On Intermediate Models for Stratified Flow

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ABSTRACT

Intermediate models contain physics between that in the primitive equations and that in the quasigeostrophic equations. The specific objective here is to investigate the absolute and relative accuracy of several intermediate models for stratified flow by a comparison of numerical finite-difference solutions with those of the primitive equations (PE) and with those of the quasigeostrophic (QG) equations. The numerical experiments involve initial-value problems for the time-dependent development of an unstable baroclinic jet on an f-plane in a doubly periodic domain with flat bottom. Although the geometry is idealized, the problem is set up so that the dynamics should be similar to that of the baroclinic jet observed off the northern California coast in the Coastal Transition Zone (CTZ) field program. Three numerical experiments are conducted where the flow fields are characterized by local Rossby numbers that range from moderately small to O(1). The unstable jet develops finite amplitude meanders that grow in amplitude until they pinch off to form detached eddies on either side of the jet. The instability process is characterized by the transfer of potential to kinetic energy accompanied by a large increase in the barotropic component of the flow. Although the initial jet velocity profiles are symmetric about the jet centerline, as the Rossby number of the jet increases the meander growth and eddy detachment process becomes more asymmetrical about the jet axis. A meander on the positive vorticity side of the jet pinches off first to form a relatively large anticyclonic eddy followed in time by the detachment of a smaller cyclonic eddy on the negative vorticity side. The intermediate models that we consider are the balance equations (BE), the balance equations based on momentum equations (BEM), the iterated geostrophic models (IG2 and IG3), the linear balance equations (LBE), the linear BEM (L-BEM), and the geostrophic momentum approximation (GM). We also include a second-order quasigeostrophic approximation (QG2) and a primitive equation model with semi-implicit time differencing (PESI). The results of the numerical experiments for moderate Rossby number flow show that the QG, QG2, LBE, and GM models give large errors and produce flow fields that have substantial qualitative differences from the PE. The L-BEM model is somewhat better, while IG2 gives considerably smaller errors. The BE, BEM, IG3, and PESI models give highly accurate approximate solutions to PE, and that result holds also for those models applied to the O(1) Rossby number flow.

1. Introduction

We continue the study of intermediate models (McWilliams and Gent 1980) for possible application to continental shelf and slope flow fields. Initial work (Allen et al. 1990a,b; Barth et al. 1990) involved investigations of intermediate models for flows governed by the f-plane shallow-water equations. Subsequent study has focused on the use of intermediate models for continuously stratified fluids. In Allen (1991) a BEM model (balance equations based on momentum equations) that possesses global invariants of both energy and potential enstrophy is formulated. In the preceding companion paper (Allen 1993), the derivation of a new set of iterated geostrophic (IG) intermediate models is presented.

A basic objective of this paper is to assess the accuracy of the new BEM and IG models by a comparison of numerical solutions from these models with corresponding solutions from the primitive equations (PE) for dynamically relevant flow fields. An additional objective is to include in the numerical experiments other intermediate models, including the balance equations (BE) (Gent and McWilliams 1983) and the geostrophic momentum approximation (GM) (Hoskins 1975) along with the standard quasigeostrophic (QG) approximation (e.g., Pedlosky 1987), so that the relative accuracy of the different models can be measured.

In the present study, we examine a set of numerical experiments involving initial-value problems for the time-dependent development of an unstable baroclinic jet on an f-plane in a doubly periodic domain with flat bottom. Although the problem is idealized, it is motivated by measurements of a baroclinic jet observed off the California coast in the Coastal Transition Zone (CTZ) field program (Kosro et al. 1991). That jet, referred to as the CTZ jet, is a dominant feature of the CTZ flow field (Strub et al. 1991), and plays a major role in the CTZ quasigeostrophic (QG) model data.
assimilation studies of Walstad et al. (1991). As part of the CTZ program, the dynamics of the CTZ jet were further investigated with the QG approximation through linear stability analyses of observed jet profiles by Pierce et al. (1991) and through numerical model studies of nonlinear finite amplitude behavior by Allen et al. (1991). The present numerical experiments are similar in design to those pursued in Allen et al. (1991). The basic stratification and model vertical resolution were identical in Walstad et al. (1991), Pierce et al. (1991), and Allen et al. (1991), and were chosen to resolve the CTZ flow based on the properties of the observed density field in the CTZ region. The jet profiles used in Allen et al. (1991) and in Pierce et al. (1991) were similarly extracted from observed streamfunction fields constructed in Walstad et al. (1991). We utilize the same stratification and similar jet profiles in the numerical experiments here. Thus, although the problem is idealized, the dynamics should be similar to that found in the coastal transition zone offshore of the continental slope of northern California.

Initial-value problems in which the initial conditions involve a uniform jet plus a small perturbation are examined in a set of three numerical experiments. The jet is unstable and develops finite amplitude meanders that grow in time and that eventually pinch off to form detached eddies. Numerical finite-difference solutions are obtained for the PE, the intermediate, and the QG models using equivalent initial conditions. All models are formulated in the same variables and utilize identical finite-difference grids. The initial jet profiles vary in strength to give three experiments characterized by different magnitudes for the maximum and minimum values of vorticity attained in the field, that is, to give three experiments corresponding to flows with local Rossby numbers that range from moderately small to O(1).

The models that we include in the evaluations are listed in Table 1 and include the PE, the BE (Gent and McWilliams 1983), the balance equations based on momentum equations (BEM) (Allen 1991), the GM approximation (Hoskins 1975), the iterated geostrophic models (IG2, IG3) (Allen 1993), the linear balance equations (LBE) (Gent and McWilliams 1983), the linear BEM model (LBEM), the QG approximation, and a second-order quasigeostrophic approximation (QG2). In addition, we include an additional primitive equation model (PESI) solved numerically with a semi-implicit scheme (Kwizak and Robert 1971).

The outline of this paper is as follows. In section 2, we record the primitive equations. In section 3, the QG and intermediate models are defined. The numerical experiments are described in section 4. The results of the numerical experiments are presented in section 5, and a summary is given in section 6.

2. Primitive equations

We consider the motion of a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, adiabatic, primitive equations with a constant Coriolis parameter and weak biharmonic momentum diffusion. A brief summary of the governing equations is given in this section. The notation is identical to that in the preceding companion paper (Allen 1993), which includes a more detailed description. Dimensionless variables are utilized so that in Cartesian coordinates \((x, y, z)\) the PE are

\[
\nabla_{3D} \cdot \mathbf{u}_{3D} = 0, \tag{2.1a}
\]

\[
\epsilon \frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -\nabla \phi - \epsilon \nu \nabla^2 \mathbf{u}, \tag{2.1b}
\]

\[
0 = -\phi_z + \theta, \tag{2.1c}
\]

\[
\frac{D\theta}{Dt} + S_w = 0, \tag{2.1d}
\]

where

\[
x = (x, y, z), \tag{2.2a}
\]

\[
\mathbf{u}_{3D} = (u, v, w), \quad \mathbf{u} = (u, v, 0), \tag{2.2b,c}
\]

\[
\nabla_{3D} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \quad \nabla = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right), \tag{2.2d,e}
\]

\[
\frac{D}{Dt} = \left( \frac{\partial}{\partial t} + \mathbf{u}_{3D} \cdot \nabla_{3D} \right), \tag{2.2f}
\]

and \(\mathbf{k}\) is the unit vector in the vertical \(z\) direction. The full dimensionless density and pressure fields are

\[
\rho = \tilde{\rho}(z) - \theta, \tag{2.2g}
\]

\[
\rho = \tilde{\rho}(z) + \phi, \tag{2.2h}
\]

where

\[
\tilde{\rho}_z = -\tilde{\rho}, \quad S(z) = -\epsilon \tilde{\rho}_z. \tag{2.2i,j}
\]

We are primarily interested in the limit of small Rossby number,
\[ \epsilon \ll 1, \quad (2.3) \]
with, in general, \( S = O(1) \). It will also be assumed that
the Coriolis parameter \( f \) is a constant so that in the
dimensionless variables defined in Allen (1993),
\[ f = 1. \quad (2.4) \]
The horizontal velocity vector is represented as the
sum of rotational and divergent components, such that
\[ u = k \times \nabla \psi + \epsilon \nabla x, \quad (2.5a) \]
\[ = u_R + \epsilon u_D, \quad (2.5b) \]
where
\[ u_R = (u_R, v_R, 0) = (-\psi_x, \psi_x, 0), \quad (2.5c) \]
\[ u_D = (u_D, v_D, 0) = (\chi_x, \chi_y, 0), \quad (2.5d) \]
and
\[ \zeta = \psi_x - u_x = v_{Rx} - u_{Ry}, \quad \psi_y + v_{gy} = \epsilon \nabla^2 \chi, \quad (2.6) \]
\[ D = u_x + v_y = \epsilon (u_{Dx} + v_{Dy}) = \epsilon \nabla^2 \chi, \quad (2.7) \]
where \( \zeta \) is the vertical component of vorticity and \( D \)
is the horizontal divergence.
For the PE solutions obtained here, we utilize the
variables \( \psi \) and \( x \) and replace equation (2.1b) for
the horizontal velocity components \( u \) and \( v \) with equations
for the vertical component of vorticity (2.6) and
the horizontal divergence (2.7). As a result, the governing
equations are
\[ \nabla^2 \chi + w_z = 0, \quad (2.8a) \]
\[ \zeta_t + J(\psi, \zeta) + \nabla^2 \chi + \epsilon \nabla \cdot [w \nabla \psi_x + \zeta \nabla x] + \epsilon^2 J(w, \chi_x) = -\nu \nabla^3 \psi, \quad (2.8b) \]
\[ \epsilon \nabla^2 \chi_t + \nabla^2 \phi - \nabla^2 \psi - \epsilon J(\psi, \psi_y) \]
\[ + \epsilon^2 \nabla^2 \chi = -J(\psi, \nabla^2 \psi) - \epsilon J(\psi, \psi_y) \]
\[ + \epsilon^2 \nabla \cdot [w \nabla \chi_x + \frac{1}{2} \nabla (|\nabla x|^2)] = -\epsilon^2 \nu \nabla^4 \chi, \quad (2.8c) \]
\[ 0 = -\phi_t + \theta_t, \quad (2.8d) \]
\[ \theta_t + J(\psi, \theta) + S w + \epsilon \nabla \cdot (\theta \nabla x) + (w \theta)_z = 0, \quad (2.8e) \]
where the operator
\[ J(a, b) = a_x b_y - a_y b_x \quad (2.9) \]
is the Jacobian. For initial-value problems, the PE
require specification of \( \psi, \chi, \) and \( \theta \) at the initial time.

3. Intermediate models
The intermediate models considered in this study,
along with the quasigeostrophic approximation, are
described in this section with the exception that formu-
lations for the iterated geostrophic models (IG2 and
IG3) and the second-order quasigeostrophic approxi-
mation (QG2) are given in the preceding paper (Allen
1993). All of the models are listed in Table 1. Numerical
finite-difference solution procedures for the models are
discussed in appendix A. Since the hori-
zontally doubly periodic domain utilized for the
numerical solutions in sections 4 and 5 does not require
the application of sidewall boundary conditions, that aspect
of the model specification is not discussed here,
and the formulations presented in the following are
specific for \( f \)-plane doubly periodic domain problems.

a. Quasigeostrophic
The QG approximation may be readily derived from
the primitive equations written in the form (2.8) by
the retention of all \( O(1) \) terms. As a result, the govern-
ing equations are (2.8a,d) and
\[ \zeta_t + J(\psi, \zeta) + \nabla^2 \chi = -\nu \nabla^4 \chi, \quad (3.1a) \]
\[ \nabla^2 \psi = \nabla^2 \phi, \quad (3.1b) \]
\[ \theta_t + J(\psi, \theta) + S w = 0. \quad (3.1c) \]
The quasigeostrophic potential vorticity equation is
derived by concluding from (3.1b) that
\[ \psi = \phi, \quad (3.2) \]
and then by combining (3.1a) with the \( z \) derivative
of (3.1c), using (2.8a) to give
\[ \nabla^2 \phi + (S^{-1} \phi)_z \]
\[ = -J(\phi, \nabla^2 \phi) - [S^{-1} J(\phi, \phi)_z]_z - \nu \nabla^4 \phi. \quad (3.3) \]
The QG potential vorticity equation (3.3) governs the
evolution of the pressure field \( \phi (x, t) \). Initial-value
problems require the specification of \( \phi (x, 0) \) at the
initial time \( t = 0 \). After the solution for \( \phi (x, t) \)
is obtained from (3.3), \( \chi \) may be found from
\[ \nabla^2 \chi = -\nabla^2 \phi_t - J(\phi, \nabla^2 \phi) + \nu \nabla^4 \phi, \quad (3.4) \]
and subsequently \( w \) from (2.8a). Note that the high-
frequency gravity–inertial waves are filtered out of the
QG model by the approximation of the divergence
equation (2.8c) by (3.1b), which drops the time
derivative \( \epsilon^2 \nabla^2 \chi_t \).

b. Balance equations
The balance equations are derived from (2.8) (Gent
and McWilliams 1983) by truncation such that all \( O(1) \)
and \( O(\epsilon) \) terms are retained and all other terms
neglected. The governing equations are (2.8a,d) and
\[ \zeta_t + J(\psi, \zeta) + \nabla^2 \chi + \epsilon \nabla \cdot [w \nabla \psi_z + \zeta \nabla x] = -\nu \nabla^4 \chi, \quad (3.5a) \]
\[ \nabla^2 \psi = \nabla^2 \phi - \epsilon J(\psi, \psi_y), \quad (3.5b) \]
\[ \theta_t + J(\psi, \theta) + S w + \epsilon \nabla \cdot (\theta \nabla x) + (w \theta)_z = 0. \quad (3.5c) \]
Note that this procedure results in the retention of all terms in the thermodynamic equation (2.8e). It also results in the neglect of the time derivative $\epsilon^2 \nabla^2 \chi$, in (2.8c), which filters out the gravity-inertial waves.

