AN ABSTRACT OF THE THESIS OF

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Title: Critical Evaluation of Ouyang's Model Reduction Method

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A model reduction method for stable, linear, time-invariant,continuous systems is critically studied. This method, proposed by Prof. Ouyang, is based on the concept of power dispersion and the least-squared estimation technique. The reduced models obtained by this method retain the stability of the stable, high order original systems and give no steady-state error between both the reduced model and the original system. Systems having repeated poles or complex poles are discussed in this thesis. The simulations of a number of examples show that the 2nd order reduced models can give good approximations to the 4th order original systems studied here whose poles are wholly or partially located on the negative real axis of the s-plane. But, the results also show that unsatisfactory approximations occur for original systems having 2 pairs of conjugate complex poles. Ouyang did not point out how to

select the retained poles for the reduced model when the original systems have negative power dispersion components.

New methods for selecting the reduced model poles are proposed for original systems having negative power dispersion components. The methods are based on selecting reduced model poles to obtain a best fit to the original system output autocorrelation function. Some examples are given to demonstrate the effectiveness of the new methods. The results obtained by new methods show that power dispersion method is still valuable. By selecting the poles due to largest positive power dispersion, the reduced model can be determined for the case of the systems having negative power dispersion component.

CRITICAL EVALUATION OF OUYANG'S MODEL REDUCTION METHOD

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Table of Contents

List of Figures

List of Tables

List of Appendix Tables

CRITICAL EVALUATION OF OUYANG'S MODEL REDUCTION METHOD

1 INTRODUCTION

1.1 Significance of Model Reduction

Dynamic systems consist of many of interconnected components. To analyze these systems, a mathematical expression is obtained for each component. These expressions are then combined to formulate a model for the entire system. The model obtained can be expressed as a set of high dimensional state variable differential equations, or difference equations. The exact analysis of high-order systems is frequently impractical. Furthermore, many systems can not be modeled precisely because of system complexity. The human dynamic system is an example. Practical problems in modeling complicated systems have been encountered in many fields including electrical power plants, chemical processes, and aerospace systems.

Problem simplification is often desirable and sometimes necessary in the analysis and design of a high-order system. Evans' (1954) graphical estimation technique is based on this philosophy: "If the time-domain response produced by a high-order model looks very simple, use the response of a simpler transfer function which is a best fit to the original system".

The usefulness of techniques for deriving low-order approximations for high-order systems has long been recognized. There are two quite distinct motivations for finding such an approximation. The first is to reduce the computational burden for simulation analysis and control system design. The second is based on the realization that potentially a simplified model can lead to acceptable simplified control system structures (Shahriar, 1981).

This topic has been studied extensively for many years and a great variety of methods for obtaining suitable low-order approximations have been suggested. Most methods use either or both time-domain or the frequency-domain. Both deterministic and stochastic processes have been utilized as the mathematical means to derive model reduction methods.

There is no single and general method which can fit all reduction problems. The difficult task is to establish comparison criteria from which an optimal model reduction method can be obtained. For an ideal reduction method in a linear, time-invariant and continuous system, the reduced model should provide good approximation to both the transient and the steady-state output time response.

1.2 Objectives of the Study

The objectives of this study are as follows:

1) To select a model reduction method from among available published methods which is simple to use and which potentially can provide a good approximation to high-order physical systems;
2) To simulate and test the selected method;
3) To critically evaluate the performance of the selected method.

The reduction method studied here is proposed by Ouyang (1987) which is for stable, linear, time-invariant and continuous systems.

Any reduction method must have the following properties:

1) The reduced model must be stable if the original system is stable;
2) The steady-state response for both the original and the reduced models must be the same;
3) The frequency response of the reduced model must be a good approximation to that of the original system.
4) The transient response must be a good approximation to the original system.

These properties are used to critically evaluate the reduced models obtained using Ouyang's method.

1.3 Literature Review

Many methods have been suggested for the model reduction of linear, time-invariant systems.

Several reduction methods utilize the frequency domain. One of the approximation methods is a continued-fraction expansion method (Chen and Shien, 1970). Vittal-Rao and Lamba (1974) presented another method based on the least squares approximation of the frequency response. These methods have the serious drawback of failing to guarantee stable reduced models of stable systems. Shamash (1975) tried to overcome this difficulty by retaining stable poles in the reduced model, but he still failed without considering that the reduced model poles would be located in the right half of the s-plane by his method (Hutton, 1975).

Different approaches to the stability problem have been considered. Bistritz and Langlolz (1979, 1980) dealt with the stability problem from the whole transfer function. However, this method may fail due to the existence of pure imaginary poles in s-plane (Lin, 1982). Lin and Wu (1982) proposed a stability-equation method. Ouyang (1987) proposed a mixed method which involves the power decomposition method and the frequency response matching methods. The concept of power decomposition is based on the contribution of each dynamic mode (system pole) to the total power of the system output. The frequency response matching method minimizes the least squares error between the original and reduced system frequency responses.

Several model reduction techniques are based on linear transformations, matrix diagonalization. A diagonal element discarding method was proposed by Davison (1966). The dominant poles retained in the reduced model are selected from the comparison of the magnitudes of the eigenvector determinant. However, its shortcoming is that the dominant poles philosophy is not suitable for control system analysis because of a unique property of the eigenvector. It often happens that the higher order reduced model obtained by Davison's method gives worse approximation than lower order reduced model. Marshell (1968) tried to modify Davison's method. He offered only a partial remedy, but failed to change the basic philosophy (Chen, 1974). Some newer techniques, such as the optimal Hankel approximation approach (Harchavardhana et al, 1984), produce stable reduced models whose eigenvalues differ from the eigenvalues of the original system. However, each of these methods gives results with a mismatch in the steady-state values without correction (Ouyang, 1987).

Several methods have also been developed for obtaining a guaranteed stable low-order system when the original high-order system is stable. Hutton and Friedland (1975) used the stability criterion of Routh to obtain the reduced model.

Appiah (1979) employed the Hurwitz polynomial approximation as the characteristic polynomial to guarantee a stable reduced-model. In these two methods, if the dominant poles are not very close to the origin, then the Routh-Hurwitz approximation fails to produce a good lower order model.

Several model reduction methods are based on the minimization of different error criterions. By comparing the reduced model with the original system in order to obtain a given error function, the unknown elements of the reduced model can be determined by minimizing the error function. The methods due to Meier and Luenberger (1966), Wilson (1970), and Lepschy (1984) belong to this category. Meier and Luenberger minimized the mean-square error. Wilson minimized a performance criterion measuring the relative merits of retaining different sets of eigenvelues in the reduced model. Lepschy minimized the output equation error. The obvious shortcoming of Meier's and Wilson's methods is that the steady-state values of the original system cannot always be reproduced by the reduced models and the required computations are excessively complicated (Chen, 1974). Lepschy's method did not solve the problem of the steady-state value.

Model reduction techniques have also been proposed by some other authors (Shieh, 1976, Moore, 1981, Lucas, 1986). Table 1.1 summarizes the features of several typical methods. As shown in Table 1.1, it is obvious that each method has its inherent advantages and disadvantages. Even if a method gives a good approximation in the transient and steady-state response for some original systems, there are still other original systems for which the approximation is unacceptable.

Table 1.1 The Features of Typical Reduction Methods

	Stable	Steady State Error	Good Approximation	Serious Error	Multiple Input
Davison (1966) Diagonal Element Discarding Method	Yes	Yes	N.C.	N.C.	Yes
Shamash (1975) Padé Approximation	N.G	N.M.	N.C.	Yes	Yes
Hutton & Fieldland (1975) Routh Approximation	Yes	N.M.	N.C.	Yes (p.n.c.o.)	Yes
Appiah (1979) Hurwitz Polynomial Approximation	Yes	N.M.	N.C.	Yes (p.n.c.o.)	No
Lin & Wan (1982) Stability Equation Method	Yes	N.M.	N.C.	N.C.	No
Harshavardhana (1984) Hankel Approximation	Yes	N.M.	N.C.	N.C.	No
Ouyang (1987) Power Dispersion & Parameter Estimation	Yes	No	Yes (A)	Yes (B)	No

Note: N.G. -- no guarantee, N.M. -- author did not mention it.
N.C. -- no comment, p.n.c.o. -- poles are not closed to the origin.
A -- for systems studied having negative real poles
B -- for systems studied having two pairs of comples poles

2 BACKGROUND AND PROBLEM STATEMENT

For a linear n^{th}-order continuous-time system, the relationship between the output Y(S) and the input U(S) can be characterized by a transform model

$$(A_0+A_1S+\cdots+A_nS^n)Y(S)=(B_0+B_1S+\cdots+B_{n-1}S^{n-1})U(S), \tag{1}$$

or in transfer function form

$$Y(S) = H(S)\ U(S)\ . \tag{2}$$

The transfer function H(S) can be expressed as

$$H(S) = \frac{B_0 + B_1\ S + B_2\ S^2 +\cdots+ B_{n-1}\ S^{n-1}}{A_0 + A_1\ S + A_2\ S^2 +\cdots+ A_n\ S^n} = \frac{Y(S)}{U(S)}\ . \tag{3}$$

Only single input-single output linear dynamic systems are considered here. Figure 2.1 shows a basic block diagram representing the transform relationship of equation (2)

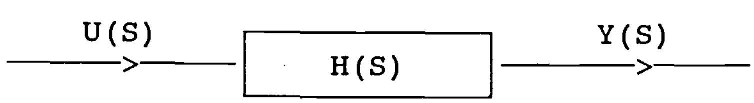


Figure 2.1 Functional Block Diagram

In many engineering investigations, it is desirable or necessary to reduce the dimension of equations used to describe a physical system. Such a reduction may be considered successful if the response of the reduced model for a given input is "close" to that of the original model for the same input. Prof. M. Ouyang proposed a method for model reduction of linear continuous systems using power decomposition and system identification methods (Ouyang, 1987). Before investigating and evaluating Ouyang's method in detail, it is necessary to discuss some problems associated with this method.

2.1 Problem Statement

The problem studied here is to critically examine Ouyang's method by applying it to many cases which are not presented in Ouyang's paper. A primary objective is to determine the conditions for which the method gives good approximation. Most of the study here involves determining the parameters of the second order "reduced" transfer function

$$R(S) = \frac{\alpha_1 S + \alpha_0}{S^2 + a_1 S + a_0}, \qquad (4)$$

as an approximation to the 4^{th} order transfer function

$$H(S) = \frac{B_3 S^3 + B_2 S^2 + B_1 S + B_0}{S^4 + A_3 S^3 + A_2 S^2 + A_1 S + A_0}, \qquad (5)$$

as discussed in following sections.

2.2 Stability of Physical System

A linear stationary continuous system is asymptotically stable if and only if all of its eigenvalues have negative real parts. The eigenvalues (or poles) of a system can be obtained by solving for the roots of characteristic function W(S),

$$W(S) = A_n S^n + A_{n-1} S^{n-1} + \cdots + A_1 S + A_0$$
$$= \sum_{i=1}^{n} (S - \mu_i) \ , \qquad (6)$$

where $A_n = 1$ and μ_i is the i^{th} eigenvalue all of which must have negative real parts for a stable system. The part of the response due to the i^{th} eigenvalue is called "the i^{th} dynamic mode".

If the original high-order model is stable, the reduction method by Ouyang guarantees that the reduced model is always stable since the reduced model utilizes only stable poles in the original high-order model.

2.3 Steady State Value For Unit-Step Input

One of the requirements of Ouyang's method is that both the original and reduced models have identical steady state values for a step input.

Consider the arrangement shown in Figure 2.2.

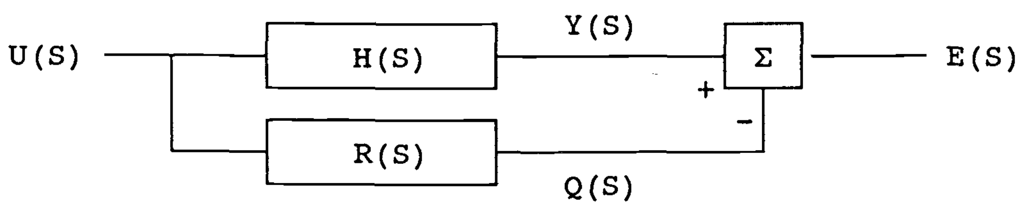


Figure 2.2 Block Diagram

R(S) is the reduced model transfer function and Q(s) is its output. E(S) is the Laplace transform of the difference between the outputs of the original and the reduced models.

Assume the transfer function R(S) has the following form

$$R(S)=\frac{\alpha_{n-1}\,S^{n-1}+\alpha_{n-2}\,S^{n-2}+\cdots+\alpha_1\,S+\alpha_0}{a_n\,S^n+a_{n-1}\,S^{n-1}+a_{n-2}\,S^{n-2}+\cdots+a_1\,S+a_0}\,, \quad (7)$$

where m is the dimension of the reduced model. Using the final value theorem, the steady state value of both the original and reduced models can be found as following

$$\lim_{t\to\infty} Y(t) = \lim_{S\to 0} S\,Y(S)\,, \quad (8)$$

and

$$\lim_{t\to\infty} Q(t) = \lim_{S\to 0} S\,Q(S)\,. \quad (9)$$

Both models must be stable. For a unit-step input, the outputs Y(S) and Q(S) are:

$$Y(S) = H(S)\ U(S)$$

$$= \frac{B_{n-1}\ S^{n-1} + B_{n-2}\ S^{n-2} + \cdots + B_1\ S + B_0}{A_n\ S^n + A_{n-1}\ S_{n-1} + \cdots + A_1\ S + A_0}\ \frac{1}{S}\ , \qquad (10)$$

and

$$Q(S) = R(S)\ U(S)$$

$$= \frac{\alpha_{m-1}\ S^{m-1} + \alpha_{m-2}\ S^{m-2} + \cdots + \alpha_1\ S + \alpha_0}{a_m\ S^m + a_{m-1}\ S^{m-1} + \cdots + a_1\ S + a_0}\ \frac{1}{S}\ . \qquad (11)$$

Substituting equations (10) and (11) into the right side of equations (8) and (9) respectively, the steady state values of two models are

$$\lim_{t \to \infty} Y(t) = \frac{B_0}{A_0}\ , \qquad (12)$$

and

$$\lim_{t \to \infty} Q(t) = \frac{\alpha_0}{a_0}\ . \qquad (13)$$

Since Ouyang's method sets the steady state values equal for both the original and reduced models, then the steady state value is

$$\frac{\alpha_0}{a_0} = \frac{B_0}{A_0}\ . \qquad (14)$$

Equation (14) determine one of the coefficients of the numerator of the reduced model.

2.4 Ouyang's Method

Ouyang's method involves two steps: a power decomposition method to obtain the reduced system denominator polynomial and the use of a least-squared identification technique to find the numerator polynomial of the reduced model.

The power spectrum of the system output is composed of a component due to the input and components due to each of the dynamic modes associated with the system eigenvalues. Thus, the power spectrum of the output is said to be "dispersed". The power dispersion, which is analyzed in detail in Chapter 3, utilizes the contribution of the i^{th} dynamic mode to the total power of the output as a measure of the relative importance of each dynamic mode inthe system output. The denominator of the reduced model is found by discarding several dynamic modes with small power contributions.

The coefficients of the numerator of the reduced model are obtained by using a least-squares parameter optimization method discussed in Chapter 4. Ouyang's method uses a frequency response function to determine the unknown parameters of the numerator.

2.5 Evaluation

In this thesis, an equivalent system is used to study and evaluate Ouyang's method. The model studied here is shown in Figure 2.3,

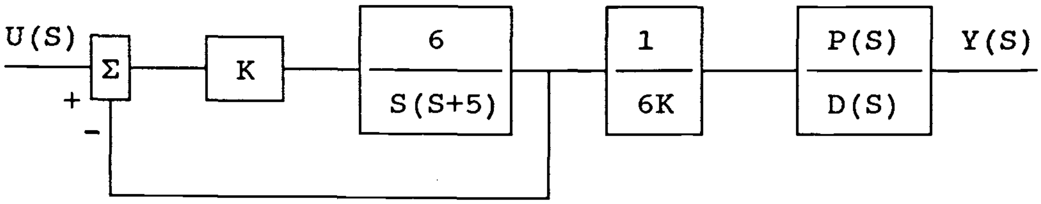


Figure 2.3 Block Diagram of Equivalent Model

where $P(S) = B_3 S^3 + B_2 S^2 + B_1 S + B_0$

$$= 13.2 S^3 + 84.8 S^2 + 167.2 S + 96.8 , \qquad (15)$$

and $D(S) = (S + 1)(S + 4),$ (16)

This model reduces to the single example case studied by Ouyang when K = 1. The transfer function of Ouyang's example is

$$H(S) = \frac{13.2 S^3 + 84.8 S^2 + 167.2 S + 96.8}{S^4 + 10 S^3 + 35 S^2 + 50 S + 24}$$

$$= \frac{13.2 S^3 + 84.8 S^2 + 167.2 S + 96.8}{(S + 1)(S + 2)(S + 3)(S + 4)} . \qquad (17)$$

For the feedback loop in Figure 2.3, the overall transfer function G(S) is

$$G(S) = \frac{6K}{S (S + 5) + 6K} = \frac{6K}{S^2 + 5 S + 6K} . \qquad (18)$$

The transfer function H(S) relating Y(S) and U(S) is

$$H(S) = \frac{P(S)\ G(S)}{6K\ D(S)}$$

$$= \frac{P(S)}{D(S)\ (S^2 + 5\ S + 6K)}$$

$$= \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 1)(S + 4)(S^2 + 5\ S + 6K)} . \quad (19)$$

When K is unity, H(S) is the example transfer function used by Ouyang given by equation (17).

Figure 2.4 shows the location of four poles and three zeros for Ouyang's example in equation (17). The locations of two of four poles in equation (19) depend upon the value of K. The loci of these two roots in the s-plane is shown in figure 2.5. The bold lines in Figure 2.5 represent the root loci of two poles depending on the various values of K. The arrows on the bold lines indicate the direction of pole movement as the value of K changes from zero to infinity.

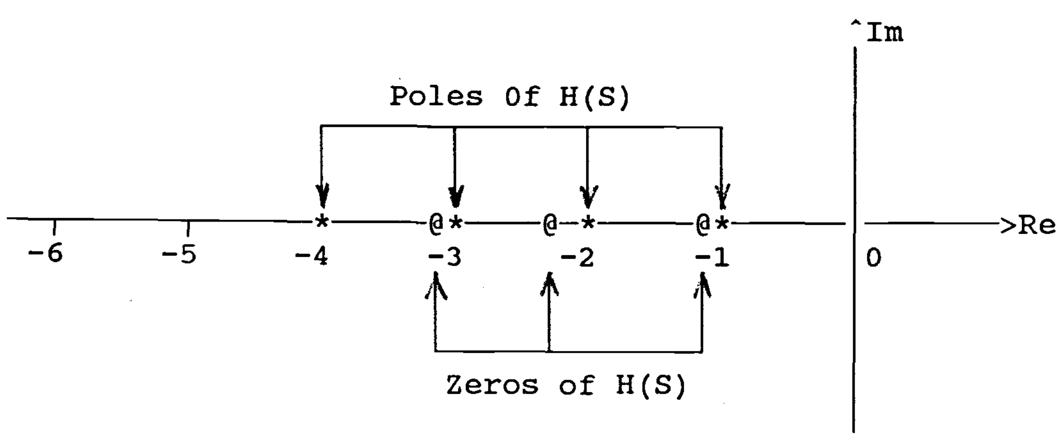


Figure 2.4 The pole and zero location of Ouyang's example

*: open loop pole of H(S) in equation (17)
@: open loop zero of H(S) in equation (17)

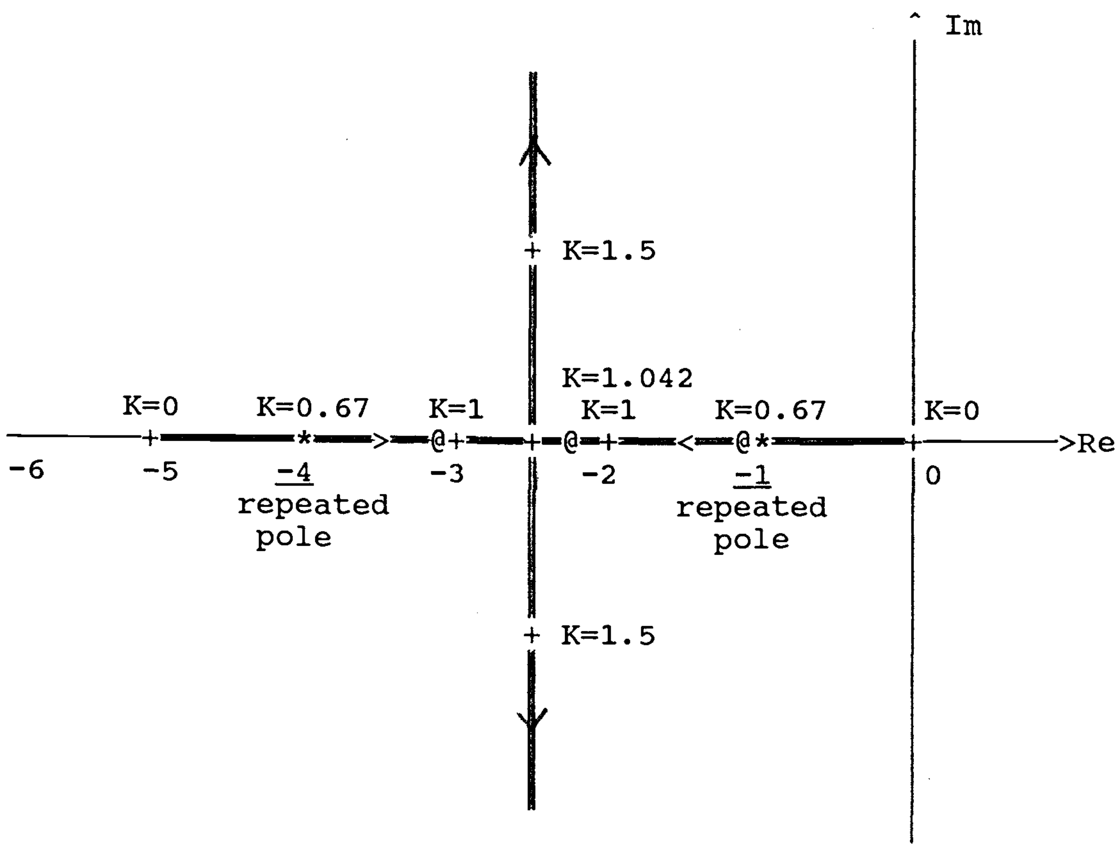


Figure 2.5 Root loci for equivalent system set

D(S) = (s + 1)(s + 4)

