (17)$$

$$z^2 = \frac{6G(1 - \gamma^4) + 3M(\gamma^3 - \gamma)}{24G\gamma^2 - 4R} \qquad (18)$$

5. BUCKLING OF INFINITELY LONG PLYWOOD PLATES UNDER COMBINED UNIFORM COMPRESSION AND SHEAR. EDGES CLAMPED.

Exactly the same procedure is followed as for plates with simply supported edges,[8] except that the form of the buckled surface is taken to be given by equation (9). As before, the shearing stress is taken to be f times the compressive stress, i.e.:

$$q = fp \qquad (19)$$

Case 1. $\theta = 0°$ or $\theta = 90°$.

Using the energy method, the constant k_c is found to be given by the formula:

$$k_c = \frac{2\pi^2}{9\lambda(1 + 2\gamma f)E_L}\left[4E_1 z^2 + 3E_1\gamma^2 + A\right] \qquad (20)$$

where z and γ are solutions of the simultaneous equations:

$$z^4 = \frac{3(E_1\gamma^4 + E_2 + 2A\gamma^2)}{16E_1} \qquad (21)$$

[8]Rept. No. 1316.

$$z^2 = \frac{3\,(E_2 f - E_1\gamma^3 - E_1 f\gamma^4 - A\gamma)}{12E_1\gamma + 12E_1 f\gamma^2 - 4Af} \tag{22}$$

After k_c has been found, k_s is calculated in accordance with (19) by the formula:

$$k_s = fk_c \tag{23}$$

Case 2. $\theta = 45°$.

The formula for k_c is found to be:

$$k_c = \frac{\pi^2}{9\lambda(1 + 2\gamma f)E_L}\left[8Gz^2 + 6G\gamma^2 + R - 3M\gamma\right] \tag{24}$$

where z and γ are solutions of the simultaneous equations:

$$z^4 = \frac{3}{16}\left[1 + \gamma^4 + \frac{R\gamma^2 - M(\gamma^3 + \gamma)}{G}\right] \tag{25}$$

$$z^2 = \frac{3M\,(3\gamma^2 + 1 + 2\gamma^3 f - 2\gamma f) - 6R\gamma - 12G\,(\gamma^3 + \gamma^4 f - f)}{48G\,(\gamma + \gamma^2 f) - 12M - 8Rf} \tag{26}$$

The values of the constants k_c and k_s as calculated by the formulas of this Supplement for infinitely long strips of plywood are given in table 8. They may be expected to be approximately correct for finite plates whose lengths are three or more times the half-wave length of the corresponding, similarly loaded, infinite strip. For shorter plates, an estimate of the values of k_c and k_s can be made with the aid of figure 12. (See Report No. 1316).

It should be clearly understood that this estimate is based on the arbitrary but somewhat plausible assumption that the effect of the ends of the plate is to increase the critical stress in approximately the same proportion for the cases under consideration as for the case for which the curve of figure 12 was constructed, namely, for a plate with simply supported edges under uniform shear.

The values of k_c and k_s are represented for several types of plywood by the curves of figures 37 to 42. An explanation of the construction and use of these figures is found in Report No. 1316.

The effect of restraint at the edges in increasing the values of the constants k_c and k_s can be seen by comparing the values of these constants in table 8 for plates with clamped edges with the corresponding values in tables 5 and 6 (see Report No. 1316) for plates with simply supported edges. The ratio of corresponding values of the constants is found to range from 1.60 to 2.09.

The remarks made in Report No. 1316 as to the approximations involved in the mathematical analysis are also applicable to this Supplement.

Explanation of Figures 27 to 34

The following figures show associated values of k_c and k_s, the constants in equations (1) and (2), for the buckling of long rectangular plates of plywood under combined uniform compression (or tension) and shear. The edges were taken to be simply supported. The constants for rotary-cut Douglas-fir plywood were used in the calculations of the coordinates of the points on these curves. Curves showing the ratio of the half-wave length b to the width a and the slope γ of the wrinkles as functions of k_s are plotted to the left in each figure.

Table 7.--Buckling of long plywood plates under combined compression or tension and shear. Edges simply supported.

[For compression $p_{cr} = k_c E_L \, h^2/a^2$. Negative k_c denotes tension.

For shear $q_{cr} = k_s E_L \, h^2/a^2$. For combined stress the shear stress is equal to f times the compressive (or tensile) stress.]