In the sense that this derivation of BE involves the retention of all O(1) and O($\epsilon$) terms in (2.8), whereas QG involves retention of all O(1) terms, the BE appear to form a systematic extension of QG to include higher-order effects in Rossby number $\epsilon$. The approximate momentum equations that correspond to the BE vorticity and divergence equations (3.5a,b) do not follow directly from approximations to the PE momentum equations, however, but involve implicitly defined force potential correction terms, and are referred to as equivalent momentum equations (Gent and McWilliams 1983). For inviscid flow ($\nu = 0$), the balance equations conserve volume integrals of energy density $E_R$ in the $J_0$-plane periodic domain problems considered here (Lorenz 1960), where

$$ E_R = \epsilon K_R + (z - \bar{z}_0)\rho, \quad K_R = \frac{1}{2} u_R^2, \quad (3.6a,b) $$

with $\bar{z}_0$ a constant reference level, but do not have an analog of potential vorticity conservation on fluid particles and thus no global invariant of potential enstrophy. Although the derivation procedure for BE by truncation appears systematic, it is not clear if the BE fit in a sequence of approximations to the PE for $\epsilon \ll 1$ that may be utilized, at least in a formal asymptotic sense, to obtain solutions of increasing accuracy in the limit $\epsilon \to 0$. The other intermediate models in this section suffer from the same criticism that, however, does not apply to the IG models formulated in Allen (1993).

For the numerical solution of BE, we form an equation for a linear approximation to the potential vorticity by eliminating $w_z = -\nabla^2 \chi$ between (3.5a) and (3.5c). With the resulting equation used in place of (3.5c) and with the definition,

$$ \phi = \psi + \epsilon \phi, \quad (3.7) $$

the complete set of governing equations for BE may be written in the form,

$$ [\nabla^2 \psi + (S^{-1} \psi_z) z]_t = -J(\psi, \zeta) - \epsilon \nabla \cdot [w \nabla \psi_z + \zeta \nabla \chi] $$

$$ - \nu \nabla^4 \psi - \{ S^{-1}[\epsilon \phi_z + J(\psi, \phi_z) $$

$$ + \epsilon \nabla \cdot (\phi_z \nabla \chi) + (w \phi_z)_z] z \}, \quad (3.8a) $$

$$ \nabla^2 \phi_z = 2 J_z(\psi, \psi_z), \quad (3.8b) $$

$$ \nabla^2 \chi = -\nabla^2 \psi_z - J(\psi, \zeta) $$

$$ - \epsilon \nabla \cdot [w \nabla \psi_z + \zeta \nabla \chi] - \nu \nabla^2 \psi, \quad (3.8c) $$

$$ \phi_z = \psi_z + \epsilon \phi_z, \quad (3.8d) $$

$$ w_z = -\nabla^2 \chi. \quad (3.8e) $$

For initial-value problems, $\psi(x, 0)$ is specified at the initial time $t = 0$. Note that for the doubly periodic domain problems considered here, only the determination of $\phi_z$, and not $\phi$, is required. Consequently, the divergence equation (3.5b) may be replaced by its $z$ derivative (3.8b).

c. Balance equations based on momentum equations

It was shown in Allen (1991) that for $\nu \ll 1$ a model very close to the balance equations can be formulated based on approximate momentum equations such that (for $\nu = 0$) an analog of potential vorticity is conserved on fluid particles and volume integrals of an approximate energy density are conserved. This approximate set of equations is referred to as the BEM model.

The governing equations for BEM may be conveniently expressed in a form similar to BE involving a vorticity and divergence equation. In this formulation, the BE and BEM models involve identical equations for continuity (2.8a), vorticity (3.5a), and density (3.5c), with differences represented only by the presence of additional higher-order terms in the divergence equation for BEM. Thus, for the numerical solution of BEM, the complete set of governing equations is (3.8a,c,d,e), the same as for BE but with (3.8b) replaced by

$$ \nabla^2 \phi_z = 2 J_z(\psi, \psi_z) $$

$$ - \epsilon [\nabla^2 K_{1Bz} - J_z(\zeta, \chi) - J_z(w, \psi_z)], \quad (3.9a) $$

where

$$ K_{1Bz} = J(\psi, \chi). \quad (3.9b) $$

As in BE, only the $z$ derivative of the divergence equation is needed. For initial-value problems, $\psi(x, 0)$ is specified, along with the added requirement that $\chi(x, 0) = 0$ (Allen 1991).

In the inviscid limit ($\nu = 0$), the BE equations imply the conservation of balance equation potential vorticity $Q_{BE}$ on fluid particles moving with velocity $u_{3D}$ (2.2b),

$$ \frac{D}{Dt} Q_{BE} = 0, \quad (3.10) $$

where

$$ Q_{BE} = (\epsilon \xi_{3D} + k) \cdot \nabla 3D \rho, \quad (3.11a) $$

$$ \xi_{3D} = (\xi_{3x}, \xi_{3y}, \xi_{3z}, \zeta) = (-v_{Rz}, u_{Rz}, v_x - u_y), \quad (3.11b) $$

and where $\rho$ is defined in (2.2g). The combination of (3.10) and (3.8e) gives a conservation equation,

$$ Q_{BE}^2 + \nabla 3D \cdot (u_{3D} Q_{BE}^2) = 0, \quad (3.12) $$

for balance equation potential enstrophy density $Q_{BE}^2$. Consequently, potential enstrophy is a global invariant (McWilliams and Gent 1980) for BEM. The following conservation equation for balance equation energy density also holds with $\nu = 0$ for the BEM model:
\[
E_R \Phi + \nabla_{3D} \cdot [\mathbf{u}_{3D}(p_T + E_R)] + \epsilon^2 \nabla \cdot [\chi \mathbf{u}_{RE}] = 0, \tag{3.13}
\]

\[
p_T = p + \epsilon^2 K_{1B}, \tag{3.14a}
\]

\[
K_{1B} = \int_{\tilde{z}_1} \int_{\tilde{z}_1} J(\tilde{\psi}, \tilde{\chi}) d\tilde{z} + g(x, y, t), \tag{3.14b}
\]

where \(\tilde{z}_1\) is an arbitrary constant reference level, \(g(x, y, t)\) is an arbitrary function, and \(p\) is defined in (2.2b). Consequently, the balance equation energy density \(E_R\) is also a global invariant for BEM. The lack of significance of the arbitrary level \(\tilde{z}_1\) and the arbitrary function \(g(x, y, t)\) in \(K_{1B}\) is discussed in Allen (1991) but may be readily seen for the problems considered here by noticing that only \(K_{1B} = J(\tilde{\psi}, \tilde{\chi})\) appears in (3.9).

The fact that, in contrast to BE, the BEM model is based on approximate momentum equations is important in providing equations for horizontal vorticity components that are necessary for the derivation of (3.10), expressing conservation of potential vorticity (3.11) on fluid particles. The existence of approximate momentum equations is also valuable in more general problems for the formulation, based on the no-normal-flow conditions, of consistent boundary conditions at rigid surfaces (Allen 1991).

d. Linear balance equations

The LBE are formulated (Gent and McWilliams 1983) by retaining the O(1) terms in the vorticity (2.8b) and the divergence (2.8c) equations, so that these are the same as (3.1a) and (3.1b) in the QG approximation, and by retaining the O(1) and O(\(\epsilon\)) terms in the thermodynamic equation (2.8e), so that it is the same as (3.5c) for BE [and (2.8e) for PE].

As a result, for LBE
\[
\nabla^2 \phi = \nabla^2 \phi, \tag{3.15a}
\]

which implies
\[
\phi = \phi, \tag{3.15b}
\]

and the equations used for the numerical solution are
\[
[\nabla^2 \phi + (S^{-1} \phi_{\perp})_z],
\]

\[
= -J(\phi, \nabla^2 \phi) - \nu \nabla^6 \phi - \{S^{-1}[J(\phi, \phi_{\perp})
\]

\[+ \epsilon[\nabla \cdot (\phi_{\perp} \nabla \chi) + (w \phi_{\perp})_z]_z\}], \tag{3.16a}

\[
\nabla^2 \chi = -\nabla^2 \phi_{\perp} - J(\phi, \nabla^2 \phi) - \nu \nabla^6 \phi, \tag{3.16b}
\]

\[
w_z = -\nabla^2 \chi. \tag{3.16c}
\]

For initial-value problems \(\phi(x, 0)\) is specified. With \(\nu = 0\), the LBE conserve volume integrals of energy density \(E_R\) but do not have an analog of the conservation of potential vorticity on fluid particles.

e. Linear BEM

The linear BEM model is defined here as having the same vorticity and heat equations as BEM and BE (3.5a) and (3.5c), but with a divergence equation (3.15a) that retains only O(1) terms. LBE is thus a simplification of either BEM or BE that involves a linear approximation to the divergence equation. However, since the terminology linear balance equation has already been taken for the previous model, we use the label LBEM here.

The resulting equations for numerical solution are
\[
[\nabla^2 \phi + (S^{-1} \phi_{\perp})_z],
\]

\[
= -J(\phi, \nabla^2 \phi) - \nu \nabla^6 \phi - \{S^{-1}[J(\phi, \phi_{\perp})
\]

\[+ \epsilon[\nabla \cdot (\phi_{\perp} \nabla \chi) + (w \phi_{\perp})_z]_z\}], \tag{3.17a}

\[
\nabla^2 \chi = -\nabla^2 \phi_{\perp} - J(\phi, \nabla^2 \phi)
\]

\[\quad - \epsilon \nabla \cdot [w \nabla \phi_{\perp} + (\nabla^2 \phi) \nabla \chi] - \nu \nabla^6 \phi, \tag{3.17b}
\]

\[
w_z = -\nabla^2 \chi. \tag{3.17c}
\]

For initial-value problems, \(\phi(x, 0)\) is specified. Conservation relations with \(\nu = 0\) for either energy or potential vorticity have not been found for LBEM. The LBEM equations (3.17) are useful to include in the model evaluations because a comparison of LBEM solutions with those from BE give a direct measure of the importance of representing the vorticity by the nonlinear balance (3.5b) in BE compared to the linear QG balance (3.15a) in LBEM.

f. Geostrophic momentum

The geostrophic momentum approximation (Hoskins 1975) involves replacing (2.1b) by
\[
\epsilon \frac{D}{Dt} \mathbf{u}_G + k \times \mathbf{u} = -\nabla \phi - \epsilon \nu \nabla^4 \mathbf{u}_G, \tag{3.18}
\]

where
\[
\mathbf{u}_G = (u_G, v_G), \tag{3.19a}
\]
\[
u = \phi_y, \quad v_G = \phi_x. \tag{3.19b,c}
\]

The other equations remain as in (2.1a,c,d).

For \(\nu = 0\), the GM model conserves an analog of potential vorticity \(Q_{GM}\) on fluid particles; that is,
\[
\frac{D}{Dt} Q_{GM} = 0, \tag{3.20}
\]

where
\[
Q_{GM} = (\epsilon \mathbf{z}_{3GM} + k) \cdot \nabla_{3D} \rho, \tag{3.21}
\]
\[
\mathbf{z}_{3GM} = (\mathbf{z}^{(x)}_{GM}, z^{(y)}_{GM}, \mathbf{z}^{(z)}_{GM}), \tag{3.22a}
\]
\[
\mathbf{z}^{(x)}_{GM} = \mathbf{u}_G - \epsilon J^{(x)}(u_G, v_G), \tag{3.22b}
\]
\[
\mathbf{z}^{(y)}_{GM} = \mathbf{u}_G + \epsilon J^{(y)}(u_G, v_G), \tag{3.22c}
\]
\[
\mathbf{z}^{(z)}_{GM} = \mathbf{z} + \epsilon J(u_G, v_G), \tag{3.22d}
\]
\[
\mathbf{z} = v_{ux} - u_{vy} = \nabla^2 \phi, \tag{3.22e}
\]
where the notation for the Jacobian operators is such that, for example,
\[
J^{(x)}(a, b) = a_x b_z - a_z b_x. \tag{3.23}
\]
The Jacobian without superscripts remains defined by (2.9). When \( \nu = 0 \), GM also has integral invariants of the energy density,
\[
E_{GM} = \frac{\epsilon}{2} \mathbf{u}_0^2 + (z - \bar{z}_0)\rho. \tag{3.24}
\]
To obtain numerical solutions we write GM in a form similar to (3.8). With \( (u, v) \) given by (2.5) and \( (u_G, v_G) \) by (3.19), we define
\[
\psi = \phi + \epsilon \psi', \tag{3.25a}
\]
\[
u = u_G + \epsilon \nu', \quad u' = -\psi' + \chi_x, \quad \tag{3.26a,b}
\]
\[
\nu = v_G + \epsilon \nu', \quad v' = \psi'_x + \chi_y. \tag{3.26c,d}
\]
The governing equations for GM may then be written as
\[
[\nabla^2 \psi + (S^{-1}) \psi],
\]
\[
= -J(\psi, \zeta_G) - \epsilon \nabla \cdot [(\zeta_G \nabla x + w \nabla \psi_G] - J(u', u_G) - \epsilon J(v', v_G)
\]
\[
- \epsilon J(u', v_G) - \nu \nabla^6 \phi - \{ S^{-1} [J(\psi, \psi_G)
\]
\[
+ \epsilon [\nabla \cdot (\psi_G \nabla x) + (w \nabla \psi_G)] \}, \tag{3.27a}
\]
\[
\nabla^2 \psi = J(v_G, u) + J(v, u_G) - \epsilon J(w, \phi_G) \tag{3.27b}
\]
\[
\nabla^2 \chi = -\nabla^2 \psi - J(\psi, \zeta_G) - \epsilon \nabla \cdot [(\zeta_G \nabla x + w \nabla \psi_G]
\]
\[
- \epsilon J(u', u_G) - \epsilon J(v', v_G) - \nu \nabla^6 \phi, \tag{3.27c}
\]
\[
\tilde{w} = -\nabla^2 \chi. \tag{3.27d}
\]
For initial-value problems, \( \phi(x, 0) \) is specified.