*: pole due to D(S)
@: open loop zero of H(S) in equation (17)
+: open loop pole of G(S) in equation (16)

Selecting D(S) by using two different poles of the transfer function H(S) in equation (17), then there are six different equivalent models which can be used to evaluate the reduction method. Figure 2.6 shows another equivalent model set where D(S) is (S + 1)(S + 3). Corresponding to this D(S), the transfer function of system becomes

$$H(S) = \frac{P(S)\ G(S)}{8K\ D(S)} = \frac{P(S)}{D(S)\ (S^2 + 6\ S + 8K)}$$

$$= \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 1)(S + 3)(S^2 + 6\ S + 8K)} \quad . \qquad (20)$$

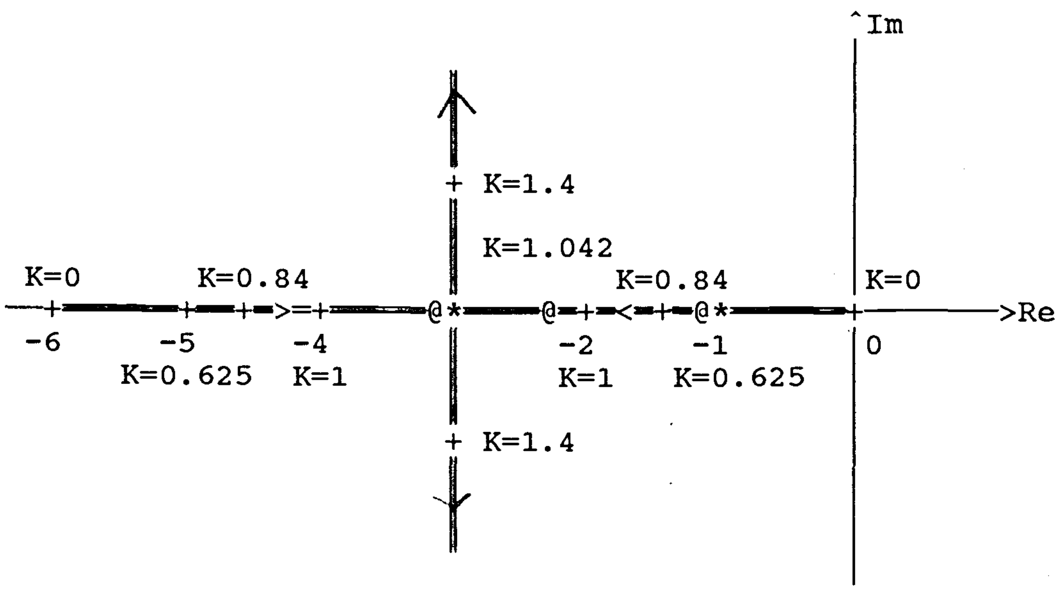


Figure 2.6 Root loci for equivalent system set

D(S) = (s + 1)(s + 3)

*: pole due to D(S)
@: open loop zero of H(S) in equation (19) or (20)
+: open loop pole of G(S) in equation (20)

Various criteria have been used to measure the quality of the reduced model approximation to the original system. To evaluate the Ouyang's method, the integrated squared step response error between the original and the reduced models will be employed here. The ISE, integrated squared error, is defined as

$$ISE = \int_{0}^{\infty} \epsilon^2(t)dt \ , \tag{21}$$

where $\epsilon(t)$ is

$$\epsilon(t) = Y(t) - Q(t) \ , \tag{22}$$

as the difference between the outputs of the original and the reduced models for unit-step input. See Figure 2.2 for the system block diagram. The Laplace transform of $\epsilon(t)$ is

$$E(S) = Y(S) - Q(S) \tag{23}$$

ISE is determined here by numerically integrating $\epsilon^2(t)$ from 0 to 10 seconds using the Euler numerical integration method with a step size of 0.1 seconds. Different cases are obtained by selecting K from 0.1 to 5.0 in increment of 0.1.

3 POWER DISPERSION ANALYSIS

The first step to find a reduced model is to determine the relative importance of each of the system eigenvalues. Refering to Figure 2.1, the n^{th} order transfer function H(S) can be written as

$$H(S) = \frac{Y(S)}{U(S)} = \sum_{i=1}^{N} \frac{h_i}{(S - \mu_i)} , \qquad (24)$$

where h_i are the coefficients of the partial fraction expansion and μ_i are the system eigenvalues which are all assumed simple and stable (Re $\mu_i < 0$). Hereafter the term "the i^{th} dynamic mode" is used to mean that part of the response due to the i^{th} eigenvalue. The unit-impulse response may then be given as

$$h(t) = \sum_{i=1}^{N} h_i e^{\mu_i \tau} , \qquad (25)$$

where in general h_i and μ_i are complex. For real poles the h_i and μ_i are always real for real poles. For complex poles which must occur in conjugate pairs, the values of h_i and μ_i for each pair of complex poles are always conjugates.

If the input, U(t), is taken as white noise with zero mean value, it can be shown that each dynamic mode in the output Y(t) is evenly weighted (Ouyang, 1987). For a linear, time-invariant system, the response of this system having white noise input is

$$Y(t) = \int_{-\infty}^{t} h(t-\tau)\ U(\tau)\ d\tau\ , \qquad (26)$$

where the input $U(\tau)$ is taken as wide-sense stationary. The autocorrelation function of $Y(t)$ is
given as

$$R_{yy}(t,\tau) = E\ [\ Y(t)\ Y(t-\tau)\ ]\ . \qquad (27)$$

For the case when $Y(t)$ is wide sense stationary (see Appendix A for more detail),

$$R(\tau) = R_{yy}(t,\tau)\ . \qquad (28)$$

Then, $R_{yy}(\tau)$ can be calculated from equations (25) and (26), and reduced to

$$R_{yy}(\tau) = E\ [\int_{-\infty}^{t} h(t-v)U(v)dv \int_{-\infty}^{t-\tau} h(t-\tau-w)U(w)dw\ ]$$

$$= E\ [\int_{-\infty}^{t} \sum_{n=1}^{N} h_n e^{\mu_n (t-v)} U(v)\, dv \int_{-\infty}^{t-\tau} \sum_{m=1}^{N} h_m e^{\mu_m (t-\tau-w)} U(w)\, dw\ ]$$

$$= \int_{-\infty}^{t-\tau} \left[\int_{-\infty}^{t} \sum_{n=1}^{N} h_n e^{\mu_n (t-v)} \sum_{m=1}^{N} h_m e^{\mu_m (t-\tau-w)} E\ [U(v)U(w)]dv \right] dw$$

$$= \int_{-\infty}^{t-\tau} \left[\int_{-\infty}^{t} \sum_{n=1}^{N} h_n e^{\mu_n (t-v)} \sum_{m=1}^{N} h_m e^{\mu_m (t-\tau-w)} \delta(v-w)\, \sigma_z^{2} dv \right] dw$$

$$= \int_{-\infty}^{t-\tau} \sum_{n=1}^{N} h_n e^{\mu_n (t-w)} \sum_{m=1}^{N} h_m e^{\mu_m (t-\tau-w)} \sigma_z^{\,2} \, dw$$

$$= \sigma_z^{\,2} \left[\sum_{n=1}^{N} h_n e^{\mu_n t} \sum_{m=1}^{N} h_m e^{\mu_m (t-\tau)} \frac{e^{-(\mu_n+\mu_m) w}}{-(\mu_n + \mu_m)} \right]_{-\infty}^{t-\tau} . \quad (29)$$

According to the assumption that the μ_n and μ_m are in the left half s-plane, the real parts of the μ_n and μ_m are negative. Then,

$$R_{YY}(\tau) = \sigma_z^{\,2} \sum_{n=1}^{N} \sum_{m=1}^{N} h_n h_m \frac{-e^{\mu_n \tau}}{\mu_n + \mu_m}$$

$$= \sigma_z^{\,2} \sum_{n=1}^{N} e^{\mu_n \tau} \left[\sum_{m=1}^{N} \frac{-h_n h_m}{\mu_n + \mu_m} \right] . \quad (30)$$

The average power of Y(t), P_Y, which is given by

$$P_Y = R_{YY}(0) = E\left[Y^2(t) \right] \quad (31)$$

$E[Y^2(t)]$ is the mean squared value of Y(t).

Let

$$p_n = \sum_{m=1}^{N} \frac{-h_n h_m}{\mu_n + \mu_m} , \quad (32)$$

If μ_n and μ_m are real then p_n is real. If μ_n or μ_m are complex, then p_n is also complex. However, there must always be another p_i, for some i, associated with the conjugate pole of μ_n, which is also the conjugate of p_n.

From equations (28), (30) and (32), it follows that

$$R_{YY}(\tau) = \sigma_z^{\,2} \sum_{n=1}^{N} p_n e^{\mu_n \tau} . \quad (33)$$

For $\tau = 0$, then

$$P_y = \sigma_z^2 \sum_{n=1}^{N} p_n \ . \tag{34}$$

Taking the input as white noise with σ_z^2, the power contribution of the n^{th} dynamic mode is equal to $\sigma_z^2 p_n$. Power dispersion for the n^{th} dynamic mode can be defined as

$$PD_n = \frac{p_n \sigma_z^2}{P_y}$$

$$= \frac{p_n}{p_1 + p_2 + \cdots + p_N} \ . \tag{35}$$

The relative power dispersion PD_n, which measures the relative power contribution of the n^{th} dynamic mode to the total power, is a measure of the relative importance of each dynamic mode of the continuous system H(S) (Ouyang, 1987). The denominator of the reduced model is determined by discarding some dynamic modes having small power contribution.

Physical system eigenvalues can be simple (non-repeated) or repeated, and can occur in any part of the complex s-plane. If the system has only simple eigenvalues PD_n can directly be calculated using equations (35). If the system has one or more multiple eigenvalues, it is assumed that each of the repeated eigenvalues can be represented by a cluster of eigenvalues, i.e.,

$$(S - \mu_i)^{r_k} = \prod_{q=1}^{r_k} (S - (\mu_i - (q-1)\theta)), \tag{36}$$

where r_k denotes the multiplicities of the repeated eigenvalue μ_i and θ is a small complex value .

Let

$$
\begin{aligned}
S_i &= \mu_i && \text{for } 1 \le i \le N_0 \\
S_{N0+q} &= \mu_{N0+1} - (q-1)\theta && 1 \le q \le r_1 \\
S_{N0+r1+q} &= \mu_{N0+2} - (q-2)\theta && 1 \le q \le r_2 \\
&\vdots && \vdots \\
S_{N0+r1+\cdots+r(k-1)+q} &= \mu_{N0+k} - (q-1)\theta && 1 \le q \le r_k \\
&\vdots && \vdots \\
S_{N0+r1+\cdots+r(M-1)+q} &= \mu_{N0+M} - (q-1)\theta && 1 \le q \le r_M
\end{aligned}
$$

and

$$N = N_0 + \sum_{k=1}^{M} r_k , \tag{37}$$

where N_0 is the number of the simple eigenvalues, M is the number of the repeated eigenvalues, and r_k denotes the multiplicity for the k^{th} repeated pole. Note that the resulting S_i are all simple poles.

Then the transfer function (24) can be written as

$$
\begin{aligned}
H(S) &= \sum_{i=1}^{N_0} \frac{h_i}{(S - \mu_i)} + \sum_{k=1}^{M} \sum_{q=1}^{r_k} \frac{g_{kq}}{(S - (\mu_{N0+k} - (q-1)\theta))} \\
&= \sum_{i=1}^{N} \frac{h_i}{(S - s_i)} .
\end{aligned} \tag{38}
$$

where g_{kq} are complex. Consequently the impulse response (25) can be written as

$$h(t) = \sum_{i=1}^{N} h_i e^{s_i \tau} \tag{39}$$

Replacing μ_n by s_i in equation (25), then the calculation of p_n and PD_n can be accomplished for a system with repeated eigenvalues.

When eigenvalues of a system are complex, p_n in equation (32) appears in a pair of conjugate complex values. For each conjugate complex eigenvalue μ_i with corresponding p_i, there exists another p corresponding to the conjugate complex mode of μ_i which is the complex conjugate of p_i. Here, $R_{yy}(0)$ can be written as

$$R_{yy}(0) = p_1 +\cdots+ p_{NO} + 2\ p_{ri} +\cdots+ 2\ p_{rk} \tag{40}$$

where $p_{ri}\cdots p_{rk}$ are the real part of the coefficients of the autocorrelation function, $p_i\cdots p_k$, for the $i^{th}\cdots k^{th}$ pair of complex conjugate eigenvalues. The magnitude of the power dispersion of either one of a pair of complex eigenvalues has the same value. Consequently, in determining the denominator of the reduced model, the conjugate pairs of the complex eigenvalues either are retained or discarded simultaneously.

The denominator of the reduced model transfer function, W(S), is found as the following

$$W(S) = \sum_{i=1}^{m} (S - \beta_i) \tag{41}$$

where m is the dimension of the reduced model and β_i is the value of the retained eigenvalue. The eigenvalues β_i are selected by assigning the pole with the largest PD to β_1, the next largest PD to β_2, and so on so forth until β_m is assigned.

Consider the 4^{th} order systems from Ouyang's example and the equivalent model set proposed in Section 2.5. The transfer functions for each of these examples are given by equations (17) and (19), separately.

The relative power dispersion analysis of H(S) for various values of K is listed in Table 3.1. The coefficient of white noise input, σ_z^2, is normalized to unity, and total power dispersion equals unity, i.e.,

$$\sum_{n=1}^{4} PD_n = 1 \tag{42}$$

When K=1, this model is exactly that of Ouyang's example. In Table 3.1, the values of p and PD for the four eigenvalues are shown for various values of K. From this table it can be seen that the values of PD_3 and PD_4 are greater than those for PD_1 and PD_2 except for K = 0.5. Negative power dispersion is discussed in Chapter 5. It follows, then, that the eigenvalues μ_1 and μ_2, corresponding to PD_1 and PD_2 in each model, are the candidates to be discarded.

TABLE 3.1 THE POWER DECOMPOSITION OF SOME SYSTEMS

K		(a+jb)			(c+jd)			
K=0.5	μ_i=	-4.0000	0.0000	p_i=	-38.0377	0.0000	PD_i=	-1.0955
		-1.0000	0.0000		-1.6088	0.0000		-0.0463
		-0.6972	0.0000		21.9418	0.0000		0.6319
		-4.3028	0.0000		52.4267	0.0000		1.5099
K=1.0	μ_i=	-1.0000	0.0000	p_i=	0.6033	0.0000	PD_i=	0.0240
		-3.0000	0.0000		2.0453	0.0000		0.0812
		-2.0000	0.0000		4.8666	0.0000		0.1933
		-4.0000	0.0000		17.6619	0.0000		0.7015
K=1.5	μ_i	-1.0000	0.0000	p_i=	0.1931	0.0000	PD_i=	0.0087
		-4.0000	0.0000		6.5938	0.0000		0.2985
		-2.5000	1.6583		7.6514	6.7222		0.3464
		-2.5000	-1.6583		7.6514	-6.7222		0.3464
K=2.0	μ_i=	-1.0000	0.0000	p_i=	0.1006	0.0000	PD_i=	0.0049
		-4.0000	0.0000		3.8636	0.0000		0.1876
		-2.5000	2.3979		8.3170	6.2514		0.4038
		-2.5000	-2.3979		8.3170	-6.2514		0.4038
K=2.5	μ_i=	-1.0000	0.0000	p_i=	0.0627	0.0000	PD_i=	0.0032
		-4.0000	0.0000		2.6446	0.0000		0.1340
		-2.5000	2.9580		8.5143	5.7315		0.4314
		-2.5000	-2.9580		8.5143	-5.7315		0.4314
K=3.0	μ_i=	-1.0000	0.0000	p_i=	0.0431	0.0000	PD_i=	0.0022
		-4.0000	0.0000		1.9625	0.0000		0.1023
		-2.5000	3.4278		8.5888	5.2924		0.4477
		-2.5000	-3.4278		8.5888	-5.2924		0.4477
K=3.5	μ_i=	-1.0000	0.0000	p_i=	0.0315	0.0000	PD_i=	0.0017
		-4.0000	0.0000		1.5310	0.0000		0.0814
		-2.5000	3.8406		8.6207	4.9290		0.4585
		-2.5000	-3.8406		8.6207	-4.9290		0.4585
K=4.0	μ_i=	-1.0000	0.0000	p_i=	0.0241	0.0000	PD_i=	0.0013
		-4.0000	0.0000		1.2363	0.0000		0.0667
		-2.5000	4.2131		8.6354	4.6253		0.4660
		-2.5000	-4.2131		8.6354	-4.6253		0.4660
K=4.5	μ_i=	-1.0000	0.0000	p_i=	0.0191	0.0000	PD_i=	0.0010
		-4.0000	0.0000		1.0239	0.0000		0.0559
		-2.5000	4.5552		8.6424	4.3679		0.4715
		-2.5000	-4.5552		8.6424	-4.3679		0.4715
K=5.0	μ_i=	-1.0000	0.0000	p_i=	0.0155	0.0000	PD_i=	0.0009
		-4.0000	0.0000		0.8646	0.0000		0.0476
		-2.5000	4.8734		8.6459	4.1467		0.4758
		-2.5000	-4.8734		8.6459	-4.1467		0.4758

4 FREQUENCY RESPONSE MATCHING

4.1 DETERMINATION OF THE NUMERATOR OF THE REDUCED MODEL

The second step in finding a reduced model is to determine the coefficients of the numerator of R(S) using a frequency response identification technique. If the poles of the reduced model are $\beta_1, \beta_2, \cdots, \beta_m$, then the transfer function of this reduced model, R(S) in equation (7), can be rewritten as

$$R(S) = \frac{C(S)}{W(S)}$$

$$= \frac{\sum_{j=0}^{m-1} \alpha_j S^j}{S^m + \sum_{j=0}^{m-1} a_j S^j}$$

$$= \frac{\alpha_0 + \alpha_1 S + \cdots + \alpha_{m-1} S^{m-1}}{(S - \beta_1)(S - \beta_2)\cdots(S - \beta_m)}$$

$$= \frac{\alpha_0 + \alpha_1 S + \cdots + \alpha_{m-1} S^{m-1}}{a_0 + a_1 S + \cdots + a_m S^m}, \tag{43}$$

where $a_0, a_1, \cdots, a_m$ are known values which are the coefficients of W(S), and $\alpha_0, \alpha_1, \cdots, \alpha_{m-1}$ need to be determined.

At S = jw, the transfer function frequency response of the original system and the reduced model can be expressed as

$$H(jw) = H_R(w) + j\,H_I(w)$$
$$= \frac{(B_0 - B_2w^2 + \cdots) + j(B_1w - B_3w^3 + \cdots)}{(A_0 - A_2w^2 + \cdots) + j(A_1w - A_3w^3 + \cdots)}, \qquad (44)$$

and

$$R(jw) = \frac{C(jw)}{W(jw)}$$
$$= \frac{(\alpha_0 - \alpha_2w^2 + \cdots) + j(\alpha_1w - \alpha_3w^3 + \cdots)}{(a_0 - a_2w^2 + \cdots) + j(a_1w - a_3w^3 + \cdots)}$$
$$= [F_R(w) + jF_I(w)][(\alpha_0 - \alpha_2w^2 + \cdots) + j(\alpha_1w - \alpha_3w^3 + \cdots)], \qquad (45)$$

where

$H_R(w)$ = real part of $H(jw)$

$H_I(w)$ = imaginary part of $H(jw)$

$F_R(w)$ = real part of $\{W(jw)\}^{-1}$

$F_I(w)$ = imaginary part of $\{W(jw)\}^{-1}$.

