Student Thoughts on Derivatives of Vector Fields

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Abstract

Junior level physics students are familiar with a few types of vector field derivatives, such as divergence and curl, but are typically unfamiliar with how to take a general derivative of a vector field. Three junior-level physics students were interviewed with the open-ended prompt, "How would you think about taking a derivative of a vector field?" The resulting data was analyzed using Zandieh's theoretical framework developed in 2000, along with extensions by Roundy et al. and Emigh. We also present a new idea of *confounding*, demonstrated by the students, and propose a small extension to Emigh's work.

Contents

1 Introduction

Vector fields are important objects in both mathematics and physics. In vector calculus courses, students are typically introduced to the idea of a vector field and learn about two different types of vector field derivatives: divergence and curl. Divergence and curl are used frequently in electromagnetism (E&M), but a general vector field derivative is not encountered in classes until much later, if at all. Consequently, although there have been many research studies that explore student understanding of divergence and curl, particularly in Physics Education Research, very few study student understanding of general vector field derivatives.

There are two main approaches a student can take when trying to take a general derivative of a vector field: a component-based approach, where students attempt to differentiate each component individually, and a vector-valued approach, where students subtract two nearby vectors. Within these broad categories, students can use a variety of methods and strategies to find a derivative, including graphical approaches, numerical approaches, or drawing ideas from what they know about divergence and curl. Due to the unfamiliar nature of a general vector field derivative to many undergraduate physics and mathematics students, it is of interest to study the approaches students take in attempting to differentiate a vector field.

This study aims to answer two research questions:

1. How do students attempt to take a derivative of a vector field?

2. How well does Zandieh's framework and subsequent extensions (Zandieh, 2001) describe student understanding of vector field derivatives, and, in particular, is an additional extension necessary to describe vector field derivatives?

These questions will be examined using interview data from three physics students answering open-ended prompts about differentiating vector fields. Section 2 explores current literature relating to student understanding of vector fields and derivatives, Section 3 explains the methodology used in the interviews and analysis, Section 4 outlines in detail the data from each interview, and Section 5 further analyzes the data and compares the results of the three interviews to draw conclusions and propose answers to the research questions above. Finally, Section 6 discusses the limitations of this study, possible ideas for future studies, and possible implications of instruction.

2 Literature Review

Students' understanding of derivatives and multivariate calculus is of interest to both mathematics and physics education, and the current literature shows a wide variety of concepts, misunderstandings, and ideas that students have when tasked to think about vectors and derivatives. There are three main areas where students' understandings of derivatives in vector calculus have been explored: derivatives in general, understanding of vector fields and vector field representations, and vector calculus, particularly in an electricity and magnetism setting. This review spotlights the work of several authors that study how students think about derivatives in a multivariate and vector calculus setting, and notes the similarities and patterns that show up throughout the literature explored.

2.1 Derivatives

Student understanding of derivatives has been studied extensively in both mathematics and physics education literature.

The Action-Process-Object Schema (APOS) framework has been used in the context of multivariate functions in Martinez-Planell et al.'s study on student understanding of multivariate functions and directional derivatives (Martnez-Planell, Gaisman, & McGee, 2015). The APOS framework breaks a task into a genetic decomposition and studies students' actions and processes on the objects in the decomposition. Martinez-Planell and Gaisman's study decomposed the task of finding a directional derivative of a scalar valued, two-variable function into finding a function of the change in height as a function of the x-direction, and the change in height as a function of the y direction. The authors found that in their student interviews, most of the students did not use this particular genetic decomposition, suggesting that students do not think about the derivative of a multivariate function in the same way that they would approach the derivative of a single variable function.

An earlier study by Martinez-Planell and Gaisman tested students understanding of functions of two variables as a whole. They found that students' confusions and understandings seem to correspond to historical discovery of properties of multivariate functions (Martnez-Planell & Gaisman, 2012). Students were particularly challenged by the idea of restricting domain. Students' struggles with the notion of restricting domains in multivariate functions may provide some explanation for their difficulties in finding small changes in multivariate functions and understanding directional derivatives.

2.2 Vector Fields

There have been many studies that provide insight into students' struggles and understanding of vector fields, particularly in the context of electricity and magnetism. Dray and Manogue outline the between differences in how vector calculus is taught in mathematics and the way vector calculus is used in physics, and the possible impacts this "gap" has on student understanding (Dray & Manogue, 1999). They explain that mathematics courses emphasize algebraic understanding and calculations, whereas physics courses typically use graphical understanding and symmetry, with less emphasis placed on algebra. The authors suggest that this disparity may contribute to students' difficulties in understanding vector calculus in physics contexts, and suggest more communication between mathematics and physics instructors as a possible solution to the problem. Dray and Manogue's paper does not explicitly use data, instead outlining common themes the authors noticed while teaching, but the studies referenced in this paper tend to corroborate their findings with data from surveys, exams, and interviews with students.

Bollen et al. specifically studied students' difficulties with vector calculus in electricity and magnetism by testing students' understanding of vector calculus in physical and non-physical contexts (Bollen, van Kampen, & Cock, 2015). They found from analyzing survey data that students have trouble understanding what graphs and equations actually mean, likely because traditional classes do not heavily emphasize conceptual understanding. These students struggle to understand how to use mathematics to represent physical systems, and apply mathematical and physical laws inappropriately.

Other studies have indicated that students' difficulties come from fundamental misunderstandings of vector fields and their different representations. In particular, graphical representations of vector fields prove to be a large source of difficulty for students. Graphical understanding is not often emphasized in vector calculus classes, and consequently students do not understand how to effectively create a vector field map, let alone understand what the resulting figure represents (Bollen, van Kampen, Baily, Kelly, & Cock, 2017). Gire and Price found that students see the arrows in graphical representations of vector fields as solid objects, leading to misunderstandings such as not realizing that there are infinitely many more vectors in the vector field than the picture shows, or believing that the vectors themselves take up space equal to their magnitude (Gire & Price, 2013). In an earlier study, Gire and Price found that students see variable and component as interchangeable when looking at algebraic representations of vector fields, and consequently creating a graph of a vector field from an algebraic function is exceptionally difficult for students (Gire & Price, 2011).

These studies vary widely in the size of the data set and methods, as Bollen et al. used survey data from large numbers of students in both their 2015 and 2017 studies, and Gire and Price's studies used interview data from a much smaller sample, but all four studies clearly show that students do not understand how to draw vector field graphs, nor do they understand what they mean. Rather than a tool to help students visualize and understand an abstract concept like a vector field, graphical representations seem to add to students' confusion, and lead to fundamental misunderstandings about vector fields. Because these misunderstandings are often unaddressed from the beginning, introducing calculus operations on vector fields becomes even more difficult for students to fully understand.

2.3 Vector Calculus and Electromagnetism

Studies that focus specifically on student understanding of calculus of vector fields tend to focus on the operations that are relevant to electricity and magnetism (E&M), namely divergence, curl, and gradient. Although there is merit to studying student understanding of these operations, there is a significant gap in the literature on student understanding of a general derivative of a vector field. Students typically do not encounter this idea explicitly in classes outside of advanced mathematics courses, but students have seen partial derivatives of vector fields frequently in the context of divergence, curl, and the applications of those operations.

Students' understanding and misconceptions about divergence and curl roughly agree with what studies have shown about students' understanding of vector fields in general. Students are able to calculate divergence, curl, and gradient algebraically, but struggle with graphical representations and tend not to understand what gradient, divergence, and curl mean in a physical sense. In particular, a study that used pre and post tests given to students in vector calculus and electricity and magnetism found that students who have only had vector calculus instruction in a mathematics class tend to show less graphical understanding of vector fields than students with electricity and magnetism instruction (Baily, Bollen, Pattie, van Kampen, & Cock, 2015). This further reinforces the points Dray and Manogue make about a gap in vector calculus instruction. The E&M students showed significant improvement after instruction on questions related to visual representations, but the students with only mathematics instruction had more consistency with answers. An example of an inconsistent student answer is selecting the statement "curl is a vector," but leaving the statement "curl has direction" blank.

That does not mean that physics students have a full understanding of the graphical meanings of curl and divergence. Baily and Astolfi found that E&M students have difficulties with divergence that persist even after instruction, particularly with estimating divergence from graphs (Baily & Astolfi, 2014). The authors found that students associate divergence too closely with the idea of sources and sinks. When tasked to identify if a given vector field graph had positive, negative, or zero divergence, students tended to answer that a field without obvious sources or sinks has zero divergence. This result aligns with the findings in the Section 2.2: students struggle with graphical representations of vector fields, and they consequently tend to focus all their attention on one particular aspect of the graph, ignoring potentially important information.

The Colorado Upper-Dvision E&M Instrument (CUE) has been used in many studies to test student understanding of electricity and magnetism. The test was developed by physics education researchers at CU-Boulder to measure how students think about a variety of concepts in electricity and magnetism, including vector calculus. Pepper et al. used the CUE test in their study on student difficulties with mathematics in E&M. The study used interview data in addition to the CUE test, and found that students tend to focus on one part of vector fields (either direction or magnitude) when doing calculations. Whether the students focused on magnitude or direction differed depending on the problem, but the pattern persisted throughout the exam (Pepper, Chasteen, Pollock, & Perkins, 2012). The CUE test also showed that students had difficulty with understanding the physical meaning behind vector field operations, such as gradient, divergence, and curl. The students were able to calculate the gradient, divergence, and curl, but were often unable to explain what the results meant, corroborating the results of similar studies on student understanding of vector calculus.

Despite general difficulties with vector calculus, students seem to respond well to E&M classes that have been restructured to a more student-centered approach. Another study used the CUE test to examine how students understand E&M by comparing students in traditionally structured classes with a class that had been restructured to center around student engagement. The authors found that students in the restructured class performed as well or better on the exam than the students in the traditional lecture-based class, and exhibited a higher retention rate when given the exam again after some time had passed since the students took the E&M class (Pollock & Chasteen, 2009). Though these are promising results in favor of restructured upper division physics classes, the CUE test possibly has problems with the rubric that make it an imperfect measurement tool for student understanding of electricity and magnetism in its current state (Zwolak & Manogue, 2015).

2.4 Theoretical Framework

Zandieh developed a theoretical framework to analyze students' understandings of ordinary derivatives (Zandieh, 2001). Zandieh's framework focuses on the process-object layers of derivatives: ratio, limit, and function. These process-object layers each can be expressed using several representations: graphical (slope), verbal (rate of change), physical (velocity), and symbolic (difference quotient). Zandieh organized the process-objects and layers into a chart, shown in Figure 1.

Zandieh's framework was developed using interview data from AP Calculus students. Consequently, the framework does not include representations typically implemented in physics contexts. Roundy et al. (Roundy, Dray, Manogue, Wagner, & Weber, 2015) extended this framework to include a numerical representation, and altered the physical representation to be a measurement, instead of a velocity (Roundy et al., 2015). Figure 2, recreated from Roundy et al., shows each process-object layer. Roundy et al.'s extension

	Contexts				
	Graphical	Verbal	Paradigmatic Physical	Symbolic	Other
Process-object layer	Slope	Rate	Velocity	Difference Quotient	
Ratio					
Limit					
Function					

Figure 1: A chart showing the process-object layers and representations for the concept of derivative (Zandieh, 2001).

intended to make Zandieh's framework more applicable to the analysis of physics' students approaches to derivatives. The authors also edited Zandieh's table to include examples of how the different process-objects may be implemented using the various representations. For example, a graphical representation of the "ratio" object would look like a line between two points on a curve.

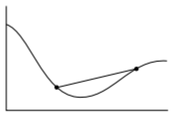
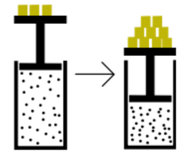
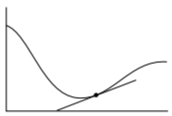
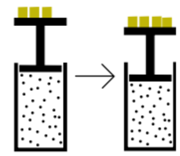
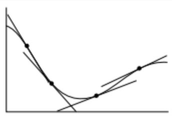
Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		"instantaneous ..."	$\lim\limits_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"... at any point/time"	$f'(x) = \cdots$	... depends on x	tedious repetition

Figure 2: A chart that shows the process-object layers and different representations for the concept of derivative, as well as examples of how a student may use them (Roundy et al., 2015).

