Contents

1 INTRODUCTION

A stochastic (or random) process satisfies the Markov property if the probability density function of the future states of the process depends only on the present state, and not on any past states. An example of this would be flipping a coin (say P(head)=P(tail)=0.5). The previous flips do not affect the current flip; if you had just flipped 100 heads in a row, you would still have a 0.50 chance to flip another head. A continuous-time *Markov process* is a stochastic process that satisfies this Markov property.

Markov processes appear in a variety of different applications. They can be used to model population dynamics in biology, genetics, carcinogenesis, AIDS epidemiology, HIV pathogenesis, and other biomedical systems. They can be used to model precipitation maps, shuffling methods (quickest way to shuffle a deck of cards), and also financial networks. These are only some of its uses. Markov processes are also commonly used in physics, especially statistical mechanics and quantum mechanics, as they are probabilistic in nature.

Markov Chain Monte Carlo (MCMC) methods are very common in physics (and indeed in a wide variety of disciplines), and were first used in statistical physics as a way to model extremely large systems while minimizing the amount of computations necessary. MCMC methods are a class of algorithms that provide a means of sampling probability distributions based on a constructed Markov chain. The premise of the idea is not too complex; if you want to sample randomly from a given probability distribution, then construct a Markov chain with that distribution and model this chain via computer simulation. The precision will get better as you take more steps in your modeled Markov chain. This simulation will allow for modeling microscopic systems of immense amounts of atoms with relative ease.

Another common model used in statistical mechanics and quantum mechanics is the Ising model, which is named after the physicist Ernest Ising. The Ising model is a d-dimensional lattice where each vertex is assigned a value of $\sigma = +1$. These values are termed spins, and although Ising models were initially intended to crudely represent ferromagnetism, it can also be used to model other systems including simple liquids, lattice gases, magnetic dipoles, and many other systems which can be represented via graphs like this.

This work answers some of the questions originating from the interdisciplinary research (physics/chemistry/biology/mathematics) done by E.Nir et al (see [4]). The occupation times for birth-and-death chains were studied by Karlin and McGregor with orthogonal polynomials (see [1], [2] and [3]).

2 OVERVIEW

Markov processes have been studied extensively for scenarios when T is taken to ∞. The motivation behind the research covered in this paper stems from the desire to be able to model the behavior of a given *Markov process* even when T is small. The goal is to solve for $f_{A,Q}(t,x)$, the continuous probability distribution function governing a *Markov process.* This function allows someone to immediately determine the probability that a process will spend a given time x at a specific state, state 0. The A in $f_{A,Q}(t,x)$ represents state 0, which is the state of interest to the observer. Q represents the *generator matrix* for the given *Markov process*(See Definitions and Examples). Since the only identifying features of a *Markov process* is the number of possible states and the rates of transfer between those states, the *generator matrix* uniquely determines any given *Markov process.* T is the interval of time that the process is allowed to run.

We begin by constructing integral equations that govern the *Markov process* from scratch. Take a look at the following equation, which is one of the two equations that represent a 2-state *Markov process* with a rate of transfer from state 0 to state 1 equal to λ

$$f_{A,Q}(t,x) = e^{-\lambda t}\delta_t(x) + \int_0^t f_{B,Q}(t-y, x-y)\lambda e^{-\lambda y} dy$$

One thing to note before going further is that the rates of transfer refer to an *exponential random variable*(See Definitions and Examples), which is where the $e^{-\lambda t}$ and $\lambda e^{-\lambda y}$ come from.

As will be mentioned later in the paper, $f_{A,Q}(t,x)$ represents the density function for the time spent in state 0 when the process started in state 0. $f_{B,Q}(t,x)$ represents the density function again for the time spent in state 0, but for the process that started in state 1. This construction not only gives us the above equation, but it allows for an intuitive understanding of it as well. It seems fairly obvious to state, but either the process will switch from state 0 to state 1 at least once or it won't switch at all.

By definition of the *exponential random variable*, the probability of not leaving state 0 in time t (i.e. not switching from state 0 to state 1) is $e^{-\lambda t}$. The delta mass, $\delta_t(x)$, acts like a point-mass that weights this probability precisely at t. Thus, $e^{-\lambda t}\delta_t(x)$ is the term that represents no switch from state 0 to state 1.

If the process switches from state 0 to state 1 at some time $y < t$, then it can now be considered a new process that started in state 1, with a new interval of interest $t - y$ (with respected occupation time being $x - y$). We must account for all possible values of y, which is where the integral comes in (since this is a continuous process). So $\int_0^t f_{B,Q}(t-y, x-y)\lambda e^{-\lambda y}dy$ is the term that represents at least one switch occurring.

This constructed integral equations allows for the solution of the density function and, as such, is the starting point for each section in this paper.

3 METHODS

As mentioned earlier, we begin solving for the occupation times by writing an integral equation governing the *Markov process*. Ultimately, we write the integral equation based on conditional probabilities. Either the process will leave its initial state or it will not leave its initial state. We weight the function either at time 0 or at time T (depending on which equation we are looking at) with the probability that no switch occurs, and then integrate over the rest of the interval with the probability that a switch does occur.

Given these integral equations containing the distribution function for a Markov process in terms of T and X, our first step will be to take the Fourier transform with respect to X. After some mathematical manipulation, we will then be able to take the Laplace transform with respect to T. It is at this point that we will be able to solve for $Ł_{\hat{f}_{A,Q}}(s_1, s_2)$, where $Ł_{\hat{f}_{A,Q}}(s_1, s_2)$ represents the *Laplace transform* of the *Fourier transform* of $f_{A,Q}(t, x)$.

We then take the inverse *Fourier transform*, and with some manipulation we will be able to evaluate the inverse *Laplace transform* and get a form for $f_{A,Q}(t, x)$. We will see that $f_{A,Q}(t, x)$ will involve Bessel functions, which are solutions to Bessel's Equation.

For convenience, I have grouped the important equations into the results section (Section 3) and included all intermediate calculations in the calculations section (Section 4) that follows it.

3.1 Definitions and Examples

Definition 3.1. *A continuous-time* Markov process *is a stochastic process that satisfies the Markov property. The Markov property states that at any time $t > 0$, the probability distribution of a given* Markov process *after time t depends only on the state of the process at time t.*

Definition 3.2. *A* Fourier transform *is a mathematical operator which is the generalization of the complex Fourier series. For the purposes of this paper, we don't need to know about this series, it is sufficient to merely know its mechanics. Given a function $f(t, x)$, we denote the* Fourier transform *of $f(t, x)$ w.r.t. x by $\hat{f}(t, s_2)$ satisfying the following equations*

$$\begin{array}{rcl} \hat{f}(t, s_2) & = & \int_{-\infty}^{\infty} f(t, x)e^{is_2x}dx \\ f(t, x) & = \frac{1}{2\pi} & \int_{-\infty}^{\infty} \hat{f}(t, s_2)e^{-is_2x}ds_2 \end{array}$$

Definition 3.3. *A* Laplace transform *is a mathematical operator that is useful for analyzing linear time-invariant systems. We denote the* Laplace transform *of $f(t, x)$ w.r.t. t by $Ł_f(s_1, x)$ satisfying the following equation*

$$Ł_f(s_1, x) = \int_0^{\infty} f(t, x)e^{-s_1t}dt$$

Definition 3.4. Cramer's rule *is a theorem in linear algebra that allows for the solution of a system of linear equations in terms of determinants. Given an* n x n *system of equations denoted by $Ax = b$, where*

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix}$$

Cramer's rule *gives us the following relation*

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is the matrix formed by replacing the i^{th} column of A with the column vector of b.

Definition 3.5. *If we have an n-state* Markov process *with rates of transfer between state* i *and state* j *written as* $\lambda_{i,j}$*, then the* generator matrix *for this* Markov process*, denoted* Q*, is defined as follows*

$$Q = \begin{pmatrix} -\sum_{j\neq 1}^{n} \lambda_{1,j} & \lambda_{1,2} & \dots & \lambda_{1,n} \\ \\ \lambda_{2,1} & -\sum_{j\neq 2}^{n} \lambda_{2,j} & \dots & \lambda_{2,n} \\ \\ \dots & \dots & \dots & \dots \\ \\ \lambda_{n,1} & \lambda_{n,2} & \dots & -\sum_{j\neq n}^{n} \lambda_{n,j} \end{pmatrix}$$

Definition 3.6. *An* exponential random variable *with parameter* λ *is a probability density function that has the following form*

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

Note that f(x)=0 everywhere else and that $\lambda > 0$.

4 RESULTS

4.1 Two-state Markov process with rates $\lambda = \mu$

In this section, we will look at a *Markov process* with state space {0,1}. We will define the process to start in state 0 and to move to state 1 with rate λ, and to move from state 1 to state 0 also with rate λ.

Again, we will use t to specify the time of the interval we are looking at and x to specify the amount of time spent in state 0. We denote the distribution function for the time spent in state 0 when the process started in state 0 as $f_{A,Q}(t,x)$ and the distribution function for the time spent in state 0 when the process started in state 1 as $f_{B,Q}(t,x)$. However, since the rates of transition are equal, we will simplify by just writing in terms of one state and call it f(t,x).

Beginning with the following integral equation

$$f(t,x) = e^{-\lambda t}\delta_t(x) + \int_0^t f(t-y, t-x)\lambda e^{-\lambda y} dy$$

We obtain the following expression for $Ł_{\hat{f}}(s_1, s_2)$, the Laplace transform of the Fourier transform of $f(t,x)$

$$Ł_{\hat{f}}(s_1, s_2) = \frac{2\lambda + s_1}{s_1(s_1 + 2\lambda) - i(s_1 + \lambda)s_2}$$

By first computing the inverse Fourier transform via the method of residues, then computing the inverse Laplace transform, the following value for $f(t,x)$ is obtained

$$\boxed{f(t,x) = e^{-\lambda t}\delta_x(t) + \lambda e^{-\lambda t} I_0(2\lambda\sqrt{x(t-x)}) + \lambda\sqrt{x}\tfrac{1}{\sqrt{t-x}}e^{-\lambda t} I_1(2\lambda\sqrt{x(t-x)})}$$

4.2 Two-state Markov process with rates $\lambda \neq \mu$

In this section, we will again look at a *Markov process* with state space {0,1}. However, this time, $\lambda \neq \mu$.

