AN ABSTRACT OF THE THESIS OF

WU-YAN CHIU for the MASTER OF SCIENCE
(Name) (Degree)

in INDUSTRIAL ENGINEERING presented on October 3, 1967
(Major) (Date)

Title: AN EVALUATION OF OPTIMIZATION TECHNIQUES FOR PRODUCTION AND EMPLOYMENT SCHEDULING

Abstract approved: [redacted]
James L. Riggs

A single item is to be produced over a given number of time periods to satisfy the future requirements which are subject to fluctuations. The production rate and work force level for each time period are to be decided in order to minimize the production cost where the costs for regular payroll, hiring and layoff, overtime, inventory and shortage are known functions of these two decisions in each period.

This thesis attempts to present and evaluate the decision models suggested in the literature for handling this type of problem. Some cost elements are added to make the problem more complete and practical. The derivations of solution methods have been simplified and examples are added to reinforce the application. Finally, comparisons among various methods and conclusions are drawn, and suggestions for future research in this area are made.

An Evaluation of Optimization Techniques for Production and Employment Scheduling

by

Wu-Yan Chiu

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

June 1968

APPROVED:

[redacted]

Professor of Mechanical and Industrial Engineering

in charge of major

[redacted]

Head of Department of Mechanical and Industrial Engineering

[redacted]

Dean of Graduate School

Date thesis is presented October 3, 1967

Typed by Clover Redfern for Wu-Yan Chiu

ACKNOWLEDGMENT

Appreciation is expressed to Dr. James L. Riggs, Professor of Industrial Engineering, for his direction and assistance in the preparation of this manuscript. The author also wishes to express gratitude to Dr. Michael S. Inoue, Professor of Industrial Engineering, for his valuable suggestions.

TABLE OF CONTENTS

Page

LIST OF FIGURES

LIST OF TABLES

AN EVALUATION OF OPTIMIZATION TECHNIQUES FOR PRODUCTION AND EMPLOYMENT SCHEDULING

INTRODUCTION

Fluctuations in customer's orders create difficult problems for a manager responsible for scheduling production and employment. Changes in ordering quantities must be absorbed by some combinations of the following actions:

1. Adjusting the size of the work force by hiring and firing.
2. Adjusting the amount of overtime or undertime.
3. Adjusting the finished goods inventory.
4. Adjusting the order backlog or the proportion of shortage.

Each of these courses of action has certain associated costs. If the fluctuations in orders can be predicted or known with certainty, the application of mathematical techniques may improve the quality of scheduling decisions and help managers make substantially better decisions than they could make by using judgment procedures. Once a general rule has been derived, the computation required to establish the optimal production quantity for each period can be computed by a clerk or on a computer without difficulty.

The purpose of this paper is to investigate mathematical approaches to deal with the production planning problem stated above. This problem is also referred to as a production smoothing problem.

It is concerned with the conflict among the costs associated with the four actions which are usually adopted in absorbing the fluctuations in orders.

The problem is defined as:

minimize:

Inventory holding cost + Shortage or backlog cost

+ Overtime and undertime costs + Hiring and layoff costs

+ Regular payroll cost

subject to:

Inventory level at the end of the production period t - 1

+ Production rate in the period t - Demand rate in the period t

= Inventory level at the end of the period t, for t = 1, 2, ... , n.

The techniques described in this paper which are applicable in solving this problem are found as:

1. Linear programming.
2. Dynamic programming.
3. Quadratic programming.

In this paper, the analysis of some important costs which are related to the decision variables are introduced first. They are followed by three separate chapters presenting three mathematical programming methods which are largely used in the field of operations research. In some of the methods, examples are employed to verify their applicability to the real situation. These examples are solved by an electronic computer for convenience and accuracy in numerical computation. The computer programs are presented in the Appendix.

COST ANALYSIS

Overview of the Chapter

In this chapter, we are concerned with various approaches for studying the costs relevant to production and work force decisions. Once the functional relation has been made of the cost structure, the next step is to evaluate the optimal policies. These will be introduced in the following three chapters.

The costs that depend on the production and employment decisions are different from factory to factory, but we will consider in general terms the costs depending upon regular payroll, hiring and layoffs, overtime and under time (spare time), inventory holding and shortages.[1]

Cost structure presented in this analysis is divided into two parts: one is linear approximation (5, 16, 22); the other is quadratic approximation (18). In the real world, however, it may neither be linear nor quadratic function of the decision variable, but it may be approximated into one of these two types of functions within some specified regions. The purpose of doing these approximations is for

[1] The costs discussed here are those which usually depend on the decision variables. Hence the material cost, etc., are not considered. The length of the decision period is also an important factor in making the decision, but will not be studied in this paper.

the convenience of establishing mathematical models.

Notation

We will first introduce some notations which will be used in this chapter and throughout this paper.

Given a planning intervals of n time periods,[2] define the following quantities for time period t $(t = 1, 2, \ldots, n)$:

P_t = Production rate (in a suitable unit),

D_t = Demand rate (in the same unit),

I_t = Net inventory level at the end of the time period = Inventory - Shortage (in the same unit),

W_t = Work force level (in man-period of time which can be obtained in regular time).

Except I_t, all the defined quantities above are non-negative values.

Regular Payroll, Hiring and Layoff Costs

When order fluctuations are absorbed by adjusting the work force, regular payroll, hiring and layoff costs are affected.

Regular Payroll Cost

With periodic adjustments in the size of work force, regular

---

[2] The time period may be in week, month, season, or year.

payroll cost per time period is approximated as a linear function of the size of work force as (18, p. 52):

$$\text{Regular payroll cost} = C_1 W_t + C_{13} \tag{2.1}$$

where C_1 may be regarded as regular time wages per unit of work force per time period and C_{13} is a fixed term; it may be considered as an indirect work force which is not changed by the scheduling decisions over a range and hence is irrelevant.[3] This cost is shown in Figure 2-1.

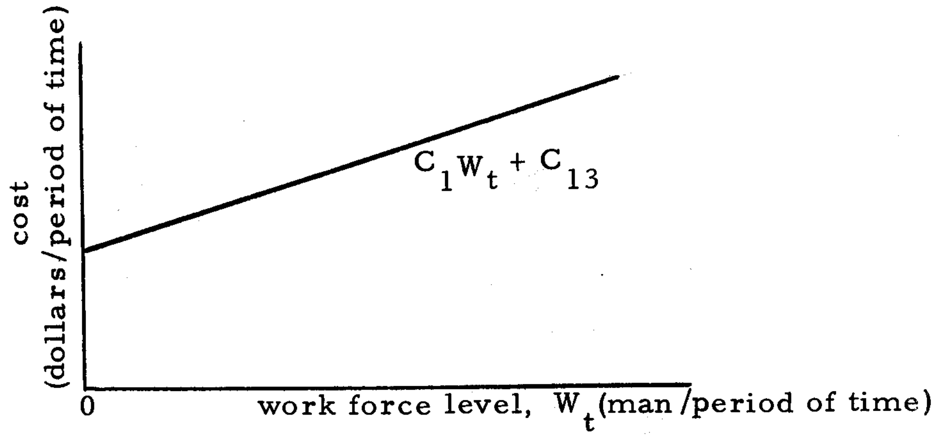

Figure 2-1. Regular payroll cost.

Hiring and Layoff Costs

These costs are associated not with the size of the work force, but with the change in its size between successive time periods.

---

[3] Since C_{13} is irrelevant to the scheduling decision, we say "variable" regular payroll cost = $C_r W_t$, where $C_r = C_1$.

<u>Linear Approximation.</u> If we consider the cost of hiring and laying off rise with the number of work force hired and laid off, then according to Hanssmann and Hess (16), these costs may be approximated as:

$$\text{Hiring cost} = \begin{cases} C_h(W_t - W_{t-1}), & \text{if } W_t - W_{t-1} > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$

$$\text{Layoff cost} = \begin{cases} C_f(W_{t-1} - W_t), & \text{if } W_{t-1} - W_t > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.3)$$

where C_h, C_f are hiring, layoff costs per unit of work force, respectively.

From the relation above, if hiring occurs, layoff may not occur. Actually, both can happen at the same time. Since this situation is caused mainly by human factor, but not by our scheduling decisions, it is not our case.

<u>Quadratic Approximation</u>. According to Holt, <u>et al</u>. (18, p. 53), if the following arguments hold, a quadratic curve may suitably represent the hiring and layoff costs as:

$$\text{Hiring and layoff costs} = C_2(W_t - W_{t-1} - C_{11})^2, \quad (2.4)$$

where C_2, C_{11} are constants obtained from the analysis of cost data[4/].

[4]For the detail derivation of this function, see (18, p. 74) "Fitting quadratic approximations to cost relations".

1. The efficiency of hiring, measured in terms of the quality of employees hired may fall when a large number of work forces are hired at one time.
2. The reorganization costs are more than proportionately larger for large layoffs than small layoffs.

Equations (2.2), (2.3), (2.4) have the common property that the smaller the work force changes, the less the hiring and layoff costs. These are shown in Figure 2-2.

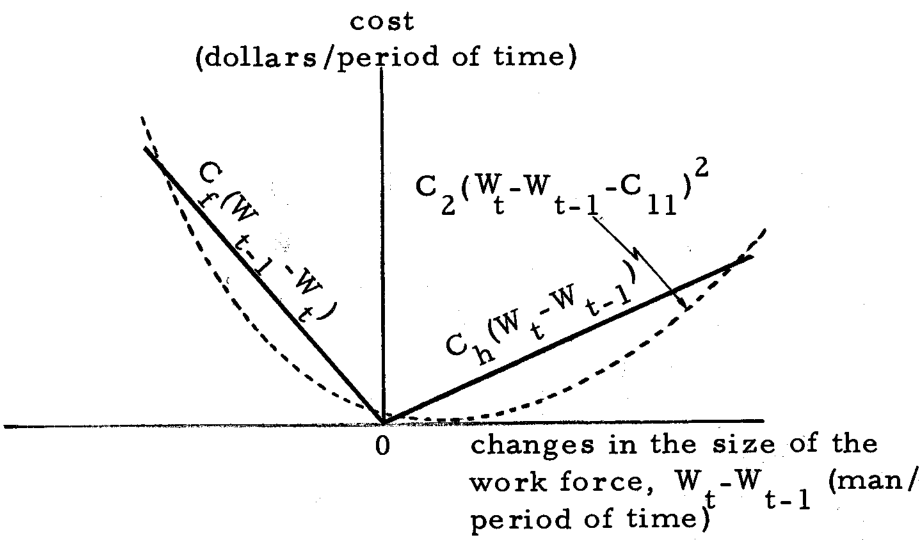

Figure 2-2. Hiring and layoff costs related to the number of work force hired and laid off.

Overtime and Under Time Costs

When order fluctuations are absorbed by increasing and decreasing production without changing the work force, overtime and under time costs are incurred. Overtime cost is an hourly wage paid at a fixed ratio (usually 50 percent) higher than that paid for in regular payments. Under time is a waste of labor time that is paid for in regular payroll, but is not used for productive activities.[5]

Linear Approximation

The overtime and under time costs depend on two decision variables, the size of work force W_t, and the production rate P_t. With an average work force needed to produce a unit of product, K (in number of direct labor per unit of product), KP_t is the number of work force required in producing the quantity P_t. But actually we have work force W_t, so $KP_t - W_t$ is the work force required to work overtime if KP_t is greater than W_t, and $W_t - KP_t$ is the number of work force idled, if W_t is greater than KP_t.

Let

C_o be the overtime cost per unit of work force,[6]

C_u be the under time cost per unit of work force.[7]

Then,

$$\text{Overtime cost (16)} = \begin{cases} C_o(KP_t - W_t), & \text{if } KP_t - W_t > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

$$\text{Under time cost} = \begin{cases} C_u(W_t - KP_t), & \text{if } W_t - KP_t > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

---

[5] It may be possible to perform some activities other than production with labor that should otherwise be wasted. If so, this possibility should be taken into account.

[6] C_o = (overtime wages-regular wages) per unit of work force.

[7] C_u = regular time wages per unit of work force (C_r), if the work forces idled are not used in other activities.

Quadratic Approximation

In most factories, a small increase in production would require only a few employees who work in bottle neck functions to work overtime. As production is increased further, more and more employees are required to work overtime until the whole work force is doing some overtime work. Also, if the random disturbances such as emergency orders, machine breakdown, quality control problems, fluctuations in productivity, etc., are taken into account, the higher the production target with a given size of work force, the greater the possibility that some disturbances will occur. Therefore, as in the case of hiring and layoffs, Holt, et al. (18, p. 54) suggested a U-shaped, possibly unsymmetrical cost curve as:

$$\text{Overtime and under time costs} = C_3(P_t - C_4W_t)^2 + C_5P_t + C_6W_t + C_{12}P_tW_t \,.^{8/} \qquad (2.7)$$

As production, P_t, exceeds or goes below C_4W_t,[9/] a level set by the size of the work force, overtime or under time costs are increased. The linear term, C_5P_t,[10/] and C_6W_t, and the cross

---

[8] C_3 is mainly obtained from the analysis of C_o and C_u.

[9] C_4 may be regarded as a measure of the "capacity" per employee.

[10] C_5 turns out to be irrelevant in making scheduling decisions. Because shifting production from one period to another will leave this component of cost unchanged.

product term $C_{12}P_tW_t$, are added to improve the approximation.

Equations (2.5), (2.6), and (2.7) reveal that the more work force required in producing the quantity P_t deviates from the size of work force W_t, the more overtime and under time costs are incurred. They are shown in Figure 2-3.

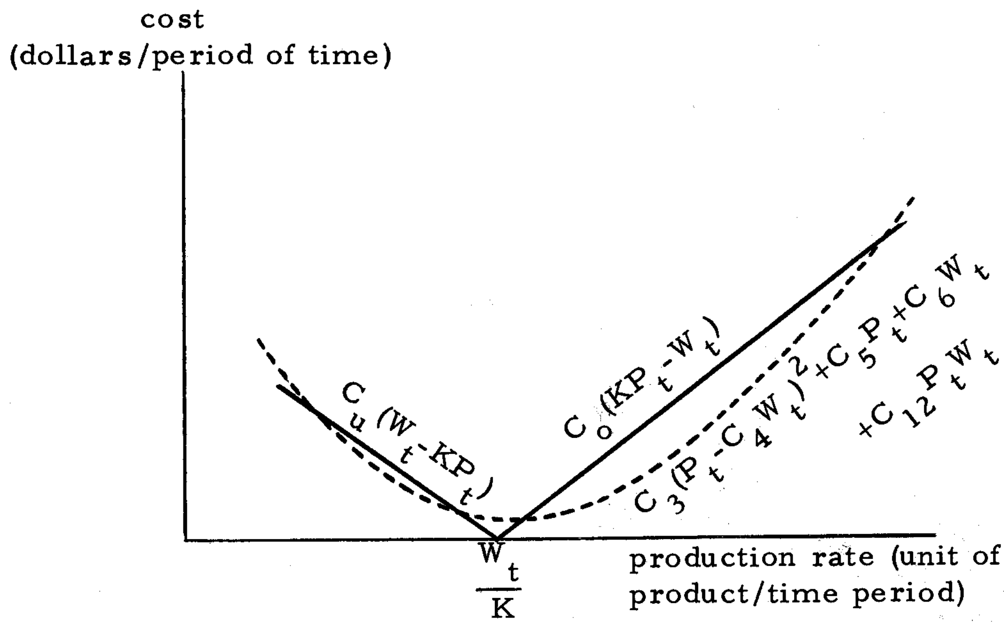

Figure 2-3. Overtime and under time costs related to the amount of production rate.

