AN ABSTRACT OF THE THESIS OF

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Abstract approved: Signature redacted for privacy.

Thomas P. Dick

This study investigated associations among teachers' pedagogical content beliefs, approaches to teaching, and their students' achievement in a high school Advanced Placement (AP) Calculus setting. The three major research questions concerned: (a) how well AP calculus teachers' pedagogical beliefs about mathematics, curriculum, and instruction aligned with a constuctivist point of view, (b) how AP teachers' pedagogical content beliefs were reflected in their approaches to teaching and goals for instruction, and (c) relationships among AP teachers' pedagogical content beliefs, approaches to teaching, and their students' achievement.

Teachers' pedagogical content beliefs were categorized as cognitively based (CB) or less cognitively based (LCB) using a belief questionnaire adapted from research with first-grade teachers. Telephone interviews with nine CB and eight LCB teachers served to provide additional insight into their beliefs and to gain information on how teachers approach teaching differential calculus. A researcher-designed Differentiation Test was administered to assess student achievement.

Interviews with nine CB teachers and eight LCB teachers revealed: (a) CB teachers were more likely to believe the role of the teacher was that of a facilitator/guide and the role of the student was to explore. LCB teachers were more likely to believe their role as that of a knowledge base, and the role of the student was to learn from the teacher. (b) CB teachers' self-reported classroom practices were found to be more student-centered and conceptual in nature than LCB teachers. (c) CB teachers were

more likely to use word problems when introducing topics, emphasize student involvement, have their students work in groups, emphasize visual approaches to topics, and consider students' knowledge when planning instruction. LCB teachers were more likely to present rules and theorems, work examples, and require students to memorize rules of differentiation.

Students of CB teachers were found to have a better conceptual understanding of differential calculus than students of LCB teachers. Students of CB teachers were better able to interpret graphical information and to interpret information given in a table. No differences were found in students' ability to work with symbolic information or to use a graphing calculator in conjunction with problem solving.

Relationships Among AP Calculus Teachers' Pedagogical Content Beliefs, Classroom Practice, and their Students' Achievement

by

Frederick W. Utter

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Dean of Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

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Frederick W. Utter, Author

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Dedicated to my parents,

Diana J. Utter
and
Frederick W. Utter

RELATIONSHIPS AMONG AP CALCULUS TEACHERS' PEDAGOGICAL CONTENT BELIEFS, CLASSROOM PRACTICE, AND THEIR STUDENTS' ACHIEVEMENT

CHAPTER ONE: INTRODUCTION

Mathematics education researchers and curriculum reformers have been developing new ideas about what it means to teach mathematics. These ideas emphasize that: (a) mathematics is not a set of procedures, but rather a changing science of quantity and patterns, (b) mathematics is learned through students construction of knowledge, not through the transmission by the teachers, and (c) mathematics students should be enabled to develop, solve, and debate interesting mathematical problems, rather than simply carry out procedures (Knapp and Peterson, 1995).

Teachers' thoughts, knowledge, judgements, and decisions have a profound effect on the way teachers teach as well as on the way students learn and achieve in their classrooms (Peterson, 1988). Research exploring the nature of beliefs has found a strong relationship between teachers' educational beliefs and their planning, instructional decisions, classroom practices, and student achievement. According to Pajares (1992), beliefs can be the single most important construct in educational research. Therefore, adopting the new constructivist theories into the classroom will require considerable change in most teachers' beliefs (Knapp and Peterson, 1995).

The focus of this study was to examine high school Advanced Placement (AP) calculus teachers' pedagogical content beliefs and to explore relationships among these beliefs, teachers' classroom practices, and student achievement. In particular, the questions addressed in this study are:

Teachers' pedagogical content beliefs

What are AP calculus teachers' pedagogical beliefs about mathematics, curriculum, and instruction in AP calculus? How well do their pedagogical content beliefs align with a constructivist point of view?

How do AP teachers' pedagogical content beliefs change with time? Do the pedagogical content beliefs of teachers become more cognitively based as they become familiar with the materials from a calculus reform project? Is there a relationship between how long a teacher has used the project materials and the degree to which they are cognitively based?

Teachers' classroom practices

How are AP teachers' pedagogical content beliefs reflected in their reports of their approaches to teaching, their concepts of the roles of the teacher and the learner, and their goals for instruction?

Student achievement

Is there a relationship between AP teachers' pedagogical content beliefs, approaches to teaching, and their students' achievement, including achievement of computational skills and problem solving?

Is there a relationship between teachers' pedagogical content beliefs and their student's ability to work in multiple representations: graphically, numerically, and symbolically?

The teachers in this study were high school Advanced Placement calculus teachers who were selected to participate in an inservice program as a part of the Calculus Connections Project, a teacher enhancement project funded by the National Science Foundation. In addition to the inservice received through the Calculus Connections Project, these teachers had three fundamental factors in common:

1. they taught AP calculus at the high school level
2. they were all using new calculus curriculum materials developed to be used with graphing calculators (*Calculus of a Single Variable*, by Dick and Patton, 1994)

3. all students in their calculus classes had access to graphing calculators (the Calculus Connections Project provided each school with a set of HP-48G calculators)

In this study teachers' pedagogical content beliefs were categorized as being cognitively based or less cognitively based. This categorization of teachers' pedagogical content beliefs, based on constructivist theories of learning, was drawn from an earlier study by Peterson, Fennema, Carpenter, and Loef (1989). Constructivism views teachers and students as active meaning-makers who are continuously giving meaning to each others' words and actions (Cobb, 1988). From this point of view, mathematical structures are not perceived, intuited, or taken in; instead, they are constructed. They are inventions of the individual's mind. On the other hand, the central assumption in the transmission view of learning is the idea that meaning is inherent in the words and actions of the teacher, or in the objects in the environment. By this account, the words and actions of teachers carry meaning in and of themselves that are waiting to be absorbed by the students.

We will begin our review of the literature in Chapter II by providing the reader with some background on calculus reform. After this context has been established, we will review the literature concerned with teachers' pedagogical content beliefs and establish a framework for studying these beliefs. In Chapter III we give a detailed account of the methods and procedures used in this study and in Chapter IV we present the results of our investigation. We will conclude this study with a discussion of results in Chapter V. In addition we will comment on the limitations of this study, and implications for future research.

CHAPTER TWO: THEORETICAL FRAMEWORK

This is a time of curricular reform in mathematics education and one of the most active areas has been calculus reform. This climate for change in the way calculus is taught has promoted the development of a number of curriculum reform projects. These curricular reforms are ultimately implemented through the classroom practices of teachers and the success of the reform is determined by students' conceptual understanding and academic achievement.

Research has shown that teachers' pedagogical content beliefs, i.e., beliefs about how a topic should be taught and how learning of the topic occurs, are closely related to both teachers' classroom practices and student achievement (Pajares, 1992; Peterson et al., 1989). This study investigated how high school Advanced Placement (AP) calculus teachers' pedagogical content beliefs relate to their classroom practices and their students' achievement. The teachers in this study were all actively engaged in implementing a specific reform curriculum in Advanced Placement calculus. A basic motivation for the study was an examination of factors that could help explain different levels of success in those teachers' efforts.

The purpose of this chapter is to provide the reader with some background on calculus reform and a theoretical framework through which we can discuss and investigate teachers' pedagogical content beliefs. We will begin this section by discussing the context of calculus reform, including:

- The Call for Reform
- Student Difficulties with Calculus Concepts
- Emerging Themes of Calculus Reform
- The Calculus Connections Project
- Calculus Reform and the Advanced Placement Calculus Program

After this context has been established, we will review the literature concerned with teachers' pedagogical content beliefs and establish a framework for studying these beliefs.

Calculus Reform

The Call for Reform

In recent years there has been a call for substantial change in precollege and college mathematics curricula and instructional methods (National Council of Teachers of Mathematics, 1989; National Research Council, 1989). Coupled with the availability of powerful and relatively inexpensive computing technology, this call presents a great opportunity for change in mathematics education.

One factor contributing to the present climate for change is the redefinition of what it means to be mathematically literate. Knowledge of facts and a facility with procedures are no longer considered adequate standards. Instead, a more conceptual definition considering mathematical literacy as a process of learning which emphasizes the social and cognitive construction of knowledge is called for (National Council of Teachers of Mathematics [NCTM], 1989). Reform efforts recommend a shift from teacher-directed transmission approaches to instruction that is learner-centered. These recommendations, based on recent developments in cognitive psychology, promote the construction of mathematical concepts by the student (Piliero & Confrey, 1992; Confrey, 1986).

Technology has also had a significant influence on the call for change. Calculators and computers can free students from algebraic manipulation, allowing for more emphasis on exploration and problem solving. The National Council of Teachers of Mathematics (NCTM), among others, highly favors the use of computing technology in mathematics instruction and testing. It is NCTM's position that teachers, authors, and test writers should integrate the use of calculators at all grade levels, and that the time presently spent on practicing computation could be used to further students' understanding and their ability to apply mathematics (NCTM, 1989).

The NCTM (1989) has recommended a more conceptually oriented curriculum that:

1. reduces the amount of time devoted to drill-and-practice,
2. involves students in challenging problem-solving situations,
3. creates a classroom atmosphere where questioning, exploration, reasoning, and justification are encouraged and expected, and
4. uses the power of computing technology to free students from tedious computations, allowing them to concentrate on problem-solving processes.

Another motivating force behind the call for change in the way mathematics is taught is the lack of understanding many students have of even the most basic mathematical concepts. Next we will review the literature concerned with student understanding of key calculus concepts.

Student Difficulties with Calculus Concepts

One of the factors influencing the reform movement is the concern among mathematics educators and the business community that there will not be a sufficiently trained technological workforce for entry into the twenty-first century (Berenson & Stiff, 1989). According to Ferrini-Mundy & Graham (1991), of the 600,000 students who enroll in calculus in four-year colleges and universities in America, half are enrolled in mainstream "engineering" calculus. Of these 300,000 students, only 140,000 finish the year with a D or higher. In addition, many students completing calculus do not have an adequate understanding of the most basic concepts of the course (Vinner, 1990; Orton, 1983b; Ferrini-Mundy and Graham, 1991). In the discussion that follows, research concerned with student understanding of key calculus concepts is explored.

One of the most important ideas in mathematics is that of function. However, many students have a poor understanding of this basic idea. Students have trouble finding images and pre-images, making the transition between the tabular, algebraic and graphical representations of a function, and understanding operations involving functions, in particular the composition of functions (Vinner & Dreyfus, 1989).

In a study that investigated students' understanding of functions, Schwarz, Dreyfus, and Bruckheimer (1990) found that:

1. Students could carry out procedures they had been taught, but were unable to link their procedural knowledge with their conceptual knowledge.

2. Students take a discrete approach to functions, conceiving of them as consisting of points.

3. Students have difficulty in those aspects of functions that describe variation, and are unable to cope with variation of variation.

4. Students have difficulty in relating graphs to functions when presented with tasks on function transformations.

In addition, students' difficulties understanding the concept of function include an attachment to linearity, difficulties in transferring between representations, a lack of a dynamic conception of functions, and an inability to see a function as an object (Schwarz et al., 1990).

As with functions, researchers have found students' understanding of even the most basic concepts of differentiation and integration lacking. Conceptual difficulties included understanding the derivative as a limit of a set of secant lines, negative and zero rates of change, rate of change at a point, and average rate of change (Orton, 1983a).

Students have difficulty with the notion of Riemann sums and are often not able to define the definite integral clearly (Ferrini-Mundy and Graham, 1991; Davis, 1985). In addition, students lack connections between their procedural and conceptual knowledge (Ferrini-Mundy and Graham, 1991; Orton 1983b) and are not retaining what they do learn (Vinner, 1990).

The concept of limit seems to give students particular difficulty. In part, this may be due to the non-mathematical images the term may invoke (Davis and Vinner, 1986). Students may conceive a limit as a bound, as something unreachable, as motion, and often choose a representation which is not correct for the problem at hand (Williams, 1991). Although most students are able to determine limits graphically, they seem to have little geometric understanding of what they are doing (Ferrini-Mundy and Graham, 1989).

One of the goals put forward by the *Standards* (NCTM, 1989) is that all students should be able to translate among tabular, symbolic, and graphical representations. The ability to transfer among representations is essential to problem solving because it allows one to make progress in one representation and use the results in another. However, researchers have found that students' ability to understand concepts within the different representations, and their ability to transfer among representations, is lacking. Many students show preferences for a certain representation (Hart, 1991) and little understanding of connections with other representations (Dreyfus and Eisenberg, 1988). In fact, students have conceptual difficulties regardless of the representation used (Markovits et al., 1986).

To address this lack of student understanding it has been recommended that curricula be developed that consider (a) the body of research available on student learning of calculus-related topics, (b) the nature of the role of technology, and (c) the role of teachers and instructors in the development and dissemination process (Ferrini-Mundy & Graham, 1991). It is in this spirit that the National Science Foundation has funded a number of projects intended to explore the opportunities for revitalizing calculus instruction. We will now turn our attention to an overview of these reform efforts.

Emerging Themes of Calculus Reform

Calculus reform projects, both NSF-funded and others, have taken a variety of approaches in their attempts to revitalize the calculus curriculum. Among the calculus reform projects there is diversity in the material emphasized, the approach taken to introduce the material, and the emphasis placed on the use of technology. However, there are important common threads among these reform projects. Central to most reform efforts is the notion that students actively construct knowledge, as evidenced by the more student-centered instructional methods which many reform projects employ. This philosophy is reflected by Hughes-Hallett et al. (1994) of the Harvard project in their belief that insight into mathematical concepts is gained through the investigation of practical problems.

Another theme common to the reform projects is that conceptual understanding should be the emphasis of the mathematics curriculum. To promote conceptual understanding, teachers employ more intuitive approaches to the development of important concepts. For example, in the Harvard project (Hughes-Hallett et al., 1994) a definition is first introduced and developed in everyday language, and only after an understanding is grasped is the formal definition introduced.

The idea of a multiple representation approach to the teaching of the important concepts of mathematics is central to the call for curriculum reform (NCTM, 1989, p. 125). In many calculus reform projects, such as the *Calculus Connections Project* of Oregon State University (Dick and Patton, 1994), an emphasis on numerical, graphical, and symbolic representations of mathematical topics is evident. The *Calculus Consortium* (Hughes-Hallett & Gleason, 1994), based at Harvard University, embraces a multiple representation approach to the teaching of calculus known as "The Rule of

Three." The Rule of Three emphasizes that every topic should be presented geometrically, numerically, and algebraically. The *St. Olaf Project* (Ostebee & Zorn, 1994) also emphasizes combining, comparing, and moving among graphical, numerical, and algebraic representations of important concepts.

The use of graphing calculators and computer programs is emphasized in many of the reform projects. Some of reform projects which emphasize the use of calculators or computer programs are technology specific: the University of Illinois project uses *Mathematica*, the Purdue University project uses *ISETL*, and the Computer-Integrated Calculus Project at the University of Connecticut uses *True BASIC* as a platform for exploring calculus concepts. Another reform project that uses a specific computer program as an integral part of its calculus curriculum can be found at the University of Northern Colorado which uses *Mathematica* "notebooks" with a weekly open laboratory assignment to complement a traditional lecture/discussion format. Project CALC (Calculus as a Laboratory Course) at Duke University uses both graphing calculators and the computer program *MathCAD* in their approach to teaching calculus (Tucker & Leitzel, 1995).

Other reform curricula are designed in a way which allows for a variety of calculator or computer programs to be used. The texts written in conjunction with the Calculus Connections Project, the Harvard project, and the St. Olaf project are among the curricula written without any particular computer program or calculator in mind. While technology is heavily emphasized in many reform projects, there are exceptions. New Mexico State University (Cohen, Gaughan, Knoebel, Kurtz, & Pengelley, 1991), for example, has instituted a curriculum development program which focuses on two-week assignments called "student research projects" which do not require the use of calculators or computer programs.

The Calculus Connections Project

Oregon State University is the site of one of several NSF-funded curriculum revision programs. *Calculus of a Single Variable*, by Dick and Patton (1994) was written as a part of this project. The Calculus Connections Project (CCP) is a follow-up NSF-funded dissemination effort specifically targeting high school calculus instruction through the Advanced Placement program. Central to the learning philosophy of the Calculus Connections Project is the notion that students actively construct knowledge. This philosophy can be seen in the more student-centered instructional methods which this reform project employs.

To foster a stronger conceptual understanding, the Calculus Connections Project emphasizes a multiple representation approach in the development of important calculus topics. Promoting the abilities of students to work with graphical, numeric, and symbolic representations, and to switch among representations, are fundamental objectives guiding this reform project. Another important pedagogical strategy of the project is that of using problem situations for investigative purposes. The Project's methodology uses real world problems to develop the concepts of calculus, allowing students to develop understanding through exploration of real life applications and solutions.

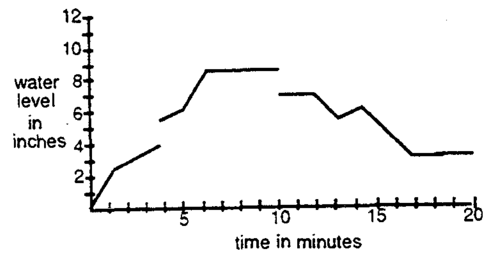
The use of technology is an important component of the Calculus Connections Project; visualization and approximation play an important role in the curriculum materials. Graphical interpretation skills take on particular importance as problem-solving aids, since they allow the student to monitor the reasonableness of results obtained numerically or symbolically. The calculating abilities of the new technologies allow estimation strategies to be approached in ways that were not possible in the pretechnology classroom.

Another characteristic of the CCP curriculum materials, which distinguishes it from traditional texts, can be found in the exercises at the end of each chapter. Traditional texts tend to have a large number of practice problems that closely follow the

Figure 1a

Types of Problems Found in *Calculus of a Single Variable (1994)*

I For exercises 1-4: Use the graph of water level as a function of time shown. When the faucet is on, the water level rises at a steady rate. Similarly, when the plug is pulled out, the water level falls at a steady rate (but slower than the faucet's rate). At various times, some events happen that affect the water level and/or the rate at which the water level changes. In the exercises below, you are asked to identify at exactly what tem the given event occurred (p. 165).



1. The plug is pulled out with the faucet turned off.
2. A large rock is pulled out of the aquarium.
3. The plug is pulled out with the faucet turned on.

II The table below gives level y (in centimeter) of fluid in a tank of industrial chemical reactants at time t (in minutes). Our purpose is to learn about the rate at which this level changes with passing time (P. 268) .

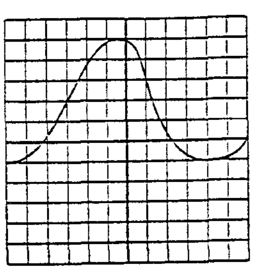
Time t	Level
0.0	0.525
0.8	4.927
1.2	6.011
1.6	6.506
2.0	6.525
2.4	6.184
2.8	5.599
3.2	4.883
3.6	4.154
4.0	3.525
4.4	3.112
4.8	3.03

1. What are the lowest and highest levels? When do they occur?
2. Are there time when the fluid is relatively or temporarily at a high or low level (even if not the overall highest or lowest)? Identify these times and levels.
3. When is the level rising most rapidly? (Try adding a column to the table to give the differences or changes in level.)

III For exercises 1-4: Zoom in (with the same factor both horizontally and vertically) on the graph of the indicated function $y = f(x)$ to estimate $f'(1)$ graphically (P. 192).

1. $f(x) = x^{2/3}$
2. $f(x) = \arctan(x)$
3. $f(x) = x^{2/3}$
4. $f(x) = \log_{10} x$

IV You are given the graph of the derivative $y = f'(x)$. Indicate the locations of the points of inflection for the graph of the original function $y = f(x)$. Where is the original function graph concave up or concave down (P. 326)?



pattern of examples given in the book. In the CCP curriculum, problems presented are intended to be more thought-provoking and open-ended. There are more problems which combine and compare the three representations, as well as problems that require students to translate between representations. The problems in Figure 1a exemplify the types of problems found in the text of the Calculus Connections Project *(Calculus of a Single Variable).*

Calculus Reform and the Advanced Placement Program

The Advanced Placement (AP) Program is intended to provide an opportunity for secondary school students to receive college credit or advanced standing for college level course work. An AP course in mathematics consists of a full academic year of work in calculus and related topics comparable to courses taught in colleges and universities (see Appendix E for AP syllabus). The AP exams are administered nationally and are open to any secondary school that elects to participate.

The AP program offers two examinations in calculus, denoted Calculus AB and Calculus BC, which are scored on a scale of 1 to 5 (1 lowest, 5 highest). Calculus AB covers at least as much material as a traditional first semester college calculus course, and Calculus BC covers at least as much material as the standard first two semesters of college calculus. It is the decision of individual colleges and universities if, and how much, credit or advanced course placement is to be given for a given score on each exam. In many institutions a full semester's credit is given for a score of 4 or 5 (and sometimes 3) on the Calculus AB exam, while a full year of college calculus credit may be awarded for a 4 or 5 on the Calculus BC exam. Students may take only one of the calculus exams per year, choosing the one more appropriate for their level of preparation.

Currently, each AP Calculus Examination has a multiple-choice section and a free-response section. The multiple-choice section contains 45 questions and the free-response section contains six questions, with 90 minutes allotted for each section. The multiple-choice questions are machine scored, while the free-response questions are scored by AP teachers and college professors who teach comparable courses. The AP exams are administered once a year in May.

Figure 1b gives examples which are indicative of the kinds of problems found in the Advanced Placement AB exam (Educational Testing Service, 1994). Problem 1 requires the use of a calculator to solve. Although this problem is given algebraically, the solution is determined graphically. Problem 2 requires the student to interpret the graph of a function. To answer this question a student must have a good conceptual understanding of the first and second derivative and have the ability to interpret information graphically. Problem 3 requires the use of a graphing calculator. The student must be able to interpret graphs and use the calculator's function to solve this question.