The method used to solve this case will be extremely similar to the method used to solve the case of the two-state process with equal rates except for this time we will not begin with the initial substitution of $f_{B,\lambda,\mu}(t,x) = f_{A,\mu,\lambda}(t, t-x)$.

Starting with the following integral equations

$$\begin{array}{rclcl} f_{A,Q}(t,x) & = & e^{-\lambda t}\delta_t(x) & + & \int_0^t f_{B,Q}(t-y, x-y)\lambda e^{-\lambda y} dy \\ f_{B,Q}(t,x) & = & e^{-\mu t}\delta_0(x) & + & \int_0^t f_{A,Q}(t-y, x)\mu e^{-\mu y} dy \end{array}$$

We obtain the value for $Ł_{\hat{f}_{A,Q}}(s_1, s_2)$

$$Ł_{\hat{f}_{A,Q}}(s_1, s_2) = \frac{(s_1 + \lambda + \mu)i}{(s_1 + \mu)(s_2 + i\frac{s_1(s_1+\lambda+\mu)}{s_1+\mu})}$$

Computing the inverse Fourier transform again via residues, then solving the inverse Laplace transform gives us

$$\boxed{f_{A,Q}(t,x) = e^{-\lambda t}\delta_t(x) + \lambda e^{-\lambda x} e^{-\mu(t-x)} I_0(2\sqrt{\lambda\mu x(t-x)}) + \sqrt{\tfrac{\lambda\mu x}{t-x}} I_1(2\sqrt{\lambda\mu x(t-x)}) e^{-\lambda x} e^{-\mu(t-x)}}$$

4.3 Three-state Markov process with equal rates

Now we move onto the three-state *Markov process* with rates of transfer equal to λ for all paths.

Here we begin with the following set of integral equations

$$\begin{aligned} f_{A,Q}(t,x) &= e^{-2\lambda t}\delta_t(x) + \int_0^t f_{B,Q}(t-y,x-y)\lambda e^{-\lambda y}dy + \\ & \quad \int_0^t f_{C,Q}(t-y,x-y)\lambda e^{-\lambda y}dy \\ f_{B,Q}(t,x) &= e^{-2\lambda t}\delta_0(x) + \int_0^t f_{C,Q}(t-y,x)\lambda e^{-\lambda y}dy + \\ & \quad \int_0^t f_{A,Q}(t-y,x)\lambda e^{-\lambda y}dy \\ f_{C,Q}(t,x) &= e^{-2\lambda t}\delta_0(x) + \int_0^t f_{A,Q}(t-y,x)\lambda e^{-\lambda y}dy + \\ & \quad \int_0^t f_{B,Q}(t-y,x)\lambda e^{-\lambda y}dy \end{aligned}$$

The *Laplace transform* of the *Fourier transform* of f(t,x) is then found to be

$$Ł_{\hat{f}_{A,Q}}(s_1,s_2) = \frac{(s_1+2\lambda)(s_2+i(s_1+\lambda))i}{s_1[s_2+i(s_1+2\lambda)][s_2+i\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}]}$$

Computing the inverse *Fourier transform* via residues and then finding a form for the inverse *Laplace transform* leads to

$$\boxed{f_{A,Q}(t,x) = e^{-2\lambda t}\delta_t(x) + 2\lambda e^{-\lambda x} I_0(2\sqrt{2\lambda^2 x(t-x)})}$$

4.4 Three-state Markov process with general rates

We now attempt to utilize the same method as earlier for a three-state *Markov process* with state space {0,1}.

The method used to solve this case will be similar to the method used in the three-state case with equal rates with the obvious difference being an increase in complexity. This increased complexity requires further manipulation, but the structure of the method remains intact. Similar to the previous cases, we will use $Ł_{\hat{f}_{A,Q}}(s_1,s_2)$ to denote the Laplace transform of the Fourier transform of $f_{A,Q}(t,x)$.

We will now switch notations slightly; we will denote by $\lambda_{(i,j)}$ the rate of transfer from state i to state j.

Beginning with the following set of equations

$$\begin{aligned}
f_{A,Q}(t,x) &= e^{-(\lambda_{(1,2)}+\lambda_{(1,3)})t}\delta_t(x) + \int_0^t f_{B,Q}(t-y,x-y)\lambda_{(1,2)}e^{-\lambda_{(1,2)}y}dy \\
&\quad + \int_0^t f_{C,Q}(t-y,x-y)\lambda_{(1,3)}e^{-\lambda_{(1,3)}y}dy \\
f_{B,Q}(t,x) &= e^{-(\lambda_{(2,3)}+\lambda_{(2,1)})t}\delta_0(x) + \int_0^t f_{C,Q}(t-y,x)\lambda_{(2,3)}e^{-\lambda_{(2,3)}y}dy \\
&\quad + \int_0^t f_{A,Q}(t-y,x)\lambda_{(2,1)}e^{-\lambda_{(2,1)}y}dy \\
f_{C,Q}(t,x) &= e^{-(\lambda_{(3,1)}+\lambda_{(3,2)})t}\delta_0(x) + \int_0^t f_{A,Q}(t-y,x)\lambda_{(3,1)}e^{-\lambda_{(3,1)}y}dy \\
&\quad + \int_0^t f_{B,Q}(t-y,x)\lambda_{(3,2)}e^{-\lambda_{(3,2)}y}dy
\end{aligned}$$

We eventually arrive at the following expressions for $Ł_{\hat{f}_{A,Q}}(s_1,s_2)$, $Ł_{\hat{f}_{B,Q}}(s_1,s_2)$, and $Ł_{\hat{f}_{C,Q}}(s_1,s_2)$ respectively

$$\begin{aligned}
Ł_{\hat{f}_{A,Q}}(s_1,s_2)(s_1^2+(\lambda_{(1,2)}+\lambda_{(1,3)}-2is_2)s_1+(\lambda_{(1,2)}-is_2)(\lambda_{(1,3)}-is_2)) &= \\
Ł_{\hat{f}_{B,Q}}(s_1,s_2)\lambda_{(1,2)}(\lambda_{(1,3)}+s_1-is_2) &+ \\
Ł_{\hat{f}_{C,Q}}(s_1,s_2)\lambda_{(1,3)}(\lambda_{(1,2)}+s_1-is_2) &+ \\
\frac{\lambda_{(1,2)}\lambda_{(1,3)}}{\lambda_{(1,2)}+\lambda_{(1,3)}+s_1-is_2} &+ (1-(\lambda_{(1,2)}+\lambda_{(1,3)}))
\end{aligned}$$

$$\begin{aligned}
Ł_{\hat{f}_{B,Q}}(s_1,s_2)(s_1^2+(\lambda_{(2,3)}+\lambda_{(2,1)})s_1+\lambda_{(2,3)}\lambda_{(2,1)}) &= \\
Ł_{\hat{f}_{C,Q}}(s_1,s_2)\lambda_{(2,3)}(\lambda_{(2,1)}+s_1) &+ \\
Ł_{\hat{f}_{A,Q}}(s_1,s_2)\lambda_{(2,1)}(\lambda_{(2,3)}+s_1) &+ \\
\frac{\lambda_{(2,3)}\lambda_{(2,1)}}{\lambda_{(2,3)}+\lambda_{(2,1)}+s_1} &+ (1-(\lambda_{(2,3)}+\lambda_{(2,1)}))
\end{aligned}$$

$$\begin{aligned}
Ł_{\hat{f}_{C,Q}}(s_1,s_2)(s_1^2+(\lambda_{(3,1)}+\lambda_{(3,2)})s_1+\lambda_{(1,2)}\lambda_{(1,3)}) &= \\
Ł_{\hat{f}_{A,Q}}(s_1,s_2)\lambda_{(3,1)}(\lambda_{(3,2)}+s_1) &+ \\
Ł_{\hat{f}_{B,Q}}(s_1,s_2)\lambda_{(3,2)}(\lambda_{(3,1)}+s_1) &+ \\
\frac{\lambda_{(3,1)}\lambda_{(3,2)}}{\lambda_{(3,1)}+\lambda_{(3,2)}+s_1} &+ (1-(\lambda_{(3,1)}+\lambda_{(3,2)}))
\end{aligned}$$

The solution for $f_{A,Q}(t,x)$ is currently being worked on.

5 CALCULATIONS

5.1 Two-state Markov process with rates $\lambda = \mu$

We can write the following integral equation to link the two functions (See Preview of Results)

$$f_{A,Q}(t,x) = e^{-\lambda t}\delta_t(x) + \int_0^t f_{B,Q}(t-y, x-y)\lambda e^{-\lambda y}dy \tag{1}$$

But now we will set $\lambda = \mu$, and the equation becomes

$$f(t,x) = e^{-\lambda t}\delta_t(x) + \int_0^t f(t-y, t-x)\lambda e^{-\lambda y}dy \tag{2}$$

We will now use the following substitutions into the above equation

$$\psi = t - y$$

$$d\psi = -dy$$

These substitutions change (2) into

$$f(t,x) = e^{-\lambda t}\delta_t(x) + \lambda e^{-\lambda t}\int_0^t f(\psi, t-x)e^{-\lambda\psi}d\psi \tag{3}$$

Now we can take the *Fourier transform* of (3) with respect to x, and the equation becomes

$$\hat{f}(t,s_2) = \int_{-\infty}^{\infty}\delta_t(x)e^{-\lambda t}e^{is_2x}dx + \lambda e^{-\lambda t}\int_0^t\int_{-\infty}^{\infty} f(\psi, t-x)e^{\lambda\psi}e^{is_2x}dxd\psi \tag{4}$$

The next substitution we can make is

$$\chi = t - x$$

$$d\chi = -dx$$

which transforms (4) into

$$\hat{f}(t,s_2) = e^{-(\lambda - is_2)t} + \lambda e^{-\lambda t}\int_0^t\int_{-\infty}^{\infty} f(\psi,\chi)e^{\lambda\psi}e^{is_2(t-\chi)}d\chi d\psi \tag{5}$$