Inventory and its Related Costs

Absorbing order fluctuations through inventory and back order buffers gives rise to new costs. Holding a good-size inventory incurs costs such as interest, obsolescence, handling, storage, and price movements (27). On the other hand, a decision to decrease these costs by operating with a smaller inventory increases the probability of running out of products and thus incurring the penalty of delaying customer shipments and possibly losing sales.

The net inventory level, I_t, has close relation to our production scheduling decisions. If the back orders can be carried over to the following periods, then the relation between I_t and the scheduling deicisions, P_t and W_t, is:

$$I_t = I_0 + \sum_{i=1}^{t} (P_i - D_i) .$$

If back orders can not be carried over to following periods, then the relation is:

$$I_t = \begin{cases} I_{t-1} + P_t - D_t, & \text{if } I_{t-1} \geq 0, \\ P_t - D_t, & \text{if } I_{t-1} < 0. \end{cases}$$

<u>Linear Approximation</u>

In this approximation, we consider inventory holding cost and shortage cost seperately. The two cost coefficients are defined as:

C_v = inventory cost per unit of product per time period,

C_s = shortage cost per unit of product.

Then,

$$\text{Inventory cost} = \begin{cases} C_v I_t, & \text{if } I_t > 0, \\ 0, & \text{otherwise,} \end{cases} \qquad (2.8)$$

$$\text{Shortage cost} = \begin{cases} -C_s I_t, & \text{if } I_t < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

The machine set up cost is ignored. Bowman (5) and Hu (22) used these relations in order to gain computing feasibility with the methods they used in their approaches.

Quadratic Approximation

From the economic lot size formula (6, p. 644), we know the optimal production quantity for the time period t, Q_t, is

$$Q_t = \sqrt{\frac{2C_p D_t}{C_v}},$$

where

C_p = set up cost for a lot,

C_v = cost of holding one unit of inventory one period of time,

D_t = demand rate for time period t.

The optimal safety stock also can be derived (34). This is a constant, say C_8, over a fixed time interval. From these two quantities, we obtain optimal average inventory level as:

$$\text{Optimal average inventory level for the time period } t = C_8 + \frac{Q_t}{2}.$$

Since the square root relation between Q_t and D_t can be approximated by a linear relation over a limited range, the optimal average inventory can be approximated as:

$$C_8 + C_9 D_t, \quad \text{where } C_9 = \sqrt{\frac{2C_p}{C_v}} \Big/ 2.$$

When actual net inventory deviates from the optimal net inventory, $(C_8 + C_9 D_t)$, in either directions, cost rises as shown in Figure 2-4 which is a U-shaped curve and has approximately the same total variable cost structure as the economic lot size problem.

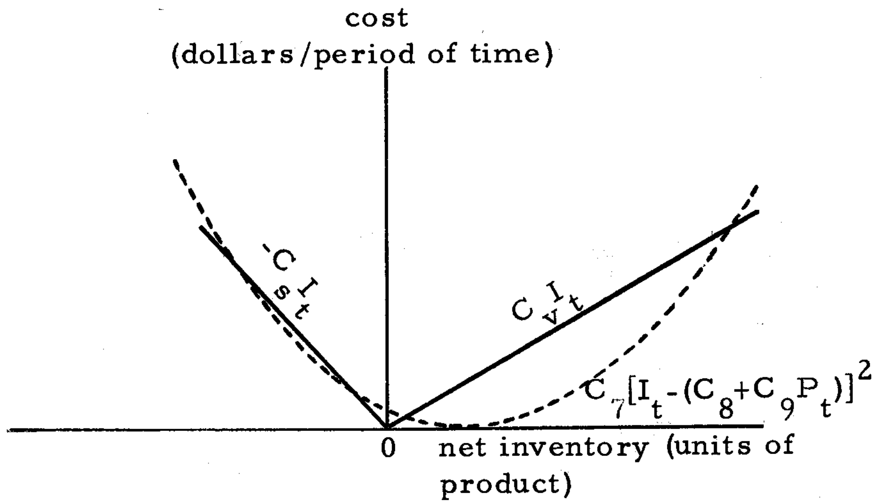

Figure 2-4. Inventory and its related costs related to the net inventory level.

According to Holt, et al. (18, p. 57),

inventory, back order, and set up costs $= C_7[I_t-(C_8+C_9D_t)]^2$, (2.10)

where C_7 is a constant derived mainly from C_v, C_s, and the machine set up cost.

Summary

Costs involved in the scheduling decisions have been analyzed and put into two possible functional forms: linear and quadratic relations. Once the cost structures are assured, some solution techniques can be applied.

In the real world, McNaughton (28) felt that a quadratic cost function would be more realistic in many cases. In our problem, the required accuracy of the estimated cost is very important. Methods of obtaining an approximated cost function are discussed in (18, Chapter 3) while the sensitivity of these cost estimations can be found in the article written by Pamne and Bosje (33).

LINEAR PROGRAMMING METHOD

Overview of the Chapter

The cost relationship to the decision variables is assumed to be linear. Two methods are introduced in this chapter under different assumptions. The first one is transportation method. This is the simplest method for developing a production schedule. Bowman (5) formulated the problem by allocating available productive capacity to various periods in such a way that sales requirements were met while combined incremental inventory and production costs were minimized. The second method is solved by the simplex algorithm (16, 29) and eliminates the constant employment restriction. This gives greater flexibility than the constant employment schedule of the transportation model. The technique of changing variables is applied in this second method in order to reach a linear programming formulation.

Transportation Method

Assumptions

(a) Fixed work force.

(b) Sales must be met.

(c) Demand in each period is known.

(d) Linear cost function.

Solution Method

If we consider the regular or overtime production in each time period as a source of supply or input, and each period's sales requirement a distination or output, and the combination of production and storage costs is considered as the cost of each possible shipment. Then the production scheduling problem may be thrown into the standard form for the transportation method and hence may be solved by this method.

Let:

a_{ij} = optimal number of units to be produced during ith time period on regular time and to be sold in jth period. So, $i \leq j$.

b_{ij} = optimal number of units to be produced during ith time period on overtime and to be sold in jth period. So, $i \leq j$.

R_i = maximum number of units which can be produced during ith time period on regular time.

0_i = maximum number of units which can be produced during ith time period on overtime.

If a unit is produced in period i and sold to the customer in period j, then the production and inventory costs become:

$C_r + C_v(j-i)$, for regular time production,

$C_o + C_v(j-i)$, for overtime production,

$j = 1, \ldots, n; \qquad i = 1, \ldots, j,$

where C_r, C_v are as defined in the Cost Analysis chapter, and C_o is the overtime wages per unit of work force per time period.[11]

Since there are a_{ij} and b_{ij} units to be produced during ith time period and to be sold in jth time period on regular time and overtime, respectively, the production and inventory cost for the delivered items on jth period is then

$$a_{ij}[C_r + (j-i)C_v] + b_{ij}[C_o + (j-i)C_v] ,$$

for $j = 1, \ldots, n; \quad i = 1, \ldots, j$.

If the starting inventory is I_0, and from I_0, I_{0j} is going to be used in the jth period, the inventory costs for I_{0j} is

$$I_{0j}jC_v, \qquad \text{for } j = 1, \ldots, n.$$

Hence the total production and inventory cost from production periods one to n can be formulated as:

---

[11]Note that $C_o = C_r(1 + \text{overtime ratio})$, it is different from what we have defined in the Cost Analysis chapter. C_r, C_v, C_o are not necessary to be constant. This model is also useful when they vary from time to time.

$$\sum_{j=1}^{n}\sum_{i=1}^{j}\{a_{ij}[C_r+(j-i)C_v] + b_{ij}[C_o+(j-i)C_v]\} + \sum_{j=1}^{n} I_{0j}jC_v . \quad (3.1)$$

The problem becomes: minimize Equation (3.1)

subject to

$$\sum_{i=1}^{n}(a_{ij} + b_{ij}) + I_{0j} \geq D_j, \quad j = 1, \ldots, n, \quad (3.2)$$

$$\sum_{j=1}^{n} a_{ij} \leq R_i, \quad (3.3)$$

$$i = 1, \ldots, n,$$

$$\sum_{j=1}^{n} b_{ij} \leq O_i, \quad (3.4)$$

$$\sum_{j=1}^{n} I_{0j} \leq I_0, \quad (3.5)$$

$$a_{ij}, b_{ij} \geq 0, \quad \text{for } j = 1, \ldots, n, \ i = 1, \ldots, j, \quad (3.6)$$

$$I_{0j} \geq 0, \quad \text{for } j = 1, \ldots, n . \quad (3.7)$$

This is a standard form of transportation problem in linear programming.

Transportation Table Representation

Table 3-1 exhibits the costs of unit production produced in the

Table 3-1. Unit costs of production and inventory holding.

Production Periods (Origins)			Sales Periods (Distinctions)							Total Capacities
			1	2	...	j	...	n	slack	
	0	Starting Inventory	I_{01} [C_v]	I_{02} [$2C_v$]	...	I_{0j} [jC_v]	...	I_{0n} [nC_v]	X_0 [0]	I_0
	1	Regular Time	a_{11} [C_r]	a_{12} [C_r+C_v]	...	a_{1j} [$C_r+(j-1)C_v$]	...	a_{1n} [$C_r+(n-1)C_v$]	X_{1a} [0]	R_1
		Overtime	b_{11} [C_o]	b_{12} [C_o+C_v]	...	b_{1j} [$C_o+(j-1)C_v$]	...	b_{1n} [$C_r+(n-1)C_v$]	X_{1b} [0]	O_1
	2	Regular Time	—	a_{22} [C_r]	...	a_{2j} [$C_r+(j-2)C_v$]	...	a_{2n} [$C_r+(n-2)C_v$]	X_{2a} [0]	R_2
		Overtime	—	b_{22} [C_o]	...	b_{2j} [$C_o+(j-2)C_v$]	...	b_{2n} [$C_o+(n-2)C_v$]	X_{2b} [0]	O_2
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	i	Regular Time	—	—	...	a_{ij} [$C_r+(j-i)C_v$]	...	a_{in} [$C_r+(n-i)C_v$]	X_{ia} [0]	R_i
		Overtime	—	—	...	b_{ij} [$C_o+(j-i)C_v$]	...	b_{in} [$C_o+(n-i)C_v$]	X_{ib} [0]	O_i
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	n	Regular Time	—	—	—	—	—	a_{nn} [C_r]	X_{na} [0]	R_n
		Overtime	—	—	—	—	—	b_{nn} [C_o]	X_{nb} [0]	O_n
Demand in Each Period			D_1	D_2	...	D_j	...	D_n		

ith period and sold in the jth period. From this table, it is readily seen that the problem may be viewed as one of allocating the available supplies I_0, R_i, 0_i (origins) to consumption points j (destinations). A dummy column is used to absorb excess capacity.

Simplex Method

Assumptions

(a) Variable work force.

(b) Shortage is allowed.

(c) Shortage can be carried over to the next period.

(d) Linear cost functions.

Description of the Problem

In this approach, analysis extends to evaluate two sets of decision variables, P_t and W_t.

For a given pair of values (P_t, W_t), the respective amounts to be produced on regular time and on overtime are determined.

From Equations (2.1), (2.2), (2.3), (2.5), (2.6), (2.8) and (2.9), the total cost incurred in period t is composed of the following elements:

Regular payroll[12/] $C_r W_t$. (3.8)

---

[12] See footnote 3 on page 5.

$$\text{Hiring} \quad \begin{cases} C_h(W_t - W_{t-1}), & \text{if } W_t - W_{t-1} > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.9}$$

$$\text{Layoff} \quad \begin{cases} C_f(W_{t-1} - W_t), & \text{if } W_{t-1} - W_t > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.10}$$

$$\text{Overtime} \quad \begin{cases} C_o(KP_t - W_t), & \text{if } KP_t - W_t > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.11}$$

$$\text{Undertime} \quad \begin{cases} C_u(W_t - KP_t), & \text{if } W_t - KP_t > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.12}$$

$$\text{Inventory} \quad C_v I_t, \qquad \text{for } I_t > 0. \tag{3.13}$$

$$\text{Shortage} \quad -C_s I_t, \qquad \text{for } I_t < 0. \tag{3.14}$$

The problem of minimizing total costs for n time periods can be formulated as:

minimize

$$C(P_1, P_2, \ldots, P_n; W_1, W_2, \ldots, W_n)$$

$$= \sum_{t=1}^{n} [\text{Eqs. } (3.8) + (3.9) + \ldots + (3.14)], \tag{3.15}$$

subject to the restrictions

$$I_t = I_{t-1} + P_t - D_t, \tag{3.16}$$

$$P_t, W_t \geq 0, \tag{3.17}$$

for $t = 1, 2, \ldots, n$.

where the demand D_t, and the initial condition I_0, and W_0 are known.

Solution of the Problem

Let

$$x_t = W_t - W_{t-1} \geq 0$$

$$y_t = W_{t-1} - W_t \geq 0,$$

$$z_t = KP_t - W_t \geq 0,$$

$$w_t = W_t - KP_t \geq 0,$$

$$u_t = I_t, \quad \text{if } I_t > 0,$$

$$v_t = -I_t, \quad \text{if } I_t < 0,$$

for $t = 1, \ldots, n$.

From Equations (3.9), (3.10), if

$W_t - W_{t-1} > 0$, hiring cost occurs,

$W_t - W_{t-1} < 0$, layoff cost occurs.

Since both these two costs can not happen in the same period, if hiring cost occurs, $x_t > 0$, $y_t = 0$; if layoff cost occurs, $x_t = 0$ $y_t > 0$ and the difference of W_t and W_{t-1} may be represented by

$$W_t - W_{t-1} = x_t - y_t.$$

The same argument holds for overtime and under time costs, and inventory and shortage costs, so,

$$KP_t - W_t = z_t - w_t,$$

$$I_t = u_t - v_t.$$

From Equation (3.16),

$$I_t = I_{t-1} + P_t - D_t.$$

Since $$I_t = u_t - v_t,$$

$$\therefore\ P_t = (u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t. \qquad (3.18)$$

Since $$P_t \geq 0,$$

$$\therefore\ (u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t \geq 0. \qquad (3.19)$$

Since $$KP_t - W_t = z_t - w_t,$$

$$\therefore\ W_t = KP_t - (z_t - w_t).$$

Use the relation in Equation (3.18), we have

$$W_t = K[(u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t] - (z_t - w_t). \qquad (3.20)$$

Since $$W_t \geq 0,$$

$$\therefore\ K[(u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t] - (z_t - w_t) \geq 0. \qquad (3.21)$$

Equation (3.21) divided by non-negative value K, we have

$$(u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t - \frac{1}{K}(z_t - w_t) \geq 0. \qquad (3.22)$$

Since $W_t - W_{t-1} = x_t - y_t$, from Equation (3.20), this relation becomes

$$K[(u_t-v_t)-(u_{t-1}-v_{t-1})+D_t] - (z_t-w_t)$$

$$- K[(u_{t-1}-v_{t-1})-(u_{t-2}-v_{t-2})+D_{t-1}] + (z_{t-1}-w_{t-1})$$

$$= K[u_t-v_t-2(u_{t-1}-v_{t-1})+u_{t-2}-v_{t-2}+D_t-D_{t-1}] - (z_t-w_t)+z_{t-1}-w_{t-1}$$

$$= x_t - y_t.$$

So,

$$K[u_t-v_t-2(u_{t-1}-v_{t-1})+u_{t-2}-v_{t-2}] - (z_t-w_t) + (z_{t-1}-w_{t-1}) - (x_t-y_t)$$

$$= K(D_{t-1}-D_t). \qquad (3.23)$$

The relations require that all variables be non-negative,

$$x_t, y_t \geq 0, \quad z_t, w_t \geq 0, \quad u_t, v_t \geq 0, \qquad (3.24)$$

$$\text{for } t = 1, \ldots, n.$$

Equations (3.19), (3.22), (3.23), (3.24) are the constraints of the problem after changing variables.

