

On Calculating Vorticity Balances in Primitive Equation Models

M. G. G. FOREMAN

Institute of Ocean Sciences, Sidney B.C., Canada

A. F. BENNETT

College of Oceanography, Oregon State University Corvallis, Oregon

3 January 1989 and 12 April 1989

ABSTRACT

A note of caution is provided to users of the widely-distributed Cox ocean circulation model. It is shown that the discrete conservation of vorticity equation associated with the finite difference approximations to the primitive equations on an Arakawa B grid contains a term that approximates $\beta\Delta y^2\partial^2u/\partial x\partial y$. Although this term has no counterpart in the continuous equation, it is consistent with a beta term in which the v velocity is evaluated by a streamfunction difference over $2\Delta x$ and must be included when model velocities are used to calculate vorticity balances.

Regions within ocean models where the spurious term might be significant are discussed and the relative magnitude of the term with respect to βv is calculated for a simple barotropic model of the North Pacific Ocean. It is shown that a similar spurious term can also arise with a C grid model.

1. Introduction

The precise form of the discrete vorticity balance in primitive equation models is governed by the finite difference approximations to the primitive equations. The finite difference equation for the conservation of vorticity is determined by forming the particular discrete curl of the momentum equations that permits cancellation of the pressure terms and a substitution for the continuity equation. When these operations are carried out on an Arakawa B grid with a variable Coriolis term, differences that approximate the term $\beta\Delta y^2\partial^2u/\partial x\partial y$ arise. Such a term is a numerical artifact; that is, it has no continuous analogue. Although this term does not arise during the primitive equation model calculations, it must be included in vorticity balance analyses that use velocities from the numerical model. The role of this term can be seen by introducing a streamfunction for the vertically integrated flow, ψ , as is done in the numerical models of Semtner (1986), Bryan (1969), and Cox (1984). When the discrete vorticity balance equation is expressed in terms of ψ , the $\beta\Delta y^2\partial^2u/\partial x\partial y$ term cancels with parts of the βv approximation to form a new beta term in which the v velocity is evaluated by a streamfunction difference over $2\Delta x$. So the presence of the spurious term is consistent with the usual discrete vorticity calculation in streamfunction form.

Bryan and Cox (1968) point out that care should be taken to follow the same numerical procedure in calculating the vorticity balance as in the primitive equations. We simply add a further note of caution with regard to the numerical procedure that one should employ if discrete vorticity balances are calculated using model velocities. With the widespread distribution of both the Cox (1984) model and results from various applications of that model, it is important that the spurious term $\beta\Delta y^2\partial^2u/\partial x\partial y$ not be overlooked in vorticity balance analyses. It can be significant in certain regions of ocean models.

In subsequent sections, we show how the spurious term arises and we calculate its magnitude, relative to the βv term, for a simple barotropic model of the North Pacific Ocean. With particular finite difference approximations to the Coriolis terms, we also show that a similar spurious term can arise with a C grid model.

2. The continuous equations

In order to compare with subsequent discrete equations, consider first the continuous equations. Without loss of generality, consider the linearized primitive equations for a homogeneous ocean with a β -plane approximation to the Coriolis parameter. The equations are

$$u_t - fv + p_x/\rho = A\nabla^2u + F^x \quad (1)$$

$$v_t + fu + p_y/\rho = A\nabla^2v + F^y \quad (2)$$

Corresponding author address: Dr. M. G. G. Foreman, Institute of Ocean Sciences, P.O. Box 6000, 9860 West Saanich Road, Sidney, British Columbia, Canada V8L 4B2.

$$u_x + v_y + w_z = 0 \tag{3}$$

$$p_z = -g\rho. \tag{4}$$

Here (u, v, w) are the velocity components in the (x, y, z) directions, p is the pressure, ρ is the density, $f = f_0 + \beta y$ is the Coriolis parameter, A is the horizontal diffusion coefficient, and (F^x, F^y) are the (x, y) components of wind stress. Boundary conditions are $u = v = 0$ along the coast; and $w = 0, w = -uH_x - vH_y$, at $z = 0$ and $z = -H$, respectively; H is the depth.