The next substitution we'll make will merely be one of convenience

$$a \equiv \lambda - is_2$$

$$\hat{f}(t, s_2) = e^{-at} + \lambda e^{-at} \int_0^t \hat{f}(\psi, -s_2) e^{\lambda\psi} d\psi \tag{6}$$

Multiplying both sides by e^{at}, we obtain the next expression

$$e^{at}\hat{f}(t, s_2) = 1 + \lambda \int_0^t \hat{f}(\psi, -s_2) e^{\lambda\psi} d\psi \tag{7}$$

Differentiating both sides of (7) with respect to t yields

$$\frac{\partial}{\partial t}(e^{at}\hat{f}(t, s_2)) = \lambda e^{\lambda t}\hat{f}(t, -s_2)$$

$$ae^{at}\hat{f}(t, s_2) + e^{at}\hat{f}_t(t, s_2) = \lambda e^{\lambda t}\hat{f}(t, -s_2)$$

Multiplying both sides by e^{-at} and plugging in for a leads to

$$(\lambda - is_2)\hat{f}(t, s_2) + \hat{f}_t(t, s_2) = \lambda e^{is_2 t}\hat{f}(t, -s_2) \tag{8}$$

Now taking the *Laplace transform* of both sides with respect to t leads to

$$(\lambda - is_2)Ł_{\hat{f}}(s_1, s_2) + s_1 Ł_{\hat{f}}(s_1, s_2) = \lambda Ł_{\hat{f}}(s_1 - is_2, -s_2) + \hat{f}(0, s_2) \tag{9}$$

But looking back at (2) we can see that $f(0, x) = \delta_0(x) \Rightarrow \hat{f}(0, s_2) = 1$ which leads to the following

$$Ł_{\hat{f}}(s_1, s_2) = \frac{\lambda}{\lambda + s_1 - is_2} Ł_{\hat{f}}(s_1 - is_2, -s_2) + \frac{1}{\lambda + s_1 - is_2} \tag{10}$$

However, by using (10) with the transformation $s_1 \to s_1 - is_2$ and $s_2 \to -s_2$, we can get an expression for $Ł_{\hat{f}}(s_1 - is_2, -s_2)$

$$Ł_{\hat{f}}(s_1 - is_2, -s_2) = \frac{\lambda}{\lambda + s_1} Ł_{\hat{f}}(s_1, s_2) + \frac{1}{\lambda + s_1}$$

Plugging the above equation into (10), we get the expression seen below

$$Ł_{\hat{f}}(s_1, s_2) = \frac{\lambda^2}{(\lambda + s_1 - is_2)(\lambda + s_1)} Ł_{\hat{f}}(s_1, s_2) + \frac{2\lambda + s_1}{(\lambda + s_1 - is_2)(\lambda + s_1)} \tag{11}$$

Solving (11) for $Ł_{\hat{f}}(s_1, s_2)$ leads to

$$Ł_{\hat{f}}(s_1, s_2) = \frac{2\lambda + s_1}{s_1(s_1 + 2\lambda) - i(s_1 + \lambda)s_2} \tag{12}$$

Now we will take the inverse *Fourier transform* of (12) with respect to s_2.

$$Ł(s_1, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(2\lambda + s_1)}{s_1(s_1 + 2\lambda) - i(s_1 + \lambda)s_2} e^{-is_2x} ds_2 \tag{13}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(2\lambda + s_1)ie^{-is_2x}}{(s_1 + \lambda)(s_2 + i\frac{s_1(s_1+2\lambda)}{s_1+\lambda})} ds_2 \tag{14}$$

Now we can make some substitutions to make our life easier.

$$\alpha \equiv \frac{s_1 + 2\lambda}{s_1 + \lambda} \quad \text{and} \quad \beta \equiv s_1\frac{s_1 + 2\lambda}{s_1 + \lambda} = s_1 \cdot \alpha$$

With these substitutions, (13) becomes

$$Ł(s_1, x) = \frac{i\alpha}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-is_2x}}{s_2 + i\beta} ds_2 \tag{15}$$

Now this is a complex integral with a simple pole at $s_2 = -i\beta$. Since this integral is already of the form $\frac{f(s_2)}{s_2-\kappa}$ where κ is the singularity, we can see that the integral we are after is just $2\pi i\mathbf{Res}(f(\kappa)) = -2\pi ie^{-x\beta}$. Which implies that

$$Ł(s_1, x) = \alpha e^{-x\beta}$$

Plugging back in for α and β gives us

$$Ł(s_1, x) = \frac{s_1 + 2\lambda}{s_1 + \lambda} e^{-xs_1(\frac{s_1+2\lambda}{s_1+\lambda})}$$

Which can be easily manipulated into the following

$$Ł(s_1, x) = (1 + \frac{\lambda}{s_1 + \lambda})e^{-x\frac{s_1^2+2\lambda s_1+\lambda^2-\lambda^2}{s_1+\lambda}}.$$

Expanding this gives us

$$Ł(s_1, x) = e^{-(s_1+\lambda)x}e^{\frac{\lambda^2 x}{s_1+\lambda}} + \frac{\lambda}{s_1 + \lambda}e^{-(\lambda+s_1)x}e^{\frac{\lambda^2 x}{s_1+\lambda}} \tag{16}$$

Now we notice the following *Laplace transforms*

$$\int_0^{\infty} I_0(2\sqrt{at})e^{-pt}dt = \frac{1}{p}e^{\frac{a}{p}}$$

and

$$\int_0^{\infty} \frac{1}{\sqrt{t}}I_1(2\sqrt{at})e^{-pt}dt = \frac{1}{\sqrt{a}}(e^{\frac{a}{p}} - 1),$$

where I_0 and I_1 are modified Bessel functions,

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{k!(k-1)!}(\frac{z}{2})^{2k} \quad , \qquad I_1(z) = \sum_{k=0}^{\infty} \frac{1}{k!k!}(\frac{z}{2})^{2k+1} \quad .$$

They can be modified as follows

$$e^{-px}\frac{1}{p}e^{\frac{a}{p}} = \int_x^{\infty} I_0(2\sqrt{a(t-x)})e^{-pt}dt$$

and

$$e^{-px}e^{\frac{a}{p}} = e^{-px} + \sqrt{a}\int_x^{\infty} \frac{1}{\sqrt{t-x}} I_1(2\sqrt{a(t-x)})e^{-pt}dt.$$

Let $a = \lambda^2 x$ and $p = s_1 + \lambda$. Plugging in, we get

$$\frac{1}{s_1+\lambda}e^{-(\lambda+s_1)x}e^{\frac{\lambda^2 x}{s_1+\lambda}} = e^{-px}\frac{1}{p}e^{\frac{a}{p}} = \int_x^{\infty} I_0(2\lambda\sqrt{x(t-x)})e^{-\lambda t}e^{-s_1 t}dt$$

and therefore the inverse *Laplace transform* of $\frac{\lambda}{s_1+\lambda}e^{-(\lambda+s_1)x}e^{\frac{\lambda^2 x}{s_1+\lambda}}$ is $\lambda e^{-\lambda t}I_0(2\lambda\sqrt{x(t-x)})$ for $0 \le x \le t$.

Similarly

$$e^{-(s_1+\lambda)x}e^{\frac{\lambda^2 x}{s_1+\lambda}} = e^{-px}e^{\frac{a}{p}} = e^{-\lambda x}e^{-s_1 x} + \lambda\sqrt{x}\int_x^{\infty} \frac{1}{\sqrt{t-x}} I_1(2\lambda\sqrt{x(t-x)})e^{-\lambda t}e^{-s_1 t}dt$$

which can be rewritten as

$$e^{-(s_1+\lambda)x}e^{\frac{\lambda^2 x}{s_1+\lambda}} = \int_0^{\infty} \delta_x(t)e^{-\lambda t}e^{-s_1 t}dt + \lambda\sqrt{x}\int_x^{\infty} \frac{1}{\sqrt{t-x}} I_1(2\lambda\sqrt{x(t-x)})e^{-\lambda t}e^{-s_1 t}dt.$$

Thus, the inverse *Laplace transform* of $e^{-(s_1+\lambda)x}e^{\frac{\lambda^2 x}{s_1+\lambda}}$ is $e^{-\lambda t}\delta_x(t)+\lambda\sqrt{x}\frac{1}{\sqrt{t-x}}e^{-\lambda t}I_1(2\lambda\sqrt{x(t-x)})$ for $0 \le x \le t$. Here we do not divide by zero when $x = t$ as the $\sqrt{t-x}$ cancels on top and the bottom.

Adding the terms together, we obtain

$$\boxed{f(t,x) = e^{-\lambda t}\delta_x(t) + \lambda e^{-\lambda t}I_0(2\lambda\sqrt{x(t-x)}) + \lambda\sqrt{x}\frac{1}{\sqrt{t-x}}e^{-\lambda t}I_1(2\lambda\sqrt{x(t-x)}).}$$

5.2 Two-state Markov process with rates $\lambda \neq \mu$

We can write the following set of integral equations:

$$\begin{aligned} f_{A,Q}(t,x) &= e^{-\lambda t}\delta_t(x) + \int_0^t f_{B,Q}(t-y,x-y)\lambda e^{-\lambda y}dy \\ f_{B,Q}(t,x) &= e^{-\mu t}\delta_0(x) + \int_0^t f_{A,Q}(t-y,x)\mu e^{-\mu y}dy \end{aligned}$$

Plugging in $\Psi = t - y$ into the top equation, with no substitution in the bottom equation, we obtain

$$\begin{aligned} f_{A,Q}(t,x) &= e^{-\lambda t}\delta_t(x) + \int_0^t f_{B,Q}(\Psi, x-t+\Psi)\lambda e^{-\lambda(t-\Psi)}d\Psi \\ f_{B,Q}(t,x) &= e^{-\mu t}\delta_0(x) + \int_0^t f_{A,Q}(\Psi, x)\mu e^{-\mu(t-\Psi)}d\Psi \end{aligned}$$