Let the difference between $H(jw)$ and $R(jw)$ be $\epsilon(jw)$, then

$$H(jw) = R(jw) + \epsilon(jw) \qquad (46)$$

The steady-state value of the reduced model is set equal to that of the original system. From equation (14),

$$\alpha_0 = a_0 \frac{B_0}{A_0} \qquad (47)$$

Substituting (44) and (45) into (46), equation (46) becomes

$$H_R(w) + jH_I(w) = [F_R(w) + jF_I(w)]$$
$$*[(\alpha_0 - \alpha_2 w^2 + \cdots) + j(\alpha_1 w - \alpha_3 w^3 + \cdots)]$$
$$+ [\epsilon_R(w) + j\epsilon_I(w)]$$
$$= F_R(w)(\alpha_0 - \alpha_2 w^2 + \cdots)$$
$$- F_I(w)(\alpha_1 w - \alpha_3 w^3 + \cdots)$$
$$+ j[F_I(w)(\alpha_0 - \alpha_2 w^2 + \cdots)$$
$$+ F_R(w)(\alpha_1 w - \alpha_3 w^3 + \cdots)]$$
$$+ [\epsilon_R(w) + j\epsilon_I(w)], \quad (48)$$

where

$$\epsilon_R(w) = \text{real part of } \epsilon(jw)$$
$$\epsilon_I(w) = \text{imaginary part of } \epsilon(jw)$$

The real and imaginary parts of both sides in equation (48) must equal for any frequency w, i.e.

$$H_R(w) = F_R(w)(\alpha_0 - \alpha_2 w^2 + \cdots)$$
$$- F_I(w)(\alpha_1 w - \alpha_3 w^3 + \cdots) + \epsilon_R(w)$$
$$H_I(w) = F_I(w)(\alpha_0 - \alpha_2 w^2 + \cdots)$$
$$+ F_R(w)(\alpha_1 w - \alpha_3 w^3 + \cdots) + \epsilon_I(w) , \quad (49)$$

Moving the two terms involving α_0 on the right side of equation (49), i.e. $F_R(w)\alpha_0$ and $jF_I(w)\alpha_0$, to the left side of equation (49) and setting

$$Y_R(w) = H_R(w) - F_R(w)\alpha_0$$
$$Y_I(w) = H_I(w) - F_I(w)\alpha_0 \quad (50)$$

Equation (49) becomes

$$Y_R(w) = F_R(w)(-\alpha_2 w^2 + \alpha_4 w^4 - \cdots)$$
$$- F_I(w)(\alpha_1 w - \alpha_3 w^3 + \cdots) + \epsilon_R(w)$$
$$Y_I(w) = F_I(w)(-\alpha_2 w^2 + \alpha_4 w^4 - \cdots)$$
$$+ F_R(w)(\alpha_1 w - \alpha_3 w^3 + \cdots) + \epsilon_I(w) . \quad (51)$$

For i = 1 to i = m - 1, set

$$X_{Ri}(w) = (-1)^{(i+1)/2}\, F_I(w)\, w^i \qquad i\text{:odd}$$
$$= (-1)^{i/2}\, F_R(w)\, w^i \qquad i\text{:even}$$
$$X_{Ii}(w) = (-1)^{(i-1)/2}\, F_R(w)\, w^i \qquad i\text{:odd}$$
$$= (-1)^{i/2}\, F_I(w)\, w^i \qquad i\text{:even}$$

Then, equation (49) can be expressed in matrix form, as

$$\begin{bmatrix} Y_R(w) \\ Y_I(w) \end{bmatrix} = \begin{bmatrix} X_{R1}(w), X_{R2}(w), \cdots, X_{R(m-1)}(w) \\ X_{I1}(w), X_{I2}(w), \cdots, X_{I(m-1)}(w) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m-1} \end{bmatrix} + \begin{bmatrix} \epsilon_R(w) \\ \epsilon_I(w) \end{bmatrix} \tag{52}$$

Let $w = w_1, \cdots w_i \cdots, w_n$. Then there are 2n equations which result when the n values of w are substituted into equation (52),

$$\begin{bmatrix} Y_R(w_1) \\ Y_R(w_2) \\ \vdots \\ Y_R(w_n) \\ Y_I(w_1) \\ Y_I(w_2) \\ \vdots \\ Y_I(w_n) \end{bmatrix} = \begin{bmatrix} X_{R1}(w_1), \cdots, X_{R(m-1)}(w_1) \\ X_{R1}(w_2), \cdots, X_{R(m-1)}(w_2) \\ \vdots \qquad\qquad \vdots \\ X_{R1}(w_n), \cdots, X_{R(m-1)}(w_n) \\ X_{I1}(w_1), \cdots, X_{I(m-1)}(w_1) \\ X_{I2}(w_2), \cdots, X_{I(m-1)}(w_2) \\ \vdots \qquad\qquad \vdots \\ X_{I1}(w_n), \cdots, X_{I(m-1)}(w_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{m-1} \end{bmatrix} + \begin{bmatrix} \epsilon_R(w_1) \\ \epsilon_R(w_2) \\ \vdots \\ \epsilon_R(w_n) \\ \epsilon_I(w_1) \\ \epsilon_I(w_2) \\ \vdots \\ \epsilon_I(w_n) \end{bmatrix} . \tag{53}$$

Let C be a vector and

$$C = [\ \alpha_1, \alpha_2, \cdots, \alpha_{m-1}\ ]^T . \tag{54}$$

The matrix equality in equation (53) can now be written as

$$Y(w) = X(w)\ C + \epsilon(w)\ . \tag{55}$$

where

$$Y(w) = \left[Y_R(w_1)\ \cdots\ Y_R(w_n)\ Y_I(w_1)\ \cdots\ Y_I(w_n)\right]^T$$

$$X(w) = \begin{bmatrix} X_{R1}(w_1) & X_{R2}(w_1) & \cdots & X_{R(m-1)}(w_1) \\ X_{R1}(w_2) & X_{R2}(w_2) & \cdots & X_{R(m-1)}(w_2) \\ \vdots & \vdots & & \vdots \\ X_{R1}(w_n) & X_{R2}(w_n) & \cdots & X_{R(m-1)}(w_n) \\ X_{I1}(w_1) & X_{I2}(w_1) & \cdots & X_{I(m-1)}(w_1) \\ X_{I1}(w_2) & X_{I2}(w_2) & \cdots & X_{I(m-1)}(w_2) \\ \vdots & \vdots & & \vdots \\ X_{I1}(w_n) & X_{I2}(w_n) & \cdots & X_{I(m-1)}(w_n) \end{bmatrix}$$

$$\epsilon(w) = \left[\epsilon_R(w_1)\ \cdots\ \epsilon_R(w_n)\ \epsilon_I(w_1)\ \cdots\ \epsilon_I(w_n)\right]^T .$$

Applying the least-squares identification method, the error criterion Q(C) is the sum of error squared (Pacut, 1988),

$$Q(C) = \sum_{w=w_0}^{w_n} |\epsilon(w)|^2 = \sum_{w=w_0}^{w_n} |Y(w) - X(w)\ C|^2\ . \tag{56}$$

The α parameters are chosen to minimize Q(C). When $(X^T(w)\ X(w))^{-1}$ exists, the value of the unknown vector C is defined and then given by

$$C_{LS} = (X^T\ X)^{-1}\ X^T\ Y\ . \tag{57}$$

Thus, the denominator and the numerator of the reduced model are now determined.

4.2 FREQUENCY RESPONSE OF THE MODELS

The reduced models are determined by utilizing first the power decomposition method and then the least-squares α parameters identification method. Appendix B-1 lists the coefficients of the denominator and numerator of some reduced models corresponding to the equivalent original systems given by equation (19).

Figure 4.1 through 4.3 show the Nyquist plots for three typical original and reduced models. Comparing these figures, it can be seen that the frequency response of the reduced model in Figure 4.3 whose original system with $K = 5$ gives larger error than those of other two models whose original systems with $K = 0.1$ and 1.0 respectively. In Figure 4.3, the maximum error between the reduced model and the original system appears at $w = 6$ with a relative magnitude of 0.02 and a phase angle of 2.6°.

$$H(S) = \frac{13.2\ S^3 + 35\ S^2 + 167.2\ S + 96.8}{(S + 0.123)(S + 4.877)(S^2 + 5\ S + 4)}$$

$$R(S) = \frac{12.62\ S + 24.19}{S^2 + 5\ S + 0.6}$$

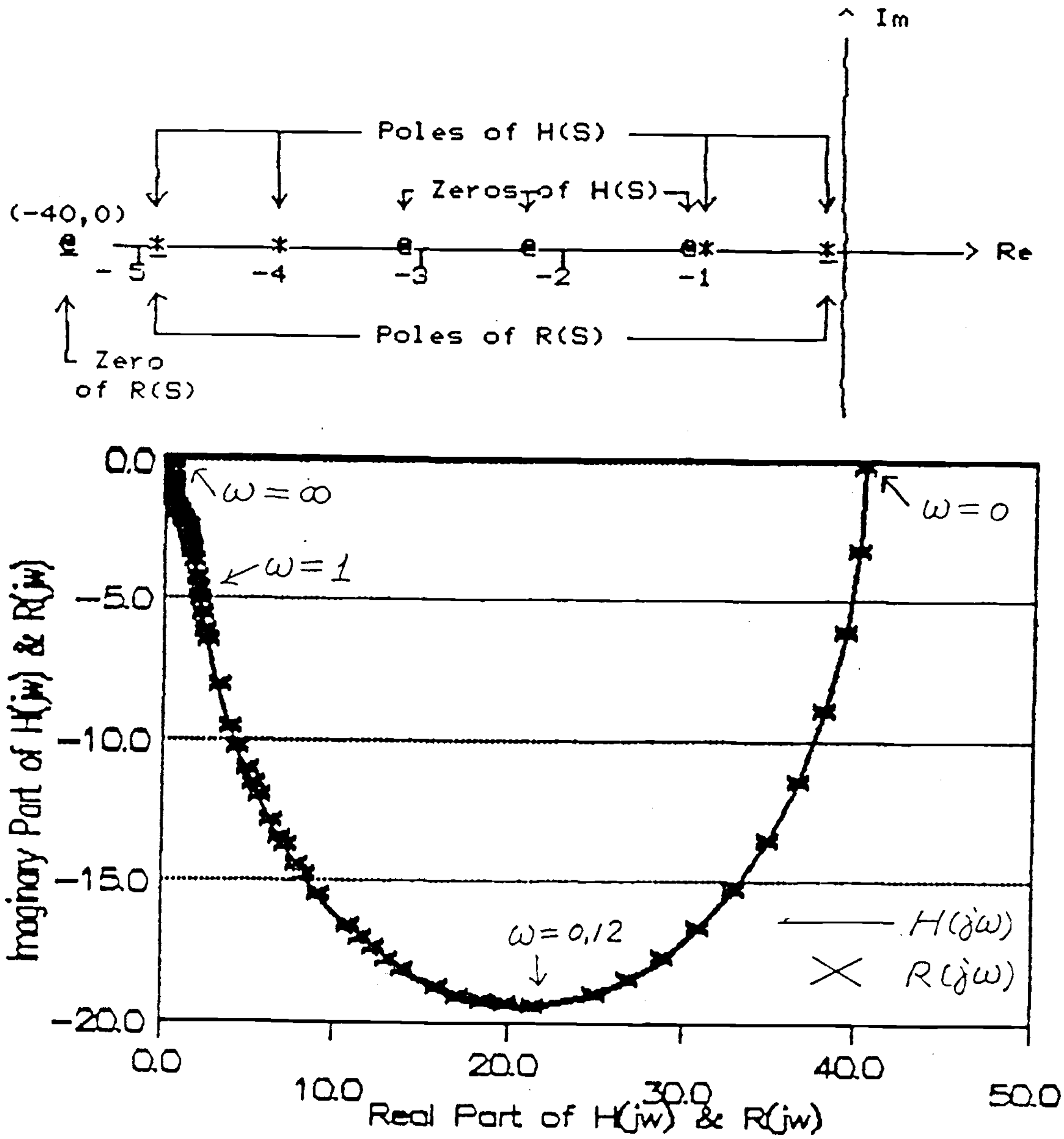


Figure 4.1 Nyquist Plot for System with K = 0.1

$$H(S) = \frac{13.2\, S^3 + 35\, S^2 + 167.2\, S + 96.8}{(S + 2.0)(S + 3.0)(S^2 + 5\, S + 4)}$$

$$R(S) = \frac{23.19\, S + 32.27}{S^2 + 6\, S + 8}$$

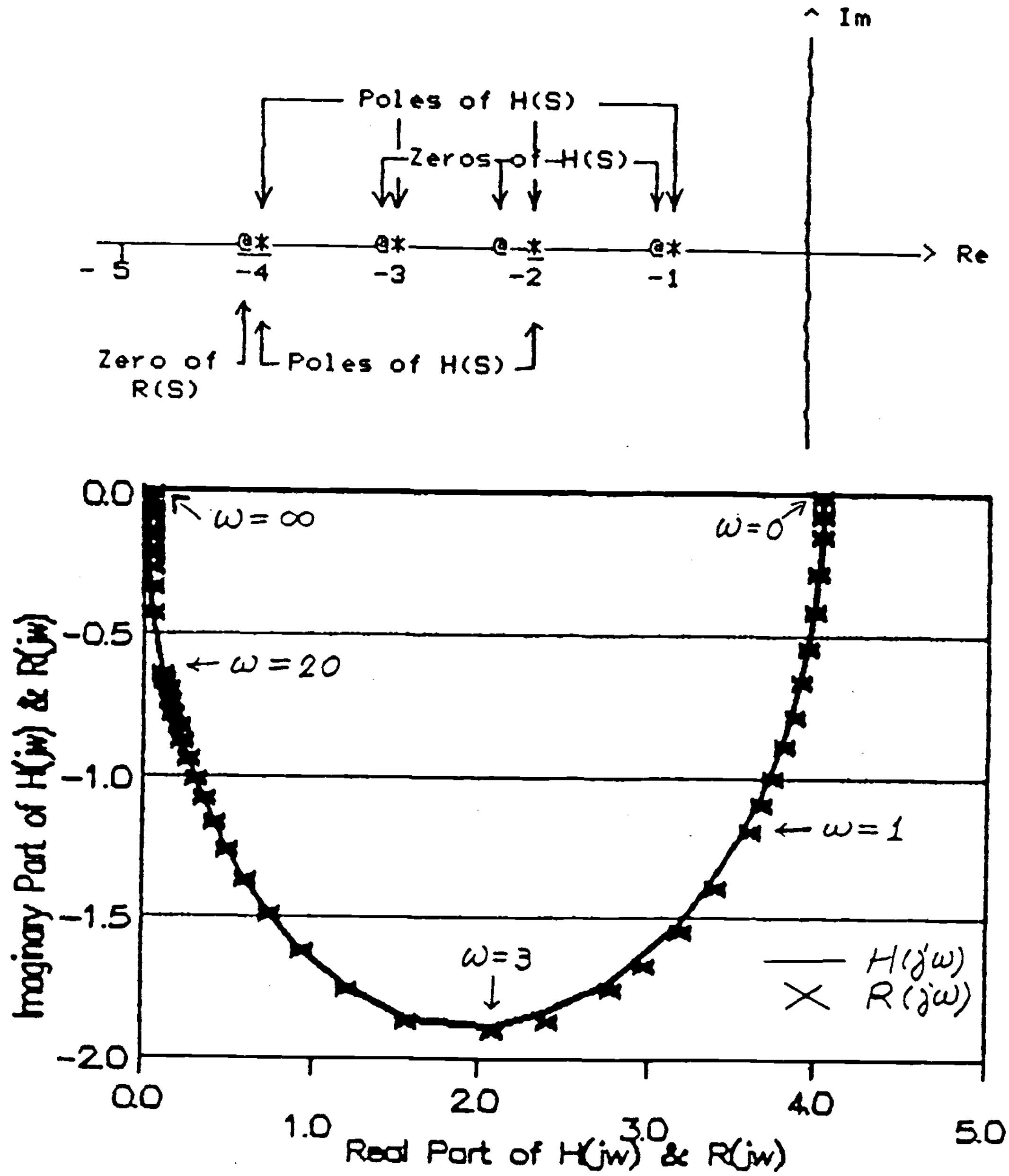
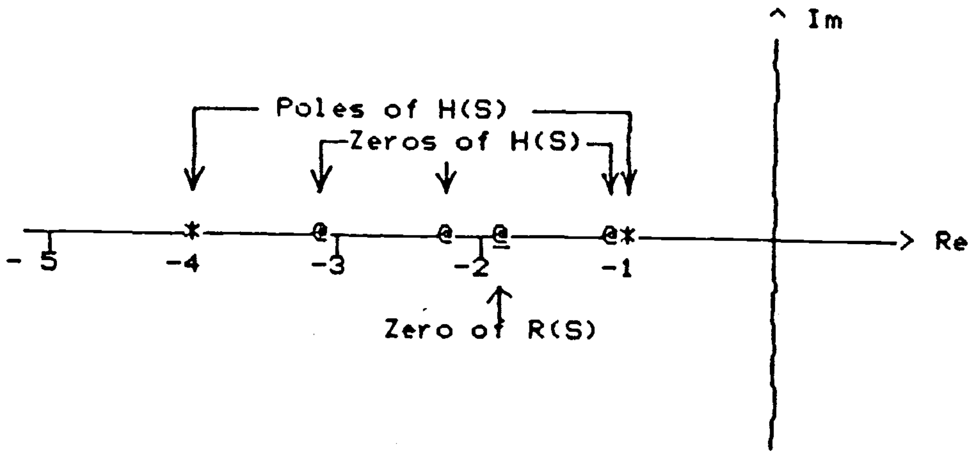


Figure 4.2 Nyquist Plot for System with K = 1.0

$$H(S) = \frac{13.2\ S^3 + 35\ S^2 + 167.2\ S + 96.8}{(S+2.5+j4.87)(S+2.5-j4.87)(S^2 + 5S + 30)}$$

$$R(S) = \frac{12.75\ S + 24.20}{S^2 + 5\ S + 12}$$

Pole of H(S) & R(S) —> $\pm$ (-2.5,4.8)



Pole of H(S) & R(S) —> $\pm$ (-2.5,-4.8)

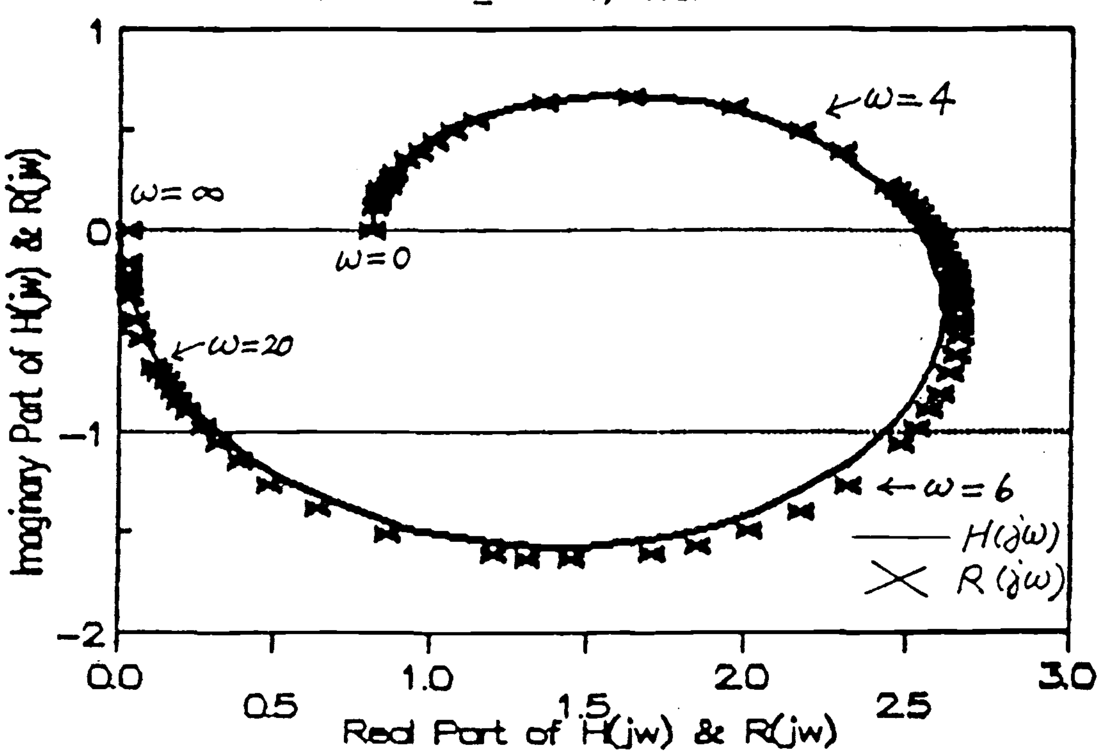


Figure 4.3 Nyquist Plot for System with K = 5

5 EVALUATION

5.1 Error Analysis for the Equivalent Models

In this chapter the integrated squared error, ISE, as defined in Section 2.4, is employed to compare the reduced model to the original system. Since the reduced models give the same steady state values to the original models, it is convenient to fix the limits on the integrating time interval. For the cases studied here the error is negligible for $t > 10$ seconds. The model used is shown in equation (19). K is varied for 0.1 to 5.0. By so doing, a variety of pole-zero configurations is obtained which strenuously tests the reduction method. Appendix B-1 contains data for the models studied here.

Figures 5.1 through 5.6 show representative unit-step response curves ranging from lightly damped to overdamped. The reduced models give good approximations to the original models for all six examples. Figure 5.7 shows a plot of maximum relative error, i.e. $|\epsilon(t_i)|/|Y(t_i)|$. From Figure 5.7, it is noted that the ratio of the maximun error $\epsilon(t_i)$ and the corresponding output $Y(t_i)$ for all cases studied is less than 6 %.

The values of ISE for the 50 models studied is shown in Figure 5.8. In this case, the denominator of the reduced model is taken as D(S) = (S + 1)(S + 4). From Figure 5.8, the ISE for systems with their poles close to the origin have larger errors than those with their poles removed from the origin.

It is possible to select D(S) in six different ways. Figure 5.9 through 5.13 show the other five choices of D(S). The equivalent model with K = 1 is the same as the example studied by Ouyang.

It is observed that the reduced models with smaller values of K have larger ISE values than the models with larger K values. This is because the response of a system with poles closer to the origin reflects a longer transient period than those systems with poles farther from the origin. Thus, the error is significant for a longer period of time.

Figure 5.14 through 5.18 show the ramp responses for some of the original systems and their reduced models. The maximum relative errors appear as the original systems have a parameter K equal to 5.0. In frequency responses of the transfer functions, the maximum error magnitude is 0.02 and phase angle is 2.6°. In unit-step response, the maximum relative error is 0.0593 as t = 0.5 second. In ramp response, the maximum relative error is 0.039 as t = 0.9 second.

5.2 Negative Power Dispersion

In Table 3.1, there are two negative power dispersion values , PD_1 and PD_2, when K = 0.5. In the equivalent model sets of the previous section, if all poles of the equivalent models are located on the negative real axis of s-plane, the coefficients of the autocorrelation function can be both positive and negative. Ouyang does not point out how to select the retained poles for the reduced model when negative power dispersion occurs.

