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Title: A Nonlinear Shallow Water Wave Equation and its Classical Solutions of the Cauchy Problem.

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Professor Ronald B. Guenther

A nonlinear wave equation is developed, modeling the evolution in time of shallow water waves over a variable topography. As the usual assumptions of a perfect fluid and an irrotational flow are *not* made, the resulting model equation is dissipative due to the presence of a viscous boundary layer at the bottom of the flow region.

The well-posedness of the Cauchy problem for classical solutions of this equation is addressed. In particular, it is established by means of various energy estimates and Sobolev space embeddings that a long time classical (C^2) solution to the Cauchy problem exists and is unique provided the initial data are small enough. An asymptotic result for the dependence of the lifespan of classical solutions upon the size of the initial data is given.

Finally, a few results of an analytical nature (e.g., explicit computation of lifespans) are given for the model equation in one dimension with the hope of providing useful parameters for experimental validation of the nonlinear wave model. Numerical results display the propagation of the nonlinear wave in one and two spatial dimensions, and a comparison is made with the waves described by the familiar linear wave equation.

A Nonlinear Shallow Water Wave Equation and its Classical Solutions of the Cauchy Problem

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Introduction

Surface water waves of lengths much greater than the depth upon which they propagate are called *shallow water waves*. To describe mathematically the dynamics of these waves, various simplifications of the Navier-Stokes have been made over the years, leading to different sets of equations. Indeed there is no shortage of shallow water wave equations, familiar ones being the Korteweg-de Vries equation, the shallow water equations [**23**], the Boussinesq equations, and those resulting from the linear potential theory. All these are described nicely in the article of Peregrine [**24**].

There is an interest in including the effects of viscosity in a shallow water wave model, and these previous models do not. In the present work, a nonlinear wave equation incorporating the effects of viscous dissipation is formulated, and the well-posedness of the associated Cauchy problem for long time classical solutions is established. Several numerical experiments have been performed displaying the rather interesting behavior of solutions of this equation.

Overview. In Chapter 1, the model equation is constructed beginning from the Navier-Stokes equations for an incompressible fluid. By integration of these equations over the depth of the fluid (water), we remove the vertical dependence of these equations and at the same time introduce explicitly the position of the free surface. Under reasonable assumptions on the flows of interest, the integrated equations lead to a simple nonlinear wave equation in two spatial dimensions, (1.6), for the evolution of the free surface.

From a mathematical point of view, the next step is to establish conditions under which a solution can be shown to exist. More generally, it is important from both mathematical and physical points of view to establish the well-posedness of problems this equation might reasonably be asked to describe, and as it is in some sense *naturally* hyperbolic, we shall inquire into the well-posedness of the simplest problem for a hyperbolic partial differential equation—the Cauchy problem—for a special case of the model equation. Specifically, we shall deal with the case when the bottom topography is flat and horizontal, and dissipation is constant. In such a case we may scale the problem to obtain the system

$$\begin{aligned} u_{tt} + \mu u_t &= \operatorname{div}(a(u)\nabla u), \\ u(0,x) &= \varphi(x), \\ u_t(0,x) &= \psi(x), \end{aligned}$$

where $a(u) = 1 + u$ and $\mu \geq 0$ is a constant. It is the well-posedness of this system we shall establish in a subsequent chapter.

Chapter 2 is preparatory material, introducing certain mathematical notation, inequalities, and results to be used in the subsequent existence and uniqueness arguments of Chapter 3. It is in Chapter 3 that the well-posedness of the Cauchy problem for classical solutions of the nonlinear wave equation is established. Instead of dealing directly with the nonlinear equation, it is more convenient to deal with a linearization of it, establishing the well-posedness of the linear problem, then showing that this well-posedness is preserved as the linearization is removed and we deal with the fully nonlinear equation. A short Chapter 4 provides a result on the asymptotic relation between the size of the initial data and the lifespan of the classical solution for the nonlinear wave equation.

Along with Chapter 1, Chapters 5 and 6 are of a more "physical" nature than the others. In Chapter 5, a variety of analytical results is presented for the nonlinear model in one spatial dimension, with an eye toward developing parameters and quantities useful for comparing theory with experiment. Chapter 6 provides some results of numerical computations in one and two spatial dimensions for the nonlinear waves being investigated, and a comparison is made between the nonlinear wave and the wave predicted by linear theory. The computations are made using centered finite differences for both space and time derivatives with spatial attention paid to preserving the divergence form of the equation. In order to mainstain stability of the iteration, a local form of the classical Courant-Friedrichs-Lewy condition is used to maintain an adequately small time step.

Previous Work. Equations similar to the one above have been derived by Lamb [**17**, §§189, 193] and Gallagher [**6**] although in both cases, the equations subsequently are linearized. In the present model, the nonlinearity is retained as physically important and interesting. In fact, one reason for retaining it is the essential difference between linear and nonlinear partial differential equations. Numerical experiments for initially smooth waves in one spatial dimension show that the solution may remain small, yet its gradient blows up, a significant departure from the usual linear behavior. Since we shall be interested in the existence of a unique, long time smooth solution to the Cauchy problem, it is of interest to survey briefly some related results. The one dimensional case is formally equivalent to the first order system

$$u_t + v_x = 0,$$
$$v_t + (1+u)u_x = -\mu v,$$

and there are important results for such systems. A "negative" result due to Lax [**18**] and John [**9**] is that in the case $\mu = 0$, smooth solutions of this system can exist for only finite time. In the present case, this is because the crest of a wave has greater local speed than its neighboring trough, and the tendency is for the wave to break. Yet, when $\mu > 0$, positive results are available. Slemrod **[29]** has shown that when μ is large enough and the initial data small enough, global classical solutions may exist. Similar results are given by Dafermos **[4]** for a variety of wave equations.

Over the past fifteen years, a great deal of attention has been directed toward the initial value and initial-boundary value problems for nonlinear wave equations in more than one spatial dimension as the equations arising in physical models naturally are set in two or three space dimensions. A very general class of these equations is of the form

$$u_{tt} - \Delta u = F(u, Du, D^2 u),$$

where $D = (\partial/\partial t, \partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $F : R^k \to R$ is regarded as a nonlinear perturbation of the usual linear wave operator appearing on the left hand side. For convenience in what follows immediately, λ shall be a k-tuple, $(\lambda_1, \ldots, \lambda_n)$ and $|\lambda| = (\sum |\lambda_j|^2)^{1/2}$; also, we shall write the spatial gradient as $D_x = \nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Particular perturbations have proved to be important as, for example, $F = F_1(Du)D^2 u$ arises in elasticity theory and $F = F_2(u)$ gives rise to nonlinear Klein-Gordon equations.

Local existence of solutions of the Cauchy problem for nonlinear hyperbolic equations (in more and less abstract settings) has been established by Schauder **[26]**, Kato **[12]**, and others, so attention currently is focused upon obtaining nonlocal results. Global existence results, which are much more valuable, guarantee the existence of a solution for all $t \geq 0$ in the spatial domain of interest. Such a result might be, for example, that for all initial data "suitably small," a solution exists for all time. Other nonlocal results deal with long time existence of solutions; that is, for any $T > 0$ we can find a solution for $0 \leq t \leq T$ provided we choose the initial data small enough. This size restriction depends upon the given T, and we can reasonably suspect that larger values of T require smaller initial data. There are cases in which a nonlinear wave equation is known to have global solutions when the spatial dimension of the problem, n, is high enough while long time solutions are known to exist for lower dimensions. It seems to be the trend that, roughly speaking, the likeliness of global existence of solutions improves as the number of spatial dimensions in a problem increases. When global results are known for $n \geq n_0$ and long time results for $n < n_0$, strange things may happen in the borderline case $n = n_0 - 1$ as is demonstrated by the phenomenon of "almost global" existence in which a slight reduction in the size of the initial data results in a tremendous increase in the lifespan of a solution.

In 1975, John [10] obtained estimates for the lifespan of solutions of the initial value problem for

$$u_{tt} - \Delta u = F_1(Du)D_x^2 u,$$

when $n \geq 3$, and a few years later, Klainerman [13] improved these by demonstrating that, in fact, global solutions exist for

$$u_{tt} - \Delta u = F(Du, D_x Du), \tag{I.1}$$

when $n \geq 6$ and $F(\lambda) = O(|\lambda|^2)$ near $\lambda = 0$. The case of "smoother" perturbation functions, $F(\lambda) = O(|\lambda|^{1+\alpha})$, for $\alpha = 1, 2, \ldots$ was considered by Shatah [27] and Klainerman & Ponce [14], and the "almost global" nature of solutions of (I.1) when $n = 3$ was demonstrated by John & Klainerman [11].

Most recently, these results have been improved by Li & Chen [20] and Lindblad [21]. In particular, Lindblad extends the work of John & Klainerman on (I.1) in three dimensions by allowing the perturbations to dependent explicitly upon u,

$$u_{tt} - \Delta u = F(u, Du, D_x Du), \tag{I.2}$$

under the restriction $F_{uu}(0,0,0) = 0$.

Little work, though, has been done in the case of two spatial dimensions. Hörmander [8] and Kovalyov [15] obtain long time results for perturbations of the form $F = F_3(Du, D^2u)$. The perturbation of interest in the present nonlinear wave model is of the form

$$F(u, Du, D^2u) = uD_x^2u + |D_xu|^2, \tag{I.3}$$

so that $F(\lambda) = O(|\lambda|^2)$ near $\lambda = 0$. From [20] we know that a dependence of $O(|\lambda|^5)$ will guarantee global existence of a classical solution, but we are far from meeting this requirement. We need also to keep in mind the results of Sideris [28] that for perturbations of the form (I.3), classical solutions of the nonlinear wave equation must break down if the initial data are too large.

Chapter 1

The Model Equation

Derivation.† Imagine a body of water bounded below by a known topography, and above by a free surface in contact with air, as shown in Fig. 1. The positive z-direction is upward. Let $z = h_0(x, y)$ describe the bottom and $z = h(x, y, t)$ the free surface. Then the thickness of the column of water at (x, y) is $H = h - h_0$. Let $\mathbf{u} = (u, v, w)$ be the velocity field of the flow, ϱ the density of the fluid, μ its viscosity, p the pressure field, and $\mathbf{g} = (0, 0, -g)$ the acceleration due to gravity.

The Navier-Stokes equations for an incompressible fluid then may be written as

$$
\begin{aligned}
&\operatorname{div} \mathbf{u} = 0, \\
&\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\varrho}\nabla p - \nu \operatorname{curl}\boldsymbol{\omega} + \mathbf{g},
\end{aligned} \tag{1.1}
$$

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$ is the vorticity and $\nu = \mu/\varrho$ is the kinematic viscosity.

In the following, $\mathbf{n} = (-h_x, -h_y, +1)$, and $\mathbf{n}_0 = (h_{0x}, h_{0y}, -1)$ are normals to the free surface and the bottom, respectively. At the free surface we impose the kinematic boundary condition $h_t = \mathbf{u} \cdot \mathbf{n}$, and the dynamic boundary condition that the stresses are continuous across the air-water interface; in the absence of surface tension and significant wind stresses, this

† The results of this chapter appear in [7].

reduces to the continuity of pressure across the interface. At the bottom, we have the no-slip condition $\mathbf{u} = 0$, and trivially, the condition for no normal flow $\mathbf{u} \cdot \mathbf{n}_0 = 0$.

