AN ABSTRACT OF THE THESIS OF

Robert H. Willie for the degree of Master of Science in Mechanical Engineering presented on June 7, 1996. Title: Fully Developed Laminar Natural Convection In A Vertical Parallel Plate Channel With Symmetric Uniform Wall Temperature.

Abstract approved: ___

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A. Murty Kanury

Described in this thesis is an investigation of the fully developed natural convection heat transfer in a vertical channel formed by two infinitely wide parallel plates maintained at a uniform wall temperature. Closed-form solutions for the velocity and temperature profiles are developed along with local and averaged Nusselt numbers. The local Nusselt number based on bulk temperature is found to be 3.77. This result is an analog corresponding to 7.60 for fully developed laminar forced convection in a parallel plate channel with uniform wall temperature boundary condition. The local Nusselt number based on the ambient temperature is deduced as a function of flowwise location. Results are compared with existing numerical and experimental data to find good agreement.

Fully Developed Laminar Natural Convection In A Vertical Parallel Plate Channel
With Symmetric Uniform Wall Temperature

by

Robert H. Willie

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

Presented June 7, 1996
Commencement June 1997

Master of Science thesis of <u>Robert H. Willie</u> presented on <u>June 7, 1996</u>

APPROVED:

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Major Professor, representing Mechanical Engineering

Redacted for Privacy

Chair of Department of Mechanical Engineering

Redacted for Privacy

Dean of Graduate School

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Robert H. Willie, Author

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NOMENCLATURE

b Channel width, m

c_p Specific heat, $J/kg \cdot k$

g Gravitational acceleration, m/s^2

h Convection coefficient, $W/m^2 \cdot k$

k Thermal conductivity, $W/m \cdot k$

l Channel length, m

T Fluid temperature, k

U Vertical velocity, m/s

V Horizontal velocity, m/s

ρ Fluid density, kg/m^3

β Volumetric thermal expansion coefficient, $1/k$

ν Kinematic viscosity, m^2/s

ξ Length coordinate, x/b

η Width coordinate, y/b

Gr Grashof number, $g\beta b^3 (T_w - T_\infty) / \nu^2$

Pr Prandtl number, ν/α

Nu Nusselt number, hb/k

$\overline{Nu}$ Average Nusselt number

Ra' Channel Rayleigh number, $Gr \, Pr \, b/l$

Θ Temperature, $(T - T_\infty)/(T_w - T_\infty)$

Φ Velocity, Ub/ν

Π Pressure, $P'b^2/(\rho\nu^2)$

SUBSCRIPTS

b based on bulk

∞ based on ambient

w value at wall

FULLY DEVELOPED LAMINAR NATURAL CONVECTION IN A VERTICAL PARALLEL PLATE CHANNEL WITH SYMMETRIC UNIFORM WALL TEMPERATURE

1. INTRODUCTION

The physical phenomenon considered in this thesis is that of temperature induced buoyant flow. In a situation where a fluid such as air is in contact with a heated surface, the fluid near the surface will be heated by conduction and begin to rise. This motion is a result of the decrease in density of the fluid when heated. Common examples of this type of flow are found in heat exchangers of all types.

If a vertical channel is formed from two parallel heated walls, a channel flow will be induced. As the lower density fluid rises, cooler and therefore denser fluid is drawn into the channel from below. This net fluid motion from the lower reservoir to the upper reservoir is referred to as a thermosyphon. A graphical depiction of this geometry is shown in figure 1.1. Once fluid motion has begun, heat transfer from the hot walls to the moving fluid is convective. As a steady state is reached, a constant heat transfer with time is achieved. It is an understanding of this steady state heat transfer from the hot walls to the cool fluid which is the goal of this work.

Natural convection in vertical channels and tubes is of great importance in many practical applications. A greater understanding of natural convection is expected to be beneficial in the design and use of arrangements involving chimneys, heaters, nuclear reactors, microchips, power supplies and other devices. Other possible applications include problems in combustion and flammability of materials, either to retard or enhance combustion. Vertical or inclined ducts may be of various cross-sections. Ducts of circular cross-section and those formed by two parallel plates of infinite width placed a known distance apart are the most convenient for

study. It is the latter of these two geometries upon which we focus attention in this study.
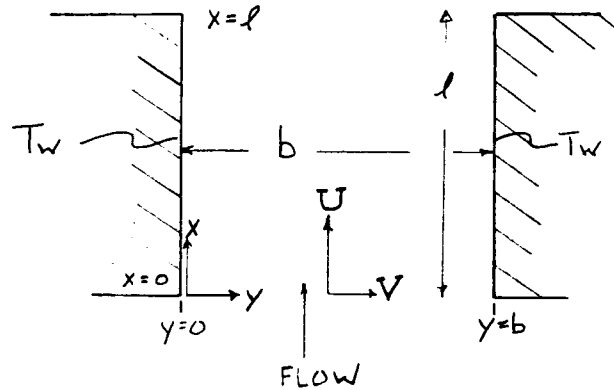

Figure 1.1 Flow Geometry

Within chimneys and heaters, natural convection occurs between a high temperature fluid and the walls of the channel. Safety and reliability of nuclear reactors could be enhanced by more effective natural convection for cooling which allows for higher power output without resorting to noisy and unreliable pumps. Effective natural convection in electronic equipment promises to allow for an increase in the component density while prolonging service life. Improvements in convective heat transfer within heaters may allow more fuel-efficient and safer operation. Ignition and combustion characteristics of household and aircraft components may also benefit from a greater understanding of the natural convection phenomenon.

Due to the importance of natural convective flows, many investigators have devoted considerable effort in the past to quantifying and predicting heat transfer results for this geometry. Consequently, much work exists for both forced convection and natural convection within channels and tubes. Natural convection between two plates with a temperature elevated with respect to the surroundings was

first studied experimentally and analytically by Elenbaas [1]. The first experiments reported by Elenbaas and later corroborated numerically by Bodia and Osterle [2], have established a foundation on which to judge all subsequently obtained results. The fluid studied by Elenbaas was air (Pr = 0.7), and experiments were conducted over a range of channel Rayleigh numbers (Ra') from 0.2 to 3,000. The experiments were performed with two elevated temperature plates. Air was assumed to enter the channel from a quiescent uniform temperature room at the bottom of the channel as a result of the buoyancy effects from the two channel wall plates.

