CAYLEY'S ABSOLUTE
AND
THE NON-EUCLIDEAN GEOMETRIES

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Introduction

A large part of Euclid's geometry depends on his fifth (parallel) postulate, the modern, less cumbersome version of which is: Through a given point one and only one line can be drawn parallel to a given line. Because so much depended on this one postulate, and because of its lack of simplicity, it was not acceptable to some followers of Euclid and many ingenious but unsuccessful attempts were made to deduce it from the other axioms. G. Saccheri (1667-1733) proposed two alternative hypotheses, one leading to two parallels to a line through a point and to hyperbolic geometry while the other leads to elliptic geometry in which two lines always meet. In his attempt to vindicate Euclid, he "proved" his hypotheses false and consequently failed to discover that each would lead to a self-consistent geometry.

G. F. Gauss (1777-1855) and some of his students, including Wolfgang Bolyai (1775-1856) of Hungary, worked on the problem but failed to publish the results. It remained for John Bolyai (1802-1860), son of Wolfgang Bolyai, and N. I. Lobachevski (1793-1856) of Russia to announce their independent and almost simultaneous discovery of hyperbolic geometry. G. F. B. Riemann (1826-1866) in 1854 showed the existence of a consistent geometry without (real) parallel lines. His geometry was essentially that in which lines were great circles on a sphere, so that two lines always intersect in two points; however, Felix Klein (1849-1925) conceived the elliptic (Riemann's) geometry in which two lines meet in a single point (12). Arthur Cayley (1821-1895) introduced the notion of the absolute conic and showed the connection between non-Euclidean ideas and projective geometry (2).

The existence of non-Euclidean geometry proves beyond a doubt the impossibility of deriving the parallel postulate from the other axioms, and as Coxeter says (6, p. 3) "Nowadays, anyone who tries to prove Postulate V is classed with circle-squarers and angle-trisectors".

Axiom A,3 of section I obviously does not necessarily hold in Euclidean geometry. An axiom analgous to it was first used by Moritz Pasch (1843-1931) in 1882. It put into writing what Euclid had tacitly assumed and used in proofs. Briefly, it amounts to this: a line passing through one side of a triangle and not passing through any vertex, must pass through another side of the triangle. It will be the aim in this thesis to introduce the notion of the absolute, to show briefly how it functions in deriving hyperbolic, parabolic, and elliptic geometry from projective geometry, and then, to direct specific attention upon Pasch's axiom. The stress placed upon this axiom is justified because of the place it takes in making clear the fundamental concept of betweenness.

I. The Axioms of Projective Geometry and Some of Their Consequences

Perhaps the two chief methods for approaching the subject of non-Euclidean geometry are those of Gauss, Lobachevski, Bolyai, and Riemann, who began by modifying the postulates of ordinary Euclidean geometry, and that of Cayley (2, pp. 561-592) and Klein (12, pp. 573-625 and 13, pp. 112-145) who regarded the real projective plane as a subspace of the complex projective plane and introduced the notion of the absolute conic as a possible means of deducing the other geometries from the projective. More will be said about the absolute later, after we have defined complex projective space and have considered a few of its more elementary properties. In the following assumptions of Veblen and Young (17, vol. 2, ch. 1) the point and the line are undefined elements, the line being regarded as an undefined class of points. "Belonging to a class" is an undefined relation.

Assumptions of Alignment, A.

A,1. If A and B are distinct points, there is at least one line on both A and B.

A,2. If A and B are distinct points, there is not more than one line on both A and B.

A,3. If A, B, C are points not all on the same line, and D and E ($D \neq E$) are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.

Assumptions of Extension, E.

E,0. There are at least three points on every line.

E,1. There exists at least one line.

E,2. All points are not on the same line.

E,3. All points are not on the same plane.

E,3'. If S_3 is a three-space, every point is on S_3.

Assumption J. A geometric number system (17, vol. 1, ch. 6) is isomorphic (17, vol. 1, pp. 149-150) with the complex number system of analysis.

This set of postulates, A, E, and J, is sufficient for complex projective geometry. That assumption J is a sweeping assumption is quite obvious, and for a thorough study of it one should consult Veblen and Young (17, vol. 2, ch. 1). Had we desired to set up a real projective space we could have used assumptions A and E, but instead of assumption J we would have needed the following assumption.

Assumption K. A geometric number system is isomorphic with the real number system of analysis.

Spaces satisfying A, E, and J or K will also satisfy the following axioms of order (7, p. 22).

Assumptions of Order, O.

O,1. If A, B, C are three distinct collinear points, there is at least one point D such that A and B separate C and D (written AB//CD).

O,2. If AB//CD, then A, B, C, D are distinct.

0,3. If AB//CD, then AB//DC.

0,4. If AB//CD and AC//BE, then AB//DE.

Assumptions A and E are sufficient to prove many important theorems of projective geometry. Among these theorems is that of Desargues (1593-1662), which is stated here for future reference. For a complete proof see Veblen and Young (17, vol. 1, p. 41).

The Theorem of Desargues. If two triangles ABC, A'B'C' are situated in the same plane or in different planes and are such that BC, B'C' meet in L, CA, C'A' meet in M, and AB, A'B' meet in N where L, M, N are collinear, then AA', BB', CC' are concurrent, and conversely.

Two ranges (sets of points on different lines) are perspective from a point P if they are in (1,1) correspondence and lines joining corresponding points meet at P. The two ranges are then said to be in perspective. A projectivity may be defined as the product of several perspectivities, i.e., the result of applying a number of perspectivities.

Assumption of Projectivity, P. If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.

Theorem A. Assumption P is valid in any space satisfying assumptions A and E and such that multiplication is commutative in a geometric number system (17, vol. 2, p. 3).