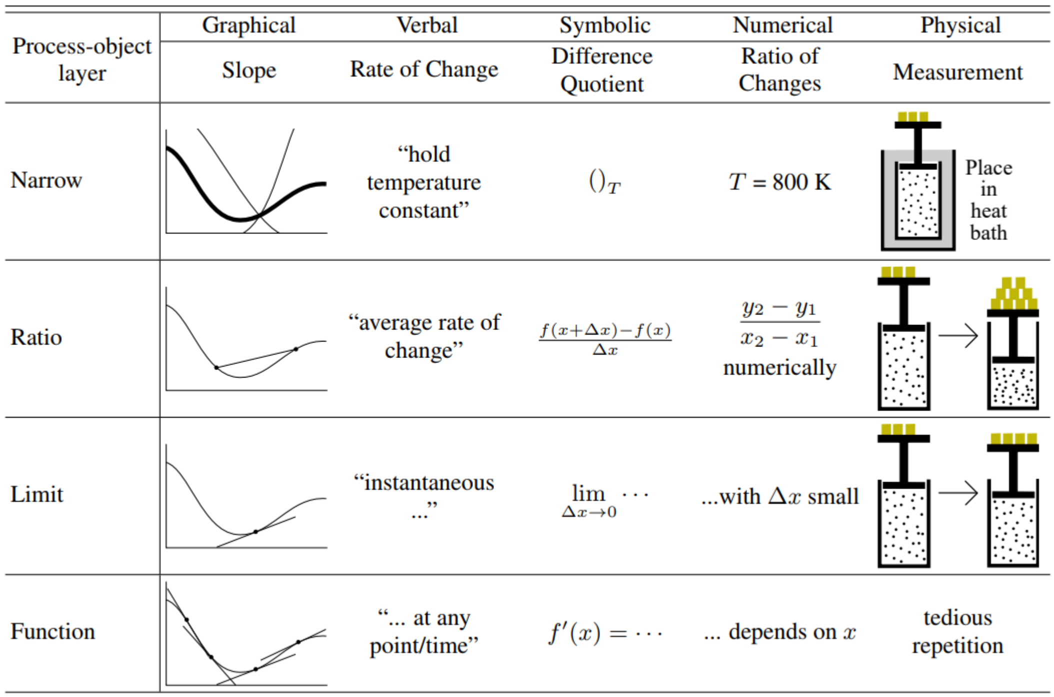


Figure 3: Another extension of Zandieh's framework chart, extended to include narrowing (Emigh & Manogue, 2019)

Roundy et al.'s extension made the framework more useful to describe the learning progression for physics students, but the framework is still rather restricted to single variable derivatives. Emigh and Manogue suggest adding an additional process-object layer of "narrowing" to Zandieh's framework (Emigh & Manogue, 2019). The layer of narrowing would represent how students choose which variable(s) to hold constant or take a derivative with respect to when taking derivatives of multivariate functions. The definition of narrowing is intentionally broad, and includes numerous different approaches students can take when determining what to hold constant.

The order of the layers in the theoretical framework takes a roughly chronological order based on when students typically learn and implement them. The *narrow* layer is presented first, as students generally determine what to hold constant prior to taking a derivative, and the *function* layer is presented last, as students typically first learn the ratio and limit layers prior to a function definition of derivative (Zandieh, 2001),(Emigh & Manogue, 2019).

Smith created a framework similar to Zandieh's focused on describing student understanding of divergence (Smith, 2014). Like Zandieh, Smith uses three process-object layers with three different representations (called “contexts”) with which the layers are presented. The process-object layers Smith uses are the finite layer, limit layer, and function layer. The finite layer “describes divergence in terms of finite volumes,” (Smith, 2014). The limit and function layer are the same as the limit and function layers in Zandieh. The representations (“contexts”) Smith uses in her framework are the descriptive context, which is similar to Zandieh’s verbal representation, the symbolic context, which uses mathematical formulas to represent divergence, and the example context. The example context is broad, and includes physical examples as well as descriptions of qualities.

3 Methods

3.1 Interview Protocol

Individual interviews were conducted with four students at the end of the Static Fields Paradigms course at Oregon State University by Paul Emigh. The interviews aimed to determine how students think about partial derivatives of functions. The first phase of the interviews asked students to think about the partial derivative of a scalar-valued function, and the second phase of the interview prompted the students to think about the partial derivative of a vector field. Three of the four students completed both phases of the interview. This paper only focuses on the vector field phase of the protocol, although some students reference their work on the scalar field during the vector field phase. Students were encouraged to say their thoughts and processes out loud, and write/draw on the provided paper throughout the interview.

Each interview lasted approximately 90 minutes, with the vector field phase lasting approximately 45 minutes. The vector field phase of the interview had two main parts: (1) a general question about vector field derivatives and (2) questions about partial derivatives. The third part of the protocol consisted of questions about divergence and curl, but the protocol dictated using these prompts only if time allowed, or if the students mentioned divergence and curl themselves. As such, only one interviewee was given the divergence and curl prompts. The prompts were open ended, and the students were provided with paper and some visual aids, as dictated by the protocol.

In Part 1, students were not provided any visual aid, and nor were they asked to take the derivative of any particular vector field. Instead, they were asked how they would think about taking a derivative of a vector field. The students were not given a written version of the prompt. The prompt was deliberately vague, and students were only asked to find the derivative of a vector field. The protocol mentions explicitly the motive for this approach: this enables the interviewer to see if the students focused on components, magnitude, direction, or simply discussed divergence and curl. In the typical physics courses at Oregon State University, divergence and curl are the only vector field derivatives

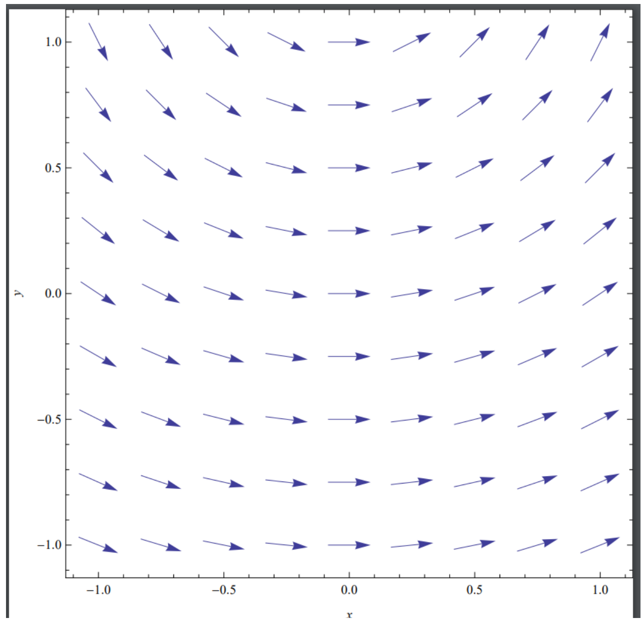


Figure 4: A vector field graph, provided to students in protocol.

the students encounter, prior to taking a differential geometry/general relativity class. Because the students were interviewed in the fall at the end of the second paradigms course, it is unlikely that any of them had encountered general vector field derivatives in their classes.

In Part 2, the students were given a printout of a vector field from Mathematica (Figure 4) and asked to find the partial derivative of the given vector field with respect to x. Because the Part 1 prompt was open ended, and did not have a definitive answer, the interviewer would give the Part 2 prompt when the student seemed to run out of things to say. The chosen vector field had constant magnitude, and the axes were labeled as shown in the figure. The prompts the students were given were more specific than in Part 1, yet were still deliberately vague in the protocol. For example, the interviewees were not given a particular location to find the derivative unless they asked for one. Throughout the interview, the interviewer asked clarifying questions and gave more personalized prompts based on the students approaches. Although not part of the original protocol, the interviewer would typically ask the students to find the partial derivative of the vector field with respect to y.

The Physics Education Research Group at Oregon State University uses open-ended interview questions frequently, particularly in the Paradigms project. Open-ended prompts allow students the freedom to develop their own unique approaches to the problems presented, and their approaches adapt and change as they work through the prompt. Students are also free to use and create their own notation to represent their mathematical ideas.

Open-ended prompts come with negative aspects. For instance, students may become flustered or confused by the nature of the question, particularly when the subject of the prompt is unfamiliar. Students may also lose track of what they are doing, or get stuck when their initial attempts do not work. To account for this, the interview progressed to Part 2 when the students appeared to get stuck or become flustered. Part 2 had follow-up questions included in the protocol. The interviewer also asked clarifying questions or prompted the interviewees to consider a particular idea they had brought up previously if the students appeared to become confused or flustered.

Each interview was recorded with video and audio, and the students' written work was collected and scanned. The parts of the interviews relating to vector fields were transcribed in their entirety, and the resulting transcripts and videos were analyzed qualitatively, in the style of Thematic Analysis (Aronson, 1995). Common themes throughout the interviews were noted and discussed. We present the results of each interview individually, then discuss the similarities, differences, and common themes further in the Discussion, where we also identify some of the particular concept images and representations that the students used in their interviews. The three interviews used were anonymized, and the pseudonyms Alex, Bailey, and Cam are used for the three participants. The alphabetical order of the pseudonyms corresponds to the chronological order in which the interviews were taken.

3.2 Interview Population

The students interviewed were enrolled in the Paradigms in Physics sequence at Oregon State University in 2016. At the time of the interview, the students would have completed the entire calculus sequence, including vector calculus 1 and 2, and the general physics with calculus sequence. The students also likely had completed Intro to Modern Physics, which was a sophomore level course introducing ideas such as relativity, quantum physics, statistical physics, and other physical ideas from the 20th century. This course was typically taken immediately after completion of the general physics with calculus sequence, though if any of the interviewees were transfer students, they may have not taken Intro to Modern Physics or were taking it concurrently with Paradigms.

3.3 Uniqueness of Paradigms Program

The Paradigms in Physics sequence is the junior-level physics curriculum at Oregon State University. The Paradigms project began in 1996 with the aim of restructuring the upper-division physics curriculum to more closely resemble the organization used by professional physicists, and to make the junior-level curriculum more accessible and improve information retention among juniors compared to the previous structure (Manogue et al., 2001). Under the Paradigms curriculum, rather than taking two courses in different subdisciplines simultaneously for ten weeks each term, each of which meets three days per week, the Paradigms students at this time took one physics course for three weeks at a time, meeting five days per week. This structure allows students to focus all of their attention on a single subdiscipline, and not become overwhelmed from dividing their attention between difficult concepts in different courses. Additionally, the Paradigms courses have a student-centered approach to teaching, and students engage in group work, discussion, and experiments throughout the courses. In the fall term, these students took the Static Fields paradigm, the Oscillations paradigm, and the Energy and Entropy paradigm. At the time of these interviews, the students had completed the Static Fields paradigm.

The Paradigms program was additionally altered in 2017 to make further improvements, but the students interviewed here did not take the newer courses. The changes included but were not limited to changing the courseload from three 3-week courses per term to two 5-week courses per term, and moving the Static Fields paradigm to spring term (Roundy, Gire, Minot, Van Zee, & Manogue, 2017).

4 Interviews

4.1 Alex

When Alex is asked about the derivative of a vector field, his first remark is, "gradient of you know some scalar valued function like this," and writes $\nabla f(x, y)$, shown in Figure 6. He then realizes that the prompt asked about the derivative of a vector field, rather than a derivative that is a vector field, and draws a simple vector field graph on the page in Figure 5.