Sparrow and Bahrami [3] reported critically on Elenbaas and other's earlier work. This work noted that for Elenbaas' report at small values of interplate spacing, the corrections for extraneous heat losses were much larger than the actual heat transfer results.

Bar-Cohen and Rohsenow [4] reported empirical relationships for Nusselt number that support Elenbaas' results. Their report emphasized the maximization of heat transfer for banks of parallel plates for symmetric and asymmetric constant wall temperature and isoflux boundary conditions. Results were provided for an optimal spacing for efficient use of space when designing heat exchangers.

Seban and Shimazaki [5] compared results for uniform heat flux boundary to those of uniform wall temperature to predict that the uniform heat flux boundary condition results in a greater heat transfer by approximately 12% to 13.5%. The current results presented here, when compared to the companion work of Aktan [6] support this general finding.

Aung [7] presented analytical results for asymmetrically heated walls with either a uniform temperature or heat flux. Quintiere and Mueller [8] report approximate analytical solutions for the uniform wall temperature boundary as well. Martin, Raithby and Yovanovich [9] discuss the existence of the asymptotic behavior of an average Nusselt number at low channel Rayleigh numbers.

Sparrow and Bahrami [3] reported results critical of previous work. Experimental investigation using a naphthalene sublimation technique led to potentially more accurate results in the low Ra' region where edge effects are the greatest. Sparrow also discussed the effect of property evaluation temperature on final heat transfer results, indicating the relative importance of evaluating fluid properties in a standardized manner. Colwell and Welty [10] experimentally investigated the low Ra' regime by use of a low Pr fluid (mercury). In this report Colwell and Welty discuss the effects aspect ratio and a low Pr fluid have on heat transfer and entrance effects. This investigation somewhat avoids this issue by neglecting the third dimension of depth in the governing equations.

In the analysis of both forced and natural convection in channel flows, two boundary condition categories can be said to exist. The first is when the walls are maintained at a constant temperature, this is referred to as UWT (uniform wall temperature boundary). The second is UHF (uniform heat flux boundary). This paper deals with the UWT boundary condition. Analysis of these two categories of problems is similar, with results for the local Nusselt number being somewhat greater for the UHF condition [5,11].

Heat transfer is generally greater in forced convection. Results are presented in terms of Nusselt numbers, either averaged over the channel length or local. Local Nusselt numbers can be defined on the basis of either the ambient temperature or bulk temperature.

Table 1.1 shows a brief compilation of existing work of relevance to this study. This table contains information cited from Kays [12]. Values shown in Table 1.1 are for local Nusselt numbers. The value listed for natural convection between parallel plates with the UWT boundary condition is a result of the analytical study presented in this paper. A companion paper by Aktan [6] provided a result of 4.12 for natural convection between parallel plates under the UHF boundary condition.

Nusselt Numbers: $Nu_b = h_b b/k$

	Circular Tubes		Parallel Plates	
	UWT	UHF	UWT	UHF
Forced	3.66	4.36	7.60	8.24
Natural			3.77*	4.12**

Table 1.1 Local Nusselt Numbers

The goal of this work is to provide analytical results for uniform wall temperature (UWT), fully developed, symmetric boundary condition, natural convection in a vertical parallel plate channel. It is desirable to fill in the above table in order to have a complete understanding of the relationship between natural and forced flows, UWT and UHF boundary conditions for different geometries. In Table 1.1, the result from Aktan[6] is marked with two asterisks(**), while the result from this work is marked with a single asterisk(*). Analytical solutions are inherently desirable because they embody the physics of the problem while clearly displaying any simplifications used in arriving at the solution. This allows easier adaptation of the results presented within this paper to other situations. Numerical solutions tend to conceal simplifications. Additionally computer code must be copied or rewritten to be adapted to situations not anticipated by the original author of the code. We hope to contribute a closed form analytical solution to the well known Elenbaas natural convection problem that is easily adaptable to other situations.

2. PROBLEM FORMULATION

2.1   Problem Statement

Two vertical parallel flat plates separated by a distance b, having length l, are maintained at a uniform temperature T_w and situated in a room of uniform quiescent air at temperature T_∞. As shown in Figure 1.1 the setup resembles a thermosyphon with both the top and bottom open to an infinite reservoir. The walls are assumed to be at an elevated temperature($T_w > T_\infty$). The plates are presumed to be of infinite width normal to the plane of the paper and of large vertical height l relative to plate spacing b. The temperature difference creates a buoyant effect, causing the fluid between the walls to rise against gravity. This causes fluid to be drawn in at the inlet(bottom opening) from the room conditions. The fluid is taken to enter the channel at T_∞. The plates are taken to be infinitely wide, thus eliminating any edge effects associated with infiltration of cool air. The local gravitational acceleration is g.

Flow within a vertical channel can be divided into three distinct regions. The first, near the channel entrance, is characterized by developing thermal and momentum boundary layers. In this region fluid velocity is high near the walls and much of the fluid remains at the ambient temperature. The heat transfer problem in the first regime is much like that of a single vertical flat plate. In the second region, boundary layers have merged at the plane of symmetry, although the velocity is still high near the wall. This is the transition regime between flow characteristic of a single flat plate and that of channel flow. The third region is the fully developed domain, in which the velocity profile has become parabolic. In the fully developed region, boundary layers are no longer present. Both the velocity and temperature

profiles are fully developed. Fully developed conditions for the velocity profile mean that velocity no longer changes in the direction of flow. Fully developed conditions for the temperature profile are somewhat different. The temperature profile does not attain a situation where it remains constant, rather the fluid is heated continuously and warms. The fully developed conditions for the temperature profile requires that, although the fluid continues to warm, the general shape of the profile remains the same. More precisely, as a consequence of the fully developed temperature profile we know there exists a generalized temperature distribution which is unchanging in the direction of flow. These fully developed assumptions will be discussed in greater detail in the following section.

If the channel is long with respect to plate spacing, much of the channel will conform to the fully developed region explained above. It is therefore convenient to assume that the entire channel conforms to the fully developed conditions, provided that it is long enough with respect to plate spacing. This assumption will be made.