Since the complex number system, as well as the real number system, obeys the commutative law of multiplication, complex projective geometry satisfies P, and all the theorems of (17, vol. 1) apply (17, vol. 2, p. 7).

A congruent transformation (a point-to-point correspondence which preserves length) is a collineation (a projective transformation effecting a rearrangement of points) which preserves the absolute (6, p. 126), and may be described by virtue of theorem B as a projectivity of the conic itself.

Theorem B. Any projectivity on a conic determines a collineation of the whole plane (6, p. 60).

Theorem C. Any projectivity on a line may be expressed in the form $x = \frac{ax+b}{cx+d}$. Conversely, every equation of this form represents a projectivity if $ad-bc \neq 0$ (17, vol. 1, p. 134).

In view of Steiner's theorem (7, pp. 75-76) or (17, vol. 1, p. 111), which states that by joining all the points on a conic to any two fixed points on the conic, we obtain two projectively related pencils, the whole theory of projectivities on a line can be carried over to projectivities on a conic. Hence, to determine the nature of the absolute we must find and examine the fixed points of a one-dimensional projectivity in the complex projective plane, since the absolute constitutes the locus of such points. Since x and x' are coordinates of corresponding points in theorem C, we will have fixed points when $x = x'$. Thus, the fixed points are given by the roots of the quadratic equation $cx^2+(d-a)x-b = 0$. Collineations may consequently be classified according to the nature of the roots of this equation. If the fixed points are real and distinct, the collineation is hyperbolic; if they are real and coincident, the collineation is parabolic; and if they are conjugate imaginary, the collineation is elliptic. We are thus led to the three types of absolute, a real conic in hyperbolic geometry, a degenerate conic in parabolic geometry, and an imaginary conic in elliptic geometry.

Before discussing the degenerate conic of the parabolic case certain fundamentals are needed. The general equation of a circle in homogeneous coordinates is given by the equation $k(x^2+y^2)+2fyz+2gzx+cz^2 = 0$, where the coefficients are any complex numbers. If $k\neq 0$ this circle cuts the line at infinity ($z=0$) in points given by $x^2+y^2 = 0$, or $(x+iy)(x-iy) = 0$. The coordinates of intersection are thus $(1,i,0)$ and $(1,-i,0)$. If $k=0$ the line at infinity is part of the locus and thus contains these same points. Hence, since the equation taken for the circle was general, all circles in the plane pass through the same two conjugate imaginary points at infinity.

Further, if the general conic $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$ passes through the points $(1,i,0)$ and $(1,-i,0)$, $a+2ih-b = 0$ and $a-2ih-b = 0$. Hence, $a = b$ and $h = 0$. If a, b, and h are real then either of these equations requires that $a = b$ and $h = 0$, thus a conic containing one of the points must contain both. Hence, a necessary and sufficient condition for a conic to be a circle is that it be on the points $(1,i,0)$ and $(1,-i,0)$. These points are called the <u>circular points</u>, and any two conjugate imaginary lines which connect a point with the circular points are called <u>circular rays</u>. For example, $x+iy = 0$ and $x-iy = 0$ are the circular rays from the origin. More generally, any line on either of the circular points is called an isotropic line.

The absolute in parabolic geometry consists of a point pair - the circular points. It is therefore a degenerate line conic which as a point conic appears as a repeated line (the line at infinity). Hence, its equation must be of the form $\lambda_0 \equiv x^2+y^2 = 0$. If we let $\lambda \equiv x^2+y^2-kz^2 = 0$ be the equation of the absolute, then for it to be real we must have $k > 0$; for it to be imaginary we must have $k < 0$; and for it to be degenerate we must have $k = 0$, in which case $\lambda = \lambda_0$. Hence, in setting up the determinant D of λ, $D = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -k \end{vmatrix} = -k$ we see that for $D < 0$, $= 0$, or > 0 we obtain the absolutes for hyperbolic, parabolic, and elliptic geometry respectively. Thus, it appears that parabolic geometry is the limiting case between hyperbolic and elliptic geometry.

Before studying the three geometries separately, some fundamental properties of distances and angles should be made clear. Among the properties commonly attributed to distances on a line are:

1. The additive property, i.e., $D(XY)\ D(YZ) = D(XZ)$.

2. The distance from a point to itself is zero, i.e., $D(XX) = 0$.

3. The distance between any two points on a line is invariant under a translation.

The parabolic measure of distance is a consequence of property 3, but to establish a hyperbolic or an elliptic scale along a line we resort to an extension suggested by Edmond Laguerre.

Laguerre's Theorem: The angle between two lines is a definite multiple of the logarithm of the cross-ratio[1] of the two lines and the circular points.

Proof: Let the two intersecting lines l and m have their vertex at the origin forming the angles, as in figure 1. The equation of l is $y = x \tan\alpha$ and the equation of m is $y = x \tan\beta$, while the equations of the circular rays through the origin are $y = ix$ and $y = -ix$. Thus the cross-ratio of the four lines is