The issue of calculator use in the AP exam has been a difficult one to resolve. The NCTM calls for students to learn when and how to use calculators to:

1. concentrate on the problem-solving process rather than on the calculations associated with problems,

2. gain access to mathematics beyond their level of computational skills,

3. explore, develop, and reinforce concepts, including estimation, computation, approximation, and properties,

4. experiment with mathematical ideas and discover patterns, and

5. perform those tedious computations that arise when working with real data in problem-solving situations (NCTM 1986).

Many educators believe that the non-use of calculators on standardized examinations keeps calculators from being widely used in mathematics instruction. AP standardized examinations provides increased motivation for teachers to begin using calculators in their classrooms. Teachers are, to one degree or another, motivated to help their students be successful on the AP exam and are often held accountable for their success. When students are required to use calculators on the exam, teachers will be motivated to provide calculator instruction and practice.

In 1983 the College Board permitted the use of calculators on the AP exams; however, their use was discontinued in 1985 because of issues of equity. Scientific calculators were allowed on the 1993 and 1994 exams, and beginning with the May 1995 examination, graphing calculators were required.

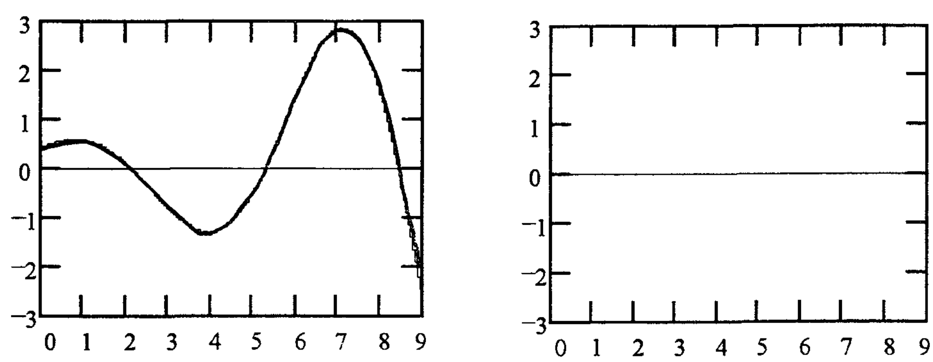
Figure 1b

<u>Examples of Problems from the Advanced Placement AB Exam</u>

---

1. Two particles start at the origin and move along the x-axis. For $0 \le t \le 10$, their respective position functions are given by $x_1 = \cos(t)$ and $x_2 = \ln(2t)^2$. For how many values of t do the particles have the same velocity?

2. Consider the graph of $y = f'(x)$ given below (note that this is the graph of the derivative of f).



A) Sketch the graph of $f''(x)$ in the space provide.
B) Over which intervals is the graph of f concave up? Justify your answer.
C) Find the x-coordinates of all relative minimums for the function $f(x)$.

Figure 1b (continued)

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3. Suppose $f(x)$ is continuous over the interval $1 < x < 7$ and $f'(x) = (\ln x)^2 + 2(\sin x)^4$ for $1 < x < 7$. Answer the following to three decimal places.

   A) For what values of x is $f'(x) = 0$?
   B) Find the x-values for all relative minima for the function $f(x)$. Justify your answer.
   C) For what values of x is $f''(x) = 0$? Justify your answer.

Teachers' Pedagogical Content Beliefs

An important consideration of this study is in the exploration of relationships between teachers' pedagogical content beliefs and student achievement. Fundamental to this inquiry are the questions: (a) how does learning occur? and (b) what roles do teachers' pedagogical content beliefs play in the educational process? One basic assumption of current research is that children actively construct knowledge by interacting with their environment and reorganizing their own mental constructs (Romberg & Carpenter, 1986). Instruction, according to this assumption, affects what children learn, but does not determine it. Research has found a strong relationship between teachers' pedagogical content beliefs and their planning, instructional decisions and classroom practices (Carpenter, Fennema, Peterson, & Carey, 1988; Peterson et al., 1989a). Therefore, if current reform efforts which reflect constructivist theories are going to affect a change in teachers' instructional methods, a corresponding change of beliefs will be required. In this section, we will examine the literature concerning these ideas. In particular, we will consider:

- Features of a Belief System

- Characteristics of Teachers' Beliefs
- Constructivism in the Classroom

Features of a Belief System

The literature offers little agreement among researchers on a definition for teachers' beliefs. Kagan (1992) points out that some researchers refer to the term "teacher belief" as the teacher's "principles of practice," "personal epistemologies," "perspectives," "practical knowledge," or "orientations." Pajares (1992) maintains that "teacher belief" should include socio-cultural factors such as attitudes, values, judgments, axioms, opinions, ideology, perceptions, conceptions and conceptual systems. In this study, teachers' beliefs were characterized as cognitively based (CB) or less cognitively based (LCB) on the basis of their scores on a belief questionnaire that was adapted from the research of Peterson et al. (1989).

To better understand the meaning of teachers' beliefs, it is helpful to consider the differences between beliefs and knowledge (Pajares, 1992; Nespor, 1987; Abelson, 1979). Nespor (1987) suggests that beliefs have stronger affective and evaluative components than knowledge, and operate independently of the cognition associated with knowledge. According to Nespor, beliefs do not require group consensus regarding validity, whereas knowledge systems are open to evaluation and critical examination. Because of this, Nespor claims, beliefs can defy logic.

Abelson (1979) identifies seven features which he believes distinguish a belief system from a knowledge system:

1. The elements (concepts, propositions, rules, etc.) of a belief system are not consensual.

2. Belief systems are in part concerned with the existence or nonexistence of certain conceptual entities.

3. Belief systems often include representations of "alternative worlds," typically the world as it is and the world as it should be.

4. Belief systems rely heavily on evaluative and affective components.

5. Belief systems are likely to include a substantial amount of episodic material.

6. The content set to be included in a belief system is usually highly "open."

7. Beliefs can be held with varying degrees of certitude.

Characteristics of Teachers' Beliefs

Although there is no consensus on a definition of teachers' beliefs, some generalizations can be made concerning their nature and effects. Pajares (1992) lists definitional inferences and generalizations drawn from the literature on the assigned meaning of "beliefs." Among them are:

1. Beliefs are formed early, are self perpetuating, and resistant against contradictions.

2. Individuals develop a belief system that houses all beliefs acquired through the process of cultural transmission.

3. Beliefs have an adaptive function which help individuals define and understand the world and themselves.

4. Beliefs and knowledge are intertwined; beliefs are a filter through which new information is interpreted.

5. Thought processes may be a precursor to beliefs; however, beliefs screen thinking.

7. Beliefs are prioritized according to their connections with other beliefs.

8. Beliefs are instrumental in defining tasks and selecting the cognitive tools with which to interpret, plan, and carry out tasks. They play a critical role in defining behavior and organizing knowledge.

9. Beliefs influence perception.

10. Beliefs affect behavior.

11. Beliefs must be inferred.

12. Beliefs about teaching are well established by the time a student gets to college.

Pajares found that researchers exploring the nature of beliefs report a strong relationship between teachers' educational beliefs and their planning, instructional decision, and classroom practices. He states, "Attention to the beliefs of teachers and teacher candidates can inform educational practice in ways which prevailing research agendas have not and cannot." (Pajares, 1992 p. 328) He contends that understanding teachers' beliefs is important in order to understand how teachers make their decisions.

Kagan (1992) reported consistent findings with regard to two generalizations about teachers' beliefs. First, teachers' beliefs appear to be relatively stable and resistant to change. Second, teachers' beliefs tend to be associated with a congruent style of teaching. The relationship between teacher beliefs and teacher behavior is a particularly significant finding, according to Kagan, because of the difficulties inherent in capturing teachers' beliefs. One difficulty the author cites in capturing teachers' beliefs is that they cannot be inferred directly from teacher behavior. This is because teachers can behave in similar ways for very different reasons. Other reasons teachers' beliefs are difficult to study are (a) teachers are often unaware of their own beliefs, (b) teachers often do not have a language to describe their own beliefs, and (c) teachers may be reluctant to explain their beliefs publicly.

Two special forms of teachers' beliefs are *self-efficacy* and *content-specific beliefs*. Kagan describes self-efficacy as a teacher's expectancy concerning a personal ability to influence students, as well as the teachers' own beliefs concerning how well they can perform certain professional tasks. Kagan found that self-efficacy has been positively related to specific classroom behaviors, such as using praise instead of criticism, persevering with low achievers, accepting student opinions, and raising the level of student achievement in reading and mathematics. He describes teachers' content-specific beliefs as their epistemological conceptions of the field to be taught. These conceptions include the teachers' judgments about appropriate instructional activities, goals, forms of evaluation, and the nature of student learning. Kagan found content-specific beliefs to be correlated with a number of instructional and noninstructional variables.

In an unpublished paper, Simonsen (1993) reviews the literature concerned with changing teachers' subject matter and pedagogical conceptions and beliefs. Simonsen's review indicates the most important factor involved in the development of teachers' subject matter and pedagogical conceptions and beliefs is their past experience in mathematics classes. Because of this, the author suggests a necessity to examine what conceptions and beliefs participants bring to teacher education programs and workshops. Although Simonsen found that staff development interventions can affect teachers' conceptions and beliefs, she noted that teachers' pedagogical thoughts and actions change gradually with experience.

Constructivism in the Classroom

Constructivism views teachers and students as active meaning-makers who are continuously giving meaning to each others' words and actions (Cobb, 1988). From this point of view, mathematical structures are not perceived, intuited, or taken in; instead, they are constructed. They are inventions of an individual's mind. On the other hand, the central assumption in the transmission view of learning is the idea that meaning is inherent in the words and actions of the teacher. The meanings they carry in and of themselves are transmitted to and absorbed by the students.

Cobb (1988) gives two reasons why a constructivist view should be considered as an alternative to the transmission perspective. The first reason is that the goal of mathematics instruction should be to help students build structures that are more powerful, complex, and abstract. The second reason is that the constructivist view of learning considers what is worth knowing in terms of conceptual development, rather than in terms of skills acquired.

While favoring constructivist theory, Cobb claims that deep-rooted problems arise when attempts are made to apply constructivist principles in the classroom. One problem is that teachers will not adopt constructivist methodology because it requires more of them, demanding both a deep relational understanding of the subject and a familiarity with the conceptual development of students in specific areas of mathematics. Another problem is that deriving precise instructional recommendations from constructivist theories is difficult. This, Cobb says, is due to the fact that we cannot explain how students construct concepts which are more advanced than the ones with which they started. According to Cobb, the best that can be done to help teachers implement constructivist mathematics instruction would be to propose general instructional heuristics and to suggest a variety of activities and interventions that might work with some students. The solution, he says, is for the teacher to become "a reflective pedagogical problem solver who, in effect, conducts an informal research program."(p. 101)

Romberg and Carpenter's (1986) findings from their review of the literature supports Cobb's findings that constructivist theories of learning are not being reflected in current classroom teaching practices. The predominant pattern Romberg and Carpenter found in elementary school mathematics pedagogy was extensive teacher-directed explanation and questioning, followed by seatwork and paper-and-pencil assignments. Romberg and Carpenter cite three limitations to this type of instruction. First, it gives the impression that mathematics is a static and bounded discipline, supporting the view that mathematics is separated from other disciplines and is composed of independent subjects. Second, the acquisition of knowledge becomes an end in itself, requiring that students spend their time learning what others have done, rather than having experiences of their own; and third, the role of the teacher becomes that of a manager.

These recent calls for a more constructivist approach to teaching, coupled with the importance of teachers' beliefs, are fundamental to the present study. A central question associated with the present study is whether AP calculus teachers' beliefs can be categorized according to the degree to which they are constructivist in nature. In addition, if teachers' beliefs can be categorized in this way, are there any relationships between these beliefs, the teachers' classroom practices, and their students' achievement? To explore these questions further it is necessary to establish a framework for studying teachers' beliefs. This is the intent of the next section.

Framework for Studying Pedagogical Content Beliefs of AP Calculus Teachers

For more than ten years a group of researchers based at the University of Wisconsin has been studying teachers' pedagogical content knowledge and teachers' pedagogical content beliefs, and their association to student achievement. Four of the key researchers are Penelope L. Peterson, Elizabeth Fennema, Thomas P. Carpenter, and Megan Loef. Their research has culminated in a program intended to promote classroom practices which have been found to be associated with improved student performance. This program, called Cognitively Guided Instruction (CGI), informs elementary teachers of current research on how children learn, and of children's knowledge of solution strategies for simple addition and subtraction problems. The aim of the CGI program is for teachers to use this information to adapt their classroom teaching practices in ways which will improve student understanding.

This study adopts the framework underlying the research undertaken at the University of Wisconsin to the AP calculus setting, and uses this framework to research relationships among teachers' pedagogical content beliefs, classroom practices, and their students' understanding of differentiation in calculus. We will begin this section by examining the philosophies that have guided the development of the CGI program. This will be followed by a review of key studies done by the researchers at Wisconsin and a detailed look at a study by Peterson, Fennema, Carpenter, and Loef on which the current study is based. In particular we will consider:

- Overview of the Cognitively Guided Instruction Program
- Development and Review of the CGI Program
- The Key Study of Peterson, Fennema, Carpenter, and Loef

Overview of the Cognitively Guided Instruction Program

In an article in the *Educational Psychologist*, Carpenter and Peterson (1988a) called for a combining of research on teaching and research on learning. According to the authors, researchers who have focused on teaching have primarily concentrated on *how* instructors teach and not on *what* is being taught, while researchers who have focused on children's cognition have been primarily concerned with *what* is learned and not with *how* learning occurs. In particular, researchers have not considered questions regarding the knowledge children bring to the classroom, how this knowledge influences what children learn, and how this prior knowledge changes with instruction. The authors believe that new paradigms for research on learning and instruction in mathematics which draws on the strengths of both are needed.

According to Fennema, Carpenter, and Peterson (1989), a prominent feature of current research is the emphasis on the content being studied. The purpose of this research has been to identify features related to the ways children think about content, and to understand the processes children use to solve specific problems within a fixed mathematical domain. According to the authors, this research provides maps of how children move from relatively intuitive knowledge to more sophisticated and abstract knowledge. In particular, they point to research on addition and subtraction that has provided a structured and detailed analysis on how children develop addition and subtraction concepts and skills with respect to the children's solutions of different types of word problems. This research gives a way of categorizing problem types and strategies used by children and provides a model of the major levels in the development of addition and subtraction concepts and skills. The framework for asking the appropriate questions and understanding student answers has been an important result of research on problem types and solution strategies.

Fennema et al. (1989) claim, ". . . understanding can be achieved in a variety of classrooms, being taught by teachers using a variety of styles. The critical component appears to be the teacher's pedagogical content knowledge. Such knowledge includes knowledge of mathematics, knowledge of instructional techniques, and knowledge of children's cognition in specific subject areas."(p. 217)

In keeping with the current curricular recommendations, the authors developed a new paradigm for curriculum development. The Cognitively Guided Instruction Project (CGI) is an investigation designed to improve student learning. In this project, the researchers have been particularly concerned with teachers' knowledge and beliefs about students' learning and thinking processes. Fennema et al. list the guiding principles of CGI as:

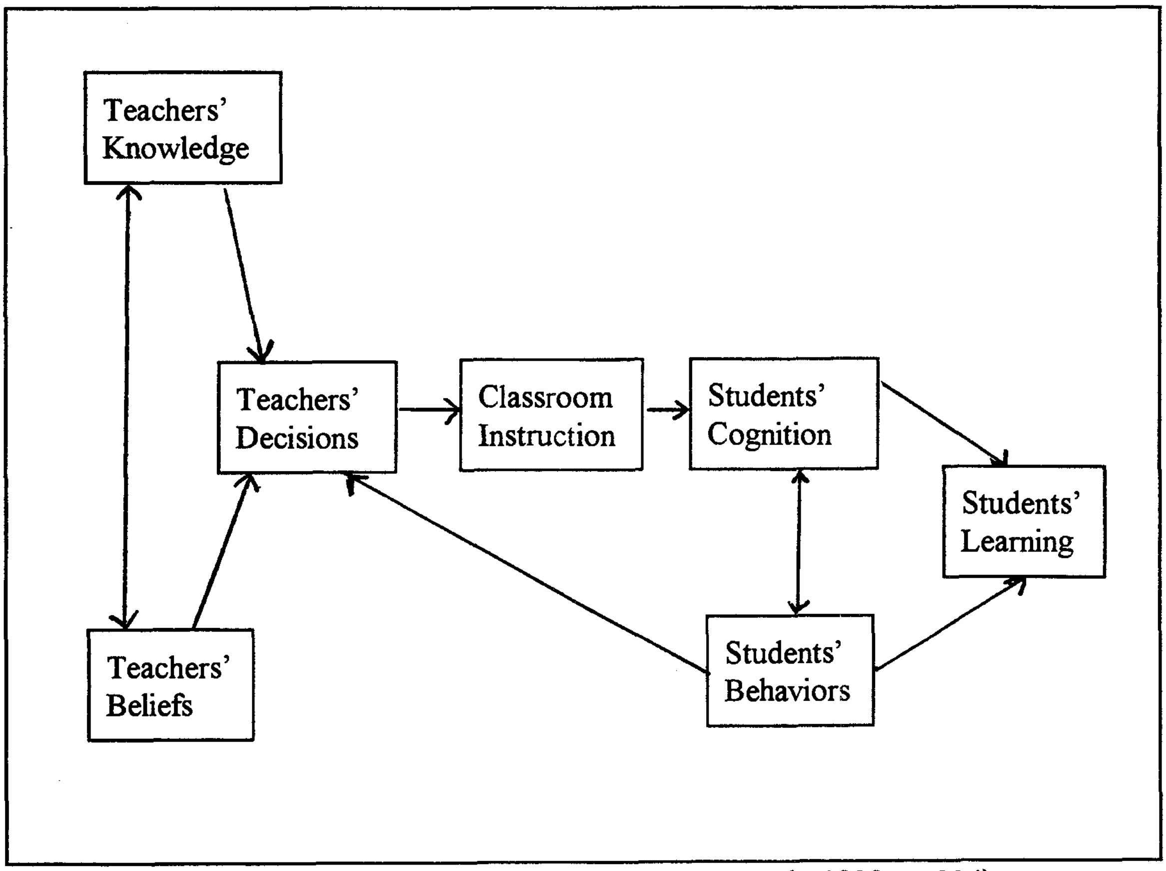
1. Instruction must be based on what each learner knows.
2. Instruction should take into consideration how children's mathematical ideas develop naturally.
3. Children must be mentally active as they learn mathematics.

The development of the CGI program was based on two assumptions (Peterson et al., 1991). First, teachers can benefit from research-based knowledge of children's thinking about addition and subtraction problems. Second, children make sense of new knowledge in light of existing knowledge, *as do teachers.* In the CGI program, research-based knowledge about children's mathematical knowledge is shared with teachers, and the teachers are given the opportunity to interpret for themselves what it means to their classroom instruction. In other words, the teachers are allowed to adapt the research-based knowledge to their existing knowledge and belief systems.

Fennema, Carpenter, and Peterson (1989) offer the following model for research and curriculum development (see Figure 2a).

Figure 2a

Model for Research and Curriculum Development



Note. Arrows indicate direction of influence. (Fennema, et al., 1989, p. 204).

The authors stated that the CGI workshop did not in and of itself change the teachers' beliefs. Peterson et al. (1991) found that teachers changed most when they began to listen, and attend seriously, to their own students' thinking as the students solved mathematical problems. In fact, they reported that the impact of the CGI program was directly related to how carefully the teachers listened to the way their students solved mathematical problems. According to the authors, "The research-based knowledge of the problem framework and children's strategies gave teachers a context for thinking about children's knowledge and for helping teachers make sense of their children's thinking." (Peterson et al., 1991, p. 89)

According to Peterson et al., adults have not always taken children's knowledge seriously. Many teachers, the authors claim, have assumed that children begin school without much knowledge. In part, the authors say this is because children do not come to school with much knowledge of formal algorithms, leading to the assumption that they do not have much mathematical knowledge. This, the authors have found, is generally a false assumption.

According to Peterson et al., children's solutions to mathematical problems demonstrate two things. First, children can solve a variety of problems by analyzing the information given instead of looking for key words or using other tricks. Second, the authors assert, children's solutions to mathematical problems demonstrate their ability to be creative in problem solving, and to solve mathematical problems based on an understanding of fundamental number concepts.

Development and Review of the CGI Program

Several studies have influenced the development and refinement of the CGI framework. Carpenter and Moser (1984) explored children's ability to solve addition and subtraction problems and defined five levels of student problem-solving ability. At level zero, the children were unable to solve any addition or subtraction problems. At level one, children were limited to direct modeling strategies. Level two was a transitional period where children used both modeling and counting strategies. At level three, students depended primarily on counting strategies. At level four, children primarily used number facts.

A closer look at the strategies that children used to solve addition problems showed that they initially solved problems with a count-all strategy and that this was gradually replaced with a counting-on and the use of number facts. This research provided a principled framework for selecting problems and analyzing students' thinking.

Having developed a framework for selecting problems and analyzing students' thinking, Peterson (1989b) and Carpenter (1988b) present an initial conceptualization of how teachers' and students' cognition and knowledge mediate effective teaching and investigate relationships between teachers' pedagogical knowledge and student achievement. In these studies there was found a significant correlation between teachers' knowledge of their own students' thinking and student performance.