The associated vorticity balance equation, formed by taking the curl of the momentum equations and substituting the continuity equation, is

$$\zeta_t + \beta v - fw_z = A\nabla^2\zeta + F_x^y - F_y^x \tag{5}$$

where

$$\zeta = v_x - u_y. \tag{6}$$

3. The finite difference equations on a B grid

The horizontal and vertical stencils for an Arakawa B grid (e.g., see Mesinger and Arakawa 1976) are shown in Fig. 1. Given the values of a variable s at adjacent points $x - \Delta x/2$ and $x + \Delta x/2$, or $x - \Delta x$ and $x + \Delta x$, in the x direction, the difference and average values midway between the two points are denoted as

$$\delta_x s = \frac{s(x + \Delta x/2) - s(x - \Delta x/2)}{\Delta x} \tag{7}$$

$$\delta_{2x} s = \frac{s(x + \Delta x) - s(x - \Delta x)}{2\Delta x} \tag{8}$$

$$\bar{s}_x = \frac{s(x + \Delta x/2) + s(x - \Delta x/2)}{2} \tag{9}$$

Using this notation, and defining similar operations in the $y, z,$ and t coordinates, the discrete approximations to the momentum and continuity equations are

$$\delta_t u - fv + \delta_x \bar{p}^y / \rho = A(\delta_x \delta_x u + \delta_y \delta_y u) + F^x \tag{10}$$

$$\delta_t v + fu + \delta_y \bar{p}^x / \rho = A(\delta_x \delta_x v + \delta_y \delta_y v) + F^y \tag{11}$$

$$\delta_x \bar{u}^y + \delta_y \bar{v}^x + \delta_z w = 0. \tag{12}$$

Defining the finite difference curl by

$$\text{curl}_z(q_1, q_2) = \delta_x \bar{q}_2^y - \delta_y \bar{q}_1^x, \tag{13}$$

the discrete analogue to equation (5) is then

$$\begin{aligned} \delta_t \zeta - \bar{f}^y \delta_z w + \beta \bar{v}^{xy} + \frac{1}{4} \beta \Delta y^2 \delta_y \delta_x u \\ = A(\delta_x \delta_x + \delta_y \delta_y) \zeta + \delta_x \bar{F}^{yy} - \delta_y \bar{F}^{xx}, \end{aligned} \tag{14}$$

where

$$\zeta = \delta_x \bar{v}^y - \delta_y \bar{u}^x. \tag{15}$$

Comparing (14) with (5), it is seen that the term $\frac{1}{4} \beta \Delta y^2 \delta_y \delta_x u$ has no continuous counterpart. This term is a numerical artifact that arises from averaging fu values in the y -direction before taking their difference in the x -direction.

However, the $\frac{1}{4} \beta \Delta y^2 \delta_y \delta_x u$ term must be included in the discrete vorticity balance equation. Its role can be seen by introducing a streamfunction for the vertically integrated flow, ψ , (defined at the center of a grid cell as shown in Fig. 1) as

$$\bar{u} = -\delta_y \bar{\psi}^x / H, \tag{16}$$

$$\bar{v} = \delta_x \bar{\psi}^y / H, \tag{17}$$

where (\bar{u}, \bar{v}) are the vertically averaged velocities. When Eqs. (10)–(12) are vertically averaged and the

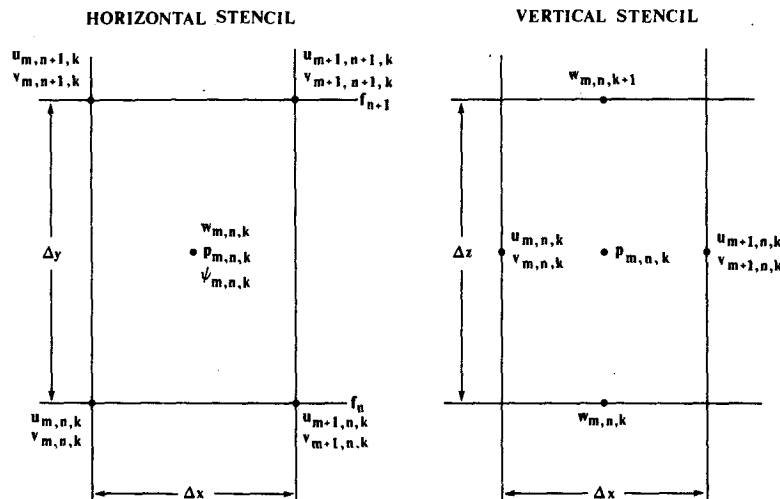


FIG. 1. Horizontal and vertical stencils for an Arakawa B grid.