2.2   Governing Equations

We will begin by considering the momentum equation in the x-direction(vertical). The Boussinesq approximation has been made so that density variations with temperature are considered negligible except in the buoyant force term.

$$\rho\left(U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}\right)=-\frac{\partial P}{\partial x}+\mu\nabla^2 U-\rho g \qquad (2.1)$$

We will utilize the notion of fully developed flow to simplify Eq. (2.1). For fully developed flow, the change in the x component of velocity is zero($\partial U/\partial x = 0$). By examination of the two dimensional continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{2.2}$$

we can see that the change in the y component of velocity must also be zero($\partial V / \partial y = 0$). From this observation and the boundary condition that the walls are impermeable to flow it is known that

$$V = const = 0 \tag{2.3}$$

With these simplifications the momentum equation becomes

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{d^2 U}{dy^2} - \rho g \tag{2.4}$$

Equation (2.4) can be applied outside the flow channel to get

$$0 = \frac{-\partial P_\infty}{\partial x} - \rho_\infty g \tag{2.5}$$

If we subtract Eq. (2.5) from Eq. (2.4) we get

$$0 = \frac{-\partial (P - P_\infty)}{\partial x} + \mu \frac{d^2 U}{dy^2} + (\rho_\infty - \rho)g \tag{2.6}$$

It is assumed that the fluid behaves as an ideal gas ($\rho T = const.$), and that the thermal expansion coefficient β can be expressed as $1/T_\infty$. It is also convenient to reckon the pressure relative to the hydrostatic pressure ($P' = P - P_\infty$). With these simplifications, Eq. (2.6) becomes

$$\frac{d^2U}{dy^2} = \frac{1}{\mu}\frac{dP'}{dx} - \frac{g\beta}{v}(T - T_\infty)$$

(2.7)

The energy equation can be reduced in a similar manner. We consider initially the two dimensional energy equation with constant fluid conductivity, no internal generation and negligible compressibility.

$$\rho c_p\left(\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

(2.8)

As before, we will make simplifying assumptions consistent with fully developed conditions. From [12,5], fully developed conditions for the temperature can be stated as

$$\frac{\partial}{\partial x}\left(\frac{T_w - T}{T_w - T_b}\right) = 0$$

(2.9)

where T_b in Eq. (2.9) is the bulk mean or average temperature across the channel at any point x. A more rigorous definition of the bulk temperature can be found from Kays and Crawford [12] as

$$T_b = \frac{\int_0^b \rho U c_p T dy}{\int_0^b \rho U c_p dy}$$

The impact of Eq. (2.9) on the velocity profile will be investigated in greater detail later in the analysis. It is known, from the simplification of the momentum equation, that V is zero. If the fluid examined is air, axial conduction can be

neglected, thus $\partial^2 T/\partial x^2$ is zero. If conditions are taken to be steady, $\partial T/\partial t = 0$. With these simplifications, the energy equation becomes

$$\frac{\partial^2 T}{\partial y^2} = U \frac{\rho c_p}{k} \frac{\partial T}{\partial x} \qquad (2.10)$$

With the reduction outlined above the velocity and temperature distributions are described by Eq. (2.7) and (2.10) respectively. For the current geometry the boundary conditions are as follows.

@ y = 0	@ y = b
$U = 0$	$U = 0$
$T = T_w$	$T = T_w$

Table 2.1 Boundary Conditions

Next we turn our attention to nondimensionalizing Eq. (2.7) and (2.10). We will introduce five dimensionless parameters to this end.

velocity	$\Phi = Ub/\nu$
temperature	$\Theta = (T - T_\infty)/(T_w - T_\infty)$
pressure	$\Pi = P'b^2/\rho \nu^2$
x coordinate	$\xi = x/b$
y coordinate	$\eta = y/b$

Table 2.2 Dimensionless Parameters

In terms of these variables the momentum Eq. (2.7) takes the following form.

$$\frac{\partial^2 \Phi}{\partial \eta^2} = \frac{\partial \Pi}{\partial \xi} - \frac{g\beta b^3}{\nu^2} \Theta \left(T_w - T_\infty \right) \qquad (2.11)$$

We recognize the dimensionless Grashof number as the following.

$$Gr = \frac{g\beta b^3}{\nu^2} \left(T_w - T_\infty \right) \qquad (2.12)$$

Equation (2.11) states that the velocity Φ is only a function of η, we can therefore surmise that the ξ functionality of $d\Pi/d\xi$ must exactly cancel the ξ functionality of Θ the nondimensional temperature. This relationship between the pressure term and the temperature term will be illuminated in later analysis. So we have as our non-dimensional x momentum equation.

$$\frac{d^2 \Phi}{d\eta^2} = \frac{\partial \Pi}{\partial \xi} - Gr\Theta \qquad (2.13)$$

And for the energy Eq. (2.10), we can rewrite with substitutions as

$$\frac{\partial^2 \Theta}{\partial \eta^2} \frac{\left(T_w - T_\infty \right)}{b^2} = \Phi \frac{\nu}{b} \frac{\rho c_p}{k} \frac{\partial \Theta}{\partial \xi} \frac{\left(T_w - T_\infty \right)}{b} \qquad (2.14)$$

Cancellation and rearrangement of Eq. (2.14), as well as recognition of the Prandtl number as ($Pr = \nu/\alpha$) with ($\alpha = k/\rho c_p$) we get

$$\frac{\partial^2 \Theta}{\partial \eta^2} = \frac{\partial \Theta}{\partial \xi} \Phi Pr \qquad (2.15)$$

The problem indicated by equations (2.7) and (2.10) is now reformulated to equations (2.13) and (2.15). In dimensionless form the boundary conditions are

@ $\eta = 0$	@ $\eta = 1$
$\Phi = 0$	$\Phi = 0$
$\Theta = 1$	$\Theta = 1$

Table 2.3 Dimensionless Boundary Conditions

2.3 Separation of Variables

In order to proceed with further analysis, it will be necessary to introduce a separation of variables for the nondimensional temperature. The function Θ is expected to be a function of both the non dimensional coordinates ξ and η. Two new functions will be introduced to allow for this separation.

$$r = r(\eta) \qquad (2.16)$$

$$s = s(\xi) \qquad (2.17)$$

To correctly define the temperature Θ in terms of $r(\eta)$ and $s(\xi)$, a further examination of the definition of fully developed flow from Eq. (2.9) is required. As defined from table 2.2

$$\Theta(\eta, \xi) = \left(\frac{T - T_\infty}{T_w - T_\infty} \right) \qquad (2.18)$$

