AN INEQUALITY FOR THE NUMBER OF INTEGERS IN THE SUM OF TWO SETS OF INTEGERS

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1. INTRODUCTION

The main purpose of this paper is to prove two theorems, which are stated below, by different methods. Before stating the theorems, we shall first define some of the notation which will be used later.

Let $A, B, \cdots\cdots$ denote sets of non-negative integers. Define $A + B = \{a + b \mid a \in A, b \in B\}$. Let $A(n), B(n), \cdots\cdots$ be the numbers of positive integers, which do not exceed n, in $A, B, \cdots\cdots$, respectively. For $n \geq m$, let $A(m,n) = A(n) - A(m)$, $B(m,n) = B(n) - B(m)$, and so on. The modified Besicovitch density α_1 is defined as follows:

$$\alpha_1 = \operatorname*{glb}_{n \geq k} \frac{A(n)}{n+1},$$

where k is the smallest positive integer missing from A. A gap of a set is defined as a positive integer missing from the set.

To avoid unnecessary repetition, we shall state the two theorems here.

Theorem 1: If $0 \in A$, $0 \notin B$, $1 \in B$, $C = A + B$, then $C(n) \geq \alpha_1 n + B(n)$ for all $n \notin C$.

Theorem 2: If $0 \in A$, $0 \in B$, $C = A + B$, then

$$C(n) \geq \alpha_1(n + 1) + B(n) \quad \text{for all} \quad n \notin C.$$

Theorem 1 was first proved by Mann [4] in 1951, using the same transformation he used to prove the $\alpha\beta$ Theorem in 1942 [5]. Later in 1955, Peter Scherk [9] proved Theorem 1 with a slight alteration on α_1, using a modification of a method of Besicovitch [1].

The proofs of Mann and Scherk will be presented. In addition, four original proofs will be presented.

We shall confine ourselves to proving Theorem 1, for once Theorem 1 is established, Theorem 2 can be readily obtained. The proof will be presented in the next chapter. Conversely, we could derive Theorem 1 from Theorem 2 by this same method.

The equalities of the theorems are necessary as shown by the following examples:

Let $A = \{0, 1, 3, 6, 9, 12, \cdots\cdots\}$,

and $B = \{1, 4, 7, 10, 13, \cdots\cdots\}$.

Then $C = \{1, 2, 4, 5, 7, 8, 10, 11, \cdots\cdots\}$.

Hence $3n \notin C$ for $n = 1, 2, \cdots\cdots$, and $C(3n) = 2n$.

Since $\alpha_1 = \frac{1}{3}$ and $B(3n) = n$, we have

$$\alpha_1(3n) + B(3n) = 2n$$

and equality in Theorem 1 holds.

Now let $A = \{0, 1, 4, 8, 12, 16, \cdots\}$,

and $B = \{0, 1, 4, 5, 8, 9, 12, 13, \cdots\}$.

Then $C = \{0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, \cdots\}$.

Hence $4n - 1 \notin C$ for $n = 1, 2, \cdots$, and $C(4n - 1) = 3n - 1$. Since $\alpha_1 = \frac{1}{4}$ and $B(4n - 1) = 2n - 1$, we have $\alpha_1(4n) + B(4n - 1) = 3n - 1$, and equality holds in Theorem 2.

Mann [5] in 1942, proved the following theorem from which the $\alpha\beta$ Theorem follows:

Theorem A. If $C = A + B$, $0 \in A$, $0 \in B$, then for any $n \notin C$,

$$(1) \quad \frac{C(n)}{n} \geq \min_{n_i \leq n} \frac{A(n_i) + B(n_i)}{n_i}, \quad n_i \notin C.$$

Mann [6] in 1960, proved the following theorem which is stronger than Theorem A and therefore also gives the $\alpha\beta$ Theorem.

Theorem B. If $C = A + B$, $0 \in A$, $0 \in B$, then for any $n \notin C$, there exist m and m_1 such that $m \leq n$, $m \notin C$, $m_1 \leq \max(m, n - m - 1)$, $m_1 \notin C$, and

$$(2)\quad \frac{C(n)+1}{n+1} \geq \frac{A(m)+B(m)+1}{m+1} + \left|\frac{C(n)+1}{n+1} - \frac{C(m_1)+1}{m_1+1}\right|.$$

We compare these theorems of Mann with Theorem 2.

First let us prove that Theorem B is stronger than Theorem A. Mann states this fact without proof. We shall begin by proving the following lemma:

Lemma 1.1. If

$$(3)\quad b \geq d > 1,\ b \geq a > 0,$$

and

$$(4)\quad \frac{a}{b} \geq \frac{c}{d},$$

then

$$\frac{a-1}{b-1} \geq \frac{c-1}{d-1}.$$

Proof. From (4), we get

$$-\frac{a}{b} \leq -\frac{c}{d},$$

$$1 - \frac{a}{b} \leq 1 - \frac{c}{d},$$

$$\frac{b - a}{b} \leq \frac{d - c}{d},$$

and with (3),

$$\frac{b - a}{b(b - 1)} \leq \frac{d - c}{d(d - 1)},$$

$$\frac{a - b}{b(b - 1)} \geq \frac{c - d}{d(d - 1)}.$$

Hence with (4),

$$\frac{a}{b} + \frac{a - b}{b(b-1)} \geq \frac{c}{d} + \frac{c - d}{d(d - 1)}$$

$$\frac{ab - a + a - b}{b(b - 1)} \geq \frac{cd - c + c - d}{d(d - 1)}$$

$$\frac{ab - b}{b(b - 1)} \geq \frac{cd - d}{d(d - 1)},$$

$$\frac{a - 1}{b - 1} \geq \frac{c - 1}{d - 1}.$$

The proof is complete.

From Theorem B, we have

$$\frac{C(n) + 1}{n + 1} \geq \frac{A(m) + B(m) + 1}{m + 1}$$

Applying Lemma 1.1, we get

$$\frac{C(n)}{n} \geq \frac{A(m) + B(m)}{m}$$

Since $m \leq n$, $m \notin n$, we get

$$\frac{C(n)}{n} \geq \min_{n_i \leq n} \frac{A(n_i) + B(n_i)}{n_i},$$

which is Theorem A. Hence, Theorem B is stronger than

Theorem A.