He mentions that there are many different types of derivatives, like divergence and curl, saying, "I mean I'm sure there's, we mentioned in class that there was some ridiculous amount, but the two useful ones we talk about are the grad- er the curl and um um uh kay so we got this guy [referring to divergence], and the divergence thank you that's the word." He also brings up the "del" operator, and the possibility of applying this to different components. He refers to the "del" operator as partial derivatives on all the different components:

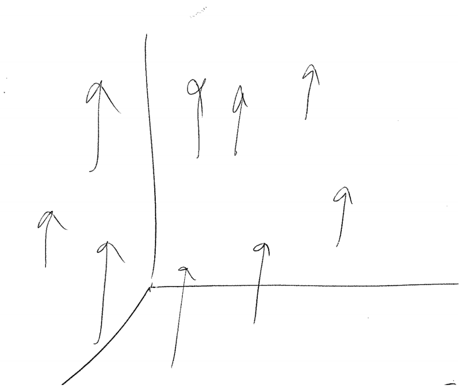


Figure 5: A vector field drawn by Alex

$\nabla f(x,y)$ — vector

Figure 6: Expression For Gradient

> *A: So these are the ones that um I mean, those have been the relevant ones so far, the derivatives that we've been asked to think about. Um, so I guess one this just comes from applying so we've got our del guy here which is partial of F and dot dot dot, all of those different directions. Cause I suppose we could have more than just three.* (A1)

From the previous quotes, Alex's first response seems to be to bring up as much information as he can remember about vector fields and derivatives, then he attempts to piece it together. This seems to show that he does not seem to have a clear definition in mind for a vector field derivative. He brings up the idea of gradient, divergence, and curl repeatedly, and repeatedly mentions that there are many different types of derivatives. Perhaps because he is trying to say as much as he can about the subject, but does not have much experience with vector field derivatives, Alex ends up expressing that there are many types of partial derivatives in multiple ways.

Alex starts to think about the components as individual functions, and begins to think about direction:

> *A: But then this is also like F of x [sic], is also gonna be a function of x and y and [F_y] is gonna be a function of x and y [writes expression in Figure 7]. So, I suppose when you're thinking about derivatives it's, it's a little more complicated than when we just had the like the scalar thing, like the height [points at scalar valued function graph]. Um, because, you've got these these directions [points at "$F_x\ \hat{x}$" and "$F_y\ \hat{y}$" on paper] that each of these are going in...* (A2)

The above quote and figure show that Alex understands that both F_x and F_y are functions of x and y, but he does not immediately say that each of these functions hes noticed

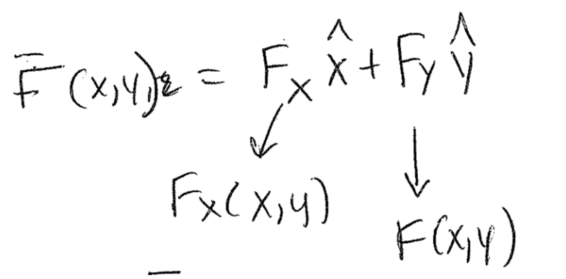


Figure 7: Alex's vector field equation

are differentiable with respect to both x and y. Instead, Alex seems to focus on the idea of direction and component.

He does seem to recognize the existence of partial derivatives of vector fields, after initial confusion about the idea of components. He recalls from classes that different operations incorporate partial derivatives of vector fields:

> *A: I think when we've mentioned in class, we've talked about how there's, you know, a ton of different ways we can consider applying, you know, these different partial derivatives, um, some of them are more useful than others so, uh, but you know, I guess the two that come to mind are you know the gradient [sic], and the curl.* (A3)

The above quote shows that not only does Alex consciously know that a vector field is a function of both x and y, as shown in the previous quote, but he also understands that there are many ways to potentially take partial derivatives. He notes that gradient and curl are types of partial derivatives. It is possible that Alex meant to say divergence when he said Gradient. However, Alex sees the partial derivatives used in calculating divergence and curl as different from simply taking a partial derivative of a function.

This is further shown when Alex is prompted to just take the partial derivative of the vector field, rather than a more complex partial derivative operation like divergence or curl. He expresses that he is uncomfortable with the unfamiliarity of the question. His immediate response is that he cannot simply take the partial derivative, saying "my gut response was no but that's without really thinking about it," but he decides to give the question more consideration. He initially decides to try to take the partial derivative with respect to x of each component:

A: So, if I was to find, if I was trying to find this I'm not, I'm not quite sure if that would mean I was to go like the partial of F_x um with respect to x in $\hat{x}$ plus the partial of F_y with respect to y in the $\hat{y}$ direction [writes expression in Figure 8] um, that seems reasonable? I'm just not sure what that would tell you I guess about the actual, like, about what this whole field is doing, because... (A4)
I: You mean by dy here or dx? [points to $\frac{\partial F_y}{\partial y}\hat{y}$]
A: Oh sorry yeah x. That was x. [changes $\frac{\partial F_y}{\partial y}\hat{y}$ to $\frac{\partial F_y}{\partial x}\hat{y}$]

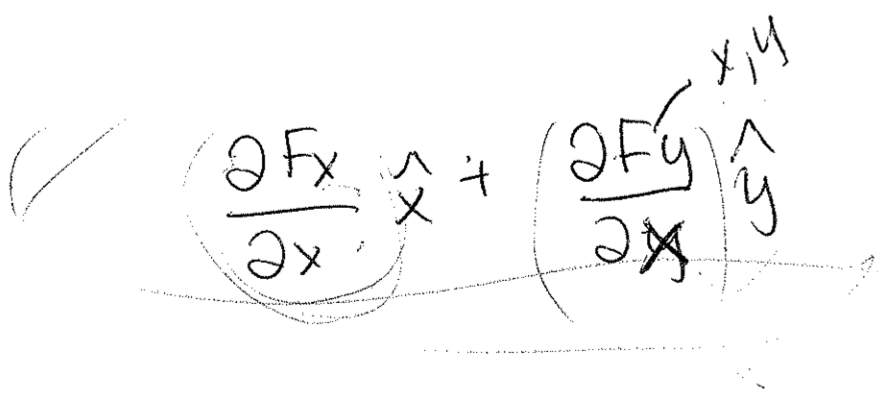


Figure 8: Alex's first definition for a partial derivative of a vector field

Alex recognizes that the y component of the vector field (F_y) has a dependence on x as does the x component (F_x), and comes to the correct conclusion that computing the partial derivatives of each component with respect to x results in another vector. He appears uncertain about the partial derivative as hes defined it, because he cannot find any physical meaning or relationship to the original function:

A: Yeah, this just seems kind of this seems kind of strange I think because uh [pause] [unclear] the concept of what like the what a slope would mean for all these, these fields. Yeah, so I guess when I'm thinking about like a derivative... (A5)

Here, Alex seems to be attempting to justify this interpretation with either an analog to the derivative of a single variable function, which can give information on the behavior of the original function, or a physical application that would back up this mathematical idea hes introduced. In particular, he is searching for an example of a slope.

It is also notable that Alex accidentally refers to the x variable as y when considering the partial derivative with respect to the y component, and writes y in the expression (see

Figure 8) initially before correcting to x after prompting. Early in the interview, Alex shows a clear understanding that variable and component are distinct objects. However, the mistake he made in the figure, writing y instead of x, shows that he is beginning to confound the idea of variable with component. This may also explain his initial response of "no" when asked if he can take the partial derivative of the y component with respect to x, as he may have been thinking of holding y constant, and confounding the y component with the y variable.

Alex's confounding of variable and component seems to become much more evident when Alex is able to think about the idea using a graphical representation, given by the interviewer as part of the protocol. Alex again brings up divergence and curl, and says that divergence and curl show how a vector field is "expanding or coming together," but is unsure what just taking the partial derivative would describe physically. He draws horizontal lines on the vectors at the top of the graph (see Figure 9), seemingly to show the magnitude of the x component, saying:

> *A: My inkling is like okay, I know that each of these is itself got a component and so, I'd imagine that this is all in that direction right. So I'd imagine that this this derivative of–*
> *I: When you say this derivative*
> *A: Sorry, I mean the derivative of this guy, the vector of the field in a certain direction, in this case x. I feel like this is gonna tell me how just [the x] component is changing, but I feel like I would need to hold this, this like I feel like I would need to choose kind of [points at different rows of vectors] well yeah this is interesting.* (A6)

Figure 9: Alex draws the magnitude of the x component of the vectors.

In the above quote, Alex realizes from looking at the graph that the elements of the vector field have an x component in the x direction, and a y component in the y direction. This realization seems to cause Alex to doubt the validity of his previous theory. Instead Alex seems to be saying that the partial derivative with respect to x shows how the function changes "in the x direction," and the y component of a vector relates to the y direction. He appears to be considering the idea that the y component is held constant when taking the partial derivative with respect to x. He knows that he needs to hold y constant, but seems unsure if y refers to variable, component, or both.

Alex's second idea of partial derivative with respect to x, taking the partial derivative of just the x component, does not necessarily have direction or two dimensions. Alex does not recognize that taking the partial derivative of both components instead of just the x component are two different operations, but recognizes that their results are different objects:

> *A: No, I suppose that that does sort of tell you [points at vector valued definition in Figure 8]. Well no this is weird, cause I don't know why would [the derivative] be a vector then. If I just apply [the partial derivative] here [to the entire vector], then this is gonna be a vector that has an x component and a y component. So I suppose what I've written here [Figure 8] does not necessarily match what I've drawn here, right [points to horizontal lines, Figure 9].* (A7)
> *I: What did you draw there?*
> *A: ...So on each of these arrows [points to paper] I was just drawing like, for example like the x component.*

This change in Alex's thinking seems to suggest that when he is looking at a vector field map, he begins to focus on direction rather than variable dependence. He knows that the partial derivative with respect to x shows how the function is changing in the x direction, holding the y direction constant, but he has not made a distinction between the y component and the y variable, and is instead holding both the y component and the y variable constant. This shows that he does not see these two objects as entirely separate. The two ideas hes come up with differ in precisely that sense: the first idea is the partial derivative of the function with respect to the x variable, holding the y variable constant, and the second idea is the partial derivative of the x component, holding the y component constant. Alex recognizes that these are different operations, as one outputs a vector and the other outputs a scalar, but since he does not see variable and component as separate objects, he does not understand why these two "partial derivatives" are different, and he cannot find an obvious reason why one would be incorrect.

He expands further on this new idea:

A: Right, yeah, yeah, that would be F_x. I was thinking maybe a useful thing to know would be, okay, how is this, how is the size of this F_x changing. But then I would probably want to fix a y, [moves hand down page] otherwise like what I would be you know doing is a change in this way or am I going this way [moves pen diagonally toward bottom left of paper from top right and top left corners]
I: Okay
A: Or this way or this way [moves pen towards bottom right corner from top right and top left corners]. So like I would want, like, I would want to know maybe what [pause, makes circling motion with pen] like would I be fixing a fixing a y on all of these I guess that's sort of my like my like my "question" [finger quotes] cause I would think so this [pointing at vector field map] would seem like it might be something useful to know so I don't know that this is really the correct interpretation. (A8)

Here, Alex explains the importance of fixing a y value by demonstrating that without a specific fixed y, it is unclear that he is taking the partial derivative of F_x along a line parallel to the $\hat{x}$ vector. Hes brought up the possibility of taking the partial derivative in different directions, but does not expand further on how that operation would be done.

Alex did not express these same concerns when he defined the partial derivative with respect to x of the vector field as taking the partial derivative of each component. This is possibly because with a visual representation of a vector field, Alex is able to focus more easily on direction than when he drew a simple vector field and was thinking mostly abstractly, resulting in a confounding that was not present when he did not have a specific vector field to consider.

The interviewer prompted Alex to further clarify what he thinks is useful:

$$\left(\frac{\partial F_x}{\partial x}\right)_y$$

Figure 10: Alex's second idea of a vector field derivative.

> *A: It might be useful to know how the um the magnitude of F_x here so the x component of the vector field so– not just F_x I suppose yeah no this is sorry F_x so how the size of F_x is changing um say that again so it might be useful it seems like it would be useful to know how the the size of F_x is changing...*
> *I: That you drew yep.*
> *A: Is changing um as a function, uh, for this like fixed y, I guess. So to me that sounds like uh I'm wanting to know how the size of this, like I've picked a y, that sounds like I'm taking F_x with respect to x, like holding y constant, right [writes expression in Figure 10]. That sounds like it might be useful to know, um, but I don't, yeah, I don't know if, like, how you would apply a derivative to this entire thing. Because if I like what I have written here would be like a vector itself, right. This would have an x, an x component and a y component and I don't know what that would be telling me visually, I guess.* (A9)

Alex claims that the partial derivative of the x component with respect to x would be "useful," but is reluctant to claim that the new operation he has defined is a partial derivative. He notices that there is no way to perform this operation on the entire vector field in a meaningful way, as this definition requires both the y variable and the y component of the vector field to be fixed. Alex seems to consider a derivative to be an operation that must be defined on all or most of a functions domain, possibly explaining his hesitation with defining a partial derivative this way. He has previously throughout the interview made comments such as "some are more useful than others," (see Quote A.3) and said that he did not know what information taking the partial derivative of each component would provide about the whole field (Quote A4). These comments seem to indicate that Alex believes a derivative must be informative and useful. He does not believe that a vector field could give information about the "slope of another vector field, or at least cannot visualize what a vector field could show about how another vector field is changing. Alex is forgetting that curl, which he has mentioned a few times throughout the interview, is a type of vector field derivative which results in another vector field.