The bulk temperature is defined as the average value of the temperature profile at any value of ξ. A nondimensional bulk temperature can be difined in a similar manner to that of Eq. (2.18) as

$$\Theta_b(\xi) = \left(\frac{T_b - T_\infty}{T_w - T_\infty} \right) \tag{2.19}$$

In order to satisfy the definition of a fully developed temperature profile from Eq. (2.9), we can observe that

$$\left(\frac{T_w - T}{T_w - T_b} \right) = \frac{1 - \Theta}{1 - \Theta_b} \tag{2.20}$$

We can then introduce our separation variables in a convenient manner

$$1 - \Theta = r(\eta) \cdot s(\xi) \tag{2.21}$$

$$1 - \Theta_b = s(\xi) \tag{2.22}$$

From these definitions we can see that the ξ functionality of the ratio defined by Eq. (2.9) has been canceled out, this is a requirement for the condition of a fully developed temperature profile.

$$\frac{\partial}{\partial x} \left(\frac{T_w - T}{T_w - T_b} \right) = \frac{\partial}{\partial x} \left(\frac{1 - \Theta}{1 - \Theta_b} \right) = \frac{\partial}{\partial x} \left(\frac{rs}{s} \right) = 0 \tag{2.23}$$

The goal is to obtain a solution to the energy Eq. (2.15). The temperature profile as described by Eq. (2.21) must be differentiated in order to allow for a substitution into

Eq. (2.15). It is necessary to take the second derivative of Eq. (2.21) with respect to η and the first derivative with respect to ξ.

$$\frac{\partial^2 \Theta}{\partial \eta^2} = -s \cdot \frac{d^2 r}{d\eta^2} \qquad\qquad \frac{\partial \Theta}{\partial \xi} = -r \cdot \frac{ds}{d\xi} \qquad\qquad (2.24)$$

We can now rewrite Eq. (2.15) with these substitutions to get

$$-s \cdot \frac{d^2 r}{d\eta^2} = \Pr \Phi (-r) \cdot \frac{ds}{d\xi} \qquad\qquad (2.25)$$

A simple rearrangement allows us to separate the variables in this differential equation.

$$\frac{1}{\Pr} \cdot \frac{1}{\Phi} \cdot \frac{1}{r} \cdot \frac{d^2 r}{d\eta^2} = \frac{1}{s} \cdot \frac{ds}{d\xi} \qquad\qquad (2.26)$$

The left side of Eq. (2.26) is only a function of η, and the right side is only a function of ξ. Both sides must therefore be equal to a constant denoted by $-\Lambda$.

$$\frac{1}{\Pr} \cdot \frac{1}{\Phi} \cdot \frac{1}{r} \cdot \frac{d^2 r}{d\eta^2} = \frac{1}{s} \cdot \frac{ds}{d\xi} = -\Lambda \qquad\qquad (2.27)$$

Two distinct ordinary differential equations can thus be obtained from Eq. (2.27). We will examine first the equation in s.

$$\frac{1}{s} \frac{ds}{d\xi} = -\Lambda \qquad\qquad (2.28)$$

By performing a single integration step and taking the exponent of both sides we get

$$s = e^{-\Lambda\xi + C} \tag{2.29}$$

where C in Eq. (2.29) is an integration constant. From Eq. (2.22), boundary conditions can be developed for s. It is known that the bulk temperature, as defined by Eq. (2.19), is zero at the inlet to the channel where $T_b = T_\infty$, therefore s must be one when ξ is zero. The constant in Eq. (2.29) is therefore zero. So we have

$$s = e^{-\Lambda\xi} \tag{2.30}$$

For the second ordinary differential equation in r from Eq. (2.27), we have

$$\frac{1}{Pr} \cdot \frac{1}{\Phi} \cdot \frac{1}{r} \cdot \frac{d^2 r}{d\eta^2} = -\Lambda \tag{2.31}$$

In order to proceed with a solution to Eq. (2.31), it is necessary to obtain a solution for the dimensionless velocity Φ. We will examine Eq. (2.13) in light of the separation variables that have been introduced. Because the velocity in Eq. (2.13) is only a function of η, we know that the ξ functionality of the two remaining terms must cancel, therefore

$$\frac{\partial^2 \Pi}{\partial \xi^2} = Gr \frac{\partial \Theta}{\partial \xi} = Gr\left(-r\frac{ds}{d\xi}\right) \tag{2.32}$$

By integrating twice we get an expression for the pressure Π as

$$\Pi = -rGr\int s\,d\xi + C_1\xi + C_2 \tag{2.33}$$

From Eq. (2.30) the indication integration can be performed and substituted to obtain

$$\Pi = \frac{rGr}{\Lambda} e^{-\Lambda \xi} + C_1 \xi + C_2 \qquad (2.34)$$

In order to solve for the integration constants in Eq. (2.34), boundary conditions must be developed. The nondimensional pressure as defined from table 2.2 contains $P' = (P - P_\infty)$. P', and therefore Π, must be zero at the entrance and exit of the channel, where $P = P_\infty$. From this, the constants of Eq. (2.34) can be evaluated to get

$$\Pi = \frac{rGr}{\Lambda} \left(e^{-\Lambda \xi} - 1 + \frac{\xi}{\xi_l} - \frac{\xi}{\xi_l} e^{-\Lambda \xi_l} \right) \qquad (2.35)$$

Where $\xi @ l = \xi_l = l/b$ is the dimensionless x coordinate evaluated at the channel exit, referred to as the channel aspect ratio. Once this pressure function is known, we can take the first derivative with respect to ξ for substitution into Eq. (2.13). The first derivative of Eq. (2.35) with respect to ξ is

$$\frac{d\Pi}{d\xi} = Gr \left(-rs + r \left(\frac{1}{\Lambda \xi_l} - \frac{e^{-\Lambda \xi_l}}{\Lambda \xi_l} \right) \right) \qquad (2.36)$$

With substitution of Eq. (2.36) into (2.13) we have

$$\frac{d^2 \Phi}{d\eta^2} = Gr \left(-rs + r \left(\frac{1}{\Lambda \xi_l} - \frac{e^{-\Lambda \xi_l}}{\Lambda \xi_l} \right) - 1 + rs \right) \qquad (2.37)$$