The above research culminated in an important study by Peterson, Fennema, Carpenter, and Loef (1989b) which found relationships among first-grade teachers' pedagogical content beliefs, teachers' pedagogical content knowledge, and students' achievement in mathematics. This study is key to the present research and will be reviewed in detail at the end of this section.

Having found teacher characteristics associated with improved student achievement, the CGI program was developed. The CGI program was designed to familiarize teachers with the findings of research on the learning and development of addition and subtraction concepts in young children. Several studies investigated whether providing teachers with information derived from this research would influence the teachers' instruction and their students' achievement (Carpenter et al., 1989b; Peterson, Fennema, Carpenter, and Loef, 1989a; Knapp and Peterson, 1995). It was found that teachers participating in the CGI program had beliefs and instructional practices which were more in keeping with the principle that children construct their own knowledge. In addition, students of teachers who participated in the CGI program had a significantly greater understanding of mathematics than students of teachers who did not participate in the CGI program.

The Key Study of Peterson, Fennema, Carpenter, and Loef

The framework and methodology employed in the current study is based closely on research carried out by Peterson, Fennema, Carpenter, and Loef (1989b). Their research investigated relationships among first grade teachers' pedagogical content beliefs, teaching strategies, and student achievement. To investigate relationships among AP calculus teachers' pedagogical content beliefs, teaching strategies and student achievement the framework and methodology of Peterson et al. was adapted to the high school AP calculus setting. This section describes in detail the study of Peterson, Fennema, Carpenter, and Loef which is central to the current study.

Peterson, Fennema, Carpenter, and Loef (1989b) conducted a study which found that teachers' pedagogical content beliefs was strongly connected to their teaching strategies and students' learning. This study involved 39 first grade teachers from 27 schools in Wisconsin. The purposes of the study were to conceptualize teachers' pedagogical content beliefs, give examples of how teachers' pedagogical content beliefs in mathematics might be analyzed, and to describe how teachers' pedagogical content beliefs influence teachers' thinking, decision making, and teaching; and how these beliefs influence students learning and achieving. Four basic principles of the framework used to analyze these questions were:

1. Children construct their own mathematical knowledge.
2. Mathematical curriculum should be organized to facilitate children's construction of knowledge.
3. Children's development of mathematical ideas should provide the basis for sequencing topics.

4. Mathematical skills should be taught in relation to understanding and problem solving.

A 48-item belief questionnaire was developed by the researcher to assess teachers' pedagogical content beliefs according to these four constructs. The questionnaire was divided into the following subscales:

1. how children learn math,
2. relationship between skills and understanding,
3. what should provide basis for sequencing of topics, and
4. how addition and subtraction should be taught.

Teachers whose responses to the questionnaire agreed with the above framework were considered cognitively based (CB), while those with a low agreement with the above framework were considered less cognitively based (LCB). Structured interviews were used to support information obtained from the questionnaire and to obtain specific information on the content and techniques that teachers used to teach addition and subtraction in their classrooms.

Peterson found that teachers differed in the degree to which their pedagogical content beliefs correspond to a cognitively-based perspective. The highest agreement was found with the idea that math skills should be taught in relationship to problem solving. The lowest agreement was found with the perspective that children "construct" mathematical knowledge rather than "receive" mathematical knowledge. Peterson also found that teachers that scored high on one construct tended to score high on the other three constructs, while teachers that scored low on any given construct tended to score low on the other three constructs.

Evidence was found indicating that the teachers' pedagogical content beliefs were related to their knowledge of addition and subtraction word problems and their ability to distinguish between different types of problems. Teachers' general knowledge of children's strategies was significantly positively related to a cognitively-based belief structure. It was reported that the CB teachers were more likely to introduce word problems early when teaching addition and subtraction, while the LCB teachers were more apt to emphasize manipulation. All teachers indicated that understanding was most important. However, the CB teachers related learning number facts as least important. The CB teachers tended to consider the role of the teacher and learner as being actively engaged with one another in the construction of mathematical knowledge and understanding, while the LCB teachers viewed the teacher's role as that of organizer and presenter and the children's role as that of receiver. It was also found that teachers' total scores on the belief questionnaire were significantly positively related to their students scores on a problem solving test. No significant difference was found in their students' scores on a computational test.

The results of this study suggest that content belief and content knowledge are linked to teachers' actions and students' learning. However, an analysis of the seven teachers who scored highest and the seven teachers that scored lowest on the four constructs found that the CB teachers had more years' teaching experience than the LCB teachers ($M = 14.57$ and $M = 8$ respectively, $p<.05$). Since no other significant demographic differences were found, it was suggested that this might indicate that the contrasts found in this study are due to experience.

Summary

The present call for reform suggests developing curricula which promote the construction of mathematical concepts by the student and incorporates the use of technology. A number of calculus reform projects, as well as the AP calculus program, have taken up the challenge of initiating these changes. Ultimately, the success of these projects lies in the hands of the teachers implementing these changes. The literature that has been reviewed indicates that if these reform projects are to be successful, teachers' pedagogical content beliefs must be considered. This research has shown that there is a strong relationship between teachers' pedagogical content beliefs and their planning, instructional decisions, and classroom practices. To promote a change in teachers' classroom practices, corresponding changes are going to have to occur in teachers' pedagogical content beliefs.

The work of Peterson, Fennema, Carpenter, and Loef (1989b) has shown that teachers' pedagogical content beliefs are closely related to both teachers' classroom practices and student achievement. These researchers have found strong relationships among first-grade teachers' pedagogical content beliefs and students' achievement in mathematics. Their research has given valuable insight into an important factor, teachers' beliefs, associated with improved student achievement. Finally, the CGI program has demonstrated that inservice workshops can be designed which will help affect teachers' beliefs in a way which promotes improved student achievement.

The population of students and teachers studied as part of the CGI program are quite different than the population of students and teachers in the AP calculus program. Nevertheless, the goals of calculus reform and of the Cognitively Guided Instruction program are strikingly similar in spirit. What is the role that teachers' pedagogical content beliefs play in determining the success of implementing calculus reform in the AP calculus setting? The purpose of the present study is to study relationships among high school AP calculus teachers' pedagogical content beliefs, their classroom practices, and student achievement.

CHAPTER THREE: RESEARCH DESIGN

Purpose of the Study

This study explores relationships among teachers' pedagogical content beliefs, teachers' classroom practices, and student achievement. Specifically, we examine:

Teachers' pedagogical content beliefs

What are AP calculus teachers' pedagogical beliefs about mathematics, curriculum, and instruction in AP calculus? How well do their pedagogical content beliefs align with a constuctivist point of view?

How do AP teachers' pedagogical content beliefs change with time? Do the pedagogical content beliefs of teachers become more cognitively based as they become familiar with the materials from a calculus reform project? Is there a relationship between how long a teacher has used the project materials and the degree to which they are cognitively based?

Teachers' classroom practices

How are AP teachers' pedagogical content beliefs reflected in their reports of their approaches to teaching, their concepts of the roles of the teacher and the learner, and their goals for instruction?

Student achievement

Is there a relationship between AP teachers' pedagogical content beliefs, approaches to teaching, and their students' achievement, including achievement of computational skills and problem solving?

Is there a relationship between teachers' pedagogical content beliefs and their student's ability to work in multiple representations: graphically, numerically, and symbolically?

Method

Subjects

Subjects for this study were drawn from teachers participating in the Calculus Connections Project. Each teacher in the Calculus Connections Project (CCP) attends intensive inservice workshops (more than 60 hours in duration) during the summer immediately before their first use of the curriculum materials. These workshops are intended to provide the teachers with a thorough orientation to the CCP curriculum materials and instruction in the use of graphing calculators.

Teachers chosen to participate in the Calculus Connections Project already show a high degree of motivation and leadership skills. Participation in workshops and other activities that demonstrated interest, motivation, and leadership were deemed desirable in the competitive selection process. The teachers, from more than 40 states, teach at both public and private schools. They reside in urban, suburban, and rural school districts and have a wide range of student populations.

A new group of teachers was selected to participate in the Calculus Connections Project in each year from 1992 to 1995. The first group consisted of 102 teachers and attended one-week workshops in the summer of 1992. This group used a preliminary version of the project materials and the HP 48SX graphing calculator.

The next three groups of teachers (123 in 1993, 113 in 1994, and 73 in 1995) attended two-week workshops, used a revised version of the CCP curriculum materials, and used the newer model HP 48G graphing calculator. Because of the differences in workshop duration, curriculum version, and calculator model, the 1992 group of teachers was not included as subjects in the study except for piloting the instruments and interview protocols.

The 1995 group played a central role in this study. The teachers in this group became involved in this study at a point before they had received any inservice instruction. This allowed for instruments to be administered to these teachers and their students before they received any instruction in the CCP curriculum materials and again after they had been using the project materials for six or seven months. Relationships among the 1995 teachers' pedagogical content beliefs, approaches to teaching, and their students' achievement were examined in depth.

Instruments

Teachers' Belief Questionnaire. To determine teachers' pedagogical content beliefs, the Teachers' Belief Questionnaire developed by Peterson, Carpenter, Fennema, and Loef for use with first grade teachers was adapted to the AP calculus setting (see Appendix A for the Teachers' Belief Questionnaire used in this study). Questions were reworded in a way that made the language appropriate to high school AP calculus.

The Belief Questionnaire uses a 5-point Likert scale. Four subscales to the Belief Questionnaire measure agreement with the following constructs:

1. Students construct their own knowledge.
2. Mathematical skills should be taught in relation to understanding and problem solving.
3. A math curriculum should be organized to facilitate students' construction of knowledge.
4. Students' development of mathematical ideas should provide the basis for sequencing topics.

Strong agreement with these constructs indicates cognitively based (CB) beliefs. Weaker agreement indicates less cognitively based (LCB) beliefs.

The first subscale is concerned with how students learn mathematics. A high score suggests a belief that students construct their own knowledge, while a low score suggests a belief that students receive knowledge. Items comprising subscale 1 are shown in Figure 3a.

The second construct explores the relationship between mathematical skills, understanding, and problem-solving. A high score suggests a belief that skills should be taught in relationship to understanding and problem solving, while a low score suggests a belief that skills should be taught in isolation. Items comprising subscale 2 are shown in Figure 3b.

The third subscale was concerned with teachers' beliefs about what should provide the basis for sequencing topics in mathematics instruction. A high score on this construct suggests a belief that students' natural development of mathematical ideas should be the basis for instruction. A low score on this construct suggests a belief that formal mathematical ideas should provide the basis for sequencing topics for instruction. Items comprising subscale 3 are shown in Figure 3c.

The fourth subscale addresses teachers' beliefs about how mathematics should be taught. A high score on this construct suggests a belief that mathematics instruction should facilitate students' construction of knowledge. A low score on this construct suggests a belief that instruction should be organized to facilitate teachers' presentation of material. Items comprising subscale 4 are shown in Figure 3d.

In each of the four subscales, six of the 12 questions were written so that a positive response indicated an agreement with CB beliefs. The remaining six questions were worded in a way that a negative response indicated CB beliefs.

Figure 3a

Students Construct/Receive Knowledge

12+	Students learn calculus best by exploring problem situations.
25+	Students can figure out ways to solve many calculus problems without formal instruction.
26+	Most students can figure out a way to solve many calculus problems without teacher help.
39+	Most students can figure out a way to solve simple calculus problems.
42+	It is important for a student to discover how to solve elementary calculus problems for him/herself.
43+	Students usually can figure out for themselves how to solve simple calculus problems.
2-	Most students have to be shown a method of solving elementary calculus problems.
4-	It is important for a student to know how to follow directions to be a good problem solver.
10-	Students learn calculus best from the teachers' demonstrations and explanations.
13-	To be successful in mathematics, a student must be a good listener.
17-	It is important for a student to be a good listener in order to learn how to do mathematics.
23-	Students learn math best by attending to the teachers' explanations.

Note. Agreement with "+" item indicates CB beliefs. Disagreement with "-" item indicates CB beliefs. Numbers indicate the order items appeared in the Belief Questionnaire.

Figure 3b

Skills Taught in Relation To/In Isolation from Problem Solving

8+	The instructional sequence of math topics should be determined by the order in which students naturally acquire math concepts.
9+	The natural development of students' mathematical ideas must be considered in making instructional decisions.
11+	When selecting the next topic to be taught, a significant consideration is what students already know.
14+	The natural development of student's mathematical ideas should determine the sequence of topics used for instruction.
28+	In planning for instruction, it is important to know how student's mathematical ideas develop naturally.
33+	It is more important to use student's concept development in planning an instructional sequence than to use a mathematically determined sequence.
1-	Time should be spent practicing computational procedures before students are expected to understand the procedures.
30-	Recall of basic rules of differentiation should precede the introduction of word problems involving differentiation.
32-	Students should master computational procedures before they are expected to understand how those procedures work.
37-	Time should be spent practicing computational procedures before students spend much time solving problems.
40-	Students will not really understand differentiation until they have mastered the basic rules of differentiation.
41-	Students should not solve basic differentiation word problems until they have mastered some basic differentiation facts.

Note. Agreement with "+" item indicates CB beliefs. Disagreement with "-" item indicates CB beliefs. Numbers indicate the order items appeared in the Belief Questionnaire.

Figure 3c

<u>Instruction Sequenced to Facilitate Student Development of Mathematical Ideas/Structure of Mathematics</u>

8+	The instructional sequence of math topics should be determined by the order in which students naturally acquire math concepts.
9+	The natural development of students' mathematical ideas must be considered in making instructional decisions.
11+	When selecting the next topic to be taught, a significant consideration is what students already know.
14+	The natural development of student's mathematical ideas should determine the sequence of topics used for instruction.
28+	In planning for instruction, it is important to know how student's mathematical ideas develop naturally.
33+	It is more important to use student's concept development in planning an instructional sequence than to use a mathematically determined sequence.
5-	The natural development of mathematical topics should determine the sequence of topics which is used for instruction.
19-	The mathematically logical sequence of topics must be considered in planning for instruction.
22-	The instructional sequence of math topics should be determined by the formal organization of mathematics rather than by the natural development of student's math ideas.
29-	It is more important to teach in a mathematically sequenced way than to use student's concept development in planning an instructional sequence.
36-	When selecting the next topic to be taught, one must carefully follow the mathematically logical sequencing of topics.
44-	The structure of mathematics is more important in making instructional decisions than is the natural development of student's ideas.

<u>Note</u>. Agreement with "+" item indicates CB beliefs. Disagreement with "-" item indicates CB beliefs. Numbers indicate the order items appeared in the Belief Questionnaire.

Figure 3d.

Instruction Organized to Facilitate Student Construction/ Teacher Presentation

15+	Teachers should allow students to figure out their own ways to solve calculus problems.
20+	Students should be allowed to invent ways to solve simple word problems before the teacher demonstrates how to solve the problems.
24+	Calculus should be presented to students in such a way that they can discover relationships for themselves.
31+	Teachers should facilitate student's invention of their own ways to solve calculus problems.
38+	Teachers should encourage students who are having difficulty solving a word problem to continue to try to find a solution.
48+	It is best to teach students how to solve a variety of word problems at one time rather than one type of problem at a time.
7-	Teachers should teach exact procedures for solving word problems.
16-	Students should be encouraged to solve problems in the same way the teacher has modeled them.
18-	The best way to teach problem solving is to show students how to solve one kind of problem at a time.
34-	Teachers should tell students who are having difficulty solving a word problem how to solve the problem.
45-	The teacher should demonstrate how to solve calculus problems before students are allowed to solve them.
46-	It is better to teach students how to solve one kind of word problem at a time.

Note. Agreement with "+" item indicates CB beliefs. Disagreement with "-" item indicates CB beliefs. Numbers indicate the order items appeared in the Belief Questionnaire

A pilot version of the adapted Belief Questionnaire was administered to ten teachers participating in a Calculus Connections Project meeting in January 1995. This allowed the researcher to administer the questionnaire in person and discuss the contents of the questionnaire with the teachers after they had completed it. The discussion was primarily concerned with how well the individual statements on the Belief Questionnaire addressed the four belief constructs (i.e. the face validity of the questionnaire) and whether the wording of the Belief Questionnaire was appropriate to the high school AP calculus setting.

Several statements in the initial version of the adapted Belief Questionnaire were subsequently revised and two versions of the Belief Questionnaire were developed: Version 1 maintained the same length as the original Peterson questionnaire, and Version 2 was shortened to 40 questions.

These two versions of the Belief Questionnaire were then piloted with the 1992 teachers. Half of the 102 teachers were randomly selected to receive Version 1, while the other half received Version 2. One week prior to mailing the two preliminary versions of the Belief Questionnaires, a letter was sent to each teacher informing them that a questionnaire was going to be mailed to them and requesting their cooperation. Two weeks after mailing the Belief Questionnaire a reminder, which included another copy of the Belief Questionnaire, was sent to teachers who had not responded. The final return of 64 completed questionnaires included 30 returns of version 1 and 34 returns of version 2.

Version 1 was chosen as the instrument for the study for two reasons: 1) it appeared to provide a sharper discrimination for determining teachers beliefs, and 2) it maintained a closer parallel to the instrument in the Peterson study. Summary statistics for Version 1 are given in Table 3a.

When the 1995 teachers completed the Belief Questionnaire at the beginning of their inservice workshops they were also asked to fill out a form which gathered demographic information. Figure 3e shows the demographic information requested and a summary of the collected demographic information.

Table 3a

Piloted Belief Questionnaire Scores for 1992 Teachers

	Scale 1	Scale 2	Scale 3	Scale 4	Total
Mean	35.23	28.33	29.57	26.87	120.00
Median	36	28	29.5	28.5	122
SD	1.46	1.35	0.87	1.19	4.06
Range	32	34	17	26	94
Minimum	14	12	21	12	64
Maximum	46	46	38	38	158

Note. n = 30. Scale 1, students construct/receive mathematical knowledge.
Scale 2, skills taught in relation to/isolation from problem solving.
Scale 3; instruction organized to facilitate development of ideas/structure of mathematics.
Scale 4, instruction sequenced by students' construction/clear teacher presentation.

Figure 3e

Demographic Information

1. Gender a) female 33 b) male 32
2. Age a) 20 - 29 3 b) 30 - 39 12 c) 40 - 49 26 d) 50 - 59 24 e) 60 or older 0
3. Highest degree
   a) Bachelor's in Mathematics 11 b) Bachelor's not in Mathematics 5
   c) Master's in Mathematics 28 d) Master's not in Mathematics 19
   e) Doctorate in Mathematics 1 f) Doctorate not in Mathematics 1
4. Number of years teaching $\mu = 19$
5. Number of years teaching Mathematics $\mu = 18$
6. Number of years teaching calculus $\mu = 8.4$
8. Number of students in your school $\mu = 19$

Note. 65 teachers from the 1995 group of participants gave demographic information.

Interview Protocol. The interview protocol served several purposes (see Appendix C for the complete interview protocol). First, the interview protocol served to gain information on how teachers approach teaching differential calculus. In addition, the interview protocol provided additional insight into teachers' pedagogical content beliefs. Figure 3f shows questions from the interview protocol which were designed to gain insight into teachers' classroom practices. Figure 3g shows the questions from the interview protocol which explored teachers' pedagogical content beliefs.

The interview protocol used in the present study was adapted for the AP calculus setting from the Knowledge and Belief Interview Protocol developed by Peterson, Carpenter, Fennema, and Loef (1989). Selected teachers from the 1995 group were interviewed. Interviews were conducted by telephone during the fall term of 1995.

To determine if teachers' self-reported approaches to teaching AP calculus accurately reflected their actual teaching practices, five AP calculus teachers were both observed while teaching their calculus class and asked to describe their teaching practices during a telephone interview. Four of the teachers were observed twice, and one teacher was observed once. These observations took place during the fall and winter quarters of the 1995-1996 school year. During classroom observations, a trained observer would alternate between observing the teacher for 30 seconds, and recording those observations for 30 seconds.

Approximately two weeks after each teacher's last observation, the researcher interviewed these teachers by telephone using the interview protocol. The teachers' observed classroom practices were compared with their self-reported classroom practices by the researcher and independently by a mathematics educator. Both concluded that the teachers very accurately reported their classroom practices during the telephone interview. Hence, using teachers' self-reported classroom practices during the telephone interview was deemed a reliable source of data for the study and no other classroom observations were conducted.

In addition to determining whether teachers accurately reported their classroom practices, the telephone interviews with the local AP teachers served to pilot the interview protocol. Several questions on the interview protocol were modified and additional questions were added after analyzing the interviews with these teachers.

Figure 3f

Interview Protocol Questions Related to Teachers' Classroom Practices

1A "Describe as specifically as you can the lesson in which you introduce differentiation to your class. We are interested in the way you organize and present the mathematics content as well as the specific teaching methods and strategies that you use."

2A "Describe as specifically as possible a typical lesson involving differentiation in your class. Again, we are interested in the way you organize and present the mathematics content as well as the specific teaching methods and strategies that you use. Also, please explain how the typical lesson differs from the introductory lesson on differentiation and whether and how the typical lesson might change over the course of the school year."

1D "Do you use a graphing calculator when introducing differentiation? If so, How?"

2C "How do you use the graphing calculator in a typical lesson on differentiation?"

1E "Do your students use a graphing calculator during the introductory lesson on differentiation? If so, How"

1B "How do you use the math textbook in your introductory lesson(s)?"

2B "How do you use the math textbook in your typical math lesson on differentiation?"

4A "Do you have the students memorize rules of differentiation sometime during the school year?"

5A "Do you have the students work on word problems involving differentiation at any time during the first few weeks of the school year?"

8C "Students have different abilities and knowledge about differentiation, slopes, and rates of change, how do you find out about these differences?"

8D "Do you use this knowledge in planning instruction?"