Now taking the Fourier transform with respect to x and plugging in $\chi = x - t + \Psi$ for the top equation only, we arrive at

$$\begin{aligned} \hat{f}_{A,Q}(t,s_2) &= e^{-(\lambda - is_2)t} + \lambda e^{-(\lambda - is_2)t}\int_0^t \hat{f}_{B,Q}(\Psi, s_2)e^{(\lambda - is_2)\Psi}d\Psi \\ \hat{f}_{B,Q}(t,s_2) &= e^{-\mu t} + \mu e^{-\mu t}\int_0^t \hat{f}_{A,Q}(\Psi, s_2)e^{\mu\Psi}d\Psi \end{aligned}$$

Making the quick substitution $a_1 \equiv \lambda - is_2$, then multiplying both sides by $e^{a_1 t}$ and $e^{\mu t}$ respectively, we arrive at

$$\begin{aligned} e^{a_1 t}\hat{f}_{A,Q}(t,s_2) &= 1 + \lambda\int_0^t \hat{f}_{B,Q}(\Psi, s_2)e^{a_1\Psi}d\Psi \\ e^{\mu t}\hat{f}_{B,Q}(t,s_2) &= 1 + \mu\int_0^t \hat{f}_{A,Q}(\Psi, s_2)e^{\mu\Psi}d\Psi \end{aligned}$$

Now we will differentiate both sides with respect to t

$$\begin{aligned} a_1 e^{a_1 t}\hat{f}_{A,Q}(t,s_2) + e^{a_1 t}\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2) &= \lambda e^{a_1 t}\hat{f}_{B,Q}(t,s_2) \\ \mu e^{\mu t}\hat{f}_{B,Q}(t,s_2) + e^{\mu t}\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2) &= \mu e^{\mu t}\hat{f}_{A,Q}(t,s_2) \end{aligned}$$

Now multiply both sides by $e^{-a_1 t}$ and $e^{-\mu t}$ respectively, then take the Laplace transform of both sides with respect to t, which will give you

$$\begin{aligned} a_1 Ł_{\hat{f}_{A,Q}}(s_1,s_2) + s_1 Ł_{\hat{f}_{A,Q}}(s_1,s_2) &= \lambda Ł_{\hat{f}_{B,Q}}(s_1,s_2) + \hat{f}_{A,Q}(0,s_2) \\ \mu Ł_{\hat{f}_{B,Q}}(s_1,s_2) + s_1 Ł_{\hat{f}_{B,Q}}(s_1,s_2) &= \mu Ł_{\hat{f}_{A,Q}}(s_1,s_2) + \hat{f}_{B,Q}(0,s_2) \end{aligned}$$

But $\hat{f}_{A,Q}(0,s_2) = \hat{f}_{B,Q}(0,s_2) = 1$ which leads to

$$Ł_{\hat{f}_{A,Q}}(s_1,s_2)(\lambda + s_1 - is_2) = \lambda Ł_{\hat{f}_{B,Q}}(s_1,s_2) + 1 \quad (17)$$

$$Ł_{\hat{f}_{B,Q}}(s_1,s_2)(\mu + s_1) = \mu Ł_{\hat{f}_{A,Q}}(s_1,s_2) + 1 \quad (18)$$

Substitution yields the following

$$Ł_{\hat{f}_{A,Q}}(s_1,s_2) = \frac{\lambda\mu}{(\lambda+s_1-is_2)(\mu+s_1)}Ł_{\hat{f}_{A,Q}}(s_1,s_2) + \frac{\lambda+\mu+s_1}{(\lambda+s_1-is_2)(\mu+s_1)}$$
$$Ł_{\hat{f}_{A,Q}}(s_1,s_2) = \frac{(s_1+\lambda+\mu)i}{(s_1+\mu)(s_2+i\frac{s_1(s_1+\lambda+\mu)}{s_1+\mu})}$$

Now computing the inverse Fourier transform via the method of residues we obtain

$$Ł(s_1,x) = \alpha e^{-x\beta}$$

where $\alpha = \frac{s_1+\lambda+\mu}{s_1+\mu}$ and $\beta = s_1\alpha$

$$\begin{aligned} Ł(s_1,x) &= (1+\tfrac{\lambda}{s_1+\mu})e^{-x\frac{(s_1+\mu)(s_1+\lambda)-\mu\lambda}{s_1+\mu}} \\ &= e^{-x(s_1+\lambda)}e^{\frac{\lambda\mu x}{s_1+\mu}} + \tfrac{\lambda}{s_1+\mu}e^{-x(s_1+\lambda)}e^{\frac{\lambda\mu x}{s_1+\mu}} \end{aligned}$$

Again we take notice of the Laplace transforms of the modified Bessel functions $I_0(z)$ and $I_1(z)$.

However, using a similar manipulation to that used in the above case where $\lambda = \mu$, we notice the following important results:

$$\int_x^\infty I_0(2\sqrt{a(t-x)})e^{-\lambda x}e^{-\mu(t-x)}e^{-s_1 t}dt = \frac{e^{-(s_1+\lambda)x}}{s_1+\mu}e^{\frac{a}{s_1+\mu}}$$

Thus, the inverse *Laplace transform* of $\frac{\lambda}{s_1+\mu}e^{-x(s_1+\lambda)}e^{\frac{\lambda\mu x}{s_1+\mu}}$ is $\lambda e^{-\lambda x}e^{-\mu(t-x)}I_0(2\sqrt{\lambda\mu x(t-x)})$ for $0 \leq x \leq t$.

Similar manipulations will show that

$$e^{-x(s_1+\lambda)}e^{\frac{\lambda\mu x}{s_1+\mu}} = \sqrt{\lambda\mu x}\int_x^\infty \frac{I_1(2\sqrt{\lambda\mu x(t-x)})}{\sqrt{t-x}}e^{-\lambda x}e^{-\mu(t-x)}e^{-s_1 t}dt + \int_0^\infty e^{-\lambda t}\delta_t(x)e^{-s_1 t}dt$$

Thus, the inverse *Laplace transform* of $e^{-x(s_1+\lambda)}e^{\frac{\lambda\mu x}{s_1+\mu}}$ is $\sqrt{\frac{\lambda\mu x}{t-x}}I_1(2\sqrt{\lambda\mu x(t-x)})e^{-\lambda x}e^{-\mu(t-x)} + e^{-\lambda t}\delta_t(x)$

Which finally leads to

$$\boxed{f_{A,\lambda,\mu}(t,x) = e^{-\lambda t}\delta_t(x) + \lambda e^{-\lambda x}e^{-\mu(t-x)}I_0(2\sqrt{\lambda\mu x(t-x)}) + \sqrt{\frac{\lambda\mu x}{t-x}}I_1(2\sqrt{\lambda\mu x(t-x)})e^{-\lambda x}e^{-\mu(t-x)}}$$

5.3 Three-state Markov process with equal rates

For the three-state *Markov process* with equal rates, we can write the following set of integral equations:

$$\begin{aligned}
f_{A,Q}(t,x) &= e^{-2\lambda t}\delta_t(x) + \int_0^t f_{B,Q}(t-y,x-y)\lambda e^{-\lambda y}dy \\
&\quad + \int_0^t f_{C,Q}(t-y,x-y)\lambda e^{-\lambda y}dy \\
f_{B,Q}(t,x) &= e^{-2\lambda t}\delta_0(x) + \int_0^t f_{C,Q}(t-y,x)\lambda e^{-\lambda y}dy \\
&\quad + \int_0^t f_{A,Q}(t-y,x)\lambda e^{-\lambda y}dy \\
f_{C,Q}(t,x) &= e^{-2\lambda t}\delta_0(x) + \int_0^t f_{A,Q}(t-y,x)\lambda e^{-\lambda y}dy \\
&\quad + \int_0^t f_{B,Q}(t-y,x)\lambda e^{-\lambda y}dy
\end{aligned}$$

Plugging in $\Psi = t - y$ into the top equation we obtain

$$\begin{aligned}
f_{A,Q}(t,x) &= e^{-2\lambda t}\delta_t(x) + \int_0^t f_{B,Q}(\Psi,x-t+\Psi)\lambda e^{-\lambda(t-\Psi)}d\Psi \\
&\quad + \int_0^t f_{C,Q}(\Psi,x-t+\Psi)\lambda e^{-\lambda(t-\Psi)}d\Psi \\
f_{B,Q}(t,x) &= e^{-2\lambda t}\delta_0(x) + \int_0^t f_{C,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi \\
&\quad + \int_0^t f_{A,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi \\
f_{C,Q}(t,x) &= e^{-2\lambda t}\delta_0(x) + \int_0^t f_{A,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi \\
&\quad + \int_0^t f_{B,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi
\end{aligned}$$

Now taking the Fourier transform with respect to x we arrive at

$$\begin{aligned}
\hat{f}_{A,Q}(t,s_2) &= e^{-(2\lambda - is_2)t} + \int_{-\infty}^{\infty}\int_0^t f_{B,Q}(\Psi,x-t+\Psi)\lambda e^{-\lambda(t-\Psi)}d\Psi e^{is_2x}dx \\
&\quad + \int_{-\infty}^{\infty}\int_0^t f_{C,Q}(\Psi,x-t+\Psi)\lambda e^{-\lambda(t-\Psi)}d\Psi e^{is_2x}dx \\
\hat{f}_{B,Q}(t,s_2) &= e^{-2\lambda t} + \int_{-\infty}^{\infty}\int_0^t f_{C,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi e^{is_2x}dx \\
&\quad + \int_{-\infty}^{\infty}\int_0^t f_{A,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi e^{is_2x}dx \\
\hat{f}_{C,Q}(t,s_2) &= e^{-2\lambda t} + \int_{-\infty}^{\infty}\int_0^t f_{A,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi e^{is_2x}dx \\
&\quad + \int_{-\infty}^{\infty}\int_0^t f_{B,Q}(\Psi,x)\lambda e^{-\lambda(t-\Psi)}d\Psi e^{is_2x}dx
\end{aligned}$$

