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Title: ADMISSIBILITY IN LINEAR MODELS

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We consider the linear model $Y = X\beta + e$ where Y is an $n \times 1$ random vector with mean vector $X\beta$ and covariance matrix in a set $\mathcal{V}$ of covariance matrices. For a given estimable linear parametric function $\lambda'\beta$, the class $\mathcal{A}$ of all admissible linear unbiased estimators is characterized under different assumptions on $\mathcal{V}$.

We first consider the case when the smallest closed convex cone, $[\mathcal{V}]$, containing $\mathcal{V}$ has the polyhedral structure $\{\Sigma_{j=1}^{k} \rho_j W_j : \rho_j \geq 0\}$. This allows us to consider fixed, mixed and random ANOVA models.

The case when $\mathcal{V} \subset \mathcal{V}_0$ where $\mathcal{V}_0$ is a set of non-negative definite matrices such that $[\mathcal{V}_0]$ has a polyhedral structure and $\text{sp}\,\mathcal{V} = \text{sp}\,\mathcal{V}_0$ is also considered, which allows us to apply our results to the two-variance component problem considered in Olsen, Seely and Birkes (1976).

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ADMISSIBILITY IN LINEAR MODELS

I. INTRODUCTION

In this thesis consideration is given to estimating linear parametric functions in the linear model

$$Y = X\beta + e,$$

where Y is an $n \times 1$ random vector having expectation $X\beta$ and covariance matrix in a set $\mathcal{V}$ of covariance matrices with X a known $n \times p$ matrix and β a vector of p unknown parameters. Under varying assumptions about the structure of $\mathcal{V}$, the problem of choosing "good" estimators of a given estimable linear parametric function is investigated. The available class of estimators is restricted to those estimators that are unbiased and linear in Y.

The estimation problem described above is widely considered in the literature. An extensive presentation of the early work in the general theory of linear models up to the year 1935 is contained in a review article by Seal (1967). In this article, Seal mentioned that the classical approach to this estimation problem is the principle of least squares introduced by Legendre in 1805. This principle says that the least squares estimator of β is the vector $\hat{\beta}$ that minimizes the sum of squares of the residuals. He also noted that it was Gauss in 1823 who proved that $\lambda'\hat{\beta}$ is the blue (best linear unbiased estimator) of $\lambda'\beta$ when X has full column rank and $\mathcal{V} = \{\rho I : \rho > 0\}$. The extension of Gauss's Theorem to $\mathcal{V} = \{\rho V : \rho > 0\}$, where V is a known p.d. (positive definite) matrix, and the first formulation of the problem in terms of matrices were published by Aitken (1935, 1945). In order to treat X not full column rank, Bose (1944) introduced the concept of an estimable linear parametric function. Goldman and Zelen (1964) extended Gauss's Theorem to linear models with constraints on the parameters and with singular covariance matrices. Zyskind (1967) characterized blue's as those linear estimators $b'Y$ for which Vb is in the range of X.

Zyskind (1967), Rao (1967) and Kruskal (1968) obtained conditions under which least squares estimators and blue's coincide. Thomas (1968) studied the question of when blue's under different covariance matrices coincide. It was recognized by Seely and Zyskind (1971) that this question is the same as asking when blue's exist. Using the coordinate-free approach introduced by Kruskal (1961), Seely (1970b) noted that linear model theory can be applied to quadratic unbiased estimation of the variance components. An application of these results to the one-way random model implies that blue's for the mean or the variance components exist if and only if the design is balanced.

In cases when blue's do not exist it is not clear which lue (linear unbiased estimator) to use. Olsen, Seely and Birkes (1976) characterized the minimal complete class of all admissible lue's of $\lambda'\beta$. Under a general covariance structure $\mathcal{V}$, they proved that the class of all admissible lue's is contained in the class of all lue's that are best at some non-zero V in $[\mathcal{V}]$, the smallest closed convex cone containing $\mathcal{V}$. They also proved that equality holds if $XX' + V$ is p.d. for all non-zero $V \epsilon [\mathcal{V}]$. LaMotte (1977) dealt with the same problem.

In this thesis we will characterize the class of all admissible lue's of $\lambda'\beta$ when $[\mathcal{V}] = \{\rho_1 W_1 + \ldots + \rho_k W_k : \rho_i \geq 0\}$ where $W_1, \ldots, W_k$ are n.n.d. (non-negative definite) matrices. This allows us to consider ANOVA models which cannot be treated by the results of Olsen, Seely and Birkes (1976). We will also extend these results to the case when $\mathcal{V} \subset \mathcal{V}_0 \subset \text{sp}\,\mathcal{V}$ and $[\mathcal{V}_0]$ has a polyhedral structure. These results will lead to a direct treatment of the two-variance component problem considered in Olsen, Seely and Birkes (1976).

The general linear model assumed throughout this thesis together with some basic definitions and notation are introduced in Chapter II.

Some general facts concerning admissibility are established in Chapter III. These facts are used extensively throughout the remaining chapters.

The case when $\mathcal{V}$ is such that $[\mathcal{V}] = \{\Sigma_{i=1}^{k} \rho_i W_i : \rho_i \geq 0\}$ is considered in Chapter IV. The results of this chapter cover all fixed, mixed and random ANOVA models.

Generalizations of the results of Chapter IV are given in Chapters V and VI. This allows us, in Chapter VII, to characterize the class of all admissible quadratic unbiased estimators of a linear combination of the variance components in a two-component model.

II. MODEL, DEFINITIONS AND NOTATION

2.1. Model and Definitions

Consider the general linear model

$$\mathcal{M} : E(Y) = X\beta, \quad Cov(Y) \in \mathcal{V}$$

where Y is an $n \times 1$ random vector, X a known $n \times p$ matrix, β a vector of p unknown parameters and $\mathcal{V}$ a given set of covariance matrices. A parametric function of the form $\lambda'\beta$ is said to be estimable if and only if it has a lue, i.e., if and only if $\lambda = X'a$ for some a. Throughout this thesis let λ be a fixed vector in the range of X' and let $\mathcal{B}_0 = \{a'Y : X'a = \lambda\}$ be the set of all lue's of $\lambda'\beta$. We are concerned with characterizing the class $\mathcal{Q}(\mathcal{V})$ of all admissible lue's of $\lambda'\beta$ when $\mathcal{V}$ assumes a given structure. The elements of $\mathcal{B}_0$ will be compared according to their possible variances w.r.t. (with respect to) $\mathcal{V}$. Let $b'Y$ and $h'Y$ be both in $\mathcal{B}_0$. We say that $b'Y$ is as good as $h'Y$ w.r.t. $\mathcal{V}$ if and only if $b'Vb \leq h'Vh$ for all $V \in \mathcal{V}$; $b'Y$ is better than $h'Y$ w.r.t. $\mathcal{V}$ if and only if $b'Vb \leq h'Vh$ for all $V \in \mathcal{V}$ and $b'Vb < h'Vh$ for some $V \in \mathcal{V}$. Also, we say that $b'Y$ is admissible w.r.t. $\mathcal{V}$ if and only if no element in $\mathcal{B}_0$ is better than $b'Y$ w.r.t. $\mathcal{V}$.

For a subset D of $\mathcal{B}_0$, we say that D is a complete class if and only if for every lue $b'Y$ in $\mathcal{B}_0$ but not in D there exists a lue in D which is better than $b'Y$; D is an essentially complete class if and only if for every lue $b'Y$ in $\mathcal{B}_0$ there exists a lue in D which is as good as $b'Y$; D is minimal (essentially) complete if and only if D is (essentially) complete and for any $b'Y$ in D the set $D \setminus \{b'Y\}$ is not (essentially) complete.

Finally, for a set B in a finite dimensional vector space, we say that B is a cone if and only if $b \in B$ implies that $\alpha b \in B$ for all $\alpha \geq 0$; B is a convex cone if and only if $b_1, \ldots, b_m \in B$ implies that $\Sigma_i \alpha_i b_i \in B$ for all $\alpha_i \geq 0$. Also, we say that B is a closed half-space if and only if $B = \{b : \langle b, c \rangle \leq a\}$ where $\langle -, - \rangle$ is an inner product.

2.2. Notation

Concerning notation, R^n is used to denote an n-dimensional Euclidean space. For a linear transformation or a matrix A, $\underline{R}(A)$, $\underline{N}(A)$, $\underline{r}(A)$, $\underline{n}(A)$, $|A|$ and $\mathrm{tr}(A)$ denote the range, null space, rank, nullity, determinant and trace of A respectively. For vectors $a, b \in R^n$, $a'b$ is used to denote the usual inner product with a' as the transpose for a. For a set $B \subset R^n$, $B^{\perp}$ is used to denote the orthogonal complement, w.r.t. the usual inner product, of B. For a function W and a set D, $W(D)$, $W^{-1}(D)$ and D^c

denote the image, the inverse image and the set complement of D respectively. For a set C in a finite dimensional vector space, $\bar{C}$, C^{o}, $[C]$ and sp C denote the closure of C, the relative interior of C, the smallest closed convex cone containing C, i.e., the intersection of all closed convex cones containing C, and the linear span of C respectively. Finally it is assumed throughout that all vector spaces are real finite dimensional vector spaces.

III. PRELIMINARY FACTS

In this chapter some general facts concerning $\mathcal{Q}(\mathcal{V})$ will be established. These facts are needed to establish the main results in the remaining chapters. No assumption about the structure of $\mathcal{V}$ is required.

3.1. A Minimal Essentially Complete Class and a Reformulated Model Having a p.d. Covariance Matrix

Let $\mathcal{C}_0 = \{a : X'a = \lambda\}$. Notice that $a'Y \in \mathcal{B}_0$ if and only if $a \in \mathcal{C}_0$. Olsen, Seely and Birkes (1976) proved that their definitions of "as good as", "better than" and "admissible" in $\mathcal{C}_0$ are the same w.r.t. $\mathcal{V}$ as w.r.t. $[\mathcal{V}]$. These two remarks enable us to conclude that

$$(3.1.1) \qquad \mathcal{Q}(\mathcal{V}) = \mathcal{Q}([\mathcal{V}]).$$

This fact will be used extensively to establish the results we need. Depending on which is more convenient, $\mathcal{Q}(\mathcal{V})$, $\mathcal{Q}([\mathcal{V}])$ or $\mathcal{Q}$ will be used to denote the class of all admissible lue's of $\lambda'\beta$.

Now, let V_M be a maximal element in $[\mathcal{V}]$ (see LaMotte, 1977). Thus V_M is such that $\underline{R}(V) \subset \underline{R}(V_M)$ for all $V \in [\mathcal{V}]$. Note that $\underline{N}(V_M) \subset \underline{N}(V)$ for all $V \in [\mathcal{V}]$. Let $\mathcal{R} = \underline{R}(X) + \underline{R}(V_M)$ and define $\mathcal{Q}_{\mathcal{R}} = \{t'Y : t'Y \in \mathcal{Q} \text{ with } t \in \mathcal{R}\}$ and $\mathcal{F} = \{f'Y : f \in \mathcal{R}^{\perp}\}$. Let $\mathcal{Q}_0$ be the class of all admissible lue's of the parametric function zero. Then,

Proposition (3.1.2). $\mathcal{A}_0 = \mathcal{F}$.

Pf. The estimator zero is a lue of the parametric function zero and $Var(0|V) = 0$ for all $V \in [\mathcal{V}]$ and so $a'Y \in \mathcal{A}_0$ if and only if $X'a = 0$ and $a'Va = Var(a'Y|V) = 0$ for all $V \in [\mathcal{V}]$, i.e., if and only if $a \in \underline{N}(X')$ and $a \in \underline{N}(V)$ for all $V \in [\mathcal{V}]$, i.e., if and only if $a \in \underline{N}(X') \cap \underline{N}(V_M) = \mathcal{R}^{\perp}$. []

Proposition (3.1.3). $\mathcal{A} = \mathcal{A}_{\mathcal{R}} + \mathcal{A}_0$.

Pf. Let $b'Y \in \mathcal{A}$. Since $b \in R^n = \mathcal{R} \oplus \mathcal{R}^{\perp}$, then there exist $t \in \mathcal{R}$ and $f \in \mathcal{R}^{\perp}$ such that $b = t + f$. Then $\lambda = X'b = X't + X'f = X't$ and $b'Vb = t'Vt + 2t'Vf + f'Vf = t'Vt$ for all $V \in [\mathcal{V}]$, since $X'f = 0$ and $Vf = 0$ for all $V \in [\mathcal{V}]$. Thus $t'Y \in \mathcal{A}$ (see (3.2.13)). Hence $b'Y = t'Y + f'Y$ with $t'Y \in \mathcal{A}_{\mathcal{R}}$ and $f'Y \in \mathcal{A}_0$, i.e.,

$$\mathcal{A} \subset \mathcal{A}_{\mathcal{R}} + \mathcal{A}_0 .$$

Now, let $f'Y \in \mathcal{A}_0$ and $t'Y \in \mathcal{A}_{\mathcal{R}}$, i.e., $f'Y$ is such that $f \in \mathcal{R}^{\perp}$ and $t'Y \in \mathcal{A}$ with $t \in \mathcal{R}$. Let $b = t + f$. Then $b'Y = t'Y + f'Y$ is such that $X'b = X't = \lambda$, and $b'Vb = t'Vt$ for all $V \in [\mathcal{V}]$ with $t'Y \in \mathcal{A}$ which implies that $b'Y \in \mathcal{A}$, i.e.,

$\mathcal{A}_{\mathcal{R}} + \mathcal{A}_0 \subset \mathcal{A}$. □

Proposition (3.1.4). $\mathcal{A}_{\mathcal{R}}$ is essentially complete.

Pf. Let $a'Y \in \mathcal{B}_0$. Since $\mathcal{A}$ is complete (see Olsen, Seely and Birkes, 1976), there exists $b'Y \in \mathcal{A}$ such that $b'Y$ is as good as $a'Y$. By (3.1.3), there exists $t \in \mathcal{R}$ and $f \in \mathcal{R}^{\perp}$ such that $b'Y = t'Y + f'Y$ with $t'Y \in \mathcal{A}_{\mathcal{R}}$ and $f'Y \in \mathcal{A}_0$. Then $t'Y$ is as good as $a'Y$ since $Var(t'Y|V) = Var(b'Y|V)$ for all $V \in [\mathcal{V}]$, i.e., $\mathcal{A}_{\mathcal{R}}$ is essentially complete. □

Let $q = \dim \mathcal{R}$, let H be an $n \times q$ matrix such that its columns form a basis for $\mathcal{R}$ and consider the linear model

$$\mathcal{M}_Z : E(Z) = G\beta, \quad Cov(Z) \in \mathcal{V}_Z$$

with $Z = H'Y$, $G = H'X$ and $\mathcal{V}_Z = \{H'VH : V \in \mathcal{V}\}$. Then

Proposition (3.1.5). $\lambda'\beta$ is estimable under $\mathcal{M}$ if and only if it is estimable under $\mathcal{M}_Z$.

Pf. It suffices to show that $\underline{R}(X') = \underline{R}(G')$. Note that $\underline{R}(G') = \underline{R}(X'H) \subset \underline{R}(X')$. Also, $\underline{r}(G') = \underline{r}(G) = \underline{r}(H'X) = \underline{r}(X) - \dim[\underline{R}(X) \cap \underline{N}(H')]$. Since $\underline{N}(H') = \mathcal{R}^{\perp}$ and $\underline{R}(X) \subset \mathcal{R}$ it follows that $\underline{R}(X) \cap \underline{N}(H') = \{0\}$ so that $\underline{r}(G) = \underline{r}(X)$. □

Proposition (3.1.6). $\underline{R}(G, H'V_M H) = R^q$.

Pf. Let $L = (X, V_M)$ and note that $\underline{R}(L) = \mathcal{Q} = \underline{R}(H)$. Then $\underline{R}(G, H'V_M H) = \underline{R}(H'X, H'V_M) = \underline{R}(H'L) = \underline{R}(H'H) = \underline{R}(H') = R^q$, since H' is $q \times n$ and $\underline{r}(H') = \underline{r}(H) = q$. []

Now let $\mathcal{A}_Z$ be the class of all admissible lue's of $\lambda'\beta$ under $\mathcal{M}_Z$. Then

Proposition (3.1.7). $\mathcal{A}_{\mathcal{Q}} = \mathcal{A}_Z$.