From the autocorrelation function in equation (33), it can be seen that $R_{yy}(\tau)$ is a summation of the terms $\sigma_z^2 p_n e^{\underline{\mu}_n \tau}$, n = 1, 2, $\cdots$, N. For the discussion that follows, $\sigma_z^2 = 1$. The autocorrelation function curve is composed of several curves corresponding to the $p_n e^{\underline{\mu}_n \tau}$. For real poles, if the value of p_n is negative, the curve represented by the term $p_n e^{\underline{\mu}_n \tau}$ is always negative as demonstrated in Figure 5.19.

A new method is now introduced to treat the case where negative power dispersion occurs. The method is based on fitting the autocorrelation function of the reduced system output to that of the original system for a set of time difference values. Ouyang's method uses only one time difference, i.e. $\tau = 0$. The new method involves the following steps:

Step 1- Select m, the number of poles of the reduced system where $m < n$.

Step 2- Select m poles from the original system H(S).

Step 3- Determine the α parameters of the reduced system to minimize the frequency response error between the original and reduced systems using the method previously discussed.

Step 4- Calculate the summed squared error between the original and reduced system autocorrelation functions for a set of time differences, i.e. τ_i, i= 1,2,...,N.

Step 5- Repeat Step 2-4 for other combinations of m poles selected from H(S).

Step 6- The best reduced model is chosen as the one having the least summed squared error between output autocorrelation functions as calculated in Step 4.

This method leads to the best choice of poles for the reduced system which are contained in the original system.

A variant to this method is to modify Steps 3 and4 as follows:

Step 3,4- Determine the α parameters of the reduced system to minimize the summed squared error between the original and reduced system output autocorrelation functions for a set of time differences, i.e. τ_i, i=1,2,..., using a minimum search method.

This variation determines the α parameters directly which minimize the autocorrelation function error instead of matching frequency responses. It should be noted that the α parameters for the two method are not necessarily identical.

The error criterion of the output autocorrelation function, J, is defined as

$$J = \sum_{i=1}^{N} |R_{yy}(\tau_i) - R_{qq}(\tau_i)|^2 \tag{59}$$

where $R_{yy}(\tau_i)$ and $R_{qq}(\tau_i)$ are the output autocorrelation functions of the original system and the reduced model separately. If a white noise is applied to a reduced system, $R_{qq}(\tau_i)$ has the same form as in equation (33)

$$R_{qq}(\tau_i) = \sigma_z^2 \sum_{j=1}^{m} q_j e^{\beta_j \tau} \tag{60}$$

where β_j are the retained poles for the reduced model and q_j is the coefficient of $R_{qq}(\tau_i)$. The value of q_j is

$$q_j = \sum_{i=1}^{m} \frac{-g_j g_i}{\beta_j + \beta_i} \tag{61}$$

where g_j and g_i are the residues of the reduced model transfer function which can be given by the following transfer function

$$R(S) = \frac{\alpha_{m-1} S^{m-1} + \cdots + \alpha_1 S + a_0}{S^m + a_{m-1} S^{m-1} + \cdots + a_1 S + a_0}$$

$$= \sum_{j=1}^{m} \frac{g_j}{(S - \beta_j)} \tag{62}$$

The coefficients, α_1, $\alpha_2, \cdots$, α_{m-1}, can be obtained by the least-squared identification method for each possible combination of the original system poles. For example, if a system is 4^{th}-order, there are four poles. First, six possible pole combinations can be taken if a 2^{nd} order reduced model is desired. Second, by substituting six pole combinations into equation (52) separately, six groups of the α coefficients can be determined using least-squared frequency identification method. Third, six values of J can be calculated using equations (59) - (62). The best fit collection of poles are those which results the smallest value of J.

The second way to obtain the best fit poles is to minimize the error criterion J using minimum search method (Press, 1986). For example, there are six pole combinations for a desired 2^{th} order reduced model due to a 4^{th} order original system. In equation (62), g_j can be as a function of unknown parameters α_j. Thus, J is also a function of α_j. By using a minimum search method, six minimum values of Js corresponding to different pole combinations, and six group corresponding values of α_j can be calculated out. Comparing six minimum values of Js, the best fit poles due to the smallest J can be determined.

The two methods provide the same result: the best fit poles for the reduced model are identical. Appendix B-3 lists the values of Js yielded by the above two ways. In

Appendix B-3, μ_i are the poles of the equivalent original systems in equation(19); P_i are the coefficients of the output autocorrelation function corresponding to each μ_i; and β_1 and β_2 are the possible retained poles for the 2^{nd} order reduced systems. J(FD) and J(MS) are the values obtained by the above two methods respectively. For the case when K = 0.5, it can be seen that the best fit poles for the reduced model are -0.6972 and -4.3028 due to the smallest values of J(FD) or J(MS). This result is the same as that obtained by power dispersion method by which the reduced model poles are due to the larger positive power dispersion.

5.3 Study of System with All Complex Poles

So far the original systems studied here had only real poles located on the negative real axis, or a single pair of conjugate complex poles and two negative real poles.

Consider the case when the original system has two pairs of conjugate complex poles. A system has the following transfer function

$$H(S) = \frac{1}{S^4 + 0.8\ S^3 + 1.49\ S^2 + 0.532\ S + 0.3016}, \quad (63)$$

with poles located on

$$S_{1,2} = 0.2 \pm j\ 0.5$$

$$S_{3,4} = 0.2 \pm j\ 1.0$$

Using Ouyang's method, the time response for unit-step input is shown in Figure 5.20. The response of the reduced model has undershoot for t < 2.5 second. The error between both peak overshoot is 15.625%. The transient response of the reduced model does not approach that of the original system. Ouyang's method gives unsatisfactory result for this example.

Here is an another example. A system has the following transfer function

$$H(S) = \frac{S^3 + 7.5\ S^2 + 18.25\ S + 13.875}{S^4 + 6\ S^3 + 15.5\ S^2 + 21.5\ S + 13.8125}. \quad (64)$$

The equivalent model of this system has the following transfer function form:

$$R(S) = \frac{S^3 + 7.5\ S^2 + 18.25\ S + 13.875}{(S^2 + 4\ S + 4.25K)(S^2 + 2\ S + 3.25)}, \qquad (65)$$

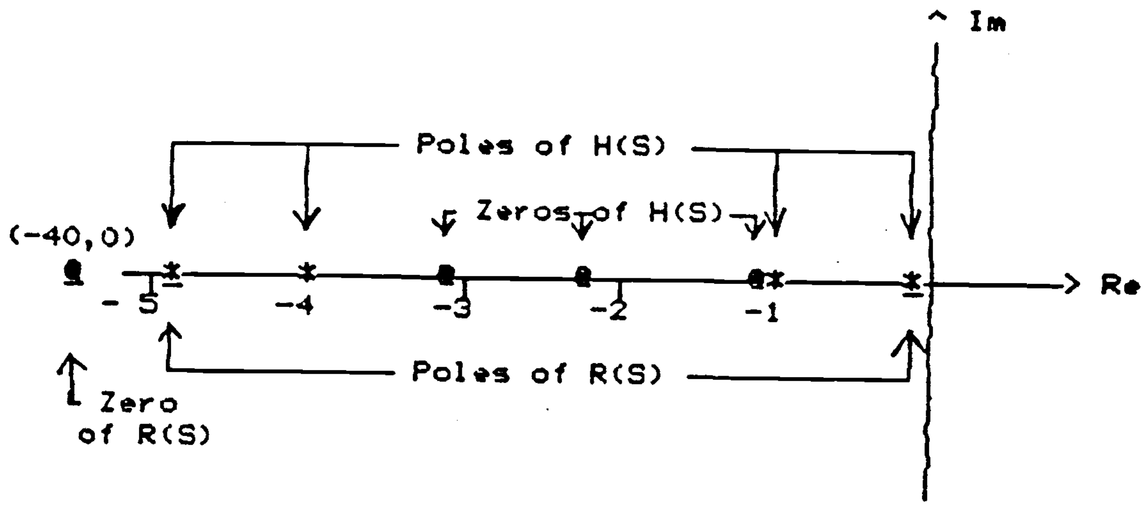
where

$$D(S) = S^2 + 2\ S + 3.25$$
$$= (S - (-1+j1.5))(S-(-1+j1.5)). \qquad (66)$$

Figure 5.21 shows the values of ISE as a function of parameter K. Only when K is close to one does the reduced model have a good approximation. Comparing Figure 5.21 and Figure 5.8, it can be seen that most of the reduced models corresponding to complex poles offer errors 5 - 10 times as great as those system with all or partial real poles. From above two examples, it can also be seen that Ouyang' method can give unsatisfactory results for systems having two pairs of complex poles. Hence, one should be careful when applying Ouyang's method to original systems having all complex poles.

$$H(S) = \frac{13.2\ S^2 + 35\ S^2 + 167.2\ S + 96.8}{(S + 0.123)(S + 4.877)(S^2 + 5\ S + 4)}$$

$$R(S) = \frac{12.62\ S + 24.19}{S^2 + 5\ S + 0.6}$$



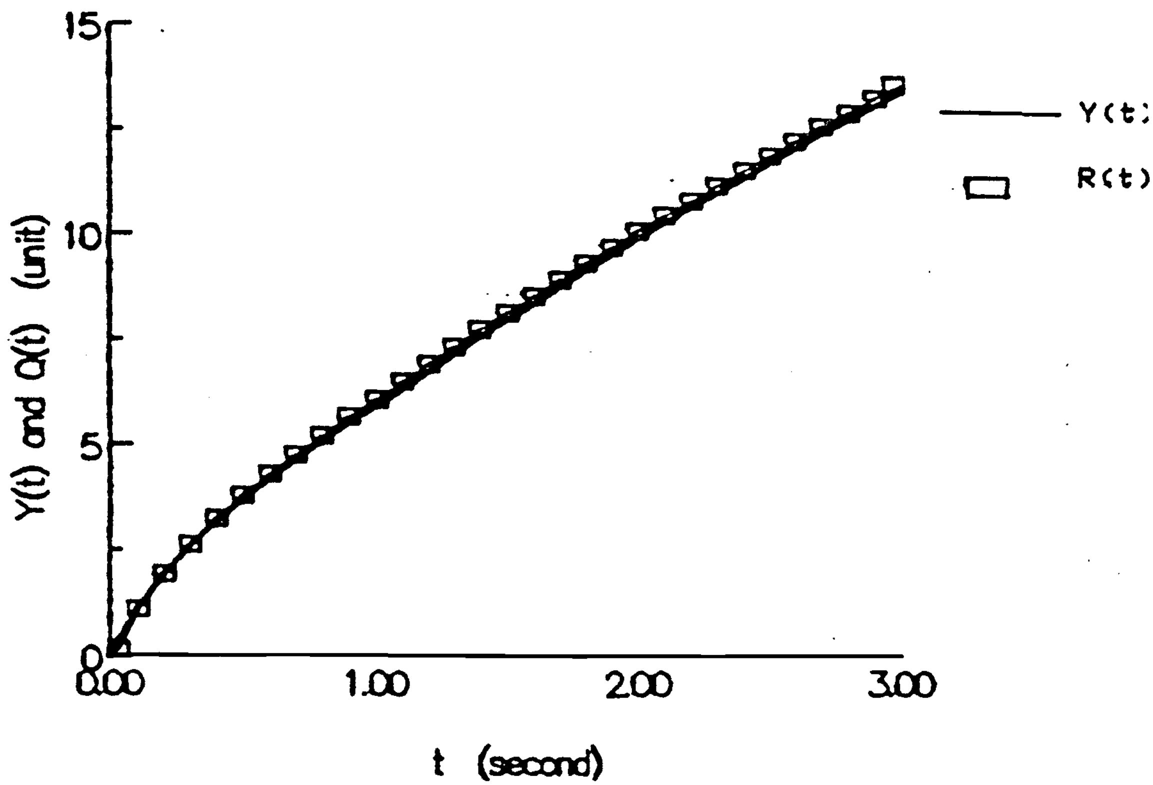
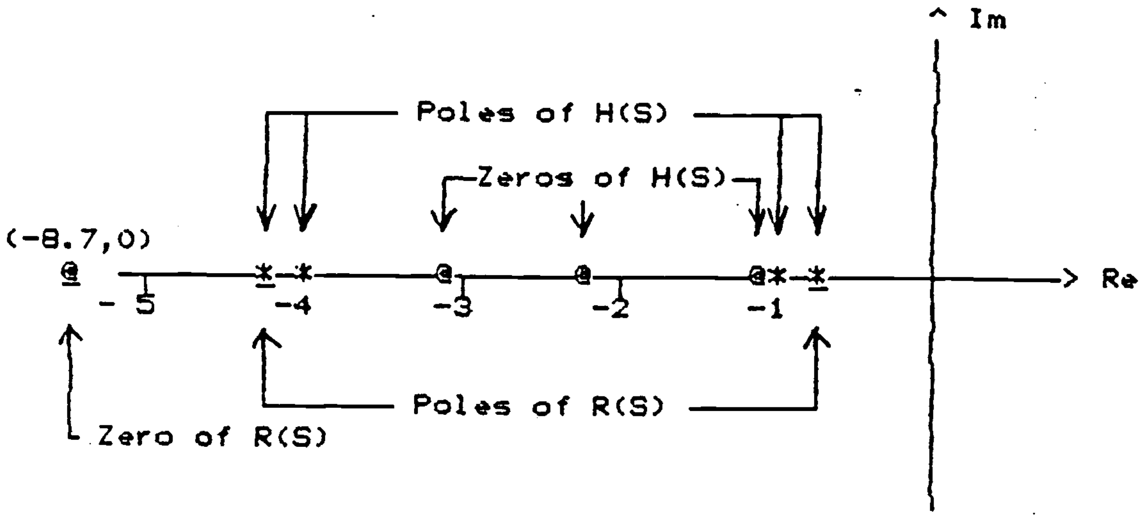


Figure 5.1 Unit Step Response with K = 0.1

$$H(S) = \frac{13.2\,S^3 + 35\,S^2 + 167.2\,S + 96.8}{(S + 0.6972)(S + 4.3028)(S^2 + 5\,S + 4)}$$

$$R(S) = \frac{12.63\,S + 24.20}{S^2 + 5\,S + 3.0}$$



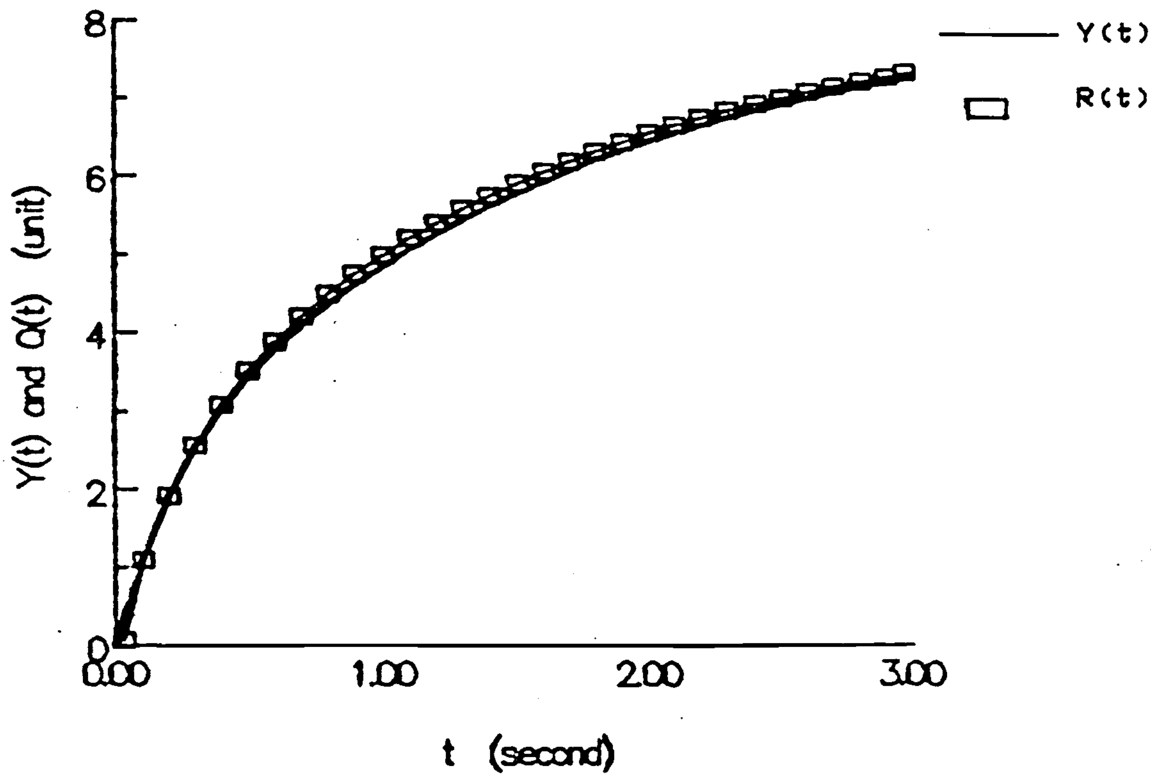


Figure 5.2 Unit Step Response with K = 0.5

$$H(S) = \frac{13.2\ S^3 + 35\ S^2 + 167.2\ S + 96.8}{(S + 2.0)(S + 3.0)(S^2 + 5\ S + 4)}$$

$$R(S) = \frac{23.19\ S + 32.27}{S^2 + 6\ S + 8}$$

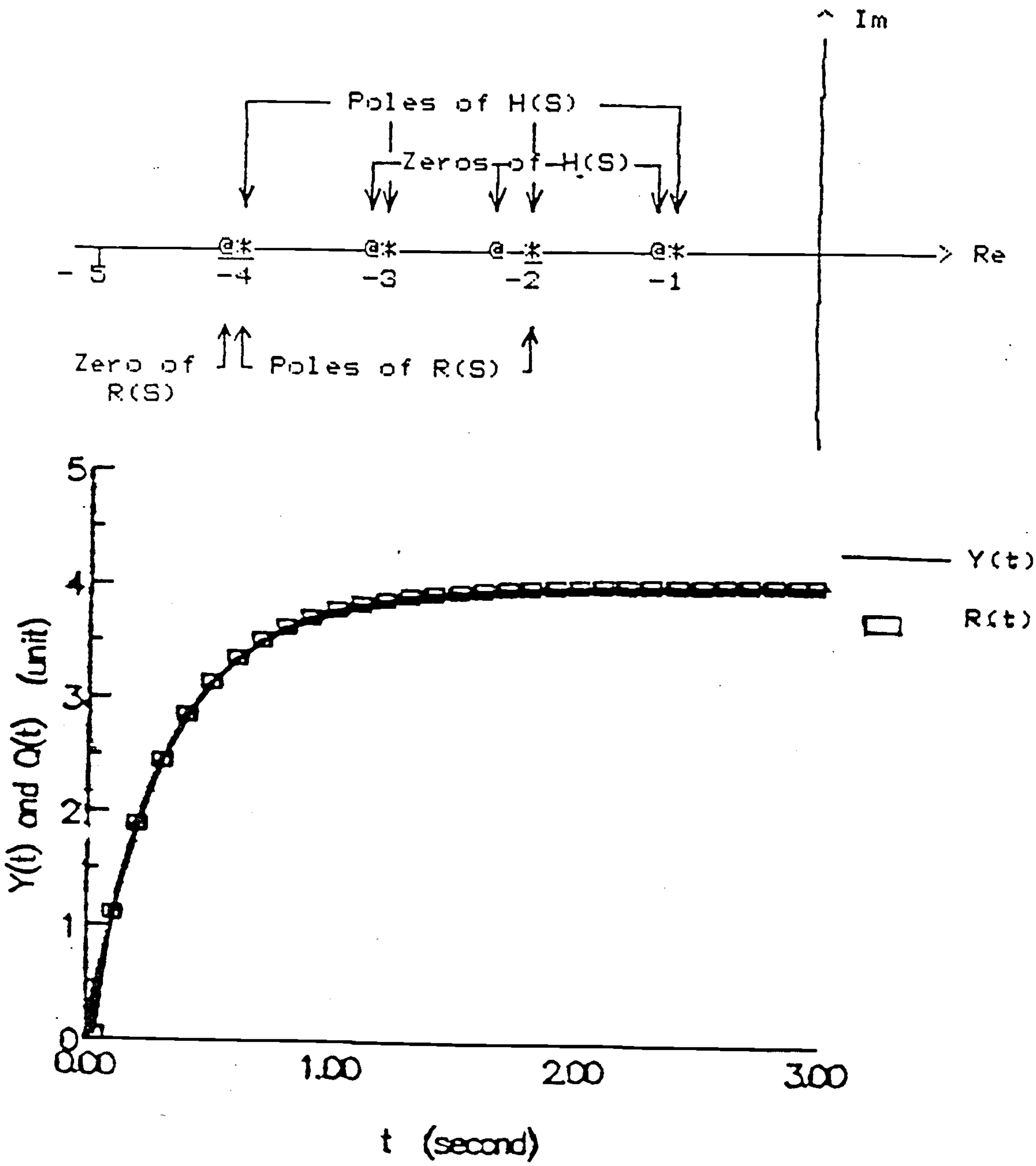
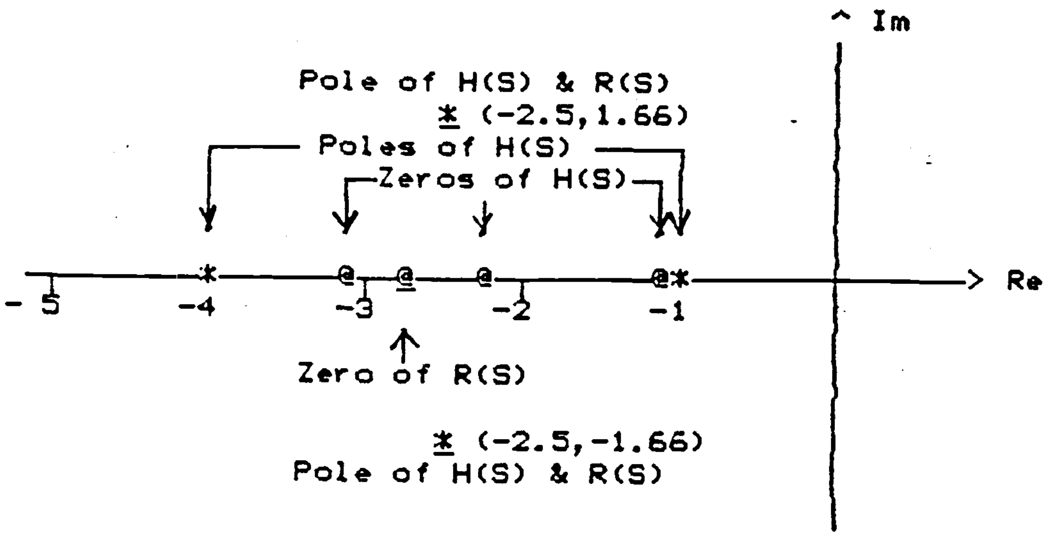