Since Mann claimed that these theorems are very strong results and since the $\alpha\beta$ Theorem can be derived from them, we would like to know whether or not Theorem 2 will also follow from them. The following examples show that Theorem 2 is sometimes stronger, and sometimes weaker, than Theorems A and B. Thus it cannot be derived from these theorems. Let

$$A = B = \{0, 1\}.$$

Then

$$C = \{0, 1, 2\}.$$

Rewriting (1) of Theorem A, we get

$$(5) \quad C(n) \geq n \cdot \min_{n_i \leq n} \frac{A(n_i) + B(n_i)}{n_i} .$$

Take $n = 3$. Then the right-hand side of (5) is

$$3(\tfrac{2}{3}) = 2 .$$

However,

$$\alpha_i(n + 1) + B(n) = 0(4) + 1 = 1 .$$

Hence, in this case, Theorem A, and thus Theorem B, is stronger than Theorem 2.

Now let

$$A = \{0, 1, 3, 6, 7, 8, 9, 12, 13, \cdots\cdots\},$$

and

$$B = \{0, 1, 6, 7, 9, 12, 13, \cdots\cdots\}.$$

Then

$$C = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, \cdots\cdots\}.$$

Rewriting (2) of Theorem B, we get

$$(6) \quad C(n) \geq \frac{A(m) + B(m) + 1}{m + 1}(n + 1) + \left|\frac{C(n) + 1}{n + 1} - \frac{C(m_1) + 1}{m_1 + 1}\right|(n + 1) - 1.$$

Take $n = 11$. It follows that $m = 5$ and $m_1 = 5$. Then the right-hand side of (6) is

$$\frac{4}{6}(12) + \left|\frac{10}{12} - \frac{5}{6}\right|(12) - 1 = 7.$$

However,

$$\alpha_1(n + 1) + B(n) = \frac{1}{3}(12) + 4 = 8.$$

Hence, in this case, Theorem 2 is stronger than Theorem B, and thus is also stronger than Theorem A. Therefore we conclude that Theorem 2 cannot be derived from Theorem A or Theorem B.

A natural approach for the proof of Theorems 1 and 2 is by induction where the induction hypothesis is used by adding a positive integer less than n to B or deleting a positive integer less than n from B. But this has not proved fruitful. Hence we proceed to other methods.

2. FUNDAMENTAL RESULTS

We shall now derive Theorem 2 from Theorem 1. Let $0 \in A$, $0 \in B$ and $C = A + B$. Suppose Theorem 1 has been proved.

Let $B_1 = \{b + 1 | b \in B\}$ and $C_1 = A + B_1$. It is clear that $0 \notin B_1$ and $1 \in B_1$. Hence, applying Theorem 1, we have

(1) $C_1(m) \geq \alpha_1 m + B_1(m)$ for all $m \notin C_1$.

For any $n \notin C$, $n + 1 \notin C_1$, for if $n + 1 \in C_1$, then $n + 1 = a + (b + 1)$, $a \in A$, $b \in B$, and $n = a + b$ which is a contradiction. Hence (1) becomes

(2) $C_1(n + 1) \geq \alpha_1(n + 1) + B_1(n + 1)$ for all $n \notin C$.

But $b \in B$ and $0 \leq b < n$ if and only if $b + 1 \in B_1$ and $0 < b + 1 < n + 1$. Hence $B_1(n + 1) = B(n) + 1$. Similarly, $C_1(n + 1) = C(n) + 1$. Hence (2) becomes

$$C(n) \geq \alpha_1(n + 1) + B(n) \quad \text{for all } n \notin C,$$

which is Theorem 2.

From now on we shall confine ourselves to proving Theorem 1. Therefore, in what follows, it will

be assumed that $0 \in A$, $0 \notin B$ and $1 \in B$, in order to avoid repetitions of this assumption. Also, for the same reason, $n_1 < n_2 < \cdots\cdots$ will be understood to be all the positive gaps in C and K the least positive integer missing from $\underline{A}$.

A few lemmas which will be referred to over and over again later will be proved here.

Lemma 2.1. If $0 \leq g \leq h < n$ where $n \notin C$, then

(3) $h - g \geq A(n - h - 1, n - g - 1) + B(g, h)$.

Proof. For each $b \in B$ such that $g < b \leq h$, we have $n - b \notin A$ and $n - h - 1 < n - b \leq n - g - 1$. Since $(n - g - 1) - (n - h - 1) = h - g$, then (3) follows.

Lemma 2.2. If $0 \leq g \leq h$ where $h + 1 \notin C$, then $h - g \geq A(h - g) + B(g, h)$.

Proof. Set $n = h + 1$ in Lemma 2.1 and Lemma 2.2 follows.

Lemma 2.3. $C(n_1) \geq A(n_1 - 1) + B(n_1)$.

Proof. Set $h = n_1 - 1$ and $g = 0$ in Lemma 2.2 to obtain

$$n_1 - 1 \geq A(n_1 - 1) + \cancel{B(n_1)}. \; B(n_1 - 1).$$

Hence since $B(n_1 - 1) = B(n_1)$ and $C(n_1) = n_1 - 1$, the proof is complete.

Lemma 2.4. $C(n_1) \geq \alpha_1 n_1 + B(n_1)$.

Proof. Since $n_1 \notin C$, $1 \in B$ and $0 \in A$, we have $n_1 - 1 \notin A$, $n_1 - 1 > 0$ and $A(n_1 - 1) \geq \alpha_1 n_1$. Thus Lemma 2.4 follows from Lemma 2.3.

Lemma 2.5. $C(n_{r-1}, n_r) \geq A(n_r - n_{r-1} - 1)$

$$+ B(n_{r-1}, n_r).$$

$r \geq 2$

Proof. Set $h = n_r - 1$ and $g = n_{r-1}$ in Lemma 2.2 to obtain

$$n_r - n_{r-1} - 1 \geq A(n_r - n_{r-1} - 1) + B(n_{r-1}, n_r - 1).$$

Hence since $B(n_r - 1) = B(n_r)$ and $C(n_{r-1}, n_r)$

$$= n_r - n_{r-1} - 1,$$

the proof is complete.