Cost elements:

Regular payroll $\quad C_r W_t = C_r K[(u_t-v_t) - (u_{t-1}-v_{t-1}) + D_t] - C_r(z_t-w_t),$

Hiring $\quad C_h(W_t-W_{t-1}) = C_h x_t,$

Layoff $\quad C_f(W_{t-1}-W_t) = C_f y_t,$

Overtime	$C_o(KP_t - W_t) = C_o z_t$,
Undertime	$C_u(W_t - KP_t) = C_u w_t$,
Inventory	$C_v I_t = C_v u_t$,
Shortage	$-C_s I_t = C_s v_t$.

The cost function after changing variables is:

$$C = \sum_{t=1}^{n} \{C_r K[(u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t] - C_r(z_t - w_t) + C_h x_t + C_f y_t + C_o z_t + C_u w_t + C_v u_t + C_s v_t\}. \quad (3.25)$$

Since $I_t = u_t - v_t$, where I_t is net inventory at the end of the time period t.

$$\therefore \sum_{t=1}^{n} \{C_r K[u_t - v_t - (u_{t-1} - v_{t-1}) + D_t] = C_r K(u_n - v_n - I_0 + \sum_{t=1}^{n} D_t).$$

$$\therefore C = C_r K(u_n - v_n - I_0 + \sum_{t=1}^{n} D_t) + \sum_{t=1}^{n} \{-C_r(z_t - w_t) + C_h x_t + C_f y_t + C_o z_t + C_u w_t + C_v u_t + C_s v_t\}$$

$$= C_r K(u_n - v_n - I_0 + \sum_{t=1}^{n} D_t) + \sum_{t=1}^{n} [C_h x_t + C_f y_t + (C_o - C_r) z_t + (C_u + C_r) w_t + C_v u_t + C_s v_t].$$

The problem becomes

minimize

$$C(x_t, y_t, z_t, w_t, u_t, v_t; \quad \text{for } t = 1, \ldots, n)$$

$$= \sum_{t=1}^{n} [C_h x_t + C_f y_t + (C_o - C_r) z_t + (C_u + C_r) w_t + C_v u_t + C_s v_t]$$

$$+ C_r K(u_n - v_n - I_0 + \sum_{t=1}^{n} D_t) \qquad (3.26)$$

subject to

$$(u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t \geq 0 \qquad (3.27)$$ [13]

$$(u_t - v_t) - (u_{t-1} - v_{t-1}) + D_t - \frac{1}{K}(z_t - w_t) \geq 0 \qquad (3.28)$$

$$K[u_t - v_t - 2(u_{t-1} - v_{t-1}) + u_{t-2} - v_{t-2}] - (z_t - w_t) + (z_{t-1} - w_{t-1}) - (x_t - y_t)$$

$$= K(D_{t-1} - D_t) \qquad (3.29)$$

$$x_t, y_t, z_t, w_t, u_t, v_t \geq 0 \qquad (3.30)$$

$$\text{for} \quad t = 1, \ldots, n$$

This is a linear programming problem and can be solved by Simplex Algorithm. Each period t contributes six variables and three constraints. If the total time periods are six, then we have 36 variables, 18 constraints.

---

[13] Equations (3.27) through (3.30) are the same as Equations (3.19), (3.22), (3.23), (3.24), respectively.

According to Equations (3.27) and (3.28), for each period, the original decision variables P_t and W_t are slack variables of these equations.

If demand is stochastic, this approach may still be useful. Replacing D_t by its expected value can obtain an approximate solution. (Proof is shown on page 69).

Summary

Two linear programming methods are applied in this chapter. In transportation method, it is assumed that in each period the product may be made in two types of operations, each of which has a constant unit cost combining with the unit inventory holding cost for the span of some fixed periods, the unit cost in each cell of the transportation table is thus established.

For more complicated cost conditions, the Simplex method is used. In the approach, the work force is also under control. This model is established by introducing a set of new non-negative variables, and hence a new set of cost functional and restrictions which are in the form of linear programming is made. This can be solved by Simplex algorithm.

The weakness of these two methods is the assumption of linear cost function. This is not met by most of the cases. Some more practical models will be discussed in the later chapters.

DYNAMIC PROGRAMMING METHOD

Overview of the Chapter

In this chapter we apply the functional equation approach of dynamic programming to the scheduling problem. This method requires no restriction in cost structure, and is very useful in formulating the case where demands are uncertain. The method is based on factoring complex problems in several variables into a series of simple problems (4). This will be found in the construction of recurrence relations in this chapter.

Three cases are considered. Sections beginning on pages 29 and 37 treat the case of demand under certainty and uncertainty, where the work force levels are not under the control. Each case is followed by an illustrative example. The section beginning on page 44 treates the case of probablistic demand while the work force levels are also taken into consideration.

Throughout this chapter where the involved costs are not necessarily linear or quadratic in relation to the decision variables, we use only functional name to represent the cost relation. For example, the function $h_t(I_{t-1}, P_t, D_t)$ is defined as an inventory cost function in the time period t, and is continuous for all finite I_{t-1}, P_t, and D_t, where I_{t-1}, P_t, D_t are the independent variables. We say the inventory cost in each period is a function of the beginning inventory, production rate, and demand of that period. Once these three quantities are known, the inventory cost can be determined.

The Case of Deterministic Demand

Assumptions

(a) Demand in each time period is known as certainty.

(b) The inventory cost per period is proportional to the closing inventory level.

(c) The components of the cost function may or may not be linear.

(d) Hiring, layoff, and overtime costs are proportional to the change in production levels of the consecutive periods.

Formulation of the Problem

Let

Inventory cost $= h_t(I_{t-1}, P_t, D_t)$, if $I_{t-1} + P_t \geq D_t$

Shortage cost $= s_t(I_{t-1}, P_t, D_t)$, if $I_{t-1} + P_t < D_t$

Overtime, hiring and layoff costs

$$= g1_t(P_t, P_{t-1})$$

Regular payroll cost

$$= g2_t(P_t, P_{t-1})$$

$g1_t$ and $g2_t$ can be combined into $g_t(P_t, P_{t-1})$

The cost function is then

$$C(P_1, P_2, \ldots, P_n) = \sum_{t=1}^{n} [g_t(P_t, P_{t-1}) + h_t(I_{t-1}, P_t, D_t) + s_t(I_{t-1}, P_t, D_t)] \tag{4.1}$$

subject to the restrictions

$$I_{t-1} + P_t - D_t = I_t \tag{4.2}$$

$$P_t, I_t \geq 0 \tag{4.3}$$

$$\text{for } t = 1, \ldots, n$$

where the initial conditions P_0, I_0 are known.

Construction of the Recurrence Relation

From the problem:

minimize Equation (4.1)

subject to Equations (4.2),(4.3).

We use backward induction, set nth period as first stage, (n-1)th stage as second stage, etc., and let

$f_i(P_{i+1}, I_{i+1})$ = cost of optimal policy for the last i stages where P_{i+1} units are produced in the (i+1)th stage and I_{i+1} units are carried over into the ith stage in inventory.

Then

$$f_1(P_2, I_2) = \min_{P_1} \begin{cases} g_1(P_1, P_2) + h_1(I_2, P_1, D_1) + f_0(P_1, I_2+P_1-D_1), \\ \qquad \text{if } I_2 + P_1 > D_1 \\ g_1(P_1, P_2) + s_1(I_2, P_1, D_1) + f_0(P_1, 0), \\ \qquad \text{if } I_2 + P_1 \leq D_1 \end{cases}$$

$$\vdots \qquad\qquad\qquad \vdots \qquad\qquad (4.4)$$

$$f_i(P_{i+1}, I_{i+1}) = \min_{P_i} \begin{cases} g_i(P_i, P_{i+1}) + h_i(I_{i+1}, P_i, D_i) + f_{i-1}(P_i, I_{i+1}+P_i-D_i), \\ \qquad \text{if } I_{i+1} + P_i > D_i \\ g_i(P_i, P_{i+1}) + s_i(I_{i+1}, P_i, D_i) + f_i-1(P_i, 0), \\ \qquad \text{if } I_{i+1} + P_i \leq D_i \end{cases}$$

This becomes a dynamic programming problem, where the optimal production levels $P_1, P_2, \ldots, P_n$ can be solved. An illustrative example will show the procedure of solving the production scheduling problem by this method.

Example

The manufacturing process for a perishable commodity is such that the cost of changing the level of production from one month to the next is twice the square of the difference in production levels. Any production not sold by the end of the month is wasted at a cost of $20 per unit. Given the sales forecast below, which must be met, determine a production schedule to minimize costs. Assume the December production was 200 units.

Month	Jan.	Feb.	Mar.	Apr.
Sales forecast	210	220	195	180

Solution:

Let April be the first stage, March be the second, . . . , and January be the fourth stage. We have

$$h_i(P_i, D_i) = 20(P_i - D_i), \quad \text{for } P_i \geq D_i$$

$$g_i(P_i, P_{i+1}) = 2(P_i - P_{i+1})^2$$

$$\text{for } i = 1, 2, 3, 4$$

The problem is then

minimize

$$\sum_{t=1}^{4} \{2(P_i - P_{i+1})^2 + 20(P_i - D_i)\}$$

subject to

$$P_i \geq D_i, \quad i = 1, 2, 3, 4$$

Let

$f_n(P_{n+1})$ = minimum achievable cost for the last n stages of the process given that P_{n+1} units are produced in the $(n+1)$th stage.

Then

$$f_i(P_{i+1}) = \min_{P_i \geq D_i} \{2(P_i - P_{i+1})^2 + 20(P_i - D_i) + f_{i-1}(P_i)\}$$

Set

$$f_0(P_1) = 0, \quad \text{for } P_1 \geq 0$$

In the first stage, when $P_2 = 195$

$$f_1(195) = \min \begin{cases} P_1 = 180, & 2(180-195)^2 + 20(180-180) + f_o(180) = 450 \\ P_1 = 181, & 2(181-195)^2 + 20(181-180) + f_o(181) = 412 \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ P_1 = 188, & 2(188-195)^2 + 20(188-180) + f_o(188) = 258 \\ P_1 = 189, & 2(189-195)^2 + 20(189-180) + f_o(189) = 252 \\ P_1 = 190, & 2(190-195)^2 + 20(190-180) + f_o(190) = 250 \leftarrow \\ P_1 = 191, & 2(191-195)^2 + 20(191-180) + f_o(191) = 252 \\ P_1 = 192, & 2(192-195)^2 + 20(192-180) + f_o(192) = 258 \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

From the result above we know that if the production level in March is 195, the optimal production level for April is 190 units, the total cost up to this stage (first stage) is 250 and is represented as:

$$f_1(195) = 250, \text{ where } P_1 = 190$$

With the same technique we find that

$$f_1(196) = 270 \qquad P_1 = 191$$

$$f_1(197) = 290 \qquad P_1 = 192$$

. . . . .

. . . . .

$$f_1(208) = 510 \qquad P_1 = 203$$

$$f_1(209) = 530 \qquad P_1 = 204$$

$$f_1(210) = 550 \qquad P_1 = 205$$

$$f_1(211) = 570 \qquad P_1 = 206$$

. . . . .

. . . . .

$$f_1(220) = 750 \qquad P_1 = 115$$

We now continue to work on the second stage:

When $P_3 = 220$

$$f_2(220) = \min \begin{cases} \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ P_2 = 208 & 2(208-220)^2 + 20(208-195) + f_1(208) = 1058 \\ P_2 = 209 & 2(209-220)^2 + 20(209-195) + f_1(209) = 1052 \\ P_2 = 210 & 2(210-220)^2 + 20(210-195) + f_1(210) = 1050 \leftarrow \\ P_2 = 211 & 2(211-220)^2 + 20(211-195) + f_1(211) = 1052 \\ P_2 = 212 & 2(212-220)^2 + 20(212-195) + f_1(212) = 1058 \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

So, $f_2(220) = 1050$ where $P_2 = 210$

and $f_2(221) = 1090$ $\quad P_2 = 211$

$f_2(222) = 1130$ $\quad P_2 = 212$

. . . . . .

. . . . . .

In the third stage:

$f_3(210) = 1250$ $\quad P_3 = 220$

$f_3(211) = 1212$ $\quad P_3 = 220$

$f_3(212) = 1178$ $\quad P_3 = 220$

. . . . . .

. . . . . .

In the fourth stage, since the production level in December was 200. So we consider only $f_4(200)$. It is found that

$$f_4(200) = 1450 \qquad P_4 = 210$$

From the results we have for each stage, we conclude that the optimal production level for January (4th stage) is 210.

Tracing back to the third stage, the policy for the production level in this stage when the production level in the fourth stage is 210 is

$$f_3(210) = 1250 \qquad P_3 = 220$$

So the optimal production level for February (3rd stage) is 220.

Since $$f_2(220) = 1050 \qquad P_2 = 210$$

So production level for March (2nd stage) is 210. From

$$f_1(210) = 550 \qquad P_1 = 205$$

the production level for April (1st stage) is 205. List the results as follows.

Month	Jan.	Feb.	March	Apr.
Demand	210	220	195	180
Optimal Production Level	210	220	210	205

Total cost = $f_4(200) = 1450$

A FORTRAN II program for solving this problem is shown in the Appendix.

The Case of Stochastic Demand

Assumptions

(a) The demand for the product in any time interval is treated as a random variable and such random variables in different periods are independent.

(b) The inventory cost is charged at the beginning of each period and is proportional to that inventory level.