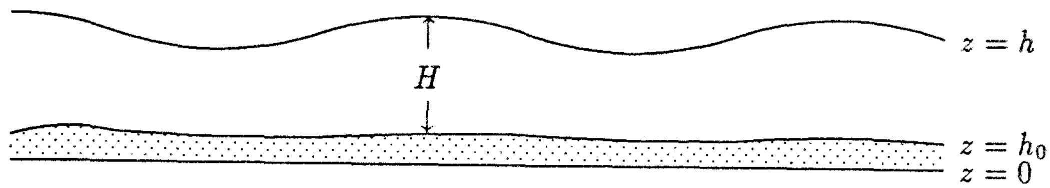

Fig. 1. Shallow Water Model Geometry.

Now integrate both expressions of (1.1) with respect to z from the bottom, $z = h_0(x, y)$, up to the free surface, $z = h(x, y, t)$. Using Leibniz' rule for interchanging differentiation and integration and these boundary conditions, the continuity equation becomes

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_{h_0}^{h} u\, dz + \frac{\partial}{\partial y}\int_{h_0}^{h} v\, dz = 0, \tag{1.2}$$

and the conservation of momentum equation becomes

$$\begin{aligned}\frac{\partial}{\partial t}\int_{h_0}^{h} \mathbf{u}\, dz = &-\frac{1}{\varrho}\int_{h_0}^{h} \nabla p\, dz - \nu \,\mathrm{curl}\int_{h_0}^{h} \boldsymbol{\omega}\, dz \\ &- \nu\, (\mathbf{n} \times \boldsymbol{\omega})\big|_{z=h} + \nu\, (\mathbf{n}_0 \times \boldsymbol{\omega})\big|_{z=h_0} - \mathrm{g}H.\end{aligned} \tag{1.3}$$

Here, in (1.3), nonlinear terms of the form $\frac{\partial}{\partial x}\int_{h_0}^{h} u^2\, dz$, $\frac{\partial}{\partial y}\int_{h_0}^{h} uv\, dz$, etc. have been neglected in linearizing about a still water base flow.

In inviscid shallow water theory, it is shown that the vorticity of the flow is, to lowest order, vertical. In the present case, although the vorticity cannot be strictly vertical throughout the flow (and not, in particular, at the bottom), we shall assume it is nearly vertical in the vicinity of the free surface, and more precisely, in the direction of $\mathbf{n}$; then $\mathbf{n} \times \boldsymbol{\omega} \approx 0$ at the free surface.

Another consequence of inviscid shallow water theory is that the pressure in the flow may be taken as hydrostatic. We assume in the flows of present interest that this approximation is still valid. Since the atmospheric pressure may be taken to be a constant along the free surface, say zero, and since p is continuous across the interface, we get an explicit expression for the pressure:

$$p(x, y, z, t) \approx \varrho g \, \big(h(x, y, t) - z\big).$$

Using this expression, the second integral in (1.3) becomes

$$-\frac{1}{\varrho}\int_{h_0}^{h} \nabla p \, dz = gH\mathbf{n}. \tag{1.4}$$

Equations (1.3) and (1.4) yield

$$\frac{\partial}{\partial t}\int_{h_0}^{h} \mathbf{u} \, dz = gH\mathbf{n} - \nu \text{ curl} \int_{h_0}^{h} \boldsymbol{\omega} \, dz - \mathbf{g}H - \nu \, \mathbf{n}_0 \times \boldsymbol{\omega}\big|_{z=h_0}. \tag{1.5}$$

The final approximation involves estimating the vorticity at the bottom. To do this, we assume the vertical gradients of the velocity field dominate so that

$$\boldsymbol{\omega}\big|_{z=h_0} \approx (-v_z, u_z, 0).$$

By the mean value theorem for integrals, we have

$$\frac{1}{H}\int_{h_0}^{h} v \, dz = v(x, y, z^*, t)$$

for some $h_0 < z^* < h$ so that by Taylor expansion about $z = h_0$,

$$v(x, y, z^*, t) = v(x, y, h_0, t) + v_z(x, y, h_0, t)(z^* - h_0).$$

Again, $\mathbf{u} = 0$ at $z = h_0$, whence

$$v_z\big|_{z=h_0} \approx \frac{1}{(z^* - h_0)H}\int_{h_0}^{h} v \, dz = \frac{1}{H_0\delta}\int_{h_0}^{h} v \, dz,$$

where $\delta = (z^* - h_0)H/H_0$ is a parameter independent of z; it is expected δ would be experimentally obtained. Similarly,

$$u_z\Big|_{z=h_0} \approx \frac{1}{H_0\delta}\int_{h_0}^{h} u\,dz.$$

For small enough beach slopes, $\partial H_0/\partial x$, $\partial H_0/\partial y \approx 0$ and are negligible. Further, since we expect δ to vary with H_0, it follows that its derivatives should also be negligible. So

$$\frac{\partial}{\partial x}\left(\frac{1}{H_0\delta}\right), \frac{\partial}{\partial y}\left(\frac{1}{H_0\delta}\right) \approx 0$$

for small beach slopes. Specifically, if D and L are length scales characterizing the flow depth and changes in the horizontal velocity field (u, v), respectively, we need $|\nabla H_0| \ll \mathrm{D}/\mathrm{L}$ to guarantee the "smallness" of these derivatives.

Now take the divergence of the momentum equation, (1.5), use the fact that

$$\operatorname{div}\frac{\partial}{\partial t}\int_{h_0}^{h} \mathbf{u}\,dz = \frac{\partial}{\partial t}\operatorname{div}\int_{h_0}^{h} \mathbf{u}\,dz = -\frac{\partial^2 h}{\partial t^2},$$

and that

$$\begin{aligned}\operatorname{div}(\mathbf{n}_0 \times \boldsymbol{\omega}) &= \frac{\partial}{\partial x}\left(\frac{1}{H_0\delta}\int_{h_0}^{h} u\,dz\right) \\ &\quad + \frac{\partial}{\partial y}\left(\frac{1}{H_0\delta}\int_{h_0}^{h} v\,dz\right) \\ &\approx -\frac{1}{H_0\delta}\frac{\partial h}{\partial t}\end{aligned}$$

to get an expression governing the evolution of the free surface:

$$\frac{\partial^2 h}{\partial t^2} + \frac{\nu}{H_0\delta}\frac{\partial h}{\partial t} = g\left[\frac{\partial}{\partial x}\left(H\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial h}{\partial y}\right)\right]. \tag{1.6}$$

Note that if we interpret the quantity $\sqrt{gH}$ as the local wave speed, (1.6) describes waves in which wave crests travel faster than the troughs, since $H_{\text{crest}} \geq H_{\text{trough}}$. This suggests the possibility of the eventual presence of shocks or wave breaking.

When the effects of viscosity are insignificant and the water is of constant depth, take $h_0 = 0$; then $H = h$ and this expression becomes

$$\frac{\partial^2 h}{\partial t^2} = \frac{1}{2} g \Delta h^2.$$

If there is a steady underlying flow, say a longshore current $\mathbf{U}$, approximations to the neglected nonlinear terms may be retained, and a similar argument shows

$$\frac{\partial^2 h}{\partial t^2} + \left(\frac{\nu}{H_0 \delta} + \mathbf{U} \cdot \nabla\right) \frac{\partial h}{\partial t} = g \left[\frac{\partial}{\partial x}\left(H \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(H \frac{\partial h}{\partial y}\right)\right].$$

From a physical standpoint, we expect solutions of (1.6) to be "wave like" and to mimick the behavior of the linear wave equation obtained by replacing H with H_0 for at least a short time. As mentioned though, the nonlinearity of (1.6) and the attendant possibility of breaking makes the eventual divergence of waveforms certain. A hopeful note is the presence of the damping term $(\nu / H_0 \delta) h_t$ which provides a mechanism for the dissipation of wave energy. Perhaps this dissipation (causing a continuous decrease in wave amplitude), coupled with the dispersion which occurs in two spatial dimensions, is sufficient to offset the tendency toward breaking caused by local height (hence, wave speed) differences.

Recalling that δ represents a sort of bottom boundary layer depth, or more precisely, an effective depth through which the vorticity present at the bottom affects the flow, it is evident that the damping coefficient $\nu / \delta H_0$ represents an energy dissipation mechanism in which wave energy is lost to bottom vorticity. Presumably, if this effective vorticity is concentrated (i.e., δ is small), the dissipation can be rather high and the eventual breaking of the wave delayed.

The Scaled Equation. Let us now restrict our attention to a special case of (1.6), namely, the case in which the bottom is horizontal and the viscous damping coefficient is constant. If we suppose H_0 is the still water depth, we can write $h = H_0 + \eta$ to get the following expression for η:

$$\eta_{tt} + \alpha\eta_t = g\,\mathrm{div}\,((H_0 + \eta)\nabla\eta).$$

If we scale all lengths by H_0 and the time by $\sqrt{H_0/g}$, we obtain an equation of the form

$$u_{tt} + \mu u_t = \mathrm{div}\,((1 + u)\nabla u). \tag{1.7}$$

It is this expression which we shall investigate in the remainder of this work with the hope that results obtained for it will lend insight to the more general case (1.6). The quantity u appearing in (1.7) is given by $u = \eta/H_0$ and should not be confused with the first component of the velocity field of which we no longer shall have need to consider.

Chapter 2

Some Mathematical Preliminaries

In the following analyses, it shall prove useful to consider functions defined on an interval $[0,T]$ with values in a function space. Thus for example, $f \in C([0,T], L^2(R^2))$ is a function assigning to each time $t \in [0,T]$ a function in $L^2(R^2)$; moreover, f is continuous on $[0,T]$. To reinforce this notion we write $\dot{f}$ instead of f_t for time derivatives. At times though it will be convenient to revert to a more usual notation: Instead of writing $f(t)(x)$ for the value of $f(t)$ when evaluated at the spatial point $x \in R^2$, we write simply $f(t,x)$. Moreover, notational abuses such as ∇f, where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, are to be interpreted in the obvious manner.

Norms and Function Spaces. The following norms and seminorms for the Lesbegue and Sobolev spaces shall be used: the L^p norms

$$\|f\|_{L^2} = \left(\int_{R^2} |f|^2\, dx\right)^{1/2},$$
$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{R^2} |f|;$$

the "integer" Sobolev space norms and seminorms,

$$\|f\|_{H^m} = \left(\sum_{0\le|\alpha|\le m} \int_{R^2} |D^\alpha f|^2 \, dx \right)^{1/2},$$

$$\begin{aligned} |f|_{H^m} &= \left(\sum_{1\le|\alpha|\le m} \int_{R^2} |D^\alpha f|^2 \, dx \right)^{1/2}, \\ &= \|\nabla f\|_{H^{m-1}}; \end{aligned}$$

and for functions of time, t, with values in a function space, the related quantities

$$\begin{aligned} \|f\|_{L^2,T} &= \max_{0\le t\le T} \|f(t)\|_{L^2}, \\ \|f\|_{L^\infty,T} &= \max_{0\le t\le T} \|f(t)\|_{L^\infty}, \\ \|f\|_{H^m,T} &= \max_{0\le t\le T} \|f(t)\|_{H^m}, \\ |f|_{H^m,T} &= \max_{0\le t\le T} |f(t)|_{H^m}. \end{aligned}$$

In Theorems 4 and 5, use shall be made of the L^2 "fractional" Sobolev spaces H^ρ generated by Bessel potentials [**5**, **31**]. These spaces coincide with those just introduced whenever ρ is an integer; the norms are equivalent, but not equal, yet this inequality of norms poses no real problem for us, and no notational distinction shall be made between the integer space norms, $\|\cdot\|_{H^m}$, and the fractional space norms, $\|\cdot\|_{H^\rho}$, in these theorems.