Figure 5.3 Unit Step Response with K = 1.0

$$H(S) = \frac{13.2\ S^3 + 35\ S^2 + 167.2\ S + 96.8}{(S+2.5+j1.66)(S+2.5-j1.66)(S^2 + 5S + 4)}$$

$$R(S) = \frac{12.71\ S + 24.20}{S^2 + 5\ S + 9}$$



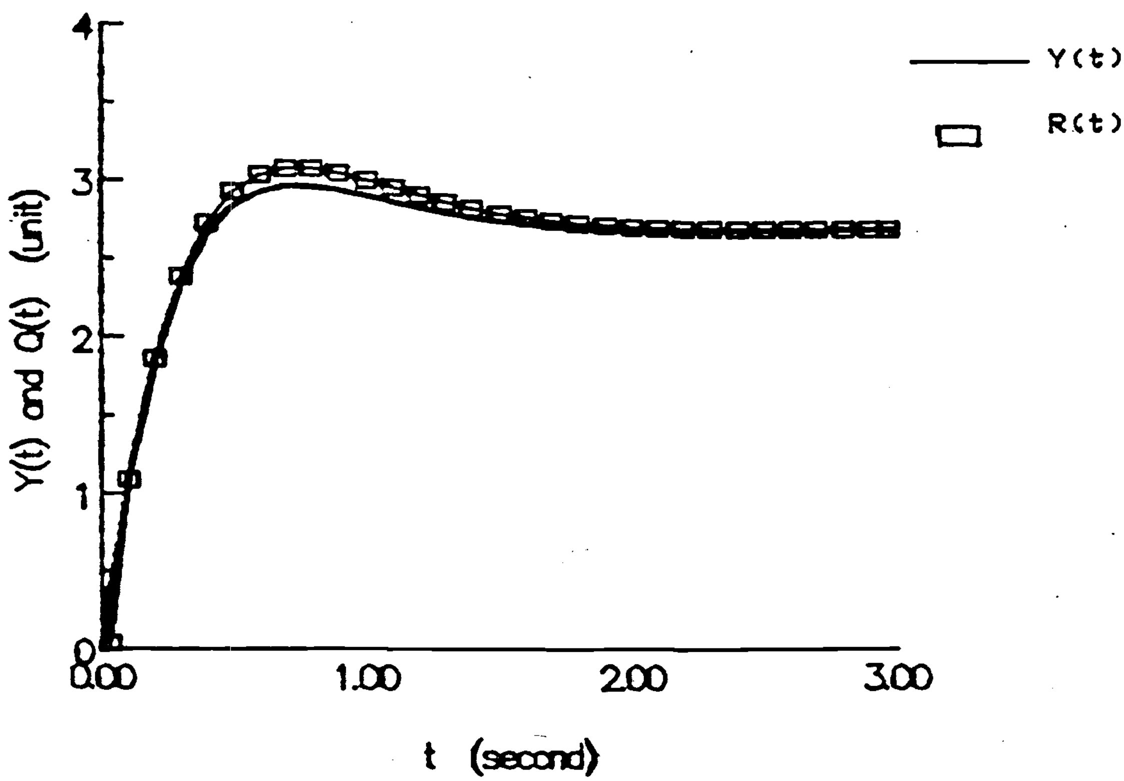
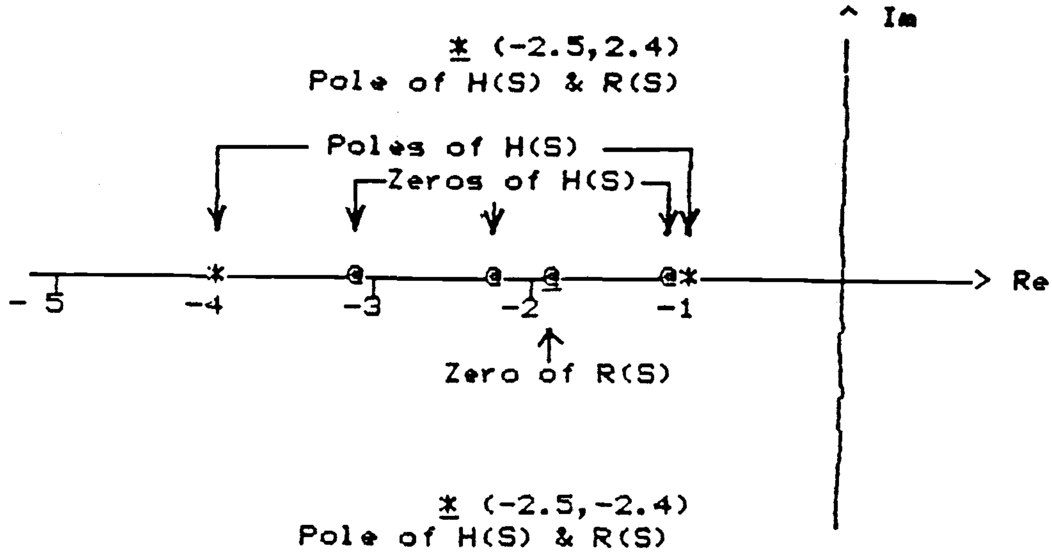


Figure 5.4 Unit step Response with K = 1.5



$$H(S) = \frac{13.2\ S^3 + 35\ S^2 + 167.2\ S + 96.8}{(S+2.5+j2.4)(S+2.5-j2.4)(S^2 + 5S + 4)}$$

$$R(S) = \frac{12.75\ S + 24.20}{S^2 + 5\ S + 12}$$

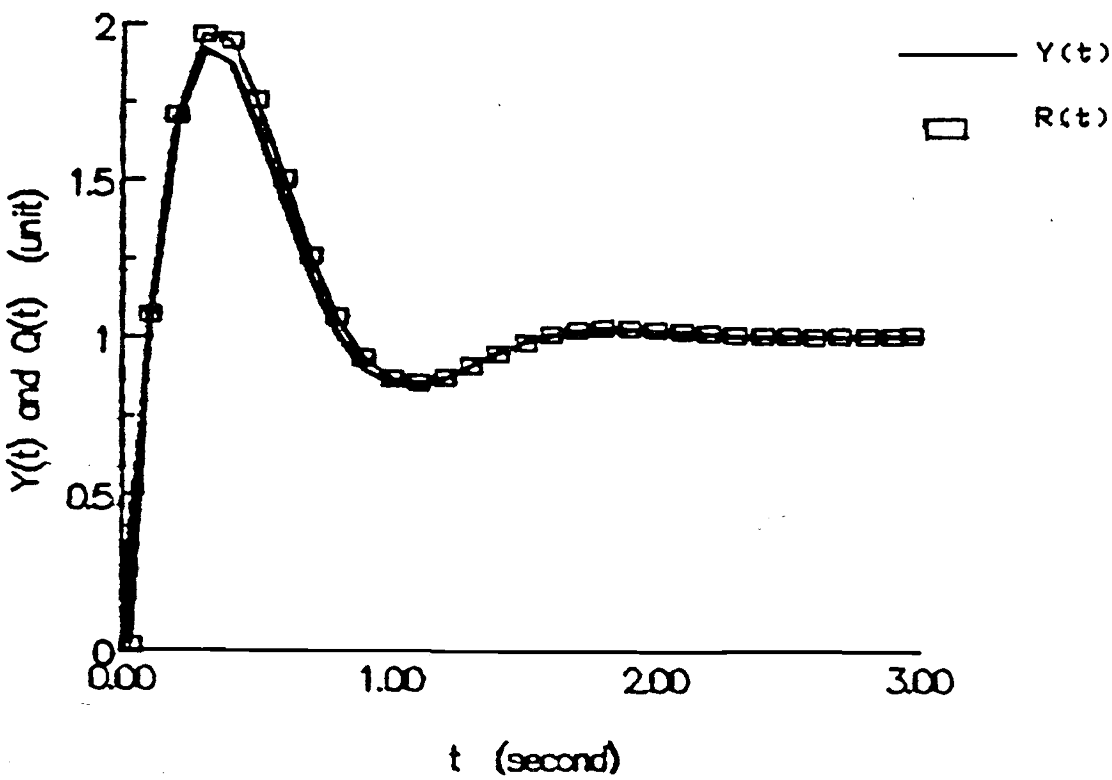
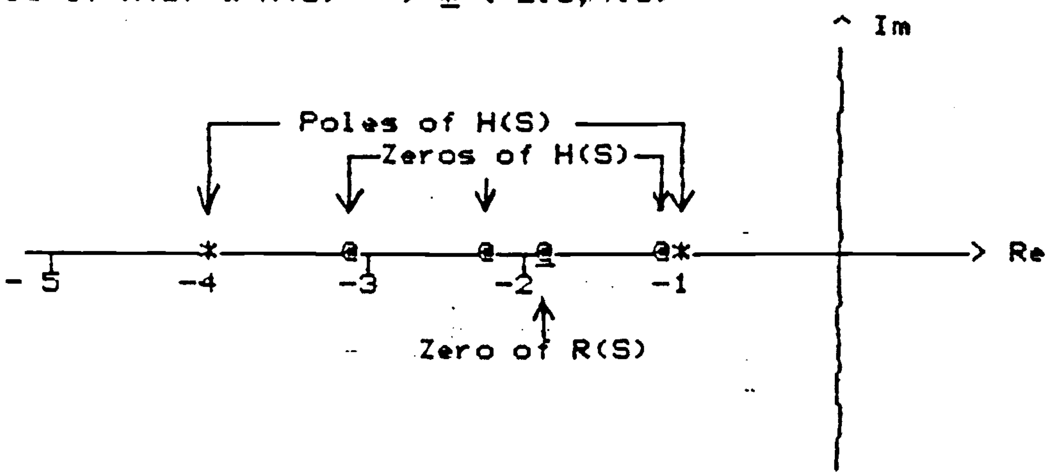


Figure 5.5 Unit step Response with K = 2.0

$$H(S) = \frac{13.2\ S^3 + 35\ S^2 + 167.2\ S + 96.8}{(S+2.5+j4.87)(S+2.5-j4.87)(S^2 + 5S + 30)}$$

$$R(S) = \frac{12.75\ S + 24.20}{S^2 + 5\ S + 12}$$

Pole of H(S) & R(S) —> ± (-2.5,4.8)



Pole of H(S) & R(S) —> ± (-2.5,-4.8)

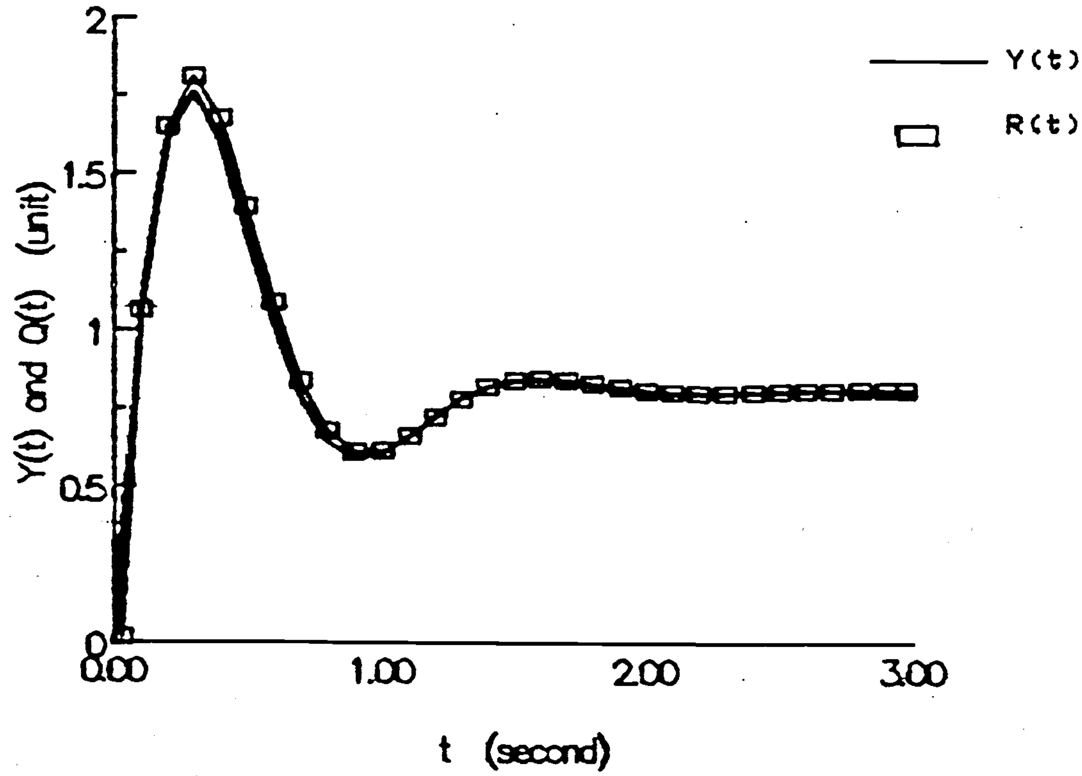


Figure 5.6 Unit Step Response with K = 5

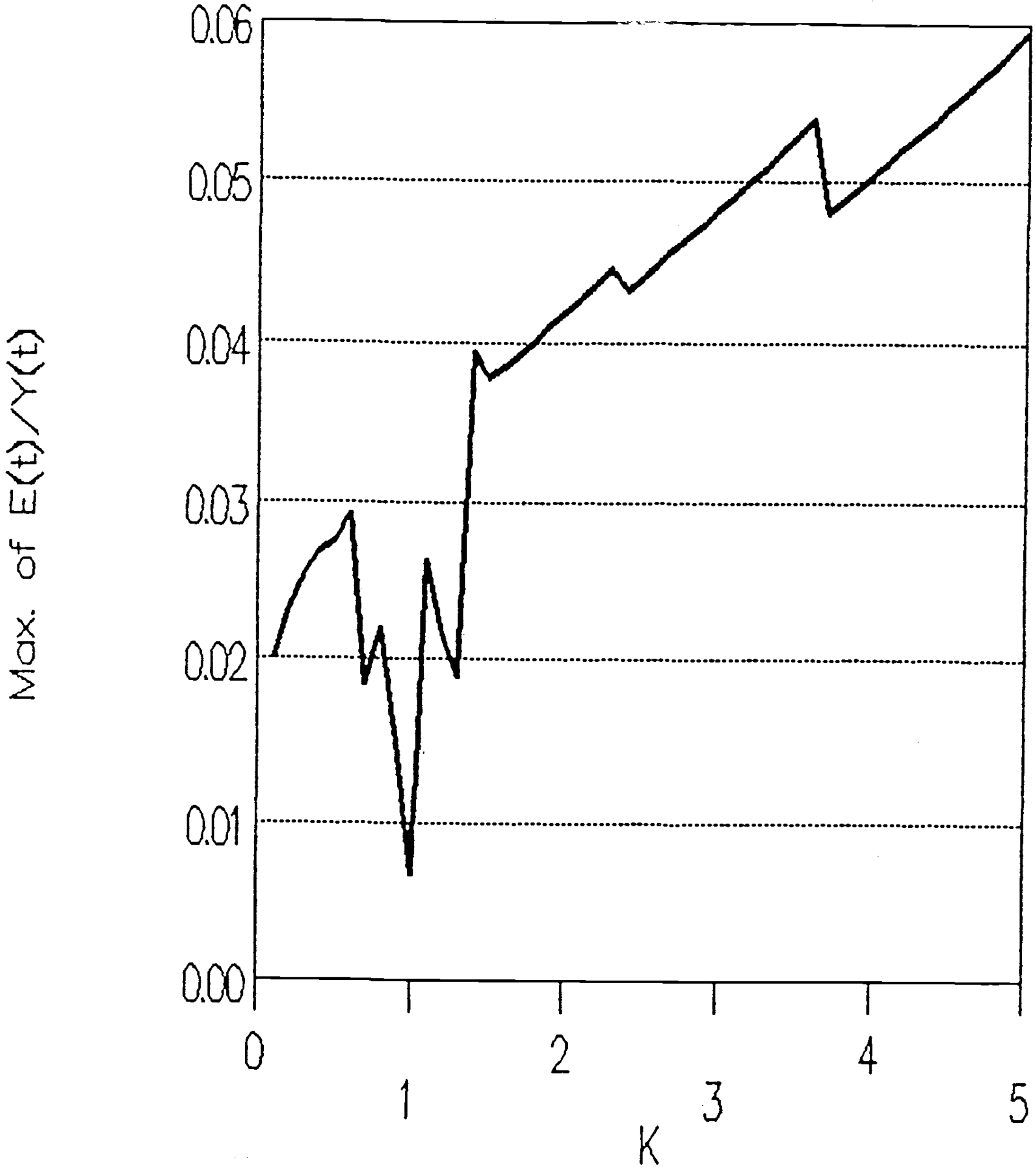


Figure 5.7 Maximum of E(t)/Y(t) VS K

$$H(S) = \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 1)\ (S + 4)\ (S^2 + 5\ S + 6\ K)}$$

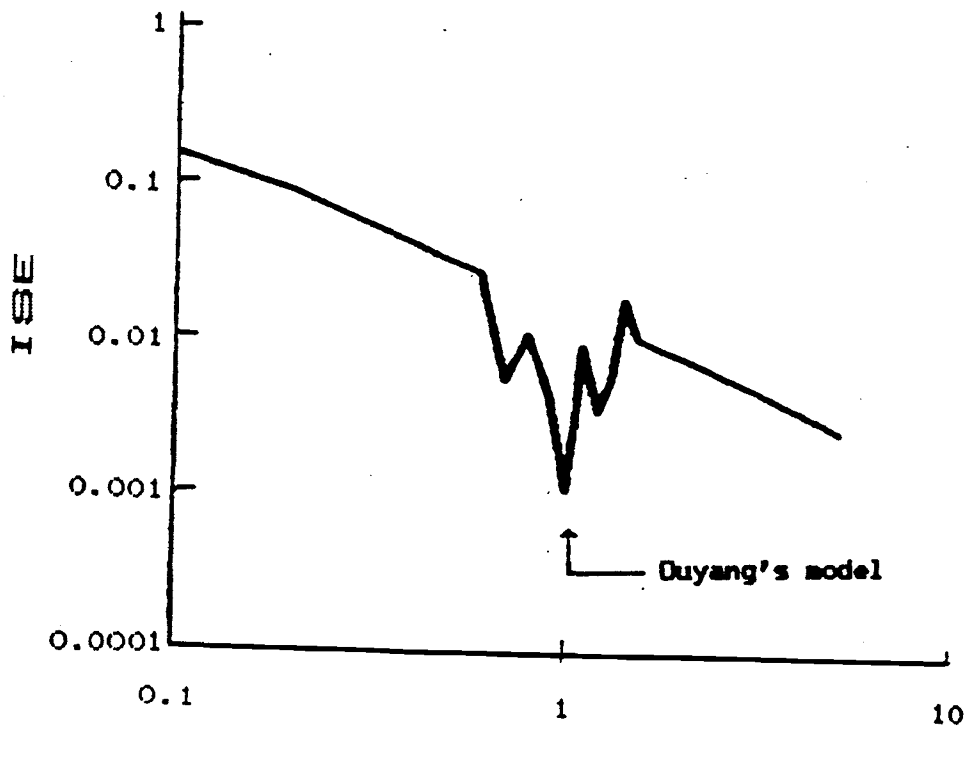


Figure 5.8 ISE as a function of K for case 1

$D(S) = (S + 1)\ (S + 4)$

$$H(S) = \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 1)\ (S + 2)\ (S^2 + 7\ S + 12\ K)}$$