From (3.25) and (3.27b) the vertical component of vorticity of the horizontal advection velocity components \( (u, v) \) is
\[
\zeta = \nabla^2 \psi = \zeta_G - \epsilon J(u, v_G) + J(u_G, v)
\]
\[
- \epsilon^2 J(w, \phi_G), \tag{3.28a}
\]
which may be written as
\[
\zeta = \zeta_G - \epsilon^2 J(u_G, v_G) + O(\epsilon^2). \tag{3.28b}
\]
This is consistent to \( O(\epsilon) \) with the balance implied by the PE divergence equation (2.8c). Note that \( O(\epsilon) \) terms in \( \zeta \) (3.28b) and in the vertical component of the advected vorticity \( \zeta_G \) (3.22d) involve the same function \( J(u_G, v_G) \) but that they differ in sign and magnitude. An implication of this difference is that \( \zeta_G \) differs at \( O(\epsilon) \) from \( \zeta \) in (3.28b) with error \( |\zeta - \zeta_G| > |\zeta - \zeta_G| \).

4. Description of numerical experiments

Numerical solutions to finite-difference approximations to the models listed in Table 1 are obtained for three initial-value problems on an \( f \)-plane. The domain is doubly periodic in the horizontal directions \( (x, y) \) and is of constant depth in the vertical direction \( (z) \). The finite-difference methods are discussed in appendix A. All models utilize the same variables \( (\psi, x, w, \phi, \theta) \) on the same finite-difference grid.

The numerical experiments are similar to the basic case experiment analyzed in the CTZ study of Allen et al. (1991), where a \( QG \) \( f \)-plane model in a flat bottom, periodic channel geometry was employed. The same basic stratification and vertical resolution (six levels) used in Allen et al. (1991) [and in Pierce et al. (1991) and Walstad et al. (1991)] are utilized here. Since the initial conditions in Allen et al. (1991) were based directly on observations, the QG solutions there were obtained in dimensional variables. We find it convenient to use dimensional variables here also. This requires conversion of the model equations in sections 2 and 3, presented there in dimensionless variables to clarify the approximations involved, to dimensional form. That conversion is straightforward and we forgo rewriting the equations in dimensional variables.

In all the experiments we use a Coriolis parameter \( f_0 = 9.20 \times 10^{-3} \text{ s}^{-1} \), a total depth of \( H_T = 3172 \text{ m} \), and \( K = 6 \) vertical grid cells. The values of the vertical grid cell dimensions \( \Delta z_k \) and the basic vertical stratification, represented by the square of the Brunt–Väisälä frequency \( N^2(z_{k+1/2}) \), are given in Table 2, where \( k \) is the vertical grid index (appendix A). The Rossby radius of the first baroclinic vertical mode is \( \delta R_1 = 24.6 \text{ km} \). The horizontal biharmonic friction coefficient \( \nu' = 8 \times 10^8 \text{ m}^4 \text{ s}^{-1} \) is chosen to be small so that dissipative processes play a nearly negligible role in the time-dependent dynamics. The three experiments are characterized by different velocity magnitudes in the initial basic jet profiles, as described below. The maximum velocity increases with experiment number from 1 to 3. The horizontal domain is
\[
0 \leq x \leq L(x), \quad 0 \leq y \leq L(y), \tag{4.1a,b}
\]

<p>| Table 2. Parameters used in experiments 1, 2, and 3. Total vertical levels ( K = 6 ) (see appendix A and Fig. 17). Total depth is ( H_T = 3172 \text{ m} ). |
|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>Vertical index ( k )</th>
<th>Depth, ( -z_k )</th>
<th>( \Delta z_k )</th>
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<tr>
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<td>100</td>
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<td>1672</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Vertical index ( k + \frac{1}{2} )</th>
<th>Depth, ( -z_{k+1/2} )</th>
<th>( N^2(z_{k+1/2}) \cdot 10^8 )</th>
</tr>
</thead>
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<tr>
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<td>100</td>
<td>10.655</td>
</tr>
<tr>
<td>2.5</td>
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<td>0.782</td>
</tr>
<tr>
<td>5.5</td>
<td>1500</td>
<td>0.319</td>
</tr>
</tbody>
</table>
where the basic jet flow is parallel to the x axis. For all experiments \( L(x) = 250 \text{ km} \) with \( L(y) = 640 \text{ km} \) in experiment 1 and \( L(y) = 810 \text{ km} \) in experiments 2 and 3. The horizontal grid spacing is \( \Delta x = \Delta y = 5 \text{ km} \). In experiments 1 and 2, time steps of \( \Delta t = 1 \text{ h} \) are used for all of the models except the PE (explicit) model, where \( \Delta t = 225 \text{ s} \). In experiment 3, a time step of \( \Delta t = 30 \text{ min} \) is used for all models except again \( \Delta t = 225 \text{ s} \) for PE (explicit).

The experiments are initialized with a uniform, geostrophically balanced, baroclinic jet in the x direction plus a small perturbation. The uniform jet is symmetric in y about its centerline at \( y = y_c = L(y)/2 \). The jet velocity profiles for experiment 2 are shown as a function of \( y \) and \( z_k \) in Fig. 1. The x-direction velocities in the basic uniform jet are given by

\[
 u_{jk}(y) = U_{0k} \exp[-(y - y_c)^2/L_x^2] + U_{ik},
\]

(4.2)

where the jet half-width \( L_y = 30 \text{ km} \). The velocity magnitudes \( U_{0k} \) for the three experiments are listed in Table 3. For experiments 1 and 2 the values of \( U_{0k} \) and the magnitude of \( L_y \) are chosen to be similar to those in the two observed CTZ jet profiles presented in Pierce et al. (1991). The values of \( U_{0k} \) for experiment 3 are proportional to those in experiment 1, but increased to give a strong jet case with resulting local Rossby numbers \( \epsilon = O(1) \). The magnitude of \( U_{ik} \) is determined by the condition that

\[
 \int_0^{L(y)} u_{jk}(y) \, dy = 0,
\]

(4.3)

which is required in the doubly periodic domain. The condition (4.3) results in a weak vertical shear away from the jet core region, as shown in Fig. 1. This weak shear appears to have little effect on the time-dependent development of finite amplitude jet instabilities, as indicated by QG solution comparisons from a doubly periodic domain using (4.2) and (4.3) and from a periodic channel as in Allen et al. (1991) with \( U_{ik} = 0 \).

For the perturbation we utilize a small amplitude disturbance proportional to the most unstable linear QG mode with wavelength \( L(x) = 250 \text{ km} \). The unstable QG mode was found for the profiles (4.2), as specified in Table 3, by the method described in Pierce et al. (1991). To ease computational requirements, the unstable modes were calculated in the domain \( |y - y_c| \leq 55 \text{ km} \), which was found to be sufficient to represent the across-jet modal structure. The wavelength of 250 km is close to the wavelength of the fastest-growing linear mode for the jet profiles in both experiments 1 and 2, consistent with the results in Pierce et al. (1991). The amplitudes of the perturbations are chosen to be small so that the initial disturbance is weak. An indication of the magnitude of the initial disturbance is given by the maximum magnitudes of the perturbation velocities, which occur for the x-direction component near the surface and are 0.033, 0.031, and 0.031 m s\(^{-1}\) for experiments 1, 2, and 3, respectively.

Given this quasigeostrophically balanced initial state, we initialize all of the models in a consistent manner such that the initial pressure fields are equivalent. The pressure field calculated as described above is assumed to be equal to the BE initial streamfunction \( \psi_{BE}(x, 0) \). The BE initialization procedure (appendix A) is used to find the consistent BE initial fields \( \phi_{BE}(x, 0) \) and \( \chi_{BE}(x, 0) \). [Since the amplitude of the initial (x dependent) perturbation is small, \( \psi_{BE}(x, 0) \) and \( \psi_{BE}(x, 0) \) actually differ by very little.] The QG, LBE, LBEM, IG2, IG3, and GM models are initialized with \( \phi_{BE}(x, 0) \). The BEM model is initialized with \( \psi_{BE}(x, 0) \). The PE and PESI models are initialized with \( \psi_{BE}(x, 0) \), \( \chi_{BE}(x, 0) \), and \( \phi_{BE}(x, 0) \).

Spatial boundary conditions are periodicity for all variables in x and in y over distances \( L(x) \) and \( L(y) \), respectively. In addition, the vertical velocity \( w \) is zero at the top and bottom rigid surfaces; that is,

\[
 w(z = 0) = w(z = -H_T) = 0.
\]

(4.4)
Although the QG study in Allen et al. (1991) examined the finite amplitude behavior of a jet with characteristics similar to experiment 1, for the comparison of model solution accuracy in the next section it is useful to concentrate on the results from experiment 2. The somewhat larger Rossby number flow in experiment 2 provides a good test for the intermediate models and clearly indicates relative model accuracy. Thus, we refer to experiment 2 as a basic case, and to experiments 1 and 3 as the weak and the strong jet cases, respectively.

Quantitative measures of the errors in the intermediate and QG model solutions, compared to PE, are found as a function of time by calculating normalized rms differences between the corresponding variables from the model and PE solutions as described in appendix B. Prior to comparison with the intermediate model results and calculation of errors, the PE solutions are averaged over an inertial period to eliminate high-frequency variability.

5. Results of numerical experiments

The function $|\xi(x, y, z, t)|/f_0$ indicates the magnitude of the local, flow-determined Rossby number. In Fig. 2, the magnitudes of the maximum and minimum values of $\xi/f_0$, denoted by $|\xi|_{M}/f_0(\pm)$, are plotted as a function of time for each of the three experiments. Note that the magnitudes of $|\xi|_{M}/f_0(\pm)$ increase from the initial values as the flow develops in time and the maximum values of the positive vorticity are greater in magnitude than the minimum values of negative vorticity, that is, that $|\xi|_{M}/f_0(+)$ > $|\xi|_{M}/f_0(\sim)$. For the weak jet, basic case, and strong jet experiments, the initial values of $|\xi|_{M}/f_0(\sim)$ are 0.174, 0.287, and 0.404, while the maximum values are 0.264, 0.555, and 1.052, respectively. Thus, the experiments cover a range of flow regimes characterized by maximum local Rossby numbers that are moderately small (weak jet), moderate (basic case), and $O(1)$ (strong jet). A measure of the magnitude of the density fluctuations relative to the basic stratification is given by the function $-\theta_{z}(x, y, z, t)/\rho_{z} = d\theta_{z}/S$. As indicated, this function is assumed to be $O(\epsilon)$ with the present scaling. To quantify the density variations, the magnitudes of the maximum and minimum values of $-\theta_{z}/\rho_{z}$ at $z = z_2$, denoted by $|\theta_{z}|_{m}/|\rho_{z}|(\pm)$, are also plotted in Fig. 2 as a function of time for each of the three experiments. For the weak jet, basic case, and strong jet experiments, the largest magnitudes are 0.275, 0.644, and 0.772, respectively. These values are in reasonable agreement with the maximum local Rossby numbers.

a. Weak jet

The time-dependent development of the unstable jet flow field is illustrated in Fig. 3 by contour plots of the near-surface streamfunction fields $\psi(x, y, z_1, t) = \psi_1(x, y, t)$ from the PE and QG solutions every 20 days from day 30 to day 110. Recall that the jet flow is toward positive $x$. The growth of the initial linear instability of the jet into finite amplitude meanders that increase in amplitude until about $t = 90$ days when they deform and pinch off to form detached eddies on either side of the jet is clearly shown. To facilitate the description of the jet, we will refer to that part of the jet meander at small $y$ as the trough and that part at large $y$ as the crest. The location of the eddies that pinch off from the crest (trough) will be referred to as above (below) the jet.

Note from the $\psi_1$ fields in Fig. 3 that the QG solution, while qualitatively correct, is visibly inaccurate for
at the troughs. In the PE solution, a relatively larger anticyclonic eddy pinches off at the crest first before day 90, followed by the detachment of a smaller cyclonic eddy at the trough. The QG solution, on the other hand, shows identical meander growth and eddy pinch-off on either side of the jet. The asymmetric behavior of the PE instability will be discussed further in connection with the results from the basic case.

The errors in the QG and the intermediate model solutions relative to PE, calculated as described in appendix B, are shown in Fig. 4 for the $\psi$, $\chi$, and $\theta$ fields. First, we note that there is a set of models (BE, BEM, IG3, PESI) whose accuracy is much greater than QG, QG2, LBE, GM, and LBE that their errors are not visible on the same scale plot. Consequently, an expanded scale is used in the bottom plots to show the relative errors of the accurate models. For the less accurate models, the errors generally increase in the order IG2, LBE, GM, QG, and LBE consistently for all of the variables $\psi$, $\chi$, and $\theta$. The QG2 errors in $\psi$ and $\theta$ are smaller than those of QG and, for short time ($t < 30$ days), are similar to those of IG2. (Note that the QG2 errors in $\chi$ are the same as for QG.) For larger times ($t > 40$ days), however, the QG2 errors in $\psi$ and $\theta$ grow substantially so that they are much greater than the errors from IG2. The QG solutions are quantitatively inaccurate, as expected from the $\psi$ fields in Fig. 3. It is interesting that the LBE solutions are more inaccurate than QG. The GM model also gives relatively large errors of the same order as QG and LBE. This is perhaps surprising, but is consistent with the results found for GM by Snyder et al. (1991) and is also con-

$\psi_{\text{PE}}$

$\psi_{\text{QG}}$

$\chi$

$\theta$

Fig. 3. Contour plots of the $\psi_i$ fields from PE and QG as a function of $(x, y)$ every 20 days from $t = 30$ to $t = 110$ days for the weak jet experiment. The distance between tick marks on the axes is 50 km. In all contour plots of fields the units are mks. Positive (negative) values are contoured by solid (dashed) lines with the zero contour a heavy solid line. The contour interval is denoted by CI and applies to all unlabeled panels below and to the right.