Formulation of the Problem

Since the inventory is charged at the beginning of each period, the inventory cost for the period t is a function of the closing inventory level of period $t - 1$, that is

$$h_t(I_{t-1}), \quad \text{for } I_{t-1} \geq 0$$

The demand during each period is random variable and has the probability density function of

$$\phi_t(D_t), \quad \text{for } D_t \geq 0$$

Therefore, the shortage cost in period t is the function of the value of the integration

$$\int_{I_{t-1}+P_t}^{\infty} [D_t-(I_{t-1}+P_t)]\phi_t(D_t)dD_t \tag{4.4}$$

and is denoted by

$$s_t[\text{Eq. (4.4)}]$$

The overtime cost, hiring and layoff cost and regular payroll cost can be represented by the function as we defined in the section beginning on page 29, which is

$$g_t(P_t, P_{t-1})$$

So the cost function is

$$C(P_1, P_2, \ldots, P_n) = \sum_{t=1}^{n} \{h_t(I_{t-1}) + s_t[\int_{I_{t-1}+P_t}^{\infty} (D_t - (I_{t-1}+P_t))\phi_t(D_t)dD_t]$$

$$+ g_t(P_t, P_{t-1})\} \qquad (4.5)$$

where the initial conditions I_0, P_0 are known, and has the following restrictions:

$$I_t = I_{t-1} + P_t - D_t \qquad (4.6)$$

$$P_t \geq 0 \qquad (4.7)$$

$$\text{for } t = 1, 2, \ldots, n$$

Construction of the Recurrence Relation

Again, we use the backward induction. Let $f_i(P_{i+1}, I_{i+1})$ be defined as for the provious case, the cost is taken by expected value in this case, then the recurrence relations take the form:

$$f_i(P_{i+1}, I_{i+1}) = \min_{P_i} \{g_i(P_i, P_{i+1}) + h_i(I_{i+1})$$

$$+ s_i[\int_{I_{i+1}+P_i}^{\infty} [D_i - (I_{i+1}+P_i)]\phi_i(D_i)dD_i]$$

$$+ \int_0^{I_{i+1}+P_i} [f_{i-1}(P_i, I_{i+1}+P_i-D_i)\phi_i(D_i)dD_i]$$

$$+ f_{i-1}(P_i, 0)\int_{I_{i+1}+P_i}^{\infty} \phi_i(D_i)dD_i\} \tag{4.8}$$

This is a two state parameters dynamic programming problem. In its solutions, the optimal value for the number of product in first period $(t = 1)$ will be determined. Since the demands are probabilistic, the optimal production level of each production period from production periods two to n will be a set of decision policy, with the production and inventory in the previous period, the optimal production quantity can be found from the policy.

<u>Example</u>

Demand distributions for February, March, and April are as follows:

	Probability		
Demand	February	March	April
0	1/4	0	2/3
1	1/2	1/4	1/3
2	1/4	1/2	0
3	0	1/4	0

Production cost for producing 0, 1, 2 units in each period is

Production rate	0	1	2
Cost	15, 000	20, 000	35, 000

Inventory cost is charged according to the closing inventory level at the end of each month (except April) as following:

Inventory level	0	1	2	3
Cost	2, 000	5, 000	9, 000	15, 000

End of April inventory cost.

Inventory level	0	1	2	3
Cost	10, 000	0	5, 000	10, 000

Shortage cost = $10, 000/unit. Shortage is not carried over. If demand is not met, the customer goes to another place. The inventory level at the end of January is 1.

Find the optimal policy.

Solution:

Let April be the first stage, March the second, and February the third stage.

Charge the inventory cost at the beginning of each stage according to the beginning inventory level. This will cause no difference with charging at the end of each stage. The cost elements for each stage are:

$$\text{Production cost: } g_i(P_i) = \begin{cases} 15, & \text{for } P_i = 0, \\ 20, & P_i = 1, \\ 35, & P_i = 2, \\ \infty, & \text{otherwise.} \end{cases}$$

$$\text{Inventory cost: } h_i(I_{i+1}) = \begin{cases} 2, & \text{for } I_{i+1} = 0, \\ 5, & I_{i+1} = 1, \\ 9, & I_{i+1} = 2, \\ 15, & I_{i+1} = 3, \\ \infty, & \text{otherwise.} \end{cases}$$

$$\text{Shortage cost: } s_i(D_i - I_{i+1} - P_i) = \begin{cases} 10(D_i - I_{i+1} - P_i), \\ \text{for } (D_i - I_{i+1} - P_i) > 0, \\ 0, \text{ otherwise.} \end{cases}$$

Let $f_n(I_{n+1})$ = Expected cost of optimal policy for the last n stages where I_{n+1} units are in inventory at the beginning of the stage.

So

$$f_n(I_{n+1}) = \min_{P_n = 0,1,2.} E \{g_n(P_n) + h_n(I_{n+1}) + s_n(D_n - I_{n+1} - P_n)$$

$$+ f_{n-1}(I_{n+1} + P_n - D_n)\}$$

$$= \min_{P_n = 0,1,2.} \{g_n(P_n) + h_n(I_{n+1})$$

$$\sum_{D_n} \phi_n(D_n)[s_n(D_n - I_{n+1} - P_n) + f_{n-1}(I_{n+1} + P_n - D_n)]\}$$

Therefore,

$$f_0(0) = 10. \quad f_0(1) = 0. \quad f_0(2) = 5. \quad f_0(3) = 10.$$

$$f_1(0) = \min \begin{cases} P_1 = 0, & 15 + 2 + \frac{2}{3}(0+10) + \frac{1}{3}(10+10) = 30.33 \\ P_1 = 1, & 20 + 2 + \frac{2}{3}(0+0) + \frac{1}{3}(0+10) = 25.33 \leftarrow \\ P_1 = 2, & 35 + 2 + \frac{2}{3}(0+5) + \frac{1}{3}(0+0) = 40.33 \end{cases}$$

From which, we have

$$f_1(0) = 25.33, \quad P_1 = 1.$$

Accordingly,

$$f_1(1) = 23.33, \quad P_1 = 1.$$

$$f_1(2) = 27.33, \quad P_1 = 0.$$

$$f_1(3) = 38.33, \quad P_1 = 0.$$

In the second stage

$$f_2(0) = 57.33, \quad P_2 = 1.$$

$$f_2(1) = 52.33, \quad P_2 = 1.$$

$$f_2(2) = 51.33, \quad P_2 = 0.$$

$$f_2(3) = 54.83, \quad P_2 = 0.$$

In the third stage, since the beginning inventory level in the fourth stage is 1. Hence,

$$f_3(1) = 78.33, \quad P_3 = 1.$$

The policy is:

-- produce 1 unit in February.

-- if the inventory levels at the end of February are

0 unit then produce 1 unit in March.

1 unit then produce 1 unit in March.

-- if the inventory levels at the end of March are

0 unit then produce 1 unit in April.

1 unit then produce 1 unit in April.

2 units then produce 0 unit in April.

Total expected cost = 78.33.

A FORTRAN II computer program input data and results are listed in the Appendix.

The Case of Considering the Work Force

Assumptions

(a) The quantity to be produced does not uniquely determine the work force. Therefore, the work force level at each period becomes a decision variable.

(b) Unfilled orders can be carried over into next period. Therefore, negative inventory occurs.

(c) Stochastic demand.

Cost Function

The inventory cost and shortage cost can be represented as

$$B_t(I_{t-1})$$

if this cost is charged at the beginning of each time period. I_{t-1} has no restriction in sign. Overtime cost is a function of labor force which deviates from the size established as ideal for an output P_t, represented by

$$E_t(W_t, P_t)$$

Regular payroll cost: $H_t(W_t)$

Hiring and layoff cost: $F_t(W_t, W_{t-1})$

The total cost function is then

$$C(P_1, P_2, \ldots, P_n; W_1, W_2, \ldots, W_n)$$

$$= \sum_{t=1}^{n} \{B_t(I_{t-1}) + E_t(W_t, P_t) + H_t(W_t) + F_t(W_t, W_{t-1})\} \tag{4.9}$$

Recurrence Relation

Use backward induction. Set

$f_i(W_{i+1}, I_{i+1}) =$ cost of optimal policy for the last i stages where W_{i+1} is the size of work force in the $(i+1)$th stage and I_{i+1} units are carried over into the ith stage in inventory.

Then, in this case

$$f_i(W_{i+1}, I_{i+1}) = \min_{P_t, W_t} \{B_i(I_{i+1}) + E_i(W_i, P_i) + F_i(W_i, W_{i+1}) + H_t(W_t)$$

$$+ \int_0^\infty f_{i-1}(W_i, I_{i+1} + P_i - D_i)\phi_i(D_i)dD_i\}$$

$$\text{for } i = 1, \ldots, n \tag{4.10}$$

Because of the necessity of minimizing over two variables, P_t

and W_t, and at each stage, the numerical solution of this problem would require much more time than in the previous two cases. However, it could be handled on a large-scale digital computer.

Summary

The application of dynamic programming methods to our scheduling problem has been studied and classified into three major cases. The main contribution of this method is its applicability to any type of cost structure. But from the examples we can find that the method can be evaluated only in the discrete case and without the computer as a tool, it is nearly impossible to obtain the solution when the cost structures are complex and the planning horizon spans a larger number of time periods. Owing to the rapid development of computer science, this approach is gradually being accepted by management.

QUADRATIC PROGRAMMING METHOD

Overview of the Chapter

From the cost analysis in the Linear Programming Method Chapter, we know the components of the production cost for each period can be approximated by quadratic forms. If we assume the cost elements for period t are

Regular payroll $\quad C_1W_t + C_{13}$ (5.1)

Hiring and layoff $\quad C_2(W_t-W_{t-1}-C_{11})^2$ (5.2)

Overtime $\quad C_3(P_t-C_4W_t)^2 + C_5P_t - C_6W_t + C_{12}P_tW_t$ (5.3)

Inventory and shortage $\quad C_7(I_t-C_8-C_9D_t)^2$ (5.4)

where C's are constants, the initial work force W_0, inventory level I_0, and D_t's are known and the inventory level I_t obeys the rule

$$I_t = I_{t-1} + P_t - D_t, \quad t = 1, 2, \ldots, n \tag{5.5}$$

then, according to Holt, Modinglian, Muth and Simon (18), the solution which minimize the cost function may reach an optimal decision, P_t, W_t, for $t = 1, 2, \ldots, n$, in particular, P_1, W_1 are of the form

$$P_1 = \sum_{t=1}^{n} a_t D_t + bW_0 + cI_0 + d$$

$$W_1 = \sum_{t=1}^{n} a'_t D_t + b'W_0 + c'I_0 + d'$$

The decision rules for P_1, W_1 are referred to as linear decision rules.

The linear decision rules can be applied only for the first period. Since the model is designed for the optimization over a shifting horizon, we are only interested in the quantities P_1 and W_1. P_2 and W_2 can be solved by next n periods from which we regard P_2, W_2 as the decision variables for the starting period. So the linear decision rules are all that is needed for sequential decision making.

This chapter will mainly treat the mathematical derivation of the decision rule. The conditions for minimum total cost may be found by taking derivatives with respect to the decision variables. Since the cost function is quadratic, the first derivative yields a set of linear equations. The section beginning on page 65 gives a step-by-step computational procedure for obtaining these decision rules by hand calculations. Alternatively an electronic computer will reduce the computation time to a few minutes. A computer program written in FORTRAN II for solving the example problem is given in the Appendix. Page 69 gives a simple proof of certainty equivalence from which we know that if future demands are uncertain, we merely insert the expected future demand replacing the actual demands, that are used in the case of certainty also yield the best decision in the presence of uncertain future conditions. No new analysis or calculations are needed.

The Derivation of the Decision Rules

Define Problem

From Equations (5.1) through (5.5), we define the problem as:

minimize

$$C(P_1, P_2, \ldots, P_n; W_1, W_2, \ldots, W_n)$$

$$= \sum_{t=1}^{n} [(C_1-C_6)W_t+C_2(W_t-W_{t-1}-C_{11})^2+C_3(P_t-C_4W_t)^2+C_5P_t$$

$$+C_{12}P_tW_t+C_7(I_t-C_8-C_9D_t)^2+C_{13}] \qquad (5.6)$$

subject to

$$P_t-D_t = I_t - I_{t-1} \quad \text{for } t = 1, 2, \ldots, n$$

Establish the Relationship Between the Optimal Production Rate and the Optimal Work Force

(a) Partially differentiating the cost function Equation (5.6) with respect to W_t for $t = 1, 2, \ldots, n-1$.

$$\frac{\partial C}{\partial W_t} = \frac{\partial}{\partial W_t}[(C_1-C_6)W_t+C_2(W_t-W_{t-1}-C_{11})^2+C_3(P_t-C_4W_t)^2+C_5P_t$$

$$+C_{12}P_tW_t+C_7(I_t-C_8-C_9D_t)^2+C_{13}+(C_1-C_6)W_{t+1}$$

$$+C_2(W_{t+1}-W_t-C_{11})^2+C_3(P_{t+1}+C_4W_{t+1})^2+C_5P_{t+1}+C_{12}P_tW_{t+1}$$

$$+C_7(I_{t+1}-C_8-C_9D_{t+1})^2+C_{13}$$

$$+ \text{ the cost in the periods other than } t \text{ and } t+1]$$

$$= (C_1-C_6) + 2C_2(W_t-W_{t-1}-C_{11}) - 2C_3C_4(P_t-C_4W_t)$$

$$+ C_{12}P_t - 2C_2(W_{t+1}-W_t-C_{11})$$

(b) Set $\frac{\partial C}{\partial W_t} = 0$ solve for P_t

$$(C_1-C_6) + 2C_2(W_t-W_{t-1}-C_{11}) - 2C_3C_4P_t + 2C_3C_4^2W_t$$

$$+ C_{12}P_t - 2C_2(W_{t+1}-W_t-C_{11}) = 0$$

$$P_t(2C_3C_4-C_{12}) = (C_1-C_6) - 2C_2(W_{t-1}+W_{t+1}) + (2C_3C_4^2+4C_2)W_t$$

$$\therefore P_t = \frac{1}{2C_3C_4-C_{12}}[(C_1-C_6)-2C_2(W_{t-1}+W_{t+1})+(2C_3C_4^2+4C_2)W_t]$$

If we let

$$K_1 = \frac{C_1 - C_6}{2C_3C_4 - C_{12}}, \quad K_2 = \frac{-2C_2}{2C_3C_4 - C_{12}}, \quad K_3 = \frac{2C_3C_4 + 4C_2}{2C_3C_4 - C_{12}}$$

Then

$$P_t = K_1 + K_2(W_{t-1} + W_{t+1}) + K_3W_t$$

$$\text{for } t = 1, 2, \ldots, n-1 \qquad (5.7)$$

The production rate of each period is a linear function of the size of the work force in the same and its adjacent periods. So if we know the work force decisions, we could readily determine the production decisions.