The interviewer prompts Alex to think about the definition in Figure 10, and Alex begins to explain that when taking the partial derivative of F_x with respect to x, y is held constant. He then considers the possibility of taking the partial derivative of the y component of the vector field with respect to x, saying "so here I've taken the partial derivative of F_x with respect to x holding y constant, so I was just thinking well what if I did that to F_y as well, what would that tell me about here?" Alex is still confused by the notion of holding y constant while focusing on the component of the vector field pointing in the y direction, but has again started thinking about the y variable and the y component as distinct objects. He is beginning to separate, or "unconfound" the idea of variable and component.

Although he is beginning to distinguish variable and component, Alex seems to still be uncomfortable with the idea of holding the y variable constant but not F_y:

> *A: ...I'm just gonna like pull it up here real quick, so it goes like this [draws line] just a little bit magnified. So we've got, this way we can say that this is F_x and this is F_y right. F_y I would need, mmm. [Pause] Yeah unless, I guess, I'm just, I'm less certain what that would look like if I said I wanted to take the derivative of that of, of this whole thing [vector field] holding if this was holding y constant. That's probably what I would be doing right?*
> *I: Sure*
> *A: That seems like thats usually how we do these partial derivatives. And then I wanna know how this is changing. First of all, changes in, I wanna know how F the vect- yet Im thi- yeah its the whole vector field thing thats kinda throwing me off, I guess.* (A10)

The above quote seems to indicate that while Alex is still fully aware of the dependence of the x and y components of the vector field on the x and y variables in the plane, he still tends to confound the two concepts and implement a relationship that is not actually present. As a result, he doubts his conclusions. The interviewer prompts him to think about just x component of the vector field. Alex feels more comfortable with the physical and mathematical meaning of the operation hes defined, namely the partial derivative of F_x with respect to x.

Alex begins to estimate the magnitudes of the x components of the vectors at the fixed y value of -0.75 (see Figures 11 and 12) and plots them on a graph:

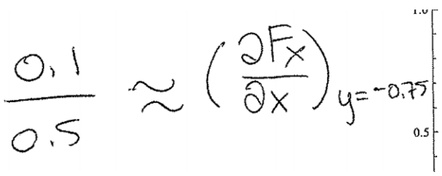


Figure 11: Alex approximates the partial derivative of F_x with respect to x.

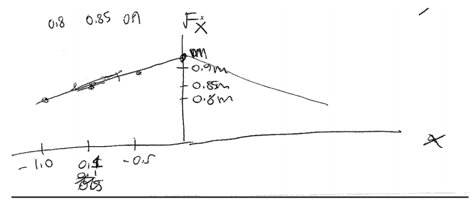


Figure 12: Alex's graph approximating the magnitude of the x component of vectors with respect to the x variable.

> *A: Okay, so the total length of the vector is* 1*, um, so each of these this length is* 1*, and this length is* 1 *and this length is* 1*, so I suppose, um [pause] interesting. [pause] I guess I would want, I would just want a value for each of these. So, maybe this is something, like, I mean over here this is gonna be* 1*,* F_x *is* F_x *this guy's gonna be* 1 *over here, so I would think that, um, if I were to assign these values. If this is like, you know* 0.8 *and then like* 0.85 *and then* 0.9 *which seems about right cause if it's you know, steadily going then maybe then we can say that over this [clears throat] this region we've got, um [pause], trying to think about what this would look like if I do this as sort of, like a like a function I guess.* (A11)

Here, Alex creates a single-variable linear function that shows the magnitude of the x component of the vector field with respect to the x variable, at a fixed y of -0.75. This allows Alex to estimate the derivative of F_x with respect to x at that specific y value by finding the slope of the line hes drawn.

After he finds a measurable approximation for the partial derivative of F_x with respect to x, Alex again decides to think about the y component of the vector field. He has managed to convince himself that the y component of the vector field is involved in the partial derivative of the entire vector field function, and therefore cannot justify ignoring it anymore. However, he is still confused by the idea of holding y constant while taking the partial derivative of the y component of the vector field, showing that he is still in the process of separating the two objects:

> *A: ...That was the one I was having a little bit more trouble with. Trying to imagine I think, so, this is, would be so we have the partial of* F_y *with respect to* x*. Shouldn't this be zero everywhere? [Chuckles] um...* (A12)

Although he initially thinks the partial derivative of the y component of the vector field with respect to x should be zero, when prompted to explain why, he remembers that F_y

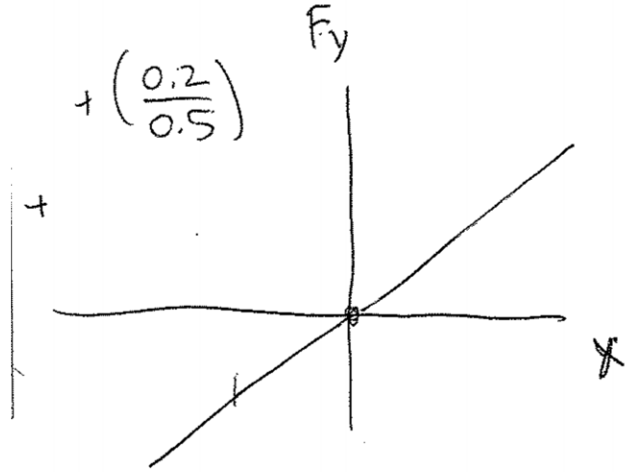


Figure 13: A graph approximating the magnitude of the y component of the vector field with respect to the x variable.

depends on the x variable, and is not being held constant, so he claims that the behavior of the partial derivative function of F_y with respect to x would behave similarly to the partial derivative of F_x with respect to x, presumably because the magnitude of the vector field is constant. He makes another linear graph to approximate the magnitude of the y component with respect to x at the same fixed y value (figure 13).

Interestingly, Alex again inadvertently calls the x variable "y" when referring to the partial derivative of F_y, and corrects his error when the interviewer calls attention to it:

A: ...This depends on x and y so it's gonna change, right the size of this this this something that um, oh no no no. So, if I'm thinking of just this by itself, uh, if I'm just thinking of F_y, then I suppose that what I'm thinking now is how it is, this would be similar to if we like instead went in this direction. This would be looking at how this component here is chang-so this would almost be like the um, like, similar graph, but slightly different. Right, so this is y vs F_y instead at um [Interviewer points at paper] no this is x sorry. (A13)
I: x
A: Yeah thank you, that's what I meant.

This is the second time in the interview Alex makes the mistake of referring to the y variable as x and is corrected by the interviewer. This previously occurred early in the interview, prior to the introduction of the vector field graph, when Alex initially defined the partial derivative of the vector field with respect to x by applying the partial derivative operator to the x and y components (see Quote A4). It is unclear whether this mistake is a simple slip of the tongue, and Alex did indeed mean x when he said y, or if he really did mean y and only realized the error when the interviewer questioned his statement.

If this is a simple slip of the tongue, it is noteworthy because this error may indicate that while Alex may consciously understand that both the x and y components of the vector field can depend on both the x and y variables, because he has already begun the process of distinguishing the two objects, he still subconsciously has a strong association of the x component with the x variable and the y component with the y variable, impeding his ability to separate the variables and components. As an isolated incident, this would not be so, but Alex has made the same mistake twice during the interview, and he has demonstrated other signs of confounding variable and component even after showing evidence that he is once again beginning to separate variable and component.

He then notices that the y component of the vector field is positive at certain points and negative at others, whereas the x component is always positive. Alex uses this information to think about how the x component of the vector field is always positive, but the y component changes. He specifically notices that the y component changes from negative to positive direction at the line $x = 0$, where the vector field is parallel to the vector $\hat{x}$:

> *A: Wait a second, these are pointing you know sort of in terms of y, this is pointing down and now it's pointing up. After we get to the zero line so that doesn't really change here um for for "x" [finger quotes] but y kind of does change [drums fingers] Yeah, I suppose if I was thinking about you know these two things that's probably how I'd think about it. The vector field.* (A14)

Alexs strategy of focusing purely on the change in sign of the y component, rather than the general idea of changing, seems to help him make sense of the idea that the derivative has direction. This approach gives him something more visually obvious to focus on than trying to find a semblance of slope in the vector field plot, as he tried to do when he was first given the visual. He seems to have a better understanding of how the y component can change with respect to x, and he is fully recognizing the y variable and the y component as separate objects. He has also begun to accept that the derivative of the vector field object is another vector field, a conclusion hes struggled to wrap his head around throughout the interview. He still struggles with the idea of differentiability and the idea of direction in the context of a derivative:

A: I mean, I guess, I guess in the way that I've I sort of guessed that this is linear on both sides, that sort of makes a problem if this is a sharp point here, cause that's sort of hard to define what a derivative is at a sharp point.
I: That's true. So it's probably not that.
A: So if it's not that yeah, if it's nice and round then it looks like that would be if um zero slope cause it's gonna you know if it's something like... (A15)

Here, Alex seems to be making sense of the idea of change in direction by associating it with the idea of a single-variable function changing from increasing to decreasing or decreasing to increasing at a critical point in calculus. He recognizes that because the y component changes from negative to positive, there must be either a point where the derivative is undefined, or there is a point where the derivative is zero in order for the direction to change. He assumes that the derivative is likely defined everywhere, so rather than a "sharp point" or point where the derivative is undefined, the direction changes at a point where the slope is zero:

A: I'm thinking about this, this slope of at, so I'm thinking of the derivative of this of F_y with respect to x. We were saying this is gonna look you know something like this. It might look a little you know more curvy but um [drums fingers] to me this looks like this has got a positive slope everywhere. But that that this the change in this is always positive.
I: Does that agree with the picture?
A: Um the y component is very negative negative negative. It's starting very negative and getting less and less so. it's not–it's zero then it's increasing right. So um yeah so I suppose that agrees with the picture. At this point y is going to be changing. Yeah. I mean I have to I have to consider just kind of you know if I was just looking at a single arrow it would be kind of impossible to think about that right? (A16)

Here, Alex again reinforces his association between the vector field having a positive component and an increasing function, but now hes begun to consider the fact that not only does the direction of the y component go from negative to positive, but the change in the vectors y component is gradual and always positive as the x variable changes. Alex now recognizes not only that the y component changes, but the change has dependence on the x variable, not just the y variable. He draws the "derivative" of the vectors as arrows on the vector field map (see Figure 14), indicating the change in the vectors with respect to the x variable.

The interviewer asks Alex about the entire vector field, or the "whole thing," and Alex appears to have returned to his initial idea of the partial derivative of a vector field, applying the partial derivative with respect to x to each component. He draws arrows on

Figure 14 that show the partial derivative with respect to x at each of the points hes chosen, but Alex still struggles with the lack of a clear physical example of the phenomenon, and how this concept relates to what he already knows:

$$\text{\textit{A: I dunno, I dunno what this is like I guess, I don't know what this would like describe ph-I mean, okay, so it seems like this now is gonna be pointing like a little bit like this more? So, it seems like we've taken, and this field looks like it definitely has some curl but it doesn't look like it really is diverging? I guess those sort of things, so somehow it looks like I've taken something that that had curl but no divergence and given it some kind of negative divergence.}} \tag{A17}$$

Even though hes accepted that the derivative of a vector field is another vector field, he struggles to find a meaningful relationship between the vector field and its partial derivative. In particular, the derivative he has drawn (see Figure 14) indicate that the derivative he has found is a vector field that behaves very differently than the original vector field. The original vector field has zero divergence, but the derivative has at least one clear "sink," giving it a negative divergence in at least one point, so Alex is hesitant to accept his ideas without an associated physical meaning:

A: Like I understand sort of the process I've done so far, I just don't understand like what I would use this to like to describe I guess if I was like trying to think about useful like if I had a vector field...
I: Right (A18)
A: Some kind of I dunno, I guess now I'd be thinking about maybe about like sort of like a temperature gradient I guess? I don't know what this would be telling me I guess.