$\psi_{\text{QG}}$

$\chi$

$\theta$

Fig. 4. The normalized rms errors (appendix B) for $\psi$, $\chi$, and $\theta$ as a function of time from the different models compared to PE for the weak jet experiments. Note the different scales for the errors in the top and bottom panels. The type of line representing the errors from each model is consistent in Figs. 4, 7, 8, 14, and 15.
Concentrating now on the BEM results, we show contour plots of the $\xi / f_0$ and $w_{1.5}$ fields as a function of time in Fig. 6. The instability growth is more rapid than in the weak jet case, with an anticyclonic eddy pinching off at the crest and detaching between days 50 and 70. The eddy detachment process is again asymmetric about the jet centerline, with a large anticyclonic eddy formed first at the crest on the positive vorticity side of the jet followed in time by the detachment between days 70 and 90 of a smaller cyclonic eddy at the trough on the negative vorticity side. For $t \leq 50$ days the vorticity field $\xi / f_0$ shows an increase in magnitude of negative vorticity in the crests and positive vorticity in the troughs as $t$ increases and as the curvature of the jet increases. In addition, positive $\xi / f_0$ develops outside of the jet core behind the crests. The vertical velocity $w_{1.5}$ is generally positive for fluid particles moving from troughs to crests and is negative for particles moving from crests to troughs. For $t = 10$ days, the largest magnitudes of $w_{1.5}$ are found near the midpoints between the crests and troughs, whereas for $t = 30$ days, the maximum magnitudes increase and occur closer to the crests and troughs. At $t = 50$ days the largest negative and positive values of $w_{1.5}$ are found near the trough. The vorticity field $\xi / f_0$ at days 70 and 90 contains considerable small-scale structure and

...
is characterized by the concentration of positive vorticity (with \( \zeta_1/\zeta_0 > 0.1 \)) into either thin, elongated regions or relatively small patches and by the spread of negative vorticity (with \( \zeta_1/\zeta_0 < -0.1 \)) over regions of greater spatial extent. The \( w_{1.5} \) field for days 70 and 90 likewise shows increased small-scale variability and loss of the coherent spatial structure that was associated with the meandering jet flow field at smaller times.

The basic case model errors for \( \psi, x, \theta \) and for \( \zeta, w \) are shown in Figs. 7 and 8, respectively. The results are similar to the weak jet case with the magnitude of the errors for all models generally larger here. The BE, BEM, IG3, and PESI models have very small errors. The QG, QG2, LBE, and GM solutions are totally inaccurate for times greater than about 40 days, while LBEM is somewhat better. The IG2 model is again substantially more accurate than QG, QG2, LBE, GM, and LBEM, although it has larger errors here than in the lower Rossby number weak jet case. Note that the magnitudes of the errors of IG2 and LBEM relative to the accurate models increase for \( \zeta \) and \( w \) compared to the \( \psi \) and \( x \) errors. That behavior is not so evident with the weak jet, and presumably results from the development of more small-scale structure in \( \zeta \) and \( w \) in the basic case.

The errors for the accurate models BE, BEM, IG3, and PESI, while all being extremely small, are generally largest for BE and smallest for PESI. For \( t \leq 50 \) days, IG3 is more accurate than BEM, with the reverse true for larger time. To illustrate the small magnitude of the errors for BE, BEM, IG3, and PESI, we show contour plots of \( \psi_1 \) at day 70 from most of the models in Fig. 9. The \( \psi_1 \) fields for PE, PESI, BEM, BE, and IG3 are visually indistinguishable. The IG2 \( \psi_1 \) is close to PE, while the LBEM \( \psi_1 \) has the correct qualitative features. The \( \psi_1 \) fields for LBE, GM, and QG are all in-

Fig. 7. The normalized rms errors (appendix B) for \( \psi, x, \) and \( \theta \) as a function of time from the different models compared to PE for the basic case. Note the different error scales in the top and bottom panels.

Fig. 8. As in Fig. 7 but for \( \zeta \) and \( w \).
Since from the results in Figs. 7–10, LBEM gives more accurate approximate solutions than GM, it appears that the extension of LBEM to GM by the inclusion of additional higher-order physical effects, represented by (3.27b) and the extra Jacobian terms in (3.27a,c), actually causes an increase in solution errors. This is presumably related to the fact that the vertical component of advected vorticity $\tilde{\zeta}_{\text{GM}}$ in GM is actually a less accurate approximation to $\zeta$ than $\tilde{\zeta}_F$ (section 3F).

The substantial difference in error between LBEM and BE (Figs. 7–10) is directly attributable to the approximation in the PE divergence equation (2.8c) of the linear QG balance (3.15a) for the vorticity in LBEM compared to the nonlinear balance (3.5b) in BE. Evidently, the nonlinear vorticity balance in BE, which retains an $O(\epsilon)$ correction to the QG relation, is important for quantitative agreement with PE at these Rossby numbers. We note, however, that the LBEM $\psi_1$, $\tilde{\zeta}_1/\tilde{f}_0$, and $w_{1.5}$ fields in Fig. 9 have much better qualitative agreement with PE than do the corresponding GM and QG fields. That is likely due to the retention in LBEM of both $O(1)$ and $O(\epsilon)$ terms in the vorticity equation (3.17a,b), similar to BE. To further assess the importance of the representation of the $O(\epsilon)$ terms in the PE vorticity equation (2.8b), we use the BE model (3.8) and calculate solutions with different components of the $O(\epsilon)$ terms, $\epsilon \nabla \cdot [\mathbf{w} \nabla \psi_2 + \zeta \nabla \chi]$, in (3.8a) and (3.8c) omitted. The term $\epsilon (\nabla \cdot \nabla \zeta + w_2 \zeta)$ represents advection of $\zeta$ by the di-

accurate, qualitatively as well as quantitatively, and, in fact, appear to be from a different experiment. In Fig. 10, we show the $\tilde{\zeta}_1/\tilde{f}_0$ and $w_{1.5}$ fields at day 70 from PE, IG3, IG2, LBEM, GM, and QG. [The PESI, BEM (compare Fig. 6), and BE fields are indistinguishable from the PE results.] It may be seen that the complex PE $\tilde{\zeta}_1/\tilde{f}_0$ and $w_{1.5}$ fields are also extremely well represented by IG3, although some minute differences can be detected between the PE and IG3 $w_{1.5}$ contours. [Note: the IG3+ contours for $w_{1.5}$ (not shown) are visually identical to those from PE.] The IG2 $\tilde{\zeta}_1/\tilde{f}_0$ and $w_{1.5}$ fields are reasonably close to PE, while the LBEM $\tilde{\zeta}_1/\tilde{f}_0$ and $w_{1.5}$ have qualitative similarity to PE. The increased relative errors of both IG2 and LBEM for $\zeta$ and $w$ compared to $\psi$ (Figs. 7 and 8) are fairly clear in a comparison of the contour plots of $\tilde{\zeta}_1/\tilde{f}_0$, $w_{1.5}$, and $\psi_1$. The $\tilde{\zeta}_1/\tilde{f}_0$ and $w_{1.5}$ fields from GM and QG have completely different spatial distributions than the corresponding PE fields.

The poor performance of GM in resolving the correct qualitative features of the $\tilde{\zeta}_1/\tilde{f}_0$ and $w_{1.5}$ fields in Fig. 10 is rather striking. In particular, the different spatial distributions of positive and negative vorticity given by PE are largely missed by GM. We note from the GM model formulation (3.27) that if the terms $[J(u', u_G) + J(v', v_G)]$ are omitted from (3.27a,c) and if (3.27b) is replaced by $\psi' = 0$, GM reduces to LBEM.

**FIG. 9.** Contour plots of the $\psi_1$ fields from most of the models at day 70 for the basic case. Scaling of axes as in Fig. 3.

**FIG. 10.** Contour plots of the $\tilde{\zeta}_1/\tilde{f}_0$ and $w_{1.5}$ fields at day 70 from PE, IG3, IG2, LBEM, GM, and QG for the basic case. Scaling of axes as in Fig. 3. Contour lines as in Fig. 6.
vergent component of the horizontal velocity field and by $w$ and is omitted in BE-1. The term $\varepsilon \nabla^2 \psi$ represents stretching of the vertical component of relative vorticity and is omitted in BE-2. The term $\varepsilon \nabla w \cdot \nabla \psi_z$ represents tilting of the horizontal components of vorticity and is omitted in BE-3. The errors for $\psi$ from these models and from LBEM and BE are shown in Fig. 11. The errors from BE-2 and BE-3 are somewhat larger than those from LBEM, and much larger than those from BE. It is interesting that the error in BE-1 from the omission of the advection terms is the smallest of these models, but is still considerably greater than the error from BE. Overall, the results indicate that all of the $O(\epsilon)$ terms in the BE vorticity equation (3.8c) are important for obtaining accurate approximate solutions. We note that IG2 includes an approximation to these $O(\epsilon)$ terms in the PE vorticity equation (2.8b) and also an approximation to the $O(\epsilon)$ terms in the divergence equation (2.8c). This is probably a major reason why IG2 gives reasonably accurate approximate solutions. The increase in accuracy of IG3 compared to IG2 appears to be due to the more accurate determination of the advecting $\chi$ field in IG3 (Allen 1993).

The dynamics of the jet instability processes of meander growth, pinching off, and eddy detachment in the basic case are characterized by a transfer of potential to kinetic energy and by large increases of lower-level kinetic energy, qualitatively similar to the results found in the QG jet instability study of Allen et al. (1991). In Fig. 12, we plot the time variability of the volume integrated kinetic energy ($K$), potential energy ($A$), and total $(K + A)$ energy from BE for the basic case along with the time variations of the volume-averaged kinetic energies associated with the $x$-averaged component of velocity ($\bar{K}$) and with the perturbations about the $x$-averaged velocities ($\bar{K}$) (top) and contributions $K_k$ to the volume-averaged kinetic energy $K$ from each level (bottom) calculated as described in appendix C.

![Fig. 11. The normalized rms errors (appendix B) for $\psi$ as a function of time from BE-1, BE-2, BE-3, LBEM, and BE for the basic case. The models BE-1, BE-2, and BE-3 are defined in section 5 and involve the BE model with omission of different $O(\epsilon)$ terms in the BE vorticity equation (3.8c).](image1.jpg)

![Fig. 12. Time variation of the volume-averaged kinetic ($K$), potential ($A$), and total ($K + A$) energy from BE for the basic case along with the time variations of the volume-averaged kinetic energies associated with the $x$-averaged component of velocity ($\bar{K}$) and with the perturbations about the $x$-averaged velocities ($\bar{K}$) (top) and contributions $K_k$ to the volume-averaged kinetic energy $K$ from each level (bottom) calculated as described in appendix C.](image2.jpg)
component of the flow. The asymmetry of the flow development about the initial jet centerline is also clearly reflected in the $\tilde{u}_k(y, t)$ plots.

c. Strong jet

In the strong jet case, the maximum local Rossby number varies from 0.4 to values greater than 1.0 (Fig. 2). This experiment thus involves an unstable $O(1)$ Rossby number flow, which should provide a rather severe test for the intermediate models derived under the assumption that $\epsilon \ll 1$.

The time development of the jet instability in the strong jet case (not shown) is qualitatively similar to that in the basic case, except that the meander growth and pinching-off processes occur more rapidly in the strong jet with detachment of the anticyclonic eddy completed before day 40. We apply only the more accurate models BE, BEM, IG3, and PESI and, of course, PE to the strong jet experiment. The errors from these models for $\psi$, $\chi$, $\theta$, and for $\zeta$, $w$ are shown in Figs. 14 and 15, respectively. Again, the errors for all of these models are small, although for IG3, BE, and BEM they are somewhat larger than in the lower Rossby number basic case. The most noticeable change from the basic case results is the relative increase in error for IG3 compared to the other models. This is perhaps not surprising since IG3 is the only one of these models that is derived in a consistent asymptotic manner for $\epsilon \ll 1$. The PESI solutions appear to remain extremely accurate with BEM somewhat more accurate than BE. (The somewhat smaller errors for BE compared to BEM at early times are presumably due to the initialization of PE with BE variables, rather than with the slightly different BEM fields.) Compared to the basic case, the increase of errors from BEM, BE, and IG3 for the $\zeta$ and $\omega$ fields is relatively greater than for $\psi$ and $\chi$. This is similar to the change in errors found in going from the weak jet to the basic case and presumably is related to the development of increased small-
The three numerical experiments cover a range of Rossby numbers including moderately small (weak jet), moderate (basic case), and $O(1)$ (strong jet). Recall that the Rossby radius of deformation of the first baroclinic mode is $\delta_{R1} = 24.6$ km, which is close to the value $L_j = 30$ km of the initial jet half-width. Consequently, these $f$-plane flow fields are characterized by substantial variations on spatial scales $O(\delta_{R1})$. The results show that for all of the experiments the models BE, BEM, IG3, and PESI give accurate approximate solutions. Although QG, QG2, LBE, and GM have some qualitative agreement with the PE solutions for the moderately small Rossby number weak jet experiment, they are inaccurate quantitatively in that case. These models lose even qualitative similarity and produce large errors in the moderate Rossby number basic case. The LBEM model gives considerably smaller errors than QG, QG2, LBE, and GM in the basic case and does much better in properly representing the asymmetric spatial features of the jet instability. The IG2 model is much more accurate than QG, QG2, LBE, GM, and LBEM. It gives reasonably small errors in the weak jet case, but the errors increase to somewhat larger values relative to the most accurate models in the basic case. The fact that IG2 gives fairly good accuracy and a general performance that is substantially better than QG, LBE, and GM is a rather interesting result given the relative simplicity of IG2. Recall from Allen (1993) (also appendix A) that in a periodic domain IG2 is similar to a two-step QG model, requiring only two inversions per time step of the QG linear operator on $\phi$.