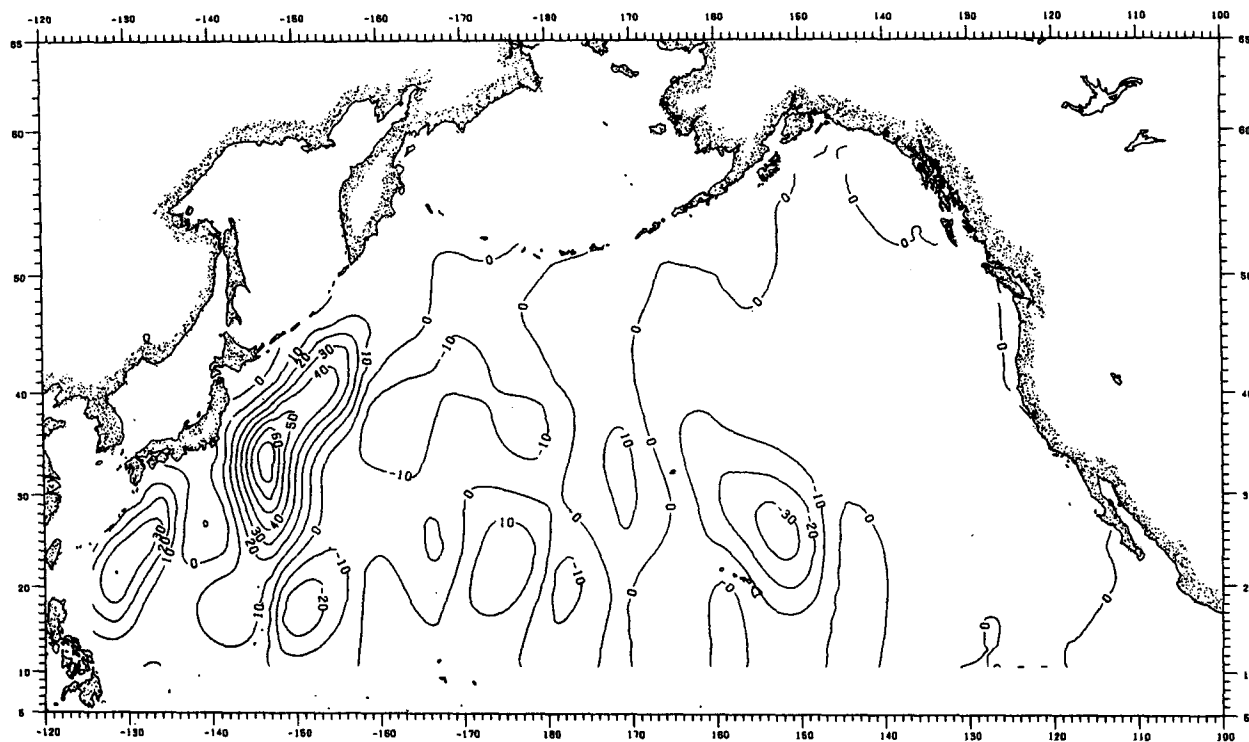


FIG. 2. Steady state contours of $\beta \bar{V}^{xy}$ ($m s^{-2} \times 10^{+11}$).