Pf. Let $b'Z$ $(=b'H'Y)$ be in $\mathcal{A}_Z$. Then $\lambda = G'b = X'Hb$ and $\nexists$ $a'Z$ such that $\lambda = G'a$ with $a'Z$ better than $b'Z$ w.r.t. $\mathcal{V}_Z$. Let $t = Hb$, so that $t'Y = b'Z$, and assume that $t'Y \notin \mathcal{A}_{\mathcal{Q}}$. Thus $t \in \underline{R}(H) = \mathcal{Q}$ is such that $E(t'Y) = \lambda'\beta$ and $t'Y \notin \mathcal{A}_{\mathcal{Q}}$ which implies that $t'Y \notin \mathcal{A}$. By the completeness of $\mathcal{A}$, there exists $h'Y \in \mathcal{A}$ such that $h'Vh \leq t'Vt$ for all $V \in \mathcal{V}$ and $h'Vh < t'Vt$ for some $V \in \mathcal{V}$. Note that $h'Y \in \mathcal{A} \subset \mathcal{B}_0$ and $\mathcal{A}_{\mathcal{Q}}$ is essentially complete. Then there exists $a'Y \in \mathcal{A}_{\mathcal{Q}}$ such that $a'Y$ is as good as $h'Y$ w.r.t. $\mathcal{V}$ with $a = Hc$ for some c. Hence $a'Va \leq t'Vt$ for all $V \in \mathcal{V}$ and $a'Va < t'Vt$ for some $V \in \mathcal{V}$ which implies that $c'H'VHc \leq b'H'VHb$ for all $V \in \mathcal{V}$ and $c'H'VHc < b'H'VHb$ for some $V \in \mathcal{V}$. Thus $c'V_Z c \leq b'V_Z b$ for all $V_Z \in \mathcal{V}_Z$ and $c'V_Z c < b'V_Z b$ for some $V_Z \in \mathcal{V}_Z$, i.e., $b'Z \notin \mathcal{A}_Z$, which is a contradiction. Thus $t'Y \in \mathcal{A}_{\mathcal{Q}}$, i.e.,

$$\mathcal{A}_Z \subset \mathcal{A}_{\mathcal{R}}.$$

Now, let $t'Y \in \mathcal{A}_{\mathcal{R}}$. Thus $t = Hb$ for some b and $t'Y = b'Z$. Assume $b'Z \notin \mathcal{A}_Z$. Thus there exists $a'Z$ with $\lambda = G'a = X'Ha$ such that $a'V_Z a \leq b'V_Z b$ for all $V_Z \in \mathcal{V}_Z$ and $a'V_Z a < b'V_Z b$ for some $V_Z \in \mathcal{V}_Z$. Let $h = Ha$. Then $h'Vh \leq t'Vt$ for all $V \in \mathcal{V}$ and $h'Vh < t'Vt$ for some $V \in \mathcal{V}$, i.e., $t'Y \notin \mathcal{A}_{\mathcal{R}}$ which is a contradiction. Hence

$$\mathcal{A}_{\mathcal{R}} \subset \mathcal{A}_Z. \quad \square$$

Proposition (3.1.8). If there exists a set of covariance matrices $\mathcal{V}_0$ such that $\mathcal{V} \subset \mathcal{V}_0 \subset \mathrm{sp}\,\mathcal{V}$ and $[\mathcal{V}_0] = \{\Sigma_{j=1}^k \rho_j V_j : \rho_j \geq 0\}$, then $\mathcal{A}_{\mathcal{R}}$ is minimal essentially complete.

Pf. Let $a'Y$ and $b'Y$ be both in $\mathcal{A}_{\mathcal{R}}$ such that $a'Va = b'Vb$ for all $V \in \mathcal{V}$. We should prove that $a = b$. By the definition of $\mathcal{A}_{\mathcal{R}}$, a and b are both in $\mathcal{R}$. Hence there exist c and d such that $a = Hc$ and $b = Hd$. Consider the linear model $\mathcal{M}_Z$ and notice that $c'Z$ $(=a'Y)$ and $d'Z$ $(=b'Y)$ are both in $\mathcal{A}_Z$. Assume there exists a set of covariance matrices $\mathcal{V}_0$ such that $\mathcal{V} \subset \mathcal{V}_0 \subset \mathrm{sp}\,\mathcal{V}$ and $[\mathcal{V}_0] = \{\Sigma_{j=1}^k \rho_j V_j : \rho_j \geq 0\}$. Hence $H'\mathcal{V}H \subset H'\mathcal{V}_0 H \subset \mathrm{sp}\, H'\mathcal{V}H$ and $H'[\mathcal{V}_0]H = \{\Sigma_{j=1}^k \rho_j H'V_j H : \rho_j \geq 0\}$, i.e., $\mathcal{V}_Z \subset \mathcal{V}_1 \subset \text{sp}\,\mathcal{V}_Z$ and $[\mathcal{V}_1] = \{\Sigma_{j=1}^{k} \rho_j H_j : \rho_j \geq 0\}$ where $H_j = H'V_jH$, $\mathcal{V}_Z = H'\mathcal{V}H$, $\mathcal{V}_1 = H'\mathcal{V}_0H$ and $[\mathcal{V}_1] = H'[\mathcal{V}_0]H$ (see (3.3.7)). By (3.1.6), $\underline{R}(G, H'V_MH) = R^q$ which implies that $\underline{R}(G, H_1, \ldots, H_k) = R^q$ since $H'V_MH \in [\mathcal{V}_Z] \subset [\mathcal{V}_1]$ and $\Sigma_{j=1}^{k} H_j$ is a maximal element in $[\mathcal{V}_1]$. Hence $\mathcal{V}_Z \subset \mathcal{V}_1 \subset \text{sp}\,\mathcal{V}_Z$ and $[\mathcal{V}_1] = \{\Sigma_{j=1}^{k} \rho_j H_j : \rho_j \geq 0\}$ where $\underline{R}(G, H_1, \ldots, H_k) = R^q$. Apply (5.2.3) to conclude that $c = d$. []

Remark (3.1.9). By (3.1.4) it suffices to consider $\mathcal{A}_{\mathcal{Q}}$ when estimating $\lambda'\beta$. Proposition (3.1.7) says that $\mathcal{A}_{\mathcal{Q}} = \mathcal{A}_Z$ where $\mathcal{A}_Z$ is the class of all admissible lue's of $\lambda'\beta$ under the transformed model $\mathcal{M}_Z$. Therefore, estimation problems concerning model $\mathcal{M}$ can be translated in terms of model $\mathcal{M}_Z$. The advantage of $\mathcal{M}_Z$ is that it satisfies (3.1.6) which is useful in Chapter IV.

Now let $\mathcal{V} + XX' = \{V + XX' : V \in \mathcal{V}\}$. Then

Proposition (3.1.10). $\mathcal{A}(\mathcal{V}) = \mathcal{A}(\mathcal{V} + XX')$.

Pf. Note that for all $a'Y \in \mathcal{B}_0$ we have

$$\text{Var}(a'Y \mid V + XX') = a'Va + a'XX'a = \text{Var}(a'Y \mid V) + \lambda'\lambda,$$

for all $V \in \mathcal{V}$. Hence the notions of "as good as" and "better than" w.r.t. $\mathcal{V}$ are the same as w.r.t. $\mathcal{V} + XX'$. []

Remark (3.1.11). As remarked in (3.1.9) we can always change from model $\mathcal{M}$ to model $\mathcal{M}_Z$. By (3.1.10), $\mathcal{Q}_Z = \mathcal{Q}_Z(\mathcal{V}_Z + GG')$. Note that $\mathcal{V}_Z + GG'$ contains a p.d. matrix $H'V_M H + GG'$. This gives us the interesting result that we can always reformulate the estimation problem so that there exists a p.d. covariance matrix, as noted by LaMotte (1977).

3.2. Best lue's and Admissibility

In this section Zyskind's Theorem (see Zyskind, 1967) will be used to establish some more general facts concerning admissibility. A restatement of this theorem is given without proof.

Zyskind's Theorem. Let Σ be a n.n.d. matrix. Then for any matrix A and vector h such that $A'h = \delta$, we have $h'\Sigma h = \min_{A'b=\delta} b'\Sigma b$ if and only if $\Sigma h \in \underline{R}(A)$.

Proposition (3.2.1). Let Σ be any n.n.d. matrix and A and F be any matrices such that $\underline{R}(F) = \underline{N}(A')$. Then $\underline{R}(A, \Sigma F) = \underline{R}(A, \Sigma)$.

Pf. Note that $\underline{R}(A, \Sigma F) \subset \underline{R}(A, \Sigma)$. It suffices to show $\underline{r}(A, \Sigma F) = \underline{r}(A, \Sigma)$. But $\underline{r}(A, \Sigma) = \underline{r}(A) + r(\Sigma) - \dim[\underline{R}(A) \cap \underline{R}(\Sigma)]$. Also $\underline{r}(A, \Sigma F) = \underline{r}(A) + \underline{r}(\Sigma F)$ because $\underline{R}(A) \cap \underline{R}(\Sigma F) = \{0\}$. (If $Ab = \Sigma F d$, then $0 = F'Ab = F'\Sigma F d$, so $\Sigma F d = 0$.) Note that

$$\underline{r}(\Sigma F) = \underline{r}(F'\Sigma) = \underline{r}(\Sigma) - \dim[\underline{R}(\Sigma) \cap \underline{N}(F')]$$
$$= \underline{r}(\Sigma) - \dim[\underline{R}(\Sigma) \cap \underline{R}(A)]. \quad \square$$

Now let $V_1, \ldots, V_r$ be non-zero matrices in $\mathcal{V}$ and define

$$\mathcal{B}(V_1) = \{b'Y \in \mathcal{B}_0 : b'V_1 b = \min_{a'Y \in \mathcal{B}_0} a'V_1 a\},$$

$$\mathcal{B}(V_1, V_2) = \{b'Y \in \mathcal{B}_1 : b'V_2 b = \min_{a'Y \in \mathcal{B}_1} a'V_2 a\},$$

$$\mathcal{B}(V_1, \ldots, V_m) = \{b'Y \in \mathcal{B}_{m-1} : b'V_m b = \min_{a'Y \in \mathcal{B}_{m-1}} a'V_m a\},$$

where $\mathcal{B}_m = \mathcal{B}(V_1, \ldots, V_m)$ for $m = 1, 2, \ldots, r$. Note that whenever $a'Y$ and $b'Y \in \mathcal{B}_r$ then $a'V_i a = b'V_i b$, $i = 1, 2, \ldots, r$.

Let $F_1, \ldots, F_r$ be such that $\underline{R}(F_1) = \underline{N}(X')$ and $\underline{R}(F_i) = \underline{N}(X') \cap \underline{N}(V_1) \cap \cdots \cap \underline{N}(V_{i-1})$, $i = 2, 3, \ldots, r$. Then

Proposition (3.2.2). $\underline{R}(X, V_1F_1, \ldots, V_rF_r) = \underline{R}(X, V_1, \ldots, V_r)$.

Pf. The result will be proved by induction. For $r = 1$, the result is established using (3.2.1) with $A = X$, $\Sigma = V_1$ and $F = F_1$. For $r \geq 2$, assume the result holds for $r-1$, i.e., assume $\underline{R}(X, V_1F_1, \ldots, V_{r-1}F_{r-1}) = \underline{R}(X, V_1, \ldots, V_{r-1})$. Thus we have $\underline{R}(X, V_1F_1, \ldots, V_{r-1}F_{r-1}, V_rF_r) = \underline{R}(X, V_1, \ldots, V_{r-1}, V_rF_r)$ and the result holds using (3.2.1) with $A = (X, V_1, \ldots, V_{r-1})$, $\Sigma = V_r$ and $F = F_r$ together with the fact that

$\underline{N}(A') = \underline{N}(X') \cap \underline{N}(V_1) \cap \cdots \cap \underline{N}(V_{r-1}) = \underline{R}(F_r)$. □

Proposition (3.2.3). $\mathcal{B}_r = \{b'Y \in \mathcal{B}_{r-1} : V_r b \in \underline{R}(X, V_1, \ldots, V_{r-1})\}$.

Pf. The result will be proved by induction. For $r = 1$, use Zyskind's Theorem with $A = X$, $\Sigma = V_1$ and $\delta = \lambda$ to obtain

$$\mathcal{B}_1 = \{b'Y \in \mathcal{B}_0 : b'V_1 b = \min_{a'Y \in \mathcal{B}_0} a'V_1 a\}$$

$$= \{b'Y \in \mathcal{B}_0 : V_1 b \in \underline{R}(X)\}.$$

Thus,

$$\mathcal{B}_1 = \{b'Y \in \mathcal{B}_0 : F_1' V_1 b = 0\} = \{b'Y : (X, V_1 F_1)' b = (\lambda', 0)'\}.$$

For $r = 2$, the proof will be given for clarification.

$$\mathcal{B}_2 = \{b'Y \in \mathcal{B}_1 : b'V_2 b = \min_{a'Y \in \mathcal{B}_1} a'V_2 a\}$$

$$= \{b'Y \in \mathcal{B}_1 : b'V_2 b = \min_{A'a = \delta} a'V_2 a\},$$

where $A = (X, V_1 F_1)$ and $\delta = (\lambda', 0)'$. Apply Zyskind's Theorem with $\Sigma = V_2$ to have

$$\mathcal{B}_2 = \{b'Y \in \mathcal{B}_1 : V_2 b \in \underline{R}(X, V_1 F_1)\},$$

and the result holds using (3.2.1). Note that

$$\mathcal{B}_2 = \{b'Y \in \mathcal{B}_1 : F_2'V_2 b = 0\}$$

$$= \{b'Y : (X, V_1F_1, V_2F_2)'b = (\lambda', 0, 0)'\} .$$

For $r \geq 3$, assume the result holds for $r-1$, i.e., assume that

$$\mathcal{B}_{r-1} = \{b'Y \in \mathcal{B}_{r-2} : V_{r-1}b \in \underline{R}(X, V_1, \ldots, V_{r-2})\} .$$

Then

$$\mathcal{B}_{r-1} = \{b'Y \in \mathcal{B}_{r-2} : F_{r-1}'V_{r-1}b = 0\}$$

$$= \{b'Y : (X, V_1F_1, \ldots, V_{r-1}F_{r-1})'b = (\lambda', 0, \ldots, 0)'\} .$$

Thus, we have

$$\mathcal{B}_r = \{b'Y \in \mathcal{B}_{r-1} : b'V_r b = \min_{a'Y \in \mathcal{B}_{r-1}} a'V_r a\}$$

$$= \{b'Y \in \mathcal{B}_{r-1} : b'V_r b = \min_{A'a = \delta} a'V_r a\} ,$$

where $A = (X, V_1F_1, \ldots, V_{r-1}F_{r-1})$ and $\delta = (\lambda', 0, \ldots, 0)'$. Applying Zyskind's Theorem with $\Sigma = V_r$ we have

$$\mathcal{B}_r = \{b'Y \in \mathcal{B}_{r-1} : V_r b \in \underline{R}(X, V_1F_1, \ldots, V_{r-1}F_{r-1})\} ,$$

and the result holds using (3.2.2). $\square$

Remark (3.2.4). Note that from the proof of Proposition (3.2.3) we have

$$\mathcal{B}_r = \{b'Y: (X, V_1F_1, \ldots, V_rF_r)'b = (\lambda', 0, \ldots, 0)'\} .$$

Proposition (3.2.5). Let b_0 be any vector such that $b_0'Y \in \mathcal{B}_r$. Then,

$$\mathcal{B}_r = \{b'Y: b \in b_0 + \underline{N}(X')\cap\underline{N}(V_1)\cap\cdots\cap\underline{N}(V_r)\} .$$

Pf. Since $b_0'Y \in \mathcal{B}_r \subset \mathcal{B}_0$, then $X'b = \lambda$ if and only if $b \in b_0 + \underline{N}(X')$. Remark (3.2.4) implies that

$$\mathcal{B}_r = \{b'Y: (X, V_1F_1, \ldots, V_rF_r)'b = (\lambda', 0, \ldots, 0)'\}$$

$$= \{b'Y: X'b = \lambda, (V_1F_1, \ldots, V_rF_r)'b = (0, \ldots, 0)'\}$$

$$= \{b'Y: b \in (b_0+\underline{N}(X'))\cap\underline{N}(F_1'V_1)\cap\cdots\cap\underline{N}(F_r'V_r)\}$$

$$= \{b'Y: b \in b_0 + \underline{N}(X')\cap\underline{N}(F_1'V_1)\cap\cdots\cap\underline{N}(F_r'V_r)\} ,$$

since $b_0 \in \underline{N}((V_1F_1, \ldots, V_rF_r)') = \underline{N}(F_1'V_1)\cap\cdots\cap\underline{N}(F_r'V_r)$. Using (3.2.2) we have

$$\underline{N}(X')\cap\underline{N}(F_1'V_1)\cap\cdots\cap\underline{N}(F_r'V_r) = \underline{R}(X, V_1F_1, \ldots, V_rF_r)^{\perp}$$

$$= \underline{R}(X, V_1, \ldots, V_r)^{\perp} = \underline{N}(X')\cap\underline{N}(V_1)\cap\cdots\cap\underline{N}(V_r). \quad \square$$

Proposition (3.2.6). $\mathcal{B}_r$ has only one element if and only if $\underline{R}(X, V_1, \ldots, V_r) = R^n$.