Plugging in $\chi = x - t + \Psi$ into the top equation only yields

$$\begin{array}{rcccl}
\hat{f}_{A,Q}(t,s_2) & = & e^{-(2\lambda - is_2)t} & + & \int_0^t \int_{-\infty}^{\infty} f_{B,Q}(\Psi,\chi)\lambda e^{is_2(\chi+t-\Psi)} e^{-\lambda(t-\Psi)} d\chi d\Psi \\
 & & & + & \int_0^t \int_{-\infty}^{\infty} f_{C,Q}(\Psi,\chi)\lambda e^{is_2(\chi+t-\Psi)} e^{-\lambda(t-\Psi)} d\chi d\Psi \\
\hat{f}_{B,Q}(t,s_2) & = & e^{-2\lambda t} & + & \int_0^t \int_{-\infty}^{\infty} f_{C,Q}(\Psi,x)\lambda e^{is_2 x} e^{-\lambda(t-\Psi)} dx d\Psi \\
 & & & + & \int_0^t \int_{-\infty}^{\infty} f_{A,Q}(\Psi,x)\lambda e^{is_2 x} e^{-\lambda(t-\Psi)} dx d\Psi \\
\hat{f}_{C,Q}(t,s_2) & = & e^{-2\lambda t} & + & \int_0^t \int_{-\infty}^{\infty} f_{A,Q}(\Psi,x)\lambda e^{is_2 x} e^{-\lambda(t-\Psi)} dx d\Psi \\
 & & & + & \int_0^t \int_{-\infty}^{\infty} f_{B,Q}(\Psi,x)\lambda e^{is_2 x} e^{-\lambda(t-\Psi)} dx d\Psi
\end{array}$$

Which simplifies to

$$\begin{array}{rcccl}
\hat{f}_{A,Q}(t,s_2) & = & e^{-(2\lambda - is_2)t} & + & \lambda e^{-(\lambda - is_2)t} \int_0^t \hat{f}_{B,Q}(\Psi,s_2) e^{(\lambda - is_2)\Psi} d\Psi \\
 & & & + & \lambda e^{-(\lambda - is_2)t} \int_0^t \hat{f}_{C,Q}(\Psi,s_2) e^{(\lambda - is_2)\Psi} d\Psi \\
\hat{f}_{B,Q}(t,x) & = & e^{-2\lambda t} & + & \lambda e^{-\lambda t} \int_0^t \hat{f}_{C,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi \\
 & & & + & \lambda e^{-\lambda t} \int_0^t \hat{f}_{A,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi \\
\hat{f}_{C,Q}(t,x) & = & e^{-2\lambda t} & + & \lambda e^{-\lambda t} \int_0^t \hat{f}_{A,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi \\
 & & & + & \lambda e^{-\lambda t} \int_0^t \hat{f}_{B,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi
\end{array}$$

Multiplying both sides of the top equation by $e^{(\lambda - is_2)t}$ and the bottom two equations by $e^{\lambda t}$ yields the following

$$\begin{array}{rcccl}
e^{(\lambda - is_2)t} \hat{f}_{A,Q}(t,s_2) & = & e^{-\lambda t} & + & \lambda \int_0^t \hat{f}_{B,Q}(\Psi,s_2) e^{(\lambda - is_2)\Psi} d\Psi \\
 & & & + & \lambda \int_0^t \hat{f}_{C,Q}(\Psi,s_2) e^{(\lambda - is_2)\Psi} d\Psi \\
e^{\lambda t} \hat{f}_{B,Q}(t,s_2) & = & e^{-\lambda t} & + & \lambda \int_0^t \hat{f}_{C,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi \\
 & & & + & \lambda \int_0^t \hat{f}_{A,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi \\
e^{\lambda t} \hat{f}_{C,Q}(t,s_2) & = & e^{-\lambda t} & + & \lambda \int_0^t \hat{f}_{A,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi \\
 & & & + & \lambda \int_0^t \hat{f}_{B,Q}(\Psi,s_2) e^{\lambda\Psi} d\Psi
\end{array}$$

Now differentiating with respect to t yields

$$
\begin{array}{cccccc}
(\lambda - is_2)e^{(\lambda-is_2)t}\hat{f}_{A,Q}(t,s_2) & + & e^{(\lambda-is_2)t}\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2) & = & & \\
-\lambda e^{-\lambda t} & + & \lambda e^{(\lambda-is_2)t}\hat{f}_{B,Q}(t,s_2) & + & \lambda e^{(\lambda-is_2)t}\hat{f}_{C,Q}(t,s_2)
\end{array}
$$

$$
\begin{array}{cccccc}
\lambda e^{\lambda t}\hat{f}_{B,Q}(t,s_2) & + & e^{\lambda t}\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2) & = & & \\
-\lambda e^{-\lambda t} & + & \lambda e^{\lambda t}\hat{f}_{C,Q}(t,s_2) & + & \lambda e^{\lambda t}\hat{f}_{A,Q}(t,s_2)
\end{array}
$$

$$
\begin{array}{cccccc}
\lambda e^{\lambda t}\hat{f}_{C,Q}(t,s_2) & + & e^{\lambda t}\frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2) & = & & \\
-\lambda e^{-\lambda t} & + & \lambda e^{\lambda t}\hat{f}_{A,Q}(t,s_2) & + & \lambda e^{\lambda t}\hat{f}_{B,Q}(t,s_2)
\end{array}
$$

Next we multiply both sides of the top equation by $e^{-(\lambda-is_2)t}$ and the bottom two equations by $e^{-\lambda t}$

$$
\begin{array}{ccccccccc}
(\lambda - is_2)\hat{f}_{A,Q}(t,s_2) & + & \frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2) & = & -\lambda e^{-(2\lambda-is_2)t} & + & \lambda\hat{f}_{B,Q}(t,s_2) & + & \lambda\hat{f}_{C,Q}(t,s_2) \\
\lambda\hat{f}_{B,Q}(t,s_2) & + & \frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2) & = & -\lambda e^{-2\lambda t} & + & \lambda\hat{f}_{C,Q}(t,s_2) & + & \lambda\hat{f}_{A,Q}(t,s_2) \\
\lambda\hat{f}_{C,Q}(t,s_2) & + & \frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2) & = & -\lambda e^{-2\lambda t} & + & \lambda\hat{f}_{A,Q}(t,s_2) & + & \lambda\hat{f}_{B,Q}(t,s_2)
\end{array}
$$

Taking the *Laplace transform* w.r.t. t leads to the following expressions

$$
\begin{array}{cccccccc}
(\lambda + s_1 - is_2)\text{Ł}_{\hat{f}_{A,Q}}(s_1,s_2) & - & \lambda\text{Ł}_{\hat{f}_{B,Q}}(s_1,s_2) & - & \lambda\text{Ł}_{\hat{f}_{C,Q}}(s_1,s_2) & = & \frac{s_1+\lambda-is_2}{s_1+2\lambda-is_2} \\
-\lambda\text{Ł}_{\hat{f}_{A,Q}}(s_1,s_2) & + & (\lambda+s_1)\text{Ł}_{\hat{f}_{B,Q}}(s_1,s_2) & - & \lambda\text{Ł}_{\hat{f}_{C,Q}}(s_1,s_2) & = & \frac{s_1+\lambda}{s_1+2\lambda} \\
-\lambda\text{Ł}_{\hat{f}_{A,Q}}(s_1,s_2) & - & \lambda\text{Ł}_{\hat{f}_{B,Q}}(s_1,s_2) & + & (\lambda+s_1)\text{Ł}_{\hat{f}_{C,Q}}(s_1,s_2) & = & \frac{s_1+\lambda}{s_1+2\lambda}
\end{array}
$$

Using *Cramer's rule* gives us the following expression for $Ł_{\hat{f}_{A,Q}}(s_1, s_2)$

$$\frac{\det\begin{pmatrix} \frac{s_1+\lambda-is_2}{s_1+2\lambda-is_2} & -\lambda & -\lambda \\ \frac{s_1+\lambda}{s_1+2\lambda} & s_1+\lambda & -\lambda \\ \frac{s_1+\lambda}{s_1+2\lambda} & -\lambda & s_1+\lambda \end{pmatrix}}{\det\begin{pmatrix} s_1+\lambda-is_2 & -\lambda & -\lambda \\ -\lambda & s_1+\lambda & -\lambda \\ -\lambda & -\lambda & s_1+\lambda \end{pmatrix}}$$

$$Ł_{\hat{f}_{A,Q}}(s_1, s_2) = \frac{\frac{s_1+\lambda-is_2}{s_1+2\lambda-is_2}((s_1+\lambda)^2-\lambda^2)+2\lambda(s_1+\lambda)}{(s_1+\lambda-is_2)[(s_1+\lambda)^2-\lambda^2]-2\lambda[\lambda^2+\lambda(s_1+\lambda)]}$$

Which simplifies to

$$Ł_{\hat{f}_{A,Q}}(s_1, s_2) = \frac{(s_1+2\lambda)(s_2+i(s_1+\lambda))i}{s_1[s_2+i(s_1+2\lambda)][s_2+i\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}]}$$

This has two poles at $s_2 = -i(s_1+\lambda)$ and $s_2 = -i\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}$

The inverse *Fourier transform* is then given by

$$Ł_{f_{A,Q}}(s_1, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(s_1+2\lambda)(s_2+i(s_1+\lambda))i}{s_1[s_2+i(s_1+2\lambda)][s_2+i\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}]} e^{-is_2x} ds_2$$

Again using the method of residues leads to

$$Ł_{f_{A,Q}}(s_1, x) = -i\left[\frac{(s_1+2\lambda)(-i(s_1+2\lambda)+i(s_1+\lambda))i}{s_1[-i(s_1+2\lambda)+i\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}]}e^{-ix(-i(s_1+2\lambda))} + \frac{(s_1+2\lambda)(-i\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}+i(s_1+\lambda))i}{s_1[-i\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}+i(s_1+2\lambda)]}e^{-is_2x}\right]$$

Which simplifies to

$$Ł_{f_{A,Q}}(s_1, x) = e^{-x(s_1+2\lambda)} + \frac{2\lambda}{s_1}e^{-x\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}}$$

The first term, $e^{-x(s_1+2\lambda)}$, can also be written as

$$e^{-x(s_1+2\lambda)} = \int_0^{\infty} e^{-2\lambda t}\delta_t(x)e^{-s_1t}dt$$

Which implies that the inverse *Laplace transform* of $e^{-x(s_1+2\lambda)}$ is $e^{-2\lambda t}\delta_t(x)$.