3. MANN'S TRANSFORMATION

Part 1.

This is a rewriting of Mann's proof[4].

Lemma 3.1. If $n_r - n_{r-1} < n_1$, then there exists sets B_1 and C_1, with $C_1 = A + B_1$, such that

(i) $0 \notin B_1$, $1 \in B_1$

(ii) $n_r \notin C_1$,

(iii) $B_1(n_r) - B(n_r) = C_1(n_r) - C(n_r) > 0$.

Proof. Define $d_i = n_r - n_i$, $i < r$. Since $n_r - n_{r-1} < n_1$, we have $d_{r-1} < n_1$ and $0 < n_1 - d_{r-1} < n_1$. Hence $n_1 - d_{r-1} \in C$ which implies $n_1 - d_{r-1} = a_o + b_o$, $a_o \in A$, $b_o \in B$. Hence

(1) $a_o + b_o + d_{r-1} = n_1$.

Let $B^* = \{b_o + d_i | a + b_o + d_i = n_j, a \in A, i < r, j < r\}$, $B_1 = B \cup B^*$, and $C_1 = A + B_1$. The definition of B^* ensures that B^* is not empty since $b_o + d_{r-1} \in B^*$ as implied by (1). In other words,

(2) $B^*(n_r) > 0$.

Also $B \cap B^*$ is empty. For if $b_o + d_i$ is any element in B^*, then $a + b_o + d_i = n_j$ for some $a \in A$, and $b_o + d_i = n_j - a \notin B$. Hence

(3) $B \cap B^*$ is empty

Proof of (i). Since $0 \notin B$, we have $b_o > 0$ and $0 \notin B^*$. Hence $0 \notin B_1$. Since $1 \in B$, then $1 \in B_1$.

Proof of (ii). Assume $n_r \in C_1$. Then $n_r = a + b_1$, $a \in A$, $b_1 \in B_1$. Since $b_1 \in B_1$, then $b_1 \in B$ or $b_1 \in B^*$. But $b_1 \notin B$ otherwise $n_r \in C$. Hence, $b_1 \in B^*$, $b_1 = b_o + d_i$, $n_r = a + b_o + d_i = a + b_o + n_r - n_i$, and $n_i = a + b_o$ which is a contradiction. Hence $n_r \notin C_1$.

Proof of (iii). From $B_1 = B \cup B^*$ and (3) it follows that $B_1(n_r) = B(n_r) + B^*(n_r)$. Hence with (2) we have $B_1(n_r) - B(n_r) = B^*(n_r) > 0$.

To prove that $C_1(n_r) - C(n_r) = B_1(n_r) - B(n_r)$, it suffices to show that the number of $n_i \in C_1$, $i < r$, is equal to the number of elements in B^*, i.e. there is a one-to-one correspondence between $n_i \in C_1$, $i < r$, and $b_o + d_i \in B^*$.

For each $t < r$, if $b_o + d_t \in B^*$, then $a + b_o + d_t = n_s$ for some $a \in A$, $s < r$. Hence

$$a + b_o + n_r - n_t = n_s,$$

$$a + b_o + n_r - n_s = n_t,$$

$$a + b_o + d_s = n_t,$$

and $n_t \in C_1$.

Conversely, for each $t < r$, if $n_t \in C_1$, then

$$n_t = a + b_o + d_s \text{ for some } a \in A,\ s < r.$$

Hence

$$n_t = a + b_o + n_r - n_s,$$

$$n_s = a + b_o + n_r - n_t,$$

$$n_s = a + b_o + d_t,$$

and $b_o + d_t \in B^*$. Thus the one-to-one correspondence is established.

Proof of Theorem 1. The theorem is true for the first gap of C as has been proved in Lemma 2.4.

Assume that the theorem is true for all j^{th} gaps, $j < r$, for all sets A, B and C.

Case 1. $n_r - n_{r-1} \geq n_1$.

Then $n_r - n_{r-1} - 1 \geq n_1 - 1 \notin A$. Since $n_1 - 1 > 0$, we have:

(4) $A(n_r - n_{r-1} - 1) \geq \alpha_1(n_r - n_{r-1})$.

From Lemma 2.5, we get

(5) $C(n_{r-1}, n_r) \geq A(n_r - n_{r-1} - 1) + B(n_{r-1}, n_r)$.

Combining (4) and (5), we get

(6) $C(n_{r-1}, n_r) \geq \alpha_1(n_r - n_{r-1}) + B(n_{r-1}, n_r)$.

By our induction hypothesis,

(7) $C(n_{r-1}) \geq \alpha_1 n_{r-1} + B(n_{r-1})$.

Adding (6) and (7), we get the theorem.

Case 2. $n_r - n_{r-1} < n_1$.

Then the Lemma 3.1 supplies us with A, B_1 and C_1 satisfying the hypothesis of the theorem. Since $C_1(n_r) > C(n_r)$ as implied by (iii), and since $n_r \notin C_1$ by (ii), then n_r must be some j^{th} gap of C_1, $j < r$. Hence by our induction hypothesis,

(8) $C_1(n_r) \geq \alpha_1 n_r + B_1(n_r)$.

But (iii) implies

(9) $C(n_r) - C_1(n_r) = B(n_r) - B_1(n_r)$.

Adding (8) and (9), we get the theorem and the proof is complete.

Part 2.

This contains a proof similar to the one in Part 1. However, we shall use a different transformation of Mann [10].

Lemma 3.2. If $n_r - n_{r-1} < n_1$, then there exist sets B_1 and C_1, with $C_1 = A + B_1$, such that

(i) $0 \notin B_1$, $1 \in B_1$,

(ii) $n_r \notin C_1$,

(iii) $B_1(n_r) - B(n_r) \geq C_1(n_r) - C(n_r) > 0$.

Proof. Define $d_i = n_r - n_i$, $i < r$. Since $n_r - n_{r-1} < n_1$, it follows that $d_{r-1} < n_1$, $0 < n_1 - d_{r-1} < n_1$, $n_1 - d_{r-1} \in C$, and $n_1 - d_{r-1} = a_o + b_o$ for some $a_o \in A$, $b_o \in B$. Hence

(10) $n_1 = a_o + b_o + d_{r-1}$ and $b_o + d_{r-1} \notin B$.