The spaces

$$\mathfrak{C}_m(0,T) = C([0,T], H^m(R^2)) \cap C^1([0,T], H^{m-1}(R^2))$$

shall be of particular interest, especially for the cases $m = 1$ and $m = 4$; the norms on these spaces are given by

$$|||f|||_{m,T} = \left(\|\dot{f}(t)\|^2_{H^{m-1},T} + \|f(t)\|^2_{H^m,T} \right)^{1/2}.$$

The generalization of $\mathfrak{C}_m(0,T)$ to $\mathfrak{C}_\rho(0,T)$ and its norm for fractional ρ is obvious.

Families of Functions. There will be an abundance of sequences, subsequences, subsubsequences, etc., so we employ the following convention: N shall always denote the natural numbers $1, 2, 3, \ldots$ so a sequence may be written as $\{f_k : k \in N\}$; moreover, N' shall always be an infinite subsequence of N, N'' an infinite subsequence of N', etc., so we write subsequences as $\{f_k : k \in N'\}$, subsubsequences as $\{f_k : k \in N''\}$, and so forth. This removes the burden of subsequences from subscripting and places it upon modifying the index set.

The following version of the Arzela-Ascoli theorem for mappings of $[0, T]$ into a metric space $\mathfrak{X}$ shall be used [**25**]:

THEOREM. (Arzela-Ascoli). *Suppose $f_1, f_2, \ldots$ is an equicontinuous family of functions from $[0, T]$ into a metric space $\mathfrak{X}$. If $\{f_k(t) : k \in N\}$ is sequentially compact for each $t \in [0, T]$, then there is a subsequence $\{f_k(t) : k \in N'\}$ converging pointwise to a continuous function f, and the convergence is uniform on $[0, T]$.*

A Result on Equicontinuity. Suppose $f_1, f_2, \ldots$ is a sequence of functions with $\dot{f}_k \in C([0, T], L^2)$, and the $\dot{f}_k$ uniformly bounded, say $\|\dot{f}_k\|_{L^2,T} \leq M$ for $k = 1, 2, \ldots$. To see that $f_1, f_2, \ldots$ is an equicontinuous family in $C([0, T], L^2)$, use the fact that for $t \geq s$,

$$f_k(t) - f_k(s) = \int_s^t \dot{f}_k(\tau)\, d\tau.$$

Then

$$\begin{aligned} \|f_k(t) - f_k(s)\|_{L^2}^2 &\leq \int_{R^n} \int_s^t \int_s^t |\dot{f}_k(\tau)\dot{f}_k(\sigma)|\, d\tau d\sigma\, dx \\ &\leq \int_s^t \int_s^t \int_{R^n} |\dot{f}_k(\tau)\dot{f}_k(\sigma)|\, dx\, d\tau d\sigma \\ &\leq \int_s^t \int_s^t \|\dot{f}_k(\tau)\|_{L^2} \|\dot{f}_k(\sigma)\|_{L^2}\, d\tau d\sigma \end{aligned}$$

where the Cauchy-Schwarz inequality has been used. Then clearly

$$\|f_k(t) - f_k(s)\|_{L^2}^2 \leq (t - s)^2 \|\dot{f}_k\|_{L^2,T}^2$$

so

$$\|f_k(t) - f_k(s)\|_{L^2} \leq M|t-s|$$

which shows the family is equicontinuous.

Mollifiers. The technique of mollification introduced by K. O. Friedrichs allows us to construct smooth approximations to functions while preserving bounds on the norm of the function. Suppose $j^0 : R \to R$ and $j : R^2 \to R$ are nonnegative C_0^∞ functions with supports contained in the unit balls of R and R^2, respectively, and $\|j^0\|_{L^1(R)} = \|j\|_{L^1(R^2)} = 1$. For $\varepsilon > 0$, we define the mollifier J_ε by

$$(J_\varepsilon f)(t,x) = \int_0^T \int_{R^2} j_\varepsilon^0(t-\tau) j_\varepsilon(x-\xi) f(\tau,\xi)\, d\xi d\tau$$

where $j_\varepsilon^0(t) = \varepsilon^{-1} j^0(t/\varepsilon)$ and $j_\varepsilon(x) = \varepsilon^{-2} j(x/\varepsilon)$. Then $J_\varepsilon f$ is in $C^\infty(R \times R^2)$ whenever, for example, $f \in L^\infty([0,T] \times R^2)$. We have the following important properties of J_ε:

(2.1) $$\|J_\varepsilon f\|_{L^p,T} \leq \|f\|_{L^p,T},$$

for $p = 2, \infty$;

(2.2) $$\|J_\varepsilon f\|_{H^m,T} \leq \|f\|_{H^m,T},$$

and when $f \in C([0,T], L^\infty)$ and uniformly continuous,

(2.3) $$\|J_\varepsilon f - f\|_{L^\infty,T} \to 0$$

as $\varepsilon \to 0$.

To verify these, notice that (2.2) follows directly from (2.1) in the case $p = 2$. It is clear that (2.1) holds in the case $p = \infty$, and (2.3) is shown in [5], so we need establish only (2.1) in the case $p = 2$. To this end, we have

$$\begin{aligned}\|J_\varepsilon f(t)\|_{L^2} &= \left\|j_\varepsilon * \textstyle\int_0^T j_\varepsilon^0(t-\tau) f(\tau)\, d\tau\right\|_{L^2} \\ &\leq \left\|\textstyle\int_0^T j_\varepsilon^0(t-\tau) f(\tau)\, d\tau\right\|_{L^2}\end{aligned}$$

by the Young inequality. Now

$$
\begin{aligned}
\left\|\int_0^T j_\varepsilon^0(t-\tau)f(\tau)\,d\tau\right\|_{L^2}^2 &= \int_0^T\int_0^T j_\varepsilon^0(t-\tau)j_\varepsilon^0(t-\sigma)\int_{R^2} f(\tau)f(\sigma)d\xi\,d\tau d\sigma \\
&\le \int_0^T\int_0^T j_\varepsilon^0(t-\tau)j_\varepsilon^0(t-\sigma)\|f(\tau)\|_{L^2}\|f(\sigma)\|_{L^2}\,d\tau d\sigma \\
&\le \|f\|_{L^2,T}^2
\end{aligned}
$$

where the Cauchy-Schwarz inequality has been used, and it follows that $\|J_\varepsilon f\|_{L^2,T} \le \|f\|_{L^2,T}$.

Inequalities of Poincaré and Sobolev. The Poincaré inequality shall prove to be essential. For $f \in C_0^\infty(R^n)$ it may be verified that $\|f\|_{L^2} \le (d/n)|f|_{H^1}$ where $d = \operatorname{diam}\operatorname{supp} f$, so in the present case with $n = 2$ we have

$$\|f\|_{L^2(R^2)} \le \tfrac{1}{2}d|f|_{H^1(R^2)}.$$

It will be convenient to use C_f to denote the "Poincaré coefficient" $d/2$ associated with f. For instance, C_u (for the function u) appears in Lemma 2. Two exceptions are C_s which is the constant in the Sobolev inequality

$$\|f\|_{L^\infty(R^2)} \le C_s|f|_{H^2(R^2)},$$

and C_p which is an upper bound on a family of constants for the Poincaré inequality, and introduced in Lemma 2 also.

Chapter 3

Well-Posedness of the Cauchy Problem

Of the mathematical questions surrounding a partial differential equation, perhaps the most fundamental are those involving the existence and uniqueness of its solutions. In this chapter, we address these problems for the Cauchy problem for (1.7), and show that for sufficiently small initial data, a long time classical solution exists and is unique, and more generally, the Cauchy problem is well-posed.† Such results are, of course, not without practical implications, especially if we view this equation as a nonlinear extension of the usual linear wave equation. Specifically what shall be shown is that, given $T > 0$, there always is a unique classical solution, $u \in C^2([0,T] \times R^2)$, of

$$\begin{aligned} \ddot{u} + \mu\dot{u} &= \operatorname{div}(a(u)\nabla u), \\ u(0) &= \varphi, \\ \dot{u}(0) &= \psi, \end{aligned} \tag{3.1}$$

whenever φ and ψ are supported compactly on R^2 and $|\varphi|^2_{H^4(R^2)} + \|\psi\|^2_{H^3(R^2)}$ is small enough. Here, and in what follows, $a(u) = 1 + u$. First, though, an overview of the approach.

Overview. The existence proof proceeds along the lines of the classical iteration scheme in which the nonlinear equation is replaced by $\ddot{u}^i + \mu\dot{u}^i =$

† The results of this chapter appear in [3].

$\operatorname{div}(a(u^{i-1})\nabla u^i)$, with the hope that $u^i \to u$ in the proper function space, and u is a solution. It was found more convenient, however, to consider instead the iteration

$$\ddot{u}^i + \mu\dot{u}^i = \operatorname{div}(a(J_\varepsilon u^{i-1})\nabla u^i), \tag{3.2}$$

where J_ε is a mollifier in space and time. The convergence $u^i \to u$ is established for every $\varepsilon > 0$, then we let $\varepsilon \to 0$. Lemma 1 provides sufficient conditions for the well-posedness of the Cauchy problem for a linear wave equation with smooth (C^∞) coefficients, and establishes the finite speed of propagation of the support of its solution. This shows (3.2) can be solved. In Lemma 2 it is shown that the growth of the solution of (3.2) can be controlled by bounding the coefficients, and that in fact there is a bound, say M, such that u^i is norm-bounded by M whenever u^{i-1} is. In Lemma 3, the convergence $u^i \to u$ is established for each $\varepsilon > 0$, and in Propositions 4 and 5 we remove the mollifier by letting $\varepsilon \to 0$ which establishes the existence of a classical solution of the fully nonlinear problem. In Proposition 6 we show the classical solution is unique, and in Proposition 7 we show that this classical solution depends continuously upon the initial data thereby establishing the well-posedness of the Cauchy problem (3.1).

Linear Equations with Smooth Coefficients. We begin now to show that (3.1) always has a long time classical solution provided the initial data can be chosen small enough. Since the proof requires linearization of the nonlinear equation, we state without proof a result from the theory of linear hyperbolic equations [**30**]:

LEMMA 1. *Suppose $v \in C^\infty([0,T] \times R^2)$ and $\|v\|_{L^\infty,T} < 1$. Then the Cauchy problem*

$$\begin{aligned} \ddot{u} + \mu\dot{u} &= \operatorname{div}(a(v)\nabla u), \\ u(0) &= \varphi, \\ \dot{u}(0) &= \psi, \end{aligned}$$

with initial data $\varphi \in H^m(R^2)$ *and* $\psi \in H^{m-1}(R^2)$, $m \geq 2$, *always has a unique solution, and*

$$u \in C^2([0,T], H^{m-2}(R^2)) \cap C^1([0,T], H^{m-1}(R^2)) \cap C([0,T], H^m(R^2)).$$

Further, $\operatorname{supp} u(t)$ *grows in a bounded fashion, and specifically, if*

$$d(t) = \operatorname{diam}\{\operatorname{supp} u(t) \cup \operatorname{supp} \dot{u}(t)\},$$

then $d(t) \leq d(0) + 2t(1 + \|v\|_{L^\infty,T})^{1/2}$.