Pf. Proposition (3.2.5) implies that $b_0'Y$ is the only element in $\mathcal{B}_r$ if and only if $\underline{N}(X')\cap\underline{N}(V_1)\cap\cdots\cap\underline{N}(V_r) = \{0\}$. ▯

Corollary (3.2.7). If $b'Y \in \mathcal{B}_r$ with $\underline{R}(X, V_1, \ldots, V_r) = R^n$, then $b'Y$ is admissible.

Pf. Let $a'Y$ be another element in $\mathcal{B}_0$ such that $a'Y$ is better than $b'Y$ w.r.t. $\mathcal{V}$. By the definition of $\mathcal{B}_r$, $a'Y \in \mathcal{B}_r$. By (3.2.6), $a'Y = b'Y$ which is a contradiction. ▯

Corollary (3.2.8). If $b'Y \in \mathcal{B}_r$ with V_r p.d., then $b'Y$ is admissible.

Proposition (3.2.9). $\mathcal{B}(V_1, \ldots, V_{r-1}, V_r) = \mathcal{B}(V_1, \ldots, V_{r-1})$ if $\underline{R}(V_r) \subset \underline{R}(X, V_1, \ldots, V_{r-1})$.

Pf. By (3.2.3) we have $b'Y \in \mathcal{B}_r$ if and only if $b'Y \in \mathcal{B}_{r-1}$ and $V_r b \in \underline{R}(X, V_1, \ldots, V_{r-1})$. Thus $b'Y \in \mathcal{B}_r$ if and only if $b'Y \in \mathcal{B}_{r-1}$ since $V_r b \in \underline{R}(V_r) \subset \underline{R}(X, V_1, \ldots, V_{r-1})$ for all $b \in R^n$. ▯

Proposition (3.2.10). If $a'Y, b'Y \in \mathcal{B}(V_1, \ldots, V_r)$ and if V is n.n.d. such that $\underline{R}(V) \subset \underline{R}(X, V_1, \ldots, V_r)$, then $a'Va = b'Vb$.

Pf. Proposition (3.2.9) implies that $\mathcal{B}(V_1, \ldots, V_r, V) = \mathcal{B}(V_1, \ldots, V_r)$, and the result holds by the definition of $\mathcal{B}(V_1, \ldots, V_r, V)$. ▯

For $i = 1, 2, \ldots, h$ with $h \leq r$, let $K_i \subset \{1, 2, \ldots, r\}$. Let $J_1 = K_1$ and $J_m = K_m \setminus \cup_{i=1}^{m-1} K_i$, $m = 2, \ldots, h$.

Proposition (3.2.11).

$$\mathcal{B}(\Sigma_{j \in K_1} V_j, \ldots, \Sigma_{j \in K_h} V_j) = \mathcal{B}(\Sigma_{j \in J_1} V_j, \ldots, \Sigma_{j \in J_h} V_j).$$

Pf. The result will be proved by induction. For $h = 1$ the result is obvious since $K_1 = J_1$. To continue the proof, define $L_m = K_m \cap (\cup_{i=1}^{m-1} K_i)$, $m = 2, \ldots, h$. For $h = 2$ the proof will be given for clarification. By (3.2.3) we have

$$b'Y \in \mathcal{B}(\Sigma_{j \in K_1} V_j, \Sigma_{j \in K_2} V_j)$$

$$<=>$$

$$b'Y \in \mathcal{B}(\Sigma_{j \in K_1} V_j)$$

$$\text{and} \quad (\Sigma_{j \in K_2} V_j) b \in \underline{R}(X, \Sigma_{j \in K_1} V_j)$$

$$<=>$$

$$b'Y \in \mathcal{B}(\Sigma_{j \in J_1} V_j)$$

$$\text{and} \quad (\Sigma_{j \in J_2} V_j) b \in \underline{R}(X, \Sigma_{j \in J_1} V_j)$$

since $J_1 = K_1$ and $K_2 = L_2 \cup J_2$ with $L_2 \subset K_1$ which implies that

$$\Sigma_{j \in J_1} V_j = \Sigma_{j \in K_1} V_j \quad \text{and} \quad \underline{R}(\Sigma_{j \in L_2} V_j) \subset \underline{R}(\Sigma_{j \in K_1} V_j).$$

Thus

$$\mathcal{B}(\Sigma_{j \in K_1} V_j, \Sigma_{j \in K_2} V_j) = \mathcal{B}(\Sigma_{j \in J_1} V_j, \Sigma_{j \in J_2} V_j).$$

For $h \geq 3$ assume the result holds for $h-1$, i.e., assume that

$$\mathcal{B}(\Sigma_{j\in K_1} V_j, \ldots, \Sigma_{j\in K_{h-1}} V_j) = \mathcal{B}(\Sigma_{j\in J_1} V_j, \ldots, \Sigma_{j\in J_{h-1}} V_j) .$$

Hence

$$b'Y \in \mathcal{B}(\Sigma_{j\in K_1} V_j, \ldots, \Sigma_{j\in K_h} V_j)$$

$$<=>$$

$$b'Y \in \mathcal{B}(\Sigma_{j\in K_1} V_j, \ldots, \Sigma_{j\in K_{h-1}} V_j)$$

and

$$(\Sigma_{j\in K_h} V_j)b \in \underline{R}(X, \Sigma_{j\in K_1} V_j, \ldots, \Sigma_{j\in K_{h-1}} V_j)$$

$$<=>$$

$$b'Y \in \mathcal{B}(\Sigma_{j\in J_1} V_j, \ldots, \Sigma_{j\in J_{h-1}} V_j)$$

and

$$(\Sigma_{j\in J_h} V_j)b \in \underline{R}(X, \Sigma_{j\in J_1} V_j, \ldots, \Sigma_{j\in J_{h-1}} V_j)$$

since $K_h = L_h \cup J_h$ with $L_h \subset \cup_{i=1}^{h-1} K_i$ which implies that $\underline{R}(\Sigma_{j\in L_h} V_j) \subset \underline{R}(\Sigma_{j\in K_1} V_j, \ldots, \Sigma_{j\in K_{h-1}} V_j)$. $\square$

This last proposition says, for example, that

$$\mathcal{B}(V_1+V_2+V_3, V_1+V_3+V_4) = \mathcal{B}(V_1+V_2+V_3, V_4) ,$$

which might be a useful simplification.

The next two propositions will be used extensively in Chapters V and VI.

Proposition (3.2.12). Let $a'Y \in \mathcal{B}_0$ and $b'Y \in \mathcal{A}(\mathcal{V})$. If $a'Y$ is as good as $b'Y$ w.r.t. $\mathcal{V}$, then $a'Va = b'Vb$ for all V in $\mathcal{V}$.

Pf. Note that $a'Va \leq b'Vb$ for all $V \in \mathcal{V}$. Assume there exists V_1 in $\mathcal{V}$ such that $b'V_1b \neq a'V_1a$. Thus $a'V_1a < b'V_1b$ which implies that $b'Y \notin \mathcal{A}(\mathcal{V})$, contradiction. []

Proposition (3.2.13). If $a'Y \in \mathcal{B}_0$ and $b'Y \in \mathcal{A}(\mathcal{V})$ are such that $a'Va = b'Vb$ for all V in $\mathcal{V}$, then $a'Y \in \mathcal{A}(\mathcal{V})$.

Pf. Assume $a'Y \notin \mathcal{A}(\mathcal{V})$. Then, by the completeness of $\mathcal{A}$, there exists $h'Y$ admissible such that $h'Vh \leq a'Va$ for all $V \in \mathcal{V}$ and $h'Vh < a'Va$ for some $V \in \mathcal{V}$. But $a'Va = b'Vb$ for all $V \in \mathcal{V}$. Hence $h'Y$ is better than $b'Y$ w.r.t. $\mathcal{V}$, i.e., $b'Y \notin \mathcal{A}(\mathcal{V})$ which is a contradiction. []

3.3. Alternative Representations for $[\mathcal{V}_Z]$, $[\mathcal{V}]$ and $[\mathcal{V}]^o$

Consider the linear model $\mathcal{M}_Z$ introduced in Section 3.1. Recall that $\mathcal{V}_Z = \{H'VH : V \in \mathcal{V}\}$ and $\underline{R}(V) \subset \underline{R}(H) = \underline{R}(X, V_M)$ for all $V \in [\mathcal{V}]$. In this section we will prove that $[\mathcal{V}_Z] = H'[\mathcal{V}]H$ and obtain alternative representations for $[\mathcal{V}]$ and $[\mathcal{V}]^o$ where $[\mathcal{V}]^o$ is the relative interior of $[\mathcal{V}]$. To begin the proof, let B be a subset of a real finite dimensional vector space. Then

Proposition (3.3.1). If B is a convex cone, then $\overline{B}$ is.

Pf. Since the closure of a convex set is convex (see Eggleston, 1963), it suffices to show that $\overline{B}$ is a cone. Let $b \in \overline{B}$ and $\alpha \geq 0$. We should prove that $\alpha b \in \overline{B}$. Since $b \in \overline{B}$, then $b = \lim_{n\to\infty} b_n$ with $b_n \in B$ for all n. Thus $\alpha b = \lim_{n\to\infty} \alpha b_n \in \overline{B}$ since $\alpha b_n \in B$. []

Now, let B^{scc} be the smallest convex cone containing B, i.e., the intersection of all convex cones containing B. Then

Proposition (3.3.2). $B^{scc} = \{\Sigma_i \alpha_i b_i : \alpha_i \geq 0, b_i \in B\}$.

Pf. It can be verified that the right hand set, G say, is a convex cone. Note $B \subset G$. Let C be any other convex cone containing B. Note that $C = \{\Sigma_i \alpha_i c_i : \alpha_i \geq 0, c_i \in C\}$. Now if $b \in G$, then $b = \Sigma_i \alpha_i b_i$ with $\alpha_i \geq 0$ and $b_i \in B \subset C$ and hence $b \in C$. Thus $G \subset C$ and so $G = B^{scc}$. []

Proposition (3.3.3). $[B] = \overline{B^{scc}}$.

Pf. Note that $[B]$ is a convex cone containing B. Thus,

$$B^{scc} \subset [B]$$

which implies that $\overline{B^{scc}} \subset \overline{[B]} = [B]$ since $[B]$ is closed. Note also that $B \subset B^{scc} \subset \overline{B^{scc}}$ and $\overline{B^{scc}}$ is a closed convex cone by (3.3.1), so

$$[B] \subset \overline{B^{scc}} . \square$$

<u>Proposition (3.3.4)</u>. If Γ is a compact convex set not containing zero, then $[\Gamma] = \{\alpha\gamma : \alpha \geq 0, \gamma \in \Gamma\}$.

<u>Pf</u>. Propositions (3.3.3) and (3.3.2) say that $[\Gamma] = \overline{\Gamma^{scc}}$ with $\Gamma^{scc} = \{\Sigma_i \alpha_i \gamma_i : \alpha_i \geq 0, \gamma_i \in \Gamma\}$. Let $G = \{\alpha\gamma : \alpha \geq 0, \gamma \in \Gamma\}$. Thus, it suffices to show that $\Gamma^{scc} = G$ and that G is closed. Clearly $G \subset \Gamma^{scc}$. Now, let $\Sigma_i \alpha_i \gamma_i \in \Gamma^{scc}$. Hence $\Sigma_i \alpha_i \gamma_i = (\Sigma_i \alpha_i)\Sigma_i(\alpha_i / \Sigma_j \alpha_j)\gamma_i = \alpha\gamma$, with $\alpha = \Sigma_i \alpha_i > 0$ and $\gamma = \Sigma_i (\alpha_i / \Sigma_j \alpha_j)\gamma_i \in \Gamma$ since Γ is convex, $\gamma_i \in \Gamma$ and $(\alpha_i / \Sigma_j \alpha_j) \geq 0$ for all i with $\Sigma_i (\alpha_i / \Sigma_j \alpha_j) = 1$. Hence, $\Sigma_i \alpha_i \gamma_i \in G$, i.e., $\Gamma^{scc} \subset G$. Thus $\Gamma^{scc} = G$. To prove that G is closed, let $\{\alpha_n \gamma_n\}$ be a convergent sequence in G. Then $\alpha_n \geq 0$ and $\gamma_n \in \Gamma$ are such that $\alpha_n \gamma_n \to h$. We should prove that $h \in G$, i.e., $h = \alpha\gamma$ with $\alpha \geq 0$ and $\gamma \in \Gamma$. Note that Γ is a compact set not containing zero. Hence there exists $a > 0$ such that $a \leq \|\gamma_n\| < \infty$ where $\| \ \|$ is the Euclidean norm. Since $\{\alpha_n \gamma_n\}$ is convergent, then there exists $c < \infty$ such that $|\alpha_n| \, \|\gamma_n\| = \|\alpha_n \gamma_n\| \leq c$ which implies that $|\alpha_n| \leq c/a < \infty$, i.e., $\{\alpha_n\}$ is a bounded sequence. Note that $\{\gamma_n\}$ is an infinite sequence in the compact set Γ. Hence

there exists a subsequence $\{\gamma_{n_j}\}$ such that $\gamma_{n_j} \to \gamma \in \Gamma$. Also, since $\{\alpha_n\}$ is bounded, then the subsequence $\{\alpha_{n_j}\}$ is bounded. Hence there exists a sub-subsequence $\{\alpha_{n_{j_i}}\}$ such that

$\alpha_{n_{j_i}} \to \alpha$ with $\alpha \geq 0$ since $\{\alpha_{n_{j_i}}\} \geq 0$. Note that $\gamma_{n_{j_i}} \to \gamma$.

Since $\alpha_n \gamma_n \to h$, then $\alpha_{n_{j_i}} \gamma_{n_{j_i}} \to h$. But $\alpha_{n_{j_i}} \gamma_{n_{j_i}} \to \alpha\gamma$ and hence $h = \alpha\gamma$ with $\alpha \geq 0$ and $\gamma \in \Gamma$. []

<u>Proposition (3.3.5)</u>. $\mathcal{V}_Z^{scc} = H'\mathcal{V}^{scc}H$.

<u>Pf</u>. By (3.3.2) we have $\mathcal{V}_Z^{scc} = \{\Sigma_j \alpha_j H_j : \alpha_j \geq 0, H_j \in \mathcal{V}_Z\}$. We can write $H_j = H'V_jH$ for some $V_j \in \mathcal{V}$. Using (3.3.2) we have

$$\mathcal{V}_Z^{scc} = \{H'(\Sigma_j \alpha_j V_j)H : \alpha_j \geq 0, V_j \in \mathcal{V}\}$$

$$= H'\{\Sigma_j \alpha_j V_j : \alpha_j \geq 0, V_j \in \mathcal{V}\}H$$

$$= H'\mathcal{V}^{scc}H\ .$$ []

<u>Proposition (3.3.6)</u>. $H'VH = 0$ if and only if $V = 0$, for all $V \in [\mathcal{V}]$.

<u>Pf</u>. Clearly if $V = 0$, then $H'VH = 0$. Now, suppose $H'VH = 0$. Then $VH = 0$ which implies that $\underline{R}(H) \subset \underline{N}(V)$. But $\underline{R}(V) \subset \underline{R}(H)$ for all $V \in [\mathcal{V}]$. Hence we have $\underline{R}(V) \subset \underline{N}(V)$ which implies that $V^2 = 0$ and the result holds. []

Theorem (3.3.7). $[\mathcal{V}_Z] = H'[\mathcal{V}]H$.