With cancellation we are left with, as the momentum equation

$$\frac{d^2\Phi}{d\eta^2} = Gr\left(r\left(\frac{1}{\Lambda\xi_l} - \frac{e^{-\Lambda\xi_l}}{\Lambda\xi_l}\right) - 1\right) \tag{2.38}$$

Further simplification is possible if we observe that, for channels that are long with respect to plate spacing

$$\Lambda\xi_l \gg 1 \Rightarrow \left(\frac{1}{\Lambda\xi_l} - \frac{e^{-\Lambda\xi_l}}{\Lambda\xi_l}\right) \approx 0 \tag{2.39}$$

This simplification is consistent with previous simplifications that the channel must be sufficiently long so that fully developed flow can be assumed to prevail for the entire channel. The unknown constant Λ is yet to be determined, however, for sufficiently large aspect ratio channels the momentum equation can be simplified to

$$\frac{d^2\Phi}{d\eta^2} = -Gr \tag{2.40}$$

The result of two integration steps on Eq. (2.40) is

$$\Phi = \frac{-Gr}{2}\eta^2 + C_1\eta + C_2 \tag{2.41}$$

By applying the boundary conditions for Φ from table 2.3, both constants can be solved for to get

$$\Phi = \frac{-Gr}{2}\left(\eta^2 - \eta\right) \tag{2.42}$$

Equation (2.42) describes the fully developed velocity profile as parabolic. This velocity profile is based on the assumption that the channel aspect ratio is sufficiently large that fully developed conditions can be assumed to exist for the entire channel. Eq. (2.42) is the desired velocity function that is sought for substitution into Eq. (2.31). By substituting we get

$$\frac{d^2 r}{d\eta^2} = \frac{-\Lambda Gr \, \mathrm{Pr}}{2} \left(\eta - \eta^2\right) r \tag{2.43}$$

2.4   Application of a Series

The function r can be determined by means of a series solution. It is proposed that the function r is given by a convergent series in η .

$$r = \left(A_0 + A_1 \eta + A_2 \eta^2 + A_3 \eta^3 \ldots\ldots A_m \eta^m\right) \tag{2.44}$$

With two differentiations, Eq. (2.44) becomes

$$\frac{d^2 r}{d\eta^2} = \left(2A_2 + 6A_3 \eta \ldots (m-1)(m) A_m \eta^{(m-2)}\right) \tag{2.45}$$

Equation (2.44) and (2.45) can be substituted into Eq. (2.42) to get

$$\left(2A_2 + 6A_3 \eta \ldots (m-1)(m) A_m \eta^{(m-2)}\right) = \frac{\Lambda Gr \, \mathrm{Pr}}{2} \left(\eta^2 - \eta\right)\left(A_0 + A_1 \eta + \ldots A_m \eta^m\right) \tag{2.46}$$

For convenience of notation the following term is introduced for substitution into Eq. (2.46).

$$\Omega = \frac{\Lambda Gr \, \mathrm{Pr}}{2} \qquad (2.47)$$

By making this simplifying substitution and equating the coefficients of equal powers of η from the left and right sides, the following recurrence relation is obtained.

$$A_m = \frac{\Omega}{(m)(m-1)}\left(A_{(m-4)} - A_{(m-3)}\right) \qquad (2.48)$$

This relationship applies only for $m \geq 4$. The first four coefficients must be determined by use of boundary conditions as discussed in the following section.

3. SOLUTION

3.1 Series Solution

With the recurrence relation obtained in Eq. (2.48), any coefficient A_m can be defined in terms of lower order coefficients. This will be useful in evaluating as many terms in the series function $r = r(\eta)$ as desired. The first four terms must be examined without the use of the recurrence formula (2.48), but by use of the boundary conditions. From the boundary condition for Θ from table 2.3 we know

$$r(\eta = 0) = 0$$

Gives

$$A_0 = 0$$

and by examination of Eq. (2.46)

$$2A_2 = 0 \therefore A_2 = 0$$

$$6A_3 = -\Omega A_0 \therefore A_3 = 0$$

In order to evaluate any further terms, an expression relating the constant Ω to the constant A_1 will be required, as both appear within the recurrence relation. To this end the mass flow through a control volume must be evaluated. For this situation the energy added to fluid flowing between two heated walls is equal to the energy convected to the fluid from the two walls in a differential length dx. This can be written for a plate width W.

$$\dot{m} \cdot c_p \cdot dT_b = -k \frac{\partial T}{\partial y}\bigg|_0 \cdot 2dx \cdot W \qquad (3.1)$$

The factor of two is to account for both walls being hot. By dividing though by the plate width, mass flow is written per unit width. Rearranging we get

$$\frac{\dot{m}'c_p}{2k} \frac{dT_b}{dx} = -\frac{\partial T}{\partial y}\bigg|_0 \qquad (3.2)$$

Making substitutions with the dimensionless forms introduced earlier from table 2.2 and Eq. (2.19) we get

$$\frac{\dot{m}'c_p}{2k} \frac{d\Theta_b}{d\xi} = -\frac{\partial \Theta}{\partial \eta}\bigg|_0 \qquad (3.3)$$

By making substitutions using the separation variables from equations (2.21) and (2.22), Eq. (3.3) can be rewritten as

$$\frac{\dot{m}'c_p}{2k}\left(-\frac{ds}{d\xi}\right) = s\frac{dr}{d\eta}\bigg|_0 \qquad (3.4)$$