The second term, $\frac{2\lambda}{s_1}e^{-x\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}}$, can also be written as

$$\frac{2\lambda}{s_1}e^{-x\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}} = 2\lambda e^{-\lambda x}[\frac{1}{s_1}e^{-xs_1}e^{\frac{2\lambda^2 x}{s_1}}] = \int_x^\infty 2\lambda e^{-\lambda x} I_0(2\sqrt{2\lambda^2 x(t-x)})e^{-s_1 t}dt$$

Which implies that the inverse *Laplace transform* of $\frac{2\lambda}{s_1}e^{-x\frac{(s_1+2\lambda)(s_1-\lambda)}{s_1}}$ is $2\lambda e^{-\lambda x} I_0(2\sqrt{2\lambda^2 x(t-x)})$.

Thus, for the 3-state *Markov process*,

$$\boxed{f_{A,Q}(t,x) = e^{-2\lambda t}\delta_t(x) + 2\lambda e^{-\lambda x} I_0(2\sqrt{2\lambda^2 x(t-x)})}$$

5.4 Three-state Markov process with general rates

For the three-state *Markov process* with general rates, we can write the following set of integral equations:

$$\begin{aligned}
f_{A,Q}(t,x) &= e^{-(\lambda_{(1,2)}+\lambda_{(1,3)})t}\delta_t(x) + \int_0^t f_{B,Q}(t-y,x-y)\lambda_{(1,2)}e^{-\lambda_{(1,2)}y}dy \\
&\quad + \int_0^t f_{C,Q}(t-y,x-y)\lambda_{(1,3)}e^{-\lambda_{(1,3)}y}dy \\
f_{B,Q}(t,x) &= e^{-(\lambda_{(2,3)}+\lambda_{(2,1)})t}\delta_0(x) + \int_0^t f_{C,Q}(t-y,x)\lambda_{(2,3)}e^{-\lambda_{(2,3)}y}dy \\
&\quad + \int_0^t f_{A,Q}(t-y,x)\lambda_{(2,1)}e^{-\lambda_{(2,1)}y}dy \\
f_{C,Q}(t,x) &= e^{-(\lambda_{(3,1)}+\lambda_{(3,2)})t}\delta_0(x) + \int_0^t f_{A,Q}(t-y,x)\lambda_{(3,1)}e^{-\lambda_{(3,1)}y}dy \\
&\quad + \int_0^t f_{B,Q}(t-y,x)\lambda_{(3,2)}e^{-\lambda_{(3,2)}y}dy
\end{aligned}$$

Plugging in $\Psi = t - y$ into the top equation we obtain

$$\begin{aligned}
f_{A,Q}(t,x) &= e^{-(\lambda_{(1,2)}+\lambda_{(1,3)})t}\delta_t(x) + \int_0^t f_{B,Q}(\Psi,x-t+\Psi)\lambda_{(1,2)}e^{-\lambda_{(1,2)}(t-\Psi)}d\Psi \\
&\quad + \int_0^t f_{C,Q}(\Psi,x-t+\Psi)\lambda_{(1,3)}e^{-\lambda_{(1,3)}(t-\Psi)}d\Psi \\
f_{B,Q}(t,x) &= e^{-(\lambda_{(2,3)}+\lambda_{(2,1)})t}\delta_0(x) + \int_0^t f_{C,Q}(\Psi,x)\lambda_{(2,3)}e^{-\lambda_{(2,3)}(t-\Psi)}d\Psi \\
&\quad + \int_0^t f_{A,Q}(\Psi,x)\lambda_{(2,1)}e^{-\lambda_{(2,1)}(t-\Psi)}d\Psi \\
f_{C,Q}(t,x) &= e^{-(\lambda_{(3,1)}+\lambda_{(3,2)})t}\delta_0(x) + \int_0^t f_{A,Q}(\Psi,x)\lambda_{(3,1)}e^{-\lambda_{(3,1)}(t-\Psi)}d\Psi \\
&\quad + \int_0^t f_{B,Q}(\Psi,x)\lambda_{(3,2)}e^{-\lambda_{(3,2)}(t-\Psi)}d\Psi
\end{aligned}$$

Now taking the Fourier transform with respect to x we arrive at

$$\begin{aligned}
\hat{f}_{A,Q}(t,s_2) &= e^{-(\lambda_{(1,2)}+\lambda_{(1,3)}-is_2)t} &+& \int_{-\infty}^{\infty}\int_0^t f_{B,Q}(\Psi,x-t+\Psi)\lambda_{(1,2)}e^{-\lambda_{(1,2)}(t-\Psi)}d\Psi e^{is_2x}dx \\
& &+& \int_{-\infty}^{\infty}\int_0^t f_{C,Q}(\Psi,x-t+\Psi)\lambda_{(1,3)}e^{-\lambda_{(1,3)}(t-\Psi)}d\Psi e^{is_2x}dx \\
\hat{f}_{B,Q}(t,s_2) &= e^{-(\lambda_{(2,3)}+\lambda_{(2,1)})t} &+& \int_{-\infty}^{\infty}\int_0^t f_{C,Q}(\Psi,x)\lambda_{(2,3)}e^{-\lambda_{(2,3)}(t-\Psi)}d\Psi e^{is_2x}dx \\
& &+& \int_{-\infty}^{\infty}\int_0^t f_{A,Q}(\Psi,x)\lambda_{(2,1)}e^{-\lambda_{(2,1)}(t-\Psi)}d\Psi e^{is_2x}dx \\
\hat{f}_{C,Q}(t,s_2) &= e^{-(\lambda_{(3,1)}+\lambda_{(3,2)})t} &+& \int_{-\infty}^{\infty}\int_0^t f_{A,Q}(\Psi,x)\lambda_{(3,1)}e^{-\lambda_{(3,1)}(t-\Psi)}d\Psi e^{is_2x}dx \\
& &+& \int_{-\infty}^{\infty}\int_0^t f_{B,Q}(\Psi,x)\lambda_{(3,2)}e^{-\lambda_{(3,2)}(t-\Psi)}d\Psi e^{is_2x}dx
\end{aligned}$$

Plugging in $\chi = x - t + \Psi$ into the top equation only yields

$$\begin{aligned}
\hat{f}_{A,Q}(t,s_2) &= e^{-(\lambda_{(1,2)}+\lambda_{(1,3)}-is_2)t} &+& \int_0^t\int_{-\infty}^{\infty} f_{B,Q}(\Psi,\chi)\lambda_{(1,2)}e^{is_2(\chi+t-\Psi)}e^{-\lambda_{(1,2)}(t-\Psi)}d\chi d\Psi \\
& &+& \int_0^t\int_{-\infty}^{\infty} f_{C,Q}(\Psi,\chi)\lambda_{(1,3)}e^{is_2(\chi+t-\Psi)}e^{-\lambda_{(1,3)}(t-\Psi)}d\chi d\Psi \\
\hat{f}_{B,Q}(t,s_2) &= e^{-(\lambda_{(2,3)}+\lambda_{(2,1)})t} &+& \int_0^t\int_{-\infty}^{\infty} f_{C,Q}(\Psi,x)\lambda_{(2,3)}e^{is_2x}e^{-\lambda_{(2,3)}(t-\Psi)}dxd\Psi \\
& &+& \int_0^t\int_{-\infty}^{\infty} f_{A,Q}(\Psi,x)\lambda_{(2,1)}e^{is_2x}e^{-\lambda_{(2,1)}(t-\Psi)}dxd\Psi \\
\hat{f}_{C,Q}(t,s_2) &= e^{-(\lambda_{(3,1)}+\lambda_{(3,2)})t} &+& \int_0^t\int_{-\infty}^{\infty} f_{A,Q}(\Psi,x)\lambda_{(3,1)}e^{is_2x}e^{-\lambda_{(3,1)}(t-\Psi)}dxd\Psi \\
& &+& \int_0^t\int_{-\infty}^{\infty} f_{B,Q}(\Psi,x)\lambda_{(3,2)}e^{is_2x}e^{-\lambda_{(3,2)}(t-\Psi)}dxd\Psi
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\hat{f}_{A,Q}(t,s_2) &= e^{-(\lambda_{(1,2)}+\lambda_{(1,3)}-is_2)t} &+& \lambda_{(1,2)}e^{-(\lambda_{(1,2)}-is_2)t}\int_0^t \hat{f}_{B,Q}(\Psi,s_2)e^{(\lambda_{(1,2)}-is_2)\Psi}d\Psi \\
& &+& \lambda_{(1,3)}e^{-(\lambda_{(1,3)}-is_2)t}\int_0^t \hat{f}_{C,Q}(\Psi,s_2)e^{(\lambda_{(1,3)}-is_2)\Psi}d\Psi \\
\hat{f}_{B,Q}(t,x) &= e^{-(\lambda_{(2,3)}+\lambda_{(2,1)})t} &+& \lambda_{(2,3)}e^{-\lambda_{(2,3)}t}\int_0^t \hat{f}_{C,Q}(\Psi,s_2)e^{\lambda_{(2,3)}\Psi}d\Psi \\
& &+& \lambda_{(2,1)}e^{-\lambda_{(2,1)}t}\int_0^t \hat{f}_{A,Q}(\Psi,s_2)e^{\lambda_{(2,1)}\Psi}d\Psi \\
\hat{f}_{C,Q}(t,x) &= e^{-(\lambda_{(3,1)}+\lambda_{(3,2)})t} &+& \lambda_{(3,1)}e^{-\lambda_{(3,1)}t}\int_0^t \hat{f}_{A,Q}(\Psi,s_2)e^{\lambda_{(3,1)}\Psi}d\Psi \\
& &+& \lambda_{(3,1)}e^{-\lambda_{(3,2)}t}\int_0^t \hat{f}_{B,Q}(\Psi,s_2)e^{\lambda_{(3,2)}\Psi}d\Psi
\end{aligned}$$