Loading	θ	f	k_c or k_s	9-ply plies of equal thickness	9-ply face plies one-half as thick as remaining plies	Infinite-ply plies of equal thickness
	Degrees					
Compression	0	.....	k_c	1.08	1.12	1.12
Shear	0	.....	k_s	2.06	1.87	1.84
Compression and shear	0	0.5	k_c	1.01	1.03	1.03
			k_s	.51	.52	.51
"	0	1.0	k_c	.88	.87	.86
			k_s	.88	.87	.86
"	0	1.5	k_c	.75	.73	.72
			k_s	1.12	1.09	1.08
"	0	4.0	k_c	.40	.38	.37
			k_s	1.62	1.51	1.49
Tension and shear	0	.5	k_c	-18.73	-15.19	-14.61
			k_s	-9.36	-7.60	-7.30
"	0	1.0	k_c	-5.14	-4.26	-4.12
			k_s	-5.14	-4.26	-4.12
"	0	1.5	k_c	-2.60	-2.21	-2.14
			k_s	-3.90	-3.31	-3.21
"	0	4.0	k_c	-.66	-.58	-.57
			k_s	-2.63	-2.33	-2.27

(Sheet 1 of 4)

Table 7.--Buckling of long plywood plates under combined compression or tension and shear. Edges simply supported. (Continued)

Loading	θ	f	k_c or k_s	9-ply plies of equal thickness	9-ply face plies one-half as thick as remaining plies	Infinite-ply plies of equal thickness
	Degrees					
Compression	90	.....	k_c	1.08	1.12	1.12
Shear	90	.....	k_s	1.52	1.80	1.84
Compression and shear	90	0.5	k_c	.97	1.02	1.03
			k_s	.48	.51	.51
"	90	1.0	k_c	.78	.86	.86
			k_s	.78	.86	.86
"	90	1.5	k_c	.64	.71	.72
			k_s	.96	1.07	1.08
"	90	4.0	k_c	.32	.37	.37
			k_s	1.27	1.46	1.49
Tension and shear	90	.5	k_c	-10.48	-14.02	-14.61
			k_s	-5.24	-7.01	-7.30
"	90	1.0	k_c	-3.06	-3.97	-4.12
			k_s	-3.06	-3.97	-4.12
"	90	1.5	k_c	-1.64	-2.07	-2.14
			k_s	-2.45	-3.10	-3.21
"	90	4.0	k_c	-.46	-.55	-.57
			k_s	-1.82	-2.21	-2.27
Compression	45	.....	k_c	1.66	1.76	1.76
Shear	45	.....	k_s	1.62	2.05	2.12
Shear	45	.....	k_s	-2.61	-2.19	-2.12

(Sheet 2 of 4)

Table 7.--Buckling of long plywood plates under combined compression or tension and shear. Edges simply supported. (Continued)

Loading	θ	f	k_c or k_s	9-ply plies of equal thickness	9-ply face plies one-half as thick as remaining plies	Infinite-ply plies of equal thickness
	Degrees					
Compression and shear	45	0.5	k_c	1.28	1.51	1.54
			k_s	.64	.76	.77
"	45	1.0	k_c	.96	1.18	1.22
			k_s	.96	1.18	1.22
"	45	1.5	k_c	.75	.94	.97
			k_s	1.13	1.41	1.46
"	45	4.0	k_c	.35	.44	.46
			k_s	1.41	1.78	1.84
"	45	-.5	k_c	1.72	1.58	1.54
			k_s	-.86	-.79	-.77
"	45	-1.0	k_c	1.44	1.25	1.22
			k_s	-1.44	-1.25	-1.22
"	45	-1.5	k_c	1.17	1.00	.97
			k_s	-1.76	-1.50	-1.46
"	45	-4.0	k_c	.56	.48	.46
			k_s	-2.26	-1.90	-1.84
Tension and shear	45	0.5	k_c	-12.50	-11.08	-10.83
			k_s	-6.25	-5.54	-5.42
"	45	1.0	k_c	-4.30	-3.70	-3.59
			k_s	-4.30	-3.70	-3.59
"	45	1.5	k_c	-2.46	-2.10	-2.03
			k_s	-3.70	-3.15	-3.05
"	45	4.0	k_c	-.75	-.63	-.61
			k_s	-3.00	-2.52	-2.44

(Sheet 3 of 4)

Table 7.--Buckling of long plywood plates under combined compression or tension and shear. Edges simply supported. (Continued)

Loading	θ	f	k_c or k_s	9-ply plies of equal thickness	9-ply face plies one-half as thick as remaining plies	Infinite-ply plies of equal thickness
	Degrees					
Tension and shear	45	-.5	k_c	-9.04	-10.59	-10.83
			k_s	4.52	5.29	5.42
"	45	-1.0	k_c	-2.84	-3.49	-3.59
			k_s	2.84	3.49	3.59
"	45	-1.5	k_c	-1.58	-1.97	-2.03
			k_s	2.37	2.96	3.05
"	45	-4.0	k_c	-.47	-.59	-.61
			k_s	1.87	2.36	2.44

(Sheet 4 of 4)

Table 8.--Buckling of long plywood plates under combined compression or tension and shear. Edges clamped.