Pf. By (3.3.3) and (3.3.5), $[\mathcal{V}_Z] = \overline{\mathcal{V}_Z^{scc}} = \overline{H'\mathcal{V}^{scc}H}$. Applying (3.3.3) again we have $H'[\mathcal{V}]H = H'\overline{\mathcal{V}^{scc}}H$. Hence we should prove that $\overline{L(\mathcal{V}^{scc})} = L(\overline{\mathcal{V}^{scc}})$, where $L(V) = H'VH$. Note that L is linear and continuous. By Dugundji (1966), $L(\overline{\mathcal{V}^{scc}}) \subset \overline{L(\mathcal{V}^{scc})}$. To prove the other containment, it suffices to show that $L(\overline{\mathcal{V}^{scc}})$ is closed since $L(\mathcal{V}^{scc}) \subset L(\overline{\mathcal{V}^{scc}})$. By Lemma 3.5 in Olsen, Seely and Birkes (1976) there exists a compact convex set Γ not containing zero such that $[\mathcal{V}] = [\Gamma]$. Hence, by (3.3.3), we have

$$L(\overline{\mathcal{V}^{scc}}) = L([\mathcal{V}]) = L([\Gamma]).$$

Using (3.3.4) twice together with the linearity of L we have

$$\begin{aligned} L([\Gamma]) &= L(\{\rho V : \rho \geq 0, V \in \Gamma\}) \\ &= \{\rho L(V) : \rho \geq 0, L(V) \in L(\Gamma)\} = [L(\Gamma)] , \end{aligned}$$

which is closed. For this last equality, notice that $L(\Gamma)$ is compact because L is continuous, convex because L is linear and does not contain zero by (3.3.6). []

Now, let $S_1, \ldots, S_k$ be $n \times n$ real matrices and let Ω be a subset of R^k such that $W(\rho) = \Sigma_{j=1}^k \rho_j S_j$ is n.n.d. for all $\rho \in \Omega$. Define

$$\mathcal{V}_{\Omega} = \{W(\rho) : \rho \in \Omega\} = W(\Omega) ,$$

and notice that W is a linear mapping from R^k to $\mathcal{K}$, the vector space of all $n \times n$ real matrices.

Proposition (3.3.8). $W([\Omega]) \subset [W(\Omega)]$.

Pf. We will begin by proving that $W^{-1}([W(\Omega)])$ is a closed convex cone containing Ω. Clearly, $\Omega \subset W^{-1}([W(\Omega)])$ because $W(\Omega) \subset [W(\Omega)]$. Also, $W^{-1}([W(\Omega)])$ is closed in R^k since W is continuous and $[W(\Omega)]$ is closed in $\mathcal{K}$. Now, let $\rho_1, \ldots, \rho_m \in W^{-1}([W(\Omega)])$ and $\alpha_1, \ldots, \alpha_m$ be nonegative numbers. Note that $\Sigma_i \alpha_i W(\rho_i) \in [W(\Omega)]$. By the linearity of W, $\Sigma_i \alpha_i W(\rho_i) = W(\Sigma_i \alpha_i \rho_i)$. This implies that $W(\Sigma_i \alpha_i \rho_i) \in [W(\Omega)]$, so $\Sigma_i \alpha_i \rho_i \in W^{-1}([W(\Omega)])$. Hence $W^{-1}([W(\Omega)])$ is a closed convex cone containing Ω which implies that $[\Omega] \subset W^{-1}([W(\Omega)])$. Therefore, $W([\Omega]) \subset [W(\Omega)]$. []

Proposition (3.3.9). If $W([\Omega])$ is closed, then $[W(\Omega)] \subset W([\Omega])$.

Pf. Since $\Omega \subset [\Omega]$, then $W(\Omega) \subset W([\Omega])$ where $W([\Omega])$ is a convex cone because W is linear and $[\Omega]$ is a convex cone. Hence $W(\Omega)^{scc} \subset W([\Omega])$. Now assume that $W([\Omega])$ is closed. Then, by (3.3.3), we have

$$[W(\Omega)] = \overline{W(\Omega)^{scc}} \subset \overline{W([\Omega])} = W([\Omega]) \ . \ \square$$

Theorem (3.3.10). If $\mathcal{V}_{[\Omega]}$ is closed, then $\mathcal{V}_{[\Omega]} = [\mathcal{V}_{\Omega}]$.

Pf. Recall that $\mathcal{V}_{[\Omega]} = W([\Omega])$ and $[\mathcal{V}_{\Omega}] = [W(\Omega)]$ and apply (3.3.8) and (3.3.9). $\square$

Proposition (3.3.11). If there exists a compact convex set $G \subset [\Omega]$ such that $[G] = [\Omega]$ and $0 \notin W(G)$, then $\mathcal{V}_{[\Omega]} = [\mathcal{V}_{\Omega}]$.

Pf. Notice that the zero vector is not in G since the zero matrix is not in $W(G)$. Hence G is a compact convex set not containing zero. Also, $W(G)$ is a compact convex set not containing zero. It is compact because W is continuous and G is compact, convex because W is linear and G is convex, and not containing zero by assumption. Applying (3.3.4) twice together with the linearity of W we have

$$\begin{aligned} [W(G)] &= \{\alpha W(\gamma) : \alpha \geq 0, W(\gamma) \in W(G)\} \\ &= \{W(\alpha\gamma) : \alpha \geq 0, \gamma \in G\} \\ &= \{W(\alpha\gamma) : \alpha\gamma \in [G]\} \\ &= W([G]) = W([\Omega]). \end{aligned}$$

The last equality holds because $[G] = [\Omega]$. Hence $\mathcal{V}_{[\Omega]} = W([\Omega])$ is closed. Apply (3.3.10). $\square$

Proposition (3. 3. 12). If $S_1, \ldots, S_k$ are linearly independent, then $\mathcal{V}_{[\Omega]} = [\mathcal{V}_\Omega]$.

Pf. Assume that $S_1, \ldots, S_k$ are linearly independent. Then $W : R^k \to W(R^k)$ is one-to-one and onto linear mapping. Hence, $\mathcal{V}_{[\Omega]} = W([\Omega])$ is closed since $[\Omega]$ is closed and W is a one-to-one and onto continuous map. Apply (3. 3. 10). []

Assumption. Hereafter we will assume that $S_1, \ldots, S_k$ are linearly independent, i. e., $W : R^k \to W(R^k)$ is a one-to-one and onto linear mapping which implies that its inverse W^{-1} exists and is linear. Hence W is a continuous map whose inverse is continuous, i.e., W is a homeomorphism and hence preserves topology as well as linear structure. Of course W^{-1}, also, preserves topology and linear structure.

Now for any set B in a real finite dimensional vector space let $\text{aff}(B)$ and B^o denote the affine hull of B and the interior of B relative to its affine hull. Hence B^o is the largest set contained in B and open in $\text{aff}(B)$.

Proposition (3. 3. 13). $W(\text{aff}(B)) = \text{aff}(W(B))$.

Pf. Apply the linearity of W. []

Proposition (3. 3. 14). $W(B^o) = (W(B))^o$.

Pf. Note that B^o is the largest set contained in B and open in aff(B). Since W preserves topology, then $W(B^o)$ is the largest set contained in $W(B)$ and open in $W(\text{aff}(B)) = \text{aff}(W(B))$ (see (3.3.13)). Also, by definition, $(W(B))^o$ is the largest set contained in $W(B)$ and open in aff(W(B)). Hence $W(B^o) = (W(B))^o$. ▯

Theorem (3.3.15). $\mathcal{V}_{[\Omega]^o} = [\mathcal{V}_\Omega]^o$.

Pf. Recall that $\mathcal{V}_{[\Omega]^o} = W([\Omega]^o)$ and, by (3.3.12), $[\mathcal{V}_\Omega]^o = (\mathcal{V}_{[\Omega]})^o = (W([\Omega]))^o$. Apply (3.3.14) with $B = [\Omega]$ to conclude that $W([\Omega]^o) = (W([\Omega]))^o$. ▯

Let $\mathcal{N}$ be the set of all $n \times n$ n.n.d. matrices. Then

Proposition (3.3.16). If $W(\rho) \in \mathcal{N}$ for all $\rho \in \Omega$, then $W(\rho) \in \mathcal{N}$ for all $\rho \in [\Omega]$.

Pf. Recall that $W : R^k \to W(R^k)$. Hence
$\Omega \subset \{\rho : W(\rho) \in \mathcal{N}\} = \{\rho : W(\rho) \in \mathcal{N} \cap W(R^k)\} = W^{-1}(\mathcal{N} \cap W(R^k))$, which is a closed convex cone. It is closed because $\mathcal{N}$ is closed in $\mathcal{K}$, so $\mathcal{N} \cap W(R^k)$ is closed in $W(R^k)$, and W^{-1} preserves topology. It is a convex cone because $\mathcal{N}$ and R^k are convex cones and W and W^{-1} preserve linear structure. Hence
$[\Omega] \subset W^{-1}(\mathcal{N} \cap W(R^k))$. ▯

3.4. Topological Properties of $\mathcal{Q}$

In this section, some topological properties of $\mathcal{Q}$ will be studied. The topology of the set of linear estimators $b'Y$, $b \in R^n$, will be taken to be the topology of the coefficient vectors b, i.e., the topology of R^n. We prove through a counter example that $\mathcal{Q}$ may not be compact. However, $\mathcal{Q}$ is shown to be compact when $(XX'+V)$ is p.d. for all non-zero $V \in [\mathcal{V}]$.

Proposition (3.4.1). If V is p.d. for all non-zero $V \in [\mathcal{V}]$, then $\mathcal{Q}$ is compact.

Pf. If V is p.d. for all $V \in [\mathcal{V}]\setminus\{0\}$, then

$$\mathcal{Q} = \cup_{V \in [\mathcal{V}]\setminus\{0\}} \mathcal{B}(V) .$$

In this case, $b'Y \in \mathcal{B}(V)$ if and only if $b = V^{-1}X(X'V^{-1}X)^{-1}\lambda$. By Lemma 3.5 in Olsen, Seely and Birkes (1976), there exists a compact set Γ not containing zero such that $[\mathcal{V}] = \{\alpha V : \alpha > 0, V \in \Gamma\} \cup \{0\}$. Note $\mathcal{B}(\alpha V) = \mathcal{B}(V)$ for $\alpha > 0$, so $\mathcal{Q} = \cup_{V \in \Gamma} \mathcal{B}(V)$. Define $f(V) = V^{-1}X(X'V^{-1}X)^{-1}\lambda$. Thus f is a continuous map from Γ onto the coefficient vectors of $\mathcal{Q}$. The result holds since the image of a continuous function on a compact set is compact. $\square$

Proposition (3.4.2). If $\underline{R}(X, V) = R^n$ for all non-zero V in $[\mathcal{V}]$, then $\mathcal{A}$ is compact.

Pf. Olsen, Seely and Birkes (1976) proved that $\mathcal{A} = \cup_{V \in [\mathcal{V}] \setminus \{0\}} \mathcal{B}(V)$. Proposition (3.1.10) says that $\mathcal{A} = \mathcal{A}(\mathcal{V} + XX')$. Note that $(XX' + V) = V_X$ is p.d. for all non-zero $V_X \in [\mathcal{V}_X]$ and the result holds if we apply (3.4.1) to $\mathcal{A}(\mathcal{V} + XX')$. $\square$

Proposition (3.4.3). Suppose $\lambda \neq 0$. If $\mathcal{V}$ is the class of all p.d. matrices, then $\mathcal{A} = \mathcal{B}_0$.

Pf. Clearly $\mathcal{A} \subset \mathcal{B}_0$. Now let $b \in R^n$ be such that $X'b = \lambda$. Then bb' is n.n.d. matrix such that $\underline{R}(bb') = \underline{R}(b)$, $\underline{N}(bb') = \underline{N}(b')$, $\underline{r}(bb') = 1$ and $\underline{n}(bb') = n-1$. Let Q be such that its columns form a basis for $\underline{N}(bb')$ and consider the matrix $(XX'+QQ')$. Then $(XX'+QQ')$ is n.n.d. with

$$\begin{aligned} \underline{r}(XX'+QQ') &= \underline{r}((X, Q)(X, Q)') = \underline{r}(X, Q) \\ &= \underline{r}(X) + r(Q) - \dim[\underline{R}(X) \cap \underline{R}(Q)] \\ &= \underline{r}(X) + \underline{n}(b') - \dim[\underline{R}(X) \cap \underline{N}(b')] = n, \end{aligned}$$

since $\underline{n}(b') = n-1$ and $1 = \underline{r}(b'X) = \underline{r}(X) - \dim[\underline{R}(X) \cap \underline{N}(b')]$. Thus $(XX'+QQ')$ is p.d. Note that

$$(XX'+QQ')b = XX'b + QQ'b = XX'b \in \underline{R}(X),$$

since $b'Q = 0 = Q'b$. Thus $b \in R^n$ is such that $X'b = \lambda$ and $(XX'+QQ')b \in \underline{R}(X)$. Zyskind's Theorem implies that $b'Y \in \mathcal{B}(XX'+QQ')$. Note that $b'Y$ is the only element in $\mathcal{B}(XX'+QQ')$ since $(XX'+QQ')$ is p.d. Thus $b'Y \in \mathcal{Q}$, which implies that $\mathcal{B}_0 \subset \mathcal{Q}$. □

Note that (3.4.3) gives an example where $\mathcal{Q}(\mathcal{V})$ is not compact.

Remark. If $\mathcal{V}$ is the class of all n.n.d. matrices then $\mathcal{Q} = \mathcal{B}_0$ since the closure of the class of all p.d. matrices is the class of all n.n.d. matrices.

IV. ADMISSIBILITY WHEN $[\mathcal{V}]$ ASSUMES A POLYHEDRAL STRUCTURE

4.1. Model and Notation

Throughout this chapter let $\mathcal{V}$ be such that

$$[\mathcal{V}] = \{\Sigma_{j=1}^{k} \rho_j W_j : \rho_j \geq 0\} ,$$

where the W_j's $(j = 1, 2, \ldots, k)$ are known $n \times n$ n.n.d. matrices. In the first four sections we assume that $\underline{R}(X, W_1, \ldots, W_k) = R^n$. This range condition is relaxed in Section 5. Note that fixed, mixed and random effect ANOVA models all have covariance structure like the one assumed in this chapter.

Suppose $J_1, J_2, \ldots, J_h$ are nonempty pairwise disjoint subsets of $J = \{1,2, \ldots, k\}$. Let $J_h' = J \setminus \cup_{i=1}^{h} J_i$. Suppose J_h' is nonempty. Set

$$(4.1.1) \quad \mathcal{W}_h = \{\Sigma_{j \in J_h'} \rho_j W_j : \rho_j \geq 0\} .$$

Consider a given set of positive numbers α_j, $j = 1, 2, \ldots, k$. Set $V_i = \Sigma_{j \in J_i} \alpha_j W_j$ $(i = 1, 2, \ldots, h)$ and $\mathcal{B}_h = \mathcal{B}(V_1, \ldots, V_h)$. Note that the results of Section 3.2 hold for $\mathcal{B}_h$ and recall that $\mathcal{Q} = \mathcal{Q}(\mathcal{V}) = \mathcal{Q}([\mathcal{V}])$.

4.2. A Characterization of $\mathcal{A}$

Suppose $b'Y$ is in $\mathcal{B}_h$. We say that $b'Y$ is admissible in $\mathcal{B}_h$ w.r.t. $\mathcal{V}$ if and only if no element in $\mathcal{B}_h$ is better than $b'Y$ w.r.t. $\mathcal{V}$; and we say that $b'Y$ is V-best in $\mathcal{B}_h$ if and only if $b'Vb \leq a'Va$ for all $a'Y \in \mathcal{B}_h$. Thus $b'Y$ is an element in $\mathcal{B}_h$ if and only if $b'Y$ is V_h-best in $\mathcal{B}_{h-1}$.

Proposition (4.2.1). Let $b'Y$ and $a'Y$ be both in $\mathcal{B}_h$. Then,

(i) $b'Y$ is as good as $a'Y$ w.r.t. $[\mathcal{V}]$ if and only if $b'Y$ is as good as $a'Y$ w.r.t. $\mathcal{W}_h$.

(ii) $b'Y$ is better than $a'Y$ w.r.t. $[\mathcal{V}]$ if and only if $b'Y$ is better than $a'Y$ w.r.t. $\mathcal{W}_h$.

(iii) $b'Y$ is admissible in $\mathcal{B}_h$ w.r.t. $[\mathcal{V}]$ if and only if $b'Y$ is admissible in $\mathcal{B}_h$ w.r.t. $\mathcal{W}_h$.

Pf. By (3.2.10), $b'W_jb = a'W_ja$ for all $j \in \cup_{i=1}^{h} J_i$ because whenever $j \in J_i$, we have

$$\underline{R}(W_j) \subset \underline{R}(V_i) \subset \underline{R}(X, V_1, \ldots, V_h).$$

So

$$b'(\Sigma_{j=1}^{k} \rho_j W_j)b \underset{(<)}{\leq} a'(\Sigma_{j=1}^{k} \rho_j W_j)a$$

if and only if

$$b'(\Sigma_{j \in J_h'} \rho_j W_j)b \underset{(<)}{\leq} a'(\Sigma_{j \in J_h'} \rho_j W_j)a .$$

This proves (i) and (ii). Part (iii) holds using (ii) together with the definition of admissibility in $\mathcal{B}_h$. ▯

Proposition (4.2.2). If $b'Y$ is admissible in $\mathcal{B}_h$ w.r.t. $[\mathcal{V}]$, then $b'Y$ is V-best in $\mathcal{B}_h$ for some non-zero $V \in \mathcal{W}_h^\rho$.