Thus, for example, if we take $v = 0$, we get a nice solution, unique and defined on $[0,T]$. As we shall see now, if v is small this is still the case, and in fact if the initial data are small we can find a number bounding u in $\mathfrak{C}_4(0,T)$ whenever v is similarly bounded. This provides the first step toward setting up a well-defined iteration scheme. Indeed, a very real concern in the solution by iteration is the loss of hyperbolicity in (3.2): Given u^0, can we be sure that the iteration continues to produce a new hyperbolic equation? By controlling the size of the initial data, we can answer this in the affirmative.

Lemma 2 ("High norm" boundedness). *Let* $v \in C^\infty([0,T], C_0^\infty(R^2))$ *with* $\|v\|_{L^\infty,T} < 1$ *be given, and suppose* u *is the solution of the Cauchy problem with compactly supported initial data*

$$\begin{aligned} \ddot{u} + \mu\dot{u} &= \operatorname{div}(a(v)\nabla u), \\ u(0) &= \varphi, \\ \dot{u}(0) &= \psi, \end{aligned}$$

for $0 \leq t \leq T$. *Whenever the "initial energy"*

$$E_0 = |\varphi|^2_{H^4(R^2)} + \|\psi\|^2_{H^3(R^2)}$$

is small enough, there are constants κ *and* λ *depending only upon* T *and the size of the support of the initial data such that*

$$\|u\|_{L^\infty,T} \leq \kappa < 1$$

and

$$\left|u\right|^2_{H^4,T} + \left\|\dot{u}\right\|^2_{H^3,T} \le \lambda E_0$$

whenever v *satisfies the same inequalities.*

Proof. By Lemma 1, $u \in \mathfrak{C}_4(0,T)$ and satisfies the linear equation $\ddot{u} + \mu\dot{u} = \operatorname{div}\left(a(v)\nabla u\right)$ in $[0,T]$. Let D^α represent partial differentiation of order $|\alpha|$ in the spatial variables, and for convenience, set $u^\alpha = D^\alpha u$, $v^\alpha = D^\alpha v$, etc. Then for $|\alpha| \le 3$ we have upon differentiation

$$\ddot{u}^\alpha + \mu\dot{u}^\alpha = \operatorname{div}\left(a(v)\nabla u^\alpha\right) + \sum_{\substack{\beta+\beta'=\alpha\\ |\beta|\ge 1}} C^\alpha_\beta \operatorname{div}\left(v^\beta \nabla u^{\beta'}\right) \tag{3.3}$$

where the summation (which is empty for $|\alpha| = 0$) results from application of the product rule and $C^\alpha_\beta = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}$. When $|\alpha| \le 3$ we have $C^\alpha_\beta \le 3$; this shall appear in a bound below. We get an energy inequality by the usual procedure: multiply through (3.3) by $\dot{u}^\alpha$ and integrate over R^2. Using the fact that u is supported compactly, we obtain

(3.4)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{R^2}\left[(\dot{u}^\alpha)^2 + a(v)|\nabla u^\alpha|^2\right]dx + \mu\int_{R^2}(\dot{u}^\alpha)^2\,dx\\ = \frac{1}{2}\int_{R^2}\dot{v}|\nabla u^\alpha|^2\,dx + \sum_{\substack{\beta+\beta'=\alpha\\ |\beta|\ge 1}} C^\alpha_\beta\int_{R^2}\left[\nabla v^\beta\cdot\nabla u^{\beta'} + v^\beta\Delta u^{\beta'}\right]\dot{u}^\alpha\,dx.\end{aligned}$$

This equation leads to an inequality if we drop the term involving μ (since $\mu \ge 0$) and bound the terms on the right hand side. Using the Sobolev embedding theorem ($\|f\|_{L^\infty(R^2)} \le C_s\|f\|_{H^2(R^2)}$) the first of these is easy,

$$\begin{aligned}\frac{1}{2}\int_{R^2}\dot{v}|\nabla u^\alpha|^2\,dx &\le \frac{1}{2}\|\dot{v}\|_{L^\infty}\|\nabla u^\alpha\|^2_{L^2}\\ &\le \frac{1}{2}C_s\|\dot{v}\|_{H^2}\|\nabla u^\alpha\|^2_{L^2}\end{aligned} \tag{3.5}$$

while the second term requires a look at different cases. When $|\beta| = 1$, $|\beta'|$

can take values 0, 1, and 2:

$$\int_{R^2}\left[\nabla v^{\beta}\cdot\nabla u^{\beta'}+v^{\beta}\Delta u^{\beta'}\right]\dot{u}^{\alpha}\,dx$$
$$\le\left\{\|\nabla v^{\beta}\|_{L^\infty}\|\nabla u^{\beta'}\|_{L^2}+\|v^{\beta}\|_{L^\infty}\|\Delta u^{\beta'}\|_{L^2}\right\}\|\dot{u}^{\alpha}\|_{L^2}$$
$$\le C_s\|\dot{u}^{\alpha}\|_{L^2}\left\{\|\nabla v^{\beta}\|_{H^2}\|\nabla u^{\beta'}\|_{L^2}+\|v^{\beta}\|_{H^2}\|\Delta u^{\beta'}\|_{L^2}\right\}$$

and by introducing the $|\cdot|$-norm, we get

$$\int_{R^2}\left[\nabla v^{\beta}\cdot\nabla u^{\beta'}+v^{\beta}\Delta u^{\beta'}\right]\dot{u}^{\alpha}\,dx\le C_s\|\dot{u}^{\alpha}\|_{L^2}\left\{|v|_{H^4}|u|_{H^3}+2|v|_{H^3}|u|_{H^4}\right\}$$
$$\le 3C_s\|\dot{u}^{\alpha}\|_{L^2}|u|_{H^4}|v|_{H^4}.$$

When the cases $|\beta|=2$ and $|\beta|=3$ are treated in such a manner, the same bound results; that is,

$$\int_{R^2}\left[\nabla v^{\beta}\cdot\nabla u^{\beta'}+v^{\beta}\Delta u^{\beta'}\right]\dot{u}^{\alpha}\,dx\le 3C_s\|\dot{u}^{\alpha}\|_{L^2}|u|_{H^4}|v|_{H^4}\tag{3.6}$$

whenever $\beta+\beta'=\alpha$, $|\alpha|\le 3$, and $|\beta|\ge 1$. Using (3.5) and (3.6) in the equation (3.4) we obtain an inequality

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}\left[(\dot{u}^{\alpha})^2+a(v)|\nabla u^{\alpha}|^2\right]dx\le\tfrac{1}{2}C_s\|\dot{v}\|_{H^2}\|\nabla u^{\alpha}\|_{L^2}^2$$
$$+36C_s\|\dot{u}^{\alpha}\|_{L^2}|u|_{H^4}|v|_{H^4},$$

which when summed over $|\alpha|\le 3$ gives

$$\frac{1}{2}\frac{d}{dt}\|\dot{u}\|_{H^3}^2+\frac{1}{2}\frac{d}{dt}\sum_{0\le|\alpha|\le 3}\int_{R^2}a(v)|\nabla u^{\alpha}|^2\,dx$$
$$\le\tfrac{1}{2}C_s\|\dot{v}\|_{H^2}|u|_{H^4}^2+360C_s\|\dot{u}\|_{H^3}|u|_{H^4}|v|_{H^4}$$
$$\le\tfrac{1}{2}C_s\|\dot{v}\|_{H^2}\left\{\|\dot{u}\|_{H^3}^2+|u|_{H^4}^2\right\}+180C_s|v|_{H^4}\left\{\|\dot{u}\|_{H^3}^2+|u|_{H^4}^2\right\}.$$

Using the Cauchy-Schwarz inequality,

$$\frac{1}{2}\frac{d}{dt}\|\dot{u}\|_{H^3}^2+\frac{1}{2}\frac{d}{dt}\sum_{0\le|\alpha|\le 3}\int_{R^2}a(v)|\nabla u^{\alpha}|^2\,dx$$
$$\le\left\{\tfrac{1}{2}C_s\|\dot{v}\|_{H^3}+180C_s|v|_{H^4}\right\}\left\{\|\dot{u}\|_{H^3}^2+|u|_{H^4}^2\right\}\tag{3.7}$$
$$\le 181C_s\left\{\|\dot{v}\|_{H^3}^2+|v|_{H^4}^2\right\}^{1/2}\left\{\|\dot{u}\|_{H^3}^2+|u|_{H^4}^2\right\}.$$

Let E_u and E_v denote the "maximum energy" of u and v:

$$E_u(t) = \|\dot{u}\|^2_{H^3,t} + |u|^2_{H^4,t},$$
$$E_v(t) = \|\dot{v}\|^2_{H^3,t} + |v|^2_{H^4,t}.$$

Integration of (3.7) over $[0,t]$ yields

$$\begin{aligned}
&\frac{1}{2}\|\dot{u}(t)\|^2_{H^3} + \frac{1}{2}\sum_{0\le|\alpha|\le 3}\int_{R^2} a(v(t))|\nabla u^\alpha(t)|^2\,dx \\
&\qquad \le \frac{1}{2}\|\dot{u}(0)\|^2_{H^3} + \frac{1}{2}\sum_{0\le|\alpha|\le 3}\int_{R^2} a(v(0))|\nabla u^\alpha(0)|^2\,dx \\
&\qquad + 181 C_s E_v^{1/2}(T) E_u(T) t,
\end{aligned} \tag{3.8}$$

and since

$$\max_{R^2} |a(v(t))| \le 1 + \|v(t)\|_{L^\infty}$$
$$\min_{R^2} |a(v(t))| \ge 1 - \|v(t)\|_{L^\infty}$$

we get

$$\sum_{0\le|\alpha|\le 3}\int_{R^2} a(v(t))|\nabla u^\alpha(t)|^2\,dx \ge \left(1 - \|v(t)\|_{L^\infty}\right)|u(t)|^2_{H^2}$$

and

$$\sum_{0\le|\alpha|\le 3}\int_{R^2} a(v(0))|\nabla u^\alpha(0)|^2\,dx \le \left(1 + \|v(0)\|_{L^\infty}\right)|u(0)|^2_{H^2}.$$

From these bounds, (3.8) yields

$$\begin{aligned}
\frac{1}{2}(1 - \|v(t)\|_{L^\infty})(\|\dot{u}(t)\|^2_{H^3} + |u(t)|^2_{H^4}) &\le \frac{1}{2}(1 + \|v(0)\|_{L^\infty})E_u(0) \\
&+ 181 C_s E_v^{1/2}(T) E_u(T) t,
\end{aligned}$$

and since this holds over $0 \le t \le T$, it follows that

$$\frac{1}{4}(1 - \|v\|_{L^\infty,T})E_u(T) \le \frac{1}{2}(1 + \|v(0)\|_{L^\infty})E_u(0) + 181 C_s E_v^{1/2}(T) E_u(T) T. \tag{3.9}$$

Expression (3.9) relates the energy of u to the energies of v and the initial data. Supposing for a moment that $v = 0$, this expression shows that $E_u(T) \leq E_u(0)$. Thus, the energy of the evolving solution is limited to that which is initially available. It is reasonable to suspect that a similar result holds when v is "small." To show that this is indeed the case, let A and B be the constants

$$A = \frac{724}{1 + C_p} T, \tag{3.10a}$$

$$B = \frac{6 + 8C_p}{1 + C_p} T, \tag{3.10b}$$

and choose γ and λ so that

$$\max\left\{\frac{A}{1+A}, \frac{B}{1+B}\right\} < \gamma < 1 \tag{3.10c}$$

and

$$\lambda = \frac{4 - 2\gamma}{\gamma - (1-\gamma)A}.$$

In these expressions, $C_p = \frac{1}{2}d + 2T$, and d is the diameter of the initial data, $d = \operatorname{diam}\{\operatorname{supp}\varphi \cup \operatorname{supp}\psi\}$. Selection of γ in this manner guarantees $\lambda > 2$. Suppose

$$E_0^{1/2} \leq \frac{A}{724 C_s T} \frac{1-\gamma}{\lambda^{1/2}}. \tag{3.11}$$