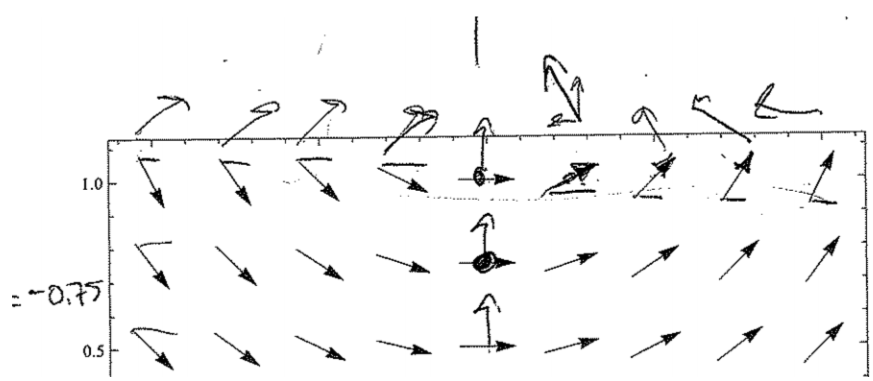


Figure 14: The provided vector field map with added vectors showing the derivative at each point of the vector field.

Here, Alex again brings up the idea of usefulness, which hes expressed throughout the interview is an important quality of a derivative or derivative-like operation to him. Alex seems to believe that a derivative needs to provide insight into the behavior of the original function, and he does not see that happening with the quantity hes defined, and he cant find a physical example to prove to himself that the derivative is correct. He cant seem to come up with a good physical example, and says that he wants to try to use Mathematica to get a better visualization of the derivative hes just found.

4.2 Bailey

When prompted to explain his thoughts on a vector field derivative, Bailey immediately brings up the idea of gradient, noting that a scalar field does not inherently have a direction like a vector field does:

> *B: So we're like, your derivative of a vector field which is the gradient if I'm-if I recall right.*
> *I: Mhm*
> *B: Um, and that gradient is going to have a direction um uh in terms of like if you have a vector field in like the x, y, and z [points to either side of him, then up] then your your derivative is gonna have your gradient is gonna have direction.* (B1)
> *I: Okay.*
> *B: So it tells you in what direction your vector field is changing.*

Despite incorrectly claiming that the derivative of a vector field is the gradient, Bailey understands that direction is a defining factor in a vector field, so the derivative of a vector field needs to include an element of direction. However, since Bailey is thinking specifically about the gradient rather than a derivative in general, he possibly confounds the idea of the derivative having a direction with the gradients property of showing the direction of greatest increase. He says that the "gradient" or derivative of a vector field, points in the direction that the vector field changes. Bailey seems to understand the idea of gradient and its relation to the direction of greatest change, but he is misremembering the exact definition. More importantly, Bailey is misremembering the object the gradient operation is applied to. It is also possible that Bailey is incorrectly referring to the derivative of a vector field as the gradient.

The interviewer presents a vector field graph as part of the interview protocol, and Bailey shifts from thinking about the gradient as the derivative of a vector field to thinking about divergence. He specifically implements the box method he learned in the Paradigms class by drawing a box around a certain point on the graph (see Figure 15). Bailey also says that each vector has a "certain amount" in the x and the y directions, and draws lines showing the projections of each vector onto the basis vectors, also shown in the figure. He explains how this method is used to estimate the divergence of a vector field at a particular point:

B: So, like if you look at like, how much is coming in that box in terms of just the x component of each arrows, then you look out. So if I make my box like right here, [draws lines] so like this those vector x right here, we can see that they're a lot shorter, so those at this point the vector field is less um has less like strength in the x direction than it does in the y direction. I think. So, the way I see it is that like youre your, I don't know if it's the derivative, but like it's... it's negative, for the x right. (B2)
I: Okay.
B: Um whereas the y um if we pick this one and this one, we can see that like this value is much greater than this one.
I: Okay.
B: So, in that case, it would be positive.

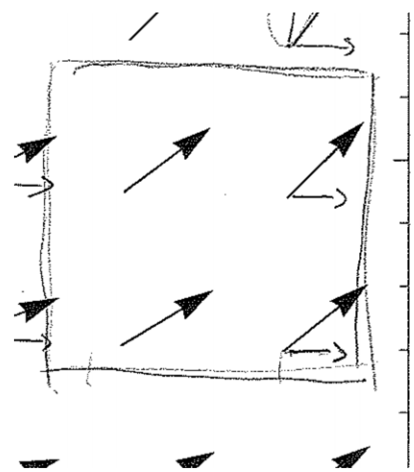


Figure 15: Bailey's box to estimate the divergence of the vector field

Bailey is using the box to visualize the "strength" of the x and y components of the vectors, like a tool to estimate the magnitude of each component. He seems to be saying that the vectors point more in the direction of the horizontal lines than the vertical lines, and he uses the fact that the magnitude of the x component of the vector field decreases to justify his claim. He seems to be saying that the magnitude of the y component increases as the x variable increases, but the magnitude of the x component decreases as the x variable increases. The box tool seems to help him visualize the change by looking at how the vectors cross the lines.

Bailey also begins thinking more broadly about vector field derivatives other than gradient and divergence:

> *B: And then the idea of the derivative of the vector field at that point with respect to x. I don't know if they're really the same thing, I think that like I picture the same thing cause here we're looking at how much is here, and how much is here after.*
> *I: Okay.*
> *B: Um so there's the same idea of change.* (B3)

Here, Bailey is using the concept of divergence to find a visual representation that clearly shows that the vector field is changing, and that the change in the vector field depends on the position. He recognizes that divergence is not the same as simply taking the partial derivative, but divergence shows a way the vector field changes with respect to position.

Bailey also recognizes that the derivative of the vector field has both a magnitude and a direction, and emphasizes several times that there is a directional element to the derivative:

> *B: Yeah, so like the derivative of um the vector field at that point...*
> *I: Mhm*
> *B: Is gonna have a magnitude with respect to x which I can't, I don't know what it is, but it it's gonna have direction [follows the way the vectors point with hand] and like, since it looks like kind of like a circle or um um vector field like [makes circle with hands] your direction of your derivative of your gradient is gonna be pointing towards like in that direction somehow like here. [moves pencil from center of map towards top right corner]*
> *I: How did you figure that out?*
> *B: Because your next value on the vector field.*
> *I: Mhm*
> *B: Is gonna be like pointing more in that [points to upper right of paper] direction. [laughs]* (B4)

In the above quote, Bailey specifically mentions that the vector field looks like a circle, and seems to be attempting to explain that the direction of the vectors is changing to be more positive in the y direction, so the derivative must be pointing in the direction the vectors are moving. He does not draw much on the paper, but points toward the top of the paper to show where he thinks the derivative would be pointing at a particular point.

He also begins to refer to the derivative as the gradient again, and attempts to explain where the gradient points in this context:

B: ...Yeah your vector field is like, well it looks like in magnitude it's not changing so that's good, but like the direction is constantly changing, right?
I: Yep.
B: So you have like at each point at each point you're gonna have um a direction that is gonna tell you like the gradient of it is gonna tell you that the way it's changing it's kind of like it's direction is gonna be changing. [makes circles and points with hands] (B5)

Here, Bailey recognizes that the derivative must have a magnitude and a direction. Because the example vector field used has constant magnitude, Bailey explains that the only change comes from the direction, so the derivative must have an element of direction. He tries to explain precisely where the direction of the "gradient" points, but cannot find the words to explain. In the process, he seems to realize that gradient and derivative are not synonyms:

I: You said "gradient" there, is that the same thing in this case as derivative? Or is that something different?
B: Um well, I think the gradient is like the um oh wait is the gradient.. well I think the gradient is like when you take the derivative, like the partial of the vector field with respect to each variable, right?
I: Okay.
B: So, if you want just a derivative of your variable with respect to x, I guess in that sense they're different. (B6)

Here, Bailey seems to be confusing the notion of derivative with gradient, and he appears to mistakenly believe that the gradient is defined to be the partial derivative of a vector field with respect to each variable. He says that the "gradient," as he has incorrectly defined it, is different from a partial derivative. He seems to be saying that the gradient involves partial derivatives of all independent variables, which distinguishes the gradient from a general partial derivative.

Earlier in the interview, Bailey says "so if we call this vector field 'h'." This is incidentally the same name he assigned the scalar field in the first part of the interview. He may be attempting to find an analog between vector fields and scalar fields to help him answer the question of what a vector field derivative would look like, and giving the fields the same name will help him with that task. He seems to be confusing himself in his attempt to generalize the notion of partial derivative to a vector field, and his thinking seems to change from seeing variable and component as distinct to interchangeable, by only focusing on the x component:

B: ...So I guess this the derivative of the vector field with respect to x would tell you how the x component of your vector field is gonna be is gonna be changing in the x direction.
I: So it only tells me how the x component is changing in the x direction? Or does it also tell me about the y component?
B: Umm about the y component? [pause] Um I guess, I guess it tells you also how the y component is changing? I don't know. Because if you have a function of h that's like x and y that's like a function where like um when you take the partial of h with respect to x you still have a y in [the derivative], right? That happens. Then you would have um like y would be influenced, like would have an influence in how um your vector field is changing, but I don't think it's going to tell you how y is changing. (B7)

In the block quote above, it is unclear if the "h" Bailey refers to when he says, "if you have a function of h thats like x and y" is the vector field or the scalar field. He seems to be most likely referring to a general function of two variables. He notes that the partial derivative of a function typically is a function of ("influences" as Bailey says) all independent variables. He says that y would influence the derivative, but the partial derivative with respect to x would not show how the y component changes. He refers to both component and variable simply as "y" and begins to show evidence of confounding.

The interviewer specifically asks how one can find how the y component changes with respect to x, and Bailey becomes noticeably flustered, shown by an increase in stammering, and he apologizes for lack of understanding, saying:

B: Um [pause] Um yes I think. I'm not entirely sure.
I: That's okay. Wanna think about it a little more?
B: Yeah, um so like, when when we talk about change I try to always see like it's hard to hard to think about like when we talk about like partials or like cause it's um...
I: Yeah sure
B: Really small so like, like between like this point and this point, for example. (B8)
I: Mhm
B: So I can see that my y was $-.1$ and then it's 0 here. So I can tell what my y change in y is. But I don't know if I can tell that just by taking the derivative of this with respect to x cause that's gonna give me, that's gonna give me a value um I I don't know, I'm sorry.

Here, Bailey seems to recognize that the y component is changing with respect to the x variable, but he struggles to understand a relationship between existence of change and an existence of a derivative. More precisely, Bailey is unsure if the partial derivative of the vector field with respect to x would give information about the change in the y component with respect to the x variable. There is some aspect of a derivative that stops Bailey from considering a derivative as a representation of change in the y component. As he elaborates further, he clarifies precisely what aspect of a vector field derivative is troubling him:

> *B: It's just, I'm having a hard time trying to picture um how um from what we learned like how this like how this y is changing. That I know just by looking at it right, but if I take the derivative, the partial of h with respect to x with like the derivative of h with respect to x, then um it's gonna... [quietly] holding y constant right? I'm gonna have two different values of y in there.* (B9)
> *I: Can you say that again?*
> *B: So, I think I'm getting kinda myself tangled with the things we learned, but like um if we take the derivative of your vector field h with respect to x, um that implies right that you're holding y constant when you take that derivative.*

Baileys confusion here apparently stems from "two different values of y" in the vector field, a problem that is not present when taking partial derivatives of scalar valued functions. He understands that a partial derivative of x must hold y constant, but does not see that holding the y variable constant does not mean that the y component is also held constant. Bailey is troubled by the notion of two different objects having the same name, and believes incorrectly that "y" must refer to both the y variable and the y component. It is the notion of "holding y constant" that is confusing Bailey, leading him to believe that despite evidence of change in the y component as the x variable changes, the partial derivative of the vector field with respect to x would give no information about the behavior of the y component, because y is held constant.