Let $B^* = \{b_o + d_i \mid b_o + d_i \notin B,\ i < r\}$, $B_1 = B \cup B^*$, and $C_1 = A + B_1$. The definition of B^* ensures that B^* is not empty as implied by (10), i.e.

(11) $B^*(n_r) > 0$,

and

(12) $B \cap B^*$ is empty.

Proof of (i). Since $0 \notin B^*$, then $0 \notin B_1$. Since $1 \in B$, then $1 \in B_1$.

Proof of (ii). Assume $n_r \in C_1$. Then $n_r = a + b_1$, $a \in A$, $b_1 \subset B_1$. Since $b_1 \in B_1$, then $b_1 \in B$ or $b_1 \in B^*$. But $b_1 \notin B$, otherwise $n_r \in C$. Hnece, $b_1 \subset B^*$, $b_1 = b_o + d_i$, $n_r = a + b_o + d_i = a + b_o + n_r - n_i$, and $n_i = a + b_o$ which is a contradiction. Hence $n_r \notin C_1$.

Proof of (iii). From the definitions of B_1 and C_1, C is a subset of C_1. From (10), we have $n_1 \in C_1$. Thus $C_1(n_r) - C(n_r) > 0$.

If $n_j \in C_1$, $j < r$, then $n_j = a + b_o + d_i$ for some $a \in A$, $i < r$. Hence,

$$n_j = a + b_o + n_r - n_i ,$$

$$n_i = a + b_o + n_r - n_j ,$$

$$n_i = A + b_o + d_j ,$$

and $b_o + d_j \in B^*$. Hence, $B^*(n_r) \geq C_1(n_r) - C(n_r)$. By (12) we have $B_1(n_r) = B(n_r) + B^*(n_r)$, and so $B_1(n_r) - B(n_r) \geq C_1(n_r) - C(n_r)$. The proof is complete.

Note that the main difference in the proofs of

Lemmas 3.1 and 3.2 lies in the proof of part (iii) of each lemma, and that the proof of part (iii) of Lemma 3.2 is simpler than the proof of part (iii) of Lemma 3.1.

Proof of Theorem 1. The proof is exactly the same as in Part 1, except that for the case $n_r - n_{r-1} < n_1$, we use Lemma 3.2, noting that (iii) implies that

$$C(n_r) - C_1(n_r) \geq B(n_r) - B_1(n_r) .$$

4. BESICOVITCH'S COUNTING PROCESS

In this chapter, we shall present four proofs using Besicovitch's counting process [1].

Part 1.

This is a rewriting of Scherk's proof [9] with the following revisions. Whereas he performed the induction on n, considering it as a natural number, we perform the induction on the gaps of C, in the proof of Theorem 1. Also, his definition of α_1 is slightly different from ours.

Lemma 4.1. If $C(n) < A(n-1) + B(n)$, $n \notin C$, then there exists $n_t < n - k$ such that

(1) $C(n_t, n) \geq A(n - n_t - 1) + B(n_t, n)$.

Proof. Let b_o be the largest element in B less than n. Then

(2) $B(b_o, n) = 0$

Now for each $a \in A$ with $0 < a \leq n - b_o$, we have $a + b_o \in C$ and $b_o < a + b_o \leq n$. Hence,

(3) $C(b_o, n) \geq A(n - b_o) \geq A(n - b_o - 1)$.

Adding (2) and (3), we get

(4) $C(b_o, n) \geq A(n - b_o - 1) + B(b_o, n)$.

Let n_t be the largest gap in C less than b_o. Such an n_t exists. For if not, then $C(b_o) = b_o$, and by Lemma 2.1 with $g = 0$ and $h = b_o$ we have

$$C(b_o) \geq A(n - b_o - 1, n - 1) + B(b_o).$$

Adding this and (4), we get

$$C(n) \geq A(n - 1) + B(n),$$

which contradicts our hypothesis.

Since $b_o + 1, b_o + 2, \cdots\cdots, b_o + k - 1 \in C$,

we have

$$n > b_o + k - 1 \geq n_t + k,$$

and $n_t < n - k$. Since $n_t < b_o < n$, from Lemma 2.1 with $h = b_o$ and $g = n_t$, we get

$$C(n_t, b_o) = b_o - n_t \geq A(n - b_o - 1, n - n_t - 1)$$
$$+ B(n_t, b_o).$$

Adding this and (4), we obtain (1). The proof is complete.

Proof of Theorem 1.

The theorem is true for the first gap of C from Lemma 2.4. Assume that the theorem is true for all n_i, $i < r$, $r \geq 2$.

Case 1. $C(n_r) \geq A(n_r - 1) + B(n_r)$.

Since $n_r - 1 \notin A$, then $A(n_r - 1) \geq \alpha_1 n_r$. Hence,

$$C(n_r) \geq \alpha_1 n_r + B(n_r).$$

Case 2. $C(n_r) < A(n_r - 1) + B(n_r)$.

Set $n = n_r$ in Lemma 4.1 to obtain

(5) $C(n_t, n_r) \geq A(n_r - n_t - 1) + B(n_t, n_r)$,

where $n_t < n_r - k$. Thus $t < r$. Since $n_t < n_r - k$, then $n_r - n_t - 1 \geq k$. Hence $A(n_r - n_t - 1) \geq \alpha_1(n_r - n_t)$, and (5) becomes

(6) $C(n_t, n_r) \geq \alpha_1(n_r - n_t) + B(n_t, n_r)$.

By our induction hypothesis,

(7) $C(n_t) \geq \alpha_1 n_t + B(n_t)$.

Adding (6) and (7), we get

$$C(n_r) \geq \alpha_1 n_r + B(n_r).$$

The proof is complete.

Part 2.

In this proof we make use of a result of Scherk [10].