[For compression $p_{cr} = k_c E_L \, h^2/a^2$. Negative k_c denotes tension. For shear $q_{cr} = k_s E_L \, h^2/a^2$.

For combined stress the shear stress is equal to f times the compressive (or tensile) stress.]

Loading	θ	f	k_c or k_s	3-ply 1:2:1	5-ply 1:2:2:2:1
	Degrees				
Compression	0	......	k_c	1.83	2.32
Shear	0	......	k_s	3.76	3.48
Compression and shear	0	0.5	k_c	1.73	2.10
			k_s	.86	1.05
"	0	1.0	k_c	1.52	1.72
			k_s	1.52	1.72
"	0	1.5	k_c	1.31	1.42
			k_s	1.96	2.12
"	0	4.0	k_c	.72	.72
			k_s	2.89	2.86
Tension and shear	0	.5	k_c	-37.92	-26.89
			k_s	-18.96	-13.44
"	0	1.0	k_c	-10.23	-7.60
			k_s	-10.23	-7.60
"	0	1.5	k_c	-5.08	-3.96
			k_s	-7.62	-5.94
"	0	4.0	k_c	-1.24	-1.06
			k_s	-4.94	-4.26
Compression	45	......	k_c	2.12	2.85
Shear	45	......	k_s	1.66	3.08
Shear	45	......	k_s	-5.17	-3.94
Compression and shear	45	0.5	k_c	1.41	2.34
			k_s	.71	1.17

Loading	θ	f	k_c or k_s	3-ply 1:2:1	5-ply 1:2:2:2:1
	Degrees				
Compression and shear	45	1.0	k_c	1.02	1.80
			k_s	1.02	1.80
"	45	1.5	k_c	.78	1.42
			k_s	1.18	2.14
"	45	4.0	k_c	.36	.67
			k_s	1.45	2.68
"	45	-.5	k_c	2.85	2.72
			k_s	-1.42	-1.36
"	45	-1.0	k_c	2.71	2.22
			k_s	-2.71	-2.22
"	45	-1.5	k_c	2.28	1.79
			k_s	-3.42	-2.68
"	45	-4.0	k_c	1.12	.86
			k_s	-4.47	-3.42
Compression	90	......	k_c	1.83	2.32
Shear	90	......	k_s	1.68	2.94
Compression and shear	90	0.5	k_c	1.46	2.02
			k_s	.73	1.01
"	90	1.0	k_c	1.06	1.60
			k_s	1.06	1.60
"	90	1.5	k_c	.82	1.28
			k_s	1.23	1.93
"	90	4.0	k_c	.37	.62
			k_s	1.49	2.49

Loading	θ	f	k_c or k_s	3-ply 1:2:1	5-ply 1:2:2:2:1
	Degrees				
Tension and shear	90	.5	k_c	-8.34	-19.52
			k_s	-4.17	-9.76
"	90	1.0	k_c	-2.72	-5.74
			k_s	-2.72	-5.74
"	90	1.5	k_c	-1.55	-3.09
			k_s	-2.32	-4.63
"	90	4.0	k_c	-.47	-.87
			k_s	-1.89	-3.49
"	45	0.5	k_c	-23.02	-18.90
			k_s	-11.51	-9.45
"	45	1.0	k_c	-8.23	-6.48
			k_s	-8.23	-6.48
"	45	1.5	k_c	-4.78	-3.71
			k_s	-7.17	-5.57
"	45	4.0	k_c	-1.48	-1.13
			k_s	-5.90	-4.52
"	45	-.5	k_c	-10.54	-15.90
			k_s	5.27	7.95
"	45	-1.0	k_c	-3.02	-5.22
			k_s	3.02	5.22
"	45	-1.5	k_c	-1.64	-2.95
			k_s	2.46	4.42
"	45	-4.0	k_c	-.48	-.89
			k_s	1.92	3.54