What shall be shown now is that if

$$E_v(T) \leq \lambda E_0 \tag{3.12a}$$

and

$$\|v\|_{L^\infty, T} \leq 1 - \gamma, \tag{3.12b}$$

then u also satisfies the same inequalities. While these inequalities perhaps are very restrictive, they always hold when we choose $v = 0$. This is important for setting up the iterative scheme described earlier, since the initial value for the iteration may then be taken to be the zero function. Assuming (3.12) holds, the energy inequality (3.9) gives

$$\tfrac{1}{4}\gamma E_u(T) \le (1 - \tfrac{1}{2}\gamma)E_0 + 181 C_s \lambda^{1/2} E_0^{1/2} T E_u(T)$$

and by rearrangement, $E_u(T)$ is bounded as

$$(\gamma - 724 C_s \lambda^{1/2} E_0^{1/2} T) E_u(T) \le (4 - 2\gamma) E_0.$$

Since $E_0^{1/2}$ is bounded as in (3.11), we have

$$(\gamma - (1-\gamma)A) E_u(T) \le (4 - 2\gamma) E_0.$$

Division yields

$$\begin{aligned} E_u(T) &\le \frac{4 - 2\gamma}{\gamma - (1-\gamma)A} E_0 \\ &= \lambda E_0 \end{aligned}$$

so we have established that (3.12a) also holds for u. To show the same for (3.12b), note that by use of the Sobolev and Poincaré inequalities,

$$\begin{aligned} \|u\|_{L^\infty(R^2)} \le C_s \|u\|_{H^4(R^2)} &= C_s \left\{ \|u\|^2_{L^2(R^2)} + |u|^2_{H^4(R^2)} \right\}^{1/2} \\ &\le C_s \left\{ C_u^2 + 1 \right\}^{1/2} |u|_{H^4(R^2)} \\ &\le C_s (1 + C_u) E_u^{1/2}(T) \end{aligned}$$

where C_u is a constant depending only upon the size of the support of u. And in particular, since the support of u propagates at finite speed, Lemma 1 guarantees

$$\begin{aligned} C_u &\le \tfrac{1}{2} d + T(1 + \|v\|_{L^\infty,T})^{1/2} \\ &\le \tfrac{1}{2} d + \sqrt{2 - \gamma} T \\ &\le \tfrac{1}{2} d + (2 - \gamma) T, \end{aligned}$$

so

$$\begin{aligned} \|u\|_{L^\infty,T} &\le C_s (1 + \tfrac{1}{2} d + (2 - \gamma) T) E_u^{1/2} \\ &\le C_s (1 + \tfrac{1}{2} d + 2T) E_u^{1/2} \\ &= C_s (1 + C_p) E_u^{1/2} \end{aligned}$$

Finally, since $E_u \leq \lambda E_0$ as just shown, we have

$$\begin{aligned} \|u\|_{L^\infty,T} &\leq C_s(1+C_p)\lambda^{1/2}E_0^{1/2} \\ &= C_s(1+C_p)\frac{A}{724C_sT}(1-\gamma) \\ &= 1-\gamma, \end{aligned}$$

so (3.12b) holds also for u. Setting $\kappa = 1-\gamma$, we have shown the common bounds. *Q.E.D.*

Convergence of the Iterative Scheme. With this result, we now turn our eyes toward setting up the linearization/iteration of the mollified equation (3.2). The following theorem shows that the iteration scheme is not only well-defined, but also convergent.

LEMMA 3 ("Low norm" convergence). *Let $T > 0$ be given and suppose J_ε is a mollifier for every $\varepsilon > 0$. If φ and ψ are supported compactly and the initial energy $E_0 = |\varphi|^2_{H^4(R^2)} + \|\psi\|^2_{H^3(R^2)}$ is small enough, then the sequence $u^1, u^2, \ldots$ generated by the iteration*

$$\begin{aligned} \ddot{u}^i + \mu\dot{u}^i &= \operatorname{div}(a(J_\varepsilon u^{i-1})\nabla u^i), \\ u^i(0) &= \varphi, \\ \dot{u}^i(0) &= \psi, \end{aligned}$$

for $i = 1, 2, \ldots$ with $u^0 = 0$ is well-defined and there is a unique $u \in \mathfrak{C}_1(0,T)$ such that $|||u^i - u|||_{1,T} \to 0$ as $i \to \infty$.

Proof. The fact that the sequence is well-defined follows immediately from Lemma 2, as long as we choose the initial energy small enough, according to (3.11). To see the ith step of the iteration is defined, set $v = J_\varepsilon u^{i-1}$. If

$$\|u^{i-1}\|_{L^\infty,T} \leq \kappa < 1$$

and

$$|u^{i-1}|^2_{H^4,T} + \|\dot{u}^{i-1}\|^2_{H^3,T} \leq \lambda E_0,$$

then properties (2.1) and (2.2) of mollification guarantee

$$\|v\|_{L^\infty,T} \leq \kappa < 1$$

and

$$|v|^2_{H^4,T} + \|\dot{v}\|^2_{H^3,T} \leq \lambda E_0,$$

and by Lemma 2, u^i has the properties required to continue the iteration. Clearly the argument above holds when $i = 1$ (i.e., $v = 0$), so the induction is complete. To show the convergence of the sequence $u^1, u^2, \ldots$ in $\mathfrak{C}_1(0, T)$, we begin by establishing an energy estimate for $w^i = u^i - u^{i-1}$. For the iterative procedure to work, we need at least $w^i \to 0$. The energy estimate to be developed presently shows that this is the case, and in fact provides enough control over this convergence as to render $u^1, u^2, \ldots$ a Cauchy sequence when the initial data are small enough.

Since u^i and u^{i-1} satisfy the differential equation, we have

$$\begin{aligned} \ddot{u}^i + \mu \dot{u}^i &= \operatorname{div}(a(J_\epsilon u^{i-1})\nabla u^i), \\ \ddot{u}^{i-1} + \mu \dot{u}^{i-1} &= \operatorname{div}(a(J_\epsilon u^{i-2})\nabla u^{i-1}), \end{aligned}$$

and by subtraction,

$$\ddot{w}^i + \mu \dot{w}^i = \operatorname{div}(a(J_\epsilon u^{i-1})\nabla w^i + (J_\epsilon w^{i-1})\nabla u^{i-1}).$$

Multiplying by $\dot{w}^i$ and integrating over R^2, manipulations as in the previous lemma yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{R^2} \left[(\dot{w}^i)^2 + a(J_\epsilon u^{i-1})|\nabla w^i|^2\right] dx &\leq \frac{1}{2}\int_{R^2} (J_\epsilon \dot{u}^{i-1})|\nabla w^i|^2\, dx \\ &+ \int_{R^2} \dot{w}^i[(J_\epsilon \nabla w^{i-1})\cdot \nabla u^{i-1} + (J_\epsilon w^{i-1})\Delta u^{i-1}]\, dx. \end{aligned}$$

Let ϵ^i denote the "energy" of the difference between successive solutions:

$$\epsilon^i(t) = \|\dot{w}^i\|^2_{L^2,t} + |w^i|^2_{H^1,t}.$$

Since $u^1, u^2, \ldots$ all satisfy the same initial conditions, we have $0 = \epsilon^2(0) = \epsilon^3(0) = \ldots$. In terms of these ϵ's, the previous inequality yields

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}\left[(\dot{w}^i)^2 + a(J_\epsilon u^{i-1})|\nabla w^i|^2\right]\,dx \le \frac{1}{2}\|\dot{u}^{i-1}\|_{L^\infty,T}\epsilon^i(t)$$
$$+\|\nabla u^{i-1}\|_{L^\infty,T}[\epsilon^i(t)]^{1/2}[\epsilon^{i-1}(t)]^{1/2}$$
$$+\|\Delta u^{i-1}\|_{L^\infty,T}[\epsilon^i(t)]^{1/2}\|w^{i-1}(t)\|_{L^2,T}^{1/2}$$

where on the right hand side we have used again the norm-bounding properties of the mollifier. We need to bound the right hand side of this inequality; by the Sobolev embedding theorem,

$$\|\nabla u^{i-1}\|_{L^\infty(R^2),T} \le C_s|u^{i-1}|_{H^4(R^2),T}$$

and by bounds established in the previous lemma,

$$\|\nabla u^{i-1}\|_{L^\infty(R^2),T} \le C_s\lambda^{1/2}E_0^{1/2}.$$

whenever E_0 is small enough. Likewise,

$$\|\Delta u^{i-1}\|_{L^\infty(R^2),T} \le 2C_s\lambda^{1/2}E_0^{1/2}$$

and

$$\|\dot{u}^{i-1}\|_{L^\infty(R^2),T} \le C_s\lambda^{1/2}E_0^{1/2}.$$

Further, the support of w^i can grow only as fast as the supports of u^i and u^{i-1}, and these have known bounds. In particular, the Poincaré inequality holds as in Lemma 2, and we have

$$\|w^i(t)\|_{L^2(R^2)} \le C_p|w^i(t)|_{H^1(R^2)} \le C_p[\epsilon^i(t)]^{1/2}$$

where $C_p = \frac{1}{2}d + 2T$. With these results, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}\left[(\dot{w}^i)^2 + a(J_\epsilon u^{i-1})|\nabla w^i|^2\right]\,dx \le \tfrac{1}{2}C_s\lambda^{1/2}E_0^{1/2}\epsilon^i(t)$$
$$+C_s\lambda^{1/2}E_0^{1/2}[\epsilon^i(t)]^{1/2}[\epsilon^{i-1}(t)]^{1/2}$$
$$+2C_sC_p\lambda^{1/2}E_0^{1/2}[\epsilon^i(t)]^{1/2}[\epsilon^{i-1}(t)]^{1/2}.$$

The crudest of estimates now suffice: integrate over $[0,t]$ where $0 < t \le T$ and use the fact that $w^i(0) = \dot{w}^i(0) = 0$ to get

$$\frac{1}{2}\int_{R^2}\left[(\dot{w}^i(t))^2 + a(J_\varepsilon u^{i-1}(t))|\nabla w^i(t)|^2\right]\,dx$$
$$\le \tfrac{1}{2}C_s\lambda^{1/2}E_0^{1/2}\epsilon^i(T)T + (1+2C_p)C_s\lambda^{1/2}E_0^{1/2}[\epsilon^i(T)]^{1/2}[\epsilon^{i-1}(T)]^{1/2}T.$$

Now from Lemma 2, $0 < \gamma < a(J_\varepsilon u^{i-1})$ where $0 < \gamma < 1$, so the integral term is bounded below:

$$\tfrac{1}{2}\gamma\int_{R^2}\left[(\dot{w}^i(t))^2 + |\nabla w^i(t)|^2\right]\,dx$$
$$\le \frac{1}{2}\int_{R^2}\left[(\dot{w}^i(t))^2 + a(u^{i-1}(t))|\nabla w^i(t)|^2\right]\,dx.$$