We first prove Scherk's result. Let n be an arbitrary gap of C. Let $m_0 = n$. Let h_j be the largest element in B less than m_j, $j \geq 0$, and let m_{j+1} be the largest integer missing from C less than h_j, $j \geq 0$. Since n is a finite number, the sequence will terminate. Thus, letting $m_{i+1} = 0$, we have

$$n = m_0 > h_0 > m_1 > \cdots\cdots > m_i > h_i > m_{i+1} = 0.$$

From Lemma 2.1 with $n = m_j$, $h = h_j$, $g = m_{j+1}$, $0 \leq j \leq i$, we get

$$(8) \quad C(m_{j+1}, h_j) = h_j - m_{j+1} \geq A(m_j - h_j - 1, m_j - m_{j+1} - 1) + B(m_{j+1}, h_j).$$

Since for each $a \in A$ with $0 < a \leq m_j - h_j$, it follows that $a + h_j \in C$ with $h_j < a + h_j \leq m_j$, we have

$$(9) \quad C(h_j, m_j) \geq A(m_j - h_j) \geq A(m_j - h_j - 1).$$

Adding (8) and (9), we get

$$(10)\quad C(m_{j+1}, m_j) \geq A(m_j - m_{j+1} - 1) + B(m_{j+1}, h_j)$$

$$= A(m_j - m_{j+1} - 1) + B(m_{j+1}, m_j).$$

This is the result of Scherk.

Proof of Theorem 1.

Now $m_j - m_{j+1} - 1 \geq k$, $0 \leq j \leq i$. For if not, then

$$\begin{aligned} k &> m_j - m_{j+1} - 1 \\ &\geq m_j - (h_j - 1) - 1 \\ &= m_j - h_j > 0. \end{aligned}$$

Hence $m_j - h_j \in A$ and $(m_j - h_j) + h_j = m_j \in C$ which is a contradiction. Hence,

$$A(m_j - m_{j+1} - 1) \geq \alpha_1(m_j - m_{j+1}),\ 0 \leq j \leq i,$$

and (10) becomes

$$(11)\quad C(m_{j+1}, m_j) \geq \alpha_1(m_j - m_{j+1}) + B(m_{j+1}, m_j).$$

Summing (11) from $j = 0$ to $j = i$, and setting $m_o = n$ and $m_{i+1} = 0$, we have

$$C(n) \geq \alpha_1 n + B(n).$$

This completes the proof.

Part 3.

In this proof we use Besicovitch's counting process in a different way.

Proof of Theorem 1.

Let $m_o = 0$. Let $k_i + 1$ be the least integer greater than m_i, $i \geq 0$, missing from C. Let $m_{i+1} + 1$ be the smallest element in B greater than k_i, $i \geq 0$. Since $m_{i+1} + 1 > k_i$, then $m_{i+1} \geq k_i$. Since $m_{i+1} + 1 \neq k_i + 1$, then $m_{i+1} > k_i$. Similarly, $k_i > m_i$.

Suppose $k_i < x < m_{i+1}, i \geq 0$. Then, since for each $a \in A$ with $k_i - m_i - 1 < a \leq x - m_i - 1$, we have $m_i + a + 1 \in C$ with $k_i < m_i + a + 1 \leq x$, it follows that

$$(12) \quad C(k_i, x) \geq A(k_i - m_i - 1, x - m_i - 1) \geq A(k_i - m_i, x - m_i - 1).$$

From Lemma 2.2 with $h = k_i$, $g = m_i$, we have

$$(13) \quad C(m_i, k_i) = k_i - m_i \geq A(k_i - m_i) + B(m_i, k_i).$$

Adding (12) and (13), and setting $B(m_i, x) = B(m_i, k_i)$, we get

(14) $C(m_i, x) \geq A(x - m_i - 1) + B(m_i, x)$.

Now $k_i - m_i \geq k$. If not, suppose $k_i - m_i < k$. Then $k_i - m_i = a \in A$ and $k_i + 1 = a + m_i + 1 \in C$ which is a contradiction to the definition of k_i. Hence, $k_i - m_i \geq k$ and $x - m_i - 1 \geq k$. Hence (14) becomes

(15) $C(m_i, x) \geq \alpha_1(x - m_i) + B(m_i, x)$.

In particular,

(16) $C(m_j, m_{j+1}) \geq \alpha_1(m_{j+1} - m_j) + B(m_j, m_{j+1})$.

Now let n be a gap in C, and let r be such that $k_r < n$, and if m_{r+1} exists then $k_r < n \leq m_{r+1}$.

If $r = 0$, set $i = 0$ and $x = n$ in (15) to obtain Theorem 1.

If $r > 0$, sum (16) for $j = 0, 1, \cdots, r - 1$, and (15) with $i = r$, $x = n$ to obtain Theorem 1. The proof is complete.

Part 4.

This proof is similar to Scherk's proof [9] which appears in Part 1 of this chapter. It is in effect a simplification of Scherk's proof. Not only do we not have to consider the two cases as Scherk did, but also it is a much shorter proof.

Proof of Theorem 1.

From Lemma 2.4, the theorem is true for the first gap of C. Assume that the theorem is true for all n_i, $i < r$, $r \geq 2$.

Let b_o be the largest element in B less than n_r.

If there is no gap of C less than b_o, then $n_1 > b_o$ and

(18) $B(n_r) = B(n_1)$.

Now for each $a \in A$ with $n_1 - 1 < a \leq n_r - 1$, we have $a + 1 \in C$ and $n_1 < a + 1 \leq n_r$, and so

(19) $C(n_1, n_r) \geq A(n_1 - 1, n_r - 1)$.

From Lemma 2.3, we get

(20) $C(n_1) \geq A(n_1 - 1) + B(n_1)$.

Adding (19) and (20), and using (18), we get

(21) $C(n_r) \geq A(n_r - 1) + B(n_r)$

Since $n_r - 1 \notin A$, then $A(n_r - 1) \geq \alpha_1 n_r$. Hence (21) becomes

$$C(n_r) \geq \alpha_1 n_r + B(n_r) .$$

If there is a gap of C less than b_o, let n_t be the largest of these gaps. Since $n_t < b_o < n_r$, we may set $g = n_t$, $h = b_o$ and $n = n_r$ in Lemma 2.1 to obtain

(22) $C(n_t, b_o) = b_o - n_t \geq A(n_r - b_o - 1, n_r - n_t - 1) + B(n_t, b_o)$. Now for each $a \in A$ with $0 < a \leq n_r - b_o - 1$, we have $a + b_o \in C$ and $b_o < a + b_o \leq n_r - 1$. Hence

(23) $C(b_o, n_r) = C(b_o, n_r - 1) \geq A(n_r - b_o - 1)$.