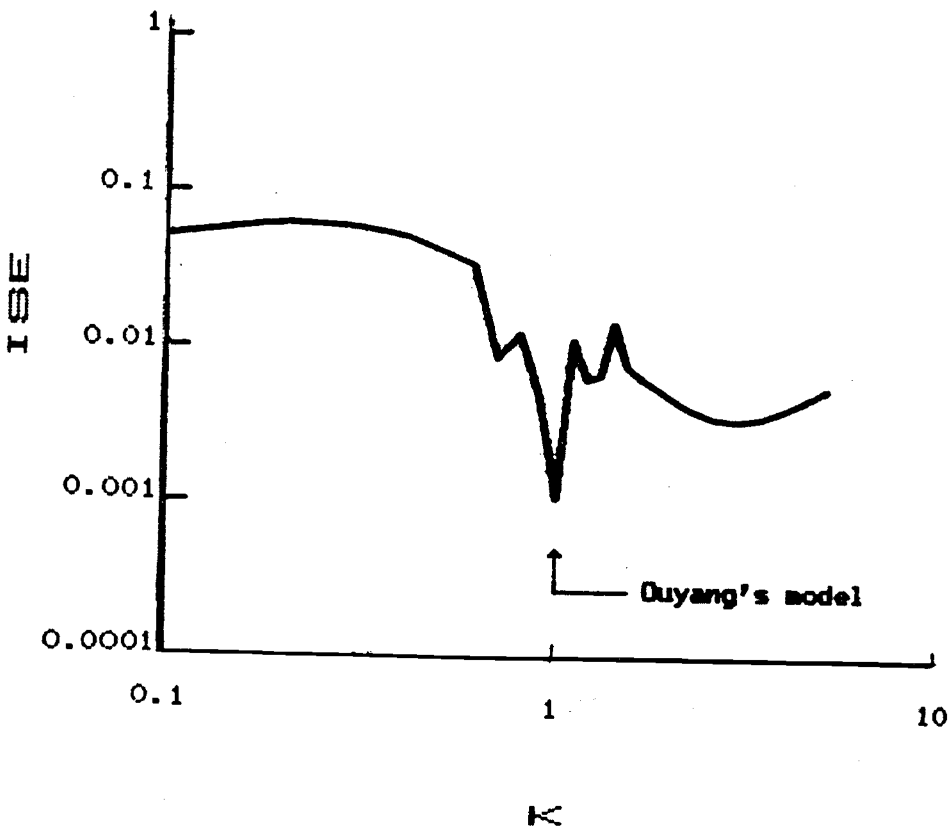


Figure 5.9 ISE as a function of K for case 2

D(S) = (S + 1) (S + 2)

$$H(S) = \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 1)\ (S + 3)\ (S^2 + 6\ S + 8\ K)}$$

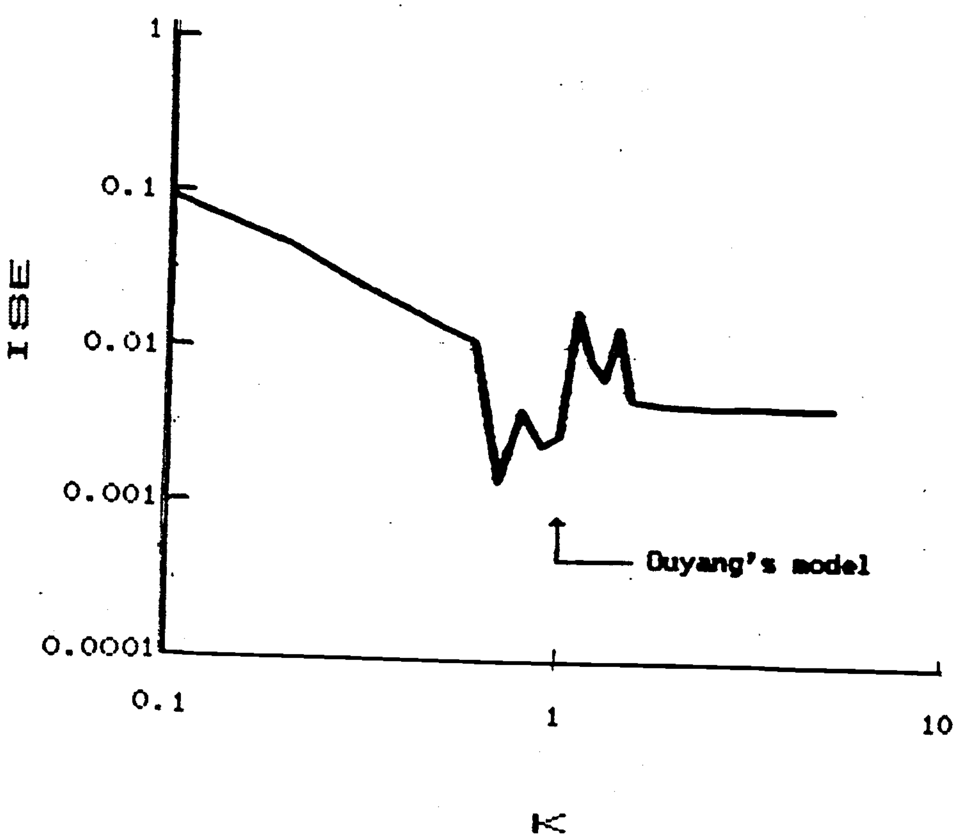


Figure 5.10 ISE as a function of K for case 3

$D(S) = (S + 1)\ (S + 3)$

$$H(S) = \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 2)\ (S + 3)\ (S^2 + 5\ S + 4\ K)}$$

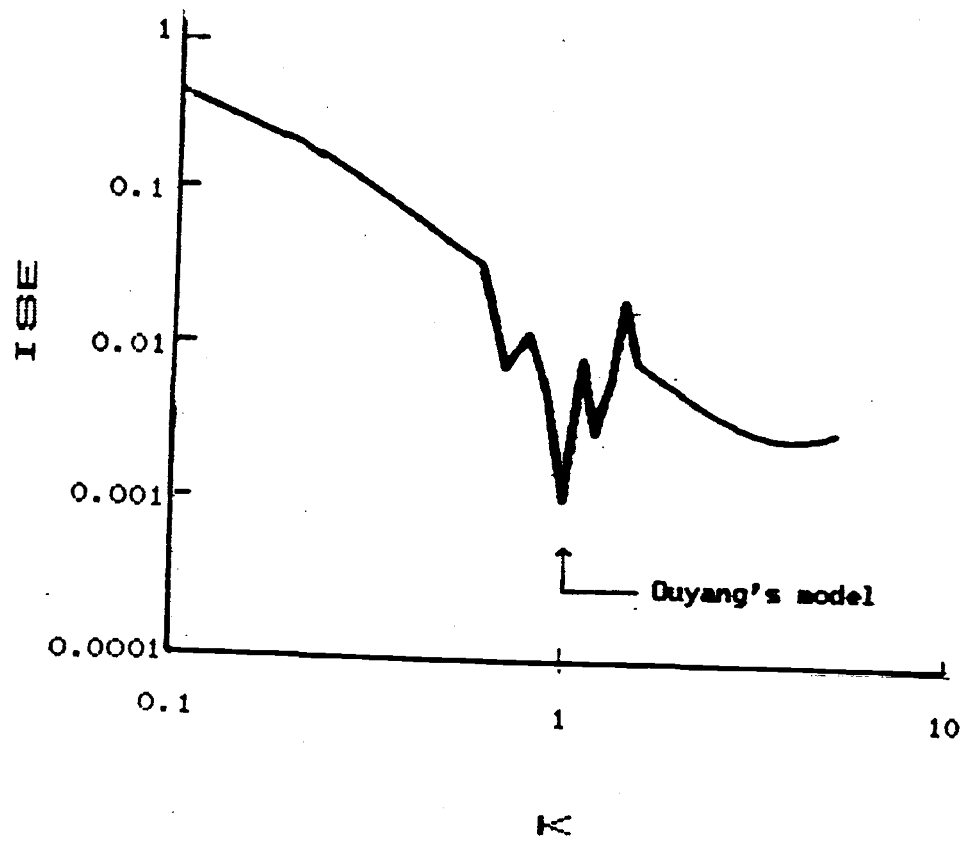


Figure 5.11 ISE as a function of K for case 4

$D(S) = (S + 2)\ (S + 3)$

$$H(S) = \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 2)\ (S + 4)\ (S^2 + 4\ S + 3\ K)}$$

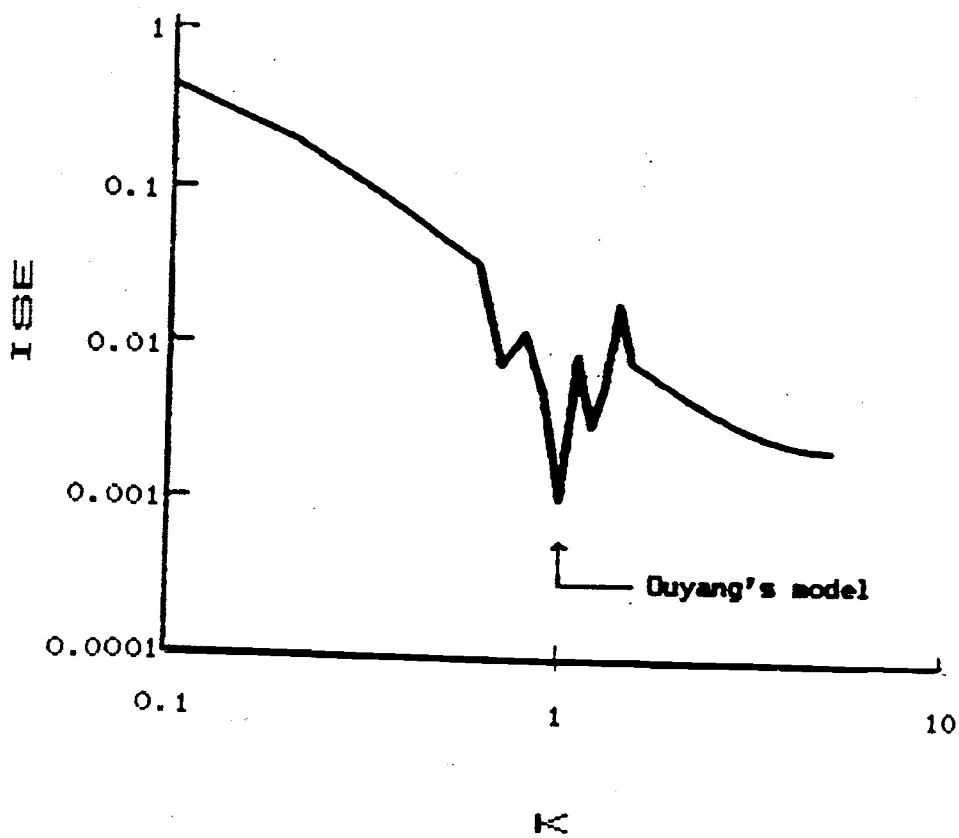


Figure 5.12 ISE as a function of K for case 5

$D(S) = (S + 2)\ (S + 4)$

$$H(S) = \frac{13.2\ S^3 + 84.8\ S^2 + 167.2\ S + 96.8}{(S + 3)\ (S + 4)\ (S^2 + 3\ S + 2\ K)}$$

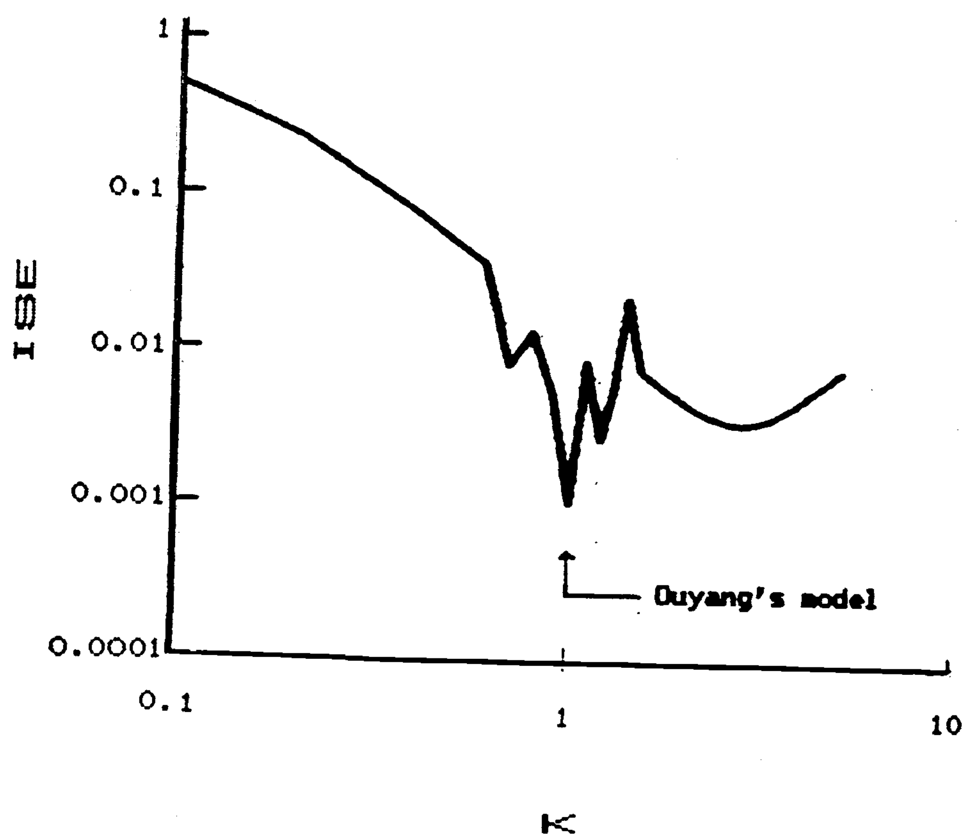
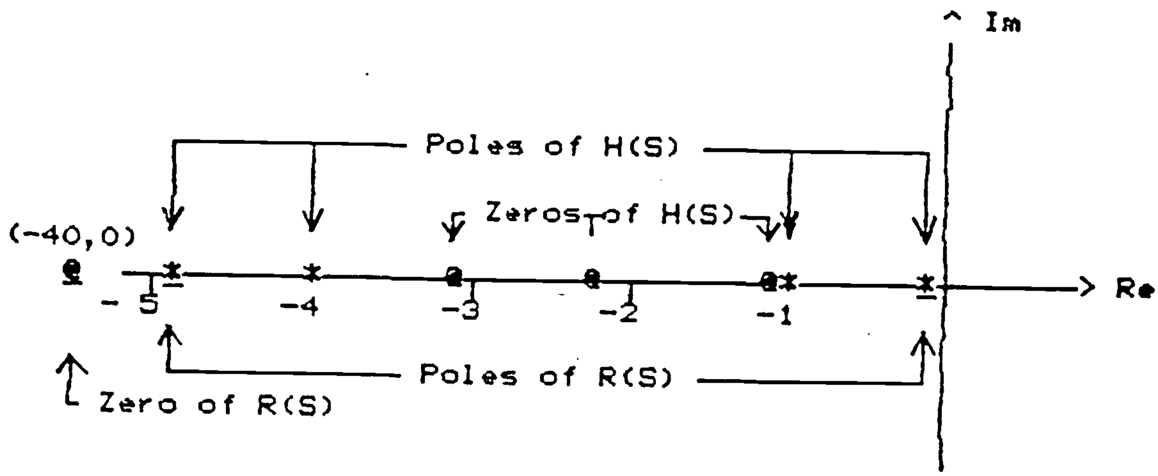


Figure 5.13 ISE as a function of K for case 6

$D(S) = (S + 3)\ (S + 4)$

$$H(S) = \frac{13.2\ S^3 + 35\ S^2 + 167.2\ S + 96.8}{(S + 0.123)(S + 4.877)(S^2 + 5\ S + 4)}$$

$$R(S) = \frac{12.62\ S + 24.19}{S^2 + 5\ S + 0.6}$$



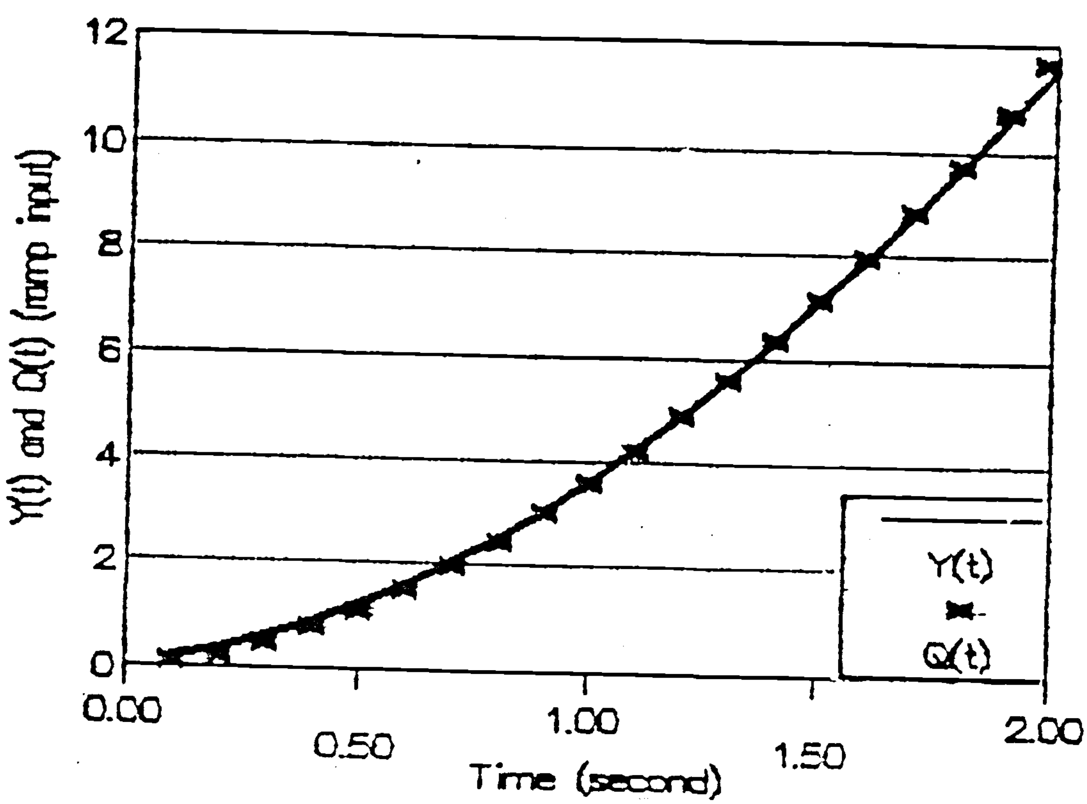
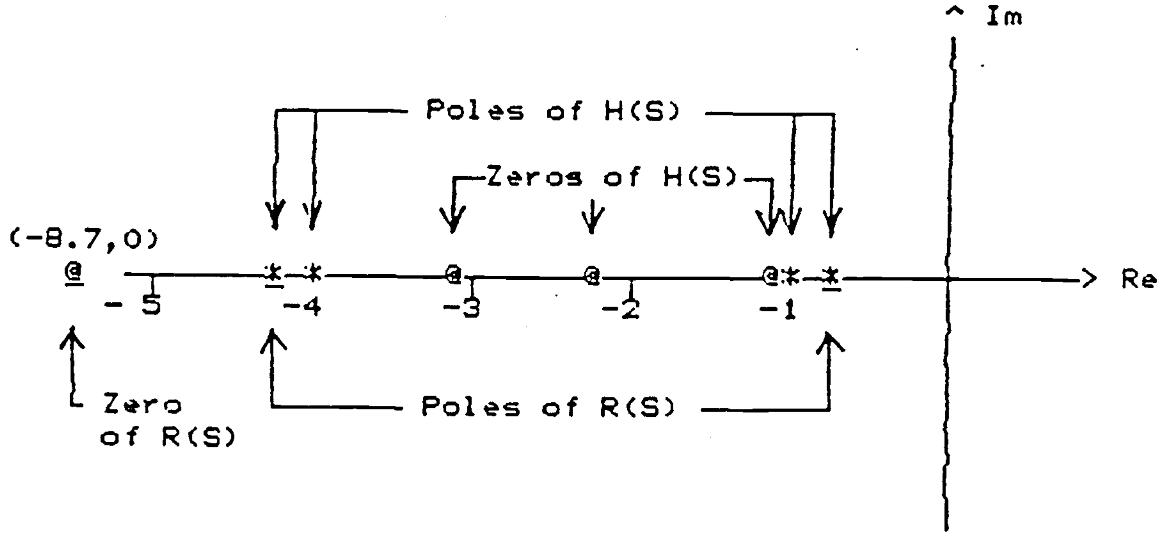


Figure 5.14 Ramp Response with K = 0.1

$$H(S) = \frac{13.2\,S^3 + 35\,S^2 + 167.2\,S + 96.8}{(S + 0.6972)(S + 4.3028)(S^2 + 5\,S + 4)}$$

$$R(S) = \frac{12.63\,S + 24.20}{S^2 + 5\,S + 3.0}$$



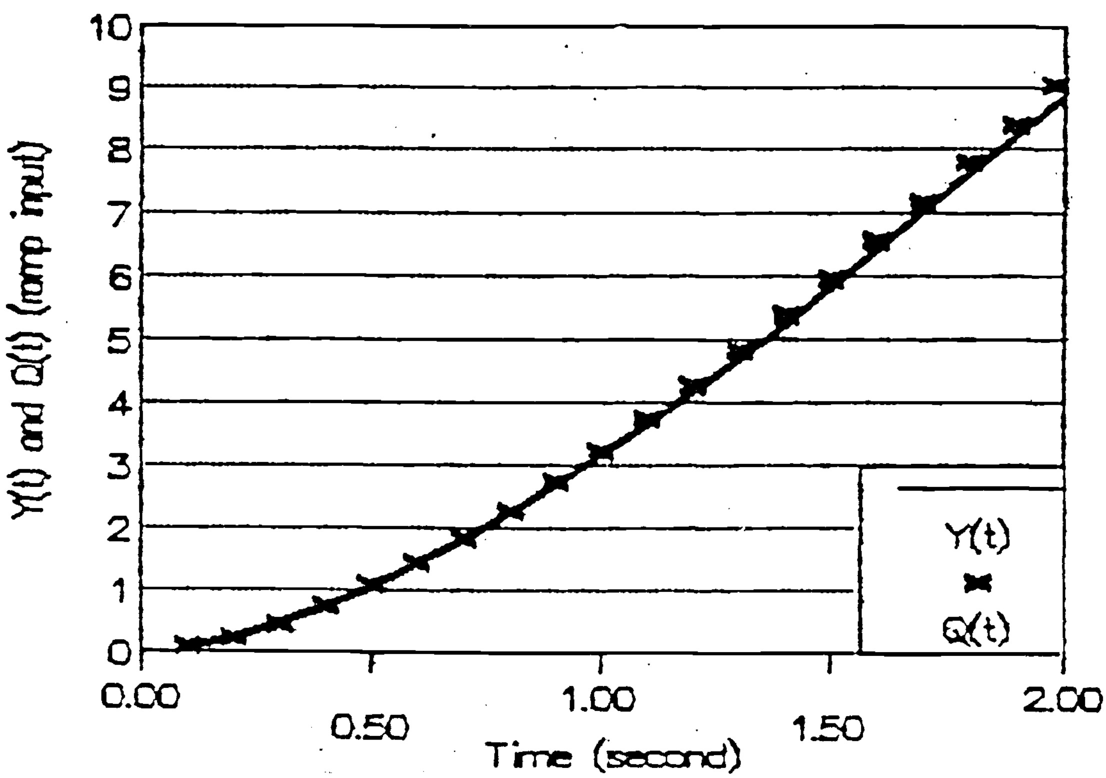
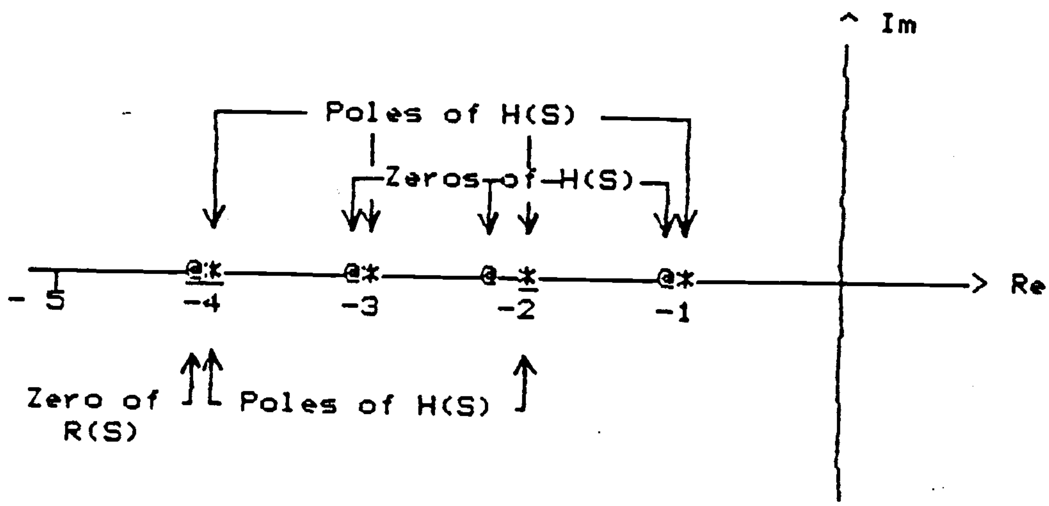