Figure 3g

Interview Protocol Questions Related to Teachers' Beliefs

2D "What do you think the role of the teacher should be in a typical lesson on differentiation in your class?"

2E "What do you think the role of learner should be in a typical lesson on differentiation in your class?"

3A "What do you try to have your students learn about differentiation during the year?"

3B "Are there certain concepts in differentiation that you want all students to learn? If so what are they?"

3D "Are there certain kinds of word problems using differentiation you believe that all students should learn to solve? If so, what are they?"

7A "What factors do you consider when determining how topics in differentiation should be sequenced?"

7B "How rigorous an understanding of limits should students have before proceeding to the study of differentiation?"

6C "hat do you see as the relationship between learning of differentiation rules, conceptual understanding differentiation, and differentiation word problems?"

7C "What do students in your calculus class know about slopes and rates of change when they start the class?"

Calculus Readiness Test. The Mathematical Association of America's *Calculus Readiness Test, form 1G (CRT)* was administered as a pretest to the students of the 1995 teachers (Mathematical Association of America [MAA], 1994). The CRT contains 25 multiple-choice questions. Students were allowed thirty minutes to take the test and calculators were permitted, but not required to answer any of the questions. Permission to use this test was granted by the Mathematical Association of America.

Differentiation Test. A researcher-designed Differentiation Test (see Appendix D) was administered to assess student achievement. This test was modeled after parts of the Advanced Placement AB exam. The Differentiation Test included questions about functions and graphs, limits and continuity, and differential calculus. However questions about integral calculus which would be found on the AP exam were excluded. The Differentiation Test was designed to assess students':

1. ability to use graphical, numeric and symbolic presentations of information,
2. computational skill with derivatives,
3. conceptual knowledge of derivatives, and
4. general knowledge of differential calculus.

The Differentiation Test was piloted with undergraduate students at Oregon State University during the summer of 1995. AP calculus teachers' who gave the Differentiation Test to their students established face validity.

The Differentiation Test contained two sections. Section 1 included two parts: Part A consisted of seven multiple choice questions where the use of a calculator was not allowed (twenty-five minutes was allowed for Part A) and Part B consisted of seven questions where a graphing calculator was required to answer some questions (twenty-five minutes was allowed for Part B). Section II contained three free-response questions, one of which required the use of a graphing calculator. In this section partial credit was possible. A scoring guide, similar to the guide used to grade AP exams, was developed by the researcher to determine partial credit (see Appendix D). Forty-five minutes was allowed for Section II.

The Differentiation Test contained four subtests designed to determine students' ability to work graphically, numerically, and symbolically, and to test their facility with the graphing calculator. The first subtest contained three questions which presented information graphically (questions 11, 14, and 15). The second subtest contained two questions which required interpreting information given in a table (questions 8 and 13). The third subtest contained three question which required the use of a graphing calculator (questions 10, 12, and 16). The fourth subtest contained nine questions which tested students' ability to work with symbolic information (questions 1,2,3,4,5,6,7,9,and 17). Figure 3h below lists the 17 questions from the Differentiation Test.

Figure 3h

<u>Questions From the Differentiation Test</u>

---

1. Given that $f(x) = 2e^{x^2}\sin(x)$, what is $f'(x)$?

   A) $4xe^{x^2}\cos(x)$
   B) $2x^2e^{x^2}\cos(x)$
   C) $2e^{x^2}\sin(x) + 2e^{x^2}\cos(x)$
   D) $4xe^{x^2}\sin(x) + 2e^{x^2}\cos(x)$
   E) $2x^2e^{x^2}\sin(x) + 2e^{x^2}\cos(x)$

---

2. The equation of the curve determined by reflecting $y = \ln x - 1$ about the x-axis is

   A) $y = e^{x-1}$
   B) $y = e^{1-x}$
   C) $y = \ln x - 1$
   D) $y = 1 - \ln(-x)$
   E) $y = 1 - \ln x$

Figure 3h (continued)

---

3. What is $\lim_{h \to 0} \frac{\sin(\pi/2+h)-\sin(\pi/2)}{h}$?

   A) $-\infty$ B) -1 C) 0 D) 1 E) $+\infty$

---

4. If r is positive and increasing, for what value of r is the rate of increase of r^4 six times the rate of increase of r^2?

   A) $\frac{3}{2}$ B) $\sqrt{3}$ C) $\frac{1}{\sqrt{3}}$ D) $\sqrt{6}$ E) $\frac{1}{\sqrt{6}}$

---

5. The slope of the tangent line to the curve $y^3x+y^2x^2=6$ at (2,1) is

   A) $\frac{-3}{2}$ B) -1 C) $\frac{-5}{14}$ D) $\frac{-3}{14}$ E) 0

---

6. Given the curve $y=2x^3-3x^4$, which of the following statements are true?

   A) The curve has no relative extrema.
   B) The curve has one point of inflection and two relative extrema.
   C) The curve has two points of inflection and one relative extremum.
   D) The curve has two points of inflection and two relative extrema.
   E) The curve has two points of inflection and three relative extrema.

---

7. For which of the following functions does the property $\frac{d^3y}{dx^3}=\frac{dy}{dx}$ hold?

   I. $y=e^x$
   II. $y=\ln x$
   III. $y=\sqrt{3}$

   A) I only
   B) II only
   C) I and II
   D) I and III
   E) I, II, and III

Figure 3h (continued)

---

8. For the following question use the information given in the tables below. What is the value of $(f \circ g)(x) + f(2x)$ for $x = 3$?

x	2	3	6	7
$f(x)$	3	5	8	2
$g(x)$	4	7	5	7

A) 43   B) 45   C) 12   D) 8   E) 10

---

9. Which of the following combination of properties could possibly describe a function?

I. $f(x)>0$ and $f'(x)>0$ for all x
II. $f(x)<0$ and $f''(x)<0$ for all x
III. $f(x)>0$ and $f''(x)<0$ for all x

A. I only   B. II only   C. I, II, and III   D. I and II   E. I and III

---

10. Let $f'(x) = \frac{x^5}{4} - x^4 + \frac{3x^2}{2}$. To three decimal places the function f has a relative maximum at $x = ?$

A) 0.00   B) -.777   C) 1.059   D) 1.572   E) 3.514

---

11. At which of the five points on the graph in the figure at the right are $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ both positive?

A) A
B) B
C) C
D) D
E) E

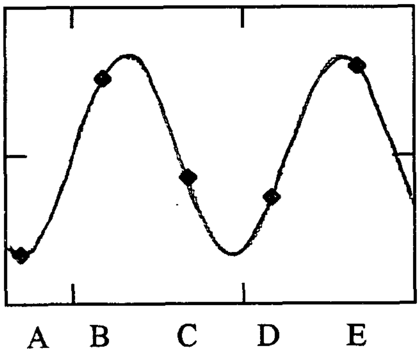


Figure 3h (continued)

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12. Two particles start at the origin and move along the x-axis. For $0 \le t \le 10$, their respective position functions are given by $x_1 = \cos t$ and $x_2 = \ln(2t) + 2$. For how many values of t do the particles have the same velocity?

A) none B) one C) two D) three E) four

---

13. For the following question use the information given in the tables below. Let $H(x) = (f \circ g)(x)$. What is the value of $H'(3)$?

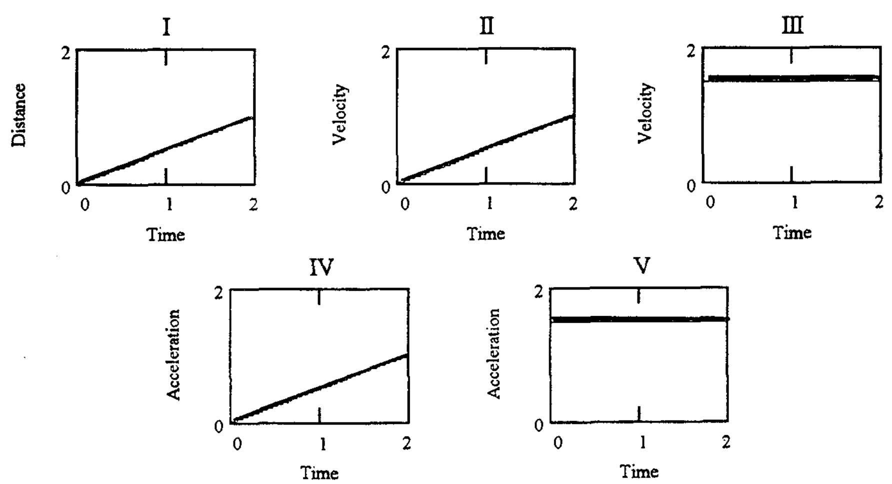
x	$f(x)$	$f'(x)$
1	2	3
2	3	4
3	4	6
7	3	8

x	$g(x)$	$g'(x)$
3	7	6
4	8	7
5	9	8
6	1	9

A) 36 B) 9 C) 46 D) 48 E) 4

---

14. Which of the following graphs below represent(s) motion at constant, non-zero acceleration?



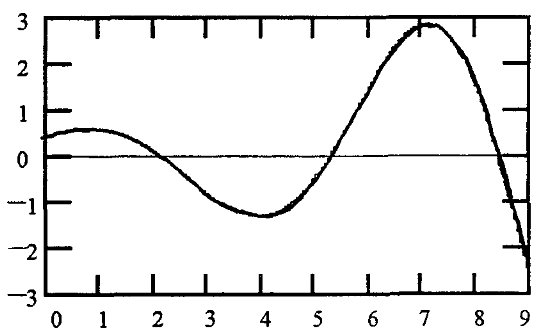
A) I, II, and IV B) I and III C) II and V D) IV only E) V only

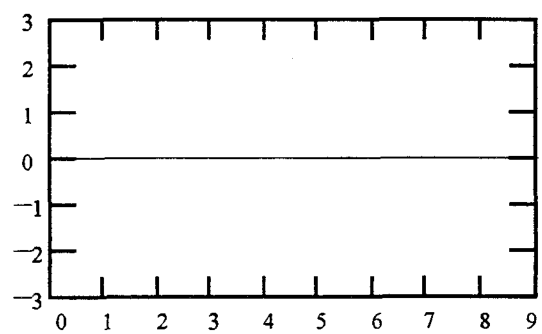
---

Figure 3h (continued)

15. Consider the graph of $y = f'(x)$ given below (note that this is the graph of the derivative of f).

    A) Sketch the graph of $f''(x)$ in the space provide.
    B) Over which intervals is the graph of f concave up? Justify your answer.
    C) Find the x-coordinates of all relative minimums for the function $f(x)$. Justify your answer.





16. Suppose f is continuous over the interval $1 < x < 7$ and $f'(x) = (\ln x)^2 - 2(\sin x)^4$ for $1 < x < 7$. Answer the following to three decimal places.

    A) For what values of x is $f'(x) = 0$?
    B) Find the x-values for all relative minima for the function $f(x)$. Justify your answer.
    C) For what values of x is $f''(x) = 0$? Justify your answer.

17. The radius r of a sphere is increasing at the constant rate of 0.05 centimeters per second. (Volume of sphere with radius r is $V = \frac{4}{3}\pi r^3$). Answer the following questions.

    A) How fast is the volume of the sphere increasing when the radius of the sphere is 10 centimeters?
    B) What is the radius of the sphere when the volume of the sphere and the radius of the sphere are increasing at the same rate?
    C) When the volume of the sphere is 39π cubic centimeters, how fast is the area of the cross section through the center of the sphere increasing?

Procedure

Teachers' Pedagogical Content Beliefs

Categorizing teachers' pedagogical content beliefs. The Belief Questionnaire was administered to the 1995 teachers during the summer of 1995. The internal consistency of teachers' scores on each of the four subscales, as well as the total score, was determined using Cronbach's alpha. Correlations between subscales were also determined using Cronbach's alpha. The median and standard deviation of 1995 teachers' scores on the four belief constructs were determined. Teachers were considered cognitively based (CB) if their scores on each of the four subscales in the Belief Questionnaire fell above the median of the total score for that construct. Teachers were considered less cognitively based (LCB) if their scores on each of the four subscales in the Belief Questionnaire fell below the median.

One of the purposes of the telephone interview was to provide an additional method of determining teachers' pedagogical content beliefs (construct validity). Immediately following the telephone interview, teachers were rated by the researcher on each of the four subscales of the Belief Questionnaire. The teachers were rated on a 5-point scale where a high score indicated cognitively based beliefs. Teachers' responses to the interview were read and rated a second time by the researcher several weeks after the final interview. Teachers' names were removed from the interview notes for the second reading so that individual teachers could not be identified. The two ratings given to each of the teachers on the four scales were compared by the researcher. In those cases where there was a disagreement between the two ratings, the notes were reread once again by the researcher and a determination was made. Correlations between the interview ratings and the Belief Questionnaire score were reported.

Changes in teachers' pedagogical content beliefs. To investigate if teachers' pedagogical content beliefs become more cognitively based as they become familiar with the CCP materials, Belief Questionnaire scores of the 1993, 1994, and 1995 were compared. The Belief Questionnaire was mailed to teachers who participated in CCP workshops in 1993 and 1994 in December of 1995. At the time the Belief Questionnaire was administered to the 1993 and 1994 teachers, the 1993 teachers had been using the project materials for more than two years and the 1994 teachers had been using the project materials for more than one year. The 1995 teachers were given the Belief Questionnaire at the beginning of their workshops, prior to any instruction in the CCP curriculum. The Extra Sum of Squares Test was used to test the Belief Questionnaire total score for consistency across the three different years.

To gain additional insight into the pedagogical content beliefs of 1993, 1994, and 1995 teachers, a One-Way ANOVA was used to test differences in means among the 1993, 1994, and 1995 CB and LCB teachers (CB and LCB distinctions for the 1993 and 1994 teachers were based on the median scores of the 1995 teachers).

To examine changes in teachers' pedagogical content beliefs over time, the 1995 teachers were given a follow-up Belief Questionnaire after having used the project materials for seven or eight months. The follow-up Belief Questionnaire (see Appendix B) consisted of the 24 positively worded questions from the Belief Questionnaire (responses to the positively-worded questions and the negatively-worded questions of the Belief Questionnaire were found to be highly correlated). The follow-up Belief Questionnaire was mailed to the 1995 teachers at the end of March in 1996. A paired t-test was used to compare the responses of the 1995 teachers to the positively worded question on the Belief Questionnaire to their responses on the follow-up Belief Questionnaire. A Paired Two-Sample t-test was to determine if the 1995 CB and LCB teachers' pedagogical content beliefs changed over time.

Teachers' Classroom Practices

Telephone interviews. Teachers' reports of their classroom practices and their goals for instruction were obtained by the interviews. All CB and LCB teachers were interviewed by the researcher in January and February of 1996. During the interview, the researcher would read a question from the interview protocol and record teachers' responses on paper. If the interviewer felt that a question had not been adequately answered, a single probe would be used. Individual interviews took from forty-five minutes to two hours. Typically, interviews took between one hour to one hour and thirty minutes.

Procedure for analyzing interviews. The handwritten comments from the Interview protocol were entered into a computer by the researcher. For each question, teachers' comments were arranged into a list. Each statement on the list was preceded by a number which identified the teacher who made the statement. CB teachers were numbered 1 through 9 and LCB teachers were numbered 11 through 18. Teacher statements to each question were then read and "cut-and-pasted" into groups of similar statements. Statements given by CB teachers to a particular question were grouped separately from statements given by LCB teachers. Categories for teachers responses to individual questions were determined from these groups of responses.

Graduate students and professors participating in a mathematics education seminar who were familiar with this research project served as sorters to confirm the categorization of teacher statements. To accomplish this, one question from the interview protocol was typed at the top of a blank document. Just below the question the category titles which had been determined by the researcher were listed. An additional category titled "OTHER" was added to the list of category titles (see Figure 3i for an example of the sorters' form). Teacher comments, preceded by their identification number, were listed below the category titles. The sorters would read a teacher comment and write the teachers' identification number next to the category title they thought it belonged to. Each question was categorized by two sorters in addition to the researcher. In those cases where both sorters and the researcher agreed on which category a statement belonged, no changes were made. In those cases where only one sorter agreed with the researcher on which category a statement belonged to, the statement was categorized by a fourth sorter. If the fourth sorter agreed with the majority, the category remained unchanged. When three out of four sorters did not agree on which category a teacher's comment belonged, the comment was put in a category titled miscellaneous, or a new category was defined for that statement.

After the teachers' responses had been categorized, the number of responses by CB teachers and LCB teachers to each category were summarized. To gain further information on teachers' pedagogical content beliefs, some questions on the interview protocol asked the teachers to provide supporting information. A summary of teachers' interviews, which considers the overall interview, is presented as a part of the data analysis.

Student Achievement

Test administration. The 1995 teachers were given copies of the Calculus Readiness Test, instructions for its administration, and the answer key during the CCP workshops they attended in the summer of 1995. A reminder, which included an additional copy of the Calculus Readiness Test, instructions for its administration, and the answer key was sent to these teachers prior to the start of the school year. Teachers were asked to administer the Calculus Readiness Test as close to the first day of class as possible.

Teachers graded the Calculus Readiness Test and returned a list of students' scores. To protect student confidentiality, teachers used the student number from their roll books for identification purposes. Teachers who did not return student scores were contacted by telephone.

Figure 3i

Sorters' Categorization Form

APPROACH TO TEACHING INTRODUCTORY LESSON

Introduce Differentiation Using Slope of Secant Line Tending to Tangent Line
Introduce Differentiation Using Average Velocity and Instantaneous Velocity
Introduce Differentiation with Word Problem
Introduce Differentiation by the Definition
Other

1 First do velocity and average velocity using a graph.

2 Discuss average rate of change, then look at instantaneous rate of change in terms of speed and velocity

3 Talk about slope of tangent line at a point, get values from curve. Use this to come up with definition of derivative

4a Talk about secant line tending to the tangent line.

4b Often use velocity and use secant line to get at the idea of tangent line.

5 Talked about average rate of speed going to Grapefalls. Ask students what average speed is then ask what the speed is at a particular place to introduce instantaneous rate of change

6 First show graph of fruit fly experiment, have data on fruit fly population Talk about average change in population. Relate this to secant line and Tangent line

7 Talk about stock prices yesterday and today. Look at change in stock prices and where the changes are the greatest.

8 Define derivative with secant line and let two points get closer together and discuss those becoming the tangent line.

9 Usually start with look at some kind of Problem, usually max/min problem. Talk about why we want to find the max point on graph of the function.

11 Start with curve on board show how secant line goes to tangent line.

12 Draw graph and talk about slope of tangent line and behavior of graph.

13 Start with slope formulas to identify rise over run then go through the formulas using the definition of derivative.

14 Begin with curve on board and look at secant line as it becomes tangent line.

15 Start with slope and rate of change. Use definition of derivative to look at this Do that with limit, start with limit.

16 Reintroduce topic by talking about slope and do secant line tending to tangent line.

17 Teach from a geometric point of view. Use a general function and talk about tangent line. Use secant line and 2 points and look at secant line as points get close together.

18 Go into definition $f(x + h)$ thing. Do drawings to get students to see secant line tending to tangent line.

Note 1-9 represent statements by CB teachers. 11-18 represent statements by LCB teachers. (Sorters were not aware of the numbering scheme.)

All 1995 teachers were sent Section I of the Differentiation Test (of the 61 teachers who returned the calculus readiness test, 51 administered and returned the Differentiation Test). CB and LCB teachers received Section II of the Differentiation Test in addition to Section I. The tests, instructions for their administration, and scantron sheets were mailed to the 1995 teachers in January 1996. Teachers were requested to administer the test immediately after completing their lessons on differentiation, prior to any instruction on integration. In order to match students' Differentiation Test scores with their Calculus Readiness Test scores, teachers identified student exams with the same student identification number used with the Calculus Readiness Test. Teachers who did not return the completed exams were contacted by telephone.

Section I of the Differentiation Test was machine-scored, Section II, the free response portion of the test, was scored by the researcher. The class average for each question was used as the unit of analysis.

Assessing student achievement. To determine relationships between the 1995 teachers' pedagogical content beliefs and student achievement correlations between teachers' scores on the Belief Questionnaire and their students' achievement on Section I Part A (computational) and Section 1 Part B (conceptual) of the Differentiation Test were analyzed using Multiple Linear Regression. Student Calculus Readiness Test scores were used as a covariate.

Relationships between teachers' pedagogical content beliefs and student achievement was explored in greater detail with the 1995 CB and LCB teachers. A t-test indicated that there was no significant difference in the pretest scores of students of CB and LCB teachers ($p < 0.05$) so that it was not necessary to adjust for Calculus Readiness Test scores when comparisons were made between students of CB and LCB teachers. Therefore, a t-test was used to test for difference in the mean scores of student of CB and LCB teachers on Section I Part A, Section I Part B, Section II, and the total score. A t-test was also used to test for a difference in means on four subtests of the

Differentiation Test which tested students' ability to work graphically, numerically symbolically, and their facility with the graphing calculator.

A t-test for regression coefficients was used to determine if the demographic variables Sex, Age, Degree, or Final Class Size were related to students' achievement after accounting for student pretest scores. The F-statistic and p-values are reported. A regression analysis was also used to determined if Years Teaching Mathematics, was related to student achievement. The t-statistic and p-values are reported.

Figure 3j gives a summary of the instruments used in this study and the dates which whey were administered. Figure 3k shows how many teachers were given each instrument and how many teachers completed each instrument. Figure 3l gives a summary of the instruments given to the 1995 CB and LCB teachers.