Pf. By (iii) of (4.2.1) we conclude that $b'Y$ is admissible in $\mathcal{B}_h$ w.r.t. $\mathcal{W}_h^\rho$. Now, consider the linear model

$$\mathcal{M}_h : E(T) = G\psi , \quad Cov(T) \in \mathcal{W}_h^\rho,$$

where T is an $n \times 1$ random vector, $G = (X, V_1F_1, \ldots, V_hF_h)$ with F_i and V_i $(i = 1, 2, \ldots, h)$ as defined in Section 3.2 and ψ a vector of g unknown parameters with g as the number of columns in G. Set $\delta = (\lambda', 0, \ldots, 0)'$. Then $\delta'\psi$ is an estimable parametric function if and only if there exists a vector b in R^n such that $G'b = (X, V_1F_1, \ldots, V_hF_h)'b = (\lambda', 0, \ldots, 0)'$. Let $\mathcal{U}_h$ be the set of all lue's of $\delta'\psi$, i.e.,

$$\mathcal{U}_h = \{b'T : (X, V_1F_1, \ldots, V_hF_h)'b = (\lambda', 0, \ldots, 0)'\} .$$

Note that (3.2.4) implies that $b'Y \in \mathcal{B}_h$ if and only if $b'T \in \mathcal{U}_h$. Moreover for $V \in \mathcal{W}_h^\rho$, $Var(b'Y|V) = b'Vb = Var(b'T|V)$. Since

$b'Y$ is admissible in $\mathcal{B}_h$ w.r.t. $\mathcal{W}_h$, $b'T$ is admissible in $\mathcal{U}_h$ under $\mathcal{M}_h$. Applying Proposition 3.6 in Olsen, Seely and Birkes (1976) to the linear model $\mathcal{M}_h$ we conclude that $b'T$ is V-best in $\mathcal{U}_h$ for some non-zero $V \in \mathcal{W}_h$. Hence $b'Y$ is V-best in $\mathcal{B}_h$. []

Definition (4.2.3). $P = (J_1, J_2, \ldots, J_s)$ is an ordered partition of $J = \{1, 2, \ldots, k\}$ if and only if

(i) J_i is nonempty for all $i = 1, 2, \ldots, s$,

(ii) $J_i \cap J_j$ is empty for all $i \neq j$,

(iii) $J = \cup_{i=1}^{s} J_i$.

Let $\mathcal{P}$ be the set of all ordered partitions of J. Then,

Proposition (4.2.4). If $(J_1, \ldots, J_s)$ is an ordered partition and $\alpha_j > 0$ for all $j = 1, 2, \ldots, k$, then

$$\mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j) \subset \mathcal{Q}.$$

Pf. Because $\alpha_j > 0$ for all j we have

$$\underline{R}(X, \Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j) = \underline{R}(X, W_1, \ldots, W_k) = R^n,$$

and the result holds from (3.2.7). []

Proposition (4.2.5).

$$\mathcal{A} \subset \cup_{P \in \mathcal{P}} \cup_{\alpha_j > 0} \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j) .$$

Pf. Assume $b'Y \in \mathcal{A}$. Then Proposition 3.6 in Olsen, Seely and Birkes (1976) implies that $b'Y \in \mathcal{B}(V_1)$ for some non-zero $V_1 \in [\mathcal{V}]$. Write $V_1 = \Sigma_{j \in J_1} \alpha_j W_j$, $\alpha_j > 0$ for all $j \in J_1$. Since $b'Y$ is in $\mathcal{B}(V_1) \subset \mathcal{B}_0$ and is admissible in $\mathcal{B}_0$ w.r.t. $[\mathcal{V}]$, then $b'Y$ is admissible in $\mathcal{B}(V_1)$ w.r.t. $[\mathcal{V}]$. If $J_1 \neq J$, we can form $\mathcal{W}_1$ as in (4.1.1). By (4.2.2) there exists a non-zero matrix $V_2 \in \mathcal{W}_1$ such that $b'Y \in \mathcal{B}(V_1, V_2)$. Write $V_2 = \Sigma_{j \in J_2} \alpha_j W_j$, $\alpha_j > 0$ for all $j \in J_2$. Then $b'Y$ is admissible in $\mathcal{B}(V_1, V_2)$ w.r.t. $[\mathcal{V}]$. Note that $J_1 \cap J_2 = \phi$. If $J_1 \cup J_2 \neq J$, we can form $\mathcal{W}_2$ as in (4.1.1). Again by (4.2.2) there exists a non-zero matrix $V_3 \in \mathcal{W}_2$ such that $b'Y \in \mathcal{B}(V_1, V_2, V_3)$. Write $V_3 = \Sigma_{j \in J_3} \alpha_j W_j$, $\alpha_j > 0$ for all $j \in J_3$. Continue for s steps until we get $J_1 \cup J_2 \cup \ldots \cup J_s = J$. So,

$$b'Y \in \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j) ,$$

where $(J_1, J_2, \ldots, J_s)$ is an ordered partition of J and $\alpha_j > 0$ for all j. $\square$

Theorem (4.2.6).

$$\mathcal{Q} = \cup_{P \in \mathcal{P}} \cup_{\alpha_j > 0} \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j) .$$

Pf. Apply (4.2.4) and (4.2.5). $\square$

Remark (4.2.7). Note that, by (3.2.9),

$$\mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_h} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j)$$

$$= \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_h} \alpha_j W_j)$$

whenever $\underline{R}(X, W_{j_1}, \ldots, W_{j_t}) = R^n$ with $\{j_1, \ldots, j_t\} = \cup_{i=1}^{h} J_i$.

Theorem (4.2.8). $\mathcal{Q} = \cup_{\substack{\underline{r}(X, V_1, \ldots, V_k)=n, \\ V_i \in [\mathcal{V}]}} \mathcal{B}(V_1, \ldots, V_k)$.

Pf. By (4.2.6) it suffices to show that

$$\cup_{\substack{\underline{r}(X, V_1, \ldots, V_k)=n, \\ V_i \in [\mathcal{V}]}} \mathcal{B}(V_1, \ldots, V_k)$$

$$= \cup_{P \in \mathcal{P}} \cup_{\alpha_j > 0} \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j).$$

Choose $P \in \mathcal{P}$ and $\alpha_j > 0$ $(j = 1, 2, \ldots, k)$ arbitrary but fixed.

Write $V_i = \Sigma_{j \in J_i} \alpha_j W_j$, $i = 1, 2, \ldots, s$. Thus,

$$\mathcal{B}(\Sigma_{j\in J_1} \alpha_j W_j, \ldots, \Sigma_{j\in J_s} \alpha_j W_j) = \mathcal{B}(V_1, \ldots, V_s).$$

Note that $\underline{r}(X, V_1, \ldots, V_s) = \underline{r}(X, W_1, \ldots, W_k) = n$. Proposition (3.2.6) says that $\mathcal{B}(V_1, \ldots, V_s)$ has only one element. Thus

$$\mathcal{B}(\Sigma_{j\in J_1} \alpha_j W_j, \ldots, \Sigma_{j\in J_s} \alpha_j W_j)$$

$$= \mathcal{B}(V_1, \ldots, V_s) = \mathcal{B}(V_1, \ldots, V_s, \ldots, V_k)$$

$$\subset \cup_{\substack{\underline{r}(X, V_1, \ldots, V_k)=n, \\ V_i \in [\mathcal{W}]}} \mathcal{B}(V_1, \ldots, V_k) .$$

For the opposite containment let $V_i = \Sigma_{j=1}^{k} \alpha_j^{(i)} W_j$, with $\alpha_j^{(i)} \geq 0$ for all i and j, be such that $\underline{r}(X, V_1, \ldots, V_k) = n$. Let

$$L_i = \{j : \alpha_j^{(i)} > 0\}, \quad i = 1, 2, \ldots, k.$$

Set $J_1 = L_1$ and $J_h = L_h \setminus \cup_{i=1}^{h-1} L_i$, $h = 2, \ldots, k$. Retain only the sets J_h that are nonempty and renumber them so that $J_1, \ldots, J_m$ are pairwise disjoint subsets of $J = \{1, 2, \ldots, k\}$. Let $J_{m+1} = J \setminus \cup_{i=1}^{m} J_i$. By (3.2.11) we have

$$\mathcal{B}(V_1, \ldots, V_k) = \mathcal{B}(\Sigma_{j\in L_1} \alpha_j^{(1)} W_j, \ldots, \Sigma_{j\in L_k} \alpha_j^{(k)} W_j)$$

$$= \mathcal{B}(\Sigma_{j\in J_1} \alpha_j^{(1)} W_j, \ldots, \Sigma_{j\in J_m} \alpha_j^{(m)} W_j) ,$$

where $\underline{r}(X, W_{j_1}, \ldots, W_{j_t}) = \underline{r}(X, V_1, \ldots, V_k) = n$ with $\{j_1, \ldots, j_t\} = \cup_{i=1}^{m} J_i$. By (4.2.7) we have

$$\mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_m} \alpha_j W_j)$$
$$= \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_m} \alpha_j W_j, \Sigma_{j \in J_{m+1}} \alpha_j W_j),$$

where $(J_1, \ldots, J_{m+1})$ is an ordered partition of $J = \{1, 2, \ldots, k\}$ $\alpha_j = \alpha_j^{(i)} > 0$ for all j, i.e.,

$$\mathcal{B}(V_1, \ldots, V_k) = \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_m} \alpha_j W_j, \Sigma_{j \in J_{m+1}} \alpha_j W_j)$$
$$\subset \cup_{P \in \mathcal{P}} \cup_{\alpha_j > 0} \mathcal{B}(\Sigma_{j \in J_1} \alpha_j W_j, \ldots, \Sigma_{j \in J_s} \alpha_j W_j). \quad \square$$

<u>4.3. Example</u>

Let $\mathcal{V}$ be such that $[\mathcal{V}] = \{\Sigma_{j=1}^{3} \rho_j W_j : \rho_j \geq 0\}$ where $W_1 = I$, the identity matrix. Note that the two-way additive or nested random model has this covariance structure. The ordered partitions of $\{1, 2, 3\}$ are:

$(\{1, 2, 3\}), (\{1, 2\}, \{3\}), (\{3\}, \{1, 2\}), (\{1, 3\}, \{2\}), (\{2\}, \{1, 3\}),$
$(\{2, 3\}, \{1\}), (\{1\}, \{2, 3\}), (\{1\}, \{2\}, \{3\}), (\{1\}, \{3\}, \{2\}), (\{2\}, \{1\}, \{3\}),$
$(\{2\}, \{3\}, \{1\}), (\{3\}, \{1\}, \{2\})$ and $(\{3\}, \{2\}, \{1\})$.

By (4.2.6) we have

$$\mathcal{Q} = \bigcup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_1 I+\alpha_2 W_2+\alpha_3 W_3) \cup \mathcal{B}(\alpha_1 I+\alpha_2 W_2, \alpha_3 W_3)$$

$$\cup\, \mathcal{B}(\alpha_3 W_3, \alpha_1 I+\alpha_2 W_2) \cup \mathcal{B}(\alpha_1 I+\alpha_3 W_3, \alpha_2 W_2)$$

$$\cup\, \mathcal{B}(\alpha_2 W_2, \alpha_1 I+\alpha_3 W_3) \cup \mathcal{B}(\alpha_2 W_2+\alpha_3 W_3, \alpha_1 I)$$

$$\cup\, \mathcal{B}(\alpha_1 I, \alpha_2 W_2+\alpha_3 W_3) \cup \mathcal{B}(\alpha_1 I, \alpha_2 W_2, \alpha_3 W_3)$$

$$\cup\, \mathcal{B}(\alpha_1 I, \alpha_3 W_3, \alpha_2 W_2) \cup \mathcal{B}(\alpha_2 W_2, \alpha_1 I, \alpha_3 W_3)$$

$$\cup\, \mathcal{B}(\alpha_2 W_2, \alpha_3 W_3, \alpha_1 I) \cup \mathcal{B}(\alpha_3 W_3, \alpha_1 I, \alpha_2 W_2)$$

$$\cup\, \mathcal{B}(\alpha_3 W_3, \alpha_2 W_2, \alpha_1 I) \quad .$$

By (4.2.7) we have, for example,

$$\bigcup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_1 I+\alpha_2 W_2, \alpha_3 W_3) = \bigcup_{\alpha_1, \alpha_2 > 0} \mathcal{B}(\alpha_1 I+\alpha_2 W_2) \;;$$

$$\bigcup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_1 I, \alpha_2 W_2+\alpha_3 W_3) = \bigcup_{\alpha_1 > 0} \mathcal{B}(\alpha_1 I) \;;$$

and

$$\bigcup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_2 W_2, \alpha_1 I, \alpha_3 W_3) = \bigcup_{\alpha_1, \alpha_2 > 0} \mathcal{B}(\alpha_2 W_2, \alpha_1 I) \;.$$

To obtain a shorter expression for $\mathcal{Q}$, we will prove

<u>Proposition (4.3.1)</u>. $\mathcal{B}(\alpha_1 V_1, \ldots, \alpha_h V_h) = \mathcal{B}(V_1, \ldots, V_h)$ if $\alpha_i > 0$ for all i.

Pf. The proof is by induction. Suppose $b'Y \in \mathcal{B}_0$. Then for $h = 1$ we have

$$b'Y \in \mathcal{B}(\alpha_1 V_1) <=> \alpha_1 b'V_1 b \le \alpha_1 a'V_1 a \quad \text{for all } a'Y \in \mathcal{B}_0$$
$$<=> b'V_1 b \le a'V_1 a \quad \text{for all } a'Y \in \mathcal{B}_0$$
$$<=> b'Y \in \mathcal{B}(V_1).$$

For $h \geq 2$, assume the result holds for $h-1$, i.e., assume that $\mathcal{B}(\alpha_1 V_1, \ldots, \alpha_{h-1} V_{h-1}) = \mathcal{B}(V_1, \ldots, V_{h-1})$. Then

$$b'Y \in \mathcal{B}(\alpha_1 V_1, \ldots, \alpha_h V_h) <=> \alpha_h b'V_h b \le \alpha_h a'V_h a,$$
$$\text{for all } a'Y \in \mathcal{B}(\alpha_1 V_1, \ldots, \alpha_{h-1} V_{h-1})$$
$$<=> b'V_h b \le a'V_h a$$
$$\text{for all } a'Y \in \mathcal{B}(V_1, \ldots, V_{h-1})$$
$$<=> b'Y \in \mathcal{B}(V_1, \ldots, V_h). \quad \square$$

Applying this result we have, for example,

$$\mathcal{B}(W_2, W_3, I) = \cup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_2 W_2, \alpha_3 W_3, \alpha_1 I) ;$$

$$\cup_{\gamma_3 > 0} \mathcal{B}(W_2 + \gamma_3 W_3, I) = \cup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_2 W_2 + \alpha_3 W_3, \alpha_1 I) ;$$

$$\cup_{\gamma_2 \geq 0} \mathcal{B}(W_3, I+\gamma_2 W_2) = \cup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_3 W_3, \alpha_1 I+\alpha_2 W_2)$$

$$\cup \cup_{\alpha_1, \alpha_3 > 0} \mathcal{B}(\alpha_3 W_3, \alpha_1 I) ;$$

and

$$\cup_{\gamma_2, \gamma_3 \geq 0} \mathcal{B}(I+\gamma_2 W_2+\gamma_3 W_3) = \cup_{\substack{\alpha_j > 0 \\ \text{for all } j}} \mathcal{B}(\alpha_1 I+\alpha_2 W_2+\alpha_3 W_3)$$

$$\cup \cup_{\alpha_1, \alpha_2 > 0} \mathcal{B}(\alpha_1 I+\alpha_2 W_2)$$

$$\cup \cup_{\alpha_1, \alpha_3 > 0} \mathcal{B}(\alpha_1 I+\alpha_3 W_3)$$

$$\cup \cup_{\alpha_1 > 0} \mathcal{B}(\alpha_1 I).$$

Applying (4.2.7) and (4.3.1) we have

$$\mathcal{A} = \cup_{\gamma_1, \gamma_3 \geq 0} \mathcal{B}(I+\gamma_2 W_2+\gamma_3 W_3) \cup \cup_{\gamma_3 > 0} \mathcal{B}(W_2+\gamma_3 W_3, I)$$

$$\cup \cup_{\gamma_3 \geq 0} \mathcal{B}(W_2, I+\gamma_3 W_3) \cup \cup_{\gamma_2 \geq 0} \mathcal{B}(W_3, I+\gamma_2 W_2)$$

$$\cup \mathcal{B}(W_2, W_3, I) \cup \mathcal{B}(W_3, W_2, I).$$

4.4. Calculating the Admissible lue's

Let $V_1, \ldots, V_h$ be matrices in $[\mathcal{V}]$ such that $\underline{R}(X, V_1, \ldots, V_h) = R^n$. Proposition (3.2.6) says that $\mathcal{B}_h = \mathcal{B}(V_1, \ldots, V_h)$ has only one element, $b'Y$ say. Corollary