Figure 5.15 Ramp Response with K = 0.5

$$H(S) = \frac{13.2\, S^3 + 35\, S^2 + 167.2\, S + 96.8}{(S + 2.0)(S + 3.0)(S^2 + 5\, S + 4)}$$

$$R(S) = \frac{23.19\, S + 32.27}{S^2 + 6\, S + 8}$$



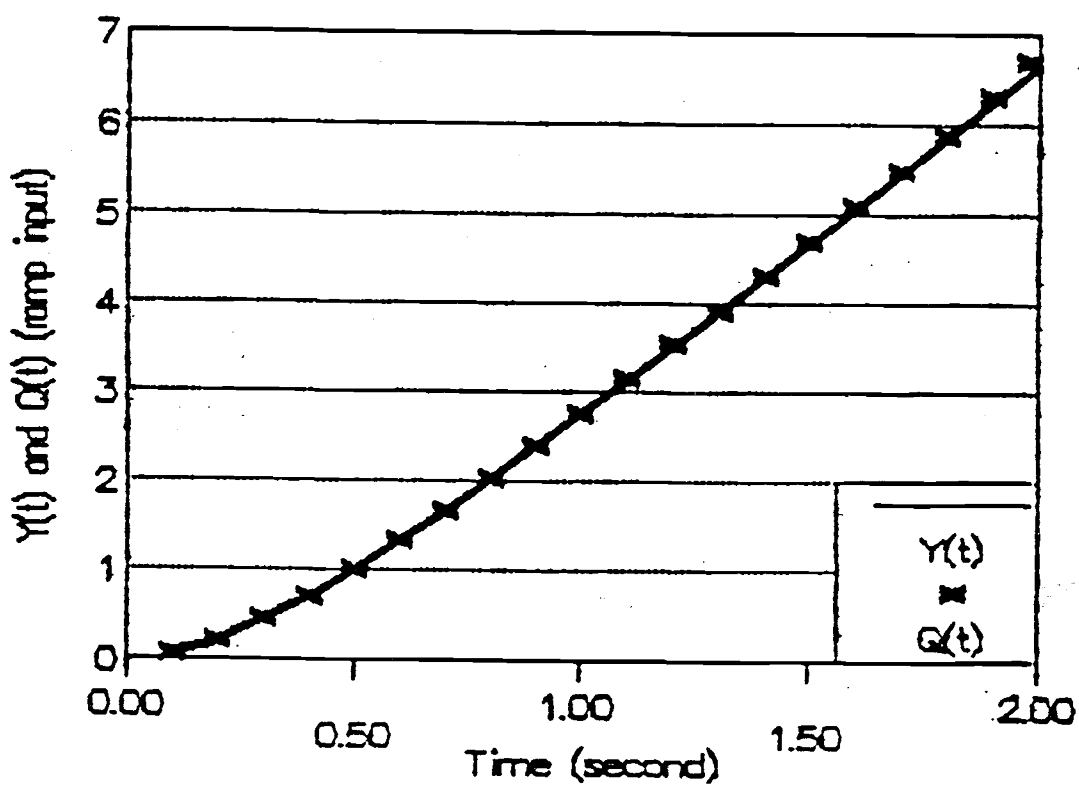


Figure 5.16 Ramp Response with K = 1.0

$$H(S) = \frac{13.2\,S^3 + 35\,S^2 + 167.2\,S + 96.8}{(S+2.5+j2.4)(S+2.5-j2.4)(S^2 + 5S + 4)}$$

$$R(S) = \frac{12.75\,S + 24.20}{S^2 + 5\,S + 12}$$

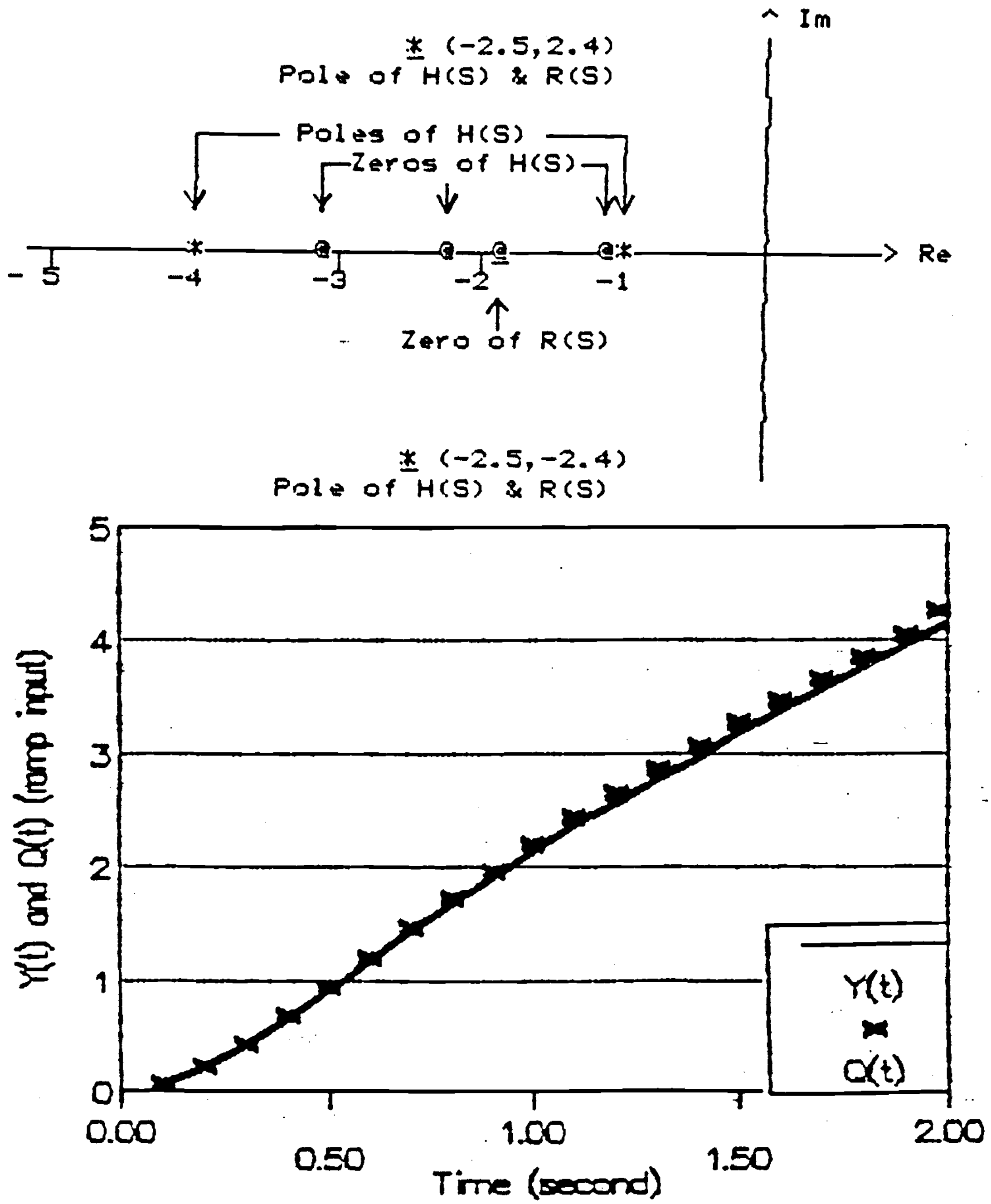
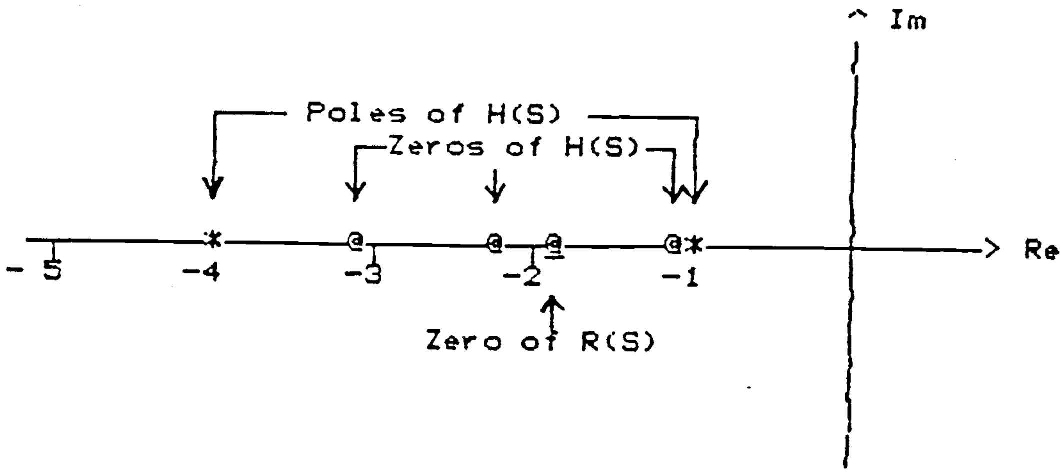


Figure 5.17 Ramp Response with K = 2.0

$$H(S) = \frac{13.2\,S^3 + 35\,S^2 + 167.2\,S + 96.8}{(S+2.5+j4.87)(S+2.5-j4.87)(S^2 + 5S + 30)}$$

$$R(S) = \frac{12.75\,S + 24.20}{S^2 + 5\,S + 12}$$

Pole of H(S) & R(S) —> * (-2.5,4.8)



Pole of H(S) & R(S) —> * (-2.5,-4.8)

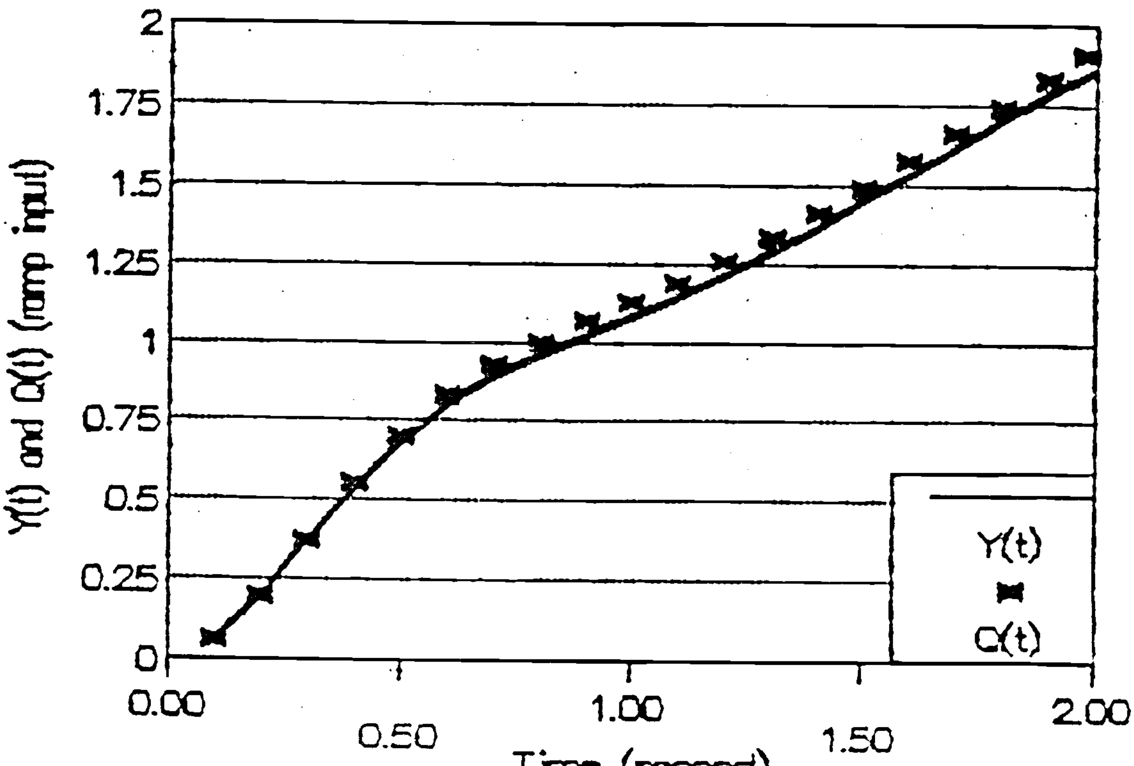


Figure 5.18 Ramp Response with K = 5

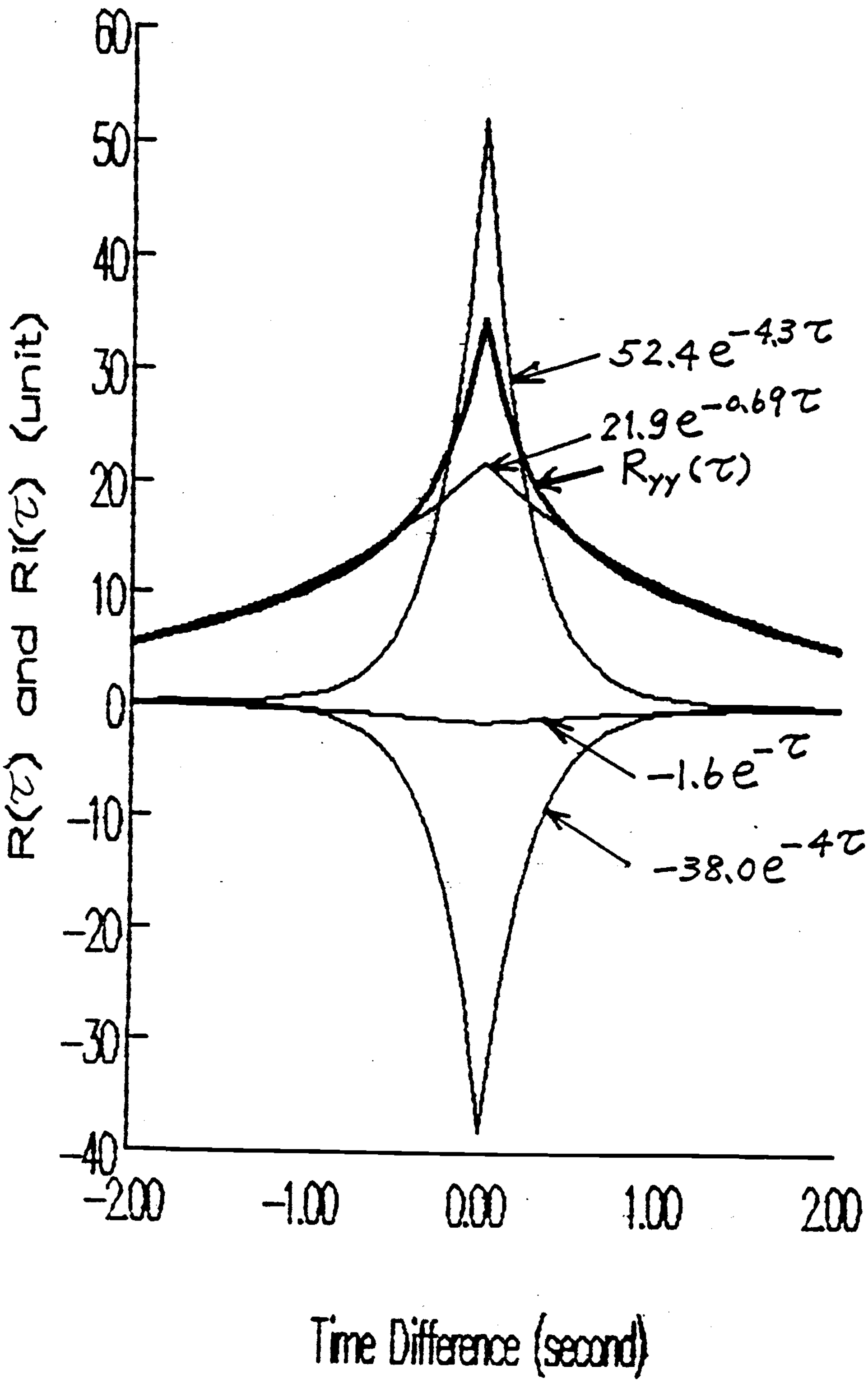
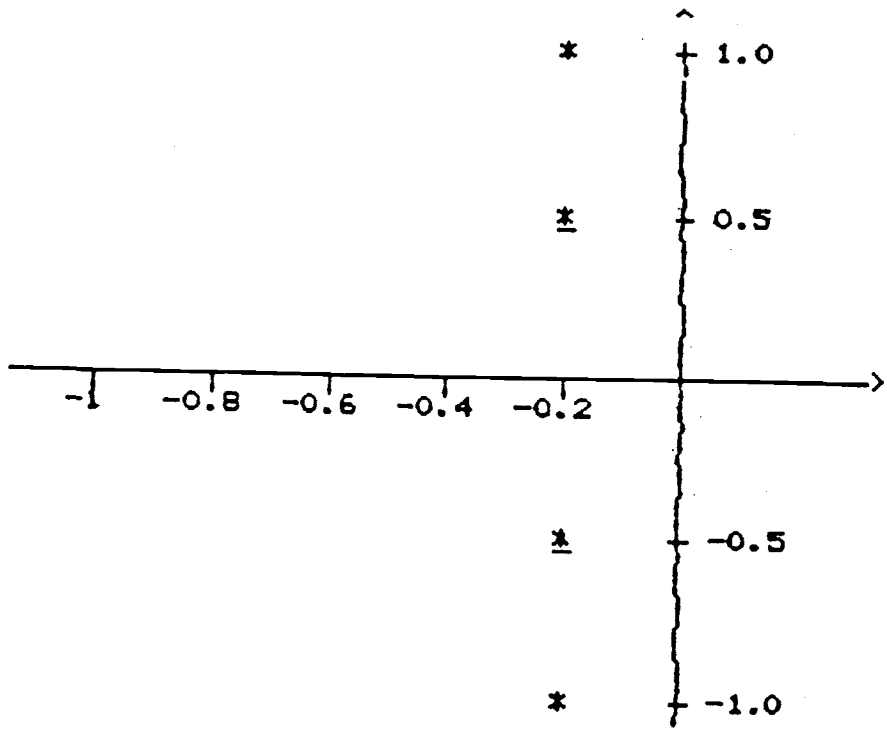