From Eq. (2.28), the constant Λ can be substituted to obtain

$$\frac{\dot{m}'c_p}{2k}\Lambda = \frac{dr}{d\eta}\bigg|_0 \qquad (3.5)$$

The mass flowrate through the channel can be written as

$$\dot{m}' = \rho v \int_0^1 \Phi d\eta \qquad (3.6)$$

Making a substitution from Eq. (2.42) for the velocity term and performing the indicated integration of the two term polynomial, we get

$$\dot{m}' = \frac{\rho v Gr}{12}$$

<div align="right">(3.7)</div>

Substitute Eq. (3.7) into (3.6) and solve for the constant

$$\Lambda = \frac{dr}{d\eta}\Big|_0 \frac{24}{Gr\,Pr}$$

<div align="right">(3.8)</div>

Finally, from the definition of the function r from Eq. (2.43), the crucial simplification can be performed to get

$$\Lambda = A_1 \frac{24}{Gr\,Pr}$$

<div align="right">(3.9)</div>

If equations (2.47) and (3.9) are combined and the constant Ω is isolated

$$\Omega = 12 A_1$$

<div align="right">(3.10)</div>

Equation (3.10) is the desired relationship between the two constants that appear within the recurrence relation Eq. (2.48). A solution has been found for three of the first four coefficients of the function r. All remaining coefficients build upon the first four($A_0 = 0, A_1 = ?, A_2 = 0, A_3 = 0$). As more terms are added the function becomes more accurate, however, r can always be reduced to a function of A_1 only. The second boundary condition can then be applied. From the boundary conditions for Θ from table 2.3 we know

$$r(\eta = 1) = 0$$

gives

$$\sum A_m = 0 \qquad (3.11)$$

The Eq. (3.11) has only one unknown, A_1. This equation can be solved for the unknown coefficient. Once we have determined the value of A_1, we will use this information to develop heat transfer results, such as local and average heat transfer coefficients. Once the complete expression for the series r is established, the temperature profile from Eq. (2.21) is known. This profile is plotted in figure (3.1) below.
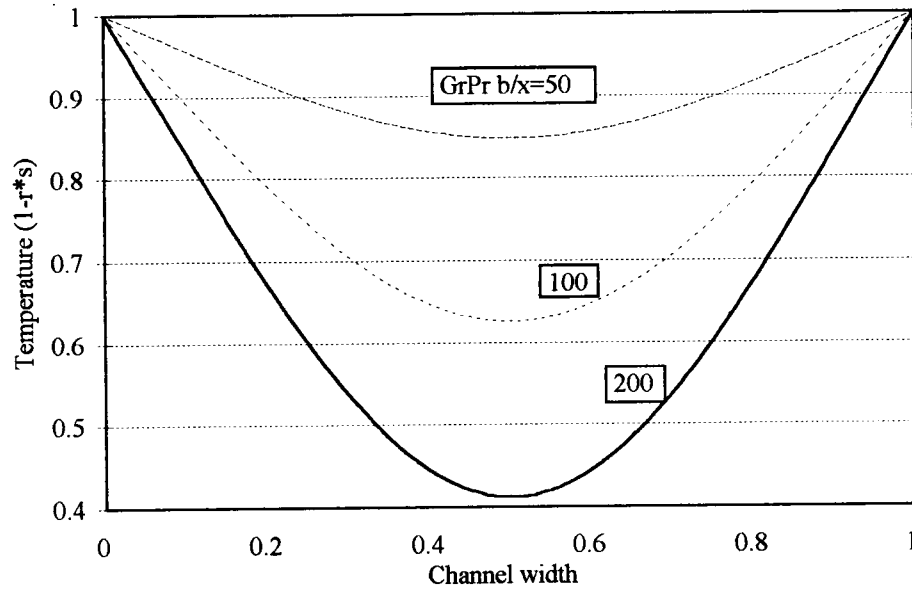

Figure 3.1 Temperature Profiles

As can be seen in figure 3.1, the temperature profile is similar as the fluid is heated. This corresponds to the fully developed condition that was discussed earlier. Although the profile is seen to shallow somewhat as x/b is increased, the form of the profile is constant.

3.2 Heat Transfer

Having determined the coefficient A_1 and the functions r and s, heat transfer results for the given conditions can be developed. Heat transfer results are presented in terms of a nondimensional Nusselt number defined as

$$Nu = \frac{hb}{k} \tag{3.12}$$

The convective heat transfer coefficient h of Eq. (3.12) can be defined in two different ways. h can be defined in terms of the temperature difference between the hot walls and the local bulk temperature

$$\dot{q}'' = h_b \left(T_w - T_b \right) \tag{3.13}$$

or in terms of the total temperature difference between the hot channel walls and the ambient fluid.

$$\dot{q}'' = h_\infty \left(T_w - T_\infty \right) \tag{3.14}$$

Defining the convective heat transfer coefficient in these two ways results in Nusselt numbers based on these two temperature differences. By defining h as in Eq. (3.13) we can write an expression that equates the energy conducted through the fluid to that convected by the fluid in order to develop a local Nusselt number based on the bulk temperature.

$$-k\frac{\partial T}{\partial y}\bigg|_0 = h_b(T_w - T_b) \tag{3.15}$$

Our first step will be to replace the temperature change at the wall with the dimensionless form from table 2.2 as follows

$$-k\frac{\partial \Theta}{\partial \eta}\bigg|_0 \frac{(T_w - T_\infty)}{b} = h_b(T_w - T_b) \tag{3.16}$$

We will rearrange, shifting terms to the RHS to form the bulk, local Nusselt number. In the same step we will replace the partial derivative at the wall with the appropriate derivative from the definition of Θ from Eq. (2.21).

$$\frac{\partial \Theta}{\partial \eta} = -s\frac{dr}{d\eta} \tag{3.17}$$

Combining Eq. (3.17) with (3.16) and rearranging we have

$$\left(\frac{T_w - T_\infty}{T_w - T_b}\right)s\frac{dr}{d\eta}\bigg|_0 = \frac{h_b b}{k} \tag{3.18}$$

From the definition of the bulk temperature in Eq. (2.19) we recognize the temperature ratio on the LHS and can make a substitution as follows

$$\frac{1}{1-\Theta_b}s\frac{dr}{d\eta}\bigg|_0 = Nu_b \qquad (3.19)$$

From Eq. (2.22) we can see that a further cancellation can be performed because

$$\frac{1}{1-\Theta_b} = \frac{1}{s} \qquad (3.20)$$

And from the definition of the series r from Eq. (2.44) we know

$$\frac{dr}{d\eta}\bigg|_0 = A_1 \qquad (3.21)$$

We can then make a substitution into Eq. (3.19) and simplify to get

$$A_1 = Nu_b \qquad (3.21)$$

The x dependency of the local Nusselt number is lost if the convective heat transfer coefficient is based on bulk temperature as in Eq. (3.13). If we perform a similar derivation as that above, but this time define our convective heat transfer coefficient on ambient temperature as in Eq. (3.14) we get

$$Nu_\infty = A_1 s \qquad (3.22)$$

For comparison with existing literature it is also desirable to derive an expression for heat transfer based on an average heat transfer coefficient

$$\dot{q} = h_{\infty,avg}(T_w - T_\infty)a \qquad (3.23)$$

The convective heat transfer coefficient in Eq. (3.23) is based on the same temperature difference as in Eq. (3.14). Consequently the Nusselt number obtained from this definition will be the average over the channel corresponding to the local value from Eq. (3.22). With the area $(a = 2l)$. Making a substitution for the LHS we have

$$\dot{m}c_p(T_{b,e} - T_\infty) = h_{\infty,avg}(T_w - T_\infty)2l \qquad (3.24)$$