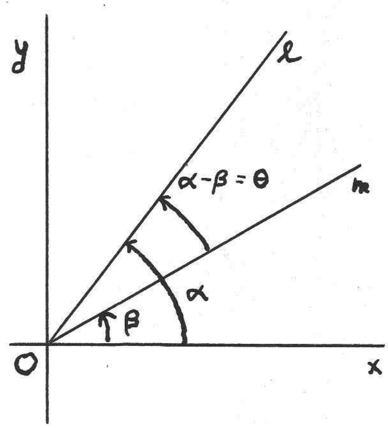

Figure 1

$$r = (\tan\alpha\,\tan\beta / i\,{-i}) = \frac{(\tan\alpha - i)(\tan\beta + i)}{(\tan\alpha + i)(\tan\beta - i)}$$

$$= \frac{\left(\frac{\sin\alpha}{\cos\alpha} - i\right)\left(\frac{\sin\beta}{\cos\beta} + i\right)}{\left(\frac{\sin\alpha}{\cos\alpha} + i\right)\left(\frac{\sin\beta}{\cos\beta} - i\right)} = \frac{\left(\frac{\sin\alpha - i\cos\alpha}{\cos\alpha}\right)\left(\frac{\sin\beta + i\cos\beta}{\cos\beta}\right)}{\left(\frac{\sin\alpha + i\cos\alpha}{\cos\alpha}\right)\left(\frac{\sin\beta - i\cos\beta}{\cos\beta}\right)}$$

$$= \frac{(\sin\alpha\sin\beta - i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + \cos\alpha\cos\beta)}{(\sin\alpha\sin\beta + i\cos\alpha\sin\beta - i\sin\alpha\cos\beta + \cos\alpha\cos\beta)} =$$

$$\frac{(\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)}{(\cos\alpha - i\sin\alpha)(\cos\beta + i\sin\beta)} = \frac{e^{i\alpha}\cdot e^{-i\beta}}{e^{-i\alpha}\cdot e^{i\beta}} = \frac{e^{i\alpha}\cdot e^{i\alpha}}{e^{i\beta}\cdot e^{i\beta}} =$$

$e^{2i(\alpha-\beta)} = e^{2i\theta}$ where e is the Naperian base of logarithms.

Thus, $\log r = \log_e e^{2i\theta} = 2i\theta$, or $\theta = \frac{1}{2i}\log r$.

1. The cross-ratio of four points x_1, x_2, x_3, x_4, in that order is defined to be the number $(x_1x_3/x_2x_4) = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}$.

The circular rays mentioned in the theorem are the fixed lines under a rotation of the pencil which accompanies the angle measurement. Thus, in defining the distance between any two points (whose coordinates are x and y) on a line, we choose two fixed points (the fixed points of a non-singular collineation of the line) whose coordinates are p and q. D(xy) is then defined to be k log (xy/pq), where k is a fixed but arbitrarily chosen constant.

If x, y, z are any three points of a line, (xy/pq)(yz/pq)

$$= \frac{(x-p)(y-q)}{(x-q)(y-p)} \cdot \frac{(y-p)(z-q)}{(y-q)(z-p)} = \frac{(x-p)(z-q)}{(x-q)(z-p)} = (xz/pq).$$

Thus taking the logarithms and multiplying through by k, k log (xy/pq) + k log (yz/pq) = k log (xz/pq) or D(xy) + D(yz) = D(xx) and we see that the new distance formula satisfies property 1. Property 2 is also satisfied, for D(xx) = k log (xx/pq) = k log $\frac{(x-p)(x-q)}{(x-q)(x-p)}$ = k log 1 = 0. The new distance formula also satisfies a requirement of which property 3 is a special case, namely: that the distance between two points is fixed under a collineation which leaves p and q fixed. This is an immediate consequence of the invariance of the cross-ratio. By an appropriate specialization (18, pp. 396-397) the new distance formula can quite easily be reduced to the Euclidean notion.

Perhaps at this point it would be well to illustrate the distance formulae of Cayley and Klein (18, pp. 406-407). Let the equation of the absolute be $A = f(x) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}x_ix_j = 0$, where

$a_{ij} = a_{ji}$, and let A_{xy} represent the polarized from of A, i.e., $A_{xy} = \frac{1}{2}(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial x_3})\, f(x) = \frac{1}{2}(y_1\frac{\partial f}{\partial x_1} + y_2\frac{\partial f}{\partial x_2} + y_3\frac{\partial f}{\partial x_3}$. Note that $A = A_{xx} = f(x)$, and that $A_{yy} = f(y)$. If x and y are taken as the base points in a parametric representation along the line, $f(tx+y) = A_{xx}t^2 + 2A_{xy}t + A_{yy} = 0$, the roots of which are the parameters t_1 and t_2 of the absolute points of the line xy. But the cross-ratio of four points equals the cross-ratio of their parameters (18, p. 94), hence, $(xy/AA') = (0\,\infty/t_1t_2) = t_1/t_2$. Then $D(xy) = \frac{k}{2}\log(t_1/t_2) =$

$\frac{k}{2}\log\frac{A_{xy}+\sqrt{A_{xy}^2 - A_{xx}A_{yy}}}{A_{xy}-\sqrt{A_{xy}^2 - A_{xx}A_{yy}}}$ which is Klein's distance formula.

But $\cos ix = \cos(-ix) = \frac{e^x + e^{-x}}{2} = \cosh x = \cosh(-x)$, or $-ix = \text{arc cos}\,\frac{e^x + e^{-x}}{2}$. In this last equation let $e^x = \sqrt{u}$ and $x = \frac{1}{2}\log u$, then $-i\cdot\frac{1}{2}\log u = \text{arc cos}\,\frac{\sqrt{u} + 1/\sqrt{u}}{2} = \text{arc cos}\,\frac{u+1}{2\cdot\sqrt{u}}$ and $\log u = \frac{2}{-i}\,\text{arc cos}\,\frac{u+1}{2\cdot\sqrt{u}} = \frac{2}{-i}\cdot\frac{i}{i}\text{arc cos}\,\frac{u^2+1}{2\cdot\sqrt{u}} = 2i\,\text{arc cos}\,\frac{u+1}{2\cdot\sqrt{u}}$. Using this form, Klein's formula becomes $D(xy) = \frac{k}{2}\log(t_1/t_2) = ik\,\text{arc cos}\,\frac{t_1+t_2}{2\sqrt{t_1t_2}}$, but $t_1 + t_2 = -2A_{xy}/A_{xx}$ and $t_1t_2 = A_{yy}/A_{xx}$. Hence, by taking the negative square root in Klein's formula we have Cayley's formula:

$$D(xy) = ik\,\text{arc cos}\,\frac{A_{xy}}{\sqrt{A_{xx}A_{yy}}}.$$

Now, if the absolute is referred to a self-polar triangle, its equation becomes $x^2 + y^2 - k^2z^2 = 0$, where the constant is introduced so as to insure that the conic be real. Then, if the two points p, p' be (x,y,z) and (x',y',z'), $D(pp') = ik\,\text{arc cos}\,\frac{(xx' + yy' - k^2zz')}{(x^2+y^2-k^2z^2)(x'^2+y'^2-k^2z'^2)}$

which is another form of Cayley's distance formula. Note the form of the equation of the absolute given here, as compared with the equation given previously. In actual history, Cayley's formula seems to have been obtained first, but Klein's log expression which followed shortly was a great improvement over the cosine expression since the additive property for lines became quite obvious.

II. Parabolic Geometry

Instead of using Hilbert's axioms (9, pp. 1-26) to characterize this geometry, I choose to use the modified list of Hilbert and Cohn-Vossen (10, pp. 239-240).

Group 1. Axioms of Incidence.

1. Two points have one and only one straight line in common.

2. Every straight line contains at least two points.

3. There are at least three points not lying on the same straight line.

Group 2. Axioms of Order.

1. Of any three points on a straight line, one and only one lies between the other two.

2. If A and B are two points, there is at least one point C such that B lies between A and C.

3. (Pasch's axiom). Any straight line intersecting the side of a triangle (i.e., containing a point lying between two vertices) either passes through the opposite vertex or intersects a second side.

Group 3. Axioms of Congruence.

1. On a straight line a given segment can be laid off on either side of a given point; the segment thus constructed is called congruent to the given segment.

2. If two segments are congruent to a third segment, then they are congruent to each other.

3. If AB and A'B' are two congruent segments, and if the points C anc C' lying on AB and A'B' respectively are such that one of the segments into which AB is divided by C is congruent to one of the segments into which A'B' is divided by C', then the other segment of AB is also congruent to the other segment of A'B'.

4. A given angle can be laid off in one and only one way on either side of a given half-line; the angle thus drawn is called congruent to the given angle.

5. If two sides of a given triangle are equal respectively to two sides of another triangle, and if the included angles are equal, the triangles are congruent.

Group 4. Axiom of Parallels.

1. Through any point not lying on a given straight line there passes one and only one straight line that does not intersect the given line.

Group 5. Axioms of Continuity.

The way in which these axioms are formulated varies a great deal. They may, for example, be stated as follows:

1. (Axiom of Archimedes) Every straight-line segment can be measured by any other straight-line segment.

2. (Cantor's axiom) Every infinite sequence of nested segments (i.e., a sequence of segments such that each contains all the following ones) has a common point.

In Euclidean metric geometry there is no line at infinity.

However, in deducing Euclidean geometry by specializing the projective, metrical ideas such as parallel lines, circle, parabola, asymptote, focus, etc., may be defined. The line at infinity is not exceptional in projective geometry since any line can be projected into any other. A parabola, for example, loses its distinctive features and becomes merely a conic tangent to a line, when the line at infinity is projected into an ordinary line. Thus, if a geometric statement involve the line at infinity, i.e., have a special relation to the absolute, it is metric, otherwise projective.

The absolute thus furnishes a basis of distinction between projective and metric properties. Further, the transition from one geometry to the other can, as will be seen shortly, actually be effected through its use. A projective theorem concerning any plane figure can, by isolating a pair of points for the circular points, or (what is more pertinent to the sequel) a line for the line at infinity, be translated into a Euclidean theorem. Similarly, some Euclidean theorems can be stated projectively by considering the circular points as an ordinary point pair or the line at infinity as an ordinary line. What is even more general will be done here, i.e., starting with the axioms of projective geometry, the axioms just given will be deduced by isolating a degenerate conic (the line at infinity repeated) for the absolute. Thus, the absolute points of a parabolic line coincide at infinity. The line is of infinite extent, but the two ends "come together at infinity", thus leaving it without ultra-infinite (ideal) points. It is not possible, however, by continuous motion in one direction to traverse the entire line. The parabolic geometry of the finite plane is thus Euclidean. Let us now see how the axioms just given may be deduced from the projective.

Group 1'. Axioms of Incidence.

The axioms of Group A hold for all points of the projective plane. By isolating the absolute we merely remove the points at infinity from the plane, leaving only ordinary points. Hence, the first part of theorem 1 (17, vol. 1, p. 17), which states that two distinct points are on one and only one line, becomes 1,1; E,1 becomes 1,2; and from A,3 and the definition of a plane (17, vol. 1, p. 17) we can deduce 1,3.

Group 2'. Axioms of Order.

By isolating the absolute cyclic order disappears and the order on a Euclidean line takes its place, i.e., betweenness will then have meaning. Thus, from Group 0 of the projective axioms, we can, by isolating the absolute, obtain 2,1 and 2,2. Pasch's axiom, which is not as obvious, will be dealt with separately in section V.

Group 3'. Axioms of Congruence.

The axioms given in section I make no reference whatsoever to congruence. We may, however, introduce an algebra of segments, based upon Desargues' theorem, which is independent of the axioms of congruence (9, pp. 79-82). Hilbert's development, though straightforward, is done with a Euclidean point of view and for this reason a number of changes would be needed in bringing it to a projective point of view so that isolating the absolute would be of some significance.