The interviewer asks Bailey if the same would be true if he took the partial derivative of the vector field with respect to y, which Bailey affirms, adding "I think that is what you have to go on. Like you're only gonna be able to know like how fully your vector field is changing when you know how it's changing with the x and the y." He does not elaborate on that statement, and is quiet for several moments until the interviewer asks about divergence. Bailey says "I'm not entirely sure how those two are connected exactly," referring to divergence and the general partial derivative hed just discussed.

4.3 Cam

Cam initially appeared bewildered by the prompt to take the derivative of a vector field. He furrowed his brows and scratched his head, saying, "Derivative for a vector field. Alright," before pausing to think about the prompt.

Figure 16: Various vector field derivatives, written by Cam.

He then brings up divergence, curl, and gradient, and the importance of considering direction in a vector field:

> *C: Yeah, so if I wanted to find the derivative of a vector field I would use, I would use something else. I would use [pause] for example, I could use curl [writes $\nabla \times f$], I could use divergence [writes $\nabla \cdot f$], or I could use gradient [writes ∇f]. Each of these incorporate the different bases' directions into the derivative, which I would need for say if we were doing a vector field.*
> *I: Okay.*
> *C: Yeah, so [long pause] whereas if you were doing a scalar field it's a lot simpler um you don't have to incorporate um different directions of, I guess...* (C1)

He also writes down the expressions for gradient and curl without using vector notation (see figure 16). He also writes ∇f, then crosses it out later in the interview. He recognizes that the operations curl, divergence, and gradient involve basis directions in their calculation, but he struggles to draw a conclusion from that bit of information. He adds that a derivative of a scalar field is a simpler operation, because directions are not involved, indicating that he understands the importance of direction in taking the derivative of a vector field. Cam does recognize that curl, gradient, and divergence are not strictly partial derivatives themselves, but rather operations that involve partial derivatives, saying, "they're all different operators that– operations that deal with partial derivatives."

Cam attempts to expand further on what he means by operations that deal with partial derivatives, saying:

$$\nabla\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

Figure 17: The formula for gradient

> *C: I'm fairly confident that these are needed, um, if you want to do an operation with, um, a vector field. For example, you want to take, um you want to find the magnetic field from the magnetic vector potential. You can just take, a you couldn't take the gradient, oh wait a minute. I can't use the gradient, you couldn't take the gradient of that [crosses out ∇f] because because, you know that the magnetic field is also a vector field right, so the gradient will give you um a scalar field from the vector field. And if I wanted to find the magnetic field from magnetic vector potential, I would use um, I believe it's the curl.*
> *I: Okay.*
> *C: Um [pause] its, its still a derivative operation, but its not a [mumbles something] its not a derivative as explicitly as the gradient is.* (C2)

Here, Cam recalls the example of magnetic field and magnetic vector potential. He is possibly trying to make sense of the unfamiliar question of a vector field derivative by using a familiar physical example as a guide. By thinking about the magnetic field and magnetic vector potential, he correctly remembers that the magnetic field is a vector field, so the gradient is not relevant. His justification is incorrect, as he states that the gradient of a vector field is a scalar field. He knows that the gradient of a vector field is not a vector field, but does not realize that the gradient is not defined on a vector field. He remembers that the magnetic field is the curl of the magnetic vector potential. He then reaffirms his previous point that the curl is a type of derivative, adding that the gradient is a more "explicit" derivative.

The interviewer asks, "Gradient's what you wrote down before, right?" Cam then writes the formula for the gradient shown in Figure 17. He explains that hes written down the formula for gradient, and that it is a scalar field. Cam then states that he got side-tracked, and the interviewer repeats the prompt, saying "if I have a vector field, and I want to take the derivative of that vector field..."

In his response, Cam switches from using algebraic representations of gradient, divergence, and curl, to a more verbal approach, saying:

> *C: Take the de-okay that would be applying the, okay so that would be the rate of change of the vector field [pause]. But the rate of change of what, see that's the thing, I, you can't, I don't think you can take just a plain old derivative of a vector field. You'd have to do some other operation to it.* (C3)

In the above quote, Cam realizes that he is looking for the rate of change, but he seems to be unsure exactly what is changing. Its unclear if he is thinking about magnitude and direction as possibly changing, or if he is simply puzzled by the entire idea of change in a vector field. He does not know if one can take a "plain old derivative" of a vector field, and likely recalls curl, divergence and (incorrectly) gradient as being operations that involve partial derivatives on a vector field.

The interviewer gives Cam the example vector field (Figure 4) and asks him to find the derivative of the vector field represented in the graph in the x direction. Cam is silent for several moments, staring at the graph.

With a graphical representation to look at, Cam seems to change his mind about not being able to find the rate of change of a vector field:

> *C: I would say yes, actually so you would be able to find [pause] the rate of change the rate of change of the [mutters something, moves hand to the right] field pointing in the x direction.*
> *I: Can you explain what you mean?*
> *C: Yeah, for example as the arrow–as the x position shifts to 1 the arrows are pointing less and less in the x direction. Okay, starting from zero. Going from zero to one the arrows are pointing less and less in the x direction and more and more in the y direction, which leads me to believe that the rate of change er okay, so that must mean that, okay so since the total change in x is going negative at each at each* (C4)

Here, Cam seems to be having difficulty expressing his thoughts in words, but he appears to be trying to say that since the vector arrows are pointing in different directions, the vector field is changing in the x direction. He brings up that the arrows are pointing less "in the x direction" and more in the y direction, and mentions that the total change in x is going negative at each. It is unclear whether Cam means there is negative change in the x direction or the x component is decreasing, or if he is incorrectly stating that the x component is negative. Given the context, the most likely interpretation is that Cam is saying the rate of change in the x component is negative.

Cam observes that the magnitude of the vectors in the vector field is constant, and explains that the x component has a maximum length:

C: And... I'm assuming that these are all the same length. They look the same length.
I: They're about the same length, you're right.
C: So this means, since this [pointing at a horizontal vector on the line $y = 0$] one's parallel, you're not gonna get any more in the x direction than that. This has no y component. It's all exclusively in x. So as you move, as it moves along to the next the next arrow [points at arrow to the right of horizontal arrow], it looks like there's a slight inclination point, which means that if these arrows were the same length, there can't, there can't be any um it can't be, it can't have the same magnitude in the x direction. Um and that goes on and on and on til say the last one where it's a significant portion in the y direction and not as much in the x. (C5)

In the previous quote, Cam observes that the points where the vector field is parallel to the $\hat{x}$ vector has the largest x component, since the magnitude of the whole vector field is constant. He uses this to explain that the magnitude of the x component decreases from the maximum at $x = 0$ as the x component increases in the positive direction. At larger y values, Cam observes that the y component has a much larger magnitude than the x component.

This observation that the magnitude of the x component decreases leads Cam to make additional operations about the y-variable:

C: So that leads me to believe that there could be a change in f or a derivative in the x. But then the rate of change in the x also depends on not the necessarily the x direction [points to the right] that you go but it also depends on y direction [points to top of the paper] because for example, arrows up at the top up higher, say $y = 1$ have more [points to a vector at the top of the page], they're pointing more [puts pen on table vertically] in the y direction than those at the bottom. (C6)
I: Okay.
C: That makes me think that the derivative, er the rate of change of the arrows is also dependent not only on the x position but also dependent on the y position.

Here, Cam notes that the direction the vectors point is different on different locations of the graph, so he conjectures that the partial derivative of the vector field with respect to x has dependence on both the x and y variable. This shows that Cam is starting to think more about how both variables influence the change in the vector field, rather than just focusing on the x variable.

Cam begins to discuss infinitesimals, and their role in finding the partial derivative of a vector field. He also begins to use invented notation and terminology:

Figure 18: Cam's expression showing the partial derivative of the vector field.

> *C: And this is only in the x–okay so, yeah the rate of change in so, let's just call that a function field, x um is some some function of x and y. Of course, it's also infinitesimal so it needs to be a dx and then a dy [writes expression shown in Figure 18].* (C7)

In the above quote, Cam refers to the rate of change as a "function field." A "function field" is a term in algebraic geometry referring to the field of objects that can be interpreted as rational functions, or a term in abstract algebra referring to a finitely generated field extension with a finite transcendence degree. Cam is most likely not referring to either of those definitions, but it is unclear exactly what he means. He could have been trying to say that the x component of the vector field is a function of x and y.

In Figure 18, Cam seems to have been writing that the derivative of the vector field with respect to x is a function of x and y. He originally names this function f, but after the interviewer asked for clarification, Cam renamed the function g. Since the derivative is a function of x and y, Cam says the derivative function, g, must have the terms dx and dy.

The interviewer asks for clarification about the infinitesimals, and Cam says "[dx, dy] is just saying that somewhere in here there also needs to be infinitesimals relating the two sides." It is important to note that he originally wrote a comma between the dx and dy, likely trying to indicate that the dx and dy are somewhere in the function g, but they may not be in the same terms of the function. He appears to be confusing differential notation with the rate of change definition of derivative. He is also unsure of how to represent an equation with differentials when the function is unknown. He also does not seem to understand how to vary one variable at a time. He is clearly taking the partial derivative with respect to x, but includes differentials dx and dy. Cam does not seem to understand the distinction between "2-ness," with two variables, and "1-ness" with one variable in the context of taking derivatives.

Though he appears to be confused about the meaning of taking the derivative, Cam is acknowledging that the x component of the vector field is a function of x and y, and therefore can be differentiated by both x and y. Here, Cam has already separated the idea of variable and component.

The interviewer asks if "dx, dy" is multiplied by the function g:

> *I: But this isn't just like multiplied or something?*
> *C: Um [pause] no it's not...*
> *I: No*
> *C: No it's not just multiplied. There's gonna be an infinitesimal related to... there should be. [pause, looks puzzled] but not necessarily because dividing an infinitesimal by an infinitesimal means that you know so I wouldnt need an infinitesimal here [crosses out "dx", "dy"] because if you divide an infinitesimal by an infinitesimal um you get a real not necessarily real but physical number. Not an infinitesimal. Yeah that um if that if that if that's helping you I don't know.* (C8)

Here, Cam is trying to explain his notation, and in the process, he seems to realize that having the quotient of differentials on the left side of the equation means the right side does not include infinitesimals. He crosses out "dx, dy" because the left side of the equation already has a "dx" in the denominator. In writing "dx, dy" initially, Cam specifies that both the x and y variables are being differentiated, but only "cancels" the differentials with dx. Cam may have realized that the y variable is not being differentiated, so the "dy" should not be there, but this is not clear. He may also be struggling with the "2-ness" on the right-hand side of the equation, and cancelling the two dimensional differentials with one differential on the left-hand side.

Cam's phrasing of "not necessarily real but physical number" to mean not infinitesimal is also interesting. He may be trying to make what he's written as general as possible by acknowledging that the derivative may be complex. Because the vector field given is certainly a vector field on the real numbers, Cam may have been transitioning to a more general case. If this is true, this seems to have been unintentional on Cam's part, as the interviewer then asks if Cam is referring to the derivative of the function in the graph with respect to x, which Cam affirms. The additional generality would likely be due to Cam being unsure of the exact function being graphed, and he does not want to claim that the function is on the real numbers without confirmation. It is also possible that Cam is simply confused, and is not sure how to express his thoughts in words.