<u>The Determination of Work Force Decisions</u>

(a) Since

$$I_t = I_0 + \sum_{i=1}^{t} P_i - \sum_{i=1}^{t} D_i,$$

I_t is a variable depending on the cumulative production of all previous periods. If we take partial derivatives of total cost, C, with respect to P_t, we obtain a very complicated expression, since I_t exists in the cost function. If we consider I_t as a second set of decision variable, the production rate for each period would then be uniquely determined tnrough the constraint Equation (5.5).

$$P_t = I_t - I_{t-1} + D_t$$

(b) Substituting P_t in terms of $I_t - I_{t-1} + D_t$ into Equation (5.6).

$$C = \sum_{t=1}^{n} [(C_1 - C_6)W_t + C_2(W_t - W_{t-1} - C_{11})^2 + C_3(I_t - I_{t-1} + D_t - C_4 W_t)^2$$
$$+ C_5(I_t - I_{t-1} + D_t) + C_{12}(I_t - I_{t-1} + D_t)W_t + C_7(I_t - C_8 - C_9 D_t)^2 + C_{13}] \qquad (5.8)$$

(c) Partially differentiating Equation (5.8) with respect to I_t for $t = 1, 2, \ldots, n-1$.

$$\frac{\partial C}{\partial I_t} = \frac{\partial}{\partial I_t}[C_3(I_t - I_{t-1} + D_t - C_4 W_t)^2 + C_5(I_t - I_{t-1} + D_t) + C_{12}(I_t - I_{t-1} + D_t)W_t$$
$$+ C_7(I_t - C_8 - C_9 D_t)^2 + C_3(I_{t+1} - I_t + D_{t+1} - C_4 W_{t+1})^2 + C_5(I_{t+1} - I_t + D_{t+1})$$
$$+ C_{12}(I_{t+1} - I_t + D_{t+1})W_{t+1} + C_7(I_{t+1} - C_8 - C_9 D_{t+1})^2]$$
$$= 2C_3(I_t - I_{t-1} + D_t - C_4 W_t) + C_5 + C_{12} W_t + 2C_7(I_t - C_8 - C_9 D_t)$$
$$- 2C_3(I_{t+1} - I_t + D_{t+1} - C_4 W_{t+1}) - C_5 - C_{12} W_{t+1} \qquad (5.9)$$

substituting $I_t - I_{t-1} + D_t$ in terms of P_t into Equation (5.9) we have

$$\frac{\partial C}{\partial I_t} = 2C_3(P_t - C_4 W_t) + 2C_7(I_t - C_8 - C_9 D_t) - 2C_3(P_{t+1} - C_4 W_{t+1})$$
$$- C_{12}(W_{t+1} - W_t) \qquad (5.10)$$

(d) Set $\frac{\partial C}{\partial I_t} = 0$, solve for I_t.

$$2C_7 I_t = 2C_3(P_{t+1}-P_t) + (C_{12}-2C_3C_4)(W_{t+1}-W_t) + 2C_7C_9D_t + 2C_7C_8$$

$$I_t = \frac{C_3}{C_7}(P_{t+1}-P_t) + \frac{C_{12}-2C_3C_4}{2C_7}(W_{t+1}-W_t) + C_9D_t + C_8$$

If we let

$$K_4 = \frac{C_3}{C_7}, \quad K_5 = \frac{C_{12}-2C_3C_4}{2C_7}, \quad K_6 = C_9, \quad K_7 = C_8$$

Then

$$I_t = K_4(P_{t+1}-P_t) + K_5(W_{t+1}-W_t) + K_6D_t + K_7$$

$$\text{for } t = 1, 2, \ldots, n-1 \tag{5.11}$$

(e) Since $P_t - D_t = I_t - I_{t-1}$

$$P_1 - D_1 = I_1 - I_0 = K_4(P_2-P_1) + K_5(W_2-W_1) + K_6D_1 + K_7 - I_0 \tag{5.12}$$

$$\begin{aligned} P_t - D_t = I_t - I_{t-1} &= K_4(P_{t+1}-P_t) + K_5(W_{t+1}-W_t) + K_6D_t + K_7 \\ &\quad - K_4(P_t-P_{t-1}) - K_5(W_t-W_{t-1}) - K_6D_{t-1} - K_7 \\ &= K_4(P_{t+1}-2P_t+P_{t-1}) + K_5(W_{t+1}-2W_t+W_{t-1}) \\ &\quad + K_6(D_t-D_{t-1}) \end{aligned}$$

$$\text{for } t = 2, 3, \ldots, n-1. \tag{5.13}$$

Using the relation between production and the size of work force in Equation (5.7), we can eliminate production from the above equations as following.

(f) From Equation (5.7) we have

$$P_1 = K_1 + K_2(W_0+W_2) + K_3W_1 \tag{5.14}$$

$$P_2-P_1 = K_1 + K_2(W_1+W_3) + K_3W_2-K_1-K_2(W_0+W_2) - K_3W_1$$

$$= -K_2W_0 + (K_2-K_3)W_1 - (K_2-K_3)W_2 + K_2W_3 \tag{5.15}$$

$$P_t-P_{t-1} = -K_2W_{t-2}+(K_2-K_3)W_{t-1}-(K_2-K_3)W_t+K_2W_{t+1} \tag{5.16}$$

$$P_{t+1}-2P_t+P_{t-1} = K_2W_{t-2}-(2K_2-K_3)W_{t-1} + (2K_2-2K_3)W_t$$

$$- (2K_2-K_3)W_{t+1} + K_2W_{t+2} \tag{5.17}$$

Substituting Equations (5.14) and (5.15) into Equation (5.12)

$$K_1+K_2W_0+K_3W_1+K_2W_2-D_1$$

$$= K_4[-K_2W_0+(K_2-K_3)W_1-(K_2-K_3)W_2+K_2W_3] + K_5(W_2-W_1)+K_6D_1+K_7-I_0$$

$$K_1+K_2W_0+K_3W_1+K_2W_2-D_1$$

$$= -K_2K_4W_0+K_4(K_2-K_3)W_1-K_4(K_2-K_3)W_2+K_2K_4W_3+K_5W_2-K_5W_1+K_6D_1+K_7-I_0$$

$$\therefore\ (K_3-K_2K_4+K_3K_4+K_5)W_1 + (K_2+K_2K_4-K_3K_4-K_5)W_2 - K_2K_4W_3$$

$$= (1+K_6)D_1 - (K_2+K_2K_4)W_0 + (K_7-K_1) - I_0 \tag{5.18}$$

Substitute Equations (5.16), (5.17) into Equation (5.13)

$$K_1 + K_2W_{t-1} + K_3W_t + K_2W_{t+1} - D_t$$

$$= K_4[K_2W_{t-2}-(2K_2-K_3)W_{t-1}+(2K_2-2K_3)W_t-(2K_2-K_3)W_{t+1}+K_2W_{t+2}]$$

$$+ K_5(W_{t+1}-2W_t+W_{t-1}) + K_6(D_t-D_{t-1})$$

$$\therefore\ (-K_2K_4)W_{t-2}+(K_2+2K_2K_4-K_3K_4-K_5)W_{t-1}+(K_3-2K_2K_4+2K_3K_4+2K_5)W_t$$

$$+ (K_2+2K_2K_4-K_3K_4-K_5)W_{t+1}+(-K_2K_4)W_{t+2}$$

$$= -K_6D_{t-1} + (1+K_6)D_t - K_1$$

$$\text{for } t = 2, 3, \ldots, n-1 \tag{5.19}$$

Equations (5.18) and (5.19) are a set of simultaneous linear equations with unknowns $W_1, W_2, \ldots, W_{n+1}$. Since this system of equations has $n + 1$ unknowns in $n - 1$ equations, we can remedy this deficiency by applying terminal condition and writing two more equations. Then the unknown employment levels for the various periods will be solved. Another method for solving this system of linear equations is stated in the following paragraph.

The Solution of Optimal Work Force Level by Applying the Infinite Set of Linear Simultaneous Equations

(a) Original system of equations represented in matrix form. From Equations (5.18) and (5.19), if we let

$$K_3 - K_2K_4 + K_3K_4 + K_5 = m_4$$

$$-(K_2+K_2K_4-K_3K_4-K_5) = m_5$$

$$-K_2K_4 = m_1$$

$$-(K_2+2K_2K_4-K_3K_4-K_5) = m_2$$

$$K_3 - 2K_2K_4 + 2K_3K_4 + 2K_5 = m_3$$

Equation (5.18) becomes:

$$m_4W_1 - m_5W_2 + m_1W_3 = (1+K_6)D_1 - (K_2+K_2K_4)W_0 + (K_7-K_1)-I_0 \tag{5.20}$$

Equation (5.19) becomes:

$$m_1W_{t-2}-m_2W_{t-1}+m_3W_t-m_2W_{t+1}+m_1W_{t+2} = -K_6D_{t-1}+(1+K_6)D_t-K_1$$

when t = 2,

$$m_1W_0-m_2W_1+m_3W_2-m_2W_3+m_1W_4 = -K_6D_1 + (1+K_6)D_2 - K_1$$

$$-m_2W_1+m_3W_2-m_2W_3+m_1W_4 = -K_6D_1+(1+K_6)D_2+K_2K_4W_0-K_1 \tag{5.21}$$

when t = 3, 4, . . . , n - 1,

$$m_1W_{t-2}-m_2W_{t-1}+m_3W_t-m_2W_{t+1}+m_1W_{t+2} = -K_6D_{t-1}+(1+K_6)D_t-K_1 \tag{5.22}$$

The set of linear equations composed of Equations (5.20), (5.21), and (5.22) can be written into the following matrix form.

$$
\begin{bmatrix}
m_4 & -m_5 & m_1 & & & & & & & & & 0 \\
-m_2 & m_3 & -m_2 & m_1 & & & & & & & & \\
m_1 & -m_2 & m_3 & -m_2 & m_1 & & & & & & & \\
 & m_1 & -m_2 & m_3 & -m_2 & m_1 & & & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & & & & & & \\
 & & & \cdot & \cdot & \cdot & \cdot & & & & & \\
 & & & & m_1 & -m_2 & m_3 & -m_2 & m_1 & & & \\
 & & & & & \cdot & \cdot & \cdot & \cdot & & & \\
 & & 0 & & & & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & & & & m_1 & -m_2 & m_3 & -m_2 & m_1
\end{bmatrix}
\begin{bmatrix}
W_1 \\ W_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ W_n \\ W_{n+1}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(1+K_6)D_1 - (K_2+K_2K_4)W_0 + (K_7-K_1) - I_0 \\
-K_6D_1 + (1+K_6)D_2 + K_2K_4W_0 - K_1 \\
-K_6D_2 + (1+K_6)D_3 - K_1 \\
\cdot \\
\cdot \\
\cdot \\
-K_6D_{t-1} + (1+K_6)D_t - K_1 \\
\cdot \\
\cdot \\
\cdot \\
-K_6D_{n-2} + (1+K_6)D_{n-1} - K_1
\end{bmatrix}
\qquad (5.23)
$$

This matrix equation has $n + 1$ unknowns in $n - 1$ equations. It has no unique solution to the unknown $W_t^{'}s$.

(b) Construction of the infinite set of linear equations: Let n approach infinity so that the terminal conditions have a negligible influence on the employment (and production) of the first few periods, then an approximate solution to Equation (5. 23) can be derived.

(c) Solution of the infinite set of linear equations: Multiply each equation by the expression λ^{t-1}, where λ is a variable and t indicates the equation. Thus the first equation is multiplied by unity (λ^0), the second is multiplied by λ, the third by λ^2, and so on. Adding the resulting system of equation, we obtain a single equation.

$$m_4W_1-m_5W_2+m_1W_3+\lambda(-m_2W_1+m_3W_2-m_2W_3+m_1W_4)$$

$$+\sum_{t=3}^{\infty}\lambda^{t-1}(m_1W_{t-2}-m_2W_{t-1}+m_3W_t-m_2W_{t+1}+m_1W_{t+2})$$

$$=(1+K_6)D_1-(K_2+K_2K_4)W_0+(K_7-K_1)-I_0$$

$$+\lambda[-K_6D_1+(1+K_6)D_2+K_2K_4W_0-K_1]$$

$$+\sum_{t=3}^{\infty}\lambda^{t-1}[-K_6D_{t-1}+(1+K_6)D_t-K_1] \qquad (5.24)$$

The left hand side of Equation (5.24) may be revised as,

$$m_4W_1 - m_5W_2 + m_2W_2 - \lambda^{-1}m_1W_2 - m_3W_1 + \lambda^{-1}m_2W_1 - \lambda^{-2}m_1W_1$$

$$+ \lambda^{-2}m_1W_1$$

$$- \lambda^{-1}m_2W_1 + \lambda^{-1}m_1W_2$$

$$+ m_3W_1 - m_2W_2 + m_1W_3$$

$$- \lambda m_2W_1 + \lambda m_3W_2 - \lambda m_2W_3 + \lambda m_1W_4$$

$$+ \lambda^2m_1W_1 - \lambda^2m_2W_2 + \lambda^2m_3W_3 - \lambda^2m_2W_4 + \lambda^2m_1W_5$$

$$+ \lambda^3m_1W_2 - \lambda^3m_2W_3 + \lambda^3m_3W_4 - \lambda^3m_2W_5 + \lambda^3m_1W_6$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= (\lambda^{-2}m_1 - \lambda^{-1}m_2 + m_3 - \lambda m_2 + \lambda^2 m_1) \sum_{t=1}^{\infty} \lambda^{t-1}W_t$$

$$+ W_1[(m_4 - m_3) + \lambda^{-1}m_2 - \lambda^{-2}m_1] - W_2[(m_5 - m_2) - \lambda^{-1}m_1]$$

The right hand side of Equation (5.24) may be revised as,

$$
\begin{aligned}
&(1+K_6)D_1 - (K_2+K_2K_4)W_0 + K_7 - K_1 - I_0 \\
&- \lambda K_6 D_1 + \lambda(1+K_6)D_2 + \lambda K_2K_4W_0 - \lambda K_1 \\
&- \lambda^2 K_6 D_2 + \lambda^2(1+K_6)D_3 \qquad\qquad - \lambda^2 K_1 \\
&- \lambda^3 K_6 D_3 + \lambda^3(1+K_6)D_4 \qquad\qquad - \lambda^3 K_1 \\
&\quad \cdot \qquad \cdot \qquad\qquad \cdot \qquad \cdot \\
&\quad \cdot \qquad \cdot \qquad\qquad \cdot \qquad \cdot
\end{aligned}
$$

$$
= (1+K_6)D_1 - (K_2+K_2K_4)W_0 + \lambda K_2K_4W_0 + K_7 - I_0
$$

$$
+ \sum_{t=2}^{\infty} \lambda^{t-1}[-K_6 D_{t-1} + (1+K_6)D_t] - K_1 \sum_{t=0}^{\infty} \lambda^t
$$

Since $\sum_{t=0}^{\infty} \lambda^t = \frac{1}{1-\lambda}$ for $0 < |\lambda| < 1$

So under the assumption that $0 < |\lambda| < 1$, the above equation becomes

$$- W_0[K_2+K_2K_4(1-\lambda)] + K_7 - I_0 - \frac{K_1}{1-\lambda}$$

$$+ \quad D_1 + \quad K_6D_1$$

$$- \lambda K_6D_1 + \lambda D_2 + \lambda K_6D_2$$

$$- \lambda^2K_6D_2 + \lambda^2D_3 + \lambda^2K_6D_3$$

$$- \lambda^3K_6D_3 + \lambda^3D_4 + \lambda^3K_6D_4$$

$$\cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot$$

$$= - W_0[K_2+K_2K_4(1-\lambda)] + K_7 - I_0 - \frac{K_1}{1-\lambda} + \sum_{t=1}^{\infty} D_t\lambda^{t-1}(1+K_6-\lambda K_6)$$

$$= - W_0[K_2+K_2K_4(1-\lambda)] + K_7 - I_0 - \frac{K_1}{1-\lambda} + \sum_{t=1}^{\infty} D_t\lambda^{t-1}[1+K_6(1-\lambda)]$$

The entire equation of Equation (5.24) now is

$$(m_1\lambda^{-2}-m_2\lambda^{-1}+m_3-M_2\lambda+m_1\lambda^2)\sum_{t=1}^{\infty}\lambda^{t-1}W_t + W_1[(m_4-m_3)+\lambda^{-1}m_2-\lambda^{-2}m_1]$$

$$- W_2[(m_5-m_2)-\lambda^{-1}m_1]$$

$$= \sum_{t=1}^{\infty}\lambda^{t-1}D_t[1+K_6(1-\lambda)] - W_0[K_2+K_2K_4(1-\lambda)] + K_7 - I_0 - \frac{K_1}{1-\lambda} \qquad (5.25)$$

Equation (5.25) holds for $0 < |\lambda| < 1$ (5.26)

In particular, we can choose values of λ which satisfy Equation (5.25) and cause

$$m_1\lambda^{-2} - m_2\lambda^{-1} + m_3 - m_2\lambda + m_1\lambda^2 = 0 \qquad (5.27)$$

Since Equation (5.27) is symmetric, if we find a solution of λ not zero or unity, we can find a solution $1/\lambda$ that also satisfies the equation. Therefore, we know that there is at least a value of λ which satisfies Equation (5.26) and Equation (5.27).