Figure 3j

Schedule for Administration of Instruments

Belief Questionnaire	Preliminary Belief Questionnaire	January 1995
	Version I and Version II of Belief Questionnaire	May 1995
	Belief Questionnaire	Summer 1995
Interview Protocol	Preliminary Version	Fall 1995
	Interview Protocol	Winter 1995
Teachers Classroom Practices	Observations	Fall 1995
	Interviews	Fall/Winter 1995
Calculus Readiness Test		August-September 1995
Differentiation Test	Preliminary Version	Summer 1995
	Differentiation Test	January-February 1996
Follow-up Belief		April 1996

Figure 3k

Summary of Data Collected From 1995 Teachers

Number of teachers in Study	69	73 teachers completed Belief Questionnaires, 1 teacher retired, 2 teachers no longer taught class, and 1 teacher missed 3 months of school due to illness.
Completed Calculus Readiness Test	61	Of the 69 original teachers, 8 teachers did not return student scores on the Calculus Readiness Test (88% return).
Completed Calculus Readiness Test and Differentiation Test	51	61 teachers who returned completed the Calculus Readiness Tests were sent the Differentiation Test, 8 teachers did not return completed Differentiation Test, 2 teachers returned scores for the differentiation test which did not include student identification numbers (82% return).
Completed Both Belief Questionnaires	57	Of the 69 teachers in the study who completed the Belief Questionnaire during the summer workshops, 12 failed to return the Belief Questionnaire which they were sent near the end of the study (83% return).

Figure 31

Summary of Data Collected From 1995 CB and LCB Teachers

Number of teachers in Study	CB	9	All teachers determined to be CB elected to participate in the study
	LCB	7	9 teachers were determined to be LCB. 1 teacher elected not to participate and 1 teacher was replaced during the course of the study.
Completed Calculus Readiness Test	CB	9	All CB teachers returned completed Calculus Readiness Tests (100% return).
	LCB	7	All LCB teachers returned completed Calculus Readiness Tests (100% return).
Completed Calculus Readiness Test and Differentiation Test	CB	8	1 CB teacher did not return the Differentiation Test (89% return).
	LCB	5	2 LCB teachers did not return the Differentiation Test (71% return).
Completed Both Belief Questionnaires	CB	9	All CB teachers completed both Belief questionnaires (100% return).
	LCB	5	2 LCB teachers failed to return the Belief Questionnaire mailed to them near the end of the study (71% return).

CHAPTER FOUR: RESULTS

This study explored relationships among teachers' pedagogical content beliefs, teachers' classroom practices, and student achievement. We begin the presentation of results by looking at AP calculus teachers' pedagogical beliefs about mathematics, curriculum, and instruction in AP calculus, and how well their beliefs align with a constuctivist point of view. This will be followed by an examination of how AP calculus teachers' pedagogical content beliefs are reflected in self-reports of their approaches to teaching, their concepts of the roles of the teacher and the learner, and their goals for instruction. This chapter is concluded with an examination of relationships among AP teachers' pedagogical content beliefs, approaches to teaching, and their students' achievement.

Teachers' Pedagogical Content Beliefs

The Belief Questionnaire used a 5-point Likert scale. High scores indicated cognitively based beliefs (5 was high). Lower scores indicated less cognitively based beliefs (LCB). Table 4a gives the mean, median and standard deviation of the 1995 teachers' responses on the Belief Questionnaire. The maximum score possible on each of the four belief constructs is 60, while the minimum score possible is 12. The maximum total score possible on the Belief Questionnaire is 240 and the minimum total score possible is 48.

The standard deviations given in table 4a indicate considerable variation existed among teachers' scores on the four belief constructs. The greatest variation occurred in Scale 1 (students construct knowledge), while the least variation is found in Scale 2 (skills should be taught in relation to problem solving). Teachers had the highest average score on Scale 4 (instruction should be organized to facilitate students' understanding of mathematics). On the other hand, teachers had the lowest average score on Scale 1 (students construct knowledge).

Teachers were characterized as cognitively based (CB) or less cognitively based (LCB) on the basis of their scores on the four subscales of the Belief Questionnaire (the same criteria used in the study of Peterson et al.). Teachers who had scores above the median on each of the four subscales were defined to be CB, while teachers whose scores fell below the median on each subscale were defined as LCB. In this way nine teachers were determined to be CB, and nine teachers were determined to be LCB. One of the LCB teachers chose not to participate leaving eight teachers in the LCB group.

Table 4a

Teachers Responses on Belief Questionnaire

	Mean	Median	SD	Range	Min 12	Max 60
Students construct or receive mathematical knowledge	37.70	38	5.29	27	23	50
Skills taught in relation to or isolated from problem solving	41.08	41	4.38	23	30	53
Instruction organized to facilitate students' development of mathematical ideas or by the structure of mathematics	40.86	42	4.45	23	26	49
Instruction sequenced by students' construction of knowledge or for clear teacher presentation	44.58	45	4.66	24	30	54
Total score	164.22	164	12.87	57	133	190

Note. Summary statistics of the 73 teachers who initially completed the Belief Questionnaire.

Table 4b gives the CB and LCB teachers' scores on the Belief Questionnaire and Table 4c compares means and standard deviations for the 1995 CB and LCB teachers.

Table 4d reports the internal consistency of the four belief constructs and correlations between subscales on the Belief Questionnaire using Cronbach's alpha. In Table 4d, the entries along the diagonal estimate the correlation between the positively worded questions and the negatively worded questions within a subscale of the Belief Questionnaire. Entries off the diagonal estimate correlations between subscales.

Table 4b

CB and LCB Teachers' Scores on Belief Questionnaire

CB Teachers' Scores on the Belief Questionnaire					
	Scale 1 **38**	Scale 2 **41**	Scale 3 **42**	Scale 4 **45**	Total **164**
1	50	44	46	50	190
2	42	49	43	46	180
3	46	45	47	49	187
4	43	48	43	49	183
5	42	42	44	47	175
6	46	49	44	47	186
7	42	50	44	48	184
8	39	44	44	46	173
9	41	44	44	47	176

LCB Teachers' Scores on the Belief Questionnaire					
	Scale 1 **38**	Scale 2 **41**	Scale 3 **42**	Scale 4 **45**	Total **164**
11	32	31	38	41	142
12	37	37	38	43	155
13	31	40	26	42	139
14	29	37	37	30	133
15	35	37	33	40	145
16	34	34	31	43	142
17	32	40	39	40	151
18	28	30	37	41	136

Note. Numbers in left hand column are teacher identification numbers. Medians for each scale and total score are given in bold.

Table 4c

Comparison of Means for 1995 CB and LCB Teachers

	Scale 1		Scale 2		Scale 3		Scale 4		Total	
	CB n = 9	LCB n = 8	CB n = 9	LCB n = 8	CB n = 9	LCB n = 8	CB n = 9	LCB n = 8	CB n = 9	LCB n = 8
Mean	43.4	32.3	46.1	35.8	44.3	34.9	46.77	40.0	181.7	143.0
SD	3.3	3.0	2.9	8.8	1.3	4.5	1.4	4.2	5.9	7.4
n	9	8	9	8	9	8	9	8	9	8

Note. Maximum score on each subscale was 60. Minimum score on each subscale was 12. Maximum total score was 240. Minimum total score was 48.

Table 4d

Belief Questionnaire Correlations using Cronbach's alpha

	Scale 1	Scale 2	Scale 3	Scale 4
Scale 1: Students construct or receive mathematical knowledge	.6776	.6162	.4652	.5918
Scale 2: Skills taught in relation to or isolated from problem solving		.7153	.4624	.3865
Scale 3: Instruction organized to facilitate students' development of mathematical ideas or by the structure of mathematics			.7603	.3151
Scale 4: Instruction sequenced by students' construction of knowledge or for clear teacher presentation				.8049

Note. Entries on the diagonal are correlations between positively worded questions and negatively worded questions on a single scale. Entries off the diagonal are correlation between scales. n = 73.

Interview Results

To establish construct validity, ratings determined by the researcher after the telephone interview were compared to the scores teachers received on the Belief Questionnaire. Table 4e shows that teachers' interview ratings and their Belief Questionnaire scores were highly correlated.

Table 4e

Interview and Belief Questionnaire Correlations using Cronbach's alpha

Scale 1: Students construct or receive mathematical knowledge	.8205
Scale 2: Skills taught in relation to or isolated from problem solving	.6269
Scale 3: Instruction organized to facilitate students' development of mathematical ideas or by the structure of mathematics	.7355
Scale 4: Instruction sequenced by students' construction of knowledge or for clear teacher presentation	.6337

Note. Correlations between interview rating and Belief Questionnaire score for the 9 CB and 8 LCB teachers.

Some interview questions were intended to shed additional light on CB and LCB teachers' pedagogical content beliefs. To explore patterns in the responses of CB and LCB teachers to questions about their pedagogical content beliefs, teachers' responses to individual questions were categorized. Table 4f gives a summary of CB and LCB Teachers' responses to questions about their pedagogical content beliefs.

Table 4f

Teachers' Beliefs: Summary of Interviews

Role of the Teacher		
Teacher allows students to explore	3-CB	1-LCB
Teacher as facilitator/guide	6-CB	5-LCB
Teacher as instructor and knowledge base	4-CB	4-LCB
Role of the Learner		
Student as explorer	3-CB	0-LCB
Students being involved	5-CB	5-LCB
Learn from teacher	1-CB	3-LCB
What Teachers Want Students to Learn		
Meaning of derivative	5-CB	3-LCB
Rules and notation	4-CB	1-LCB
Solve problems	6-CB	4-LCB
What is on AP exam	0-CB	2-LCB
Concepts Teachers Want Students to Learn		
Conceptual understanding of derivative.	7-CB	6-LCB
The ability to work problems	3-CB	3-LCB
Rules of differentiation	1-CB	4-LCB
Important Word Problems		
Typical calculus problems	8-CB	6-LCB
Real World	2-CB	0-LCB
Non-traditional	0-CB	1-LCB
No particular kind	0-CB	1-LCB
Factors Considered When Sequencing Topics		
Follow the text	8-CB	5-LCB
Ability of student	4-CB	3-LCB
Relationship of Rules, Concepts, and Word Problems		
Concepts most important	8-CB	3-LCB
Word problems most important	0-CB	1-LCB
Rules most important	0-CB	2-LCB
Equal importance	1-CB	2-LCB
Instructional Strategies		
Memorize rules	4-CB	7-LCB
Start with word problems	7-CB	1-LCB
Use student knowledge in planning	9-CB	6-LCB

Role of the teacher. Question 2D: "What do you think the role of the teacher should be in a typical lesson on differentiation in your class?" All of the CB teachers included in their answer to this question statements which indicated they believed teachers should allow students to explore or the teacher should act as a facilitator/guide. Only five of the LCB teachers made statements related to these categories. Four LCB teachers and four CB teachers made statements indicating their role to be that of instructor or knowledge base. However, all of the CB teachers who indicated that the role of the teacher was that of instructor or knowledge base also indicated a belief that the role of the teacher was to allow students to explore or act as a "facilitator/guide." Only two of the LCB teachers who believed the role of the teacher was that of instructor or knowledge base also indicated they thought their role was to allow students to explore or act as a "facilitator/guide."

Role of the learner. Question 2E: "What do you think the role of learner should be in a typical lesson on differentiation in your class?" Most teachers responded by giving statements which were categorized as "students should be involved." Four CB teachers viewed the role of the student as that of an explorer, while none of the LCB teachers made comments related to this category. On the other hand, three LCB teachers made statements indicating that the role of the student was to learn from the teacher, while only one CB teacher made a statement which fit into this category.

What teachers want students to learn. Question 3A: "What do you try to have your students learn about differentiation during the year?" Solving problems was the most common response from both CB and LCB teachers when they were asked what they would like to have their students learn about differentiation. Two-thirds of the CB teachers and half of the LCB teachers mentioned wanting students to learn how to solve problems. An interesting difference between the responses of CB and LCB teachers to this question was that CB teachers were more likely to include in their response to this question a desire to have their students learn rules and notation. However, all of the CB teachers who believed learning rules and notation was important also indicated that they thought solving problems and/or understanding the meaning of derivative was important. Only two teachers indicated that they wanted students to learn what was necessary to be successful on the AP exam and both of these teachers were LCB

Concepts teachers want students to learn. Question 3B: " Are there certain concepts in differentiation that you want all students to learn? If so what are they?" When teachers were asked if there were certain concepts they wanted all students to learn, the most frequently given response was a desire to have students understand the concept of derivative. Although teachers were specifically asked what concepts they felt were important, many teachers responded with ideas which were not conceptual. Half of the LCB teachers included in their answer to this question statements that indicated the rules of differentiation were important, while only one CB teacher gave this response.

Important word problems. Question 3D: "Are there certain kinds of word problems using differentiation you believe that all students should learn to solve? If so, what are they?" CB and LCB teachers most often reported believing that traditional calculus problems such as related rates, max/min, and application problems were most important. Interesting differences were found when the CB and LCB teachers were asked why they thought these word problems are important. Six of the CB teachers gave reasons related to future employment or success in college. For example, CB teacher 1 stated that she lived in a rural area with lots of business and industry and wanted her students to be able to use mathematics related to these employers. On the other hand, no LCB teachers gave a response to this question related to future employment or success in college. Two LCB teachers indicated that they thought application problems were important because they were common in calculus texts. No CB teacher gave this reason.

Factors influencing the sequencing of topics. Question 7A: "What factors do you consider when determining how topics in differentiation should be sequenced?" This is the first year that the 1995 teachers have used the CCP materials and it is not surprising that when they were asked what factors they consider in determining how topics should be taught, the most common response was categorized as "Follow the text." All but one CB teacher indicated the class text influenced how topics were sequenced, five of the LCB teachers gave this response. Less than half of both CB and LCB teachers gave responses to this question which were categorized as "Ability of student."

Relationship of rules, concepts, and word problems. Question 6C: "What do you see as the relationship between learning of differentiation rules, conceptual understanding of differentiation, and differentiation word problems?" Important differences were found in the responses of CB and LCB teachers to this question. All but one of the CB teachers indicated that they believed conceptual understanding was most important. The remaining CB teacher believed learning rules, conceptual understanding, and solving word problems were equally important. This is in sharp contrast to the responses given by the LCB teachers. Only three LCB teachers indicated a belief that conceptual understanding was most important. In addition, LCB teachers were more likely than CB teachers to believe that learning rules of differentiation was most important.

Comparison of Interview Results for CB and LCB Teachers

An analysis of teachers' responses to interview questions further supported the categorization of teachers' pedagogical content beliefs determined by the Belief Questionnaire. Scale 1 of the Belief Questionnaire reflects a belief that students construct their own knowledge. During the telephone interview teachers were asked to describe their beliefs about the role of the teacher and learner. More CB teachers gave responses indicating that the role of their students should be to explore and the role of the teachers to guide and facilitate. On the other hand, LCB teachers were more likely to indicate that students learn from the teacher.

Scale 2 of the Belief Questionnaire reflects a belief that mathematical skills should be taught in relation to understanding and problem solving. To gain insight into this construct during the telephone interview, teachers were asked their beliefs about what should be the relative emphasis on rules of differentiation, conceptual understanding, and solving word problems. CB teachers were more likely to indicate that a conceptual understanding of differentiation was most important. LCB teachers' belief that skills should be taught as a discrete component is indicated by the fact that LCB teachers were more likely to have their students memorize the rules of differentiation. In addition, LCB teachers more often indicated that rules were more important than conceptual understanding and solving word problems.

Scale 3 of the Belief Questionnaire indicates that a math curriculum should be organized to facilitate students' construction of knowledge. To explore this construct during the telephone interview teachers were asked how rigorous an understanding of limits they thought students should have before proceeding to the study of differentiation. Intuitive approaches were considered in keeping with the idea that a math curriculum should be organized to facilitate students' construction of knowledge. On the other hand, an approach to limits which required students to work on traditional delta-epsilon problems was considered to indicate a greater concern for the structure of mathematics. Results of the telephone interview indicated that CB teachers were more likely to approach limits intuitively and LCB teachers more often reported having their student work delta-epsilon problems.

Scale 4 of the Belief Questionnaire reflects the belief that students' development of mathematical ideas should provide the basis for sequencing topics rather than sequencing topics to insure a clear presentation by the teacher. To address this construct teachers were asked what factors they considered when determining how topics should be sequenced. Responses which indicated that teachers believed it important to take into consideration the students' ability or previous knowledge were considered in keeping with this construct. Although CB teachers were more likely to report following the text closely, they were also more likely to indicate that it was important to take into consideration the ability of their students.

Demographic Relationships

Peterson (1989) found a strong relationship between the number of years first grade teachers had been teaching and their Belief Questionnaire scores. For this reason relationships between Belief Questionnaire scores and demographic information for the 1995 AP calculus teachers was determined. An Extra Sum of Squares F-Test was used to explore possible relationships. The demographic variables analyzed were; gender, age, highest degree obtained, and number of years teaching mathematics.

A simple linear regression analysis between years teaching mathematics and years teaching provides strong evidence of a relationship (t-stat=21.11 , df = 71, and $p < .0001$) between years teaching mathematics and years teaching for this set of teachers, this relationship is described by

$$\text{Years\_Math} = -.0676 + .9559 * \text{Years\_Teaching}$$

This equation supports the obvious fact that the population of teachers considered for this study are mostly mathematics teachers. That is, each year gained in teaching experience has usually been accumulated in a mathematics program.

Results of the Extra Sum of Squares Test gives convincing evidence that the number of years teaching mathematics is related to the total belief score for the 1995 group of teachers ($F=5.13$, $p=.027$). However, there is no evidence that any of the other demographic variables, sex, age, degree, and years teaching calculus, are related to the total score on the Belief Questionnaire (p= .7975).

Relationships among the demographic information and the four subscales on the Belief Questionnaire were also explored. There is marginally suggestive evidence that the number of years the 1995 teachers have been teaching mathematics is associated with Scale 1 (p=.0726) and Scale 3 (p=.0895). There is convincing evidence that the number of years teaching mathematics and the degree obtained by the teacher were associated with the Scale 2 (p=.0104). No other demographic variables appeared related to any of the subscale measurements.

A two sample t-test was used to compare the teaching experience of CB and LCB teachers. There is strong evidence of a difference in the average number of years the CB teachers and LCB teachers have been teaching mathematics (p = .0397). The mean number of years CB teachers have been teaching mathematics was 22.13, while the mean number of years LCB teachers have been teaching mathematics is 13.44.

A regression analysis was used to determine whether CB and LCB distinctions were significant after accounting for years teaching experience. The results of this analysis gave strong evidence supporting a difference between belief scores of CB and LCB teachers after accounting for differences due to years teaching mathematics (p < .01). Table 4g gives the results of this analysis.

Table 4g

Significance of CB and LCB Distinctions Accounting for Years Teaching Mathematics

	F	p
Scale 1: Students construct or receive mathematical knowledge	40.77	.0007
Scale 2: Skills taught in relation to or isolated from problem solving	13.87	.0098
Scale 3: Instruction organized to facilitate students' development of mathematical ideas or by the structure of mathematics	37.76	.0009
Scale 4: Instruction sequenced by students' construction of knowledge or for clear teacher presentation	4.93	.0683
Total Score	197.84	.0001

Note. Results of Regression analysis for 9 CB teachers and 8 LCB teachers.

Changes in Teachers' Beliefs

Comparison of 1993, 1994, and 1995 Teachers

The Belief Questionnaire scores of the 1993, 1994, and 1995 teachers were compared to determine if there was a relationship between how long a teacher has used the CCP project materials and the degree to which they are cognitively based. Table 4h is a Stem and Leaf diagram of 1993, 1994, and 1995 teachers' scores on the Belief Questionnaire. In Table 4h, CB and LCB distinctions for 1993, 1994, and 1995 teachers were based on the median Belief Questionnaire scores of the 1995 teachers. A One-Way ANOVA found no evidence that the total scores on the Belief Questionnaire are different for these teachers ($F=.97$, $p=.3808$). There was also no significant difference among the 1993, 1994, and 1995 teachers on any of the four subscales of the Belief Questionnaire.

Changes in 1993, 1994, and 1995 teachers' pedagogical content beliefs were also explored by comparing scores of CB and LCB teachers on the Belief Questionnaire. Table 4i gives the results of a One-Way ANOVA used to test differences in means among the 1993, 1994, and 1995 CB and LCB teachers. A significant difference in means was found among the three groups of CB teachers ($F = 4.40$, $p = 0.0205$). The mean total score for the 1993 and 1994 teachers is 192.32 and 190.71 respectively, while the mean total score for the 1995 teachers was 181.56. No statistically significant difference was found in the means of the 1993, 1994, and 1995 LCB teachers.

Table 4h

Belief Questionnaire Scores of 1993, 1994, and 1995 Teachers

1993 Teachers' Scores on Belief Questionnaire

22	**1**
21	**4**
20	**4**
19	**0** 0 **1 2 3 5 6 7 9**
18	**1 1 2 2 2 2** 2 2 3 **6** 7 9
17	0 0 1 1 2 2 3 3 5 7 8
16	0 1 2 2 3 3 3 3 3 3 4 4 5 5 5 7 8 9 9 9
15	2 2 2 2 3 3 3 4 4 4 5 5 6 6 6 6 6 6 7 7 7 8 8 9
14	2 2 4 4 6 6 7 7 9 9
13	3 7
12	2 5

1994 Teachers' Scores on Belief Questionnaire

22	
21	
20	
19	**0** 1 **2 3 3 5** 6
18	0 0 0 **3 3 4 5** 5 6 **8**
17	0 0 1 1 2 2 2 2 3 3 4 5 5 5 6 6 8 8
16	0 2 3 4 4 7 9 9
15	1 4 4 4 5 5 6 7 7 8 8
14	1 2 4 5 5 7 7 7 8
13	7
12	

1995 Teachers' Scores on Belief Questionnaire

22	
21	
20	
19	**0**
18	**0** 1 **3 4** 5 **6** 7
17	2 2 **3** 4 4 5 5 5 5 **5 6** 6 6 6 7 7 9
16	0 1 1 2 2 3 3 3 4 4 4 4 4 4 5 6 7 7 7 7 8 9 9
15	0 1 2 3 4 5 5 6 7 8 8 8 9 9
14	2 2 5 6 7 8 9 9
13	3 6 9
12	

Note. 17|5 represents a score of 175 on the Belief Questionnaire. Bold type represents CB teachers. Underline type represents LCB teachers.