By the definition of b_o, we have $B(b_o, n_r) = 0$, and so (23) may be written

(24) $C(b_o, n_r) \geq A(n_r - b_o - 1) + B(b_o, n_r)$.

Adding (22) and (24), we get

(25) $C(n_t, n_r) \geq A(n_r - n_t - 1) + B(n_t, n_r)$.

Now $n_r - n_t - 1 \geq k$. For if not, then $n_r - n_t - 1 < k$, $n_r - n_t \leq k$, $n_r - b_o < k$, $n_r - b_o \in A$ and $n_r \in C$ which is a contradiction. Hence $A(n_r - n_t - 1) \geq \alpha_1(n_r - n_t)$ and (25) becomes

(26) $C(n_t, n_r) \geq \alpha_1(n_r - n_t) + B(n_t, n_r)$.

By our induction hypothesis, we have

(27) $C(n_t) \geq \alpha_1 n_t + B(n_t)$.

Adding (26) and (27), we get the theorem. This completes the proof.

5. DYSON'S TRANSFORMATION

An interesting question which remains to be answered is whether or not it is possible to prove Theorem 1 by using Dyson's transformation [3]. This inquiry is perfectly natural in view of the fact that both Mann's and Dyson's transformations have been used successfully to prove the $\alpha\beta$ Theorem [3,5] and Chowla's inequality for cyclic groups [2,7], and in addition, Mann's transformation has provided a proof of Theorem 1 [4] as presented in Chapter 3.

A version of Dyson's proof of the $\alpha\beta$ Theorem may be found in Niven and Zuckerman [8]. Let A_1 and B_1 consist of all elements of A and B not exceeding g, an arbitrary positive integer. Let $C_1 = A_1 + B_1$. Dyson actually used two transformations, the main one being for the case $A_1 \subset B_1$.

If $A_1 \not\subset B_1$, let $A' = \{a | a \in A_1, a \notin B_1\}$. Let $A_2 = A_1 - A'$, $B_2 = B_1 \cup A'$ and $C_2 = A_2 + B_2$.

If $A_1 \subset B_1$, then let b_0 be the least positive integer in B_1 for which there is an $a \in A_1$ such that $a + b_0 \notin B_1$. Let

$$A' = \{a | a \in A_1, a + b_0 \notin B_1\},$$

and

$$B' = \{a + b_0 | a \in A', a + b_0 \leq b\}.$$

Let $A_2 = A_1 - A'$, $B_2 = B_1 \cup B'$ and $C_2 = A_2 + B_2$.

The following lemma is then proved from which the $\alpha\beta$ Theorem follows.

Lemma 4.1. If for some θ such that $0 < \theta \leq 1$,

$$A_1(m) + B_1(m) \geq \theta m, \quad m = 1, 2, \cdots, g,$$

then $C_1(g) \geq \theta g$.

Suppose we were to try to use this approach in a natural way to prove Theorem 1. We would first prove the following similar lemma.

Lemma 4.2. Let A_1 and B_1 consist of all elements of A and B not exceeding n, $n \notin C$. If for some θ such that $0 < \theta \leq 1$,

$$A_1(m - 1) + B_1(m) \geq \theta m, \quad m = k + 1, \cdots, n,$$

then $C_1(n) \geq \theta n$.

Suppose we succeeded in proving Lemma 2 by means of Dyson's transformation. Then to obtain Theorem 1 from this lemma, we would take

$$\theta = \alpha_1 + \frac{B_1(n)}{n}$$

and need

$$(1) \quad A_1(m - 1) + B_1(m) \geq \left[\alpha_1 + \frac{B_1(n)}{n}\right]m, \quad m = k + 1, \cdots, n.$$

However, (1) is not true as shown by the following example.

Let

$$A = \{ 0, 1, 3, 6, 9, 10, 11, 12, \cdots\},$$

and

$$B = \{1, 4, 5, 7, 11, 12, \cdots\}.$$

Then

$$C = \{1, 2, 4, 5, 6, 7, 8, 10, 11, 12, \cdots\}.$$

Take $n = 9$. Then $A_1 = \{0, 1, 3, 6, 9\}$, $B_1 = \{1, 4, 5, 7\}$ and $C_1 = \{1, 2, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16\}$. We have $\alpha_1 = \frac{1}{3}$ and $\frac{B_1(n)}{n} = \frac{4}{9}$. Hence

$$\alpha_1 + \frac{B_1(n)}{n} = \frac{7}{9}.$$

For $m = 3$, we have

$$A_1(m - 1) + B_1(m) = 1 + 1 = 2,$$

while

$$\lfloor \alpha_1 + \frac{B_1(n)}{n} \rfloor m = \frac{21}{9}.$$

Hence (1) is not true and Theorem 1 would not follow. This does not, by any means, show that Dyson's transformation will not work. It only shows that this approach fails.

Let us try another approach. Suppose we were to try to prove Theorem 1 by induction on the gaps of C as Mann did. Using Dyson's transformation, we would get A_2, B_2, and C_2 with $C_2(n_r) \leq C(n_r)$ [8] (set $n = n_r$). A natural way to proceed would be to establish the following two inequalities:

(i) $C_2(n_r) \geq \alpha_1 n_r + B_2(n_r)$;

(ii) $C(n_r) - C_2(n_r) \geq B(n_r) - B_2(n_r)$.

Then we would add them to complete the proof. However, our induction hypothesis would be $C(n_j) \geq \alpha_1 n_j + B(n_j)$ for all $j < r$, for all sets A, B and C. Hence, since $C_2(n_r) \leq C(n_r)$, this induction hypothesis does not give us $C_2(n_r) \geq \alpha_2 n_r + B_2(n_r)$ where α_2 may be taken to be the modified Besicovitch density of $A - A'$. Hence, since $\alpha_2 \leq \alpha_1$, a fortiori it would not give us (i). So this approach also fails.

Note that in performing the transformation above, we take out elements from A_1 and add new elements to B_1. The reason is that we want to have $1 \in B_2$ to satisfy the hypothesis of Theorem 1.

From these attempts, we can see the difficulty in trying to use Dyson's transformation to prove Theorem 1. If we modify the transformation slightly, though not successful in proving Theorem 1, we get some interesting results as shown below.