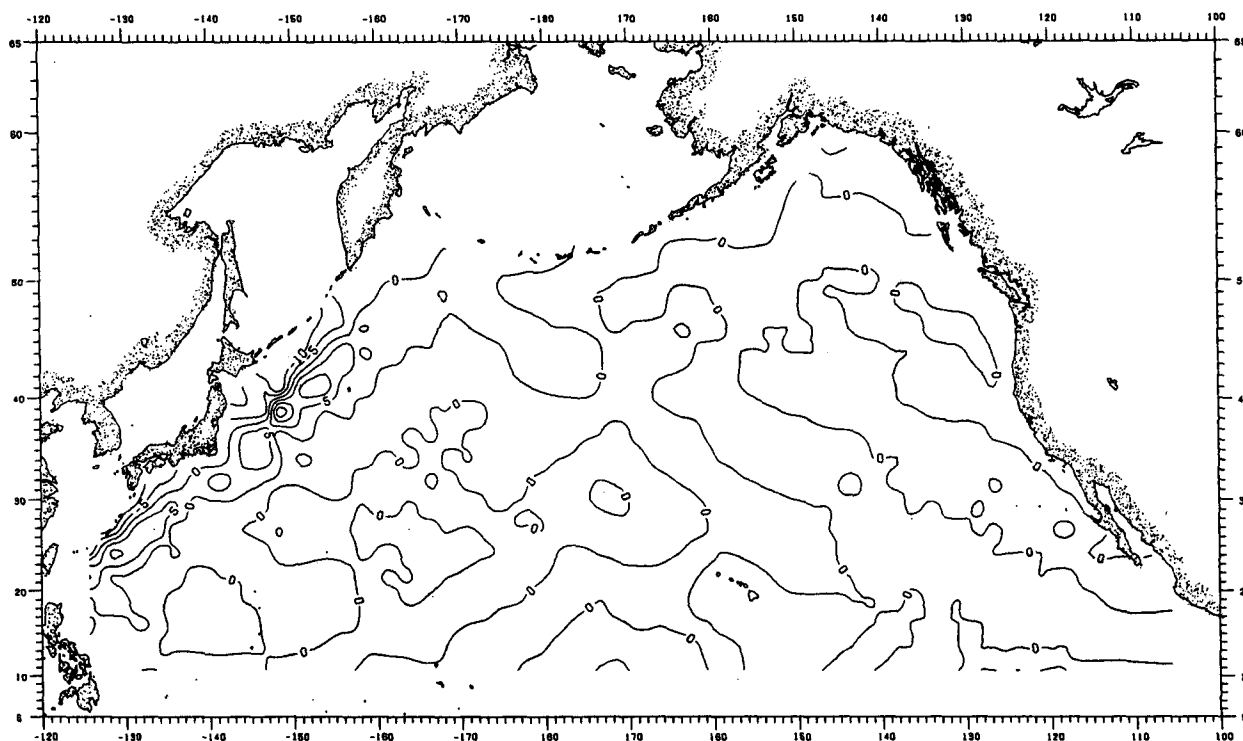


FIG. 3. Steady state contours of $\frac{1}{4} \beta \Delta y^2 \delta_y \delta_x U$ ($m s^{-2} \times 10^{+11}$).

preceding curl operations are repeated, the third and fourth terms in (14) become

$$\beta \bar{v}^{xy} + \frac{1}{4} \beta \Delta y^2 \delta_y \delta_x \bar{u} = \frac{\beta}{H} \delta_{2x} \psi \quad (18)$$

for a constant depth H . So the spurious term has been canceled by parts of the $\beta \bar{v}$ approximation and the outcome is a simple two-point finite difference approximation to $\beta \bar{v}$ in terms of ψ . With a nonconstant depth there are similar cancellations and the result involves four ψ values, namely those to the north, south, east, and west of a central point (Semtner 1986).

Although the numerical models of Semtner (1986), Bryan (1969), and Cox (1984) are formulated on an Arakawa B grid, their barotropic mode is expressed and solved in terms of a streamfunction for the vertically integrated flow. Consequently, the spurious term that is present when the discrete vorticity balance equation is written in terms of primitive equation variables, is not seen. For consistency however, it must be included in (14).

The $\frac{1}{4} \beta \Delta y^2 \delta_y \delta_x u$ term has no net effect on the global vorticity balance if the numerical model has no-slip boundary conditions. This is seen by summing along columns of any level of the model. Interior values cancel leaving only terms of the form

$$\left. \frac{\partial u}{\partial x} \right|_{\text{north}} - \left. \frac{\partial u}{\partial x} \right|_{\text{south}} \quad (19)$$

If the northern and southern boundary conditions are $u = 0$ (i.e., no-slip), there is no net contribution. If, however, the boundary conditions are free-slip, the spurious term will make a net contribution.

In most ocean regions

$$\beta \bar{v}^{xy} \gg \frac{1}{4} \beta \Delta y^2 \delta_y \delta_x u, \quad (20)$$

and the latter term will be insignificant. However in regions where the velocity is turning (e.g., where the western boundary current is leaving the coast), $\frac{1}{4} \beta \Delta y^2 \delta_y \delta_x u$ could play a significant role in the vorticity balance.

With (U, V) denoting the vertically integrated velocity components, Figs. 2 and 3 show steady-state contours of $\beta \bar{V}^{xy}$ and $\frac{1}{4} \beta \Delta y^2 \delta_y \delta_x U$ for a simple model of the North Pacific Ocean. Governing equations are the vertically integrated, spherical-coordinate versions of (1)-(4) with realistic coastline, bathymetry, and a horizontal resolution of $2\frac{1}{2}^\circ \times 2\frac{1}{2}^\circ$. Forcing is with zonally-averaged Kutsuwada (1982) winds, A is set to $0.25 \times 10^6 \text{ m}^2 \text{ s}^{-1}$, and the Coriolis parameter is represented exactly rather than with a β -plane. Even though the horizontal resolution is too coarse to represent accurately the width of the Kuroshio current (the Munk boundary layer is roughly equal to $3.6 \Delta x$), Figs. 2 and 3 show that the $\frac{1}{4} \beta \Delta y^2 \delta_y \delta_x U$ term is as large as 40% of the $\beta \bar{V}^{xy}$ term in parts of the western bound-

ary current. With finer horizontal resolution and fully nonlinear models, the importance of the spurious term should increase due to strong shears arising from eddies and meanders of the western boundary current.