Cam restates a previous statement that one can't take a derivative (see Quote C3) but acknowledges that there is clear evidence of change, saying, "just looking at the graph makes me think that there certainly is a rate of change occurring." He explains what is meant by the rate of change:

I: What does it mean when you say rate of change?
C: Rate of change means derivative in the x direction.
I: Why is a rate of change the same as derivative here?
C: Um
I: How are you using those words?
C: Right so. The derivative, that's the that's the physical meaning of rate of change. So rate of change is let's say that, in this case, rate of change is how rapidly is this arrow going from parallel to the x axis to um not parallel to the x axis. How rapidly is that shift occurring? Um that would give me the rate of change. (C9)

Here, Cam says that the derivative and the rate of change are the same concept, which seems to contradict his previous statement that there is no "basic" derivative, but there is a rate of change. Cam has also introduced a new idea of derivative that does not involve infinitesimals at all, but is instead more of an approximation of the derivative using very small increments. Interestingly, Cam's approach measures the change in x with respect to a particular change in F:

I: What does rapidly mean here?
C: Um with how small of an increment do I need to go in order to get let's say a given amount of change. (C10)

In the above quote, Cam is clearly saying that the derivative, or rate of change, shows the "amount" of change in the x direction needed to achieve a given change in f. He seems to change his approach immediately after saying the above quote, after prompting for clarification. He also begins focusing only on the change in the x component, and ignores the y component:

C: So let's just say that I go .1 from 0.5 in the x um how much does the angle of basically this shift. How much less does this arrow point in the x direction at this increment, or at this this point than it does at this point?
I: Okay how would you quantify how much less it points?
C: Um that would also be equivalent to the um distance in the x direction. [draws dotted line near vector in figure 19]
I: And when you say distance you sketched the triangle there...
C: The triangle yes. I'm trying to break it down into its, trying to break down the vector into its different components. (C11)

In the above quote, rather than determining how much change in the x direction is needed to produce a given change in the function, Cam uses the more traditional method of estimating a derivative. He finds how much the direction changes by finding the length

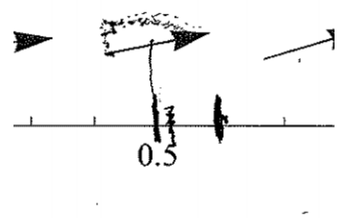


Figure 19: A triangle on the vector, seemingly indicating the magnitude of x and y components

of the x component. Cam has previously expressed understanding that the magnitude is constant (see Quote C5), so he is possibly trying to quantify the change in direction with magnitude of the x component because the change in y magnitude is equivalent. This may also be evidence that Cam is beginning to confound the x variable with the x component, despite previously showing evidence of understanding that component and variable are separate objects. He does acknowledge that he is "breaking the vector down into its different components." However, it remains to be seen whether he is doing this because he acknowledges the constant magnitude, or if he is confounding the variable and component.

When asked for clarification, Cam seems to be struggling with finding how the vector field is changing:

I: So you've only been looking at the x part of that vector?
C: Yes I'm only looking at um yes. Only looking at the x part of that vector.
I: Okay.
C: And how rapidly it changes. Um [pause] wait. [long pause] Maybe I couldn't then. Okay. So dF okay so the function for our vector field dF um dx is the rate of change of F as x is as x is changing. But F, F isn't changing. Yeah so now I'm, now I'm thinking that you couldn't use a basic derivative. (C12)
I: Can you tell me more about that?
C: Um well F is changing. F is fffff okay the, the equation for the vector at each point is changing. That. That uh the equation for a vector at each point is changing. So plug in different points to your–to your equation [long pause] So it's this that is tripping me up. So the rate of change...

Here, Cam clarifies that he is measuring the change using the x component only, but realizes there is a problem with his approach. He again says, "you couldnt use a basic derivative," a term hes used several times in the interview (See Quote C3) . He also says that the vector field is not changing, then changes his mind and says the equations of the vectors are changing at different points. The contradictions and frequently changing statements indicate that Cam is very confused. He likely saw a problem with his previous approach, and without another clear approach to the problem, Cam is likely trying to make sense of his ideas by saying whatever he can come up with.

Cam again says that he cannot take a basic derivative, and begins to consider the implications of the object being a vector:

I: What's a vector?
C: Oh it is a vector. I got vectors. Hopefully. It is a vector so that means yeah it, it just doesn't work you can't, you're not saying what you're not saying like a direction associated with that, or how it's changing, so yeah that would lead me to believe that you can't take a vector field, no you can't take a basic derivative of a vector field. In a certain direction. (C13)

Here, Cam attempts to justify his claim that he cannot take a "basic" derivative of a vector field. He seems to be saying that without a specific direction, there is no way to accurately measure the change in the vector field. He says, "youre not saying like a direction associated with that" as he is explaining why he cannot take a derivative, but moments later, Cam says "you cant take a basic derivative of a vector field in a certain direction." Cam seems to be using the word "direction" to refer to two different ideas. It is not clear what those are, but one possibility is that when he says there is not a direction associated, he is referring to the components. If that is what Cam means, then his comment in the above quote may mean he does not know if he should be taking the derivative of the x component or the y component.

Another possible explanation is that Cam is trying to say that he does not have an explicit basis with which to find the direction. Cam has mentioned basis directions previously (Quote C1) so he may be acknowledging the role the basis plays in finding a vectors direction:

I: So when you say basic derivative you mean something like this? [points to expression in Figure 18]
C: I mean yeah something like this. [points to expression in figure 18] Yes. I think. Let's see how much I remember. (C14)

Here, Cam explains what he means by a "basic" derivative (see Quote C13). He says that the expression in Figure 18 is an example of a basic derivative, so Cam seems to be saying that a basic derivative is a scalar valued derivative.

The interviewer then asks Cam if there are other derivatives of vector fields, and Cam mentions curl and divergence. He says that curl and divergence are "different arrangements

Figure 20: Scribbled out expression. The writing is $\nabla \times f = \frac{\partial f}{\partial x}\hat{x}+$.

of derivatives that allow you to find different things about [the vector field].”

The interviewer asks Cam if there are other derivatives that “live within” divergence and curl:

C: [pause] Yes that okay yeah yeah you just have to associate a direction with it. So for example the curl of f is um so I believe this is right df dx $\hat{x}$ [writes expression in Figure 20, see caption] + oh no that would be gradient [scribbles out writing]. I don’t remember curl. Um yes I so I don’t remember the exact equation for curl but I know that it incorporates different partial derivatives of F um in its equation. (C15)

Cam initially writes “$\nabla \times f$” the correct notation for curl, then begins writing the formula for gradient of a scalar valued function. He catches himself and scratches out what hes written. Cam then says that curl “incorporates different partial derivatives of F.” He seems to be starting to understand that there is some distinction between the components of the vector field that would change the value of the partial derivatives, but Cam does not realize that the components are represented by different functions entirely. He so far only recognizes that the components are needed to find the derivative.

Cam clarifies what he means by direction after the interviewer asks for an example:

I: Can you give me an example of one such partial derivative?
C: Okay, so I’m not recalling exactly what-which specific ones um are incorporated into the curl um I don’t recall exactly um but we say in the $\hat{x}$ direction [writes expression in Figure 21] um so this should be how much is the function F changing when an increment of y is taken so from 0 to 1 say. (C16)
I: Okay
C: Um in the $\hat{x}$ direction so how much..

In the above quote and Figure 21, Cam provides a correct example of one of the partial derivatives that make up the curl operation. He explains after the interviewer asks for clarification that the invented notation in figure 21 refers to the partial derivative of the

Figure 21: Can's expression for the partial derivative of the x component of the vector field with respect to y.

x component with respect to the y variable. He specifically says, "so my two different points that I'm taking for finding my derivative are moving in the y direction, but when I'm looking at the um vector field exclusively, I'm only looking in the $\hat{x}$." Cam does not seem to recall the standard notation for the x components and y components of a vector field, and uses the notation above.

It is also interesting that Cam has chosen a partial derivative taken with respect to y instead of with respect to x. Cam may be interpreting the prompt "partial derivative in the x direction" to mean "partial derivative of the x component." This could also be a coincidence, as he has previously stated that the "direction" (i.e. component) was not specified in the prompt, and Cam instead chose the first partial derivative that he recalled in the formula for curl.

Cam finds a numerical approximation for the partial derivative shown in Figure 21, and the interviewer asks if he can do the same process with the y component:

> *I: But could you have found $\frac{df}{dy}$ but with $\hat{y}$ here? Is that something that's possible to find from this field?*
> *C: [long pause] Yes yeah.*
> *I: How would you do that?*
> *C: instead of using the x difference in the vector, um you would use the y, how much the y change. You'd still pick your points moving in the y direction, but you'd just change that so that...* (C17)
> *I: How would you know that you'd still want to change in the y direction?*
> *C: Because you're assuming that this is–it's still $\frac{df}{dy}$ you're still changing f with a small change in y, so this would be $\frac{df}{dy}$ um in $\hat{y}$ [writes expression in Figure 22]. So yeah you're still um moving in the y direction you're still changing your points in the y direction.*

$$\left(\frac{\partial f}{\partial y}\right)\hat{y}$$

Figure 22: Cam's expression for the partial derivative of the y component with respect to y.

Here Cam, after a moment of consideration, claims that he could take the derivative of the y component of the vector field, indicated by the "$\hat{y}$" in the expression he wrote in Figure 22. Cam no longer has an issue with the presence of change in the vector field, and he has found an understandable notation that allows him to find the different partial derivatives one can take of a vector field.

Cam also does not have difficulty recognizing that the x and y components each depend on the x and y variables. He also seems to understand that both the x and y components are changing. Cams difficulties seem to come from having many different objects that could be differentiated in the vector field. More precisely, he does not know whether he should be looking at the x component or y component. Cam also does not seem to recognize that the derivative would have a direction, or if he does, Cam appears to believe that this is an error. Consequently, he does not consider the possibility of taking the partial derivative of both the x and y components. Moreover, Cams approach seems to aim to remove the directional element to the change, seen here:

> *C: Yeah um that makes me think that in order to take a derivative in a vector field, you need to have a specific–a more specific direction associated then just*
> *I: Can you tell me what you mean by more specific?*
> *C: Yeah yeah. If I can find the words. The uh the so for example if I wanted to just say oh I'll take $\frac{df}{dy}$ and not have um a certain direction associated with it, if I took $\frac{df}{dy}$ um it's not defined what it's not defined how the actual vector itself is changing. You're not giving it a bound to measure how much the arrow say changes in that time. You're just giving it, what's the change? That you don't know. You need a certain direction associated with it.* (C18)

In the above quote, Cam explains that a vector field changes in different directions, so to take the derivative, one needs to know the direction, or component, that should be differentiated. He still does not recognize that the derivative of the vector field would itself

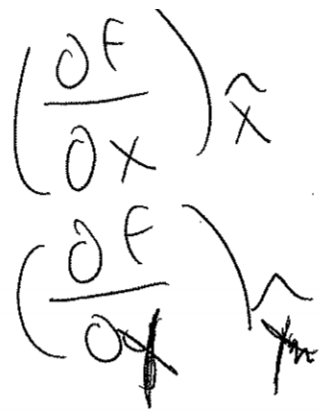


Figure 23: Cam's expressions for more possible partial derivatives.

have direction, but he recognizes that the y component can be differentiated with respect to each variable.

He explicitly writes the derivative of each component with respect to each variable:

> *I: Are there any other of these derivatives beyond what you wrote down there?*
> *C: Oh certainly um I could find the partial f of and then partial x and then just kinda and then I could find partial f partial y in the $\hat{x}$ [writes expression in Figure 23].* (C19)

Here, Cam correctly deduces that the x component of the vector field can be differentiated with respect to each component, as with the y component. From his previous discussion of curl, Cam seems to understand that these partial derivatives are combined to form derivatives of the entire vector field. Cam does not bring up any potential ways of combining them, or the type of objects these partial derivatives are. Because he chose, whether consciously or not, to find derivatives that produce scalars rather than vectors with direction, Cam did not realize that the derivative could itself be a vector and have direction.

Additionally, Cam seems to consider divergence and curl to be entirely separate from derivatives. He has previously referred to them as "arrangements" of derivatives, but does not bring them up again when asked about other derivatives:

I: Are there any others or is that it?
C: Well if you wanted to find more basis directions you could.
I: So if I had more directions I could get more?
C: Yes. (C20)
I: But without more directions there's no other derivatives that I could come up with.
C: None that are particularly useful, yes.

Here, Cam affirms that the number of partial derivatives of a vector field depends on the number of variables and components, or basis directions. Cam does not mention the arrangements of derivatives, or say that he could arrange the derivatives hes found in different ways to form new derivative operations. He seems to believe that the partial derivatives hes defined, and other objects defined similarly, are the only "derivatives" of the vector field, and anything involving them would be "arrangements."