This holds for every $t \in [0,T]$, so in fact we have

$$\tfrac{1}{4}\gamma\epsilon^i(T)$$
$$\le \gamma C_s\lambda^{1/2}E_0^{1/2}\epsilon^i(T)T + (1+2C_p)C_s\lambda^{1/2}E_0^{1/2}[\epsilon^i(T)]^{1/2}[\epsilon^{i-1}(T)]^{1/2}T.$$

Now if $\epsilon^{i-1}(T) = 0$ we've hit upon a fixed point; then $\epsilon^i(T) = \epsilon^{i+1}(T) = \cdots$ and the sequence of interest obviously converges. So supposing $\epsilon^{i-1}(T) \ne 0$, divide by it to get

$$\tfrac{1}{4}\gamma\left(\frac{\epsilon^i}{\epsilon^{i-1}}\right) \le \tfrac{1}{2}C_s\lambda^{1/2}E_0^{1/2}\left(\frac{\epsilon^i}{\epsilon^{i-1}}\right)T + (1+2C_p)C_s\lambda^{1/2}E_0^{1/2}\left(\frac{\epsilon^i}{\epsilon^{i-1}}\right)^{1/2}T$$

and by rearrangement,

$$(\gamma - 2C_s\lambda^{1/2}E_0^{1/2}T)\left(\frac{\epsilon^i}{\epsilon^{i-1}}\right)^{1/2} \le (4+8C_p)C_s\lambda^{1/2}E_0^{1/2}T.$$

The leftmost factor is positive (in fact, as already has been shown in Lemma 2, $\gamma - 724C_s\lambda^{1/2}E_0^{1/2}T$ is positive), so

$$\left(\frac{\epsilon^i}{\epsilon^{i-1}}\right)^{1/2} \le \frac{(4+8C_p)C_s\lambda^{1/2}E_0^{1/2}T}{\gamma - 2C_s\lambda^{1/2}E_0^{1/2}T}.$$

Clearly, by choosing E_0 small enough we can make the right hand side as small as we please. The particular choices and bounds made previously for γ, λ, and $E_0^{1/2}$ in fact guarantee the ratio $\epsilon^i(T)/\epsilon^{i-1}(T)$ is less than unity. Thus, $\epsilon^i(T) \leq A\vartheta^i$, where $0 \leq \vartheta < 1$ for some A. To show that this forces $u^1, u^2, \ldots$ to be a Cauchy sequence in $\mathfrak{C}_1(0,T)$, write $u^{m+k} - u^m$ as

$$u^{m+k} - u^m = \sum_{l=1}^{k}(u^{m+l} - u^{m+l-1}).$$

Then

$$\begin{aligned} \left|\!\left|\!\left| u^{m+k} - u^m \right|\!\right|\!\right|_{1,T} &\leq \sum_{l=1}^{k} \left|\!\left|\!\left| u^{m+l} - u^{m+l-1} \right|\!\right|\!\right|_{1,T} \\ &= \sum_{l=1}^{k} \left|\!\left|\!\left| w^{m+l} \right|\!\right|\!\right|_{1,T} \end{aligned}$$

and since $\left|\!\left|\!\left| w^i \right|\!\right|\!\right|_{1,T} \leq (1 + C_p)\epsilon^i(T)$,

$$\begin{aligned} \left|\!\left|\!\left| u^{m+k} - u^m \right|\!\right|\!\right|_{1,T} &\leq (1 + C_p)\sum_{l=1}^{k} \epsilon^{m+l}(T) \\ &\leq (1 + C_p)A\frac{\vartheta^m}{1-\vartheta} \end{aligned}$$

which tends to zero as $m \to \infty$. This shows $u^1, u^2, \ldots$ is a Cauchy sequence, and since $\mathfrak{C}_1(0,T)$ is complete we conclude $u^i \to u$ in $\mathfrak{C}_1(0,T)$. *Q.E.D.*

Removal of the Mollifier. We now seek to remove the mollifier in order to attack the full, nonsmoothed equation, (3.1). This shall be done in two steps: first, we use the technique of Lax [**19**], using interpolation inequalities to show that the convergence $u^i \to u$ in $\mathfrak{C}_1(0,T)$ established in the previous lemma, along with the uniform boundedness of the iterates in $\mathfrak{C}_4(0,T)$, actually yields convergence in any $\mathfrak{C}_\rho(0,T)$ for $1 \leq \rho < 4$; second, we choose $\varepsilon_1, \varepsilon_2, \ldots$ tending to zero, and use the theorems of Rellich and Arzela-Ascoli to produce a subsequence of solutions converging in $C^2([0,T] \times R^2)$ to a solution of (3.1).

A nice demonstration of Lax's technique is given by Majda in Chapter 2 of **[22]**.

PROPOSITION 4. *Let $T > 0$ be given and suppose J_ε is a mollifier for every $\varepsilon > 0$. If φ and ψ are supported compactly and the initial energy $E_0 = |\varphi|^2_{H^4(R^2)} + \|\psi\|^2_{H^3(R^2)}$ is small enough, then the Cauchy problem*

$$\begin{aligned} \ddot{u} + \mu\dot{u} &= \operatorname{div}(a(J_\varepsilon u)\nabla u), \\ u(0) &= \varphi, \\ \dot{u}(0) &= \psi, \end{aligned} \tag{3.13}$$

has a unique classical solution, $u \in C^2([0,T] \times R^2)$, and $u \in \mathfrak{C}_\rho(0,T)$ for every $\rho < 4$. Moreover, u, $\dot{u}$, and $\ddot{u}$ are bounded in $C([0,T], L^2(R^2))$ for every $\varepsilon > 0$ independently of ε.

Proof. We shall need to make a brief detour into the "fractional" Sobolev spaces, and this is only to use the interpolation inequality

$$\|f\|_{H^\rho} \le \|f\|_{H^r}^{\rho/r} \|f\|_{L^2}^{1-\rho/r} \tag{3.14}$$

for $\rho \le r$. By the previous lemma we can set up a sequence $u^1, u^2, \ldots$ of "approximate solutions" to the nonlinear equation with $u^i \to u$ in $\mathfrak{C}_1(0,T)$. Using the interpolation inequality with $r = 4$, we have

$$\|u^i - u^j\|_{H^\rho} \le \|u^i - u^j\|_{H^4}^{\rho/4} \|u^i - u^j\|_{L^2}^{1-\rho/4},$$

and consequently

$$\|u^i - u^j\|_{H^\rho,T} \le \|u^i - u^j\|_{H^4,T}^{\rho/4} \|u^i - u^j\|_{L^2,T}^{1-\rho/4}, \tag{3.15}$$

for every $0 \le \rho \le 4$. The first term on the right is always bounded (by construction of the sequence), and for ρ strictly less than four the second term on the right is tending to zero. Suppose then that $\rho = 7/2$. With the embedding $H^{7/2}(R^2) \hookrightarrow C^2(R^2)$, the interpolation inequality shows $u^i \to u$ in $C([0,T], C^2(R^2))$. In a similar fashion with $r = 3$,

$$\|\dot{u}^i - \dot{u}^j\|_{H^\rho,T} \leq \|\dot{u}^i - \dot{u}^j\|_{H^3,T}^{\rho/3} \|\dot{u}^i - \dot{u}^j\|_{L^2,T}^{1-\rho/3}. \tag{3.16}$$

With $\rho = 5/2$, and the fact that $H^{5/2}(R^2) \hookrightarrow C^1(R^2)$, we have $\dot{u}^i \to \dot{u}$ in $C([0,T], C^1(R^2))$. With these facts we conclude that, since the right hand side of

$$\ddot{u}^i = -\mu\dot{u}^i + a(J_\varepsilon u^{i-1})\Delta u^i + \nabla J_\varepsilon u^{i-1} \cdot \nabla u^i \tag{3.17}$$

is continuous and uniformly convergent to $-\mu\dot{u} + a(J_\varepsilon u)\Delta u + \nabla u \cdot \nabla u$, so the left hand side also is uniformly convergent; thus $u^i_{tt} \to u_{tt}$ in $C([0,T], C(R^2))$. And since

$$\begin{aligned} \|u^i - u\|_{C^2([0,T]\times R^2)} = &\|u^i - u\|_{C([0,T],C^2(R^2))} + \|\dot{u}^i - \dot{u}\|_{C([0,T],C^1(R^2))} \\ &+ \|\ddot{u}^i - \ddot{u}\|_{C([0,T],C(R^2))} \end{aligned}$$

it follows that $u^i \to u$ in $C^2([0,T] \times R^2)$ and u is a classical solution. Inequalities (3.15) and (3.16) establish this convergence in $\mathfrak{C}_\rho(0,T)$ for $\rho < 4$.

In order to show that this solution and its first two time derivatives are bounded functions in $C([0,T], L^2(R^2))$, and bounded independently of ε, use the fact that

$$\|u\|_{L^2,T} \leq \|u^i\|_{L^2,T} + \|u - u^i\|_{L^2,T}.$$

Since $u^i \to u$ in $C([0,T], L^2(R^2))$, the bound for u is established if we can show $\|u^i\|_{L^2,T}$ is bounded independent of ε. But this is the case since $\|u^i\|_{L^2,T} \leq C_p |u^i|_{H^1,T}$ and $|u^i|_{H^1,T}$ is bounded above by $\sqrt{\lambda E_0}$. A similar argument holds for the bound on $\|\dot{u}\|_{L^2,T}$. To show the uniform bound on $\ddot{u}$, we now show that $\ddot{u}^i \to \ddot{u}$ in $C([0,T], L^2(R^2))$ and that $\|\ddot{u}^i\|_{L^2,T}$ is bounded independent of ε. Since the support of $\ddot{u}^i(t)$ is contained in a bounded set Ω for every i and $t \in [0,T]$, it follows that

$$\|\ddot{u}^i - \ddot{u}\|_{L^2,T} \leq \|\ddot{u}^i - \ddot{u}\|_{C([0,T],L^\infty)} |\Omega|^{1/2}$$

and the right hand side tends to zero which shows that we have the convergence in $C([0,T],L^2)$. Now from (3.17) we have

$$\|\ddot{u}^i\|_{L^2,T} \le |\mu|\|\dot{u}^i\|_{L^2,T} + 2\|a(J_\varepsilon u^{i-1})\nabla u^i\|_{H^1,T}$$

and expanding the second term on the right, it follows from a little algebra that

$$\|\ddot{u}^i\|_{L^2,T} \le |u^i|_{H^2,T}\|a(J_\varepsilon u^{i-1})\|_{L^\infty,T} + |u^i|_{H^2,T}\|J_\varepsilon \nabla u^i\|_{L^\infty,T}.$$

Now

$$\|a(J_\varepsilon u^{i-1})\|_{L^\infty,T} \le 2-\gamma$$

as already established in Lemma 2, and

$$\begin{aligned}\|J_\varepsilon \nabla u^i\|_{L^\infty,T} &\le \|\nabla u^i\|_{L^\infty,T}\\ &\le C_p|u^i|_{H^3,T}.\end{aligned}$$

Since $|u^i|_{H^2,T} \le |u^i|_{H^3,T} \le \sqrt{\lambda E_0}$, the required bounded on $\|\ddot{u}\|_{L^2,T}$ follows; specifically,

$$\|\ddot{u}\|_{L^2,T} \le (2-\gamma)\sqrt{\lambda E_0} + C_p\lambda E_0$$

and the bound is independent of ε. *Q.E.D.*

Now we pass to the limit, $\varepsilon \to 0$, through a sequence $\varepsilon_1, \varepsilon_2, \ldots$ tending to zero. The properties established for the solutions of the mollified problem for each ε and the properties of the mollifier, J_ε, yield the desired convergence to a classical solution of the fully nonlinear wave equation (3.1). The special notations introduced for sequences and subsequences are used.