Changes in 1995 Teachers' Pedagogical Content Beliefs.

The 1995 teachers were administered Belief Questionnaires twice. The first time they were given the Belief Questionnaire was during the summer of 1995 prior to the CCP inservice workshop. A follow-up Belief Questionnaire, containing only the positively worded questions of the Belief Questionnaire, was mailed to teachers in April of 1996, near the end of the school year. Of the 69 teachers remaining in the study when the follow-up Belief Questionnaire was mailed, 57 returned a completed questionnaire (83%). Teachers not returning the follow-up Belief Questionnaire showed no indication of being biased toward CB or LCB.

Table 4j gives the results of a Paired Two-Sample t-test used to test for a difference in means between the scores of teachers who completed both questionnaires. Only the positively worded questions from the first Belief Questionnaire were used in the analysis. There is no evidence of a difference between the means of all teachers ($p = 0.23$), for CB teachers ($p = 0.14$), or for LCB teachers ($p = 0.22$).

Table 4i

Differences in Means Among 1993, 1994, and 1995 Teachers

	1993 Teachers			1994 Teachers			1995 Teachers		
	CB	LCB	ALL	CB	LCB	ALL	CB	LCB	ALL
Mean	192.32	142.38	166.40	190.71	147.17	168.39	181.56	142.88	164.22
SD	11.25	10.40	18.21	3.73	6.24	15.49	5.90	7.36	12.87
n	19	13	93	7	12	64	9	8	73

Note. 1993, 1994, and 1995 CB and LCB groups based on the median scores on the Belief Questionnaire of the 1995 teachers.

Table 4j

Comparison of 1995 Teachers' Belief Questionnaire Scores with their Follow-up Belief Questionnaire Scores

	All Teachers		CB Teachers		LCB Teachers	
	1st Belief Survey	2nd Belief Survey	1st Belief Survey	2nd Belief Survey	1st Belief Survey	2nd Belief Survey
Mean	86.37	87.86	93.11	95.89	77.80	75.80
SD	8.58	8.51	10.39	3.92	6.14	4.92
n	57		9		5	
t	0.74		1.17		0.88	
p	0.23		0.14		0.22	

Note. Only teachers who completed both the Belief Questionnaire and the follow-up Belief Questionnaire were included, n = 57.

Teachers' Classroom Practices

The interview protocol included questions which explored teachers' classroom practices. Three of these questions required a "Yes" or "No" answer, the remaining seven questions required detailed responses by the teachers. Similar statements made by teachers to any one question were grouped into categories. The number of CB teachers and the number of LCB teachers who made a statement related to a given category are reported in Table 4k.

CB and LCB Teachers' responses to the three "Yes" or "No" questions show some dramatic differences. When asked if they had their students memorize the rules of differentiation at any time during the school year, all but one LCB teacher said yes, while only four of the nine CB teachers reported having their students memorize rules of differentiation. CB teacher 1 was one of the teachers that did not have her students

Table 4k

Teachers' Classroom Practices: Summary of Interviews

Instructional Strategies		
Memorize rules	4-CB	7-LCB
Start with word problems	7-CB	1-LCB
Use student knowledge in planning	9-CB	6-LCB
Approach to Teaching Introductory Lesson		
Slope of secant line	3-CB	7-LCB
Average velocity	4-CB	0-LCB
Word problems	3-CB	0-LCB
Definition of derivative	0-CB	1-LCB
Approach to Teaching Typical Lesson		
Starts class with problem	3-CB	1-LCB
Students working in groups	3-CB	0-LCB
Rules and examples	4-CB	6-LCB
Discovery/intuitive approach	2-CB	1-LCB
Reinforce ideas	0-CB	1-LCB
Teachers' Calculator Use in Introductory Lesson		
Graphing functions	8-CB	3-LCB
Computational	1-CB	4-LCB
Don't use (much)	2-CB	2-LCB
Teachers' Calculator Use in Typical Lesson		
Graphing functions	8-CB	5-LCB
Computational	4-CB	1-LCB
Check work	1-CB	2-LCB
Don't use (much)	0-CB	3-LCB
Students' Use of Calculator in Introductory Lesson		
Students follow along with instructor	3-CB	2-LCB
Use calculator for graphing	3-CB	4-LCB
Use calculator for evaluation	0-CB	1-LCB
Check work	1-CB	0-LCB
Do not use	3-CB	1-LCB
Use of Text in Introductory Lesson		
Do not follow the book	4-CB	4-LCB
Follows book closely	1-CB	1-LCB
For problems and examples	7-CB	4-LCB
Assign students to read	4-CB	2-LCB
Use of Text in Typical Lesson		
Follows book closely	2-CB	1-LCB
For problems and examples	8-CB	7-LCB
Assign students to read	4-CB	1-LCB

memorize rules of differentiation. She reported having her student do "thousands of problems" and they picked up the rules as they went along. CB teacher 4 required her student to learn some of the basic rules. She believed that, "just like the times tables, knowing the rules can save time." Generally, teachers who required their students to memorize the rules of differentiation, did so as the rules were introduced.

An important difference can also be seen in the responses CB and LCB teachers gave when asked if they had their students work on word problems at any time during the first few weeks of the school year. Seven CB teachers reported working on word problems in the first few weeks while only one of the LCB teachers gave this response. LCB teachers 14 and 17 gave typical reasons for not introducing word problems in the first few weeks of the school year. LCB teacher 14 said that he did not have his students work word problems until they had enough of the rules down to be able to attack problems. LCB teacher 18 reported wanting to get the mechanical things out of the way before doing word problems. On the other hand, CB teachers who reported working word problems during the first few weeks of the school year seemed to center their instruction around problem situations. CB teacher 7 said he presented word problems on the first day and used them to introduce ideas whenever possible. CB teacher 5 reported using word problems to introduce calculus using mathematics they already know.

The third question which required a "Yes" or "No" answer asked teachers if they used what they know of their students' abilities and knowledge about slopes and rate of change in planning instruction. All of the CB teachers and six of the LCB teachers answered "yes' to this question. Two of the LCB teachers reported that they did not use what they knew of their students' knowledge about slopes and rate of change in planning for instruction. LCB teacher 17 said he would not alter a lesson unless three-fourths of his students were lost because he needed to stay on schedule. The AP exam was the driving force for him. LCB teacher 15 also said that she did not use what she knew about her students' knowledge when presenting new material. Instead, she would get back to students who had questions on material that had already been covered.

Approach to teaching introductory lesson. One of the most significant differences in the responses of CB and LCB teachers to questions about their teaching practices was found in the way teachers reported introducing differentiation. Seven LCB teachers reported using the idea of the secant line tending to the tangent line to introduce differentiation. The remaining LCB teacher reported using the definition of derivative to introduce differentiation. This is in sharp contrast to the responses given by CB teachers where only three out of nine reported introducing differentiation in this way. Three CB teachers reported using word problems in their introductory lesson. For example, CB teacher 6 reported introducing differentiation by using data from a fruit fly experiment and CB teacher 7 used stock market prices to introduce the idea of derivative. Four of the CB teachers reported using the idea of average velocity to introduce differentiation. This approach is exemplified by CB teacher 5 who reported introducing differentiation by discussing with his students the velocity associated with driving to a nearby town. Only three of the CB teachers reported using the slope of the secant line tending to the tangent line to introduce the derivative, and none of the CB teachers reported using the definition of derivative.

Approach to teaching typical lesson. In a typical lesson CB teachers were more likely to report beginning a class by presenting a problem, while LCB teachers were more likely to present rules and examples. For example, CB teacher 9 reported that she typically started class by putting a problem on the board and getting students' ideas on how to solve it. On the other hand, LCB teacher 16 reported that in a typical lesson students get down the rules and practice working problems. CB teachers were also more likely to report having their students work in groups during a typical lesson.

Teachers' calculator use in introductory lesson. Eight CB teachers reported using the calculator for graphing functions, while only three LCB teachers reported using calculators for visualization in the introductory lesson. Typical uses of the calculator for visualization were: looking at graphs of functions, using the calculator to draw secant lines, and using the zoom feature to look at local linearity. LCB teachers more often reported using the calculator for computational purposes in the introductory lesson. Altogether, four teachers reported not using the graphing calculator during the introductory lesson. Of these, only LCB teacher 18 reported that he wanted his students to do derivatives mechanically first.

Teachers' calculator use in typical lesson. Eight CB teachers reported using the calculator for graphing functions in an introductory lesson on differentiation, while only five LCB teachers reported using calculators in this way. CB teachers were also more likely to report using the calculator for computational purposes. Three LCB teachers reported not using the calculator much during the typical lesson while none of the CB teachers reported not using the graphing calculator during a typical lesson. The reason LCB teacher 14 gave for not wanting his students to use a graphing calculator was that he wanted his students to do the algebra. LCB teacher 16 gave a similar reason, saying she wanted her students to do differentiation by hand.

Students' calculator use in introductory lesson. Teachers most often reported their students using the graphing calculator to graph functions and often used the introductory lesson to acquaint their students with the use of the calculators. The most notable difference found in teachers' reports of their students' use of the calculator was that half of the LCB teachers reported their students using the calculator for computational purposes, while none of the CB teachers reported this. Three of the CB teachers reported that their students did not use a calculator in the introductory lesson.

Teachers' use of text in introductory lesson. CB and LCB teachers reported using the class text in much the same way. Five CB teachers and four LCB teachers reported not following the class text in the introductory lesson. The reasons CB and LCB teachers gave for not following the text were much the same. CB teacher 1 stated that she had taught calculus enough that she made things up on her own. LCB teacher 17 reported using ideas that had worked for him before. CB teachers were more likely to report using the text for problems and examples and to require their students to read the book than were the LCB teachers.

Teachers' use of text in typical lesson. As with the introductory lesson, CB and LCB teachers' reports on how they use the class text in a typical lesson were similar. The most commonly reported uses of the text were for homework problems and for examples to work in class. The most significant difference was that more CB teachers reported requiring their students to read the text than did LCB teachers.

Student Achievement

A researcher designed Differentiation Test was designed to assess students' achievement. This test contained two sections. Section I of the Differentiation Test was given to students of all 1995 teachers, while students of the 1995 CB and LCB teachers were also given Section II of the Differentiation Test. To adjust for possible differences in abilities of students entering the AP calculus classes, the MAA Calculus Readiness Test form 1G was administered to students at the beginning of the school year. Table 4l shows that student scores on the Calculus Readiness Test were strongly related to Section IA ($p < .05$), Section IB ($p < .05$), and the total score of Section I ($p < .05$).

Table 4m shows that when the entire group of 1995 teachers were considered, there was no conclusive evidence that student achievement was related to teachers' scores on the Belief Questionnaire ($p > .05$). In addition, no relationship between the demographic variables (sex, age, degree, and years teaching mathematics) and student achievement were found (see Table 4n).

Table 4l

Relationships of Calculus Readiness Test Scores to Differentiation Test Scores

	Section I Part A	Section I Part B	Total
t-statistic	2.60	3.35	3.56
p-value	**0.0123**	**.0026**	**.0008**

Note. The analysis included 51 teachers who completed both the Calculus Readiness Test and the Differentiation Test.

Table 4m

Relationships of 1995 Teachers' Belief Questionnaire Scores and Student Differentiation Test Scores

	Scale 1		Scale 2		Scale 3		Scale 4		Total	
	t	p	t	p	t	p	t	p	t	p
Part A	0.48	0.36	0.41	0.34	-0.62	0.27	-0.43	0.34	-0.03	0.49
Part B	1.33	0.09	0.92	0.18	1.15	0.13	-1.03	0.15	0.84	0.20
Total	0.29	0.14	0.79	0.22	0.29	0.38	-0.87	0.19	0.48	0.32

Note. The analysis included 51 teachers who completed the Belief Questionnaire, the Calculus Readiness Test, and the Differentiation Test.

Table 4n

Relationships of Student Differentiation Test Scores to 1995 Teachers' Demographic Data After Accounting for Calculus Readiness Test Scores

	Part A		Part B		Total	
	F	p	F	p	F	p
Sex	1.47	0.23	0.73	0.40	0.05	0.82
Age	0.38	0.77	1.89	0.14	1.15	0.34
Degree	0.53	0.76	0.66	0.66	0.48	0.79
Years Teaching Math	0.82	0.42	1.95	0.06	1.65	0.10

Note. Pretest scores were accounted for by using multiple linear regression.

To gain further insight into potential relationships between student achievement and teachers' pedagogical content beliefs, CB and LCB teachers were examined in greater detail. A t-test indicated that there was no significant difference of means between the students' Calculus Readiness Test scores of CB and LCB teachers ($p < 0.01$), and therefore it was not necessary to adjust for the pretest scores when comparisons were made between the CB and LCB teachers.

Table 4p gives the results of a t-test used to compare students' scores on Section I Part A, Section I Part B, Section II, and the total score on the Differentiation Test. A statistically significant difference was found in Section I Part B ($p < .05$), the conceptual part of the test, and there was suggestive evidence of a difference in the total score of the Belief Questionnaire ($p = .11$). No significant differences were found between student scores of CB and LCB teachers on Section I Part A (computational) and Section II (free response) of the differentiation test.

The Differentiation Test consisted of four subtests designed to explore relationships between teachers' pedagogical content beliefs and their student's ability to work graphically, numerically, symbolically, and to use a graphing calculator. Table 4q shows that students of CB teachers scored significantly higher on questions that required interpreting graphs ($p < .05$), and there was marginally suggestive evidence that students of CB teachers scored higher on questions which required students to interpret tabular information ($p = 0.087$). No difference was found in students' ability to work symbolically ($p = .237$) or in their ability to use a graphing calculator ($p = .481$).

Table 4p

Student Scores on Differentiation Test for CB and LCB Teachers

	Pretest		Section 1 Part A		Section 1 Part B		Section II		Total	
	CB	LCB	CB	LCB	CB	LCB	CB	LCB	CB	LCB
Mean	65	64	45	41	41	30	37	32	42	35
SD	13	03	11	11	10	10	18	14	10	08
t	0.10		0.64		1.89		0.50		1.31	
p	0.46		0.27		**0.04**		0.31		0.11	

Note. Scores are percent correct. Part A, 7 points possible. Part B, 7 points possible. Section II, 27 points possible. There were 9 CB teachers and 8 LCB teachers.

Table 4q

Students' Ability to Work in Multiple Representations

	Graph		Table		Symbolic		Calculator	
	CB	LCB	CB	LCB	CB	LCB	CB	LCB
Mean	44.2	27.6	53.1	38.2	45.0	40.8	23.1	23.3
SD	15.3	16.7	19.0	16.4	9.4	11.0	9.8	8.9
t	1.846		1.449		0.340		-0.482	
p	**0.045**		0.087		0.237		0.481	

Note. Scores reported are percent correct. Part A has 7 points possible. Part B has 7 points possible. Section II has 27 points possible. There were 9 CB teachers and 8 LCB teachers.

CHAPTER FIVE: DISCUSSION

In this chapter we will discuss the results and their implications for calculus teachers, curriculum reformers, and mathematics researchers. In addition, we compare and contrast the results of the current study with the work of Peterson, Fennema, Carpenter, and Loef, and examine the limitations of this study. We begin by restating the research questions.

Teachers' pedagogical content beliefs

> What are AP calculus teachers' pedagogical beliefs about mathematics, curriculum, and instruction in AP calculus? How well do their pedagogical content beliefs align with a constructivist point of view?
>
> How do AP teachers' pedagogical content beliefs change with time? Do the pedagogical content beliefs of teachers become more cognitively based as they become familiar with the materials from a calculus reform project? Is there a relationship between how long a teacher has used the project materials and the degree to which they are cognitively based?

Teachers' classroom practices

> How are AP teachers' pedagogical content beliefs reflected in self-reports of their approaches to teaching, their concepts of the roles of the teacher and the learner, and their goals for instruction?

Student achievement

> Is there a relationship between AP teachers' pedagogical content beliefs, approaches to teaching, and their students' achievement, including achievement of computational skills and problem solving?
>
> Is there a relationship between teachers' pedagogical content beliefs and their student's ability to work in multiple representations: graphically, numerically, and symbolically?

Summary of Results

Teachers' Pedagogical Content Beliefs

This study focused on AP calculus teachers' pedagogical beliefs about mathematics, curriculum, and instruction in AP calculus and how well their beliefs align with a constructivist point of view. We found that teachers vary greatly in their pedagogical content beliefs about mathematics, curriculum, and instruction in AP calculus. In particular, the teachers in this study differed significantly in the degree to which their beliefs corresponded to a cognitively based perspective.

The Belief Questionnaire developed for this study proved effective in distinguishing teachers' pedagogical content beliefs as cognitively based (CB) or less-cognitively based (LCB). Additional support for these categorizations was obtained by the telephone interviews. The analysis of these interviews showed CB teachers' responses to questions about their pedagogical content beliefs were more in keeping with constructivist theories. This was exemplified by the responses of CB and LCB teachers when asked to describe their beliefs about the role of the teacher and learner. CB teachers were more likely to indicate a belief that the role of the teacher was that of a facilitator/guide and that the role of the student was that of explorer. On the other hand, LCB teachers were more likely to believe the role of the teacher was that of a knowledge base, and the role of the student was to learn from the teacher.

How AP Teachers' Pedagogical Content Beliefs Change with Time

Relationships between how long teachers had used the project materials and the degree to which they were cognitively based were explored in two ways. First we compared the Belief Questionnaire scores of the 1993, 1994, and 1995 teachers. Because no data was available on 1993 and 1994 teachers' pedagogical content beliefs prior to instruction in the CCP materials, CB and LCB distinctions for these teachers were defined by the median scores of the 1995 teachers on the Belief Questionnaire. No statistically significant differences were found in the scores of the 1993, 1994, and 1995, nor were any differences found in the Belief Questionnaire scores of the 1993, 1994, and 1995 LCB teachers. However, the Belief Questionnaire scores of the 1993 and 1994 CB teachers were significantly higher than the 1995 CB teachers ($p < .05$).

The second way in which we investigated how teachers' pedagogical content beliefs change with time involved administering the Belief Questionnaire to the 1995 teachers prior to instruction with the CCP materials, and again six or seven months later. We found no statistically significant evidence of a change in 1995 teachers' pedagogical content beliefs when the 1995 teachers were considered as a group. The lack of any statistically significant change in the Belief Questionnaire scores of the 1995 teachers is not entirely unexpected. Beliefs tend to change slowly over time and no substantial change in the teachers' pedagogical content beliefs was expected in six or seven months.

It is interesting that CB teachers tended to show greater change in their beliefs. Suggestive evidence was found indicating that CB teachers' scores on the Belief Questionnaire were higher for the 1993 and 1994 groups of teachers than for the 1995 group of teachers. When the scores of the 1995 CB teachers' Belief Questionnaire scores were compared before and after their inservice instruction, they significantly increased. This implies that those teachers who were more cognitively based to begin with were most affected by their use of CCP materials.

Teachers' Classroom Practices

Teachers' self-reports of their classroom teaching practices were found to be strongly associated with their pedagogical content beliefs. Compared to LCB teachers, CB teachers' self-reported classroom practices were found to more closely reflect the calculus reform's recommendation for instructional approaches that are student centered and conceptual in nature. CB teachers were more likely to use word problems when introducing topics and to emphasize student exploration than were LCB teachers. In addition, CB teachers more often reported having their students work in groups, emphasize visual approaches to topics and consider students' knowledge when planning instruction. On the other hand, LCB teachers were more likely to present rules and theorems, work examples, and require their student to memorize rules of differentiation.

The categorization of teachers' classroom practices given in Figure 4k of Chapter Four provides strong evidence that teachers with more cognitively based beliefs tended to be more constructivist in their approaches to teaching. The difference in the number of CB and LCB teachers giving a particular response to any one question in Figure 4k was not always profound. However, the differences in the number of CB and LCB teachers reporting any particular classroom practice consistently show CB teachers reporting more responses which were in keeping with constructivist ideas.

Student Achievement

The results of this study suggest that students of teachers with cognitively based beliefs tend to have a better conceptual understanding of differential calculus than students of teachers with less cognitively based beliefs. However, no evidence was found of a difference in the computational skills of students of CB teachers and students of LCB teachers. This is particularly interesting because of the fact that the LCB teachers placed a greater emphasis on the learning of rules and procedures.

In addition to a better conceptual understanding, students of CB teachers were better able to interpret graphical information and to interpret information given in a table. The fact that students of CB teachers were better able to interpret graphical and numeric information suggests that teachers with cognitively based beliefs used a greater variety of approaches when teaching calculus than did teachers with less cognitively based beliefs.

No differences between students of CB and LCB teachers were found in their ability to work with symbolic information. The types of problems associated with assessing students' ability to work with information given symbolically and the types of problems designed to determine their computations skills are closely related since computational problems are very often given symbolically. Since no difference was found in students of CB and LCB teachers' computational abilities, it is not surprising that no difference was found in their ability to work with symbolic information. However, we note that the greater emphasis placed on the symbolic aspects of differential calculus by LCB teachers did not result in higher scores for their student on the symbolic subtest of the Differentiation Test.