Next, we will show that there are two and only two values of λ (say λ_1 and λ_2) which satisfy Equations (5.26) and (5.27).

> λ_1 and λ_2 are the roots of Equation (5.27), which is symmetric. Hence $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ are also roots. Suppose $|\lambda_1| = 1$, then $m_1 + \frac{m_3}{2} \pm m_2 = 0$. But this is not the case. Therefore, $|\lambda_1|, |\lambda_2| \gtrless 1$ and $|\frac{1}{\lambda_1}|, |\frac{1}{\lambda_2}| \lessgtr 1$, this implies that there are only two roots in Equation (5.27), which satisfy Equation (5.26).

Having obtained λ_1, λ_2 from Equations (5.26) and (5.27), we then substitute these values into Equation (5.25) and have

$$W_1[(m_4-m_3)+\lambda_i^{-1}m_2-\lambda_i^{-2}m_1] = W_2[(m_5-m_2)-\lambda_i^{-1}m_1]$$

$$= \sum_{t=1}^{\infty} \lambda_i^{t-1} D_t[1+K_6(1-\lambda_i)] - W_0[K_2+K_2K_4(1-\lambda_i)] + K_7 - I_0 - \frac{K_1}{1-\lambda_i} \quad (5.28)$$

for i = 1, 2,

Since λ_1 and λ_2 are the roots of Equation (5.27),

$$m_1\lambda_i^{-2} - m_2\lambda_i^{-1} + m_3 - m_2\lambda_i + m_1\lambda_i^2 = 0$$

or

$$-(m_1\lambda_i^{-2} - m_2\lambda_i^{-1} + m_3) = - m_2\lambda_i + m_1\lambda_i^2$$

for i = 1, 2.

So the coefficient of the first term in Equation (5.28) is

$$m_4 - (m_3-\lambda_i^{-1}m_2+\lambda_i^{-2}m_1) = m_4 - m_2\lambda_i + m_1\lambda_i^2, \quad i = 1, 2,$$

Equation (5.28) becomes:

$$W_1(m_4-m_2\lambda_i+m_1\lambda_i^2) - W_2[(m_5-m_2) - \lambda_i^{-1}m_1]$$

$$= \sum_{t=1}^{\infty} \lambda_i^{t-1} D_t[1+K_6(1-\lambda_i)] - W_0[K_2+K_2K_4(1-\lambda_i)] + K_7 - I_0 - \frac{K_1}{1-\lambda_i} \quad (5.29)$$

for i = 1, 2.

(d) The decision rules for the work force level and the production level: Equation (5.29) has two equations in two unknowns W_1 and W_2 and we can solve such a system of linear equations then obtain the decision rules for W_1 and W_2. In particular, W_1, W_2 will be in the form

$$W_1 = \sum_{t=1}^{\infty} [a'_{t1}\lambda_1^{t-1} + a'_{t2}\lambda_2^{t-1}] D_t + b'W_0 + c'I_0 + d'$$

$$\doteq \sum_{t=1}^{n} a'_t D_t + b'W_0 + c'I_0 + d'.$$

$$W_2 \doteq \sum_{t=1}^{n} a''_t D_t + b''W_0 + c''I_0 + d''.$$

Since from Equation (5.14)

$$P_1 = K_1 + K_2 W_0 + K_3 W_1 + K_2 W_2$$

it is obvious that

$$P_1 \doteq \sum_{t=1}^{n} a_t D_t + bW_0 + cI_0 + d.$$

where

$$a_t = K_3 a'_t + K_2 a''_t, \quad \text{for } t = 1, 2, \ldots, n,$$

$$b = K_3 b' + K_2 b'' + K_2,$$

$$c = K_3 c' + K'_2 c'',$$

$$d = K_3 d' + K_2 d'' + K_1.$$

Example

The components of the cost function of a single item to be produced are as follows.

Regular payroll	$350\,W_t + 3,500$
Hiring and layoff	$67\,(W_t - W_{t-1})^2$
Overtime	$0.15(P_t - 4.57W_t)^2 + 49P_t - 285W_t$
Inventory connected cost	$0.15(I_t - 325)^2$

The demands from period 1 to n are known as

$$D_1, D_2, \ldots, D_n$$

Derive the decision rules to determine the production and work force levels in the first period.

Solution:

Step 1. List of the cost data:[14]

$C_1 = 350.$	$C_5 = 49.$	$C_9 = 0.$
$C_2 = 67.$	$C_6 = 285.$	$C_{11} = 0.$
$C_3 = 0.15.$	$C_7 = 0.15.$	$C_{12} = 0.$
$C_4 = 4.57.$	$C_8 = 325.$	

[14] Since C_{13}=3500 is irrelevant to the decision analysis, it has been dropped.

Step 2. Evaluating the derived coefficients which are introduced in the sections beginning on pages 50, 51, and 55:

$$K_1 = \frac{C_1 - C_6}{2C_3C_4 - C_{12}} = \frac{350-285}{(2)(0.15)(4.57)} = 47.410$$

$$K_2 = \frac{-2C_2}{2C_3C_4 - C_{12}} = \frac{-(2)(67)}{(2)(0.15)(4.57)} = -97.738$$

$$K_3 = \frac{2C_3C_4^2 + 4C_2}{2C_3C_4 - C_{12}} = \frac{(2)(0.15)(4.57)^2 + (4)(67)}{(2)(0.15)(4.57)} = 200.047$$

$$K_4 = \frac{C_3}{C_7} = \frac{0.15}{0.15} = 1.000$$

$$K_5 = \frac{C_{12} - 2C_3C_4}{2C_7} = \frac{-(2)(0.15)(4.57)}{(2)(0.15)} = -4.570$$

$$K_6 = C_9 = 0.$$

$$K_7 = C_8 = 325.$$

$$m_1 = -K_2K_4 = 97.738$$

$$m_2 = -(K_2 + 2K_2K_4 - K_3K_4 - K_5) = 488.694$$

$$m_3 = K_3 - 2K_2K_4 + 2K_3K_4 + 2K_5 = 786.481$$

$$m_4 = K_3 - K_2K_4 + K_3K_4 + K_5 = 493.264$$

$$m_5 = -(K_2 + K_2K_4 - K_3K_4 - K_5) = 390.955$$

Step 3. Calculation of roots:

Equation (5.28) now is

$$m_1\lambda^{-2} - m_2\lambda^{-1} + m_3 - m_2\lambda + m_1\lambda^2$$

$$= 97.738\lambda^{-2} - 488.694\lambda^{-1} + 786.481 - 488.694\lambda + 97.738\lambda^2 = 0$$

From this equation we obtain

$$\lambda_1 = 2.5602$$

$$\lambda_2 = 0.3905$$

$$\lambda_3 = 1.2477$$

$$\lambda_4 = 0.8014$$

Since according to Equation (5.27) λ_i should be the values which are between 0 and 1, so we pick up two values $\lambda_2 = 0.3905$, $\lambda_4 = 0.8014$ and substitute them into Equation (5.29). In the next step, we let $\lambda_1 = 0.3905$, $\lambda_2 = 0.8014$.

Step 4. The reduced system of equations: The system of equations in Equation (5.29) now becomes:

$$317.2966W_1 - 152.4955W_2 = \sum_{t=1}^{\infty} \lambda_i^{t-1} D_t + 157.3020W_0 - I_0 + 247.2024$$

$$164.3724W_1 - 24.2101W_2 = \sum_{t=1}^{\infty} \lambda_i^{t-1} D_t + 117.1426W_0 - I_0 + 86.1879$$

for i = 1, 2

Step 5. The solution of the equation for W_1 and W_2: From the system of linear equations we obtained in Step 4, the values of W_1 and W_2 can be solved. They are

$$\begin{aligned} W_1 = {} & 0.007379D_1 + 0.006486D_2 + 0.005422D_3 + 0.004433D_4 \\ & + 0.003587D_5 + 0.002888D_6 + 0.002320D_7 + 0.001861D_8 \\ & + 0.001492D_9 + 0.001196D_{10} + 0.000959D_{11} + 0.000768D_{12} \\ & + 0.808514W_0 - 0.007379I_0 + 0.411778 \end{aligned}$$

$$\begin{aligned} W_2 = {} & 0.008796D_1 + 0.010935D_2 + 0.010281D_3 + 0.008833D_4 \\ & + 0.007311D_5 + 0.005950D_6 + 0.004804D_7 + 0.003864D_8 \\ & + 0.003102D_9 + 0.002488D_{10} + 0.001995D_{11} + 0.001599D_{12} \\ & + 0.650752W_0 - 0.008796I_0 - 0.764262 \end{aligned}$$

Step 6. Solution for P_1: Form Equation (5.14)

$$\begin{aligned} P_1 & = K_1 + K_2W_0 + K_3W_1 + K_2W_2 \\ & = 47.410 - 97.738W_0 + 200.047W_1 - 97.738W_2 \end{aligned}$$

$$= 0.616452D_1 + 0.228824D_2 + 0.079794D_3 + 0.023487D_4$$

$$+ 0.0031018D_5 - 0.003753D_6 - 0.005419D_7 - 0.005285D_8$$

$$- 0.004604D_9 - 0.003833D_{10} - 0.003128D_{11} - 0.002529D_{12}$$

$$+ 0.398764W_0 - 0.616452I_0 + 204.484090$$

The Case of Probabilistic Demand

The previous derivation of the linear decision rules was based on the assumption that the demand in each period is known with certainty. In the case of probabilistic demand, the decision rules are still applicable if we use the expected values of the demand as the value of D_t in the deterministic case. A brief proof is shown as follows:

Let $f(d_1, d_2, \ldots, d_t)$ be the joint probability density of demand in period $1, 2, \ldots, t$, as known at the beginning of period 1, and let $E(d_1), E(d_2), \ldots, E(d_t)$ be the expected demands. In a probabilistic formulation, the cost of inventory and shortage--$C_7(I_t-C_8-C_9D_t)^2$ in the deterministic case--must be replaced by

$$C_7 \sum_{d_1} \sum_{d_2}, \ldots, \sum_{d_t} (I_0+P_i-d_i-C_8-C_9d_t)^2 f(d_1, d_2, \ldots, d_t)$$

all other cost terms remain unaffected. When partial derivatives with respect to $P_1, P_2, \ldots, P_t$ are taken, this cost becomes

$$2C_7[I_0 + \sum_{i=1}^{t} P_i - \sum_{i=1}^{t} E(d_i) - C_8 - C_9 E(d_t)]$$

This result no longer depends on any probability distribution but only on the expected values of the demands.

Therefore, the same linear decision rules as in the deterministic case apply if the demands d_t's are replaced by their expectations.

Summary

When the quadratic cost function has been obtained for a particular factory, the optimal decision rule can be computed for scheduling production and work force in that factory. The step-by-step derivation of decision rules was given under the condition of certainty demand.

A brief proof of the certainty equivalence showed that the linear decision rules are optimal even when sales are subject to chance fluctuations.

COMPARISON OF METHODS

Many approaches to production and employment planning have been reported in the literature, but none is universally best. In describing several of these approaches, we shall therefore summarize their relative strengths and weaknesses. Four criteria are important in choosing an analysis:

1. Applicability.
2. The relative computational difficulty.
3. How well the analysis may be revised as operating experience accumulated.
4. The sensitivity of operating costs to errors in forecasts and data of the decision model.

Since the sensitivity analysis requires another mathematical approach, no studies will be carried out in this comparison task.

Applicability

Linear Programming Method

Transportation Method. This method has been advocated widely for seasonal demand scheduling as evidence by its presence in many production management and control texts (see, for example, (6), and (21)). However, in trying to apply it to the industries, Vergin (37) found that this method had two rather severe weaknesses:

1. There is an implicit assumption of constant employment in the model. Production above some normal capacity level can be accomplished only by overtime.
2. The model allows changes to occur in the production rate for any level below the normal capacity level without any costs assigned to making such changes. Such a solution is unrealistic, since such rate changes could only be accomplished by hiring and layoff workers, allowing idle time, etc.

Since the optimal schedules in industries include some, and often substantial, employment and production levels change with demand fluctuations, the constant employment schedules of the transportation model would be inappropriate in its application.

If we divide the production operations into some more levels (in our analysis we divided the production operations into two levels, the regular time production and the overtime production), some weaknessess of this method may be eliminated. Moreover, according to Bowman (5), this method can readily extend to several products. With its computational advantage, this method would provide decisions with considerably lower cost than other methods.

Simplex Method. This method eliminates the constant employment restriction. The remaining restrictions that production requirements are assumed known and exact and that cost relationships are linear appear to be rather rigid. The seriousness of the assumption of linear costs depends on the individual case. If the linear approximation fits the case, this method would be useful. Especially when it is too expensive to undertake a full scale non-linear programming analysis or where the necessary data are simply not available, this method can be helpful.

Dynamic Programming Method

The major contribution of this method is its lack of restrictions in the cost structures. Moreover, the cost structure is not necessarily an approximated function; we can list the actual cost value as represented by the example on page 31.

In handling the case where demands are uncertain, the dynamic programming method considers the whole demand probability distribution, but not their expected values. The result is a set of optimal policies under various conditions. This is easier to handle when actual demand deviates from the forecase demand, if there are no significant errors in the estimated demand probability distribution. The concepts of this method are clear and easy; people who possess elementary mathematical backgrounds can accept it without difficulty.

No indication of dynamic programming application to production and employment scheduling is presented in the literature. This might be because this type of problem has been used very infrequently and the dynamic programming requires too much computational effort.

Quadratic Programming

Several applications of the linear decision rules have been reported: In a paint factory, using the decision rules and a simple moving average forecast, a simulation of past operations reduced variable system costs by eight percent (18, p. 24). Simulation of a 50-man ice cream plant over a two-year period produced cost savings of one percent of $50, 000 (18, p. 35). By contrast, an application in a large fiber manufacturing company produced no savings (18, p. 34). In that plant, the size of the work force was tied closely to the number of production machines operating. Any change in production level produced sudden large increases and decreases in the work force. Therefore, the fit of the quadratic cost function was only approximate and the resulting decisions were no better than those previously made by management.

The difficulty of this method might be in obtaining the cost parameters. Fortunately, Holt, et al. (18) pointed out that fairly large errors in estimating the cost relations lead to relatively small differences in the decision. So only reasonable accuracy is required in estimating the cost relationship.

The assumption that the cost structure remains constant over many time periods is not practical. Therefore, a periodic review of the cost estimates has to be made and to determine whether or not the cost structure has changed sufficiently to require that a decision rule be computed. The constant cost structure assumption might be the reason that the reported applications did not get a significant savings in production cost.