If there is no $b \in B$ for which there is an $a \in A$ such that $a + b \notin B$, $a + b < n$ where $n \notin C$, then $1 \notin A$. For if $1 \in A$, let h be the first gap in B. Then $h - 1 \in B$ and we have $1 + (h - 1) = h \notin B$ with $h < n$, which contradicts our assumption. Hence

$1 \notin A$ and $\alpha_1 = 0$. Thus Theorem 1 is true in this case.

If, on the other hand, there is at least one $b \in B$ for which there is an $a \in A$ such that $a + b \notin B$, $a + b < n$, then let b_0 be the smallest of these b's. Let

$$A_1' = \{a | a \in A,\ a + b_0 \notin B,\ a + b_0 < n\},$$

and

$$B_1' = \{a + b_0 | a \in A_1'\}.$$

Let $A_1 = A - A_1'$, $B_1 = B \cup B_1'$ and $C_1 = A_1 + B_1$. Note that this transformation differs from Dyson's transformation in that $a + b_0 < n$ in our definition of A_1'. Repeat this process until we get to the sets A_p, B_p and C_p when there is no $b_p \in B_p$ for which there is an $a_p \in A_p$ such that $a_p + b_p \notin B_p$ and $a_p + b_p < n$. Then the following propositions are true.

(i) $\sum_{i=1}^{p} A_i'(n) = \sum_{i=1}^{p} B_i'(n)$.

(ii) $A_p(n) = A(n) - \sum_{i=1}^{p} A_i'(n)$.

(iii) $B_p(n) = B(n) + \sum_{i=1}^{p} B'_i(n)$.

(iv) $A_p(n-1) + B_p(n) = A(n-1) + B(n)$.

(v) $0 \in A_p$, $0 \notin B_p$, $1 \in B_p$.

(vi) If $x \in C_p$ and $1 \leq x \leq n$, then $x \in C$.

(vii) $C_p(n) \leq C(n)$.

(viii) $n \notin C_p$.

(ix) $C_p(n) = B_p(n)$

(x) If $C_p(n) = C(n)$, then $C(n) \leq A(n-1) + B(n)$.

Proof of (i) From the definitions of A'_i and B'_i, $A'_i(n) = B'_i(n)$ for each i. Hence (i) follows.

Proof of (ii). From the definition of A_1, we have

$$A_1(n) = A(n) - A'_1(n).$$

Similarly,

$$A_2(n) = A_1(n) - A'_2(n),$$
$$\vdots$$
$$A_p(n) = A_{p-1}(n) - A'_p(n).$$

Adding these, we get (ii).

We get (iii) by the same argument above.

Proof (iv). From (i), (ii) and (iii), we get

$$A_p(n) + B_p(n) = A(n) + B(n).$$

Since $n \notin A'_i$, then $n \in A_p$ if $n \in A$, and $n \notin A_p$ if $n \notin A$. Hence (iv) follows.

Proof of (v). From the definition of A'_i, $0 \notin A'_i$. Hence $0 \in A_1$. Similarly, $0 \in A_p$. Since $b_0 > 0$, we have $0 \notin B'_i$, and since $0 \notin B$, then $0 \notin B_1$. Similarly, $0 \notin B_p$. Since $B \subset B_p$ and $1 \in B$, then $1 \in B_p$. The proof is complete.

Proof of (vi). If $x \in C_1$ and $1 \leq x \leq n$, then $x = a_1 + b_1$, $a_1 \in A_1$, $b_1 \in B_1$. Since $b_1 \in B_1$, then either $b_1 \in B$ or $b_1 \in B'_i$. If $b_1 \in B$, then, since $a_1 \in A_1 \subset A$, $x \in C$. If $b_1 \in B'_i$, then $b_1 = a + b_0$, $a \in A$, $b_0 \in B$, and $x = a_1 + a + b_0$. But $a_1 + b_0 \in B$, otherwise $a_1 \in A'_i$ and $a_1 \notin A_1$. Hence $x \in C$. Similarly, if $x \in C_2$ and $1 \leq x \leq n$, then $x \in C_1$, and so on. Hence (vi) follows.

It is clear that (vii) and (viii) follow immediately from (vi).

Proof of (ix). Since $a_p + b_p \in B_p$ for all $a_p \in A_p$ and $b_p \in B_p$ such that $a_p + b_p < n$, and since $n \notin B_p$ as implied by (viii), then (ix) follows.

Proof of (x). If $C_p(n) = C(n)$, then with (iv) and (ix), we get

$$C_p(n) = B_p(n) = A(n - 1) + B(n) - A_p(n - 1).$$

Hence (x) follows.

The proofs are complete.

One interesting fact is that if $A_p(n - 1) = 0$, then from (iv), (vii) and (ix), we get

$$C(n) \geq C_p(n) = B_p(n) = A(n - 1) + B(n) \geq \alpha_1 n + B(n).$$

So Theorem 1 follows if $A_p(n - 1) = 0$. But if $A_p(n - 1) > 0$, then it is hard to say whether or not Theorem 1 will follow easily.

6. STRONGER RESULTS

Part 1.

Although equality may hold in Theorem 1, the result is by no means the best possible one. We can actually improve this inequality by adding to its right-hand side a non-negative term. Mann found the following stronger result.

Theorem 6.1. $C(n) \geq \alpha_1 n + B(n) + \min_{n_i \leq n} [A(n_i - 1) - \alpha_1 n_i]$.

It can be proved in exactly the same way Mann proved Theorem 1 (Part 1, Chapter 3).

Proof. By Lemma 2.3, we have

$$C(n_1) \geq A(n_1 - 1) + B(n_1)$$

$$= \alpha_1 n_1 + B(n_1) + A(n_1 - 1) - \alpha_1 n_1.$$

Hence it is true for $n = n_1$.

Assume it is true for $n = n_j$, $j < r$, $r \geq 2$, for all sets A, B and C.

Case 1. $n_r - n_{r-1} \geq n_1$.

Then $n_r - n_{r-1} - 1 \geq n_1 - 1 \notin A$.

Hence $A(n_r - n_{r-1} - 1) \geq \alpha_1(n_r - n_{r-1})$.