PROPOSITION 5. *Let $T > 0$ be given. If φ and ψ are supported compactly and the initial energy $E_0 = |\varphi|^2_{H^4(R^2)} + \|\psi\|^2_{H^3(R^2)}$ is small enough, then the Cauchy problem*

$$\begin{aligned}\ddot{u} + \mu\dot{u} &= \operatorname{div}(a(u)\nabla u),\\ u(0) &= \varphi,\\ \dot{u}(0) &= \psi,\end{aligned}$$

has a unique classical solution, $u \in C^2([0,T] \times R^2)$.

Proof. As shown in the previous theorem, the mollified nonlinear problem has a classical solution which in fact is in every $\mathfrak{C}_\rho(0,T)$ for $\rho < 4$. Suppose $\{\varepsilon_k : k \in N\}$ is a sequence of positive numbers tending to zero, and let u_k be the solution of (3.13) corresponding to ε being replaced by ε_k. It has been established that $\{u_k\}$, $\{\dot{u}_k\}$, and $\{\ddot{u}_k\}$ are uniformly bounded in $C([0,T], L^2(R^2))$, so it follows that when u_k and $\dot{u}_k$ are considered as functions of $[0,T]$ into L^2, the sequences $\{u_k\}$ and $\{\dot{u}_k\}$ are equicontinuous on $[0,T]$. Moreover, because of the finite propagation speed, the supports of all the u_k's are contained in a fixed, bounded domain. By the Rellich compactness theorem, the sequence $\{u_k(t)\}$ is sequentially compact in L^2 for each $t \in [0,T]$, so the Arzela-Ascoli theorem guarantees a subsequence $\{u_k : k \in N'\}$ converging in $C([0,T], L^2(R^2))$. Now $\{\dot{u}_k : k \in N'\}$ is equicontinuous (since $\{\dot{u}_k : k \in N\}$ is) and the theorems of Rellich and Arzela-Ascoli yield a subsequence $\{\dot{u}_k : k \in N''\}$ converging in $C([0,T], L^2(R^2))$. Thus we may use the interpolation inequality (3.14) again:

$$\|u_i - u_j\|_{H^\sigma,T} \le \|u_i - u_j\|_{H^\rho,T}^{\sigma/\rho} \|u_i - u_j\|_{L^2,T}^{1-\sigma/\rho},$$
$$\|\dot{u}_i - \dot{u}_j\|_{H^{\sigma-1},T} \le \|\dot{u}_i - \dot{u}_j\|_{H^{\rho-1},T}^{(\sigma-1)/(\rho-1)} \|\dot{u}_i - \dot{u}_j\|_{L^2,T}^{(\rho-\sigma)/(\rho-1)},$$

for $1 \le \sigma < \rho < 4$. In each of these expressions, the first term on the right is bounded and the second term tends to zero as $i,j \to \infty$ through N'', so $u_k \to u$ in $C([0,T], H^\sigma(R^2))$ and $\dot{u}_k \to \dot{u}$ in $C([0,T], H^{\sigma-1}(R^2))$ with $k \in N''$. Selecting $3 < \sigma < \rho < 4$, we have the embeddings $H^\sigma(R^2) \hookrightarrow C^2(R^2)$ and $H^{\sigma-1}(R^2) \hookrightarrow C^1(R^2)$, so it follows that $u_k \to u$ in $C([0,T], C^2(R^2))$ and $\dot{u}_k \to \dot{u}$ in $C([0,T], C^1(R^2))$ as $k \to \infty$ through N''.

Using the fact that each u_k satisfies a mollified differential equation, namely

$$\ddot{u}_k = -\mu\dot{u}_k + a(J_{\varepsilon_k} u_k)\Delta u_k + (J_{\varepsilon_k} \nabla u_k) \cdot \nabla u_k,$$

we now demonstrate $\ddot{u}_k \to \ddot{u}$ in $C([0,T], C(R^2))$. It is clear that $\dot{u}_k \to \dot{u}$, $\nabla u_k \to \nabla u$, and $\Delta u_k \to \Delta u$ in this space, so if we can show $a(J_{\varepsilon_k} u_k) \to a(u)$

and $J_{\varepsilon_k} \nabla u_k \to \nabla u$, the convergence of $\{\ddot{u}_k : k \in N''\}$ follows. Now

$$\begin{aligned}\left\| a(J_{\varepsilon_k} u_k) - a(u) \right\|_{L^\infty,T} &= \left\| J_{\varepsilon_k} u_k - u \right\|_{L^\infty,T} \\ &\leq \left\| J_{\varepsilon_k}(u_k - u) \right\|_{L^\infty,T} + \left\| J_{\varepsilon_k} u - u \right\|_{L^\infty,T}.\end{aligned}$$

Since $\left\| J_{\varepsilon_k}(u_k - u) \right\|_{L^\infty,T} \leq \left\| u_k - u \right\|_{L^\infty,T}$, the first term on the right tends to zero as $k \to \infty$ through N'', and since u is uniformly continuous on $[0,T] \times R^2$, it follows from relation (2.3) that the second term also tends to zero as $k \to \infty$ through N''.

The same argument shows $\left\| J_{\varepsilon_k} \nabla u_k - \nabla u \right\|_{L^\infty,T} \to 0$ as $k \to \infty$ through N'', so it follows that $\{\ddot{u}_k : k \in N''\}$ is a convergent sequence; and since the convergence is uniform, we have $\ddot{u}_k \to \ddot{u}$ in $C([0,T], C(R^2))$. Thus we have established the existence of $u \in C^2([0,T] \times R^2)$ satisfying the Cauchy problem for the nonlinear equation. *Q.E.D.*

Uniqueness of the Classical Solution. Assuming u and v are classical solutions satisfying the same initial data we can subtract the corresponding equations to get an equation for $w = u - v$ with $w(0) = \dot{w}(0) = 0$. A "low order" energy estimate analogous to the one in Lemma 3 may be developed showing $w = 0$ on $[0,T]$, thereby establishing the uniqueness of the classical solution. In this and the following section we shall use two sets K_ϱ and $\mathfrak{I}_\varrho$: Let K_ϱ be the closed ball of radius ϱ in R^2, and set

$$\mathfrak{I}_\varrho = \left\{ (\varphi, \psi) \in H^4(R^2) \times H^3(R^2) : \operatorname{supp} \varphi, \operatorname{supp} \psi \subset K_\varrho \right\}.$$

That is, $\mathfrak{I}_\varrho$ is to be considered a set of initial data for (3.1) vanishing outside of the ball K_ϱ.

PROPOSITION 6. *A classical solution of (3.1) with initial data in $\mathfrak{I}_\varrho$ is unique.*

Proof. Suppose u and u' are two classical solutions of (3.1) on $[0,T]$ corresponding to the initial data (φ, ψ) and (φ', ψ') which are in $\mathfrak{I}_\varrho$. Let

$w = u - u'$. We shall show by means of a "low norm" energy estimate that $w = 0$ when both $\varphi' = \varphi$ and $\psi' = \psi$. To this end, subtract the corresponding differential equations satisfied by u and u' to obtain

$$\begin{aligned}
\ddot{w} + \mu\dot{w} &= \operatorname{div}(a(u)\nabla u - a(u')\nabla u') \\
&= \operatorname{div}(a(u)\nabla u - a(u)\nabla u' + a(u)\nabla u' - a(u')\nabla u') \\
&= \operatorname{div}(a(u)\nabla w) + \operatorname{div}(w\nabla u').
\end{aligned}$$

We shall use the final term in this expression in its expanded form $\nabla w \cdot \nabla u' + w\Delta u'$. Multiply through by $\dot{w}$ to get

$$\frac{1}{2}\frac{\partial}{\partial t}(\dot{w}^2) + \mu\dot{w}^2 = \dot{w}\operatorname{div}(a(u)\nabla w) + \dot{w}\nabla w \cdot \nabla u' + \dot{w}w\Delta u'.$$

If we rewrite the divergence term as

$$\begin{aligned}
\dot{w}\operatorname{div}(a(u)\nabla w) &= \operatorname{div}(a(u)\dot{w}\nabla w) - \tfrac{1}{2}a(u)\frac{\partial}{\partial t}|\nabla w|^2 \\
&= \operatorname{div}(a(u)\dot{w}\nabla w) - \tfrac{1}{2}\frac{\partial}{\partial t}\left[a(u)|\nabla w|^2\right] + \tfrac{1}{2}\dot{u}|\nabla w|^2,
\end{aligned}$$

it follows that

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\left[\dot{w}^2 + a(u)|\nabla w|^2\right] + \mu\dot{w}^2 = \operatorname{div}(a(u)\dot{w}\nabla w) &+ \tfrac{1}{2}\dot{u}|\nabla w|^2 \\
&+ \dot{w}\nabla w \cdot \nabla u' + \dot{w}w\Delta u'.
\end{aligned}$$

Now upon integration with respect to the spatial variables of R^2, the divergence vanishes; using the fact that $\mu \geq 0$, we obtain the inequality

$$\begin{aligned}
\text{(3.18)} \qquad \frac{1}{2}\frac{d}{dt}\int_{R^2}\left[\dot{w}^2 + a(u)|\nabla w|^2\right]dx &\leq \tfrac{1}{2}\int_{R^2}\dot{u}|\nabla w|^2\,dx \\
&+ \int_{R^2}\dot{w}\nabla w \cdot \nabla u'\,dx + \int_{R^2}\dot{w}w\Delta u'\,dx
\end{aligned}$$

which shall serve as basis for the final energy estimate. Set

$$\delta(t) = \int_{R^2}\left[\dot{w}^2 + a(u)|\nabla w|^2\right]dx.$$

We may bound the right hand side of (3.18) in terms of δ as follows:

$$\begin{aligned}\tfrac{1}{2}\int_{R^2} \dot{u}|\nabla w|^2\,dx &= \tfrac{1}{2}\int_{R^2}\{a^{-1}(u)\dot{u}\}\cdot\{a(u)|\nabla w|^2\}\,dx \\ &\le \tfrac{1}{2}\|a^{-1}(u)\dot{u}\|_{L^\infty}\|a^{1/2}(u)\nabla w\|_{L^2}^2 \\ &\le \tfrac{1}{2}\|a^{-1}(u)\dot{u}\|_{L^\infty}\delta(t);\end{aligned}$$

$$\begin{aligned}\int_{R^2}\dot{w}\nabla w\cdot\nabla u'\,dx &= \int_{R^2}\{a^{-1/2}(u)\nabla u'\}\cdot\dot{w}\{a^{1/2}(u)\nabla w\}\,dx \\ &\le \|a^{-1/2}(u)\nabla u'\|_{L^\infty}\|\dot{w}\|_{L^2}\|a^{1/2}(u)|\nabla w|\|_{L^2} \\ &\le \|a^{-1/2}(u)\nabla u'\|_{L^\infty}\delta(t).\end{aligned}$$

To bound the final term, we require the Poincaré inequality to provide a bound on the rate at which w grows.

$$\begin{aligned}\int_{R^2}\dot{w}w\Delta u'\,dx &\le \|\Delta u'\|_{L^\infty}\|\dot{w}\|_{L^2}\|w\|_{L^2} \\ &\le C_p\|\Delta u'\|_{L^\infty}\|\dot{w}\|_{L^2}\|\nabla w\|_{L^2} \\ &\le C_p\|a^{-1/2}(u)\|_{L^\infty}\|\Delta u'\|_{L^\infty}\|\dot{w}\|_{L^2}\|a^{1/2}(u)\nabla w\|_{L^2} \\ &\le C_p\|a^{-1/2}(u)\|_{L^\infty}\|\Delta u'\|_{L^\infty}\delta(t).\end{aligned}$$