The most accurate models, BE, BEM, IG3, and PESI, contain representations of certain common terms in the PE vorticity, divergence, and density equations (2.8b,c,e). The retention of these terms thus seems critical for the production of accurate approximate so-
olutions to the PE in moderate to O(1) Rossby number flows with variations on scales O(δR1). These models retain all of the terms in the density equation (2.8e). In the divergence equation (2.8c), BE involves the greatest simplification, retaining O(1) and O(ε) terms only. Comparison of the much greater LBEM errors with those from BE (Figs. 7, 8) gives a direct indication of the importance of these O(ε) terms. The divergence equations for BEM (3.9a) and IG3 (A49a) contain the O(ε) terms and additional O(ε2) terms. In BEM, the O(ε2) terms are the ones that follow from the approximate momentum equations and that ensure particle conservation of potential vorticity and conservation of energy. The BEM model generally gives slightly more accurate solutions than BE. For the weak jet and basic case, IG3 likewise gives more accurate solutions than BE for ψ̃, θ̃, and ỹ (and also for the IG3+ x and w). For the O(1) Rossby number strong jet case, IG3 loses accuracy relative to BE after about 30 days. In the vorticity equation (2.8b), BE retains O(1) and O(ε) terms (3.5a). The importance of the inclusion of all of the O(ε) terms in the vorticity equation is demonstrated in experiments with different combinations of these terms omitted from BE (Fig. 11). In addition, if the vorticity equation in BE is considered the defining equation for X as in (3.8c), then we note that X in BE is determined to O(ε). It is evident that the advection X field be determined to this O(ε) accuracy. That point is demonstrated by the difference in the errors from the IG2 and IG3 solutions. It appears that one of the critical improvements of IG3 over IG2 is that the advection X field in IG3 [X2 in (A49b)] is determined to O(ε) whereas the advection X in IG2 [X1 in (A45b)] is only determined to O(1). In general, the terms retained in BE appear to be the minimum required for accuracy in the moderate to O(1) Rossby number flows studied here.

The existence of exact equations for conservation of an analog of potential vorticity on fluid particles or for conservation of volume integrals of an appropriate energy density does not appear to be of crucial importance for model solution accuracy in these particular mesoscale problems. This is perhaps most clearly illustrated by the performance of GM, which has analogs of both conservation relations but gives inaccurate solutions. The point is also illustrated by the increased accuracy of LBEM, which has no conservation equations, compared to LBE, which conserves energy. Ironically, it seems that one motivation for the neglect in LBE of the O(ε) terms in the vorticity equation that are retained in LBEM was to ensure energy conservation. Of the accurate models, BEM has conservation equations for both potential vorticity and energy but BE and IG3 do not. Nevertheless, all three of these models give accurate approximate solutions.

The fact that PESI, which when used with a large time step is in a sense a numerically based intermediate model, gives consistently accurate approximate solutions should be noted. The possible increased application of semi-implicit PE formulations in ocean models deserves consideration. We point out, however, that PESI may share with PE some of the difficulties associated with open boundary condition application (Oliger and Sundstrom 1978), data assimilation, and initialization. On the other hand, the manner in which BE, BEM, and IG3 handle the first two of those problems has not yet been determined either.

In summary, for moderate Rossby number flows the QG, QG2, LBE, and GM models give large errors. The LBEM model provides somewhat lower errors, while IG2 gives substantially smaller errors. The BE, BEM, IG3, and PESI models give highly accurate approximate solutions to PE, and that result is found also in the O(1) Rossby number strong jet flow.

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**APPENDIX A**

**Numerical Models**

The numerical finite-difference approximations utilized for the different models are discussed in this appendix. All models are formulated in terms of the same variables (ψ, X, W, φ, θ) on the same finite-difference grid. The spatial difference approximations generally follow those used by Norton et al. (1986). The time difference schemes for BEM, BE, LBEM, LBE, and GM are new, as are the methods of solution for IG2 and IG3. For consistency, the difference approximations are discussed here in terms of the dimensionless variables of sections 2 and 3, although the numerical solutions are obtained in corresponding dimensional variables. General comments on the relative computational times required for each model and on other computational issues are included at the end of this appendix.

A uniform rectangular grid (Arakawa and Lamb 1977) is used for the horizontal coordinates (x, y) with grid spacing (Δx, Δy). In the vertical coordinate z, the grid is staggered and nonuniform (Fig. 17). There are K vertical grid cells of dimension Δzk with centers at zk (k = 1, ⋯, K) and with

$$\sum_{k=1}^{K} Δz_k = 1.$$  

Cell boundaries are at $z_{k+1/2}$ (Fig. 17). Specifically, with $z_{1/2} = 0$, $z_{1} = -\frac{1}{2} Δz_1$, and $z_{K+1/2} = -1$,
The variables \( \psi, \chi, \phi \) are defined at \( z_k \), while \( w, \theta, N^2 \) are defined at \( z_{k+1/2} \).

To specify the spatial finite-difference approximations, it is useful to define the following operators. For the vertical differences, we use

\[
\bar{\theta} = \bar{\theta}(z_k) = \frac{1}{2} \left[ \theta(z_{k-1/2}) + \theta(z_{k+1/2}) \right], \quad (A2a)
\]
\[
\delta_z \theta = \delta_z \theta(z_k) = \frac{1}{\Delta z_k} \left[ \theta(z_{k-1/2}) - \theta(z_{k+1/2}) \right], \quad (A2b)
\]
\[
\check{\bar{\theta}} = \check{\bar{\theta}}(z_{k+1/2}) = \frac{1}{2} \left[ \phi(z_k) + \phi(z_{k+1}) \right], \quad (A2c)
\]
\[
\delta_z \phi = \delta_z \phi(z_{k+1/2}) = \frac{1}{2} \left[ \phi(z_k) - \phi(z_{k+1}) \right]. \quad (A2d)
\]

For the horizontal differences, we define

\[
\check{\bar{\phi}} = \frac{1}{2} \left[ \phi(x + 1/2 \Delta x) + \phi(x - 1/2 \Delta x) \right], \quad (A3a)
\]
\[
\delta_x \phi = \frac{1}{\Delta x} \left[ \phi(x + 1/2 \Delta x) - \phi(x - 1/2 \Delta x) \right], \quad (A3b)
\]
for both \( x \) and \( y \) coordinates, and

\[
\nabla^2 \phi = (\delta_x^2 + \delta_y^2) \phi, \quad \nabla^4 \phi = \nabla^2(\nabla^2 \phi), \quad (A3c,d,e)
\]

The notation \( J(A, B) \) will imply the Arakawa formulation of the Jacobian (Arakawa 1966); that is,

\[
J(A, B) = \frac{1}{3} \left[ \delta_x A \delta_y B - \delta_y A \delta_x B \right.
\]
\[
+ \Delta z_k \left( \delta_x (\delta_y B) - \delta_y (\delta_x B) \right] \quad (A4)
\]

The operator \( \nabla \cdot (A \nabla B) \) is calculated as in Norton et al. (1986) by using the identity

\[
\nabla \cdot (A \nabla B) = \frac{1}{2} \left[ \nabla^2 (AB) - B \nabla^2 A + A \nabla^2 B \right], \quad (A5)
\]

where \( \nabla^2 \) is defined in (A3c). We will also use the following notation for vertical integrals:

\[
\langle \phi \rangle = \int_{-1}^{0} \phi dz = \sum_{k=1}^{K} \phi(z_k) \Delta z_k. \quad (A6)
\]

The pressure obeys the hydrostatic relation (2.8d), which is

\[
\delta_z \phi = \theta(z_{k+1/2}), \quad (A7)
\]

It is convenient to write

\[
\phi = \langle \phi \rangle + \phi', \quad (A8a)
\]

where

\[
\phi'(z_k) = \int_{z_{k-1}}^{z_k} \theta dz - \left( \int_{z_{k-1}}^{z_k} \theta dz \right), \quad (A8b)
\]

and where we define

\[
\int_{z_{k-1}}^{z_k} \theta dz = \sum_{k'=k}^{k-1} \left( \theta(z_{k'+1/2}) \frac{1}{2} (\Delta z_{k'} + \Delta z_{k'+1}) \right), \quad 1 \leq k \leq K - 1, \quad (A8c)
\]

so that \( \phi' \), and hence \( \phi \), satisfies (A7).

The continuity equation (2.8a) is

\[
\delta_z w + \nabla^2 \chi = 0. \quad (A9)
\]

The vertical boundary conditions (4.4) are \( w(z = 0) = w(z = -1) = 0 \), or

\[
w(z_{1/2}) = w(z_{K+1/2}) = 0. \quad (A10a,b)
\]

Generally, \( w \) is calculated from the \( z \) integration of (A9):

\[
w(z) = w(z_{k+1/2}) = - \int_{z_{k-1}}^{z_k} \nabla^2 \chi dz', \quad (A11)
\]

which, with (A10), implies

\[
\langle \nabla^2 \chi \rangle = \nabla^2 \langle \chi \rangle = 0. \quad (A12)
\]

For the time difference approximations, the time variable is discretized in the usual manner \( t = n \Delta t \), where \( n \) is an integer and \( \Delta t \) is a constant time step. The notation \( \phi^n = \phi(t = n \Delta t) \) is used to denote evaluation of \( \phi \) at time level \( n \). For several of the models we use the second-order Adams–Bashforth time difference scheme, (e.g., Haltiner and Williams 1980), where

\[
\psi^{n+1} = \psi^n + \frac{1}{2} \Delta t (3 \psi^n - \psi^{n-1}). \quad (A13)
\]

a. Primitive equations

Primitive equation (2.8) solutions are obtained with the explicit time difference scheme (A13). We write (2.8b,c,e) in the form,

\[
\nabla^2 \psi^n = -\nabla^2 \chi^n - NL \psi^n, \quad (A14a)
\]
\[
\partial^2 \chi^n = -\nabla^2 \phi^n + \nabla^2 \psi^n - NL \chi^n, \quad (A14b)
\]
\[
\theta^n = -Sw^n - NL \theta^n, \quad (A14c)
\]
where

\[ \delta\sigma = -S\dot{w} - NL\theta, \]  

(A18c)

and use the notation:

\[ \delta\theta = \frac{\psi^{n+1} - \psi^{n}}{2\Delta t}, \]  

(A18d)

\[ \overline{\psi} = \frac{\psi^{n+1} + \psi^{n}}{2}, \]  

(A18e)

\[ \varsigma = \nabla^3 \psi, \quad \theta = \delta_\xi \theta. \]  

(A15b)

Using (A6), we define

\[ \psi = \langle \psi \rangle + \psi', \quad \chi = \chi', \]  

(A16a,b)

where \( \langle \chi \rangle = 0 \) is implied by (A12) (for a horizontally doubly periodic domain). It follows from (A14a,b) that

\[ \nabla^2 \langle \psi \rangle = -\langle NL\varsigma \rangle, \]  

(A17a)

\[ \nabla^2 \psi = -NL\sigma, \]  

(A17b)

\[ \nabla^2 \langle \phi \rangle = \nabla^2 \langle \psi \rangle - \langle NL\chi \rangle, \]  

(A17c)

\[ \epsilon^2 \nabla^2 \chi = -\nabla^2 \phi + \nabla^2 \psi = -NL\chi. \]  

(A17d)

With all variables known at \( t = n\Delta t \) and at previous time levels, we calculate \( \langle \psi \rangle, \psi, \chi, \theta \) from (A17a,b,d) and (A17c,a,c), respectively. The time difference scheme (A13) then gives \( \langle \psi^{n+1} \rangle, \psi^{n+1}, \chi^{n+1}, \theta^{n+1} \), whence \( \phi^{n+1} \) and \( w^{n+1} \) are found from (A8) and (A11). The calculation of \( \langle \phi^{n+1} \rangle \) is not necessary, but if desired, \( \langle \phi^{n+1} \rangle \) may be found from (A17c). At the initial time, \( t = 0 \), we specify \( \psi, \chi, \) and \( \theta \) (see section 4). A forward step is used in place of (A13) for the first time step.

b. Primitive equation models with semi-implicit time differencing

We also obtain solutions to the primitive equations with a semi-implicit time difference scheme that we formulate following Kwiat and Robert (1971) and that treats both gravity and inertial waves implicitly. This model (PESI) provides useful verification of the solutions from the explicit PE model when the time step \( \Delta t \) in PESI is taken to be as small as in PE. The main reason for the inclusion of PESI here, however, is to assess its accuracy and utility as a numerically based intermediate model when relatively large values of \( \Delta t \) are utilized (i.e., when \( \Delta t \) is chosen to be the same size as for the other intermediate models and thus when \( \Delta t \) is too large for PESI to accurately resolve the high-frequency gravity-inertial waves).