Multiplying both sides by $e^{(\lambda_{(1,2)}-is_2)t}$, $e^{\lambda_{(2,3)}t}$, and $e^{\lambda_{(3,1)}t}$ respectively yields the following

$$\begin{aligned}
e^{(\lambda_{(1,2)}-is_2)t}\hat{f}_{A,Q}(t,s_2) &= e^{-\lambda_{(1,3)}t} + \lambda_{(1,2)}\int_0^t \hat{f}_{B,Q}(\Psi,s_2)e^{(\lambda_{(1,2)}-is_2)\Psi}d\Psi \\
&+ \lambda_{(1,3)}e^{(\lambda_{(1,2)}-\lambda_{(1,3)})t}\int_0^t \hat{f}_{C,Q}(\Psi,s_2)e^{(\lambda_{(1,3)}-is_2)\Psi}d\Psi \\
e^{\lambda_{(2,3)}t}\hat{f}_{B,Q}(t,s_2) &= e^{-\lambda_{(2,1)}t} + \lambda_{(2,3)}\int_0^t \hat{f}_{C,Q}(\Psi,s_2)e^{\lambda_{(2,3)}\Psi}d\Psi \\
&+ \lambda_{(2,1)}e^{(\lambda_{(2,3)}-\lambda_{(2,1)})t}\int_0^t \hat{f}_{A,Q}(\Psi,s_2)e^{\lambda_{(2,1)}\Psi}d\Psi \\
e^{\lambda_{(3,1)}t}\hat{f}_{C,Q}(t,s_2) &= e^{-\lambda_{(3,2)}t} + \lambda_{(3,1)}\int_0^t \hat{f}_{A,Q}(\Psi,s_2)e^{\lambda_{(3,1)}\Psi}d\Psi \\
&+ \lambda_{(3,2)}e^{(\lambda_{(3,1)}-\lambda_{(3,2)})t}\int_0^t \hat{f}_{B,Q}(\Psi,s_2)e^{\lambda_{(3,2)}\Psi}d\Psi
\end{aligned}$$

Now differentiating with respect to t yields

$$\begin{aligned}
(\lambda_{(1,2)}-is_2)e^{(\lambda_{(1,2)}-is_2)t}\hat{f}_{A,Q}(t,s_2) &+ e^{(\lambda_{(1,2)}-is_2)t}\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2) = -\lambda_{(1,3)}e^{-\lambda_{(1,3)}t} + \\
\lambda_{(1,2)}e^{(\lambda_{(1,2)}-is_2)t}\hat{f}_{B,Q}(t,s_2) &+ \\
\lambda_{(1,3)}e^{(\lambda_{(1,2)}-is_2)t}\hat{f}_{C,Q}(t,s_2) &+ \lambda_{(1,3)}(\lambda_{(1,2)}-\lambda_{(1,3)})e^{(\lambda_{(1,2)}-\lambda_{(1,3)})t}\int_0^t \hat{f}_{C,Q}(\Psi,s_2)e^{(\lambda_{(1,3)}-is_2)\Psi}d\Psi
\end{aligned}$$

$$\begin{aligned}
\lambda_{(2,3)}e^{\lambda_{(2,3)}t}\hat{f}_{B,Q}(t,s_2) &+ e^{\lambda_{(2,3)}t}\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2) = -\lambda_{(2,1)}e^{-\lambda_{(2,1)}t} + \\
\lambda_{(2,3)}e^{\lambda_{(2,3)}t}\hat{f}_{C,Q}(t,s_2) &+ \\
\lambda_{(2,1)}e^{\lambda_{(2,3)}t}\hat{f}_{A,Q}(t,s_2) &+ \lambda_{(2,1)}(\lambda_{(2,3)}-\lambda_{(2,1)})e^{(\lambda_{(2,3)}-\lambda_{(2,1)})t}\int_0^t \hat{f}_{A,Q}(\Psi,s_2)e^{\lambda_{(2,1)}\Psi}d\Psi
\end{aligned}$$

$$\begin{aligned}
\lambda_{(3,1)}e^{\lambda_{(3,1)}t}\hat{f}_{C,Q}(t,s_2) &+ e^{\lambda_{(3,1)}t}\frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2) = -\lambda_{(3,2)}e^{-\lambda_{(3,2)}t} + \\
\lambda_{(3,1)}e^{\lambda_{(3,1)}t}\hat{f}_{A,Q}(t,s_2) &+ \\
\lambda_{(3,2)}e^{\lambda_{(3,1)}t}\hat{f}_{B,Q}(t,s_2) &+ \lambda_{(3,2)}(\lambda_{(3,1)}-\lambda_{(3,2)})e^{(\lambda_{(3,1)}-\lambda_{(3,2)})t}\int_0^t \hat{f}_{B,Q}(\Psi,s_2)e^{\lambda_{(3,2)}\Psi}d\Psi
\end{aligned}$$

Next we multiply both sides by $e^{(\lambda_{(1,3)}-\lambda_{(1,2)})t}$, $e^{(\lambda_{(2,1)}-\lambda_{(2,3)})t}$, and $e^{(\lambda_{(3,2)}-\lambda_{(3,1)})t}$ respectively

$$\begin{aligned}
(\lambda_{(1,2)} - is_2)e^{(\lambda_{(1,3)}-is_2)t}\hat{f}_{A,Q}(t,s_2) &+ e^{(\lambda_{(1,3)}-is_2)t}\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2) = -\lambda_{(1,3)}e^{-\lambda_{(1,2)}t}+\\
\lambda_{(1,2)}e^{(\lambda_{(1,3)}-is_2)t}\hat{f}_{B,Q}(t,s_2) &+\\
\lambda_{(1,3)}e^{(\lambda_{(1,3)}-is_2)t}\hat{f}_{C,Q}(t,s_2) &+ \lambda_{(1,3)}(\lambda_{(1,2)} - \lambda_{(1,3)})\int_0^t \hat{f}_{C,Q}(\Psi,s_2)e^{(\lambda_{(1,3)}-is_2)\Psi}d\Psi
\end{aligned}$$

$$\begin{aligned}
\lambda_{(2,3)}e^{\lambda_{(2,1)}t}\hat{f}_{B,Q}(t,s_2) &+ e^{\lambda_{(2,1)}t}\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2) = -\lambda_{(2,1)}e^{-\lambda_{(2,3)}t}+\\
\lambda_{(2,3)}e^{\lambda_{(2,1)}t}\hat{f}_{C,Q}(t,s_2) &+\\
\lambda_{(2,1)}e^{\lambda_{(2,1)}t}\hat{f}_{A,Q}(t,s_2) &+ \lambda_{(2,1)}(\lambda_{(2,3)} - \lambda_{(2,1)})\int_0^t \hat{f}_{A,Q}(\Psi,s_2)e^{\lambda_{(2,1)}\Psi}d\Psi
\end{aligned}$$

$$\begin{aligned}
\lambda_{(3,1)}e^{\lambda_{(3,2)}t}\hat{f}_{C,Q}(t,s_2) &+ e^{\lambda_{(3,2)}t}\frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2) = -\lambda_{(3,2)}e^{-\lambda_{(3,1)}t}+\\
\lambda_{(3,1)}e^{\lambda_{(3,2)}t}\hat{f}_{A,Q}(t,s_2) &+\\
\lambda_{(3,2)}e^{\lambda_{(3,2)}t}\hat{f}_{B,Q}(t,s_2) &+ \lambda_{(3,2)}(\lambda_{(3,1)} - \lambda_{(3,2)})\int_0^t \hat{f}_{B,Q}(\Psi,s_2)e^{\lambda_{(3,2)}\Psi}d\Psi
\end{aligned}$$

Again we differentiate with respect to t to obtain

$$\begin{aligned}
(\lambda_{(1,2)} - is_2)(\lambda_{(1,3)} - is_2)e^{(\lambda_{(1,3)}-is_2)t}\hat{f}_{A,Q}(t,s_2) &+ (\lambda_{(1,2)} + \lambda_{(1,3)} - 2is_2)e^{(\lambda_{(1,3)}-is_2)t}\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2)+\\
e^{(\lambda_{(1,3)}-is_2)t}\frac{\partial^2}{\partial t^2}\hat{f}_{A,Q}(t,s_2) &= \lambda_{(1,3)}\lambda_{(1,2)}e^{-\lambda_{(1,2)}t}+\\
\lambda_{(1,2)}(\lambda_{(1,3)} - is_2)e^{(\lambda_{(1,3)}-is_2)t}\hat{f}_{B,Q}(t,s_2) &+ \lambda_{(1,2)}e^{(\lambda_{(1,3)}-is_2)t}\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2)\\
\lambda_{(1,3)}(\lambda_{(1,2)} - is_2)e^{(\lambda_{(1,3)}-is_2)t}\hat{f}_{C,Q}(t,s_2) &+ \lambda_{(1,3)}e^{(\lambda_{(1,3)}-is_2)t}\frac{\partial}{\partial t}\hat{f}_{C,Q}(\Psi,s_2)
\end{aligned}$$

$$\begin{aligned}
\lambda_{(2,3)}\lambda_{(2,1)}e^{\lambda_{(2,1)}t}\hat{f}_{B,Q}(t,s_2) &+ (\lambda_{(2,3)} + \lambda_{(2,1)})e^{\lambda_{(2,1)}t}\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2)+\\
e^{\lambda_{(2,1)}t}\frac{\partial^2}{\partial t^2}\hat{f}_{B,Q}(t,s_2) &= \lambda_{(2,1)}\lambda_{(2,3)}e^{-\lambda_{(2,3)}t}+\\
\lambda_{(2,1)}\lambda_{(2,3)}e^{\lambda_{(2,1)}t}\hat{f}_{C,Q}(t,s_2) &+ \lambda_{(2,3)}e^{\lambda_{(2,1)}t}\frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2)\\
\lambda_{(2,1)}\lambda_{(2,1)}e^{\lambda_{(2,1)}t}\hat{f}_{A,Q}(t,s_2) &+ \lambda_{(2,1)}e^{\lambda_{(2,1)}t}\frac{\partial}{\partial t}\hat{f}_{A,Q}(\Psi,s_2)
\end{aligned}$$

$$\begin{aligned}
\lambda_{(3,1)}\lambda_{(3,2)}e^{\lambda_{(3,2)}t}\hat{f}_{C,Q}(t,s_2) &+ (\lambda_{(3,1)} + \lambda_{(3,2)})e^{\lambda_{(3,2)}t}\frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2)+\\
e^{\lambda_{(3,2)}t}\frac{\partial^2}{\partial t^2}\hat{f}_{C,Q}(t,s_2) &= \lambda_{(3,2)}\lambda_{(3,1)}e^{-\lambda_{(3,1)}t}+\\
\lambda_{(3,1)}\lambda_{(3,2)}e^{\lambda_{(3,2)}t}\hat{f}_{A,Q}(t,s_2) &+ \lambda_{(3,1)}e^{\lambda_{(3,2)}t}\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2)\\
\lambda_{(3,2)}\lambda_{(3,2)}e^{\lambda_{(3,2)}t}\hat{f}_{B,Q}(t,s_2) &+ \lambda_{(3,2)}e^{\lambda_{(3,2)}t}\frac{\partial}{\partial t}\hat{f}_{B,Q}(\Psi,s_2)
\end{aligned}$$