The latter will not be done here, but rather the simple development of Hilbert will be used to show how Desargues' theorem, which is valid in all three geometries, can be used in defining equality of segments.

Take two fixed straight lines in the plane intersecting at O, and consider only such line segments as have their origin at O and their extremity in one of the fixed lines. Point O is regarded as the segment o, i.e., $OO \equiv o$. If E and E' are two definite points lying on the respective lines through O (see figure 2), then define the segments OE and OE' so that $OE \equiv OE' \equiv 1$. The line EE' is called the unit line. If A and A' are points on the lines OE and OE' respectively, and if line AA' is parallel to EE' then we say that $OA \equiv OA'$.

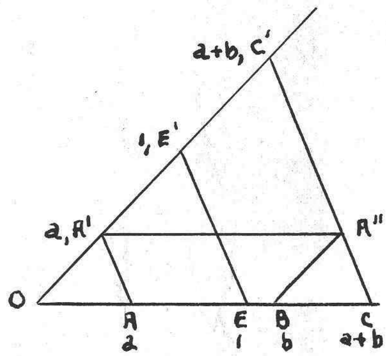

Figure 2

To define the sum of the segments $a \equiv OA$ and $b \equiv OB$, construct AA' parallel to OE, and through B a parallel to OE'. Let these last two parallels meet in A''. Through A'' draw a straight line parallel to EE', cutting OE and OE' in C and C' respectively. Then $c \equiv OC \equiv OC'$ is called the sum of the segments $a \equiv OA$ and $b \equiv OB$, or $c \equiv a + b$.

In order to determine the product of a segment $a \equiv OA$ by a segment $b \equiv OB$, the following construction is made (see figure 3): determine on OE' a point A' such that AA' is parallel to EE', and draw A'E. Draw a straight line through B parallel to A'E intersecting OE'

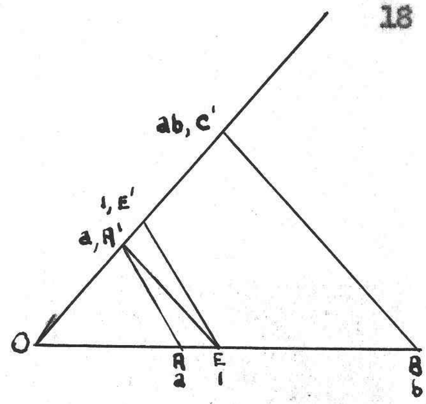

Figure 3

in point C'. Let c = OC' be called the product of segment a = OA by the segment b = OB. Indicate this relation by writing c = ab.

For a more complete discussion of this algebra of segments as well as proofs of associative and commutative laws in this algebra see Hilbert (1, pp. 79-89).

Similarly, equality of angles may be defined independently of the congruence axioms by means of Laguerre's theorem. Since the angle between two lines is a definite multiple of the logarithm of the cross-ratio of these two lines and the circular rays through the origin, two angles may be said to be equal if these respective cross-ratios are equal. Note, however, that if the angles do not have the same vertex, their respective circular rays must meet at infinity (be parallel in a sense).

Group 4'. Axiom of Parallels.

In the projective plane there are no lines through an external point which can be drawn not intersecting a given line. By isolating the absolute, the one line through an external point which previously met the given line at infinity is now parallel to it.

Group 5'. Axioms of Continuity.

For a proof of Archimedes' axiom see Coxeter's development (7, pp. 138-139). A study of deducing these axioms for parabolic, hyperbolic, and elliptic geometry from the projective will not be considered in the sequel.

III. Hyperbolic Geometry

In order to characterize hyperbolic geometry it is necessary to have a postulate system. The following five sets of axioms are sufficient for this purpose. Where the axioms are identical with those of parabolic geometry (see section II) they will not be restated.

Group 1. Axioms of Incidence.

These axioms are the same as for parabolic geometry.

Group 2. Axioms of Order.

These axioms are also the same as in the parabolic case.

Group 3. Axioms of Congruence.

All five of these axioms hold in hyperbolic geometry (10, pp. 244-245).

Group 4. Axiom of Parallels.

This axiom of Euclidean geometry is not valid in the hyperbolic plane. As a substitute for it, we have for the hyperbolic plane:

Axiom X. Through a given point, not on a given line, two lines can be drawn parallel (in the sense of Lobachevski, i.e., meeting the line in points at infinity) to the given line.

Group 5. Axioms of Continuity.

These two axioms hold in hyperbolic geometry (10, p. 245).

Definition H_1. $D(xy) = \frac{k}{2} \log (xy/pq)$, where p and q are the absolute points collinear with x and y; k is real and finite and p and q are real and distinct.

Note that $D(xp) = \frac{k}{2} \log (xp/pq) = \frac{k}{2} \log \frac{(x-p)(p-q)}{(x-q)(p-p)} = \frac{k}{2} \log \infty = \infty$ and similarly, that $D(xq) = -\infty$. Thus the hyperbolic line contains two real points at infinity, and the absolute must be the locus of such points. $(xy/pq) > 0$ and hence $D(xy)$ is real unless $xy//pq$, in which case $(xy/pq) < 0$ and $D(xy)$ is imaginary. Points interior to the absolute are called ordinary points and points exterior to it are called ultra-infinite or ideal. Similarly, there are three classes of lines: actual lines, which have two real (distinct) points in common with the absolute; isotropic lines which are tangent to the absolute; and ideal lines which lie in the ideal region and which cut the absolute in conjugate imaginary points.