For PESI, (2.8b,c,e) are approximated by

\[ \nabla^2 \delta\psi = -\nabla^2 \chi - NL\varsigma, \]  

(A18a)

\[ \epsilon^2 \nabla^2 \delta\phi = -\nabla^2 \psi + \nabla^2 \psi - NL\chi, \]  

(A18b)

\[ \delta\theta = -S\dot{w} - NL\theta, \]  

(A18c)

where

\[ \bar{\psi} = \frac{\psi^{n+1} + \psi^{n}}{2}, \]  

(A18e)

\[ \varsigma = \nabla^3 \psi, \quad \theta = \delta_\xi \theta. \]  

(A15d,e)

\[ \nabla^2 \phi = \alpha^{-1} [\nabla^2 \phi - RHSx + \epsilon^2 \Delta t^{-1} \nabla^2 \phi], \]  

(A19a)

\[ \nabla^2 \chi = \alpha^{-1} [\nabla^2 \chi - RHS\varsigma - \Delta t^{-1} \nabla^2 \phi]. \]  

(A19b)

\[ w^{n+1} = S^{-1} [-\Delta t^{-1} \theta^{n+1} + RHS\theta], \]  

(A19c)

where

\[ RHS\varsigma = -2NL\sigma + \Delta t^{-1} \nabla^2 \psi - \nabla^2 \chi, \]  

(A20a)

\[ RHSx = -2NL\chi + \epsilon^2 \Delta t^{-1} \nabla^2 \chi \]  

(A20b)

\[ RHS\theta = -2NL\theta + \Delta t^{-1} \theta^{n+1} - Sw^{n+1}, \]  

(A20c)

\[ \alpha = [1 + (\epsilon^2/\Delta t^2)]. \]  

(A20d)

The requirement that \( \nabla^2 \chi^{n+1} \) and \( w^{n+1} \) in (A19b,c) satisfy (A9) gives the following elliptic equation for \( \phi^{n+1} \):

\[ \nabla^2 \phi + \alpha \delta_\zeta (S^{-1} \delta_\xi \phi) = \Delta t [RHS\varsigma + \alpha \delta_\zeta (S^{-1} \nabla^2 \phi) + \Delta t^{-1} RHS\Theta]. \]  

(A21)

Boundary conditions in \( z \) are correctly applied to (A21) by utilizing (A10a,b) in the derivation. In solving (A21) for \( \phi^{n+1} \) (and in solving similar three-dimensional elliptic equations in the other models), it is convenient to utilize an expansion for \( \phi^{n+1} \) in terms of vertical linear normal modes; that is,

\[ \phi(z_k)^{n+1} = \sum_{j=1}^{K} g_j(z_k) \Phi_j^{n+1}(x, y), \]  

(A22)

where

\[ \delta_\zeta (S^{-1} \delta_\xi g_j) + \lambda_j^2 g_j = 0, \]  

(A23a)

and

\[ \delta_\zeta g_j = 0 \quad \text{at} \quad z = 0, -1. \]  

(A23b)

With all variables known at \( t = n\Delta t \) and previous time levels, \( \phi^{n+1} \) is found from the solution to (A21) and then \( \psi^{n+1}, \chi^{n+1} \) are obtained from (A19a,b), and \( w^{n+1} \) from (A11). [Equivalent results are obtained of course if \( w^{n+1} \) is calculated from (A19c).] At \( t = 0, \psi, \chi, \) and \( \theta \) are specified. The first time step is calculated by replacing (A18d,e) for \( n = 0 \) with \( \delta\psi = (\psi^{1} - \psi^{0})/\Delta t, \bar{\psi} = (\psi^{1} + \psi^{0})/2). \)
c. Balance equations based on momentum equations

The governing equations for BEM (3.8a,c,d,e) and
(3.9), in difference form are (A7), (A9), and
\[
\nabla^2 \psi + \delta_z(S^{-1} \delta_z \psi) = -\epsilon \delta_w[S^{-1} \delta_w \psi] + \text{RBEQ},
\]
(A24a)
\[
\nabla^2 \chi = -\nabla^2 \psi + \text{RBE} \xi,
\]
(A24b)
\[
\nabla^2 \delta \phi = 2 \delta_z J(\delta_x \check{\psi}, \delta_y \check{\psi}) - \epsilon \nabla^2 \check{J}(\delta_x \psi, \check{\chi}) - \delta_z J(\xi, \chi) - \delta_z \check{J}(w, \delta_z \psi),
\]
(A24c)
where
\[
\text{RBEQ} = \text{RBE} \xi + \text{RDZ} \theta,
\]
(A25a)
\[
\text{RBE} \xi = -J(\psi, \chi) - \epsilon \nabla \cdot [\check{\psi} \nabla \delta \phi + \check{\chi} \nabla \xi] - \nu \nabla^2 \psi,
\]
(A25b)
\[
\text{RDZ} \theta = -\delta_z[S^{-1} \{J(\check{\chi}, \delta_z \psi) + q \nabla \cdot (\delta_x \delta \phi \check{\psi} + \delta_y \nabla \chi) \}].
\]
Boundary conditions in \( z \) are correctly applied to (A24a) by using (A10a,b) in the derivation of (A24a) in difference form. (This procedure is used generally in all of the following models.) The solution of (A24a) for \( \psi \) is obtained using an expansion of \( \psi_i \) in vertical linear normal modes (A22).

Solutions to BEM may be obtained using the time difference scheme described in Allen (1991) or the implicit time difference scheme discussed below for BE. We obtained the best results for BEM (see section k of appendix A) using the following iterative scheme, which is a modification of that in Allen (1991). We assume that the variables \( \psi^n, \delta \phi^n = \phi^n, \chi^n, w^n, \psi_i^n, \) and \( \phi_i^n \) are known at time \( t = n \Delta t \) and at previous time levels. In addition, we assume that we have estimates from the iteration procedure at the previous time step for \( \psi_i^{n+1} \), \( \phi_i^{n+1} \), \( \chi^{n+1} \), and \( w^{n+1} \). First, estimate \( \phi_i^{n+1} = 2 \phi_i^n - \phi_i^{n-1} \). Use the estimates for the variables at \( t = (n + 1) \Delta t \) in the rhs of (A24a) and solve (A24a) for \( \psi_i^{n+1} \). Use this value of \( \psi_i^{n+1} \) along with the other estimates in the rhs of (A24b). Calculate \( \chi^{n+1} \) from (A24b) and then \( w^{n+1} \) from (A11). Correct \( \phi_i^{n+1} = \phi_i^n + \Delta t (\psi_i^{n+1} + \psi_i^n) / 2 \). Use the new values of \( \chi^{n+1}, w^{n+1}, \) and \( \psi^{n+1} \) in the rhs of (A24c). Calculate \( \phi_i^{n+1} \) from (A24c). Using the new values, calculate \( \psi_i^{n+1} \) from (A24a), \( \chi^{n+1} \) from (A24b), and \( w^{n+1} \) from (A11). Then calculate estimates for \( \phi^{n+2}, \chi^{n+2}, w^{n+2}, \) and \( \phi_i^{n+2} \) in the following manner: \( \phi^{n+2} = \phi^n + 2 \Delta t \psi^{n+1} + \chi^{n+2} = 2 \chi^{n+1} - \chi^n, w^{n+2} \) from (A11) using \( \chi^{n+2} \) and \( \phi_i^{n+2} \) from (A24c). Calculate \( \phi_i^{n+1} = (\phi_i^{n+2} - \phi_i^n) / 2 \Delta t \). Return to the step where (A25a) is first solved for \( \psi_i^n \) and use the most recent values for the variables at \( t = (n + 1) \Delta t \) in the rhs. Repeat the cycle until convergence is obtained for \( \psi^{n+1}, \phi_i^{n+1}, \chi^{n+1}, w^{n+1}, \psi_i^{n+1} \), and \( \phi_i^{n+1} \).

At \( t = 0, \psi = \psi(x, 0) \) is specified. In addition, we assume \( \chi^0 = 0 \) (Allen 1991). An iteration procedure is used to initialize the other variables, that is, to find \( \phi_i^0, \chi^0, w^0, \psi_i^0, \) and \( \phi_i^0 \). Estimate \( x^0 = w^0 = \psi_i^0 = 0 \). Calculate \( \phi_i^0 \) from (A24c) and then \( \psi_i^0 \) from (A24a). Use the value of \( \psi_i^0 \) in the rhs of (A24b). Calculate \( x^0 \) from (A24b) and then \( w^0 \) from (A11). Recalculate \( \phi_i^0 \) from (A24c). Obtain \( \psi_i^0 \) from (A24a) and \( x^0 \) from (A24b). Estimate \( \psi_i^1 = \psi_i^0 + \Delta t \psi_i^0 \) and set \( \chi^1 = \chi^0, w^1 = w^0 \). Calculate \( \phi_i^1 \) from (A24c) using \( \psi_i^1, \chi^1, \) and \( w^1 \) in the rhs. Estimate \( \phi_i^0 = (\phi_i^1 - \phi_i^0) / \Delta t \). Return to the step where \( \psi_i^0 \) is first calculated from (A24a). Repeat until convergence for \( \phi_i^0, \chi^0, w^0, \psi_i^0, \) and \( \phi_i^0 \) is obtained. For the first time step, estimate \( \psi_i^1 = \psi_i^0 + \Delta t \psi_i^0, \phi_i^1 = \phi_i^0 + \Delta t \phi_i^0, \chi^1 = \chi^0, w^1 = w^0, \phi_i^1 = \phi_i^0 \) and proceed with the iteration as outlined above for the general \( (n + 1) \) time level calculation.

d. Balance equations

The governing equations for BE (3.8) in difference form are the same as BEM (A7), (A9), and (A24a,b) except that (A24c) is replaced by

\[
\nabla^2 \delta \phi = 2 \delta_z J(\delta_x \check{\psi}, \delta_y \check{\psi}).
\]
(A26)

We use a new implicit time difference scheme to solve BE. This involves an iterative procedure at each time step, similar to the scheme described above for BEM except that the iteration steps here are considerably more straightforward than those in the BEM scheme.

The equations (A24a,b) are time differenced in the following implicit manner:

\[
\nabla^2 \delta \psi^{n+1/2} = \delta_z(S^{-1} \delta_z \psi^{n+1/2}) + \text{RBEQ}^{n+1/2},
\]
(A27a)
\[
\nabla^2 \psi^{n+1/2} = -\nabla^2 \delta \psi^{n+1/2} + \text{RBE} \xi^{n+1/2},
\]
(A27b)
where we use the notation

\[
\delta_i^{n+1/2} = (\psi_i^{n+1} - \psi_i^n) / \Delta t,
\]
(A28a)
\[
\psi_i^{n+1/2} = (\psi_i^{n+1} + \psi_i^n) / 2,
\]
(A28b)
and RBEQ and RBE\xi are defined in (A25a,b). The divergence equation (A26) is assumed to hold at each time level \( t = n \Delta t \).

It follows from (A27a,b) that

\[
\nabla^2 \psi^{n+1} + \delta_z(S^{-1} \delta_z \psi^{n+1}) = \text{RBIQ},
\]
(A29a)
\[
\nabla^2 \chi^{n+1} = \text{RBI} \xi,
\]
(A29b)
where

\[
\text{RBIQ} = \nabla^2 \psi^n + \delta_z(S^{-1} \delta_z \psi^n) + \Delta t \times \{ -\epsilon \delta_w[S^{-1} \delta_w \psi^{n+1/2}] + \text{RBEQ}^{n+1/2} \},
\]
(A30a)
\[
\text{RBI} \xi = -\nabla^2 \psi^n + 2[\nabla^2 \psi^{n+1/2} + \text{RBE} \xi^{n+1/2}],
\]
(A30b)

With all variables known at \( t = n \Delta t \) and at previous time levels, estimate the variables \( \psi, \chi, w, \phi \) at \( t = (n \Delta t + 1 \Delta t) \) using the iteration scheme outlined above.
+ 1)\Delta t by extrapolation; for example, \( \psi^{n+1} = 2\psi^n - \psi^{n-1} \). Use these estimates in the rhs of (A29a) and (A29b) and solve (A29a) for \( \psi^{n+1} \). Substitute the new value of \( \psi^{n+1} \) in the time derivative term \( -\nabla^2 \delta_{t+1/2} \psi \) on the rhs of (A29b) and solve (A29b) for \( \chi^{n+1} \). Calculate \( w^{n+1} \) from (A11). Substitute \( \psi^{n+1} \) in the rhs of (A26) and solve (A26) for \( \delta_z \phi^{n+1} \). Return to the step where (A29a) is solved for \( \psi^{n+1} \) and substitute the latest values for the variables in the rhs. Repeat the cycle until convergence is obtained for all variables at \( t = (n + 1)\Delta t \). This implicit scheme may be used also for BEM where (A24c) is solved in place of (A26) for \( \delta_z \phi^{n+1} \), with the latest estimates for \( \psi^{n+1} \), \( \chi^{n+1} \), \( w^{n+1} \) on the rhs of (A24c).

At \( t = 0 \), \( \psi^0 \) is specified. We use a procedure similar to that described for BEM to find initial values for \( \phi^0_0, \chi^0, \) and \( w^0 \). Estimate \( \chi^0 = w^0 = \phi^0_0 = 0 \). Calculate \( \phi^0_0 \) from (A26). Calculate \( \psi^0 \) from (A24a), \( \chi^0 \) from (A24b), and \( w^0 \) from (A11). Using the new values for \( \chi^0 \) and \( w^0 \) in the rhs, recalculate \( \psi^0 \) from (A24a) and then \( \chi^0 \) from (A24b). Calculate \( \psi^1 = \psi^0 + \Delta t\psi^0 \) and then calculate \( \phi^1_0 \) from (A24b) using \( \psi^1 \) in the rhs. Estimate \( \phi^1_0 = (\phi^1_0 - \phi^0_0)/\Delta t \). Return to the step where \( \psi^0 \) is first calculated from (A24a). Repeat until convergence for \( \chi^0, w^0, \psi^0, \) and \( \phi^0_0 \) is obtained. For the first time step, estimate \( \psi^1 = \psi^0 + \Delta t\psi^0 \), \( \phi^1_0 = \phi^0_0 + \Delta t\phi^0_0 \), \( \psi^1 \), \( w^1 = w^0 \) and proceed with the iteration outlined above for the general \((n + 1)\) time level calculation.

e. Linear balance equations based on momentum equations

The governing equations for LBEM (3.17a,b,c) in difference form are (A9) and

\[
\nabla^2 \phi_t + \delta_z (S^{-1} \delta_z \phi_t) = RLBQ, \tag{A31a}
\]

\[
\nabla^2 \chi = -\nabla^2 \phi_t + RLB\xi, \tag{A31b}
\]

where

\[
RLBQ = RLB\xi + RLDZ\theta, \tag{A32a}
\]

\[
RLB\xi = -J(\phi, \xi) - \epsilon \nabla \cdot [\frac{1}{2} \nabla \delta_z \phi_t + \xi \nabla \chi - \nu \nabla \phi_t], \tag{A32b}
\]

\[
RLDZ\theta = -\delta_z \{ S^{-1} [J(\phi_t, \delta_z \phi_t) + \epsilon \nabla \cdot (\delta_z \phi_t \nabla \xi)] \}, \tag{A32c}
\]

We use the implicit time difference scheme described above for BE to solve LBEM. The equations (A31a,b) are time differenced as in (A27a,b). The iteration procedure now involves only solutions to the two equations (A31a,b) for \( \phi^{n+1} \) and \( \chi^{n+1} \) [and (A11) for \( \phi^{n+1} \)].