Where $T_{b,e}$ is the bulk temperature at the exit of the channel. In this derivation we have need for a new temperature ratio to be

$$\Theta_{b,e} = \left(\frac{T_{b,e} - T_\infty}{T_w - T_\infty}\right) \qquad (3.25)$$

This quantity is the dimensionless bulk temperature, similar to that defined in Eq. (2.22), but applied at the exit of the channel. We can now combine equations (3.25) and (3.7) with (3.24) and rearrange to get

$$\frac{\rho c_p \sqrt{Gr}}{24}\Theta_{b,e} = h_{\infty,avg}l \qquad (3.26)$$

With the definition of the bulk temperature in terms of the separation variables from Eq. (2.22), we can see that the bulk, exit temperature as defined in Eq. (3.25) can be written as

$$\Theta_{b,e} = 1 - s(\xi_l) = 1 - e^{\left(\frac{-\Lambda l}{b}\right)} \qquad . \qquad (3.27)$$

We can then combine Eq. (3.27) with (3.26) and solve for the average Nusselt number based on fluid ambient temperature.

$$\frac{h_{\infty,avg} l}{k} = \frac{Gr \, Pr}{24}\left(1 - e^{\left(\frac{-A l}{b}\right)}\right)$$

(3.28)

If both sides of Eq. (3.28) are then multiplied by the same problem constant b/l, we arrive at the desired result

$$\overline{Nu_\infty} = \frac{h_{\infty,avg} b}{k} = \frac{Gr \, Pr \, b}{24 l}\left(1 - e^{\left(\frac{-24 A_1 l}{Gr \, Pr \, b}\right)}\right)$$

(3.29)

Equation (3.29) is plotted in figure 4.2 and will be discussed in greater detail in the following section. Defining the Nusselt number in this manner allows comparison with a many existing results. Figure 4.2 clearly shows that for a limited range of channel Rayleigh number, the analytical expression obtained here shows good agreement with previously published results.

4. RESULTS AND DISCUSSION

Using the recurrence relation, Eq. (2.48), 48 coefficients were defined as functions of A_1 only. This procedure is outlined in section 3. These terms when summed and set equal to zero create the second known boundary condition from Eq. (3.11). This equation can be solved for a value of A_1, which is also the local Nusselt number based on fluid bulk temperature. For a 48 term series

$$A_1 = 3.77 \qquad (4.1)$$

And, consequently

$$Nu_b = 3.77 \qquad (4.2)$$

The important result presented as Eq. (4.2) has been unavailable in heat transfer literature to date. This result tells us that the convective heat transfer coefficient is constant with ξ. From Eq. (3.13) we can see that the heat transfer from the hot walls to the cool fluid must then be decreasing with ξ because the temperature difference between the hot walls and the fluid is decreasing as the fluid warms. To get a result for the local Nusselt number based on fluid ambient temperature we must first use Eq. (4.1) and (3.9) to solve for s as defined in Eq. (2.30). From the result presented as Eq. (4.1) we can return to our simplification in Eq. (2.39) to observe that the results presented here should have good accuracy for channel aspect ratios (ξ_l) of ten or greater.

For a 48 term series

$$s = e^{\frac{-22.62\xi}{GrPr}} \qquad (4.3)$$

This can be used as indicated by Eq. (3.22) to get

$$Nu_\infty = 3.77e^{\frac{-22.62\xi}{GrPr}}$$

(4.4)

Equation (4.4) is plotted as figure 4.1 below. As can be seen in figure 4.1 the Nusselt number when based on the ambient temperature is not a constant with respect to ξ. When the product of Gr and Pr, called the Rayleigh number is low, the Nusselt number decays rapidly from a value of 3.77 at the channel entrance.
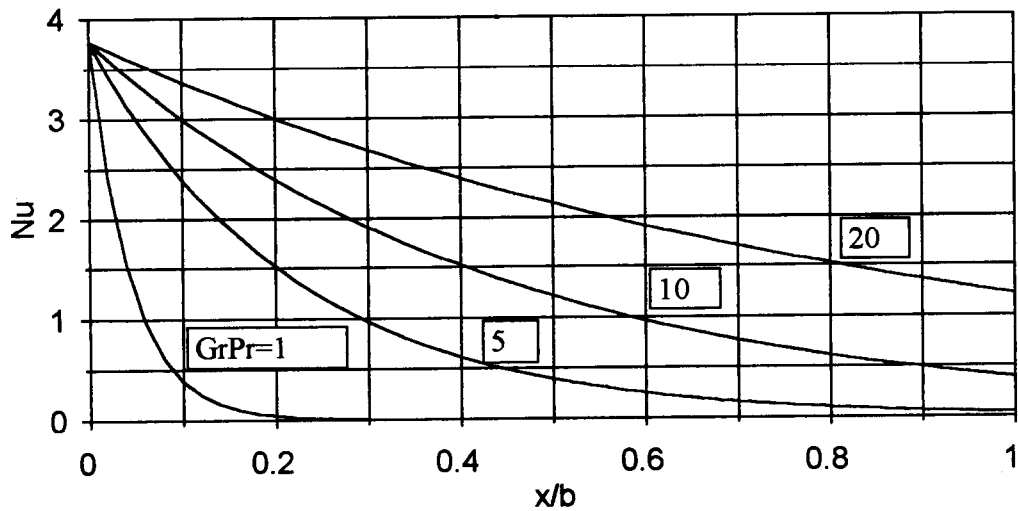