The Relative Computational Difficulty

Linear Programming Method

The computational advantages of the transportation method are fairly well known for its tabular layout providing a conveneient work sheet for doing the computation. Both transportation and simplex methods are easily handled in the computer. Furthermore, some canned programs are available for use. In the simplex method, since each time period contributes six variables and three constraints, owing to the limitation of the size of computer memory, the planning horizon is not allowed to span a large number of time periods.

Dynamic Programming Method

Compared with the other two methods, dynamic programming is the most difficult both in hand calculating and computer programming.

If the demand fluctuates in a large range, or if the decision variables are continuous, the solution of this method is too time consuming. However, using the technique of Maximum Principle (25) will reduce its computational effort to some extent.

Quadratic Programming Method

The computation requires a certain familiarity with elementary mathematics. However, once an electronic computer program has been written for a particular job, it may be used in a purely routine manner. We just insert the cost coefficients and the decision rules can be read from the computer in a few minutes.

The Revision of the Decision-Making System

The structure of the production and work force scheduling problem may change over time so that the old answer simply does not solve the new problem. Thus, the manager not only has a responsibility to control the operation, but to control the decision system itself in order to keep it current. Therefore, the study and analysis of our decision problem should not be considered as once-and-for-all operation. Instead, a process of review, testing, and revision should be a continuing responsibility of the manager in charge. In this comparison, we shall consider how well the analysis may be revised as operating experience accumulates among different approaches.

Linear Programming Method

In this method, any change in the cost coefficient and the forecast demand lead to the reformulation of the problem. It is unusual that the actual performance will confirm the estimated quantitites, hence a new decision problem has to be established in almost every time period and the same computational effort as the beginning of the planning horizon has to be applied. Compared with others, this method requires most frequently revisions and the revision technique is also complex.

Dynamic Programming Method

A change in cost function means an over-all revision. If the estimated demand distributions are fairly accurate, revision is not necessary each time actual demand deviates from the expected forecast demand.

Quadratic Programming Method

In this method, the future demands have no influence to the decision rules. If the numerical constants in the cost function of the factory should change, the numbers in the previous decision rules would need to be recomputed. However, the algebraic forms of the decision rules would remain unchanged.

Even though a wide variety of quantitative models designed to handle the production and work force scheduling problem have been developed, they are seldom used. Up to now, only a few applications of linear decision rules have been reported. In general, management should adopt the method from which the most profit can be expected.

SUMMARY AND SUGGESTIONS

Summary

Decision models in scheduling production and work force levels under conditions of variable demand have been presented and evaluated. The costs involved in scheduling decisions are approximated into linear and quadratic forms. From these cost structures and under some assumptions, three different approaches which lead to the optimal production and employment decisions are derived. Some examples are employed to reinforce the evaluations. The methods were compared as to applicability, computational difficulty, appropriateness, and the revision of the decision-making system.

Some Other Approaches

Besides the methods we have already discussed in this paper, certain other approaches to production and employment scheduling have been reported. They are:

1. Network analysis: Hu and Prager (22), linear cost function.
2. Break-even approach: Manne (30), linear cost function.
3. Horizon planning: Modigliani and Hohn (31), linear or non-linear cost function.
4. Discrete maximum principle: Hwang, Fan and Erickson (24), linear or non-linear cost function.

A method called piece-wise linear approximation which approximates the non-linear function with straight-line segments and solved by linear programming method might be useful for the case of non-linear cost functions.

Some Other Applications of the Scheduling Algorithms

Decision problems in areas outside of production would also appear to be candidates for the application of optimal decision methods. The scheduling of warehouse operations, of employment in retail stores, of working capital and of some type of transportation operations--all appear to be effective areas for research. With ingenuity, management will undoubtedly discover still other applications in the future.

For the Future

In our study, all of the models are based on the cost of production and employment changes. A factor which perhaps is equally critical is ignored. That is the "point of departure". According to McGarrah (27), this is defined as the production, employment, and inventory levels of the time change. The influence of the current rates can be shown through the following example:

> If a firm has been operating at 80 percent of normal one-shift capacity, an increase in output could be effected without additional costs of overtime or second-shift

premiums. A decrease in output might necessitate laying off key personnel who would be difficult to replace. However, if the plant has been operating at 100 percent of normal one-shift capacity, an increase very likely may involve overtime costs or second-shift premiums, and extra supervision. A decrease in production could be affected by reducing the work day or by layoff personnel with the lowest seniority and least skill, these costs would not be so great as a layoff when a plant was at 80 percent of the normal capacity.

It is found that the production smoothing decision is relatively insensitive to errors in estimating the parameters of the cost function. Holt, et al., (18, p. 68) stated that the linear decision rules calculated on the basis of cost parameter estimates, which contain errors as large as plus or minus 50 percent, yield operating results which are very near optimal, and it is concluded that the model is insensitive to cost coefficient errors. This might be caused by the ignorance of the point of departure.

Thus, it appears that the most important current requirements in improving the production and employment decisions are:

1. More empirical evidence on the magnitude of the point of departure effect.
2. Evidence on the sensitivity of the production smoothing decision to this effect.

The first task is difficult, since the pertinent data are not available in either financial or cost accounting records. The second task may be carried out by a simulation test.

BIBLIOGRAPHY

1. Antosiewicz, H. and A. Hoffman. A remark on the smoothing problem. Management Science 1: 92-95. 1954.

2. Arrow, Kenneth J., Theodore Harris and Jacab Marschak. Optimal inventory policy. Econometrica 19: 250-272. 1951.

3. Bellman, R.E. Dynamic programming and the smoothing problem. Management Science 3: 111-113. 1956.

4. Bellman, R.E. and S.E. Dreyfus. Applied dynamic programming. Princeton, Princeton University, 1962. 363 p.

5. Bowman, Edward H. Production scheduling by the transportation method of linear programming. Operations Research 4: 100-103. 1956.

6. Buffa, Elwood S. Modern production management. New York, Wiley, 1961. 636 p.

7. Bursk, Edward C. and John F. Chapman. New decision-marking tools for managers. Cambridge, Mass., Harvard University, 1963. 413 p.

8. Carr, Charles R. and Charles W. Houe. Quantitative decision procedure in management and economics: Deterministic theory and application. New York, McGraw-Hill, 1964, 383 p.

9. Charnes, A. and W.W. Cooper. Management models and industrial applications of linear programming. Vol. I. New York, Wiley, 1961. 467 p.

10. Charnes, A., W.W. Cooper and Donald Farr. Linear programming and profit preference scheduling for a manufacturing firm. Operations Research 1: 114-129. 1953.

11. Charnes, A., W.W. Cooper and B. Mellon. A model for optimizing production by reference to cost surrogates. Econometrica 23(3): 307-322. 1955.

12. Dantzig, George B. Linear programming under uncertainty. Management Science 1: 197-206. 1955.

13. Green H. Geene. Production control: System and decision. Homewood, Ill., Richard D. Irwin, 1965. 605 p.

14. Hadley, G. Nonlinear and dynamic programming. Reading, Mass., Addison-Wesley, 1964. 484 p.

15. Hanssmann, Fred. Operations research in production and inventory control. New York, Wiley, 1962. 254 p.

16. Hanssmann, Fred and Sidney W. Hess. A linear programming approach to production and employment scheduling. Management Technology 1(1):46-51. 1960.

17. Hoffman, A. J. and W. Jacobs. Smooth patterns of production. Management Science 1:86-91. 1954.

18. Holt, Charles C. et al. Planning production, inventory, and work force. Englewood, N.J., Prentice-Hall, 1960. 419 p.

19. Holt, C.C., F. Modigliani and J.F. Muth. Derivation of a linear decision rule for production and employment scheduling. Management Science 2:159-177. 1956.

20. Holt, C.C. and H. A. Simon. Linear decision rule for production and employment scheduling. Management Science 2:1-30. 1955.

21. Holzman, A.G., H. H. Schaeffer and R. Glaser. Matrices and mathematical programming: Mathematical basis for management decision making. Vol. I. Chicago, Ill., Encyclopedia Britannica. 1962.

22. Hu, Te-chiang and W. Prager. Network analysis of production smoothing. Naval Research Logistic Quarterly 6:17-23. 1959.

23. Hugli, Wilfred C., Jr. Production planning through inventory control. Management Technology 1(2):59-65. 1960.

24. Hwang, C.L. and L.T. Fan. The application of the maximum principle to industrial and management systems. The Journal of Industrial Engineering 17:589-593. 1966.

25. Hwang, C.L., L.T. Fan and L.E. Erickson. Optimum production planning by the maximum principle. Management Science 13: 751-755. 1967.

26. Johnson, S.M. Sequential production planning over time at minimum cost. Management Science 3: 435-437. 1957.

27. McGarrah, R.E. Production programming. The Journal of Industrial Engineering 7: 263-271. 1956.

28. McNaughton, Robert. Scheduling with deadline and loss function. Management Science 6: 1-12. 1959.

29. Magee, J.F. Production planning and inventory control, 2d ed. New York, McGraw-Hill, 1958. 333 p.

30. Manne, Alan S. A note on the Modigliani-Hohn production smoothing model. Management Science 3: 371-379. 1957.

31. Modigliani, Franco and Franz E. Hohn. Production planning over time and the nature of the expectation and planning horizon. Econometrica 23: 46-66. 1955.

32. Muth, John F. and Gerald L. Thompson. Industrial scheduling. Englewood Cliff, N.J., Prentice-Hall, 1963. 387 p.

33. Panne, C. Van De and P. Bosje. Sensitivity analysis of cost coefficient estimates: The case of linear decision rules for employment and product. Management Science 9: 82-107. 1962.

34. Sasieni, Maurice, Arthur Yaspan and Lawrence Friedman. Operations research - methods and problems. New York, Wiley, 1959. 316 p.

35. Schild, Albert and Irwin J. Fredman. Scheduling tasks with deadlines and non-linear loss functions. Management Science 9: 73-81. 1962.

36. Teichroew, Daniel. An introduction to management science: Deterministic models. New York, Wiley, 1964. 713 p.

37. Vergin, Roger C. Production scheduling under seasonal demand. The Journal of Industrial Engineering 17: 260-266. 1966.

APPENDIX

```
C     PRODUCTION SCHEDULING
C     DYNAMIC PROGRAMMING  THE CASE OF DETERMINISTIC DEMAND
      DIMENSION F(8,50), P(8,50), D(8), ID(8)
  100 FORMAT(I2)
  200 FORMAT(8F4.0)
  300 FORMAT (F4.0)
  400 FORMAT(I3, F6.0,F12.2,F6.0)
      READ 100,N
      READ 200,(D(I),I=2,N)
      DMIN=D(2)
      DMAX=D(2)
      DO 4 I=3,N
      IF(D(I)-DMIN)1,2,2
    1 DMIN=D(I)
    2 IF(DMAX-D(I))3,4,4
    3 DMAX=D(I)
    4 CONTINUE
      IR=DMAX-DMIN+3.
      DO 5 I=2,N
    5 ID(I)=D(I)-DMIN+1.
      IS=ID(2)
      DO 6 IP=IS,IR
      F(1,IP)=0.
    6 P(1,IP)=1.
      L=N-1
      DO 8 K=2,L
      PAUSE
      IS=ID(K+1)
      M=ID(K)
      B=M
      DO 8 IP=IS,IR
      A=IP
      F(K,IP)=2.*(B-A)**2+F(K-1,M)
      P(K,IP)=B
      DO 8 I=M,IR
      X=I
      C=2.*(X-A)**2+20.*(X-B)+F(K-1,I)
      IF (SENSE SWITCH 1)20,30
   20 PT1=A+DMIN-1.
      KS=K-1
      PT=X+DMIN-1.
      PRINT 400,KS,PT1,C,PT
   30 IF(C-F(K,IP)) 7,8,8
    7 F(K,IP)=C
      P(K,IP)=X
    8 CONTINUE
      READ 300,PO
      IP=PO-DMIN+1.
      A=IP
      M=ID(N)
      B=M
      F(N,IP)=2.*(B-A)**2+F(N-1,M)
```

```
      P(N,IP)=B
      DO 10 I=M,IR
      X=I
      C=2.*(X-A)**2+20.*(X-B)+F(N-1,I)
      IF(C-F(N,IP))9,10,10
    9 F(N,IP)=C
      P(N,IP)=X
   10 CONTINUE
      DO 15 I=1,L
      J=ID(I+1)
      DO 15 IL=J,IR
      A=IL
      KS=I-1
      PT1=A+DMIN-1.
      PT=P(I,IL)+DMIN-1.
   15 PUNCH 400,KS,PT1,F(I,IL),PT
      A=IP
      KS=N-1
      PT1=A+DMIN-1.
      PT=P(N,IP)+DMIN-1.
      PUNCH 400,KS,PT1,F(N,IP),PT
      STOP
      END


C     INPUT DATA


05
 180 195 220 210
 200
```

C RESULTS

0	180.	0.00	180.
0	181.	0.00	180.
0	182.	0.00	180.
0	183.	0.00	180.
0	184.	0.00	180.
0	185.	0.00	180.
0	186.	0.00	180.
0	187.	0.00	180.
0	188.	0.00	180.
0	189.	0.00	180.
0	190.	0.00	180.
0	191.	0.00	180.
0	192.	0.00	180.
0	193.	0.00	180.
0	194.	0.00	180.
0	195.	0.00	180.
0	196.	0.00	180.
0	197.	0.00	180.
0	198.	0.00	180.
0	199.	0.00	180.
0	200.	0.00	180.
0	201.	0.00	180.
0	202.	0.00	180.
0	203.	0.00	180.
0	204.	0.00	180.
0	205.	0.00	180.
0	206.	0.00	180.
0	207.	0.00	180.
0	208.	0.00	180.
0	209.	0.00	180.
0	210.	0.00	180.
0	211.	0.00	180.
0	212.	0.00	180.
0	213.	0.00	180.
0	214.	0.00	180.
0	215.	0.00	180.
0	216.	0.00	180.
0	217.	0.00	180.
0	218.	0.00	180.
0	219.	0.00	180.
0	220.	0.00	180.
0	221.	0.00	180.
0	222.	0.00	180.
1	195.	250.00	190.
1	196.	270.00	191.
1	197.	290.00	192.
1	198.	310.00	193.
1	199.	330.00	194.