5 Results

Here, we discuss the content of Alex, Bailey, and Cam's interviews, including the similarities and differences between the three students' approaches. We also propose answers to the research questions in Section 1:

1. How do students attempt to take a derivative of a vector field?
2. How well does Zandieh's framework and subsequent extensions (Zandieh, 2001) describe student understanding of vector field derivatives, and, in particular, is an additional extension necessary to describe vector field derivatives?

Questions 1 and 2 are addressed by introducing the new idea of confounding and unconfounding, describing the interviewees' difficulties with terminology, and using Zandieh's framework, along with Roundy and Emigh's extensions, to describe the students' processes in addressing the prompts.

5.1 Confounding and Unconfounding

Confounding involves treating distinct objects as interchangeable. In this study, we focus on confounding of variable and component, though other types of confounding may be present. Alex, Bailey, and Cam each showed evidence of confounding variable and component throughout their interviews, as well as evidence of unconfounding, or separating the ideas which they had previously confounded.

Both Alex and Bailey struggled with the idea of "holding y constant." Bailey seemed to consciously know that the y component and y variable were different, yet when trying to take a partial derivative with respect to x, Bailey did not know whether to hold the y

component or the y variable constant (see Quote B9). Alex, on the other hand, had difficulties conceptualizing a derivative with direction. He initially defined a vector-valued partial derivative (Figure 7), but expressed uncertainty about what a derivative that is a vector would mean physically. Instead, he considered what it would mean to hold y constant, and decided to only take the partial derivative of the x component with respect to x. Moreover, when Alex began to consider the partial derivative of the y component with respect to x, he initially claimed the derivative would be 0, because he was holding y constant (Quote A10).

Cam differed from both Alex and Bailey in that his difficulties with confounding were not related to holding y constant, but rather what taking a partial derivative "in the x direction" meant. Cam repeatedly said "you can't take a basic derivative" throughout his interview, meaning simply differentiating with respect to each variable (Quote C14), and emphasized the idea of direction. He eventually clarified that he interpreted "the partial derivative in the x direction" to mean "a partial derivative of the x component," which could be with respect to either x or y. Therefore, Cam's difficulties with confounding were not with variable and component, like Bailey and Alex, but rather were with variable and direction.

Despite some differences in what specifically Alex, Bailey, and Cam were confounding and why, they each took a similar approach to differentiating the vector field as a result. All three students used projections to isolate the magnitude of each component, and considered the change in the magnitude of just the x component of the vector. Alex and Bailey attempted to separate the y variable and y component to find something they could reasonably differentiate (Quotes A8, A13, and Quote B7), whereas Cam's interpretation of the prompt required him to only differentiate one component at a time.

Confounding naturally leads to an idea of "unconfounding," where students begin to recognize the previously confounded objects as distinct and/or unrelated in order to compute a partial or ordinary derivative. Alex and Cam displayed an obvious process of unconfounding, whereas Bailey was unable to complete the unconfounding process during his interview.

Alex struggled with finding the partial derivative of the y component of the vector field with respect to x, initially saying it should be zero. Upon further consideration, Alex noticed that the y component of the vector field was negative for $x < 0$ and positive for $x > 0$ (Quote A14). He had previously defined a function that output the magnitude of the x component with respect to x (Figure 12) and defined a similar function that output the magnitude of the y component with respect to x (Figure 13). This process of defining a function and drawing a linear graph helped Alex recognize the relationship between the y component and the x variable, and recognize that the y component and y variable are distinct objects.

An interesting aspect of Alex's interview was that Alex initially did not confound the x and y variable, but his doubts about a derivative being a vector object appeared to cause him to begin confounding. By defining a function that clearly showed the presence of change (Figure 13), Alex was able to accept his original idea, despite his discomfort with a vector-valued derivative. In this sense, Alex first did not confound variable and component, then began to confound, then finally unconfounded the two objects, as opposed to beginning confounded and unconfounding throughout the interview, as Cam appeared to do.

Cam's process of unconfounding took a different direction. Both Alex and Cam struggled with the idea of a derivative having direction. Alex did not understand why a derivative would be a vector, and Cam claimed having a directional element made it impossible to take a "basic" derivative. Cam's approach was to remove the directional element from the derivative by only considering one component at a time, and noting the dependence of each component on each variable. Cam eventually found four different partial derivatives that one could take of a vector field (Quotes C17 and C18), but did not conclude that the derivative itself had a direction as Alex and Bailey did.

5.2 Confusion of Types of Derivatives

The three students interviewed each struggled with using the correct language to describe the mathematical objects they were working with. The most common misnomer used by Alex, Bailey, and Cam was referring to the derivative of a vector field as a "gradient." All three students referred to a general derivative as a gradient at some point during their interview. This is likely either due to misremembering the definition, which is most likely the case for Bailey and Cam, or misspeaking, which is possibly the case for Alex. As a gradient is not actually a derivative operation on a vector field, but is rather an operation on a scalar field that results in a vector field. As the gradient is related to vector fields and is a derivative operation, the students appear to be conflating the ideas and believe that the "gradient" is the derivative of a vector field.

5.3 Use of Zandieh's Framework

All of the concept images in Zandieh's framework, as well as Emigh's extension, were used by at least one of the three students interviewed at some point in the interview process.

Narrowing was prevalent throughout all three interviews. The students each spent a significant amount of time determining what to hold constant and what to differentiate. The narrowing process seemed to overlap with the unconfounding process quite often. For example, as Bailey was attempting to determine what to hold constant, which would be considered "narrowing," he realized that he had "two different y" values (Quote B9). Although Bailey was not able to fully separate the y component and the y variable, he seemed to recognize that y was used to represent two distinct objects as a result of the narrowing process.

Cam and Alex had similar approaches to narrowing as Bailey. All three of the students had to consider not only what it meant to hold a variable constant, but also which variable/component to hold constant. In this sense, the process-object of *narrowing* appears to have some connection to confounding variable and component in vector fields. All three students opted to break the vector into components, and had some difficulty deciding what exactly they needed to hold constant. It is possible that confounding blocks students from beginning the narrowing process, or is an extra element that students need to narrow when attempting a vector field derivative.

The three interviewees also frequently used the *ratio* layer to understand how to take a vector field derivative. Alex spent a significant amount of time searching for a slope as he was finding a derivative. Moreover, the lack of a clear "slope" frequently caused Alex to doubt that his ideas were leading him on a correct path, such as when he initially defined the vector-valued partial derivative. Alex said he was looking for "what a slope would mean" for a vector field, and presumably a vector did not match his idea of a slope.

Later in the interview, Alex drew a linear approximation of the magnitude of the x-component (See Figure 12) which gave a graphical representation of a slope, which he could then estimate numerically. This process overlaps with the *function* process-object layer in Zandieh's framework. This shows that more than one process-object layer can be implemented at one time. The students in Zandieh's study also used more than one process-object layer at a given time (Zandieh, 2001). However, Alex's combination of the function layer and ratio layer is unique to multivariable functions, particularly vector fields, because Alex created a new function that would allow him to find the slope. This approach would neither be necessary nor feasible with a single-variable function.

Alex also showed the strongest preference toward graphical representations of the three. Not only did he choose to draw graphs of the functions he defined to estimate the slope (Figures 12, 13), but drawing a simple vector field graph was one of the first things he did after receiving the prompt (Figure 5). Neither Bailey nor Cam attempted to create a graphical representation of a vector field prior to being provided one. Bailey also primarily used graphical representations in the sense that he did all of his work from looking at the graph itself. He did not attempt to create his own graphs, as Alex did.

Cam used the verbal representation "rate of change" explicitly in his approach (Quotes

C3, C4), and described the rate of change as how much less the vector points in a certain direction given a certain amount of variable change. Cam and Alex both attempted a numerical approximation. Alex found an explicit number for his estimate. Cam and Bailey each referenced estimating the rate of change numerically (Quote C11 and Quote B4) by estimating the change in magnitudes of components, but neither were ultimately able find an explicit numerical value for a derivative.

Cam was the only of the three to include a discussion of infinitesimals, which would be considered the *limit* process-object layer of the framework as he attempted to understand the general behavior of the derivative of a vector field. Cam recognized that the change in the x component depended on both the x and y variable, and attempted to define a differential form using dx, dy to represent the infinitesimal change in each variable (Figure 18). Though his definition was ultimately incorrect, as Cam realized prior to scribbling out the expression he wrote, Cam made a distinction between "rate of change" and "derivative" that neither Alex nor Bailey considered, namely that a derivative involves an infinitesimal change.

Overall, Alex, Cam, and Bailey each used process-object layers and representations detailed in Zandieh's framework and its extensions, so it appears Zandieh's framework with Emigh's extension is useful in describing student understanding of vector field derivatives. However, Emigh's extension alone is not entirely sufficient to describe student understanding of vector fields. The idea of *confounding* appears to be largely connected to the process of *narrowing*; a small extension to the narrowing layer to include the confounding and unconfounding processes might make the framework sufficient to describe confounding and unconfounding in student understanding of vector field derivatives.

6 Discussion

6.1 Limitations of the Study

Many of the limitations of this work are due to the small sample size of students interviewed. Only three interviews were given, and the students were all male physics majors at Oregon State University in the same class. A larger, more diverse pool of subjects, particularly including other genders, minority students, and/or students from other schools and disciplines, might include different approaches and difficulties that did not come up in these interviews. Additionally, the Paradigms program is a unique approach to junior level physics curriculum, so the difficulties students from Oregon State University have may differ from the difficulties of students from other schools.

Another limitation in this work was due to the time constraint in the interview. The interview data used was the last phase of a two phase interview, and the interviewees may have been tired or under a time limit. It is possible that if the entire interview had been devoted to vector fields, the students may have had different answers or been able to devote more time to thinking about the prompts.

6.2 Suggestions for Future Work

This work focused primarily on confounding variable and component, but the students interviewed used other techniques and ideas that were not analyzed in detail. For instance, more analysis can be done on the students' process in determining what to differentiate, or on the graphical techniques used by the students.

Additionally, the limitations of this study outlined in Section 6.1 are possible subjects of future study. Conducting similar interviews with a larger, more diverse pool of subjects may have interesting results that were not evident in this work. Students from other disciplines, such as mathematics or engineering, may think about the prompts differently.

6.3 Implications for Instruction

Because the number of students interviewed is low, the conclusions this study draws and the implications thereof may not be indicative of the entire student body. That said, given the strong evidence that students confound variable and component in this study, greater emphasis should be placed on distinguishing between component and variable. In fact, it may need to be explicitly stated during instruction that the two objects are not interchangeable, despite similar names. The three students interviewed did not show evidence of confounding magnitude and direction, although given the small sample size, magnitude and direction should not be ruled out as a possible source of confusion.

Additionally, Alex, Bailey, and Cam appeared to struggle with the concept that the components of a vector field are themselves functions of two variables. This is possibly a cause of the confounding of variable and component, or possibly a result thereof, but further shows that instructors may need to be more explicit about the relationship between a vector field and the independent variables. The students appeared to consciously acknowledge that each component depended on both the x and y variables, but did not know what to do with that knowledge when attempting to differentiate, and the students gravitated toward only considering the x component of the vector field. Prompts from the interviewer to the effect of "could you do [the same operation] to the y component?" appeared to help the students recognize that the y component is an independent object that can also be differentiated.

Both Alex and Cam had difficulties with the idea that a derivative can itself be a vector with direction, despite acknowledging the existence of curl as an operation. Alex and Cam demonstrated that students may not consider the divergence and curl to be derivatives, but rather "arrangements" of derivatives. This is likely a problem other students in vector calculus have, and may be evidence that vector calculus classes do not explain what divergence and curl are in a way that students retain. These students appear to understand that divergence and curl show how the vector field is changing, but do not make the connection between "description of change" and "derivative." Placing emphasis not only on the fact that the curl is a derivative, but also that the curl is a vector field itself may help students with the unfamiliar idea that a derivative can itself be a vector.

Overall, the areas of confusion brought up in these studies show that students have difficulty understanding what the components in a vector field are, and what the derivative operations divergence, curl, and gradient do. Though further studies are needed to make general conclusions, it would likely benefit students' understanding if both mathematics and physics classes placed strong emphasis on what a vector field is, rather than primarily focusing on calculations on and applications of vector fields. Such an approach may alleviate some of the confusion that these three students showed in their interviews, particularly the difficulties with confounding variable and component.

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