(3.2.7) says that $b'Y$ is admissible. By (3.2.4) we have $b'Y \in \mathcal{B}_h$ if and only if $(X, V_1F_1, \ldots, V_hF_h)'b = (\lambda', 0, \ldots, 0)'$ with $F_1, \ldots, F_h$ as defined in Section 3.2. Let $U = (X, V_1F_1, \ldots, V_hF_h)$. Note that by (3.2.2), $\underline{R}(U) = \underline{R}(X, V_1, \ldots, V_h) = R^n$, i.e., U has full row rank. Then $U'b = (\lambda', 0, \ldots, 0)'$ implies that $b = (UU')^{-1}U(\lambda', 0, \ldots, 0)'$, i.e.,

$$(4.4.1) \qquad b = (XX'+V_1F_1F_1'V_1+\ldots+V_hF_hF_h'V_h)^{-1}X\lambda .$$

To calculate $F_1, \ldots, F_h$ let $H_1, \ldots, H_h$ be such that the columns of H_i form a basis of $\underline{R}(X, V_1, \ldots, V_{i-1})$ $(i = 1, 2, \ldots, h)$. Notice that $H_i(H_i'H_i)^{-1}H_i'$ is the orthogonal projection operator on $\underline{R}(X, V_1, \ldots, V_{i-1})$. Hence

$$(4.4.2) \qquad F_i = I - H_i(H_i'H_i)^{-1}H_i', \qquad i = 1, 2, \ldots, h,$$

is such that $\underline{R}(F_i) = \underline{N}(X')\cap\underline{N}(V_1)\cap\ldots\cap\underline{N}(V_{i-1})$ $(i = 1, 2, \ldots, h)$ and $F_iF_i' = F_i^2 = F_i$ for all i. Thus

$$(4.4.3) \qquad b = (XX'+V_1F_1V_1+\ldots+V_hF_hV_h)^{-1}X\lambda .$$

4.5. The Case Where $\underline{R}(X, W_1, \ldots, W_k) \neq R^n$

Let $[\mathcal{W}] = \{\Sigma_{j=1}^k \rho_jW_j : \rho_j \geq 0\}$ and suppose that $\underline{R}(X, W_1, \ldots, W_k) \neq R^n$. Let H be such that its columns form a basis of $\underline{R}(X, W_1, \ldots, W_k)$ and let $\underline{r}(H) = q$. Consider the linear model $\mathfrak{M}_Z$ introduced in Section 3.1. Recall that $\mathcal{V}_Z = \{H'VH : V \in \mathcal{V}\}$. Applying (3.3.7) we have

$$[\mathcal{V}_Z] = H'[\mathcal{V}]H = \{\Sigma_{j=1}^{k} \rho_j H_j : \rho_j \geq 0\} ,$$

where $H_j = H'W_jH$, $j = 1, 2, \ldots, k$. By (3.1.6), $\underline{R}(G, H_1, \ldots, H_k) = R^q$. Let $\mathcal{Q}_0$ be defined as in Section 3.1. Then

<u>Theorem (4.5.1)</u>.

$$\mathcal{Q} = \bigcup_{\substack{\underline{r}(H'X, H'V_1H, \ldots, H'V_kH)=q, \\ V_i \in [\mathcal{V}]}} \mathcal{B}(H'V_1H, \ldots, H'V_kH) + \mathcal{Q}_0 .$$

<u>Pf</u>. Apply (3.1.3), (3.1.7) and (4.2.8). []

V. AN INTERSECTION RESULT

5.1. Introduction and Notation

In this chapter let $S_1, \ldots, S_k$ be $n \times n$ real matrices, not necessarily linearly independent, and let Ω be a subset of R^k such that $W(\rho) = \Sigma_{j=1}^{k} \rho_j S_j$ is n.n.d. for all $\rho \in \Omega$. Note that W is a linear mapping from R^k to $\mathcal{K}$. Consider the linear model $\mathcal{M}$, defined in Chapter II, with the covariance structure

$$\mathcal{V}_\Omega = \{W(\rho) : \rho \in \Omega\} = W(\Omega).$$

Under the assumption that Ω is contained in a polyhedral convex set Λ, (see below), such that $W(\rho)$ is n.n.d. for all $\rho \in \Lambda$ we will characterize $\mathcal{Q}(\mathcal{V}_\Omega)$. The generality of this chapter and the following one lies in the fact that we do not assume that $[\mathcal{V}_\Omega]$ has a polyhedral structure.

Definition (5.1.1). For a subset A in R^k, A is a non-empty finitely generated convex set if and only if there exist vectors $a_1, \ldots, a_m$, $(m \geq 1)$, such that for a fixed integer h, $0 \leq h \leq m$,

$$A = \{\Sigma_{j=1}^{m} \lambda_j a_j : \Sigma_{j=1}^{h} \lambda_j = 1, \lambda_j \geq 0, j = 1, 2, \ldots, m\}.$$

Note that if $h = m$ then A is bounded and if $h = 0$ then A is a cone. Theorem (19.1) in Rockafellar (1970) implies that A is a non-empty finitely generated convex set if and only if A is a non-empty polyhedral convex set, i.e., if and only if A is the non-empty intersection of a finite number of closed half-spaces. Throughout, the qualifier "non-empty" will be dropped when dealing with non-empty polyhedral convex sets.

Now let Λ be a polyhedral convex set in R^k, i.e., there exist $e_1, \ldots, e_m$ and $0 \leq h \leq m$ such that

$$\Lambda = \{\Sigma_{j=1}^{m} \lambda_j e_j : \Sigma_{j=1}^{h} \lambda_j = 1, \lambda_j \geq 0, j = 1, \ldots, m\}.$$

Then

<u>Proposition (5.1.2)</u>. $[\Lambda] = \{\Sigma_{j=1}^{m} \lambda_j e_j : \lambda_j \geq 0\}$.

<u>Pf</u>. Notice that $\{\Sigma_{j=1}^{m} \lambda_j e_j : \lambda_j \geq 0\} = B$ is a convex cone containing Λ. Theorem (19.1) in Rockafellar (1970) says that B is closed. Hence $[\Lambda] \subset B$, since $[\Lambda]$ is the smallest closed convex cone containing Λ. To prove the other containment let
$D = \{\Sigma_{j=1}^{m} \nu_j e_j : \nu_j \geq 0, \text{ if } \nu_1 = \ldots = \nu_h = 0 \text{ then } \nu_1 = \ldots = \nu_m = 0\}$
and apply (3.3.2) together with the definition of Λ to obtain

$$\Lambda^{scc} \supset \{cu : c \geq 0, u \in \Lambda\} = \{\Sigma_{j=1}^{m} (c\lambda_j) e_j : \lambda_j \geq 0, \Sigma_{j=1}^{h} \lambda_j = 1, c \geq 0\} \supset D.$$

Proposition (3.3.3) says that $[\Lambda] = \overline{\Lambda^{scc}}$. Since $\overline{\Lambda^{scc}} \supset \overline{D}$, it suffices to show that $\overline{D} \supset B$. Let $b \in B$, then $b = \Sigma_{j=1}^{m} \lambda_j e_j$, $\lambda_j \geq 0$. If $\lambda_j > 0$ for any $1 \leq j \leq h$, then $b \in D \subset \overline{D}$. It remains to consider the case when $\lambda_1 = \ldots = \lambda_h = 0$, i.e., $b = \Sigma_{j=h+1}^{m} \lambda_j e_j$, $\lambda_j \geq 0$. Let $b_n = \Sigma_{j=1}^{m} \nu_{jn} e_j$ where

$$\nu_{jn} = \begin{cases} 1/n, & j = 1, \ldots, h \\ \lambda_j, & j = h+1, \ldots, m \end{cases}, \quad n = 1, 2, \ldots .$$

Then $\{b_n\}$ is a sequence in D such that $\lim_{n \to \infty} b_n = \Sigma_{j=h+1}^{m} \lambda_j e_j = b$, i.e., $b \in \overline{D}$. []

Suppose $W(\rho)$ is n.n.d. for all $\rho \in \Lambda$ and let

$$\mathcal{V}_\Lambda = W(\Lambda) = \{\Sigma_{j=1}^{m} \lambda_j W(e_j) : \Sigma_{j=1}^{h} \lambda_j = 1, \lambda_j \geq 0\} .$$

Then

<u>Proposition (5.1.3).</u> $[\mathcal{V}_\Lambda] = \{\Sigma_{j=1}^{m} \lambda_j W(e_j) : \lambda_j \geq 0\}$.

<u>Pf</u>. Note that $\mathcal{K}$ is isomorphic to R^{n^2} and apply (5.1.2). []

<u>Proposition (5.1.4).</u> $[\mathcal{V}_\Lambda] = \mathcal{V}_{[\Lambda]}$.

<u>Pf</u>. By definition $\mathcal{V}_{[\Lambda]} = W([\Lambda])$. Apply (5.1.2) together with the definition of W to obtain $W([\Lambda]) = \{\Sigma_{j=1}^{m} \lambda_j W(e_j) : \lambda_j \geq 0\}$. Then apply (5.1.3). []

5.2. The Main Result

Proposition (5.2.1). Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two sets of covariance matrices such that $\text{sp}\,\mathcal{V}_1 = \text{sp}\,\mathcal{V}_2$. Then $a'Va = b'Vb$ for all $V \in \mathcal{V}_1$ if and only if $a'Va = b'Vb$ for all $V \in \mathcal{V}_2$.

Pf. Assume that $a'Va = b'Vb$ for all $V \in \mathcal{V}_1$ and let $V_1, \ldots, V_k$ be a spanning set of $\text{sp}\,\mathcal{V}_1$ with $V_j \in \mathcal{V}_1$, $j = 1, 2, \ldots, k$. Then $a'V_ja = b'V_jb$ for all $j = 1, 2, \ldots, k$ which implies that $\alpha_j a'V_j a = \alpha_j b'V_j b$ for all $\alpha_j \in R^1$. Hence

$$a'(\Sigma_{j=1}^{k} \alpha_j V_j)a = b'(\Sigma_{j=1}^{k} \alpha_j V_j)b \qquad \text{for all} \quad \alpha_j \in R^1 ,$$

i.e., $a'Va = b'Vb$ for all $V \in \text{sp}\,\mathcal{V}_1$. Then $a'Va = b'Vb$ for all $V \in \mathcal{V}_2$ since $\mathcal{V}_2 \subset \text{sp}\,\mathcal{V}_2 = \text{sp}\,\mathcal{V}_1$. The opposite implication follows similarly. $\square$

Lemma (5.2.2). Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be such that $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \text{sp}\,\mathcal{V}_1$. Then $\mathcal{Q}(\mathcal{V}_1) \subset \mathcal{Q}(\mathcal{V}_2)$.

Pf. Let $b'Y \in \mathcal{Q}(\mathcal{V}_1)$. By the completeness of $\mathcal{Q}(\mathcal{V}_2)$ there exists $a'Y \in \mathcal{Q}(\mathcal{V}_2)$ such that $a'Va \leq b'Vb$ for all $V \in \mathcal{V}_2$. Since $\mathcal{V}_1 \subset \mathcal{V}_2$ we conclude that $a'Y$ is as good as $b'Y$ w.r.t. $\mathcal{V}_1$. But $b'Y \in \mathcal{Q}(\mathcal{V}_1)$ and so by (3.2.12) we have $a'Va = b'Vb$ for all $V \in \mathcal{V}_1$. Note that $\text{sp}\,\mathcal{V}_1 = \text{sp}\,\mathcal{V}_2$. Hence (5.2.1) implies that $a'Va = b'Vb$ for all $V \in \mathcal{V}_2$. Since $a'Y \in \mathcal{A}(\mathcal{V}_2)$, (3.2.13) implies that $b'Y \in \mathcal{A}(\mathcal{V}_2)$. □

Proposition (5.2.3). Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be as in (5.2.2). Furthermore, suppose $[\mathcal{V}_2] = \{\Sigma_{j=1}^{k} \rho_j V_j : \rho_j \geq 0\}$ such that $\underline{R}(X, V_1, \ldots, V_k) = R^n$. If $a'Y$ and $b'Y$ are both in $\mathcal{A}(\mathcal{V}_1)$ such that $a'Va = b'Vb$ for all $V \in \mathcal{V}_1$, then $a = b$.

Pf. By (5.2.2), $\mathcal{A}(\mathcal{V}_1) \subset \mathcal{A}(\mathcal{V}_2)$ which implies that $a'Y$ and $b'Y$ are both in $\mathcal{A}(\mathcal{V}_2)$. Theorem (4.2.6) says that $a'Y$ and $b'Y$ are both in $\mathcal{B}(\Sigma_{j \in J_1} \alpha_j V_j, \ldots, \Sigma_{j \in J_s} \alpha_j V_j)$ where $\alpha_j > 0$ for all j and

$$\underline{R}(X, \Sigma_{j \in J_1} \alpha_j V_j, \ldots, \Sigma_{j \in J_s} \alpha_j V_j) = \underline{R}(X, V_1, \ldots, V_k) = R^n .$$

By (3.2.5) we have $a = b + f$ with

$$f \in \underline{N}(X') \cap \underline{N}(V_1) \cap \cdots \cap \underline{N}(V_k) = \{0\} . \quad □$$

Assume there exists a polyhedral convex set Λ in R^k such that $\Omega \subset \Lambda$ and $W(\rho)$ is n.n.d. for all $\rho \in \Lambda$. By intersecting Λ with $\text{sp}\,\Omega$ if necessary, we can suppose $\Lambda \subset \text{sp}\,\Omega$. Let $\mathcal{L}$ be the set of all such Λ's. Then

Theorem (5.2.4). $\mathcal{A}(\mathcal{V}_\Omega) = \cap_{\Lambda \in \mathcal{L}} \mathcal{A}(\mathcal{V}_\Lambda)$.

Pf. Since $\Omega \subset \Lambda \subset \text{sp}\,\Omega$, $W(\Omega) \subset W(\Lambda) \subset W(\text{sp}\,\Omega)$. By the linearlity of W we have $W(\text{sp}\,\Omega) = \text{sp}\,W(\Omega)$. Thus we can apply (5.2.2) to conclude $\mathcal{Q}(\mathcal{V}_\Omega) \subset \cap_{\Lambda \in \mathcal{L}} \mathcal{Q}(\mathcal{V}_\Lambda)$. Now let $b'Y \in \cap_{\Lambda \in \mathcal{L}} \mathcal{Q}(\mathcal{V}_\Lambda)$. By the completeness of $\mathcal{Q}(\mathcal{V}_\Omega)$, there exists $a'Y \in \mathcal{Q}(\mathcal{V}_\Omega)$ such that $a'W(\rho)a \leq b'W(\rho)b$ for all $\rho \in \Omega$. Choose Λ in $\mathcal{L}$ and let $\Lambda_1 = \{\rho \in \Lambda : a'W(\rho)a \leq b'W(\rho)b\}$. Notice that $\Omega \subset \Lambda_1 \subset \Lambda$. Since Λ is a polyhedral convex set, then $\Lambda = \cap_{j=1}^{c} \mathcal{H}_j$, where $\mathcal{H}_j$ is a closed half-space $(j = 1, 2, \ldots, c)$. Note that $\mathcal{H}_{c+1} = \{\rho \in R^k : a'W(\rho)a \leq b'W(\rho)b\}$ is a closed half-space. Hence $\Lambda_1 = \cap_{j=1}^{c+1} \mathcal{H}_j$ with $\Omega \subset \Lambda_1 \subset \Lambda \subset \text{sp}\,\Omega$. Hence $\Lambda_1 \in \mathcal{L}$ is such that $a'W(\rho)a \leq b'W(\rho)b$ for all $\rho \in \Lambda_1$ and $b'Y \in \mathcal{Q}(\mathcal{V}_{\Lambda_1})$. By (3.2.12) we have $a'W(\rho)a = b'W(\rho)b$ for all $\rho \in \Lambda_1$. Since $\Omega \subset \Lambda_1$, then $a'W(\rho)a = b'W(\rho)b$ for all $\rho \in \Omega$. Since $a'Y \in \mathcal{Q}(\mathcal{V}_\Omega)$, (3.2.13) implies that $b'Y \in \mathcal{Q}(\mathcal{V}_\Omega)$, i.e., $\cap_{\Lambda \in \mathcal{L}} \mathcal{Q}(\mathcal{V}_\Lambda) \subset \mathcal{Q}(\mathcal{V}_\Omega)$. $\square$

Recall that for any polyhedral convex set Λ, defined as in Section 5.1, we have

$$[\mathcal{V}_\Lambda] = \{\Sigma_{j=1}^{m} \lambda_j W(e_j) : \lambda_j \geq 0\},$$

which has a polyhedral structure. Also, $\mathcal{Q}(\mathcal{V}_\Lambda) = \mathcal{Q}([\mathcal{V}_\Lambda])$. Hence $\mathcal{Q}(\mathcal{V}_\Lambda)$ can be characterized using (4.5.1).