Definition H_2. $\angle uv = c \log (uv/\alpha\alpha')$, where α and α' are the isotropic lines concurrent with u and v, i.e., the angle between two lines is a constant multiple of the logarithm of the cross-ratio formed by the lines and the tangents to the absolute from their points of intersection.

If u and v are perpendicular, $(uv/\alpha\alpha') = -1$. Let the unit of measure for a right angle be $\pi/2$ (not radian measure as in trigonometry), thus fixing k. Since $\log(-1) = i\pi$, a right angle formed by u and v equals $\pi/2 = k \log(-1) = ki\pi$, or $k = 1/2i$. From these properties we may prove the following two theorems (18, p. 403):

Theorem D. All lines perpendicular to a given line l meet in a point P, the pole of the line with respect to the absolute.

Theorem E. All lines through a point P are perpendicular to a unique line l, the absolute polar of P.

If P is actual, l is ideal; if P is at infinity, l is isotropic (tangent to the absolute at P); and if P is ideal, l is actual. Thus two lines always have a common perpendicular, and when these two lines are parallel, the common perpendicular is tangent to the absolute at their point of intersection. Thus parallel lines do not have an infinity of common perpendiculars as in Euclidean geometry.

Consider now the specific characteristics of the distance between two points (18, pp. 405-408). From now on, instead of p and q we shall use A and A' in the distance formula in order to emphasize the pole and polar relationships of the A's and the α's. The distance between two points conjugate with respect to the absolute is called a quadrant. If x and y are conjugate points, $(xy/AA') = -1$, thus quadrant $(Q) = \frac{k}{2} \log(-1) = \frac{ki\pi}{2}$. If x is fixed y may be anywhere on the polar of x, i.e., the locus of points a quadrant distance from a fixed point x is the absolute polar of x. Two distances whose sum is a quadrant are called complementary. The distance from a point to a line is the complement of the distance from the point to the absolute pole of the line. The distance between two points is proportional to the angle between their absolute polars, or the angle between two lines is proportional to the distance between their absolute poles.

From a theorem of projective geometry (18, pp. 132-133) we know that the polar lines of the points of a range constitute a pencil projective with the range, and dually. Hence, the polars of x, y, A, and A' form a pencil u, v, α, α' which is projective with the range. Thus $(xy/AA') = (uv/\alpha\alpha') = r$, and $D(xy) = \frac{k}{2} \log r = ik\angle uv$.

With this background in mind we shall now see how the first four groups of axioms for the hyperbolic geometry may be deduced from projective geometry. Since the absolute represents the locus of points at infinity in the hyperbolic plane, isolating it will have the same effect as isolating the line at infinity in the parabolic case. When the arguments for the hyperbolic case are the same as for the parabolic case, they will be omitted.

Group 1'. Axioms of Incidence.

Since these axioms are the same as for the parabolic case, isolating the absolute, even though it is a real conic in this case, has the same effect as the absolute of the parabolic case and hence the arguments are the same.

Group 2'. Axioms of Order.

All these axioms can be deduced from projective geometry by the same arguments as for the parabolic case. See section V concerning Pasch's axiom.

Group 3'. Axioms of Congruence.

Again, with definitions H_1 and H_2 taken into consideration, the arguments would be the same as in the parabolic case.

Group 4'. Axiom of Parallels.

Within the absolute conic there are, quite obviously, two lines through a point P which intersect a given line l in points at infinity. Isolating the absolute removes the points at infinity from the plane and consequently leaves two lines through point P which do not meet line l.

IV. Elliptic Geometry

To characterize this geometry I again choose the axioms of Hilbert and Cohn-Vossen, modifying them where necessary.

Group 1. Axioms of Incidence.

These axioms are all valid in elliptic geometry.

Group 2. Axioms of Order.

The axioms of order do not hold in the elliptic plane; for straight lines in this geometry are closed (cyclic in nature), and it cannot be said of three collinear points that one lies between the other two. Instead, the axioms of Group 0 for the projective plane (see section I) may be used.

Like the corresponding Euclidean axioms, the elliptic axioms or order (or separation) also lead to the definition of a straight-line segment and to the other concepts used in the axioms of congruence. But these definitions must be based on the fact that two points A and B always define two segments rather than just one. Only by recourse to a third point C of the straight line AB can we distinguish between the two segments defined by A and B; one segment consists of all those points that are separated from C by A and B, and the other segment consists of the remaining points of the straight line AB. Furthermore, it is necessary to stipulate that the interior angles of a triangle shall be less than a straight angle, as two sides and the included angle would otherwise determine not one triangle but two non-congruent triangles (see figures 4-a and 4-b) so that the side-angle-side theorem on congruence would be violated. This last restriction is also vital in connection

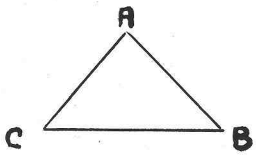

Figure 4-a

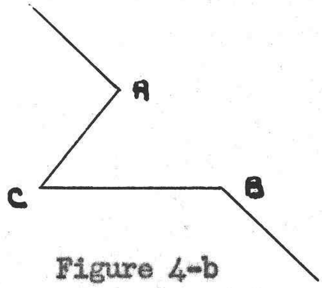

Figure 4-b

with Pasch's axiom (see section V). If these restrictions are observed, it is found (10, pp. 240-241) that the analogy with a region of the Euclidean is preserved on every sufficiently small portion of the elliptic plane, and that the Euclidean axioms of congruence, and the axioms of continuity as well, would remain valid in the elliptic plane.

Group 3. Axioms of Congruence.

With the restrictions given in the proceeding paragraph, these axioms will hold in elliptic geometry.

Group 4. Axiom of Parallels.