Figure 5.19 Autocorrelation Function Components



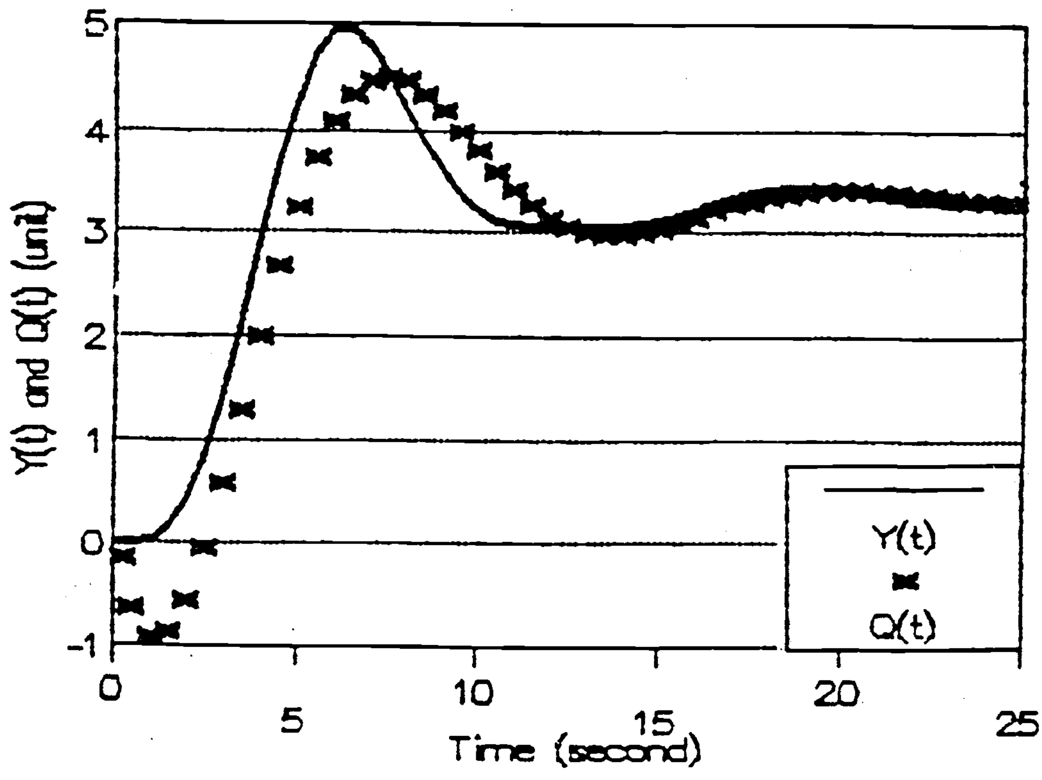


Figure 5.20 **Unit-Step Response of a System with All Complex Poles**

$$H(S) = \frac{S^3 + 7.5\,S^2 + 18.25\,S + 13.875}{S^4 + 6\,S^3 + (11.25+4.25K)\,S^2 + (13+8.5K)\,S + 13.8125K}$$

$$= \frac{S^3 + 7.5\,S^2 + 18.25\,S + 13.875}{(S^2 + 2\,S + 3.25)\,(S^2 + 4\,S + 4.25\,K)}$$

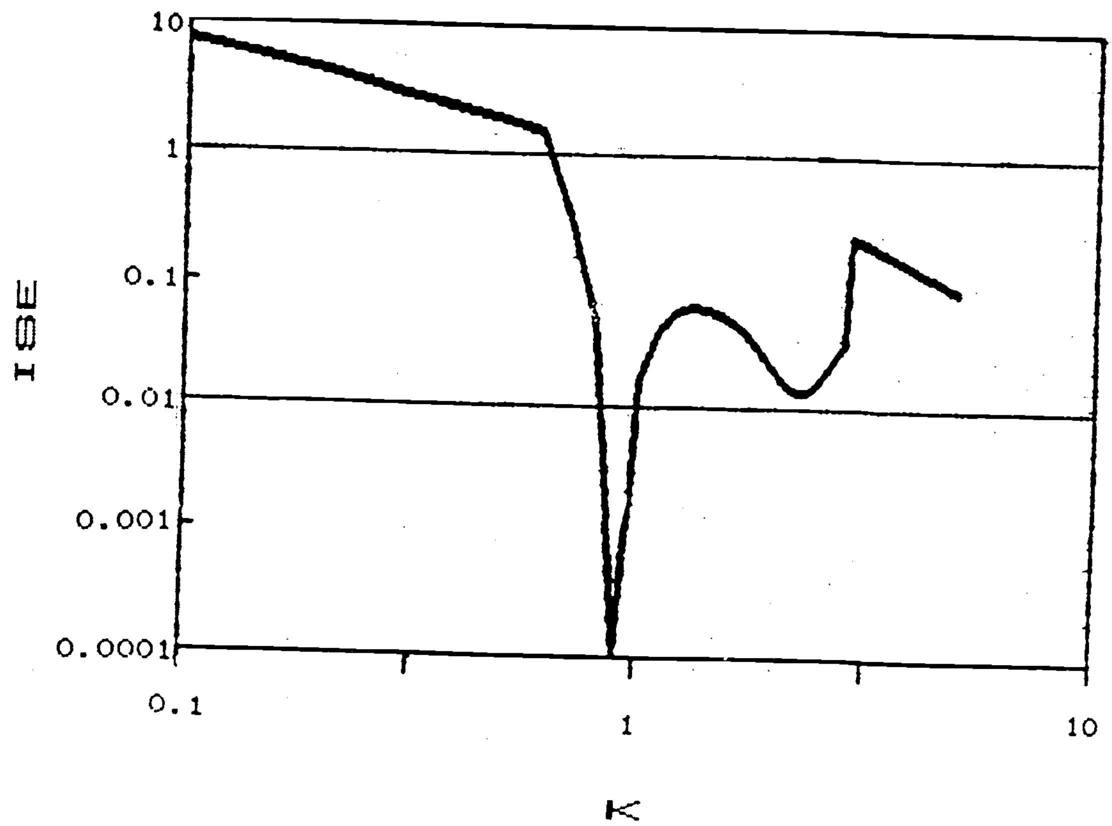


Figure 5.21 ISE as a function of K for the Equivalent Systems with All Complex Poles

6 COMMENTS AND CONCLUSIONS

In this thesis, Ouyang's model reduction method is studied and evaluated for a class of linear, time-invariant, continuous systems. Application of Ouyang's method based on the concept of power decomposition and system identification technique shows that the resulting reduced models can give a good approximation for some original systems.

In general, this method offers the following advantages:

1) It is easy to use and reduces computing costs.
2) The stability of the reduced model is guaranteed for the stable original system.
3) There is no steady state error between the reduced model and the original system.
4) The order of the reduced model can be specified by examining the power dispersion analysis.
5) Good reduced model approximations were obtained for the original systems studied which have
   (a) all negative real poles,
   (b) one pair of conjugate complex poles and other negative real poles.
6) The Ouyang's method also indicates how to determine which poles should be retained for original systems having negative power dispersion components.

This method has the following disadvantages:

1) It requires that the roots of high order polynomials must be solved.

2) The reduced model for the systems with two or more pairs of complex poles can be unsatisfactory.

By using the alternate methods proposed in Chapter 5, the best fit poles for the reduced system can be found. However, for the study included here, the reduced systems obtained by all methods were the same.

In order to apply Ouyang's method to engineering design and system analysis, several suggestions are offered for future research:

1) A standard software package should be developed to make this method attractive for practical applications. The package should include the determination of the parameters and poles of the reduced model, plots of time and frequency response.

2) Discrete systems should be studied.

3) The model reduction for the case of multiple inputs should be explored. This would make the reduction method more widespread applicable.

A feasible and simple, yet powerful technique would be of great benefit for system analysis and effective, economic design of controllers and compensators.

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Appendices

Appendix A

Autocorrelation Function of the Output

A general linear system is said to be time-invariant if the form of its impulse response $h(t,\tau)$ does not depend on the time that the impulse is applied. When an impulse $\delta(t)$, occurring at $t = 0$, causes the impulse response $h(t)$, then an impulse $\delta(t-\tau)$, occurring at $t = \tau$, must cause the response $h(t-\tau)$ if the system is time-invariant. This fact means that

$$h(t,\tau) = h(t-\tau) \tag{A-1}$$

For a linear,time-invariant system, the output response is given by the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} X(u)\ h(t - u)du \tag{A-2}$$

where $X(u)$ is the input.

If $X(t)$ ia a wide-sense stationary random signal,then the mean value of the system responde is

$$E\ [Y(t)] = E\ [\int_{-\infty}^{\infty} h(t - u)\ X(u)\ du]$$

$$= \int_{-\infty}^{\infty} h(t - u)\ E\ [X(u)]\ du \tag{A-3}$$

Assume $X(t)$ is white noise with zero mean value. Then,

$$E\ [X(t)] = 0 \tag{A-4}$$

For a causul system, $h(t)$ is zero for $t < 0$, thus

$$h(t - \tau) = 0 \qquad \text{for} \quad \tau > t \tag{A-5}$$

Then,

$$Y(t) = \int_{-\infty}^{\infty} X(u)\ h(t - u)du$$

$$= \int_{-\infty}^{t} X(u)\ h(t - u)du \qquad \text{(A-6)}$$

and

$$Y(t - \tau) = \int_{-\infty}^{\infty} X(u)\ h(t - \tau - u)du$$

$$= \int_{-\infty}^{t-\tau} X(u)\ h(t - \tau - u)du \qquad \text{(A-7)}$$

The autocorrelation function of Y(t) is

$$R_{YY}(t_1,t_2) = E\ [Y(t_1)\ Y(t_2)] \qquad \text{(A-8)}$$

Set $t = t_1$ and $\tau = t_1 - t_2$,

$$R_{YY}(t,t-\tau) = E\ [Y(t)\ Y(t - \tau)]$$

$$= E\ [\int_{-\infty}^{t} X(v)h(t-v)dv \int_{-\infty}^{t-\tau} X(u)h(t-\tau-u)du]$$

$$= \int_{-\infty}^{t}\int_{-\infty}^{t-\tau} h(t-v)h(t-\tau-u)E[X(v)X(u)]dudv \qquad \text{(A-9)}$$

According to the assumtion that X(t) is white noise which is wide-sense stationary, the autocorrelation function of X(t) is

$$R_{XX}(v,u) = E\ [X(v)\ X(u)]$$

$$= \sigma_x^{\ 2}\delta(v - u) \qquad \text{(A-10)}$$

Let $\tau = v - u$, then,

$$R_{XX}(v,v-\tau) = \sigma_x^{\ 2}\delta(\tau) \qquad \text{(A-11)}$$

and

$$R_{yy}(t,t-\tau) = \int_{-\infty}^{t}\int_{-\infty}^{t-\tau} \sigma_x^2 \delta(\tau) h(t-v) h(t-\tau-u)\,du\,dv$$

$$= \int_{-\infty}^{t}\int_{-\infty}^{t-\tau} \sigma_x^2 \delta(\tau) h(t-v) h(t-\tau-u)\,du\,dv$$

$$= \int_{-\infty}^{t-\tau} \sigma_x^2 h(t-u) h(t-\tau-u)\,du$$

$$= \int_{-\infty}^{t-\tau} \sigma_x^2 \sum_{i=1}^{N} h_i e^{\mu_i t} \sum_{j=1}^{N} h_j e^{\mu_j (t-\tau)}\,du$$

$$= \sigma_x^2 \left[\sum_{i=1}^{N} h_i e^{\mu_i t} \sum_{j=1}^{N} h_j e^{\mu_j (t-\tau)} \frac{e^{-(\mu_i+\mu_j)u}}{-(\mu_i + \mu_j)} \right]_{-\infty}^{t-\tau}$$

$$= \sigma_x^2 \sum_{i=1}^{N} \sum_{j=1}^{N} h_i h_j \frac{-\, e^{\mu_i \tau}}{\mu_i + \mu_j}$$

$$= \sigma_x^2 \sum_{i=1}^{N} e^{\mu_i \tau} \left[\sum_{j=1}^{N} \frac{-\, h_i h_j}{\mu_i + \mu_j} \right] . \qquad \text{(A-12)}$$

Since $E[Y(t)] = 0$ and $R_{yy}(t,t-\tau)$ does not depend on time t, Y(t) is also wide-sense stationary. For a wide-sense stationary process, the autocorrelation function of the output response is

$$R_{yy}(\tau) = R_{yy}(t,t-\tau) \qquad \text{(A-13)}$$

Appendix B

Table B-1 lists the coefficients of the denominator and the numerator of 50 reduced models corresponding to the equivalent original systems given by equation (19). In Table B-1, β_1 and β_2 are the retained poles in the reduced models; D_1 and D_0 are the coefficients of the denominator of the reduced models; C_1 and C_0 are the values which minimize the frequency response error function.

Table B-2 lists the transfer functions of some original systems and reduced models, and their corresponding values of ISE.

Table B-3 lists the values of error criterion of the output autocorrelation functions for several systems. The best fit poles for the reduced models are due to the minimum value of error criterion of the output autocorrelation function. In Table B-3, μ_i are the eigenvalues of the original systems; P_i are the coefficients of autocorrelation functions of the system outputs; β_1 and β_2 are the possible retained poles for the reduced models. J(FD) and J(MS) are the values of error criterion of the output autocorrelation functions obtained by the frequency match and the minimum search method individually.

TABLE B-1 THE COEFFICIENTS OF THE REDUCED MODELS

K	β_1 (a +j b)		β_2 (a +j b)		D_1	D_o	C_1	C_o
0.10	-4.88	0.00	-0.12	0.00	5.00	0.60	12.62	24.19
0.20	-4.75	0.00	-0.25	0.00	5.00	1.20	12.62	24.20
0.30	-4.61	0.00	-0.39	0.00	5.00	1.80	12.62	24.20
0.40	-0.54	0.00	-4.46	0.00	5.00	2.40	12.62	24.20
0.50	-0.70	0.00	-4.30	0.00	5.00	3.00	12.63	24.20
0.60	-0.87	0.00	-4.13	0.00	5.00	3.60	12.64	24.20
0.70	-1.00	0.00	-4.00	0.00	5.00	4.00	12.76	23.05
0.80	-1.30	0.00	-4.00	0.00	5.30	5.18	12.86	26.13
0.90	-1.58	0.00	-4.00	0.00	5.58	6.31	13.03	28.29
1.00	-2.00	0.00	-4.00	0.00	6.00	8.00	13.19	32.27
1.10	-1.00	0.00	-4.00	0.00	5.00	4.00	13.65	14.67
1.20	-1.00	0.00	-4.00	0.00	5.00	4.00	13.77	13.44
1.30	-1.00	0.00	-4.00	0.00	5.00	4.00	13.86	12.41
1.40	-1.00	0.00	-4.00	0.00	5.00	4.00	13.92	11.52
1.50	-2.50	1.66	-2.50	-1.66	5.00	9.00	12.71	24.20
1.60	-2.50	1.83	-2.50	-1.83	5.00	9.60	12.72	24.20
1.70	-2.50	1.99	-2.50	-1.99	5.00	10.20	12.73	24.20
1.80	-2.50	2.13	-2.50	-2.13	5.00	10.80	12.73	24.20
1.90	-2.50	2.27	-2.50	-2.27	5.00	11.40	12.74	24.20
2.00	-2.50	2.40	-2.50	-2.40	5.00	12.00	12.75	24.20
2.10	-2.50	2.52	-2.50	-2.52	5.00	12.60	12.75	24.20
2.20	-2.50	2.64	-2.50	-2.64	5.00	13.20	12.76	24.20
2.30	-2.50	2.75	-2.50	-2.75	5.00	13.80	12.76	24.20
2.40	-2.50	2.85	-2.50	-2.85	5.00	14.40	12.77	24.20
2.50	-2.50	2.96	-2.50	-2.96	5.00	15.00	12.78	24.20
2.60	-2.50	3.06	-2.50	-3.06	5.00	15.60	12.78	24.20
2.70	-2.50	3.15	-2.50	-3.15	5.00	16.20	12.79	24.20
2.80	-2.50	3.25	-2.50	-3.25	5.00	16.80	12.79	24.20
2.90	-2.50	3.34	-2.50	-3.34	5.00	17.40	12.80	24.20
3.00	-2.50	3.43	-2.50	-3.43	5.00	18.00	12.80	24.20
3.10	-2.50	3.51	-2.50	-3.51	5.00	18.60	12.80	24.20
3.20	-2.50	3.60	-2.50	-3.60	5.00	19.20	12.81	24.20
3.30	-2.50	3.68	-2.50	-3.68	5.00	19.80	12.81	24.20
3.40	-2.50	3.76	-2.50	-3.76	5.00	20.40	12.82	24.20
3.50	-2.50	3.84	-2.50	-3.84	5.00	21.00	12.82	24.20
3.60	-2.50	3.92	-2.50	-3.92	5.00	21.60	12.83	24.20
3.70	-2.50	3.99	-2.50	-3.99	5.00	22.20	12.83	24.20
3.80	-2.50	4.07	-2.50	-4.07	5.00	22.80	12.83	24.20
3.90	-2.50	4.14	-2.50	-4.14	5.00	23.40	12.84	24.20
4.00	-2.50	4.21	-2.50	-4.21	5.00	24.00	12.84	24.20
4.10	-2.50	4.28	-2.50	-4.28	5.00	24.60	12.85	24.20
4.20	-2.50	4.35	-2.50	-4.35	5.00	25.20	12.85	24.20
4.30	-2.50	4.42	-2.50	-4.42	5.00	25.80	12.85	24.20
4.40	-2.50	4.49	-2.50	-4.49	5.00	26.40	12.86	24.20
4.50	-2.50	4.56	-2.50	-4.56	5.00	27.00	12.86	24.20
4.60	-2.50	4.62	-2.50	-4.62	5.00	27.60	12.86	24.20
4.70	-2.50	4.69	-2.50	-4.69	5.00	28.20	12.87	24.20
4.80	-2.50	4.75	-2.50	-4.75	5.00	28.80	12.87	24.20
4.90	-2.50	4.81	-2.50	-4.81	5.00	29.40	12.87	24.20
5.00	-2.50	4.87	-2.50	-4.87	5.00	30.00	12.88	24.20

TABLE B-2 THE VALUES OF ISE FOR SOME SYSTEMS

K	ORIGINAL SYSTEM	REDUCED MODEL	ISE
0.5	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 5S^3 + 32S^2 + 35S + 12}$	$\frac{12.53\ S + 24.19}{S^2 + 5\ S + 2.99}$	.0323
1.0	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 10S^3 + 35S^2 + 50S + 24}$	$\frac{13.18\ S + 32.26}{S^2 + 6\ S + 8}$	.0010
1.5	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 15S^3 + 38S^2 + 65S + 36}$	$\frac{12.71\ S + 24.19}{S^2 + 5\ S + 9}$	.0100
2.0	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 20S^3 + 41S^2 + 80S+ 48}$	$\frac{12.74\ S + 24.1}{S^2 + 5\ S + 11.98}$	.0072
2.5	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 25S^3 + 44S^2 + 95S + 60}$	$\frac{12.77\ S + 24.19}{S^2 + 5\ S + 14.99}$	.0056
3.0	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 30S^3 + 47S^2 + 110S + 72}$	$\frac{12.80\ S + 24.19}{S^2 + 5\ S + 17.99}$	.0046
3.5	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 35S^3 + 50S^2 + 125S+ 84}$	$\frac{12.82\ S + 24.20}{S^2 + 5\ S + 21.00}$	.0038
4.0	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 40S^3 + 53S^2 + 140S+ 96}$	$\frac{12.84\ S + 24.20}{S^2 + 5\ S + 24.00}$	.0032
4.5	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 45S^3 + 56S2 + 155S+ 108}$	$\frac{12.85\ S + 24.19}{S^2 + 5\ S + 26.99}$	.0028
5.0	$\frac{13.19S^3+84.80S^2+167.19S+96.8}{S^4 + 50S^3 + 59S^2 + 170S+ 120}$	$\frac{12.87\ S + 24.20}{S^2 + 5\ S + 30}$	.0024

Table B-3 The Values of Error Criterion

K	μ_i	p_i	β_1	β_2	J (FD)	J (MS)
0.1	-1.0	-0.6453	-1.0	-4.0	6.3313	6.3118
	-4.0	-11.9218	-1.0	-0.123	2.8900	2.1069
	-0.123	99.8989	-1.0	-4.877	6.3454	6.3269
	-4.877	25.2542	-4.0	-0.123	1.4757	1.0813
			-4.0	-4.877	6.9660	6.9612
			-0.123	-4.877	0.9944	0.7510
0.2	-1.0	-0.7183	-1.0	-4.0	4.9254	4.8857
	-4.0	-14.2439	-1.0	-0.2528	2.7628	2.0529
	-0.2528	51.0002	-1.0	-4.7472	4.9401	4.9039
	-4.7472	27.7892	-4.0	-0.2528	1.3884	1.0096
			-4.0	-4.7472	5.6842	5.6766
			-0.2528	-4.7472	0.9446	0.6880
0.3	-1.0	-0.8438	-1.0	-4.0	3.9752	3.8890
	-4.0	-17.8404	-1.0	-0.3905	2.6418	2.0476
	-0.3905	34.6835	-1.0	-4.6095	3.9925	3.9105
	-4.6095	31.6263	-4.0	-0.3905	1.3539	0.9496
			-4.0	-4.6095	4.8865	4.8715
			-0.3905	-4.6095	0.8939	0.6367
0.4	-1.0	-1.0775	-1.0	-4.0	3.1714	2.9865
	-4.0	-24.1495	-1.0	-0.5379	2.5317	1.9946
	-0.5379	26.5620	-1.0	-4.4621	3.1752	3.0075
	-4.4621	38.2124	-4.0	-0.5379	1.1620	1.1593
			-4.0	-4.4621	4.2715	4.2428
			-0.5379	-4.4621	0.8485	0.5870
0.5	-1.0	-1.6088	-1.0	-4.0	2.3904	2.1758
	-4.0	-24.1495	-1.0	-0.6972	2.4287	2.0297
	-0.6972	21.9418	-1.0	-4.3028	2.3753	2.1779
	-4.3028	52.4267	-4.0	-0.6972	1.0299	0.9785
			-4.0	-4.3028	3.7449	3.6890
			-0.6972	-4.3028	0.8066	0.6067
0.6	-1.0	-3.7703	-1.0	-4.0	1.5132	1.2538
	-4.0	-93.6478	-1.0	-0.8721	2.3274	1.9097
	-0.8721	20.5021	-1.0	-4.1279	1.4776	1.3983
	-4.1279	108.4333	-4.0	-0.8721	0.8734	0.6602
			-4.0	-4.1279	3.4331	3.1453
			-0.8721	-4.1279	0.7714	0.4968
0.7	-1.0	7.0999	-1.0	-4.0	0.2440	-0.062
	-4.0	184.5742	-1.0	-1.6082	2.2253	1.8493
	-1.6082	6.8492	-1.0	-3.9318	0.3488	0.0503
	-3.9318	-169.284	-4.0	-1.6082	0.6796	0.3840
			-4.0	-3.9318	2.7687	2.5715
			-1.6082	-3.9318	0.7391	0.4569

Note: FD - frequency domain, MS - minimum search method