A scaling analysis confirms this. Consider the region where the transport turning western boundary current separates from the coast, nominally at an angle of 45° . If the current has width of order ϵ , then u and v will scale as ϵ^{-1} and Δx and Δy will scale as ϵ . Both βv and the spurious term $\beta \Delta y^2 \partial^2 u / \partial x \partial y$ will then scale as ϵ^{-1} and maintain the same order as $\epsilon \rightarrow 0$.

4. The finite difference equations on a C grid

The horizontal stencil for an Arakawa C grid is shown in Fig. 4. The C-grid finite difference equation analogous to (14) will also have a spurious term of the form $\beta \Delta y^2 \partial^2 u / \partial x \partial y$ if the Coriolis term for the $u_{m,n}$ momentum equation is approximated as

$$-\frac{1}{4} [f_{n+1}(v_{m,n+1} + v_{m-1,n+1}) + f_n(v_{m,n} + v_{m-1,n})], \quad (21)$$

and the Coriolis contribution to the $v_{m,n}$ momentum equation is handled similarly. If however, the more common approximation,

$$-\frac{1}{8} (f_{n+1} + f_n)(v_{m,n+1} + v_{m-1,n+1} + v_{m,n} + v_{m-1,n}) \quad (22)$$

is employed, no spurious term will be generated.

5. Conclusions

It has been shown that the discrete conservation of vorticity equation associated with a primitive equation

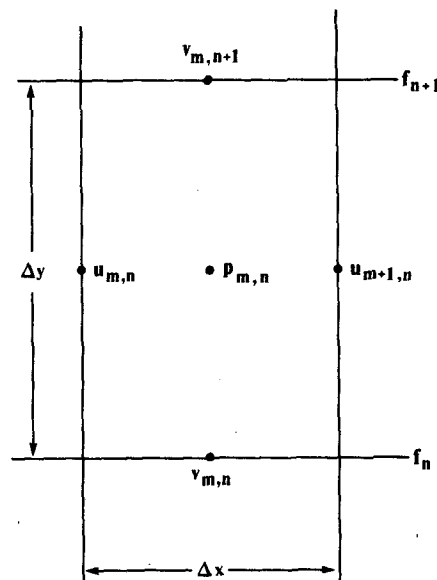


FIG. 4. Horizontal stencil for an Arakawa C grid.

model on an Arakawa B grid contains a term that approximates $\beta\Delta y^2\partial^2u/\partial x\partial y$. Though such a term does not appear in the continuous equation, it should be included in all vorticity balances that are calculated with model velocities. Its presence is consistent with the $2\Delta x$ approximation to the βv term in the Cox (1984) equation for streamfunction of the vertically integrated flow.

A coarse resolution, vertically integrated, linear model of the North Pacific Ocean was used to demonstrate that the spurious term can be significant in regions where the current is turning, such as at the coastal separation point of the western boundary current. With finer horizontal resolution and fully nonlinear models (e.g., Cox 1984), the term could play an even larger role.

It was also shown that a similar spurious term will be present in C grid models with particular Coriolis term approximations.

Acknowledgments. We thank Patrick Cummins for valuable discussions and Patricia Kimber for preparing the figures.

REFERENCES

- Bryan, K., 1969: A numerical method for the study of the circulation of the world ocean. *J. Comput. Phys.*, **4**, 347–376.
- , and M. D. Cox, 1968: A nonlinear model of an ocean driven by wind and differential heating. Part II: An analysis of the heat, vorticity and energy balance. *J. Atmos. Sci.*, **25**, 968–978.
- Cox, M. D., 1984: A primitive equation three-dimensional model of the ocean. GFDL Ocean Group Tech. Rep. No. 1 GFDL/NOAA, Princeton University, 250 pp.
- Kutsuwada, K., 1982: New computation of the wind stress over the North Pacific Ocean. *J. Oceanogr. Soc. Japan*, **38**, 159–171.
- Mesinger, F., and A. Arakawa, 1976: Numerical methods used in atmospheric models, Vol. 1, WMO-ICSU Joint Organizing Committee, GARP Publ. Ser. No. 17.
- Semtner, A. J., 1986: Finite-difference formulation of a world ocean model. *Proc. of the NATO Advanced Study Institute on Advanced Physical Oceanographic Numerical Modelling*, D. Reidel, 608 pp.