From Lemma 2.5, we get

$$C(n_{r-1}, n_r) \geq A(n_r - n_{r-1} - 1) + B(n_{r-1}, n_r).$$

Hence,

$$(1) \quad C(n_{r-1}, n_r) \geq \alpha_1(n_r - n_{r-1}) + B(n_{r-1}, n_r).$$

By our induction hypothesis,

$$(2) \quad C(n_{r-1}) \geq \alpha_1 n_{r-1} + B(n_{r-1}) + \min_{n_i \leq n_{r-1}} [A(n_i - 1) - \alpha_1 n_i].$$

Adding (1) and (2), we get

$$C(n_r) \geq \alpha_1 n_r + B(n_r) + \min_{n_i \leq n_{r-1}} [A(n_i - 1) - \alpha_1 n_i]$$

$$\geq \alpha_1 n_r + B(n_r) + \min_{n_i \leq n_r} [A(n_i - 1) - \alpha_1 n_i].$$

Case 2. $n_r - n_{r-1} < n_1$.

From Lemma 3.1, we get new sets B_1 and C_1 with $C_1 = A + B_1$, $C_1(n_r) > C(n_r)$ and

$$(3) \quad C(n_r) - C_1(n_r) = B(n_r) - B_1(n_r).$$

Thus by our induction hypothesis,

$$(4) \quad C_1(n_r) \geq \alpha_1 n_r + B_1(n_r) + \min_{n_i \leq n_r} [A(n_i - 1) - \alpha_1 n_i].$$

Adding (3) and (4), we get

$$C(n_r) \geq \alpha_1 n_r + B(n_r) + \min_{n_i \leq n_r} [A(n_i - 1) - \alpha_1 n_i].$$

The proof is complete.

We can get another stronger result for Theorem 1 with the aid of Lemma 4.1.

Theorem 6.2. $C(n) \geq \left[\min_{k<m\leq n} \frac{A(m-1)}{m}\right] n + B(n).$

Proof. From Lemma 2.3, we get

$$C(n_1) \geq A(n_1 - 1) + B(n_1)$$

$$= \frac{A(n_1 - 1)}{n_1} n_1 + B(n_1)$$

$$\geq \left[\min_{k<m\leq n_1} \frac{A(m - 1)}{m}\right] n_1 + B(n_1).$$

Hence the theorem is true for $n = n_1$.

Assume it is true for $n = n_j$, $j < r$, $r \geq 2$.

Case 1. $C(n_r) \geq A(n_r - 1) + B(n_r)$.

Then $C(n_r) \geq \frac{A(n_r - 1)}{n_r} n_r + B(n_r)$

$$\geq \left[\min_{k<m\leq n_r} \frac{A(m-1)}{m}\right] n_r + B(n_r).$$

Case 2. $C(n_r) < A(n_r - 1) + B(n_r)$.

From Lemma 4.1, there exists an n_t, $t < r$, and $n_r - n_t < k$, such that

$$(5) \quad C(n_t, n_r) \geq A(n_r - n_t - 1) + B(n_t, n_r).$$

We can change (5) to

$$(6) \quad C(n_t, n_r) \geq \frac{A(n_r - n_t - 1)}{n_r - n_t}(n_r - n_t) + B(n_t, n_r).$$

$$\geq \left[\min_{k<m\leq n_r} \frac{A(m-1)}{m}\right](n_r - n_t) + B(n_t, n_r).$$

By our induction hypothesis,

$$(7) \quad C(n_t) \geq \left[\min_{k<m\leq n_t} \frac{A(m-1)}{m}\right] n_t + B(n_t)$$

$$\geq \left[\min_{k<m\leq n_r} \frac{A(m-1)}{m}\right] n_t + B(n_t).$$

Adding (6) and (7), we get

$$C(n_r) \geq \left[\min_{k<m\leq n_r} \frac{A(m-1)}{m}\right] n_r + B(n_r).$$

The proof is complete.

Part 2.

For Theorem 2, we can get similar stronger inequalities. We can derive from Theorem 6.1 the following stronger result.

Theorem 6.3. $C(n) \geq \alpha_1(n + 1) + B(n) + \min_{n_i \leq n}$

$$[A(n_i) - \alpha_1(n_i + 1)].$$

Proof. Let $B_1 = \{b_1 | b_1 = b + 1, b \in B\}$ and $C_1 = A + B_1$. Now for any $n \notin C$, $n + 1 \notin C_1$. For if $n + 1 \in C_1$, then $n + 1 = a + b_1 = a + b + 1$ and $n = a + b$ which is a contradiction. Since $0 \in A$, $0 \notin B_1$ and $1 \in B_1$, we can apply Theorem 6.1 and get

$$(8) \quad C_1(n + 1) \geq \alpha_1(n + 1) + B_1(n + 1) + \min_{n_i \leq n}$$

$$[A(n_i) - \alpha_1(n_i + 1)].$$

But since $0 \in A$ and $0 \in B$, then $C_1(n + 1) = C(n) + 1$, $B_1(n + 1) = B(n) + 1$. Hence (8) becomes

$$C(n) \geq \alpha_1(n + 1) + B(n) + \min_{n_i \leq n} [A(n_i) - \alpha_1(n_i + 1)].$$

The proof is complete.

Theorem 6.3 can be rewritten in the form,

$$C(n) \geq \alpha_1 n + B(n) + \min_{n_i \leq n} [A(n_i) - \alpha_1 n_i],$$

which Mann obtained [3].

Using the same proof as above, we can derive another stronger inequality from Theorem 6.2.

Theorem 6.4. $C(n) \geq \left[\min_{k \leq m \leq n} \frac{A(m)}{m+1} \right] (n+1) + B(n).$

Proof. Let $B_1 = \{b_1 | b_1 = b + 1, b \in B\}$ and $C_1 = A + B_1$. Applying Theorem 6.2, we get

$$C_1(n+1) \geq \left[\min_{k < m \leq n+1} \frac{A(m-1)}{m} \right] (n+1) + B_1(n+1).$$

Hence,

$$C(n) \geq \left[\min_{k \leq m \leq n} \frac{A(m)}{m+1} \right] (n+1) + B(n).$$

The proof is complete.

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