In addition to not finding any differences in students of CB teachers and LCB teachers computational ability, we also found no statistically significant differences in students' ability to use the graphing calculator when solving calculus problems. This may, in part, be because the use of the graphing calculator was emphasized during the CCP workshops. Teachers were not only taught how to use the graphing calculator during these workshops, an emphasis was placed on how the calculator could be used in the classroom. Another factor which may have contributed to the lack of any difference in students' ability to use the graphing calculator for solving problems may be that CB and LCB teachers reported having their students use the graphing calculator in similar ways.

In this study we found that the more student-centered approaches to teaching calculus associated with cognitively based beliefs were effective in promoting improved student understanding. In addition, the results of the present study supported the results reported by Peterson et al. with first grade teachers and their students. The fact that similar results were obtained with two such very different groups of teachers has implications reaching beyond the teaching of first grade mathematics and AP calculus. In the next section we will compare and contrast the results of the present study with the results reported in the Peterson study to gain some insight into these implications.

Comparison with Peterson's Study

Both the present study and Peterson's study found notable variation in teachers responses to the Belief Questionnaire. However, teachers' Belief Questionnaire scores in the Peterson study tended to be higher than they were for teachers in the present study, indicating that the first-grade teachers in Peterson's study, as a whole, were more cognitively based than were the AP calculus teachers in the present study. In addition, there tended to be greater variation in the teachers scores on the Belief Questionnaire in the Peterson study than was found in the present study.

In the present study we found very distinct differences in CB and LCB teachers' self-reports of their teaching practices. This was also the case in the Peterson study. In both studies, CB teachers' classroom teaching practices were more in keeping with constructivist philosophies than were the teaching practices of LCB teachers. Peterson reported that CB teachers tended to use word problems as the basis for teaching mathematics with the assumption that knowledge of number-facts would follow. On the other hand, LCB teachers were more likely to teach word problems after the students had learned their number-facts. We found similar differences between CB and LCB teachers in the present study. LCB teachers in the present study were more likely to report having their students learn the rules of differentiation before they moved on to word problems. CB teachers, on the other hand, more often reported working word problems and expecting student to acquire skills as they went along.

The present study and Peterson's study obtained similar results when relationships between Belief Questionnaire scores of CB and LCB teachers were compared to their students' scores on the achievement tests. In the Peterson study, students of CB teachers outperformed students of LCB teachers on the problem-solving test ($p < .05$). However, no difference was found in the student scores of CB and LCB teachers on the number-fact test. In the present study, a statistically significant difference was found in the student scores on Section I Part B of the Differentiation Test, the conceptual part of the test ($p < .05$) and no significant difference was found student scores on the computational portion of the Differentiation Test. Not finding any difference in students' computational ability in both studies was particularly interesting because, in both studies, LCB teachers placed greater emphasis on skills than did CB teachers.

Implications for Teaching

Teachers' pedagogical content beliefs were found to be strongly associated with self-reports of their classroom teaching practices. Compared to LCB teachers, CB teachers' self-reported classroom practices were found to more closely reflect the calculus reform's recommendation for instructional approaches that are student centered and conceptual in nature. The results of this study support the calculus reform's recommendation for instructional approaches that are student-centered and conceptual in nature. We found that teachers who reported placing a greater emphasis on exploration and conceptual understanding were more effective in helping students develop a better conceptual understanding of differential calculus.

To allow for this greater emphasis on problem solving and student exploration, it is necessary to reduce the amount of time devoted to drill-and-practice. A concern expressed by some teachers is that reducing the time spent on skills will be detrimental to students' computational ability. This study suggests that placing a greater emphasis on conceptual understanding and problem-solving need not lead to corresponding deficiencies in students computational abilities.

The recommendation of the NCTM for an increase in the use of computing technology is, in part, an attempt to free students from tedious computations and allow them to concentrate on conceptual understanding and problem-solving. In this study we did not explore, in depth, students' use of the graphing calculator. However, we gained some insight into how students use the graphing calculator through the telephone interviews with CB and LCB teachers, and found students of CB and LCB teachers used the graphing calculator in similar ways.

In addition to the computational uses suggested by the NCTM, the graphing calculator has the potential to be an excellent exploratory tool. It stands to reason that allowing student to use calculators to explore and discover key calculus concepts will allow them to build for themselves a better mathematical foundation. When teachers were asked how they used the graphing calculator we found that the greatest difference between how CB teachers and LCB teachers used graphing calculators was that CB teachers were more likely to use the calculator for visualization.

The results of this study suggest that AP calculus teachers can place a greater emphasis on student exploration and problem solving, and that by doing so, they can improve students' conceptual understanding without necessarily sacrificing skill proficiencies. From a purely statistical point of view the results of this study cannot be generalized to a larger population. However, the fact that Peterson's study, involving first-grade teachers, and the present study, involving AP calculus teachers, found similar teaching strategies associated with improved student understanding lends support to the idea that constructivist based teaching strategies are sound pedagogically across the K-12 mathematics curriculum.

Ultimately, the goal of educational research is to improve student understanding and this goal will be achieved through the efforts of teachers in the classroom. It follows that to improve student achievement will require changes on the part of the teacher.

However, the results of this study supports the idea that changing teachers' classroom practices will require a corresponding change in their pedagogical content beliefs.

Implications for Calculus Reform

Teacher inservice programs are often ineffective in bringing about significant long term changes in teachers' classroom practices (Guskey, 1986). However, the CGI program has shown that a teacher intervention program can be designed which will affect first grade teachers' beliefs and have a corresponding effect on their classroom teaching practices (Carpenter et al., 1989b; Peterson, Fennema, Carpenter, and Loef, 1989a; Knapp and Peterson, 1995). Because relationships between AP calculus teachers' pedagogical content beliefs and their classroom teaching practices found in the present study were similar to the relationships found with first grade teachers, the philosophies which guided the development of the CGI program might have relevance to inservice programs associated with the calculus reform movement.

One of the guiding principles of the CGI program is that teachers can benefit from research-based knowledge of children's thinking about addition and subtraction problems and how children make sense of new knowledge in light of existing knowledge (Peterson et al., 1991). In the CGI program, research-based knowledge about children's mathematical knowledge was shared with teachers, and the teachers were given the opportunity to interpret for themselves what it meant to their classroom instruction. In other words, the teachers are allowed to adapt the research-based knowledge to their existing knowledge and belief systems.

This has important implications for the calculus reform movement. For example, one goal of the calculus reform movement is to promote an increased emphasis on conceptual understanding and problem solving. We suggest that a good first step toward changing teachers' classroom practices to attain this goal is to familiarize teachers with research demonstrating that this approach to teaching is associated with improved student understanding. However, this research-based knowledge is not enough, in and of itself. To effectively promote a greater emphasis on conceptual understanding and problem solving, teachers must come to believe that changing their teaching practices will have a positive effect on their students' learning.

Research on teachers' beliefs has shown that beliefs tend to change slowly over time (Kagan, 1992; Simomsen, 1993). Therefore, changing calculus teachers' pedagogical content beliefs about what emphasis should be placed on conceptual understanding and problem solving will be a gradual process. The effectiveness of the CGI program suggests that rather than presenting calculus teachers with a "script" to follow and expecting instant changes, it may be more effective to allow teachers to adapt this research-based knowledge to fit their own particular style of teaching. This suggests educators interested in calculus reform should make materials available to teachers which have been shown to improve student understanding and encourage teachers to adapt them to their personal teaching style. However, learning how to "fit" these new approaches, as well as coming to believe these approaches are effective, will take time and ongoing support is essential.

According to Knapp and Peterson (1995), the ongoing support the teachers received played an important part in the long term effect of the CGI program. In particular, interaction with other teachers who were attempting to use the CGI ideas was found to be particularly important. This continuing support is going to play an important role in the calculus reform movement. Teachers are going to vary in the degree to which that are successful in incorporating new approaches to teaching and are bound to run into difficulties. The opportunity to share their experiences and ideas with other teachers seems to be a key factor in the successful implementation of a new curriculum.

It is important for curriculum reformers to keep in mind that teachers participating in an inservice program take on the role of student. This provides a valuable opportunity to model constructivist teaching methods. Inservice programs promoting calculus reform may benefit by incorporating greater levels of participation by teachers, by helping them become better informed, and by paying special attention to their pedagogical content beliefs.

Limitations of the Study

The teachers selected to participate in this study should not be viewed as a representative sample of high school AP calculus teachers. No attempt should be made to make statistical inferences beyond this population to any broader population, such as all AP calculus teachers or moreover to all high school mathematics teachers. Participation in the CCP project came was determined by both self-selection (who applies) and a competitive selection (who gets selected out of those who apply).

Relationships between teachers' Belief Questionnaire scores and the demographic information (gender, age, highest degree obtained, and number of years teaching mathematics) indicate that CB teachers had more years teaching experience than LCB teachers ($M = 22.13$ and $M = 13.44$ respectively, $p < 0.05$). This relationship between years teaching and the degree to which teachers pedagogical content beliefs are cognitively based suggests that the contrasts found in this study may be influenced by experience (similar results were reported by Peterson)

Another limitation of this study was that no data was available on 1993 and 1994 teachers' pedagogical content beliefs prior to instruction with the CCP curriculum. The 1993 and 1994 teachers received their inservice instruction with the CCP materials before the start of the current study. Because of this, the determination of CB and LCB groups for the 1993 and 1994 teachers was based on the median scores on the Belief Questionnaire of the 1995. This aspect of the study must be considered observational in nature. The differences among the three groups of teachers could be attributed to factors other than the project curriculum. One alternate hypothesis is that the 1993 and 1993 teachers may have had more cognitively based beliefs when they joined the Calculus Connections Project than did the 1995 teachers. The 1995 teachers may have been motivated to join the Calculus Connections project because graphing calculators were required beginning with the 1996 AP exam and the project provided teachers instruction in their use. On the other hand, 1993 and 1994 teachers joined the project before graphing calculators were required and may have chosen to participate because of their beliefs about the projects' curriculum.

Implications for Future Research

The results of the present study support the claim that teachers' pedagogical content beliefs are indeed linked to teachers' actions and student learning. The first-grade mathematics teachers studied by Peterson et al. and high school AP calculus studied here are near opposite ends of K-12 mathematics curricular spectrum. However, the fact that similar results were found with two such different groups of teachers lends support to the idea that similar results might be obtained with other mathematics classes and teachers in other grades. Further research is needed to determine whether teachers' pedagogical content beliefs, teachers' classroom practices, and student achievement are related similarly at all levels of school mathematics.

In both the Peterson study and the present study the mean number of years teaching mathematics for CB teachers was significantly greater than the mean number of years LCB teachers had taught mathematics, A possible conclusion to draw from this is that teachers' pedagogical content beliefs simply become more cognitively based as teachers gain classroom teaching experience. Another area open to future research is to explore the factors which contribute to these changes in teachers pedagogical content beliefs over time. By determining the specific experiential factors which contribute to a more cognitively based perspective, it may be possible to develop teacher inservice programs which will help promote these beliefs earlier in a teacher's career.

Conclusions and Recommendations

This study has important implications in the current educational reform climate. It supports the claim that teachers' pedagogical content beliefs are an important consideration for teacher education programs and teacher inservice education. The results of this study have shown strong relationships among teachers' pedagogical content beliefs, their classroom practices, and student achievement. It supports previous research suggesting that teachers' beliefs, thoughts, judgements, knowledge, and decisions indeed affect how teachers perceive and think about teaching a new curriculum and to what extent they implement it (Peterson et al., 1989). If reform efforts and teacher preparation programs are to effect a meaningful change in the way mathematics is taught, teachers' pedagogical content beliefs need to be considered.

The research of Carpenter, Fennema, Peterson, and Loef (1989) has shown a positive correlation between cognitively based beliefs and student achievement in the first grade. The fact that the relationships among first-grade teachers' pedagogical content beliefs, classroom teaching practices, and their students' achievement were also found with high school Advanced Placement teachers and their students has significant implications for the calculus reform movement. It lends strong support to the idea that teachers' pedagogical content beliefs play a very important role in curricular reform. If the reform efforts are to have a significant impact on teachers' classroom practices, they will need to have a corresponding impact on teachers' pedagogical content beliefs.

In closing, we make the following recommendations for improving the effectiveness of inservice efforts aimed at helping teachers implement calculus reform curricula. First, teachers can benefit from research based knowledge concerning effective approaches to teaching mathematics. This is not to say that teachers should be expected to read educational research papers, rather reformers should summarize important findings research and present them in a way which benefit teachers.

The results of the present study suggest that AP calculus teachers can emphasize student exploration to help students develop an improve conceptual understanding without necessarily sacrificing skill development. These results are supported by other studies showing that exploration can have a positive effect on students' conceptual understanding (Orton, 1983a; Orton, 1983b; Tall, 1985). Using technology to allow students to explore has been shown to be particularly effective, and a great deal of research on the effectiveness of calculators and computer algebra systems as exploratory tools (Palmiter, 1991; Heid, 1988; Tall, 1985; Orton 1993a; Orton, 1993b). These studies show how using technology to take over computation demands and allow students to explore important calculus concepts can promote a better conceptual understanding without necessarily sacrificing computational skills.

Informal approaches to teaching calculus that include numerical and graphical exploration have also been found to improve students' understanding (Beckmann, 1988; Schwartz, Dreyfus, and Bruckheimer, 1990; Dreyfus and Eisenberg, 1984; Dreyfus and Eisenberg, 1982). These studies have shown how exploration of important calculus concepts in multiple representations: (symbolic, numerical, and graphical) can be used as effective teaching strategies.

Familiarizing teachers with the research which supports the called for change in teaching practices is the first step in effecting a change in teachers' pedagogical content beliefs. In addition, materials and activities need to be made available to the teachers which promote the desired teaching strategies. The *Calculus Connections Project* of Oregon State University, Project CALC of Duke University, The St. Olaf Project, and *The Calculus Consortium*, based at Harvard University are among many projects which provide curricula promoting the ideas of the reform effort.

Finally, ongoing support is essential. The results of this study suggest that changing teachers' classroom practices requires corresponding change in their pedagogical content beliefs. These changes in teachers' pedagogical content beliefs are realized through positive experiences in their classrooms with the new teaching strategies. The support teachers receive from other teachers play an important part in insuring a positive classroom experience.

If teachers are the true "change agents" in implementing curricular reforms, then reforms must act as "change agents" of these teachers. The results of this study has shown that cognitively based beliefs are associated with both teaching practices which are constructivist in nature, and improved student achievement. These results suggest that calculus reforms efforts need to promote these beliefs in order to effect meaningful changes in how calculus is taught and a corresponding improvement in student understanding.

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APPENDICES

Appendix A: Belief Questionnaire

INSTRUCTIONS

There are 48 statements in this survey that represent beliefs teachers may have regarding the teaching and learning of calculus. We are interested in how well these statements reflect your own feelings about the teaching and learning of calculus. Please respond to each statement with your degree of agreement or disagreement by filling in the appropriate "bubble." We really are interested in your opinion.

Your responses to this survey will be held in the strictest confidence, and no individual teacher's responses will be reported in any way. You will be provided with a summary of the survey results when they are compiled. Thank you for your help!

BACKGROUND INFORMATION

NAME ______________________________

1. Gender
   a) female　　b) male

2. Age
   a) 20 - 29　　c) 40 - 49　　e) 60 or older
   b) 30 - 39　　d) 50 - 59

3. Highest degree
   a) Bachelors in Mathematics　　d) Masters not in Mathematics
   b) Bachelors not in Mathematics　　e) Doctorate in Mathematics
   c) Masters in Mathematics　　f) Doctorate not in Mathematics

4. Number of years teaching ________

5. Number of years teaching mathematics ______

6. Number of years teaching calculus ______

7. Describe your school
   a) Urban　　b) Suburban　　c) Rural

8. Number of students in your school ___________

A=Strongly Agree B=Agree C=Undecided D=Disagree E=Strongly Disagree

A B C D E

0 0 0 0 0 1. Time should be spent practicing computational procedures before students are expected to understand the procedures.

0 0 0 0 0 2. Most students have to be shown a method of solving elementary calculus problems.

0 0 0 0 0 3. Students should understand computational procedures before they master them.

0 0 0 0 0 4. It is important for a student to know how to follow directions to be a good problem solver.

0 0 0 0 0 5. The natural development of mathematical topics should determine the sequence of topics which is used for instruction.

0 0 0 0 0 6. Students should understand computational procedures before they spend much time practicing them.

0 0 0 0 0 7. Teachers should teach exact procedures for solving word problems.

0 0 0 0 0 8. The instructional sequence of math topics should be determined by the order in which students naturally acquire math concepts.

0 0 0 0 0 9. The natural development of students' mathematical ideas must be considered in making instructional decisions.

0 0 0 0 0 10. Students learn calculus best from the teachers' demonstrations and explanations.

0 0 0 0 0 11. When selecting the next topic to be taught, a significant consideration is what students already know.

0 0 0 0 0 12. Students learn calculus best by exploring problem situations.

0 0 0 0 0 13. To be successful in mathematics, a student must be a good listener.

0 0 0 0 0 14. The natural development of student's mathematical ideas should determine the sequence of topics used for instruction.

0 0 0 0 0 15. Teachers should allow students to figure out their own ways to solve calculus problems.

0 0 0 0 0 16. Students should be encouraged to solve problems in the same way the teacher has modeled them.

0 0 0 0 0 17. It is important for a student to be a good listener in order to learn how to do mathematics.

0 0 0 0 0 18. The best way to teach problem solving is to show students how to solve one kind of problem at a time.

0 0 0 0 0 19. The mathematically logical sequence of topics must be considered in planning for instruction.

0 0 0 0 0 20. Students should be allowed to invent ways to solve simple word problems before the teacher demonstrates how to solve the problems.

0 0 0 0 0 21. Students should have many informal problem solving experiences in calculus before they are expected to memorize basic differentiation facts.

0 0 0 0 0 22. The instructional sequence of math topics should be determined by the formal organization of mathematics rather than by the natural development of student's math ideas.

0 0 0 0 0 23. Students learn math best by attending to the teachers explanations.

0 0 0 0 0 24. Calculus should be presented to students in such a way that they can discover relationships for themselves.

0 0 0 0 0 25. Students can figure out ways to solve many calculus problems without formal instruction.

A=Strongly Agree B=Agree C=Undecided D=Disagree E=Strongly Disagree

A B C D E

0 0 0 0 0 26. Most students can figure out a way to solve many calculus problems without teacher help.

0 0 0 0 0 27. Students should explore problem situations before they master computational procedures.

0 0 0 0 0 28. In planning for instruction, it is important to know how student's mathematical ideas develop naturally.

0 0 0 0 0 29. It is more important to teach in a mathematically sequenced way than to use student's concept development in planning an instructional sequence.

0 0 0 0 0 30. Recall of basic rules of differentiation should precede the introduction of word problems involving differentiation.

0 0 0 0 0 31. Teachers should facilitate student's invention of their own ways to solve calculus problems.

0 0 0 0 0 32. Students should master computational procedures before they are expected to understand how those procedures work.

0 0 0 0 0 33. It is more important to use student's concept development in planning an instructional sequence than to use a mathematically determined sequence.

0 0 0 0 0 34. Teachers should tell students who are having difficulty solving a word problem how to solve the problem.

0 0 0 0 0 35. Students should understand the meaning of differentiation before they memorize rules of differentiation.

0 0 0 0 0 36. When selecting the next topic to be taught, one must carefully follow the mathematically logical sequencing of topics.

0 0 0 0 0 37. Time should be spent practicing computational procedures before students spend much time solving problems.

0 0 0 0 0 38. Teachers should encourage students who are having difficulty solving a word problem to continue to try to find a solution.

0 0 0 0 0 39. Most students can figure out a way to solve simple calculus problems

0 0 0 0 0 40. Students will not really understand differentiation until they have mastered the basic rules of differentiation.

0 0 0 0 0 41. Students should not solve basic differentiation word problems until they have mastered some basic differentiation facts.

0 0 0 0 0 42. It is important for a student to discover how to solve elementary calculus problems for him/herself.

0 0 0 0 0 43. Students usually can figure out for themselves how to solve simple calculus problems.

0 0 0 0 0 44. The structure of mathematics is more important in making instructional decisions than is the natural development of student's ideas.

0 0 0 0 0 45. The teacher should demonstrate how to solve calculus problems before students are allowed to solve them.

0 0 0 0 0 46. It is better to teach students how to solve one kind of word problem at a time.

0 0 0 0 0 47. Time should be spent exploring problem situations before students spend much time practicing computational procedures.

0 0 0 0 0 48. It is best to teach students how to solve a variety of word problems at one time rather than one type of problem at a time.

Appendix B: Follow-up Belief Questionnaire

INSTRUCTIONS There are 24 statements in this survey that represent beliefs teachers may have regarding the teaching and learning of calculus. Please respond to each statement with your degree of agreement or disagreement by filling in the appropriate "bubble." Your responses to this survey will be held in the strictest confidence, and no individual teacher's responses will be reported in any way.

A=Strongly Agree B=Agree C=Undecided D=Disagree E=Strongly Disagree

A B C D E

0 0 0 0 0 1. Students should understand computational procedures before they master them.

0 0 0 0 0 2. Students should understand computational procedures before they spend much time practicing them.

0 0 0 0 0 3. The instructional sequence of math topics should be determined by the order in which students naturally acquire math concepts.

0 0 0 0 0 4. The natural development of students' mathematical ideas must be considered in making instructional decisions.

0 0 0 0 0 5. When selecting the next topic to be taught, a significant consideration is what students already know.

0 0 0 0 0 6. Students learn calculus best by exploring problem situations.

0 0 0 0 0 7. The natural development of student's mathematical ideas should determine the sequence of topics used for instruction.

0 0 0 0 0 8. Teachers should allow students to figure out their own ways to solve calculus problems.

0 0 0 0 0 9. Students should be allowed to invent ways to solve simple word problems before the teacher demonstrates how to solve the problems.