1	200.	350.00	195.
1	201.	370.00	196.
1	202.	390.00	197.
1	203.	410.00	198.
1	204.	430.00	199.
1	205.	450.00	200.
1	206.	470.00	201.
1	207.	490.00	202.
1	208.	510.00	203.
1	209.	530.00	204.
1	210.	550.00	205.
1	211.	570.00	206.
1	212.	590.00	207.
1	213.	610.00	208.
1	214.	630.00	209.
1	215.	650.00	210.
1	216.	670.00	211.
1	217.	690.00	212.
1	218.	710.00	213.
1	219.	730.00	214.
1	220.	750.00	215.
1	221.	770.00	216.
1	222.	790.00	217.
2	220.	1050.00	210.
2	221.	1090.00	211.
2	222.	1130.00	212.
3	210.	1250.00	220.
3	211.	1212.00	220.
3	212.	1178.00	220.
3	213.	1148.00	220.
3	214.	1122.00	220.
3	215.	1100.00	220.
3	216.	1082.00	220.
3	217.	1068.00	220.
3	218.	1058.00	220.
3	219.	1052.00	220.
3	220.	1050.00	220.
3	221.	1052.00	220.
3	222.	1058.00	220.
4	200.	1450.00	210.

```
C     PRODUCTION SCHEDULING
C     DYNAMIC PROGRAMMING  THE CASE OF STOCHASTIC DEMAND
      DIMENSION P(8,8),F(8,8),PD(8,8),D(8),T(8)
  100 FORMAT(4I2)
  110 FORMAT(8F5.3)
  120 FORMAT(8F4.0)
  130 FORMAT(F4.0)
  180 FORMAT(3I5,2F7.2)
      READ 100,N,M,ND,NI
      DO 1 I=2,N
    1 READ 110, (P(I,J), J=1,M)
      READ 120, (D(I), I=1,ND)
      READ 120, (T(I), I=1,NI)
      READ 120, (F(1,J), J=1,NI)
      READ 130, S
      DO 13 J=1, NI
   13 PD(1,J)=1.
      XNI=NI
      DO 10 I=2,N
      PAUSE
      XI=I
      DO 10 J=1,NI
      XJ=J
      F(I,J)=999.
      PD(I,J)=ND+10
      DO 10 K=1,ND
      XK=K
      EXPC=0.
      DO 7 L=1,M
      XL=L
      AS=XJ+XK-XL-1.
      IF (AS-XNI+1.)12,12,14
   14 IF(P(I,L))7,7,15
   15 L=1
      GO TO 10
   12 IF(AS)4,3,3
    3 SC=0.
      GO TO 8
    4 SC=-AS*S
    8 IA=AS+1.
      IF(IA-1)5,5,6
    5 REC=F(I-1,1)
      GO TO 16
    6 REC=F(I-1,IA)
   16 EXPC=EXPC+P(I,L)*(SC+REC)
      KA=I-1
      KB=J-1
      KC=K-1
      KD=L-1
      IF(SENSE SWITCH 1)20,7
   20 PRINT 180,KB,KC,KD,SC,REC
    7 CONTINUE
```

```
      TEMPF=D(K)+T(J)+EXPC
      IF (SENSE SWITCH 2)50,60
   50 PRINT 200,KA,KB,TEMPF,KC
  200 FORMAT (/2I4,F7.2,I4/)
   60 IF(TEMPF-F(I,J))9,10,10
    9 F(I,J)=TEMPF
      PD(I,J)=XK
   10 CONTINUE
      DO 11 I=1,N
      K=I-1
      PUNCH 140,K
      PUNCH 150
      DO 11 J=1,NI
      K=J-1
      PROD=PD(I,J)-1.
   11 PUNCH 160,K,F(I,J),PROD
  140 FORMAT(///12HSTAGE NUMBER,I2//)
  150 FORMAT(39H  INV. LEVEL(I)   F(I)        PROD. LEVEL)
  160 FORMAT(7X,I3,5X,F7.2,9X,F5.0)
      STOP
      END


C     INPUT DATA


04040304
0.6670.3330.0000.000
0.0000.2500.5000.250
0.2500.5000.2500.000
 15. 20. 35.
  2.  5.  9. 15.
 10.  0.  5. 10.
 10.
```

C RESULTS

STAGE NUMBER 0

INV. LEVEL(I)	F(I)	PROD. LEVEL
0	10.00	0.
1	0.00	0.
2	5.00	0.
3	10.00	0.

STAGE NUMBER 1

INV. LEVEL(I)	F(I)	PROD. LEVEL
0	25.33	1.
1	23.33	0.
2	27.33	0.
3	38.33	0.

STAGE NUMBER 2

INV. LEVEL(I)	F(I)	PROD. LEVEL
0	57.33	1.
1	52.33	1.
2	51.33	0.
3	54.83	0.

STAGE NUMBER 3

INV. LEVEL(I)	F(I)	PROD. LEVEL
0	80.58	1.
1	78.33	1.
2	77.33	0.
3	82.45	0.

```
C        COMPUTATION OF LINEAR DECISION RULES FOR PRODUCTION
C           AND EMPLOYMENT SCHEDULING
      DIMENSION T(4),RT(2),AE(2),BE(2),CE1(2),CE2(2),CE3(2)
     1,CE4(2),BP(2),CP(2),DP(2),AP(2,20),AT(20)
  100 FORMAT(11F5.0)
  110 FORMAT(6HSTEP 1//)
  120 FORMAT(4X,4HC1 =,F9.4,6H  C2 =,F9.4,6H  C3 =,F9.4)
  130 FORMAT(4X,4HC4 =,F9.4,6H  C5 =,F9.4,6H  C6 =,F9.4)
  140 FORMAT(4X,4HC7 =,F9.4,6H  C8 =, F9.4,6H  C9 =,F9.4)
  150 FORMAT(4X,4HC11=,F9.4,6H  C12=,F9.4///)
  200 FORMAT(6HSTEP 2//)
  210 FORMAT(4X,4HK1 =,F8.3,6H  K2 =,F8.3,6H  K3 =,F8.3,
     16H  K4 =,F8.3)
  220 FORMAT(4X,4HK5 =,F8.3,6H  K6 =,F8.3,6H  K7 =,F8.3//)
  230 FORMAT(4X,4HM1 =,F8.3,6H  M2 =,F8.3,6H  M3 =,F8.3,
     16H  M4 =,F8.3)
  240 FORMAT(4X,4HM5 =,F8.3///)
  300 FORMAT(11HFIRST ENTER)
  310 FORMAT(12HSECOND ENTER)
  320 FORMAT(11HTHIRD ENTER)
  330 FORMAT(2F14.4)
  340 FORMAT(6HSTEP 3//)
  350 FORMAT(4X,5HRT1 =,F7.4,7H  RT2 =,F7.4,7H  RT3 =,F7.4,
     17H  RT4 =,F7.4//)
  360 FORMAT(4X,4HX1 =,F9.4,6H  X2 =,F9.4///)
  400 FORMAT(6HSTEP 4//)
  410 FORMAT(4X,F12.4,4HW1 +,F12.4,4HW2 =)
  420 FORMAT(6X,F12.4,24HSUM OF RT(I)**(T-1)*D(T))
  430 FORMAT(6X,1H+,F12.4,9HW0 - I0 +,F12.4///)
  500 FORMAT (6HSTEP 5//)
  510 FORMAT(4X,2H+ ,F12.6,3H D(,I3,1H))
  520 FORMAT(4X,2H+ ,F12.6,6H W0 + ,F12.6,6H I0 + ,F12.6//)
  530 FORMAT(2X,2HW(,I2,3H) =)
  600 FORMAT(6HSTEP 6//)
  610 FORMAT(2X,6HP(1) =)
C     STEP 1
      PAUSE
      READ 100, C1,C2,C3,C4,C5,C6,C7,C8,C9,C11,C12
      PUNCH 110
      PUNCH 120,C1,C2,C3
      PUNCH 130,C4,C5,C6
      PUNCH 140,C7,C8,C9
      PUNCH 150,C11,C12
C     STEP 2
      PAUSE
      X=2.*C3*C4-C12
      AK1=(C1-C6)/X
      AK2=(-2.*C2)/X
      AK3=(2.*C3*C4*C4+4.*C2)/X
      AK4=C3/C7
      AK5=(C12-2.*C3*C4)/(2.*C7)
      AK6=C9
```

```
      AK7=C8
      AM1=-AK2*AK4
      AM2=-(AK2+2.*AK2*AK4-AK3*AK4-AK5)
      AM3=AK3-2.*AK2*AK4+2.*AK3*AK4+2.*AK5
      AM4=AK3-AK2*AK4+AK3*AK4+AK5
      AM5=-(AK2+AK2*AK4-AK3*AK4-AK5)
      PUNCH 200
      PUNCH 210,AK1,AK2,AK3,AK4
      PUNCH 220,AK5,AK6,AK7
      PUNCH 230,AM1,AM2,AM3,AM4
      PUNCH 240,AM5
C     STEP 3
      PAUSE
      A=AM1
      B=-AM2
      C=AM3-2.*AM1
      PRINT 300
      CALL ROOT(A,B,C,X1,X2)
      B1=-X1
      B2=-X2
      PRINT 330,X1,X2
      PRINT 310
      CALL ROOT (1.,B1,1.,X1,X2)
      T(1)=X1
      T(2)=X2
      PRINT 330,X1,X2
      PRINT 320
      CALL ROOT (1.,B2,1.,X1,X2)
      T(3)=X1
      T(4)=X2
      PRINT 330,X1,X2
      RT(1)=0.
      DO 3 I=1,4
      IF(ABSF(T(I))-1.)2,3,3
    2 IF(RT(1))1,1,4
    1 RT(1)=T(I)
      GO TO 3
    4 RT(2)=T(I)
    3 CONTINUE
      PUNCH 340
      PUNCH 350, (T(I),I=1,4)
      PUNCH 360, RT(1),RT(2)
      PUNCH 400
C     STEP 4
      PAUSE
      DO 10 K=1,2
      AE(K)=AM4-AM2*RT(K)+AM1*RT(K)*RT(K)
      BE(K)=-(AM5-AM2+AM1/RT(K))
      CE1(K)=1.+AK6*(1.-RT(K))
      CE2(K)=-(AK2+AK2*AK4*(1.-RT(K)))
      CE3(K)=-1.
      CE4(K)=AK7-AK1/(1.-RT(K))
```

```
      PUNCH 410,AE(K),BE(K)
      PUNCH 420,CE1(K)
   10 PUNCH 430,CE2(K),CE4(K)
C     STEP 5
      PAUSE
      X=AE(1)*BE(2)-AE(2)*BE(1)
      Y=-X
      BP(1)=(CE2(1)*BE(2)-CE2(2)*BE(1))/X
      CP(1)=(CE3(1)*BE(2)-CE3(2)*BE(1))/X
      DP(1)=(CE4(1)*BE(2)-CE4(2)*BE(1))/X
      BP(2)=(CE2(1)*AE(2)-CE2(2)*AE(1))/Y
      CP(2)=(CE3(1)*AE(2)-CE3(2)*AE(1))/Y
      DP(2)=(CE4(1)*AE(2)-CE4(2)*AE(1))/Y
      DO 20 K=1,20
      S1=RT(1)**(K-1)
      S2=RT(2)**(K-1)
      AP(1,K)=(CE1(1)*BE(2)*S1-CE1(2)*BE(1)*S2)/X
   20 AP(2,K)=(CE1(1)*AE(2)*S1-CE1(2)*AE(1)*S2)/Y
   22 PUNCH 500
      DC 24 I=1,2
      PUNCH 530,I
      DO 23 K=1,20
   23 PUNCH 510,AP(I,K),K
   24 PUNCH 520,BP(I),CP(I),DP(I)
C     STEP 6
      PAUSE
      DO 30 K=1,20
   30 AT(K)=AP(1,K)*AK3+AP(2,K)*AK2
      BT=BP(1)*AK3+BP(2)*AK2+AK2
      CT=CP(1)*AK3+CP(2)*AK2
      DT=DP(1)*AK3+DP(2)*AK2+AK1
      PUNCH 600
      PUNCH 610
      DO 31 K=1,20
   31 PUNCH 510, AT(K), K
      PUNCH 520,BT,CT,DT
      STOP
      END
```

```
      SUBROUTINE ROOT (A,B,C,X1,X2)
  250 FORMAT(12HCOMPLEX ROOT, 4F10.2)
  260 FORMAT(5HX1=X2, 4F10.2)
      D=B*B-4.*A*C
      IF(D)1,2,3
    1 PRINT 250,A,B,C,D
      PAUSE
    2 PRINT 260, A,B,C,D
      X1=-B/(2.*A)
      X2=X1
      GO TO 4
    3 X1=(-B+SQRTF(D))/(2.*A)
      X2=(-B-SQRTF(D))/(2.*A)
    4 RETURN
      END


C     INPUT DATA

350.0067.000.1504.5749.00285.0.1500325.0000.0000.0000.0
```

```
C       RESULTS

STEP 1

    C1  = 350.0000  C2  =   67.0000  C3  =     .1500
    C4  =   4.5700  C5  =   49.0000  C6  = 285.0000
    C7  =    .1500  C8  = 325.0000  C9  =    0.0000
    C11=    0.0000  C12=     0.0000

STEP 2

    K1  =   47.410  K2  = -97.738  K3  = 200.047  K4  =    1.000
    K5  =   -4.570  K6  =   0.000  K7  = 325.000

    M1  =   97.738  M2  = 488.694  M3  = 786.481  M4  = 493.264
    M5  = 390.955

STEP 3

    RT1 = 2.5602   RT2 =    .3905  RT3 = 1.2477   RT4 =    .8014

    X1  =     .3905  X2  =     .8014

STEP 4

        317.2966W1 +    -152.4955W2 =
            1.0000SUM OF RT(I)**(T-1)*D(T)
     +      157.3020W0 - I0 +     247.2024

        164.3724W1 +     -24.2101W2 =
            1.0000SUM OF RT(I)**(T-1)*D(T)
     +      117.1426W0 - I0 +      86.1879
```

STEP 5

```
W( 1) =
   +       .007379 D(  1)
   +       .006486 D(  2)
   +       .005422 D(  3)
   +       .004433 D(  4)
   +       .003587 D(  5)
   +       .002888 D(  6)
   +       .002320 D(  7)
   +       .001861 D(  8)
   +       .001492 D(  9)
   +       .001196 D( 10)
   +       .000959 D( 11)
   +       .000768 D( 12)
   +       .000616 D( 13)
   +       .000493 D( 14)
   +       .000395 D( 15)
   +       .000317 D( 16)
   +       .000254 D( 17)
   +       .000203 D( 18)
   +       .000163 D( 19)
   +       .000130 D( 20)
   +       .808514 WO +     -.007379 IO +      .411778

W( 2) =
   +       .008796 D(  1)
   +       .010935 D(  2)
   +       .010281 D(  3)
   +       .008833 D(  4)
   +       .007311 D(  5)
   +       .005950 D(  6)
   +       .004804 D(  7)
   +       .003864 D(  8)
   +       .003102 D(  9)
   +       .002488 D( 10)
   +       .001995 D( 11)
   +       .001599 D( 12)
   +       .001282 D( 13)
   +       .001027 D( 14)
   +       .000823 D( 15)
   +       .000660 D( 16)
   +       .000529 D( 17)
   +       .000424 D( 18)
   +       .000339 D( 19)
   +       .000272 D( 20)
   +       .650752 WO +     -.008796 IO +     -.764262
```

STEP 6

```
P(1) =
  +       .616452 D(  1)
  +       .228824 D(  2)
  +       .079794 D(  3)
  +       .023487 D(  4)
  +       .003018 D(  5)
  +      -.003753 D(  6)
  +      -.005419 D(  7)
  +      -.005285 D(  8)
  +      -.004604 D(  9)
  +      -.003833 D( 10)
  +      -.003128 D( 11)
  +      -.002529 D( 12)
  +      -.002035 D( 13)
  +      -.001635 D( 14)
  +      -.001311 D( 15)
  +      -.001051 D( 16)
  +      -.000843 D( 17)
  +      -.000675 D( 18)
  +      -.000541 D( 19)
  +      -.000434 D( 20)
  +       .398764 W0 +      -.616452 I0 +    204.484090
```