VI. A UNION RESULT

6.1. Introduction and Notation

For the purpose of this chapter let $W(\rho) = \Sigma_{j=1}^{k} \rho_j S_j$ be n.n.d for all $\rho \in \Omega$ where Ω is a subset of R^k and $S_1, \ldots, S_k$ are linearly independent $n \times n$ real matrices. By (3.3.16), $W(\rho)$ is n.n.d. for all $\rho \in [\Omega]$. In the linear model $\mathcal{M}$, defined in Chapter II, let

$$\mathcal{V}_\Omega = \{W(\rho) : \rho \in \Omega\} = W(\Omega).$$

By the results of Section 3.3 we have $\mathcal{V}_{[\Omega]} = [\mathcal{V}_\Omega]$ and $\mathcal{V}_{[\Omega]^o} = [\mathcal{V}_\Omega]^o$. Throughout this chapter $\mathcal{Q}(\Omega)$ and $\mathcal{Q}([\Omega])$ will be used to represent $\mathcal{Q}(\mathcal{V}_\Omega)$ and $\mathcal{Q}(\mathcal{V}_{[\Omega]}) = \mathcal{Q}([\mathcal{V}_\Omega])$ respectively. Hence $\mathcal{Q}(\Omega) = \mathcal{Q}([\Omega])$.

Now let Λ be a polyhedral convex set defined as in (5.1.1). Let $e_1, \ldots, e_m$ be vectors in R^k and $0 \leq h \leq m$ be such that

$$\Lambda = \{\Sigma_{j=1}^{m} \lambda_j e_j : \Sigma_{j=1}^{h} \lambda_j = 1, \lambda_j \geq 0, j = 1, \ldots, m\},$$

and recall (5.1.2) which says that $[\Lambda] = \{\Sigma_{j=1}^{m} \lambda_j e_j : \lambda_j \geq 0\}$. Again by the results of Section 5.1 we have

$$\mathcal{V}_\Lambda = \{\Sigma_{j=1}^{m} \lambda_j W(e_j) : \Sigma_{j=1}^{h} \lambda_j = 1, \lambda_j \geq 0, j = 1, \ldots, m\},$$

and

$$[\mathcal{V}_\Lambda] = \mathcal{V}_{[\Lambda]} = \{\Sigma^m_{j=1} \lambda_j W(e_j) : \lambda_j \geq 0\},$$

which has a polyhedral structure.

6.2. The Main Result

In this section we will assume that there exists $\rho_0 \in [\Omega]$ such that $\underline{R}(X, W(\rho_0)) = R^n$. We will also assume that there exists a polyhedral convex set Ω_1 such that $\Omega \subset \Omega_1$ with $W(\rho)$ n.n.d for all $\rho \in \Omega_1$. By intersecting Ω_1 with sp Ω if necessary, we can suppose $\Omega_1 \subset \text{sp}\,\Omega$. Let $\{\Lambda_N\}$ be an increasing sequence of polyhedral convex sets such that $\rho_0 \in \Lambda_1$, $\Lambda_N \subset [\Omega] \subset \text{sp}\,\Lambda_N$ and $[\Omega]^o \subset \cup_N \Lambda_N$. Note that $\mathcal{V}_{[\Omega]^o} = W([\Omega]^o) \subset \cup_N W(\Lambda_N)$. Set

$$\mathcal{V}_N = W(\Lambda_N) = \{W(\rho) : \rho \in \Lambda_N\}.$$

Hence

$$\mathcal{V}_N \subset \mathcal{V}_{N+1} \subset \mathcal{V}_{[\Omega]}, \qquad \text{for all } N.$$

Proposition (6.2.1). $\mathcal{Q}(\Lambda_N) \subset \mathcal{Q}(\Omega)$ for all N.

Pf. Apply Lemma (5.2.2) and the fact that $\mathcal{Q}(\Omega) = \mathcal{Q}([\Omega])$. □

Let $a'Y \in \mathcal{Q}$ and for each N define

$$K_N = \{b \in a+\underline{N}(X') : b'W(\rho)b \leq a'W(\rho)a \text{ for all } \rho \in \Lambda_N\}.$$

Hence, for all N, we have $K_{N+1} \subset K_N$ since $\Lambda_N \subset \Lambda_{N+1}$, i.e., $\{K_N\}$ is a decreasing sequence.

Proposition (6.2.2). K_N is compact for all $N = 1, 2, \ldots$.

Pf. Recall that $\{\Lambda_N\}$ is constructed such that $\rho_0 \in \Lambda_N$ and $\underline{R}(X, W(\rho_0)) = R^n$. Apply the proof of Lemma 3.2 in Olsen, Seely and Birkes (1976) with $\mathfrak{M} = \underline{N}(X')$ and $\mathfrak{J} = \underline{N}(X') \cap \underline{N}(W(\rho_0)) = \{0\}$ to get the result. []

Proposition (6.2.3). $\mathcal{Q}(\Omega) \subset \overline{\cup_N \mathcal{Q}(\Lambda_N)}$.

Pf. Let $a'Y \in \mathcal{Q}(\Omega)$. By the completeness of $\mathcal{Q}(\Lambda_N)$ there exists $b_N'Y \in \mathcal{Q}(\Lambda_N)$ such that $b_N'W(\rho)b_N \leq a'W(\rho)a$ for all $\rho \in \Lambda_N$. Note that $b_N \in K_N \subset K_1$. Thus $\{b_N\}$ is an infinite sequence in the compact set K_1. Then there exists a subsequence $\{b_{N_j}\}$ such that $b_{N_j} \rightarrow b_0$ for some $b_0 \in K_1$. Let $V \in \mathcal{V}_{[\Omega]^o}$ be artibrary but fixed. Note $V \in \cup_N W(\Lambda_N)$ and $\{W(\Lambda_N)\}$ is an increasing sequence. Then for sufficiently large N_j, $V \in W(\Lambda_{N_j})$ and so we have $b_{N_j}'Vb_{N_j} \leq a'Va$ which implies that $b_0'Vb_0 \leq a'Va$. Since V is arbitrary in $\mathcal{V}_{[\Omega]^o}$, then $b_0'Vb_0 \leq a'Va$ for all $V \in \mathcal{V}_{[\Omega]^o}$. Thus $\mathcal{V}_{[\Omega]^o} \subset \{V : b_0'Vb_0 \leq a'Va\}$ which is closed. Hence by (3.3.12), (3.3.15) and Eggleston (1963), p. 11, we have

$$\mathcal{V}_{[\Omega]} = [\mathcal{V}_\Omega] = \overline{[\mathcal{V}_\Omega]^o} = \overline{\mathcal{V}_{[\Omega]^o}} \subset \{V : b_0'Vb_0 \leq a'Va\},$$

i.e., $b_0'Vb_0 \leq a'Va$ for all $V \in \mathcal{V}_{[\Omega]}$. By (3.2.12) we have $b_0'Vb_0 = a'Va$ for all $V \in \mathcal{V}_{[\Omega]}$ since $a'Y \in \mathcal{Q}(\Omega)$. Proposition (3.2.13) says that $b_0'Y \in \mathcal{Q}(\Omega)$. It suffices to show that $a = b_0$. Let Ω_1 be a polyhedral convex set such that $\Omega \subset \Omega_1 \subset \text{sp}\,\Omega$ with $W(\rho)$ n.n.d. for all $\rho \in \Omega_1$. Then $[\Omega_1]$ has a polyhedral structure of the form $\{\Sigma_{j=1}^m \lambda_j e_j : \lambda_j \geq 0\}$ and $\rho_0 \in \Lambda_1 \subset [\Omega] \subset [\Omega_1]$ is such that $\underline{R}(X, W(\rho_0)) = R^n$. Hence $\underline{R}(X, W(e_1), \ldots, W(e_m)) = \underline{R}(X, W(\rho_0)) = R^n$ since $\Sigma_{j=1}^m W(e_j)$ is a maximal element in $\mathcal{V}_{[\Omega_1]}$ which contains $\mathcal{V}_{[\Lambda_1]}$. Apply (5.2.3) with $\mathcal{V}_1 = \mathcal{V}_{[\Omega]}$ and $\mathcal{V}_2 = \mathcal{V}_{[\Omega_1]}$ to get the result. []

<u>Proposition (6.2.4)</u>. $\overline{\cup_N \mathcal{Q}(\Lambda_N)} \subset \overline{\mathcal{Q}(\Omega)}$.

<u>Pf</u>. By (6.2.1), $\mathcal{Q}(\Lambda_N) \subset \mathcal{Q}(\Omega)$ for all N. Hence $\cup_N \mathcal{Q}(\Lambda_N) \subset \mathcal{Q}(\Omega) \subset \overline{\mathcal{Q}(\Omega)}$ and the result follows. []

<u>Theorem (6.2.5)</u>. $\overline{\mathcal{Q}(\Omega)} = \overline{\cup_N \mathcal{Q}(\Lambda_N)}$.

<u>Pf</u>. Apply (6.2.3) and (6.2.4). []

Recall again that $\mathcal{V}_{[\Lambda_N]}$ has a polyhedral structure and hence $\mathcal{Q}(\Lambda_N)$ can be characterized using (4.2.8).

VII. AN APPLICATION TO THE TWO VARIANCE COMPONENT PROBLEM

7.1. Introduction

In this chapter we let Y be an $n \times 1$ random vector having a multivariate normal distribution with zero mean and a covariance matrix $V_\theta = \theta_1 I + \theta_2 V$ where V is a known $n \times n$ n.n.d. matrix and $\theta = (\theta_1, \theta_2)'$ is a vector of unknown parameters called the variance components with $\theta_1 > 0$ and $\theta_2 \geq 0$. Our interest is to characterize the class $\mathcal{A}$ of admissible estimators of a given linear parametric function of the form $\lambda'\theta$ when attention is restricted to the class $\mathcal{U}_0$ of all quadratic unbiased estimators. Note that if Y has a non-zero vector μ, then a reduction via invariance will lead to a model with zero mean (see Olsen, Seely and Birkes, 1976).

Recently, Olsen, Seely and Birkes (1976) reduced $\mathcal{U}_0$ via sufficiency to a minimal complete class which allowed them to characterize $\mathcal{A}$ using linear model techniques. In this chapter we will consider this problem directly through $\mathcal{U}_0$. The approach we adopt will follow the general framework established in Seely (1970a), Seely (1970b) and Seely and Zyskind (1971).

Now let $\mathcal{S}$ be the vector space of all $n \times n$ real symmetric matrices and let the associated inner product be the trace inner product defined by $(A, B) = tr(AB)$ for all $A, B \in \mathcal{S}$. Hence we have

$$\mathcal{U}_0 = \{(A, YY') : E(A, YY') = \lambda'\theta, A \in \mathcal{S}\} .$$

Remark (7.1.1). Notice that $\mathcal{S}$ is isomorphic to $R^{n(n+1)/2}$ and that $\mathcal{U}_0$ may be expressed in the form

$$\mathcal{U}_0 = \{a'U : E(a'U) = \lambda'\theta, a \in R^{n(n+1)/2}\} ,$$

with $U = (Y_1^2, \ldots, Y_n^2, Y_1Y_2, \ldots, Y_1Y_n, \ldots, Y_{n-1}Y_n)'$. Hence considering quadratic unbiased estimators of the form $(A, YY') = Y'AY$, $A \in \mathcal{S}$, is equivalent to considering lue's of the form $a'U$, $a \in R^{n(n+1)/2}$, when estimating a parametric function of the form $\lambda'\theta$. Thus the results of Chapters III, IV, V and VI can be applied to the problem of estimating the variance components.

7.2. The Linear Model

Using the normality of Y and the results of Seely (1970a), Seely (1970b) and Seely and Zyskind (1971) together with the fact that YY' is a random matrix in $\mathcal{S}$, we have for arbitrary $A, B \in \mathcal{S}$

$$E(Y'AY) = E(A, YY') = (A, E(YY')) = (A, V_\theta) ,$$

and

$$\text{Cov}[Y'AY, Y'BY] = \text{Cov}[(A, YY'), (B, YY')] = 2(A, V_\theta B V_\theta)$$

$$= 2\theta_1^2(A, B) + 2\theta_1\theta_2(A, VB+BV) + 2\theta_2^2(A, VBV)$$

$$= 2\theta_1^2(A, \Gamma_1 B) + 4\theta_1\theta_2(A, \Gamma_2 B) + 2\theta_2^2(A, \Gamma_3 B) ,$$

where Γ_1, Γ_2 and Γ_3 are the linear operators on $\mathcal{S}$ defined by

$$\Gamma_1 B = B, \ \Gamma_2 B = (1/2)(VB+BV), \ \Gamma_3 B = VBV, \quad \text{for all } B \in \mathcal{S}.$$

Note that Γ_1, Γ_2 and Γ_3 are n.n.d. linear operators on $\mathcal{S}$ and Γ_1 is a p.d. linear operator. Hence we have

$$E(YY') = \theta_1 I + \theta_2 V, \quad \text{Cov}(YY') = \Sigma_\theta = 2\theta_1^2\Gamma_1 + 4\theta_1\theta_2\Gamma_2 + 2\theta_2^2\Gamma_3 .$$

Define $Z = YY'$ and let H be the linear transformation from R^2 to $\mathcal{S}$ defined by $H\alpha = \alpha_1 I + \alpha_2 V$, for all $\alpha = (\alpha_1, \alpha_2)' \in R^2$. Then we have the model

$$\mathcal{M}_1 : E(Z) = H\theta, \quad \text{Cov}(Z) = \Sigma_\theta ,$$

where $\theta_1 > 0$ and $\theta_2 \geq 0$. More precisely (θ, Σ_θ) is an element of the parameter space Θ which has the form

$$\Theta = \{[(\theta_1, \theta_2)', 2(\theta_1^2\Gamma_1 + 2\theta_1\theta_2\Gamma_2 + \theta_2^2\Gamma_3)] : \theta_1 > 0, \theta_2 \geq 0\} ,$$

and is contained in $R^2 \times \mathcal{N}$ where $\mathcal{N}$ is the set of all n.n.d linear operators on $\mathcal{S}$. Hence we have a situation where the possible θ vectors do not form an affine set and where the mean vector and the covariance structure have a known functional relationship. Let $D = \{\theta : (\theta, \Sigma_\theta) \in \Theta \text{ for some } \Sigma_\theta\}$ and note that $\text{sp } D = R^2$. Hence, by Remark (7.1.1) and by Olsen, Seely and Birkes (1976), the linear model techniques can be applied to model $\mathcal{M}_1$. Thus concerning admissibility we may act like we have the linear model

$$\mathcal{M}_2 : E(Z) = H\theta, \quad \text{Cov}(Z) \in \mathcal{V}_\Omega ,$$

where $\theta \in R^2$, H is defined as before and

$$\mathcal{V}_\Omega = \{\alpha_1\Gamma_1 + \alpha_2\Gamma_2 + \alpha_3\Gamma_3 : (\alpha_1, \alpha_2, \alpha_3)' \in \Omega\},$$

with

$$\Omega = \{2(\delta_1^2, 2\delta_1\delta_2, \delta_2^2)' : \delta_1 > 0, \delta_2 \geq 0\} .$$

Now let Δ be the triangle in R^3 with extreme points $e_1 = (1,0,0)'$, $e_2 = (0,1,0)'$ and $e_3 = (0,0,1)'$ and let us regard the line joining e_1 and e_3 as the base of the triangle. Let Ω_0 be the convex set in Δ whose boundary consists of the curve $\{(1+\rho)^{-2}[1, 2\rho, \rho^2]' : \rho \geq 0\}$ and the base of the triangle. Let Γ be the linear operator from R^3 to the vector space of all linear operators from $\mathcal{S}$ to $\mathcal{S}$ defined by

(7.2.1) $\Gamma(\delta) = \delta_1\Gamma_1 + \delta_2\Gamma_2 + \delta_3\Gamma_3 = \Gamma_\delta$ for all $\delta \in R^3$.