Figure 4.1 Local Nusselt Number based on Ambient Temperature

By comparing these two methods of obtaining heat transfer results we can see that the energy transferred from the walls to the fluid by either Eq. (3.13) or by Eq. (3.14) must decrease with increasing ξ. These two contrasting definitions simply make the decrease in heat transfer a result of the temperature difference or conversely as a

result of the convective coefficient decreasing. By making similar substitutions into Eq. (3.29), we get an expression for the average Nusselt number based on ambient or reservoir temperature. This is plotted in figure 4.2 for comparison with previous results.
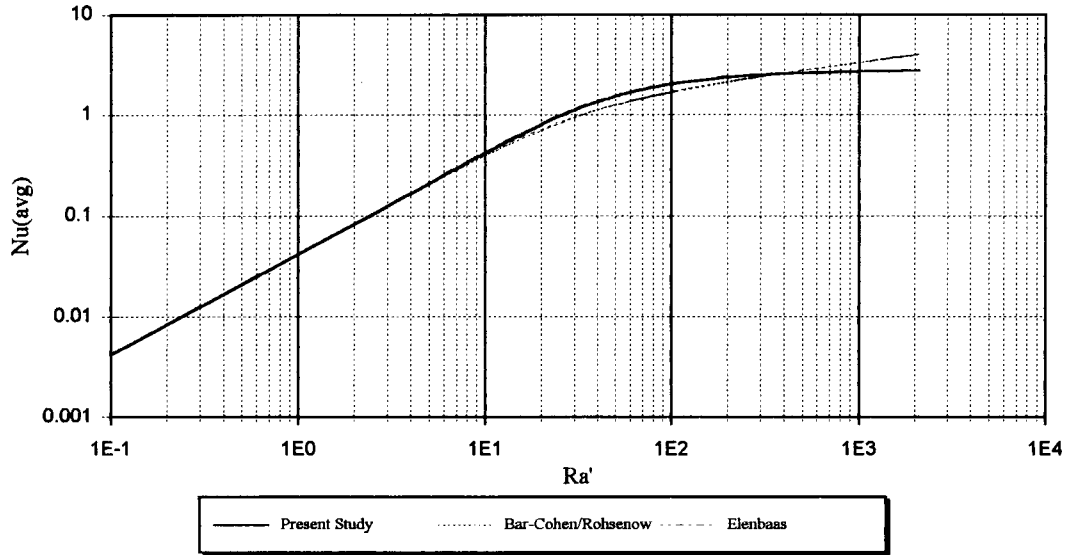
Figure 4.2 Average Nusselt Number Comparison

While applying the series solution method to determine the numerical value of A_1, the number of terms included in the series is of critical importance. As the recurrence formula from Eq. (2.48) is used to develop successive terms and then find solutions for A_1, two interesting features of the solution become apparent. First, an even number of terms must be used to avoid a meaningless and erratic solution. This is most notably true when the number of terms is less than 16. The second observation is that a computer mathematical application is required due to the fact that the solution was observed to change when the number of terms was increased from 12 to 14. A 14 term hand solution was time consuming and error prone. Hand

calculation beyond this point was therefore prohibitive. When a computer was employed the solution was seen to stabilize and the data are presented in figure 4.3. After approximately 26 terms are included, the value is stabilized at 3.77. No change was seen in the value of A_1 to 4 decimal places when the number of terms was increased to 48. This gives confidence that the series is sufficiently convergent with 48 terms that the final results presented are accurate to the precision indicated.
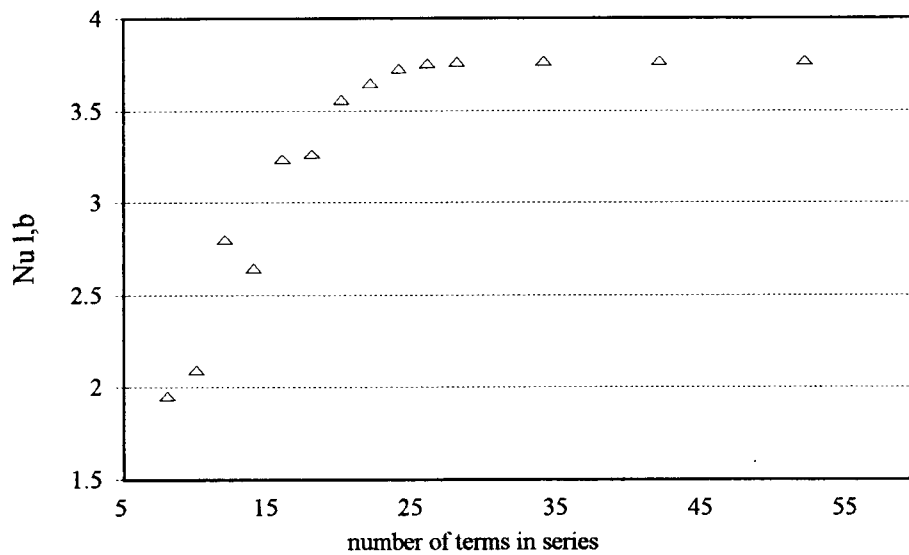

Figure 4.3 Convergence of Series

The expression for the average Nusselt number based on ambient fluid conditions is presented in figure 4.2. This value is plotted verses the channel Rayleigh number, which is traditional. As can be seen the analytical result obtained compares well over a range of channel Rayleigh number from 10 to 100. Figure 4.2 shows also the correlation presented by BarCohen and Rohsenow[2]. Results are plotted from a Rayleigh number of 0.1 to 1000. Results below this Rayleigh number are not included because of the assumption of negligible axial conduction. The

assumption of negligible axial conduction is not applicable to low Prandtl fluids such a liquid mercury, where axial conduction can be significant. Higher channel Rayleigh numbers are associated with channels that are short with respect to interplate spacing. Under these conditions, the assumption made here that the channel is sufficiently long that fully developed conditions exist everywhere within the channel is inaccurate.

5. CONCLUDING REMARKS

The number found for the local Nusselt number based on the bulk temperature is about what would be expected. This value of 3.77 has been unavailable in the literature before this report. The value of the averaged Nusselt number based on the ambient fluid temperature shows excellent agreement with previously reported data over specific ranges of the channel Rayleigh number.

The solutions obtained, being analytical in nature, offer greater flexibility and elegance compared to previous experimental results.

Further work is needed to apply the current solution technique to the entrance domain. This domain is characterized by developing boundary layers. In this analysis, simplifications used to reduce the complexity of the governing equations limits the applicability of the results for the high channel Rayleigh number regime. In addition, further work might focus on different wall conditions such as allowing one wall to be unheated or cooled.

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