Now we multiply both sides by $e^{-(\lambda_{(1,3)}-is_2)t}$, $e^{-\lambda_{(2,1)}t}$, and $e^{-\lambda_{(3,2)}t}$, respectively, to get

$$\begin{array}{ccc}
(\lambda_{(1,2)} - is_2)(\lambda_{(1,3)} - is_2)\hat{f}_{A,Q}(t,s_2) & + & (\lambda_{(1,2)} + \lambda_{(1,2)} - 2is_2)\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2) + \\
\frac{\partial^2}{\partial t^2}\hat{f}_{A,Q}(t,s_2) & = & \lambda_{(1,3)}\lambda_{(1,2)}e^{-(\lambda_{(1,2)}+\lambda_{(1,3)}-is_2)t} + \\
\lambda_{(1,2)}(\lambda_{(1,3)} - is_2)\hat{f}_{B,Q}(t,s_2) & + & \lambda_{(1,2)}\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2) \\
\lambda_{(1,3)}(\lambda_{(1,2)} - is_2)\hat{f}_{C,Q}(t,s_2) & + & \lambda_{(1,3)}\frac{\partial}{\partial t}\hat{f}_{C,Q}(\Psi,s_2)
\end{array}$$

$$\begin{array}{ccc}
\lambda_{(2,3)}\lambda_{(2,1)}\hat{f}_{B,Q}(t,s_2) & + & (\lambda_{(2,3)} + \lambda_{(2,3)})\frac{\partial}{\partial t}\hat{f}_{B,Q}(t,s_2) + \\
\frac{\partial^2}{\partial t^2}\hat{f}_{B,Q}(t,s_2) & = & \lambda_{(2,1)}\lambda_{(2,3)}e^{-(\lambda_{(2,3)}+\lambda_{(2,1)})t} + \\
\lambda_{(2,1)}\lambda_{(2,1)}\hat{f}_{C,Q}(t,s_2) & + & \lambda_{(2,3)}\frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2) \\
\lambda_{(2,1)}\lambda_{(2,3)}\hat{f}_{A,Q}(t,s_2) & + & \lambda_{(2,1)}\frac{\partial}{\partial t}\hat{f}_{A,Q}(\Psi,s_2)
\end{array}$$

$$\begin{array}{ccc}
\lambda_{(3,1)}\lambda_{(3,2)}\hat{f}_{C,Q}(t,s_2) & + & (\lambda_{(3,1)} + \lambda_{(3,1)})\frac{\partial}{\partial t}\hat{f}_{C,Q}(t,s_2) + \\
\frac{\partial^2}{\partial t^2}\hat{f}_{C,Q}(t,s_2) & = & \lambda_{(3,2)}\lambda_{(3,1)}e^{-(\lambda_{(3,1)}+\lambda_{(3,2)})t} + \\
\lambda_{(3,1)}\lambda_{(3,2)}\hat{f}_{A,Q}(t,s_2) & + & \lambda_{(3,1)}\frac{\partial}{\partial t}\hat{f}_{A,Q}(t,s_2) \\
\lambda_{(3,2)}\lambda_{(3,1)}\hat{f}_{B,Q}(t,s_2) & + & \lambda_{(3,2)}\frac{\partial}{\partial t}\hat{f}_{B,Q}(\Psi,s_2)
\end{array}$$

Now taking the *Laplace transform* w.r.t. t leads to the following expressions

$$\begin{aligned}
\text{Ł}_{\hat{f}_{A,Q}}(s_1,s_2)[s_1^2+(\lambda_{(1,2)}+\lambda_{(1,3)}-2is_2)s_1+(\lambda_{(1,2)}-is_2)(\lambda_{(1,3)}-is_2)] \quad &= \\
\text{Ł}_{\hat{f}_{B,Q}}(s_1,s_2)\lambda_{(1,2)}(\lambda_{(1,3)}+s_1-is_2) \quad &+ \\
\text{Ł}_{\hat{f}_{C,Q}}(s_1,s_2)\lambda_{(1,3)}(\lambda_{(1,2)}+s_1-is_2) \quad &+ \\
\frac{\lambda_{(1,2)}\lambda_{(1,3)}}{\lambda_{(1,2)}+\lambda_{(1,3)}+s_1-is_2} \quad &+ \quad (1-(\lambda_{(1,2)}+\lambda_{(1,3)}))
\end{aligned}$$

$$\begin{aligned}
\text{Ł}_{\hat{f}_{B,Q}}(s_1,s_2)(s_1^2+(\lambda_{(2,3)}+\lambda_{(2,1)})s_1+\lambda_{(2,3)}\lambda_{(2,1)}) \quad &= \\
\text{Ł}_{\hat{f}_{C,Q}}(s_1,s_2)\lambda_{(2,3)}(\lambda_{(2,1)}+s_1) \quad &+ \\
\text{Ł}_{\hat{f}_{A,Q}}(s_1,s_2)\lambda_{(2,1)}(\lambda_{(2,3)}+s_1) \quad &+ \\
\frac{\lambda_{(2,3)}\lambda_{(2,1)}}{\lambda_{(2,3)}+\lambda_{(2,1)}+s_1} \quad &+ \quad (1-(\lambda_{(2,3)}+\lambda_{(2,1)}))
\end{aligned}$$

$$\begin{aligned}
\text{Ł}_{\hat{f}_{C,Q}}(s_1,s_2)(s_1^2+(\lambda_{(3,1)}+\lambda_{(3,2)})s_1+\lambda_{(1,2)}\lambda_{(1,3)}) \quad &= \\
\text{Ł}_{\hat{f}_{A,Q}}(s_1,s_2)\lambda_{(3,1)}(\lambda_{(3,2)}+s_1) \quad &+ \\
\text{Ł}_{\hat{f}_{B,Q}}(s_1,s_2)\lambda_{(3,2)}(\lambda_{(3,1)}+s_1) \quad &+ \\
\frac{\lambda_{(3,1)}\lambda_{(3,2)}}{\lambda_{(3,1)}+\lambda_{(3,2)}+s_1} \quad &+ \quad (1-(\lambda_{(3,1)}+\lambda_{(3,2)}))
\end{aligned}$$

Using the identities $\hat{f}_{A,Q}(0,s_2)=\hat{f}_{B,Q}(0,s_2)=\hat{f}_{C,Q}(0,s_2)=1$ with *Cramer's Rule* leads to the following expression for $\text{Ł}_{\hat{f}_{A,Q}}(s_1,s_2)$

$$\frac{\begin{Vmatrix}
\frac{\lambda_{(1,2)}\lambda_{(1,3)}}{\lambda_{(1,2)}+\lambda_{(1,3)}+s_1-is_2}+1-(\lambda_{(1,2)}+\lambda_{(1,3)}) & -\lambda_{(1,2)}(\lambda_{(1,3)}+s_1-is_2) & -\lambda_{(1,3)}(\lambda_{(1,2)}+s_1-is_2) \\
\frac{\lambda_{(2,3)}\lambda_{(2,1)}}{\lambda_{(2,3)}+\lambda_{(2,1)}+s_1}+1-(\lambda_{(2,3)}+\lambda_{(2,1)}) & (s_1^2+(\lambda_{(2,3)}+\lambda_{(2,1)})s_1+\lambda_{(2,3)}\lambda_{(2,1)}) & -\lambda_{(2,3)}(\lambda_{(2,1)}+s_1) \\
\frac{\lambda_{(3,1)}\lambda_{(3,2)}}{\lambda_{(3,1)}+\lambda_{(3,2)}+s_1}+1-(\lambda_{(3,1)}+\lambda_{(3,2)}) & -\lambda_{(3,2)}(\lambda_{(3,1)}+s_1) & (s_1^2+(\lambda_{(3,1)}+\lambda_{(3,2)})s_1+\lambda_{(3,1)}\lambda_{(3,2)})
\end{Vmatrix}}{\begin{Vmatrix}
(s_1^2+(\lambda_{(1,2)}+\lambda_{(1,3)}-2is_2)s_1+(\lambda_{(1,2)}-is_2)(\lambda_{(1,3)}-is_2)) & -\lambda_{(1,2)}(\lambda_{(1,3)}+s_1-is_2) & -\lambda_{(1,3)}(\lambda_{(1,2)}+s_1-is_2) \\
-\lambda_{(2,1)}(\lambda_{(2,3)}+s_1) & (s_1^2+(\lambda_{(2,3)}+\lambda_{(2,1)})s_1+\lambda_{(2,3)}\lambda_{(2,1)}) & -\lambda_{(2,3)}(\lambda_{(2,1)}+s_1) \\
-\lambda_{(3,1)}(\lambda_{(3,2)}+s_1) & -\lambda_{(3,2)}(\lambda_{(3,1)}+s_1) & (s_1^2+(\lambda_{(3,1)}+\lambda_{(3,2)})s_1+\lambda_{(3,1)}\lambda_{(3,2)})
\end{Vmatrix}}$$

The remainder of the solution is currently being worked on.

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