0 0 0 0 0 10. Students should have many informal problem solving experiences in calculus before they are expected to memorize basic differentiation facts.

0 0 0 0 0 11. Calculus should be presented to students in such a way that they can discover relationships for themselves.

0 0 0 0 0 12. Students can figure out ways to solve many calculus problems without formal instruction.

0 0 0 0 0 13. Most students can figure out a way to solve many calculus problems without teacher help.

0 0 0 0 0 14. Students should explore problem situations before they master computational procedures.

A=Strongly Agree B=Agree C=Undecided D=Disagree E=Strongly Disagree

A B C D E

0 0 0 0 0 15. In planning for instruction, it is important to know how student's mathematical ideas develop naturally.

0 0 0 0 0 16. Teachers should facilitate student's invention of their own ways to solve calculus problems.

0 0 0 0 0 17. It is more important to use student's concept development in planning an instructional sequence than to use a mathematically determined sequence.

0 0 0 0 0 18. Students should understand the meaning of differentiation before they memorize rules of differentiation.

0 0 0 0 0 19. Teachers should encourage students who are having difficulty solving a word problem to continue to try to find a solution.

0 0 0 0 0 20. Most students can figure out a way to solve simple calculus problems.

0 0 0 0 0 21. It is important for a student to discover how to solve elementary calculus problems. for him/herself.

0 0 0 0 0 22. Students usually can figure out for themselves how to solve simple calculus problems.

0 0 0 0 0 23. Time should be spent exploring problem situations before students spend much time practicing computational procedures.

0 0 0 0 0 24. It is best to teach students how to solve a variety of word problems at one time rather than one type of problem at a time.

Appendix C: Interview Protocol

Belief Interview

"We are interested in how you teach differentiation in your classroom and why you teach it the way that you do. We are going to ask you some specific questions about the way you teach differentiation and why you teach as you do. There are no right or wrong answers to these questions. We are interested in your opinions and ideas. We would like you to respond to the specific questions. However, after you answer each question, please feel free to depart from the original question if you have additional comments on <u>how</u> and <u>why</u> you teach differentiation as you do."

1A. "Describe as specifically as you can the lesson in which you introduce differentiation to your class. We are interested in the way you organize and present the mathematics content as well as the specific teaching methods and strategies that you use."

--Probe (to be used once)

"Anything else?" or "Can you tell me more?" or repeat question or paraphrase question.

B. "How do you use the math textbook in your introductory lesson(s)?"

or if the teacher has already discussed how (s)he uses the textbook then "You said that you used the textbook [insert teacher's words]." "Is that correct?" "Is there anything else you would like to add?"

--Probe once if necessary.

C. "When introducing differentiation, how closely do you follow the presentation of the material in the book?"

D "Do you use a graphing calculator when introducing differentiation? If so, How?"

--Probe once if necessary: "How do you use the graphing calculator when introducing differentiation?"

E. "Do your student use a graphing calculator during the introductory lesson on differentiation? If so, how"

--Probe once of necessary: "How do you have your students use the graphing calculator when introducing differentiation?"

2A. "Describe as specifically as possible a <u>typical</u> lesson involving differentiation in your class. Again, we are interested in the way you organize and present the mathematics content as well as the specific teaching methods and strategies that you use. Also, please explain how the typical lesson differs from the introductory lesson on differentiation and whether and how the typical lesson might change over the course of the school year."

--Probe once if necessary.

B. "How do you use the math textbook in your typical math lesson on differentiation?"

or

"You said that in your typical math class the textbook [insert teacher's specific words]?" "Is that correct?" " Is there anything else you would like to add?"

--Probe once if necessary.

C. "How do you use the graphing calculator in a typical lesson on differentiation?"

--Probe once if necessary

D. "What do you think the role of the teacher should be in a typical lesson on differentiation in your class?"

--Probe once if necessary: "What do you think the teacher's responsibility should be in a typical lesson on differentiation in your class?"

E. "What do you think the role of learner should be in a typical lesson on differentiation in your class?"

--Probe once if necessary: "What do you think the learner's responsibility should be in a typical lesson on differentiation in your class?"

3A. "What do you try to have your students learn about differentiation during the year?"

--Probe once if necessary: "What is important for most students to know about differentiation by the end of the year?"

B. "Are there certain concepts in differentiation that you want all students to learn? If so what are they?"

--Probe once if necessary.

C "Why did you decide that these concepts are important?"

--Probe once if necessary.

D. "Are there certain kinds of word problems using differentiation you believe that all students should learn to solve? If so, what are they?"

--Probe once if necessary.

E. "Why did you decide that these kinds of differentiation word problems are important for all students to learn to solve?"

--Probe once if necessary.

4A. “Do you have the students memorize rules of differentiation sometime during the school year?”

Yes

B.”When?”

C.”How do you decide when?”

--Probe once

No

B. “Do you teach differentiation rules to all calculus students?”

Yes — C. “When?”

--Probe once

D. “How?”

--Probe once

No — “Why not?”

5A. “Do you have the students work on word problems involving differentiation at any time during the <u>first few weeks</u> of the school year?”

Yes

B. “When?”

C.”How do you decide when?”

--Probe once

No

B “Do have students work on word problems involving differentiation during the school year.”

Yes — C.”When?”

--Probe once

No — “Why not?”

6A. “In teaching differentiation in high school calculus, what do you believe should be the relative emphasis on rules of differentiation versus conceptual understanding versus solving of word problems?”

--Probe once if necessary.

B. “Why?”

--Probe once if necessary.

C. “What do you see as the <u>relationship</u> between learning of differentiation rules, conceptual understanding of differentiation, and differentiation word problems?”

--Probe once if necessary.

7A. "What factors do you consider when determining how topics in differentiation should be sequenced?"

--Probe once if necessary

B. "How rigorous an understanding of limits should students have before proceeding to the study of differentiation?"

--Probe once if necessary

8A. "What do students in your calculus class know about slopes and rates of change when they start the class?"

--Probe once if necessary.

B. "Where do students get this knowledge?"

--Probe once if necessary

C. "Students have different abilities and knowledge about differentiation, slopes, and rates of change. How do you find out about these differences?"

D. "Do you use this knowledge in planning instruction?"

Yes   No

E. "How?"

After you have completed interviewing the teacher on the above seven questions, think back over the teacher's responses and judge where you think that teacher falls on the continuum for each of the four beliefs below. If you are unable to make a judgement for one or more beliefs, then ask additional questions to get the information that you need to make a judgement.

1. Students receive mathematical knowledge from the teacher and others			Students construct their own mathematical knowledge	
1	2	3	4	5

2. Mathematica skills should should be taught as a discrete component			Mathematics skills should be taught in relationship to understanding and problem solving	
1	2	3	4	5

3. Structures of mathematics provides the basis of sequencing topics for instruction			Students's development of mathematical ideas providing the basics for sequencing topics for instruction	
1	2	3	4	5

4. Mathematics instruction should be organized to be organized to facilitate teacher's clear presentation of knowledge			Mathematics instruction should be ed to facilitate students's construction of knowledge	
1	2	3	4	5

Appendix D: Differentiation Test

PRACTICE AP CALCULUS EXAM
SECTION I
PART A

INSTRUCTIONS

Time allowed is 25 minutes

SECTION I PART A consists of 7 multiple-choice questions. CALCULATORS ARE NOT ALLOWED ON THIS SECTION OF THE TEST. Unless otherwise indicated, the domain of any given function is assumed to be the set of all real numbers for which the function is defined. Examine the given answers and circle the response you decide is correct on both the test and on the answer sheet.

DO NOT GO ON TO PART B

1. Given that $f(x) = 2e^{x^2}\sin(x)$, what is $f'(x)$?

A) $4xe^{x^2}\cos(x)$
B) $2x^2e^{x^2}\cos(x)$
C) $2e^{x^2}\sin(x) + 2e^{x^2}\cos(x)$
D) $4xe^{x^2}\sin(x) + 2e^{x^2}\cos(x)$
E) $2x^2e^{x^2}\sin(x) + 2e^{x^2}\cos(x)$

---

2. The equation of the curve determined by reflecting $y = \ln x - 1$ about the x-axis is

A) $y = e^{x-1}$
B) $y = e^{1-x}$
C) $y = \ln x - 1$
D) $y = 1 - \ln(-x)$
E) $y = 1 - \ln x$

---

3. What is $\lim_{h\to 0} \frac{\sin(\pi/2+h) - \sin(\pi/2)}{h}$?

A) $-\infty$ B) -1 C) 0 D) 1 E) $+\infty$

---

4. If r is positive and increasing, for what value of r is the rate of increase of r^4 six times the rate of increase of r^2?

A) $\frac{3}{2}$ B) $\sqrt{3}$ C) $\frac{1}{\sqrt{3}}$ D) $\sqrt{6}$ E) $\frac{1}{\sqrt{6}}$

---

5. The slope of the tangent line to the curve $y^3x + y^2x^2 = 6$ at $(2,1)$ is

A) $\frac{-3}{2}$ B) -1 C) $\frac{-5}{14}$ D) $\frac{-3}{14}$ E) 0

6. Given the curve $y = 2x^3 - 3x^4$, which of the following statements are true?

A) The curve has no relative extrema.
B) The curve has one point of inflection and two relative extrema.
C) The curve has two points of inflection and one relative extremum.
D) The curve has two points of inflection and two relative extrema.
E) The curve has two points of inflection and three relative extrema.

---

7. For which of the following functions does the property $\frac{d^3y}{dx^3} = \frac{dy}{dx}$ hold?

I. $y = e^x$
II. $y = \ln x$
III. $y = \sqrt{3}$

A) I only
B) II only
C) I and II
D) I and III
E) I, II, and III

PRACTICE AP CALCULUS EXAM
SECTION I
PART B

INSTRUCTIONS

Time allowed is 25 minutes

SECTION I PART B consists of 7 multiple-choice questions (numbered 8 through 14). A GRAPHING CALCULATOR MAY BE REQUIRED FOR SOME OF THE QUESTIONS IN THIS SECTION. Unless otherwise indicated, the domain of any given function is assumed to be the set of all real numbers for which the function is defined. Circle the correct response on both the test and on the answer sheet.

8. For the following question use the information given in the tables below. What is the value of $(f \circ g)(x) + f(2x)$ for $x = 3$?

x	2	3	6	7
$f(x)$	3	5	8	2
$g(x)$	4	7	5	7

A) 43 B) 45 C) 12 D) 8 E) 10

---

9. Which of the following combination of properties could possibly describe a function?

I. $f(x)>0$ and $f'(x)>0$ for all x

II. $f(x)<0$ and $f''(x)<0$ for all x

III. $f(x)>0$ and $f''(x)<0$ for all x

A. I only B. II only C. I, II, and III D. I and II E. I and III

---

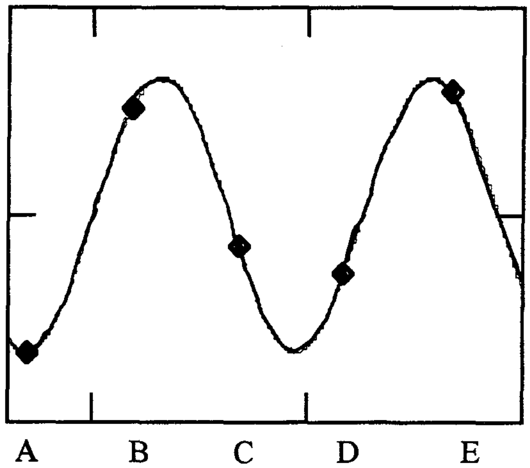
10. Let $f'(x) = \frac{x^5}{4} - x^4 + \frac{3x^2}{2}$. To three decimal places the function f has a relative maximum at $x = ?$

A) 0.00 B) -.777 C) 1.059 D) 1.572 E) 3.514

---

11. At which of the five points on the graph in the figure at the right are $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ both positive?

A) A
B) B
C) C
D) D
E) E



12. Two particles start at the origin and move along the x-axis. For $0 \le t \le 10$, their respective position functions are given by $x_1 = \cos t$ and $x_2 = \ln(2t) + 2$. For how many values of t do the particles have the same velocity?

A) none B) one C) two D) three E) four

---

13. For the following question use the information given in the tables below. Let $H(x) = (f \circ g)(x)$. What is the value of $H'(3)$?

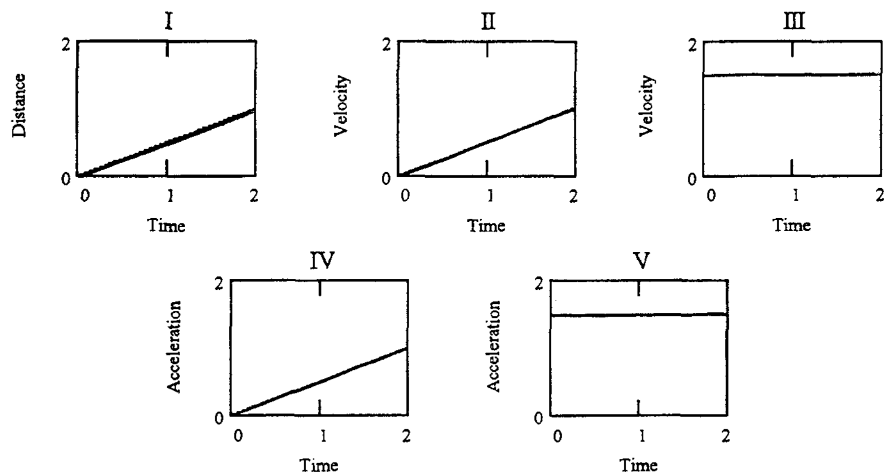
x	$f(x)$	$f'(x)$
1	2	3
2	3	4
3	4	6
7	3	8

x	$g(x)$	$g'(x)$
3	7	6
4	8	7
5	9	8
6	1	9

A) 36 B) 9 C) 46 D) 48 E) 4

---

14. Which of the following graphs below represent(s) motion at constant, non-zero acceleration?



A) I, II, and IV B) I and III C) II and V D) IV only E) V only

PRACTICE AP CALCULUS EXAM
SECTION II

INSTRUCTIONS

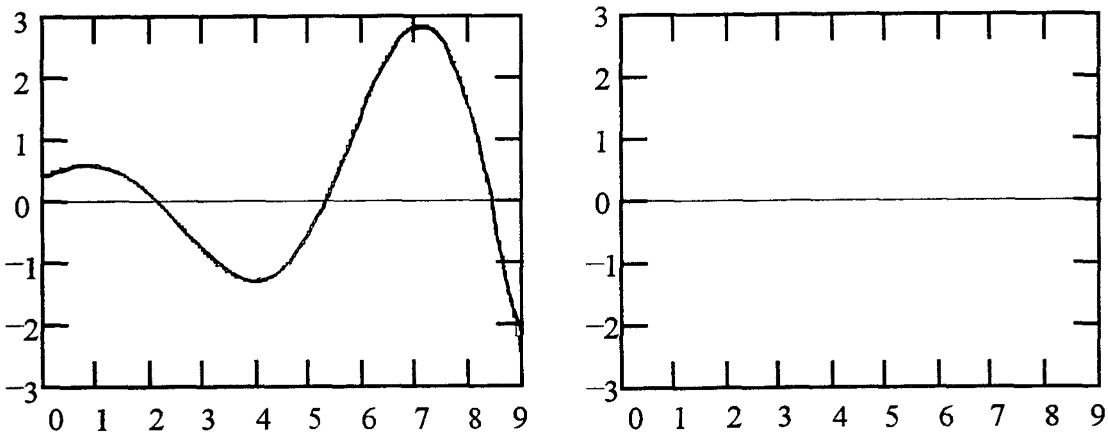
Time allowed is 45 minutes

SECTION II

Section II contains 3 free-response questions. Partial credit may be given so be sure to show your work in the space provided. A GRAPHING CALCULATOR MAY BE REQUIRED FOR SOME OF THE QUESTIONS IN THIS SECTION. Unless otherwise indicated, the domain of any given function is assumed to be the set of all real numbers for which the function is defined.

15. Consider the graph of $y = f'(x)$ given below (note that this is the graph of the derivative of f).

A) Sketch the graph of $f''(x)$ in the space provide.



B) Over which intervals is the graph of f concave up? Justify your answer.
C) Find the x-coordinates of all relative minimums for the function $f(x)$. Justify your answer.

---

16. Suppose f is continuous over the interval $1 < x < 7$ and $f'(x) = (\ln x)^2 - 2(\sin x)^4$ for $1 < x < 7$. Answer the following to three decimal places.

A) For what values of x is $f'(x) = 0$?
B) Find the x-values for all relative minima for the function $f(x)$. Justify your answer.
C) For what values of x is $f''(x) = 0$? Justify your answer.

---

17. The radius r of a sphere is increasing at the constant rate of 0.05 centimeters per second. (Volume of sphere with radius r is $V = \frac{4}{3}\pi r^3$). Answer the following questions.

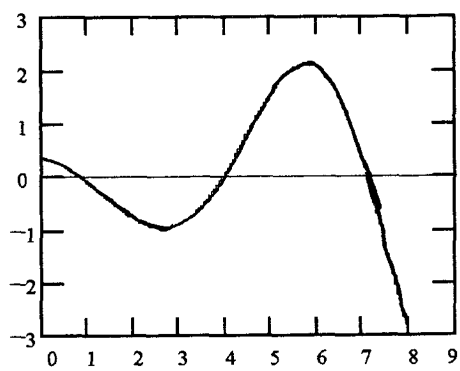
A) How fast is the volume of the sphere increasing when the radius of the sphere is 10 centimeters?
B) What is the radius of the sphere when the volume of the sphere and the radius of the sphere are increasing at the same rate?
C) When the volume of the sphere is 39π cubic centimeters, how fast is the area of the cross section through the center of the sphere increasing?

ANSWER KEY

SECTION I

	Part A	Part B
	1. D	8. E
	2. E	9. D
	3. C	10. D
	4. B	11. D
	5. C	12. D
	6. C	13. D
	7. D	14. C

SECTION II



15a. The graph of $f''(x)$ is above.

1 point for the 3 zeros in approximately the correct places

1 point if the graph is drawn above and below the x-axis in the proper intervals

1 point for relative minimum of about − 1 and relative maximum of about 2

15b f is concave up when $f''(x) > 0$
OR
f is concave up when f' is increasing

1 point for (0,1)
1 point for (4,7)
1 point for reason

15c Minimum occurs when $f'(x) = 0$ and $f''(x) > 0$
OR
Minimum occurs when $f'(x) = 0$ and $f'(x)$ changes from negative to positive

1 point for answer of $x \approx 5.25$
2 points for reason

16a. Using the calculator we find $f''(x) = 0$ when $x = 0.620$ and $x = 2.275$

2 points for $x = 2.275$

16b	When $x = 2.275$, $f'(x)$ changes signs from negative to positive	1 point for value 1 point for reason
16c.	$f''(x) = 0$ when the derivative of f' is 0	1 point for $x = 1.503$
	OR	1 point for $x = 3.625$
	$f''(x) = 0$ at the relative maximums and relative minimums of f'.	1 point for $x = 4.629$
		1 point for $x = 6.725$
	$f''(x) = 0$ when $x = 1.503$, $x = 3.625$, $x = 4.629$, and $x = 6.725$.	1 point for reason
17a	$dV/dt = 4\pi r^2(dr/dt)$	1 point for derivative
	When $r = 10$, $dV/dt = 4\pi(10^2)(0.05) = 20\pi$	1 point for answer
17b.	Need the volume of sphere increasing at 0.05 centimeters per second so,	1 point for setting up problem
	$0.05 = 4\pi r^2(0.05)$	1 point for finding r
	solving for r we get, $r = 1/(2\sqrt{\pi})$	
17c.	$39\pi = (4/3)\pi r^3 \Rightarrow r = (117/4)^{1/3}$	1 point for finding r
	Area $= \pi r^2 \Rightarrow dA/dt = 2\pi r(dr/dt)$	1 point for $A = \pi r^2$
	Evaluating dA/dt at $r = (117/4)^{1/3}$ we get $dA/dt \approx 0.968$ square centimeters per second.	1 point for dA/dt 1 point for answer

Appendix E: 1994 Advanced Placement Syllabus

A. Functions and Graphs

1. Properties of functions

   a. Domain and range
   b. Sum, product, quotient, and composition
   c. Inverse functions
   d. Odd functions
   e. Periodic Functions
   f. Zeros of a function

2. Properties of graphs

   a. Intercepts
   b. Symmetry
   c. Asymptotes
   d. Relationships between graphs

B. Limits and Continuity

1. Finite limits

   a. Limit of a constant, sum, product, or quotient
   b. One-sided limits
   c. Limits at infinity

2. Nonexistent limits

   a. Types of nonexistence
   b. Infinite limits

3. Continuity

   a. Definition
   b. Graphical interpretation of continuity and discontinuity
   c. Existence of absolute extrema of a continuous function
   d. Application of the Intermediate Value Theorem

B. Differential Calculus

1. The derivative

   a. Definition
   b. Derivative formulas

2. Statements and applications of theorems about derivatives

   a. Relationship between differentiability and continuity
   b. The Mean Value Theorem
   c. l'Hopital's Rule for indeterminate forms

3. Applications of the derivative

   a. Geometric applications
   b. Optimization problems
   c. Rate-of-change problems

C. Integral Calculus

1. Antiderivatives (indefinite integrals)

   a. Techniques of integration
   b. Applications of antiderivatives

2. The definite integral

   a. Definition of the definite integral as a limit of sums
   b. Properties
   c. Approximations to the definite integral
   d. Fundamental theorems
   e. Applications of the definite integral

2. Applications of antiderivatives

3. Techniques of integration

4. The definite integral

5. Applications of the integral