This axiom does not remain valid, and must be replaced by the following:

Axiom Y. Through a given point P not on a given line l no line can be drawn not meeting l.

Group 5. Axioms of Continuity.

With the restrictions given in Group 2 above, these axioms will hold as in the parabolic case.

Elliptic geometry also has other important properties. For example, in it we have:

Definition E_1. $\angle uv = \frac{1}{2i} \log (uv/\alpha\alpha')$, where α and α' are the isotropic lines on the vertex of the angle.

The absolute points (p and q) of the elliptic line are conjugate imaginary (see section I). Since they are imaginary we can write $(xy/pq) = e^{i\theta}$ as was done in Laguerre's theorem. Then, $D(xy) = k \log (xy/pq) = k \log e^{i\theta} = ki\theta$. For D(xy) to be real, k must be taken as imaginary. If we again use A and A' for p and q in order to point out pole and polar relationships we have:

Definition E_2. $D(xy) = \frac{k}{2i} \log (xy/AA')$, where A and A' are the absolute points collinear with x and y.

In this geometry, a quadrant $(Q) = \frac{k}{2i} \log (-1) = \frac{k}{2i} \cdot i\pi = \frac{k\pi}{2}$. Further, if u and v are the polars of x and y, then $D(xy) = k\angle uv$. The elliptic plane like the (real) elliptic line is finite but unbounded. It contains a single class of real points and real lines, the actual points and the actual lines respectively. Further, there is but one type of line pair, the intersectors. In other words, two lines always meet within a finite distance.

Two lines perpendicular to the same line l meet in a point P, the absolute pole of l. But these perpendiculars will meet in either direction because of symmetry. Rather than to consider that the lines meet in two points, assume that these two points are identical. Hence, all lines in the elliptic plane are of the same finite length namely, two quadrants:

$2\,(Q) = 2 \cdot \frac{k\pi}{2} = k\pi$ (18, pp. 413-415).

Let us now look briefly at the problem of deducing some of the elliptic axioms from the projective by use of the absolute. The absolute, while imaginary in this case, still represents the locus of points at infinity, but on a complex projective line. The real elliptic geometry must be thought of as pertaining to the geometry resulting from considering only the real part of such lines.

Group 1'. Axioms of Incidence.

Isolating an imaginary conic would not change the projective axioms, Group A and their consequences, hence, the elliptic axioms of incidence would follow directly from assumptions A for the projective plane.

Group 2'. Axioms of Order.

Isolating the absolute leaves the real elliptic line which has cyclic order. Thus, the axioms of order are the same for elliptic geometry as they are for projective geometry. See the next section for a discussion of Pasch's axiom and the restrictions mentioned previously.

Group 3'. Axioms of Congruence.

With the restrictions mentioned previously and with a proper definition of equality of segments and angles, these axioms would be deducible as mentioned in the previous section.

Group 4'. Axiom of Parallels.

This axiom would follow directly from the projective theorem that two lines always intersect.

V. Pasch's Axiom

In deducing Pasch's axiom from the axioms of projective geometry it is convenient, not to use the axiom as stated by Hilbert, but rather to reword it as follows:

(1) Pasch's axiom: If A, B, C are three non-collinear points and D, E are two points in the order BCD and CEA, then there is a point F in the order AFB such that D, E, F are collinear.

Before proceeding further, it is necessary to establish a number of properties concerning the cross-ratio. If A, B, C, D, E are any five distinct points on a line l, then $(AB/CD)(AB/DE)(AB/EC) = \frac{(A-C)(B-D)}{(A-D)(B-C)} \cdot \frac{(A-D)(B-E)}{(A-E)(B-D)} \cdot \frac{(A-E)(B-C)}{(A-C)(B-E)} = 1$, thus

(2) $(AB/CD)(AB/DE)(AB/EC) = 1$

Consider figure 5 which is used by Robinson (15, p. 101) in extending the coordinate system from a line to a plane. By applying (2) to the line A_0I_0 we get, $(A_0I_0/X'X'')(A_0I_0/X''I)(A_0I_0/IX') = 1$. Projecting these cross-ratios from X, A_2, and A_1 respectively, we get $(A_1A_2/X_0I_0)(A_0A_1/X_2I_2)(A_2A_0/XI_1) = 1$ or we may write it

(3) $(A_1A_2/X_0I_0)(A_2A_0/X_1I_1)(A_0A_1/X_2I_2) = 1$

Pasch's axiom may now be deduced from the axioms of projective geometry, along with Desargues' theorem in the plane and the assumption (15, p. 113) that the coordinate field in projective geometry is ordered. By applying (3) to figure 6, which is used by Robinson (15, p. 103) in connection with Desargues' theorem, we have

(4) $(A_1A_2/M_0I_0)(A_2A_0/M_1I_1)(A_0A_1/M_2I_2) = 1$.