By Olsen, Seely and Birkes (1976),

$$[\mathcal{V}_\Omega] = \{\alpha\Gamma_\delta : \alpha \geq 0, \delta \in \Omega_0\}.$$

By (3.3.4), $[\Omega_0] = \{\alpha\delta : \alpha \geq 0, \ \delta \in \Omega_0\}$, and so $[\mathcal{V}_\Omega] = \Gamma([\Omega_0]) = \mathcal{V}_{[\Omega_0]}$ which is closed. Apply (3.3.10) to get $[\mathcal{V}_\Omega] = [\mathcal{V}_{\Omega_0}]$. Hence

(7.2.2) $\mathcal{A} = \mathcal{A}(\Omega) = \mathcal{A}(\Omega_0)$.

Thus, concerning admissibility, we can always talk about Ω_0 instead of Ω. Notice also that Γ_δ is p.d. for all $\delta \in \Omega_1 = \Omega_0 \setminus \{e_3\}$.

7.3. The Admissible Class $\mathcal{A}$

In this section we will characterize the class $\mathcal{A}$ of all admissible quadratic unbiased estimators of $\lambda'\theta$. We will continue using the notion of the sets $\mathcal{B}_r = \mathcal{B}(V_1, \ldots, V_r)$ defined in Chapter III with quadratic unbiased estimators and their respective variances in mind. By Proposition 3.6 in Olsen, Seely and Birkes (1976) we conclude that

$$\mathcal{A} \subset \cup_{\delta \in \Omega_0} \mathcal{B}(\Gamma_\delta) = \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_\delta) \cup \mathcal{B}(\Gamma_3).$$

By (3.2.8), $\cup_{\delta\in\Omega_1} \mathcal{B}(\Gamma_\delta) \subset \mathcal{Q}$ since Γ_δ is p.d. for all $\delta \in \Omega_1$. Hence

(7.3.1) $\mathcal{Q} = \cup_{\delta\in\Omega_1} \mathcal{B}(\Gamma_\delta) \cup [\mathcal{Q} \cap \mathcal{B}(\Gamma_3)]$.

In order to investigate $\mathcal{Q} \cap \mathcal{B}(\Gamma_3)$, we use the fact that $\Omega_0 \subset \Delta$. This implies that $[\Omega_0] \subset [\Delta] = \{\Sigma_{i=1}^3 \alpha_i e_i : \alpha_i \geq 0\}$, and so $[\mathcal{V}_\Omega] = \Gamma([\Omega_0]) \subset \Gamma([\Delta]) = \{\Sigma_{i=1}^3 \alpha_i \Gamma_i : \alpha_i \geq 0\}$. We use the notation $\mathcal{V}_{[\Delta]} = \Gamma([\Delta])$. Since $sp[\Omega_0] = sp[\Delta] = R^3$, then we use the linearity of Γ to obtain

$$sp[\mathcal{V}_\Omega] = sp\Gamma([\Omega_0]) = \Gamma(sp[\Omega_0]) = \Gamma(sp[\Delta]) = sp\Gamma([\Delta]) = sp\ \mathcal{V}_{[\Delta]}.$$

Hence applying (5.5.2) we have

(7.3.2) $\mathcal{Q} = \mathcal{Q}([\mathcal{V}_\Omega]) \subset \mathcal{Q}(\mathcal{V}_{[\Delta]})$.

Since $\mathcal{V}_{[\Delta]} = \{\Sigma_{i=1}^3 \alpha_i \Gamma_i : \alpha_i \geq 0\}$ has a polyhedral structure with $\underline{R}(H) + \Sigma_{i=1}^3 \underline{R}(\Gamma_i) = \underline{R}(\Gamma_1) = \mathcal{S}$, the results of Chapter IV can be applied to $\mathcal{Q}(\mathcal{V}_{[\Delta]})$.

Let $\mathcal{L}$ be the set of all polyhedral convex sets Λ in R^3 such that $\Omega_0 \subset \Lambda$ and $sp\,\Lambda = R^3$ and let $\mathcal{L}^*$ be the subset of $\Lambda \in \mathcal{L}$ such that $\Lambda \subset \Delta$. Note that $\Delta \in \mathcal{L}^*$. By (5.2.4), $\mathcal{Q} = \cap_{\Lambda\in\mathcal{L}} \mathcal{Q}(\mathcal{V}_\Lambda)$. Given $\Lambda \in \mathcal{L}$, let $\Lambda^* = \Delta \cap \Lambda$. Hence $\Lambda^* \in \mathcal{L}^*$. Notice that $\Omega_0 \subset \Lambda^* \subset \Delta$ and

$\text{sp}\,\Lambda = \text{sp}\,\Omega_0 = R^3 = \text{sp}\,\Lambda^*$. So by (5.2.2), $\mathcal{Q}(\mathcal{V}_{\Lambda^*}) \subset \mathcal{Q}(\mathcal{V}_\Lambda)$. Thus we see

(7.3.3) $$\mathcal{Q} = \cap_{\Lambda^* \in \mathcal{L}^*} \mathcal{Q}(\mathcal{V}_{\Lambda^*}) .$$

We will begin by proving that any $\Lambda^* \in \mathcal{L}^*$ should have e_1, e_3 and $\alpha e_2 + (1-\alpha)e_3$, for some $0 < \alpha \leq 1$, as three of its extreme points. Since the base of Δ is in the boundary of Ω_0, it is clear from a picture of Δ that any $\Lambda^* \in \mathcal{L}^*$ must have e_1 and e_3 as two of its extreme points. To prove that $\alpha e_2 + (1-\alpha)e_3$ is an extreme point of Λ^*, for some $0 < \alpha \leq 1$, we will prove that the line connecting e_2 and e_3 is the tangent of Ω_0 at e_3. Recall that the curve $\{(1+\rho)^{-2}[1, 2\rho, \rho^2]' : \rho \geq 0\} = C$ is in the boundary of Ω_0. The projection of Δ from the (x, y, z) space to the (x, y) plane preserves the linear structure of Δ. In particular, tangent lines are preserved. The projection of C is $\{(1+\rho)^{-2}[1, 2\rho]' : \rho \geq 0\}$ and the points e_1, e_2 and e_3 are projected to the points $(1, 0)'$, $(0, 1)'$ and $(0, 0)'$ respectively. Thus we need to prove that the line joining $(0, 1)'$ and $(0, 0)'$ is the tangent of the projection of Ω_0 on R^2 at $(0, 0)'$. For any given $\rho \geq 0$, let $x = (1+\rho)^{-2}$ and $y = 2\rho(1+\rho)^{-2}$. Hence $y = 2\rho x$ with $\rho = (1/\sqrt{x}) - 1$ which implies that $y = 2(\sqrt{x} - x)$. The derivative of y w.r.t. x is given by $(dy/dx) = (1/\sqrt{x}) - 2$ which tends to infinity as x tends to zero. Thus the line connecting $(0,1)'$ and $(0,0)'$ is the tangent of the projection of Ω_0 on R^2 at $(0,0)'$. Thus we have proved the following:

Proposition (7.3.4). If $\Lambda^* \in \mathcal{L}^*$, then e_1, e_3 and $\alpha e_2 + (1-\alpha)e_3$, for some $0 < \alpha \leq 1$, form a subset of its extreme points.

Now, for any $\Lambda^* \in \mathcal{L}^*$, let $\mathcal{V}_{\Lambda^*} = \Gamma(\Lambda^*)$ with Γ as defined in (7.2.1). By (5.1.3), $[\mathcal{V}_{\Lambda^*}]$ has a polyhedral structure. By (7.3.4) and (7.2.1), Γ_1, Γ_3 and $\alpha\Gamma_2 + (1-\alpha)\Gamma_3$ are in the generating set of $[\mathcal{V}_{\Lambda^*}]$ with $\underline{R}(H) + \underline{R}(\Gamma_1) = \underline{R}(\Gamma_1) = \mathcal{S}$. Hence, for all $\Lambda^* \in \mathcal{L}^*$, the results of Chapter IV can be applied to $\mathcal{Q}([\mathcal{V}_{\Lambda^*}])$.

Proposition (7.3.5). $\mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1) \subset \mathcal{Q}$.

Pf. Pick any $\Lambda^* \in \mathcal{L}^*$ and recall that the results of Chapter IV are applicable to $\mathcal{Q}([\mathcal{V}_{\Lambda^*}])$. By (4.2.8) we have $\mathcal{B}(\Gamma_3, \alpha\Gamma_2+(1-\alpha)\Gamma_3, \Gamma_1) \subset \mathcal{Q}([\mathcal{V}_{\Lambda^*}])$, $0 < \alpha \leq 1$. By (3.2.11) and (4.3.1) we have

$$\mathcal{B}(\Gamma_3, \alpha\Gamma_2+(1-\alpha)\Gamma_3, \Gamma_1) = \mathcal{B}(\Gamma_3, \alpha\Gamma_2, \Gamma_1) = \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1) .$$

Now apply (7.3.3). ▯

By (7.3.2) we have $\mathcal{Q} \cap \mathcal{B}(\Gamma_3, \Gamma_2) \subset \mathcal{Q}(\mathcal{V}_{[\Delta]}) \cap \mathcal{B}(\Gamma_3, \Gamma_2)$. Proposition (4.2.2) implies that $\mathcal{Q}(\mathcal{V}_{[\Delta]}) \cap \mathcal{B}(\Gamma_3, \Gamma_2) \subset \mathcal{B}(\Gamma_3, \Gamma_2 \Gamma_1)$. Hence we conclude that

$$\mathcal{Q} \cap \mathcal{B}(\Gamma_3, \Gamma_2) \subset \mathcal{Q} \cap \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1) = \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1),$$

using (7.3.5). Notice that $\mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1) \subset \mathcal{Q} \cap \mathcal{B}(\Gamma_3, \Gamma_2)$. Thus we have proved the following:

Proposition (7.3.6). $\mathcal{Q} \cap \mathcal{B}(\Gamma_3, \Gamma_2) = \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1)$.

Proposition (7.3.7).

$$\mathcal{Q} \cap \mathcal{B}(\Gamma_3) = \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_3, \Gamma_\delta) \cup \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1).$$

Pf. Note that for any $\delta \in \Omega_1$ we have

$$\mathcal{B}(\Gamma_3, \Gamma_\delta) = \mathcal{Q} \cap \mathcal{B}(\Gamma_3, \Gamma_\delta) \subset \mathcal{Q} \cap \mathcal{B}(\Gamma_3),$$

since Γ_δ is p.d. Also, by (7.3.6)

$$\mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1) = \mathcal{Q} \cap \mathcal{B}(\Gamma_3, \Gamma_2) \subset \mathcal{Q} \cap \mathcal{B}(\Gamma_3).$$

Hence we have

$$\cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_3, \Gamma_\delta) \cup \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1) \subset \mathcal{Q} \cap \mathcal{B}(\Gamma_3).$$

To prove the other containment we use (7.3.2) to obtain

$\mathcal{A} \cap \mathcal{B}(\Gamma_3) \subset \mathcal{A}(\mathcal{V}_{[\Delta]}) \cap \mathcal{B}(\Gamma_3)$. By (4.2.2) we conclude that

$$\mathcal{A}(\mathcal{V}_{[\Delta]}) \cap \mathcal{B}(\Gamma_3) \subset \cup_{\substack{\alpha_1, \alpha_2 \geq 0 \\ \text{not both zero}}} \mathcal{B}(\Gamma_3, \alpha_1\Gamma_1 + \alpha_2\Gamma_2) .$$

Note that

$$\cup_{\substack{\alpha_1, \alpha_2 \geq 0 \\ \text{not both zero}}} \mathcal{B}(\Gamma_3, \alpha_1\Gamma_1 + \alpha_2\Gamma_2)$$

$$= \mathcal{B}(\Gamma_3, \Gamma_1) \cup \mathcal{B}(\Gamma_3, \Gamma_2) \cup \cup_{\alpha_1, \alpha_2 > 0} \mathcal{B}(\Gamma_3, \alpha_1\Gamma_1 + \alpha_2\Gamma_2) .$$

Applying (4.3.1) and (3.2.11) we have

$$\cup_{\alpha_1, \alpha_2 > 0} \mathcal{B}(\Gamma_3, \alpha_1\Gamma_1 + \alpha_2\Gamma_2)$$

$$= \cup_{\alpha > 0} \mathcal{B}(\Gamma_3, \Gamma_1 + \alpha\Gamma_2) = \cup_{\alpha > 0} \mathcal{B}(\Gamma_3, \Gamma_1 + \alpha\Gamma_2 + \alpha^2 4^{-1} \Gamma_3)$$

$$= \cup_{\alpha > 0} \mathcal{B}(\Gamma_3, (1+\alpha/2)^{-2}(\Gamma_1 + \alpha\Gamma_2 + \alpha^2 4^{-1}\Gamma_3))$$

$$\subset \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_3, \Gamma_\delta) .$$

This implies that

$$\cup_{\substack{\alpha_1, \alpha_2 \geq 0 \\ \text{not both zero}}} \mathcal{B}(\Gamma_3, \alpha_1\Gamma_1 + \alpha_2\Gamma_2) \subset \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_3, \Gamma_\delta) \cup \mathcal{B}(\Gamma_3, \Gamma_2) .$$

Thus we have

$$\mathcal{A}(\mathcal{V}_{[\Delta]}) \cap \mathcal{B}(\Gamma_3) \subset \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_3, \Gamma_\delta) \cup \mathcal{B}(\Gamma_3, \Gamma_2) ,$$

which implies that

$$\mathcal{A} \cap \mathcal{B}(\Gamma_3) \subset \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_3, \Gamma_\delta) \cup [\mathcal{A} \cap \mathcal{B}(\Gamma_3, \Gamma_2)] .$$

Apply (7.3.6) to obtain

$$\mathcal{A} \cap \mathcal{B}(\Gamma_3) \subset \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_3, \Gamma_\delta) \cup \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1) . \quad \square$$

<u>Theorem (7.3.8)</u>. $\mathcal{A} = \cup_{\delta \in \Omega_1} [\mathcal{B}(\Gamma_\delta) \cup \mathcal{B}(\Gamma_3, \Gamma_\delta)] \cup \mathcal{B}(\Gamma_3, \Gamma_2, \Gamma_1)$.

<u>Pf</u>. By (7.3.1) we have $\mathcal{A} = \cup_{\delta \in \Omega_1} \mathcal{B}(\Gamma_\delta) \cup [\mathcal{A} \cap \mathcal{B}(\Gamma_3)]$.

Apply (7.3.7). $\square$

To calculate the admissible estimators in $\mathcal{B}_r$, $r \geq 1$, apply (4.4.1) or (4.4.3) with the trace inner product in mind.

In 1976, Olsen, Seely and Birkes reduced the problem of quadratic unbiased estimation of $\lambda'\theta$ to considering linear combinations of the minimal sufficient statistics $T_1, \ldots, T_m$ where m is the number of the distinct eigenvalues of V and $T_k = Y'E_kY/r_k$, $k = 1, \ldots, m$, with Y defined in Section 1, r_k the multiplicity of λ_k, the $k^{\underline{th}}$ distinct eigenvalue of V, and E_k the orthogonal projection operator on the subspace associated with the eigenvector of λ_k. Then, for $m \geq 2$, they characterized the admissible class $\mathcal{A}$ using the linear model

$$E(T) = G\theta, \quad Cov(T) \in \mathcal{V} = \{\gamma D_\delta : \gamma \geq 0, \ \delta \in \Omega_0\}$$

where

$$G' = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \end{bmatrix},$$

$T = (T_1, \dots, T_m)'$, Ω_0 as defined in Section 2 and $D_\delta = \Sigma_{i=1}^3 \delta_i D_i$ with $D_i = \text{diag}\{\lambda_1^{i-1}/r_1, \dots, \lambda_m^{i-1}/r_m\}$, $i = 1, 2, 3$. Note that $\underline{N}(G') \cap \underline{N}(D_\delta) = \{0\}$. They proved that

$$\mathcal{A} = \cup_{\delta \in \Omega_0} \mathcal{B}(D_\delta) .$$

In our characterization of the admissible class $\mathcal{A}$, we considered the problem of the admissible quadratic unbiased estimators of $\lambda'\theta$ directly through $\mathcal{U}_0$, the class of all quadratic unbiased estimators of $\lambda'\theta$. We were able to use this direct approach because our main results in Chapters III, IV and V do not require that $\underline{N}(H') \cap \underline{N}(\Gamma_\delta) = \{0\}$ for all $\delta \in \Omega_0$.

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