At \( t = 0 \), \( \phi^0 \) is specified. Initial values of \( \chi^0 \) and \( w^0 \) are found by an iterative procedure. Estimate \( \chi^0 = w^0 = 0 \). Calculate \( \phi^0_0 \) from (A31a) and then \( \chi^0 \) from (A31b), and \( w^0 \) from (A11). Repeat the calculations until convergence for \( \chi^0, w^0, \) and \( \phi^0_0 \) is obtained. For the first time step, estimate \( \phi^1 = \phi^0 + \Delta t\phi^0 \), \( \chi^1 = \chi^0 \), \( w^1 = w^0 \), and proceed with the general implicit time difference scheme.

\[f. \text{ Linear balance equations}\]

The governing equations for LBE (3.16a,b,c) in difference form are (A9) and (A31a,b) from LBEM with RLB\xi in (A32b) replaced by

\[RLB\xi = -J(\phi, \xi) - \nu \nabla \phi_t. \tag{A33}\]

The LBE are solved implicitly by the same method used for LBEM. At \( t = 0 \), \( \phi^0 \) is specified. Initial values of \( \chi^0 \) and \( w^0 \) and estimates of \( \phi^1, \chi^1, w^1 \) for the first time step are found similarly to LBEM.

g. Geostrophic momentum approximation

The governing equations for GM (3.27) in difference form are (A9) and

\[\nabla^2 \phi_t + \delta_z (S^{-1} \delta_z \phi_t) = RGMQ, \tag{A34a}\]

\[\nabla^2 \chi = -\nabla^2 \phi_t + RGM\xi, \tag{A34b}\]

\[\nabla^2 \psi' = J(v_G, u) + J(v, u_G) - \epsilon \nabla \cdot \{ w \nabla \delta_z \phi_t + \xi \nabla \chi - \nu \nabla \phi_t \}, \tag{A34c}\]

where

\[RGMQ = RGM\xi + RDZ\theta \tag{A35a}\]

\[RGM\xi = -J(\psi, \xi_G) - \epsilon \nabla \cdot \{ w \nabla \delta_z \phi_t + \xi_G \nabla \chi - \nu \nabla \phi_t \} \]

\[+ J(u', u_G) + J(v, v_G) \} - \nu \nabla \phi_t, \tag{A35b}\]

\[RDZ\theta \text{ is defined in (A25c)}, \]

\[\psi = \phi + \epsilon \psi', \tag{A36a}\]

\[u = u_G + \epsilon u' \], \[v = v_G + \epsilon v' \tag{A36b,c}\]

\[u_G = -\delta_y \phi', \quad v_G = \delta_x \phi', \quad \xi_G = \nabla^2 \phi' \tag{A36d,e,f}\]

\[u' = -\delta_y \psi' + \delta_x \chi', \quad v' = \delta_x \psi' + \delta_y \chi'. \tag{A36g,h}\]

The GM is solved using the implicit time difference scheme described for BE. The equations (A34a,b) are time differenced as

\[\nabla^2 \delta_t^{n+1/2} \phi_t + \delta_z (S^{-1} \delta_z \delta_t^{n+1/2} \phi_t) = RGMQ^{n+1/2}, \tag{A37a}\]

\[\nabla^2 \chi^{n+1} = -\nabla^2 \delta_t^{n+1/2} \phi_t + RGM\xi^{n+1/2}. \tag{A37b}\]

The divergence equation (A34c) is assumed to hold at each time level \( t = n\Delta t \).

It follows from (A37a,b) that

\[\nabla^2 \phi^{n+1} + \delta_z (S^{-1} \delta_z \phi^{n+1}) = RGIQ, \tag{A38a}\]

\[\nabla^2 \chi^{n+1} = RGI\xi. \tag{A38b}\]
where
\[
\text{RG} \text{IQ} = \nabla^2 \phi^0 + \delta_z (S^{-1} \delta_z \phi^0) + \Delta \text{RGMQ}^{n+1}(2) \tag{A39a}
\]
\[
\text{RG} \text{I} \zeta = -\nabla^2 \chi^0 + 2[-\nabla^2 \lambda^{n+1}(1/2) \phi + \text{RGMQ}^{n+1}(2)] . \tag{A39b}
\]

With all variables known at \( t = n \Delta t \) and at previous time levels, we solve (A38a,b) and (A34c) by iteration. Estimate all variables at \( t = (n+1) \Delta t \) by extrapolation; for example, \( \phi^{n+1} = 2\phi^n - \phi^{n-1} \). Use these estimates in the rhs of (A38a) and (A38b). Solve (A38a) for \( \phi^{n+1} \). Substitute the new value of \( \phi^{n+1} \) in the time derivative term \( -\nabla^2 \delta_t^{n+1}(1/2) \) in the rhs of (A38b) and solve (A38b) for \( \chi^{n+1} \). Calculate \( w^{n+1} \) from (A11). Substitute \( \phi^{n+1} \), \( w^{n+1} \) in the rhs of (A34c) and solve (A34c) for \( \psi^{n+1} \). Return to the step where (A38a) is solved for \( \phi^{n+1} \) and substitute the latest values for the variables at \( t = (n+1) \Delta t \) in the rhs. Repeat the cycle until convergence is obtained for \( \phi^{n+1} \), \( \chi^{n+1} \), \( w^{n+1} \), and \( \psi^{n+1} \).

At \( t = 0 \), \( \phi^0 \) is specified. Initial values for \( \psi^0 \), \( \chi^0 \), and \( w^0 \) are found by an iterative procedure. Estimate \( \psi^0 = \chi^0 = w^0 = 0 \). Calculate \( \psi^0 \) from (A38c), \( \phi^0 \) from (A38a), \( \chi^0 \) from (A38b), and \( w^0 \) from (A11). Repeat the calculations until convergence for \( \psi^0 \), \( \chi^0 \), \( w^0 \), and \( \phi^0 \) is obtained. For the first time step, estimate \( \phi^1 = \phi^0 + \Delta t \phi^0 \), \( \psi^1 = \psi^0 \), \( \chi^1 = \chi^0 \), \( w^1 = w^0 \), and proceed with the general implicit time difference scheme. As a check on the GM solutions, we also solved (A34) using an appropriately modified form of the time difference scheme described for BEM, and obtained identical results. In addition, we verified in the above scheme that replacement of (A34c) with \( \psi^0 = 0 \) and omission of the \( J(u', v_c) + J(v', v_c) \) terms in (A35b) gave the LBM solutions.

**h. Quasigeostrophic approximation**

The governing equation for QG (3.3) in difference form is
\[
\nabla^2 \phi + \delta_z (S^{-1} \delta_z \phi) = -J(\phi, \nabla^2 \phi) - \delta_z [S^{-1} J(\phi, \delta_z \phi)] - \nu \nabla^6 \phi . \tag{A40}
\]

With \( \phi^0 \) known, solutions for QG are obtained by solving (A40) for \( \phi^1 \) using the Adams–Bashforth time difference scheme (A13) to find \( \phi^{n+1} \). We then may find \( \chi^n \) from (3.4) and \( w^n \) from (A11). At \( t = 0 \), \( \phi^0 \) is specified. A forward time step is used to find \( \phi^1 \) from \( \phi^0 \).

**i. Second-order quasigeostrophic approximation**

The QG2 model includes the lowest (first)-order quasigeostrophic equations and the next (second)-order correction. The equations for QG2 are given in appendix A of Allen (1993). The variables are represented by
\[
\phi = \phi(0) + \epsilon \phi(1), \quad \psi = \psi(0) + \epsilon \psi(1), \quad \chi = \chi(1), \quad w = w(1). \tag{A41a,b,c,d}
\]

The equation for \( \phi(0) \) is (A40) (with \( \phi \rightarrow \phi(0) \)). We obtain \( \chi(1) \), \( \psi(1) \), and \( w(1) \) from
\[
\nabla^2 \chi(1) = -\nabla^2 \rho(0) - J(\phi(0), \nabla^2 \phi(0)) - \nu \nabla^6 \phi(0) , \tag{A42a}
\]
\[
\nabla^2 \psi(1) = \frac{\zeta(1)}{\psi(1)} - 2J(\delta_x \phi(0), \delta_y \phi(0)), \tag{A42b}
\]
\[
\nabla^2 w(1) = -\nabla^2 \chi(1). \tag{A42c}
\]

The equation for \( \psi(1) \) is
\[
\nabla^2 \psi(1) + \delta_z (S^{-1} \delta_z \phi(1)) \quad \begin{cases} \quad -J(\psi(0), \zeta(1)) - J(\psi(1), \zeta(0)) \\ -\nabla \cdot [w(1) \nabla \phi(0)] + \nabla \chi(1) - \nu \nabla^6 \psi(1) \\ - \delta_z [S^{-1} J(\psi(0), \delta_z \phi(0)) + J(\delta_x \phi(0), \delta_y \phi(0))] \\ - \delta_z [S^{-1} J(\psi(1), \delta_z \phi(1)) + J(\psi(1), \delta_z \phi(0))] \\ - \delta_z [S^{-1} J(\psi(0), \delta_z \phi(0)) + J(\delta_x \phi(0), \delta_y \phi(0))] 
\end{cases} \tag{A43}
\]

With \( \phi(0) \) and \( \psi(1) \) known, solutions for QG2 are calculated by solving (A40) for \( \phi(0) \) and (A43) for \( \psi(1) \). The Adams–Bashforth time difference scheme (A13) is used to find \( \phi^{n+1} \) and \( \psi^{n+1} \) at \( t = 0 \), \( \phi(0) \) is specified and \( \psi(1) = 0 \). A forward time step is used to find \( \phi(0) \) and \( \psi(1) \). In the error plots in Figs. 4, 7, and 8, the QG2 variables \( \chi \) and \( w \) (A41c,d) are the same as for QG.

**j. Iterated geostrophic model IG2**

The governing equations for IG2 and IG3 are formulated in section 3 of Allen (1993). Subscripts \( n \) are used there to designate iteration number \( n = 0, 1, 2, 3 \cdots \) for the variables \( \psi_n, \chi_n, w_n, \) and \( \phi_n \). For consistency, we use that notation in the following descriptions of IG2 and IG3. These subscripts appear only in this part of the appendix and should not be confused with the use elsewhere of subscripts \( k \) to indicate evaluation at vertical level \( z_k \).

With \( \phi^0 \) known, we obtain \( \phi_1 \) from
\[
\nabla^2 \phi_0 + \delta_z (S^{-1} \delta_z \phi_0) = -J(\phi, \nabla^2 \phi) - \delta_z [S^{-1} J(\phi, \delta_z \phi)] - \nu \nabla^6 \phi, \tag{A44}
\]

similar to QG. All variables here are evaluated at \( t = n \Delta t \) and the superscript \( n \) is omitted for simplicity. We then obtain \( \psi_1 \) and \( \chi_1 \) from
\[
\nabla^2 \psi_1 = \zeta_1 = \nabla^2 \phi - 2J(\delta_x \phi, \delta_y \phi) , \tag{A45a}
\]
\[ \nabla^2 x_1 = -\nabla^2 \phi_{0t} - J(\phi, \nabla^2 \phi) - \nu \nabla^6 \phi, \quad (A45b) \]

and \( w_t \) from (A11) using \( \nabla^2 x_1 \).

Next, we obtain \( \phi^{\prime}_{it} \) from

\[ \nabla^2 \phi^{\prime}_{it} + \delta_z (S^{-1} \delta_z \phi_{0t}) 
= \text{RIG2} \frac{z}{z} \{ S^{-1} [J(\psi_{1t}, \delta_z \phi) 
+ \epsilon [\nabla \cdot (\delta_z \phi \nabla x_1) + \delta_z (w_1 \delta_z \phi)]]) \}, \quad (A46) \]

where

\[ \text{RIG2} \frac{z}{z} = -J(\psi_{1t}, \delta_z \phi) - \nu \nabla^6 \phi 
+ \epsilon (-z_{it} - \nabla \cdot [w_1 \nabla \delta_z \phi_{1t} + \delta_z \nabla x_1]) 
- \epsilon^2 \nabla \cdot (w_1, \delta_z x_1), \quad (A47a) \]

\[ \frac{z}{z}_{it} = -2 [J(\delta_z \phi_{0t}, \delta_z \phi_{1t}) (\delta_z \psi_{1t}, \delta_z \psi_{1t})] + J(\delta_z \phi_{0t}, \delta_z \phi_{0t}) + J(\delta_z \phi_{0t}, \delta_z \phi_{0t}) \}. \quad (A47b) \]

Finally, using \( \phi^{\prime}_{it} \) we calculate \( \phi^{n+1} \) from the Adams–Bashforth time difference scheme (A13) that is,

\[ \phi^{n+1} = \phi^n + \frac{1}{\Delta t} (3 \phi^{n}_{it} - \phi^{n-1}_{it}). \quad (A48) \]

At the initial time \( \phi(x, 0) \) is specified. A forward time step is used in place of (A48) for the first time step.

\( k. \) Iterated geostrophic model IG3

To obtain the IG3 solutions, we assume \( \phi^n \) is known and continue with the calculations made for IG2 [(A44)–(A47)]. To make the IG3 model more directly comparable to BE and BEM, we omit the biharmonic diffusion term involving \( x \) in the divergence equation (2.8c) in IG3 (see Allen 1993).

We find \( \psi_2 \) and \( x_2 \) from

\[ \nabla^2 \psi_2 = \delta_z \frac{z}{z} = \nabla^2 \phi - 2J(\delta_z \psi_{1t}, \delta_z \psi_{1t}) 
+ \epsilon^2 [(\nabla x_{\psi}) + J(x_1, \delta_z \psi_{1t}) + \nabla J(\psi_{1t}, x_1) 
- \nabla \cdot (w_1, \delta_z \psi_{1t})] + \epsilon^2 \nabla \cdot [w_1 \nabla \delta_z x_{\psi} \frac{z}{z}] 
+ \frac{1}{2} \nabla \cdot (\delta_x \frac{z}{z} x_{\psi}^2 + (\delta_x \frac{z}{z} x_{\psi}^2))], \quad (A49a) \]

\[ \nabla^2 x_2 = -\nabla^2 \phi_{0t} + \text{RIG2} \frac{z}{z}. \quad (A49b) \]

To calculate \( \psi_2 \) from (A49a), it is necessary to first find \( \phi_{0t} \) from

\[ \nabla^2 \phi_{0t} + \delta_z (S^{-1} \delta_z \phi_{0t}) = \text{RIG3} \frac{z}{z} \]

\[ - \delta_z \{ S^{-1} [J(\phi_{0t}, \delta_z \phi) + J(\phi, \delta_z \phi_{0t})] \}, \quad (A50a) \]

where

\[ R\frac{z}{z} = -J(\phi_{0t}, \nabla^2 \phi) - J(\phi, \nabla^2 \phi_{0t}) - \nu \nabla^6 \phi, \quad (A50b) \]

and then to obtain \( x_{1t} \) from

\[ \nabla^2 x_{1t} = -\nabla^2 \phi_{0t} + \text{RIG3} \frac{z}{z}. \quad (A51) \]

Next, \( w_2 \) is calculated from (A11) using \( \nabla^2 x_2 \) (A49b).