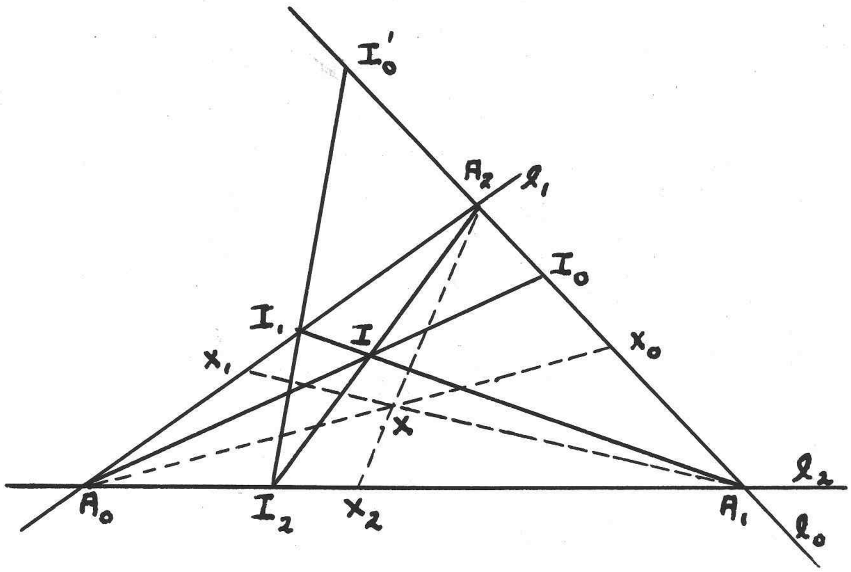

Figure 5

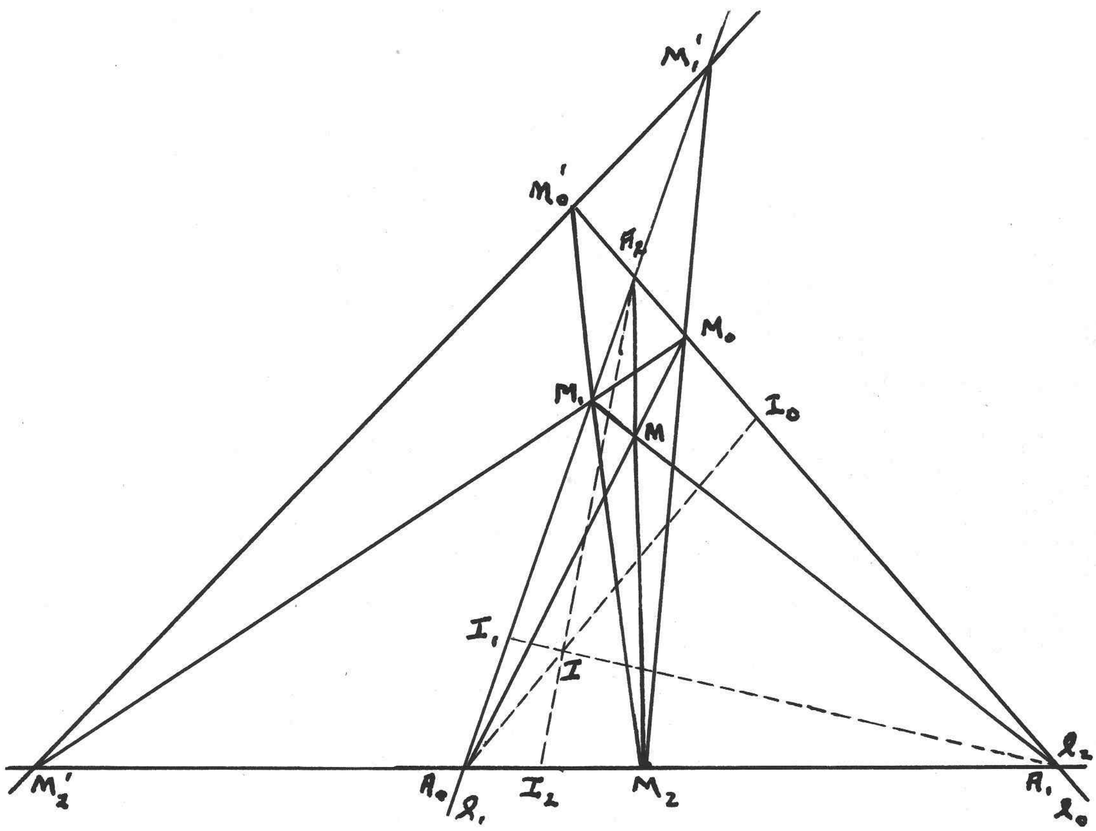

Figure 6

Now let $(A_1A_2/M_0I_0) = (A_1A_2/M_0M_0')(A_1A_2/M_0'I_0)$

$(A_2A_0/M_1I_1) = (A_2A_0/M_1M_1')(A_2A_0/M_1'I_1)$

$(A_0A_1/M_2I_2) = (A_0A_1/M_2M_2')(A_0A_1/M_2'I_2)$.

Substituting these in (4) we get,

$(A_1A_2/M_0M_0')(A_1A_2/M_0'I_0)(A_2A_0/M_1M_1')(A_2A_0/M_1'I_1)(A_0A_1/M_2M_2')(A_0A_1/M_2'I_2)$ $= 1$. But, $(A_1A_2/M_0M_0') = (A_2A_0/M_1M_1') = (A_0A_1/M_2M_2') = -1$, since the four points determined by a complete quadrangle on any diagonal form a harmonic range. Hence,

(5) $(A_1A_2/M_0'I_0)(A_2A_0/M_1'I_1)(A_0A_1/M_2'I_2) = -1$.

Thus, either (i) all the cross-ratios in (5) are negative or (ii) one is negative and the other two are positive. Then, by isolating a real conic or a degenerate conic (for both have the same effect here) we can say that either (i) all three of the points M_0', M_1', M_2' lie outside the segments A_1A_2, A_2A_0, A_0A_1 respectively, or (ii) one lies outside and two inside. Case (ii) thus gives Pasch's axiom.

Isolating an imaginary conic fails to abolish cyclic order and we are unable to say anything definite about segments, betweenness, etc. Obviously, triangles could arise (figure 4-b, section IV) so that Pasch's axiom would not always hold in elliptic geometry unless, of course, the restrictions given previously were imposed. Limiting the size of the interior angles of a triangle has the same effect as specifying what is meant by "the segment making up the side of a triangle".

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