We obtain \( \phi^{\prime}_{2t} \) from

\[ \nabla^2 \phi^{\prime}_{2t} + \delta_z (S^{-1} \delta_z \phi_{2t}) 
= \text{RIG3} \frac{z}{z} \{ S^{-1} [J(\psi_{2t}, \delta_z \phi) 
+ \epsilon [\nabla \cdot (\delta_z \phi \nabla x_{2t}) + \delta_z (w_2 \delta_z \phi)]]) \}, \quad (A52) \]

where

\[ \text{RIG3} \frac{z}{z} = -J(\psi_{2t}, \delta_z \phi) - \nu \nabla^6 \psi_2 
+ \epsilon (-z_{2t} - \nabla \cdot [w_2 \nabla \delta_z \psi_{2t} + \delta_z \nabla x_{2t}]) 
- \epsilon^2 \nabla \cdot (w_2, \delta_z x_{2t}), \quad (A53a) \]

\[ \frac{z}{z}_{2t} = -2 [J(\delta_z \phi_{0t}, \delta_z \psi_{2t}) + J(\delta_z \psi_{1t}, \delta_z \psi_{2t})] 
+ \epsilon \{ \nabla^2 x_{2t} + J(x_{1t}, \delta_z \psi_{1t}) + J(x_{1t}, \delta_z \psi_{1t}) 
+ \nabla^2 J(\psi_{1t}, x_{1t}) + \nabla^2 J(\psi_{1t}, x_{1t}) 
- \nabla \cdot (w_1, \delta_z \psi_{1t}) - \nabla \cdot (w_1, \delta_z \psi_{1t}) \} 
+ \epsilon^2 \{ \nabla \cdot [w_1 \nabla \delta_z x_{2t} \frac{z}{z} + w_1 \nabla \delta_z x_{2t} \frac{z}{z}] 
+ \nabla^2 [\delta_x \frac{z}{z} x_{1t} \frac{z}{z} + \delta_x \frac{z}{z} x_{1t} \frac{z}{z}] \}. \quad (A53b) \]

To calculate \( \psi_{2t} \) in (A53b) it is necessary to first find \( \phi_{0t} \) from

\[ \nabla^2 \phi_{0t} + \delta_z (S^{-1} \delta_z \phi_{0t}) = R\frac{z}{z} 2T 
- \delta_z \{ S^{-1} [J(\phi_{0t}, \delta_z \phi) + 2 J(\phi, \delta_z \phi) + J(\phi, \delta_z \phi)] \}, \quad (A54a) \]

where

\[ R\frac{z}{z} 2T = -J(\phi_{0t}, \nabla^2 \phi) - 2 J(\phi, \nabla^2 \phi_{0t}) 
- J(\phi, \nabla^2 \phi_{0t}) - \nu \nabla^6 \phi_{0t}, \quad (A54b) \]

and then to obtain \( x_{1t} \) from

\[ \nabla^2 x_{1t} = -\nabla^2 \phi_{0t} + R\frac{z}{z} 2T. \quad (A55) \]

In addition, \( x_3 \) may be obtained from

\[ \nabla^2 x_3 = -\nabla^2 \phi_{2t} + \text{RIG3} \frac{z}{z}, \quad (A56) \]

and then \( w_3 \) from (A11) using \( \nabla^2 x_3 \).

Finally, we calculate \( \phi^{n+1} \) as in (A48), but using \( \phi^{\prime}_{2t} \) and \( \phi^{\prime}_{2t} \).
calculations of errors for the $x$ and $w$ fields in IG3, we refer in section 5 (Figs. 4, 7, 8, 14, 15) to the $x_2$ and $w_2$ errors as from IG3 and the $x_3$ and $w_3$ errors as from IG3+.

1. General comments

The relative computation times required by the different models on an IBM RS/6000 Model 560 (expressed as a ratio to the QG time) with the time steps specified in section 4 are QG 1, PES 2, IG2 2.6, QG2 2.6, LBE 2.6, LBEM 3.0, GM 5.4, BE 6.5, IG3 7.3, BEM 7.7, and PE 26.5 (18.9). For PE, the smaller time in parentheses reflects the fact that it is possible to run PE with a larger $\Delta t = 300$ s. For the weak jet and basic case experiments, the intermediate and QG models typically also ran using a longer time step of $\Delta t = 2$ h with negligible changes in the results. The computation times relative to PE were lowered by a factor of about 0.6.

For BE, BEM, LBE, LBEM, and GM, the convergence of the iteration procedure at each time step was checked for the convergence of $\psi^{n+1}$ in BEM, $\psi^{n+1}$ in BE, and $\phi^{n+1}$ in LBE, LBEM, and GM. Convergence of these variables was found to be indicative of convergence for all variables. Tolerance levels for convergence were established by starting with strict requirements and then by relaxing the levels to values that retained negligible differences between final solutions. The aforementioned relative computation times were obtained with the relaxed tolerances. The number of iterations required varied with the model and the problem. For the basic case the number of iterations required for these models ranged from 3 to 5 with strict tolerances and from 2 to 3 with the relaxed tolerances.

Solutions for BEM obtained with the time difference scheme described in Allen (1991) and with the implicit time difference scheme discussed for BE showed evidence at times of spurious high-frequency oscillations that were most readily seen in the vertical velocity $w$. Interestingly, use of a larger time step with the implicit scheme (for example, for the basic case an increase of $\Delta t$ from 1 to 2 h) usually eliminated the oscillations. No problems concerning the occurrence of spurious high-frequency variability were found with the time difference scheme for BEM described in appendix A, section c.

APPENDIX B
Calculation of Errors

Errors in the intermediate and QG model solutions, compared to PE, are quantified by calculating normalized rms differences between corresponding variables.

We denote a horizontal area average by braces; for example,

$$\{\psi_k\} = \frac{1}{L(x) L(y)} \int_0^{L(x)} \int_0^{L(y)} \psi_k \, dx \, dy,$$  \hspace{1cm} (B1a)

where the subscript $k$ denotes evaluation at $z = z_k$. The finite-difference equivalent of (B1a) is

$$\{\psi_k\} = \frac{1}{N(x) N(y)} \sum_{i=1}^{N(x)} \sum_{j=1}^{N(y)} \psi_k(i \Delta x, j \Delta y),$$  \hspace{1cm} (B1b)

where $L(x) = N(x) \Delta x$, $L(y) = N(y) \Delta y$.

The variables $x$ and $w$ in all model solutions satisfy

$$\{x\} = \{w\} = 0.$$  \hspace{1cm} (B2)

On the other hand, the area average of (2.8e) gives

$$\{\theta\}_x + \epsilon\{w\}_x = 0,$$  \hspace{1cm} (B3)

so that, in general, $\{\theta\} \neq 0$ and the time evolution of $\{\theta\}$ is part of the PE solution. To make the notation compact we designate a particular model variable with a subscript corresponding to the model abbreviation, for example, $\psi_{PE}$. In solving PE, we set $\{\psi_{PE}\} = 0$. However, for BE and BEM we have $\{\psi_{BE}\} = \{\phi_{BE}\} \neq 0$. Variables with the area average removed will be denoted by a tilde; for example,

$$\tilde{\psi}_k = \psi_k - \{\psi_k\}. $$  \hspace{1cm} (B4)

For $\psi$, $x$, and $\xi$, we calculate the error as, for example,

$$E(\psi_{BE}) = \left[ \sum_{k=1}^{K} \left\{ \tilde{\psi}_{BEk} - \tilde{\psi}_{PEk} \right\}^2 \right]^{1/2}. $$  \hspace{1cm} (B5)

For $\theta$, we use

$$E(\theta_{BE}) = \left[ \sum_{k=1}^{K} \left\{ \theta_{BEk+1/2} - \theta_{PEk+1/2} \right\}^2 \right]^{1/2}. $$  \hspace{1cm} (B6)

Equation (B6) also holds for $w$, since $\{w\} = 0$. Note that the error measure essentially involves summation of squared differences over all grid points. It does not correspond to a volume integral, since there is no weighting by $\Delta x_k$ in (B5) and (B6).

APPENDIX C
Calculation of BEM Energy Integrals

The energy density for BEM (3.6) [see (3.13)] is

$$E_R = \epsilon K_R + (z - \bar{z}_0) \dot{\rho}, $$  \hspace{1cm} (C1)

which we write as

$$E'_R = E_R - (z - \bar{z}_0) \dot{\rho} = \epsilon K_R - (z - \bar{z}_0) \theta. $$  \hspace{1cm} (C2)
Volume averages of $E_{K}$, $\epsilon K_{R}$, and $-(z - \bar{z}_{0})\theta$ are calculated in difference form from the BEM solutions as a function of time and are presented for the basic case in Fig. 12.

The volume average of the kinetic energy $\epsilon K_{R}$ is

$$K = \{ \{ \epsilon K_{R} \} \} = \sum_{k=1}^{K} K_{k}, \quad (C3a)$$

where

$$K_{k} = \Delta z_{k} \{ \epsilon K_{Rk} \}, \quad (C3b)$$

$$\epsilon K_{Rk} = \frac{1}{2} \epsilon \left[ \left( \delta_{y} \overline{v_{k}} \right)^{2} + \left( \delta_{y} \overline{v_{k}} \right)^{2} \right] \quad (C3c)$$

and where the area average $\{ \cdot \}$ is defined in (B1).

To compute the potential energy, it is necessary to add time-dependent calculations of $\theta$ ($z = 0$) = $\theta_{1/2}$ and, of $\theta$ ($z = -1$) = $\theta_{K+1/2}$) = $\theta_{K+1/2}$. We use

$$\left\{ \theta_{1/2} \right\} + J(\psi_{1}, \theta_{1/2}) + \epsilon \left[ \nabla \cdot \left( \theta_{1/2} \nabla x_{1} \right) - \theta_{3/2} w_{3/2} \Delta z_{1} \right] = 0, \quad (C4a)$$

$$\left\{ \theta_{K+1/2} \right\} + J(\psi_{K}, \theta_{K+1/2}) + \epsilon \left[ \nabla \cdot \left( \theta_{K+1/2} \nabla x_{K} \right) + \theta_{K-1/2} w_{K-1/2} \Delta z_{K} \right] = 0. \quad (C4b)$$

With (C4a,b), $\theta$ for BEM satisfies

$$\{ \{ \theta_{i} \} \} = 0, \quad (C5a)$$

as it should, where

$$\left\{ \{ \theta_{i} \} \right\} = \frac{1}{2} \Delta z_{1} \{ \{ \theta_{1/2} \} \} + \sum_{k=1}^{K-1} \frac{1}{2} \left( \Delta z_{k} + \Delta z_{k+1} \right) \left\{ \theta_{k+1/2} \right\} \quad (C5b)$$

The volume average of the potential energy term $-(z - \bar{z}_{0})\theta$, with $\bar{z}_{0} = -1$, is

$$A = - \{ \{ (z + 1)\theta \} \}, \quad (C6a)$$

where

$$\left\{ \{ (z + 1)\theta \} \right\} = \frac{1}{2} \Delta z_{1} \left\{ \{ z_{1/2} + 1 \} \theta_{1/2} \right\} + \sum_{k=1}^{K-1} \frac{1}{2} \left( \Delta z_{k} + \Delta z_{k+1} \right) \left\{ \{ z_{k+1/2} + 1 \} \theta_{k+1/2} \right\} + \frac{1}{2} \Delta z_{K} \left\{ \{ z_{K+1/2} + 1 \} \theta_{K+1/2} \right\}. \quad (C6b)$$

With (C7) and (C8), $A$ for BEM satisfies

$$A_{i} - \{ \{ b \} \} = 0, \quad (C7a)$$

where

$$\{ \{ b \} \} = \epsilon \sum_{k=1}^{K-1} \frac{1}{2} \left( \Delta z_{k} + \Delta z_{k+1} \right) \{ w_{k+1/2} \theta_{k+1/2} \}. \quad (C7b)$$

To calculate volume-averaged kinetic energy associated with the $x$-averaged velocity components, we define

$$\overline{u}_{k}(y, t) = \frac{1}{L(x)} \int_{0}^{L(x)} u_{k}(x, y, t) dx, \quad (C8a)$$

$$= \frac{1}{N(x)} \sum_{i=1}^{N(x)} u_{k}(i \Delta x, y, t), \quad (C8b)$$

with similar definitions for $v_{k}$. Then we calculate

$$\epsilon \overline{K}_{Rk} = \frac{1}{2} \epsilon \overline{u}_{Rk}^{2}, \quad (C9)$$

$$\epsilon \overline{K}_{Rk} = \frac{1}{2} \epsilon \left( u_{Rk}^{2} + v_{Rk}^{2} \right), \quad (C10)$$

$$\{ \{ \epsilon \overline{K}_{Rk} \} \}, \quad (C12)$$

$$\{ \{ \epsilon \overline{K}_{Rk} \} \}, \quad (C13)$$

where the volume average $\{ \{ \cdot \} \}$ is the same as in (C3a), and note that

$$K = \overline{K} + \overline{K}. \quad (C14)$$

REFERENCES


Gent, P. R., and J. C. McWilliams, 1983: Consistent balanced models