Thus we have established the fact that

$$\begin{aligned}\dot{\delta}(t) \le \{&\|a^{-1}(u)\dot{u}\|_{L^\infty} + 2\|a^{-1/2}(u)\nabla u'\|_{L^\infty} \\ &+ 2C_p\|a^{-1/2}(u)\|_{L^\infty}\|\Delta u'\|_{L^\infty}\}\delta(t).\end{aligned} \tag{3.19}$$

Since we are dealing with classical solutions, the term in braces is bounded on $[0,T]$ by some constant C and it follows that $\dot{\delta}(t) \le C\delta(t)$, and so $\delta(t) \le e^{Ct}\delta(0)$. When the initial data are identical, $\delta(0) = 0$, so $\delta = 0$ on $[0,T]$ which demonstrates $u = u'$ there. *Q.E.D.*

Continuous Dependence upon the Initial Data. To conclude the proof of the well-posedness of the Cauchy problem for long time classical solutions of (3.1), we demonstrate that the solution depends in a continuous manner upon the initial data. This fact follows from another energy estimate of the variety found in Lemma 2, with a few additions. Rather than perform the detailed estimate showing $u' \to u$ in $C^2([0,T] \times R^2)$ as $(\varphi', \psi') \to (\varphi, \psi)$ in $\mathfrak{I}_\varrho$ we show the low norm estimate, $\|u' - u\|_{H^1} + \|\dot{u}' - \dot{u}\|_{L^2} \to 0$.

PROPOSITION 7. *Suppose $\varrho > 0$ and $(\varphi, \psi) \in \mathfrak{I}_\varrho$. If $\|\varphi\|_{H^4}^2 + \|\psi\|_{H^3}^2$ is small eneough, then a unique classical solution of (3.1) exists. Further, the problem depends continuously upon the initial data in the sense that $|u' - u|_{H^1} + \|\dot{u}' - \dot{u}\|_{L^2} \to 0$ as $(\varphi', \psi') \to (\varphi, \psi)$ in $\mathfrak{I}_\varrho$; u' is the classical solution of (3.1) corresponding to the initial data (φ', ψ').*

Proof. The results of Propositions 5 and 6 guarantee the existence of a unique classical solution whenever $\|\varphi\|_{H^4}^2 + \|\psi\|_{H^3}^2$ is small enough, the actual degree of smallness required depending only upon T and the diameter of the initial data. It is clear that we can base this requirement upon T and ϱ instead since the support of the initial data is contained in K_ϱ. It follows then that for fixed T (which we assume), there is a number $\mathfrak{e}_\varrho > 0$ with the property that a unique classical solution of (3.1) exists whenever we have initial data $(\varphi', \psi') \in \mathfrak{I}_\varrho$ bounded as $\|\varphi'\|_{H^4}^2 + \|\psi'\|_{H^3}^2 \leq \mathfrak{e}_\varrho$.

Now assume (φ, ψ) has been chosen in $\mathfrak{I}_\varrho$ such that $\|\varphi\|_{H^4}^2 + \|\psi\|_{H^3}^2 \leq \frac{1}{2}\mathfrak{e}_\varrho$. We are interested in letting $(\varphi', \psi') \to (\varphi, \psi)$ in $\mathfrak{I}_\varrho$ in the sense that $\|\varphi' - \varphi\|_{H^4} \to 0$ and $\|\psi' - \psi\|_{H^4} \to 0$, so clearly for all $(\varphi', \psi') \in \mathfrak{I}_\varrho$ near enough to (φ, ψ) we have $\|\varphi'\|_{H^4}^2 + \|\psi'\|_{H^3}^2 \leq \mathfrak{e}_\varrho$ and each problem (3.1) with initial data (φ', ψ') will have a unique classical solution, u'. To show the dependence upon the data, we are in a position to use (3.19) from the previous result. Since u is a classical solution, it follows that $a(u)$ is bounded below on $[0,T]$: $0 < \gamma \leq a(u)$. Let $C(t)$ denote the term in braces in (3.19). Then

$$C(t) \leq \gamma^{-1}\|\dot{u}\|_{L^\infty,T} + 2\gamma^{-1/2}\|\nabla u'\|_{L^\infty,T} + 2\gamma^{-1/2}C_p\|\Delta u'\|_{L^\infty,T}$$

and by Sobolev embedding,

$$\begin{aligned} C(t) &\le \gamma^{-1} C_s \|\dot{u}\|_{H^2,T} + 2\gamma^{-1/2} C_s \|\nabla u'\|_{H^2,T} + 2\gamma^{-1/2} C_p C_s \|\Delta u'\|_{H^2,T} \\ &\le \gamma^{-1} C_s E(T)^{1/2} + 2\gamma^{-1/2}(1 + C_p) C_s E'(T)^{1/2} \end{aligned}$$

where $E(t) = |u|^2_{H^4,t} + \|\dot{u}\|^2_{H^3,t}$ and $E'(t) = |u'|^2_{H^4,t} + \|\dot{u}'\|^2_{H^3,t}$. By the Cauchy-Schwarz inequality then,

$$C(t) \le M \cdot (E(T) + E'(T))^{1/2}$$

where

$$M = \left(\gamma^{-2} C_s^2 + 2\gamma^{-1}(1 + C_p)^2 C_s^2\right)^{1/2}.$$

Now since the initial data are all sufficiently small, we know from Lemma 2 that there is a number λ depending only upon T and ϱ with the property that $E(T) \le \lambda E(0)$ and $E'(T) \le \lambda E'(0)$ so it follows that

$$C(t) \le \lambda^{1/2} M \cdot \left(E(0) + E'(0)\right)^{1/2}$$

and thus (3.19) leads to

$$\dot{\delta}(t) \le \lambda^{1/2} M \cdot \left(E(0) + E'(0)\right)^{1/2} \delta(t),$$

therefore,

$$\delta(t) \le \exp\left(\lambda^{1/2} M \cdot (E(0) + E'(0))^{1/2} t\right) \delta(0).$$

As $(\varphi', \psi') \to (\varphi, \psi)$ in $\mathfrak{I}_\varrho$, we have $\delta(0) \to 0$; the argument of the exponential remains bounded, so $\max_{[0,T]} \delta(t) \to 0$ and demonstrates the continuous dependence upon the initial data. *Q.E.D.*

Chapter 4

An Asymptotic Result for the Lifespan of Classical Solutions

It has been established in the previous sections that we always have a unique classical solution of the nonlinear system (3.1) on $[0, T]$ provided the initial data are chosen small enough. We now consider the reverse problem: How does the magnitude of the initial data influence the lifespan of the classical solution? Recall that for a nonlinear system, a smooth solution usually can be guaranteed for only a finite time, and after this time smoothness may be lost due to the formation of shocks or the breaking of waves. For a given set of initial data, let $[0. T_*)$ be the largest interval of time on which a smooth solution of (3.1) exists. (Clearly we may have $T_* = \infty$ in the case of a global solution.) Then T_* is referred to as the *lifespan* of the classical solution. As pointed out in the introduction, John & Klainerman [11] discovered the phenomenon of "almost global" solutions to certain nonlinear wave equations in which the lifespan T_* is very sensitive to the size of the initial data. Using results developed in Lemma 2, we shall obtain a lower bound for T_*. A much more difficult and delicate problem is to find an upper bound for T_*.

PROPOSITION 8. *Consider the parameterized system in two space dimensions*

$$
\begin{aligned}
&\ddot{u} + \mu\dot{u} = \operatorname{div}(a(u)\nabla u),\\
(4.1)\qquad &u(0) = \varepsilon\varphi,\\
&\dot{u}(0) = \varepsilon\psi,
\end{aligned}
$$

where φ and ψ are of compact support, $|\varphi|_{H^4} + \|\psi\|_{H^3}$ is finite, and $\varepsilon > 0$. Let $T_*(\varepsilon)$ be the lifespan of the classical solution of (4.1). Then $T_*(\varepsilon) \to \infty$ as $\varepsilon \to 0+$, and in fact, there is a lower bound $T(\varepsilon)$ such that $T_*(\varepsilon) \geq T(\varepsilon)$ and $T(\varepsilon) = O(\varepsilon^{-1/2})$.

Proof. For convenience, set $\varepsilon C = (\|\varepsilon\varphi\|^2_{H^4} + \|\varepsilon\psi\|^2_{H^3})^{1/2}$ where we shall regard C as fixed and assume $(\varphi, \psi) \in \mathfrak{I}_\varrho$. As in (3.10), let A and B be the quantities

$$(4.2a)\qquad A = \frac{724}{1 + C_p}T,$$

$$(4.2b)\qquad B = \frac{6 + 8C_p}{1 + C_p}T,$$

where $C_p = \frac{1}{2}d + 2T$. Also let

$$(4.2c)\qquad \gamma = \tfrac{1}{2} + \tfrac{1}{2}\max\left\{\frac{A}{1 + A}, \frac{B}{1 + B}\right\},$$

and

$$(4.2d)\qquad \lambda = \frac{4 - 2\gamma}{\gamma - (1 - \gamma)A}.$$

As was shown in the previous sections, if we choose the initial data small enough, i.e., ε such that

$$(4.3)\qquad \varepsilon C \leq \frac{A}{724 C_s T}\frac{1 - \gamma}{\lambda^{1/2}},$$

then a unique classical solution of (4.1) exists on $[0, T]$. Conversely, for a given ε, a classical solution exists on $[0, T]$ for any T satisfying (4.3). Let us examine the case in which ε is given, and look for all values of T satisfying

$$\varepsilon C = \frac{A}{724 C_s T} \frac{1-\gamma}{\lambda^{1/2}}.$$

By definition of A, we may rewrite this as

$$\varepsilon C = \frac{1-\lambda}{(1+C_p) C_s \lambda^{1/2}}. \tag{4.4}$$

We really are interested in the case $\varepsilon \to 0+$, and note that for this expression to hold we need one of the following to occur:

$$C_p \to \infty, \tag{4.5a}$$

$$\gamma \to 1-, \tag{4.5b}$$

$$\lambda \to \infty. \tag{4.5c}$$

Clearly, since $C_p = \frac{1}{2}d + 2T$ (where $d = 2\varrho$ is fixed), the condition (4.5a) necessitates $T \to \infty$. If instead $\gamma \to 1-$ as in (4.5b), we have from (4.2c) the fact that $B \to \infty$ (since the quantity $A/(1+A)$ is bounded above by 362/363) and so again have $T \to \infty$. Finally, it is clear that the only way for λ to approach ∞ as in (4.5c) is to have $\gamma - (1-\gamma)A \to 0+$. But by (4.3c) this cannot happen, so it follows that in any case, for ε and T satisfying (4.4), $T \to \infty$ as $\varepsilon \to 0+$.

When ε is sufficiently small, then T is large, and eventually $B/(1+B)$ will be larger than $A/(1+A)$ since the former is not bounded as $T \to \infty$, but the latter is. Thus for ε small,

$$\gamma = \tfrac{1}{2} + \tfrac{1}{2}\frac{B}{1+B}.$$

It is straightforward to show that as $T \to \infty$,

$$A = 362 - 181(1 + \tfrac{1}{2}d)T^{-1} + O(T^{-2})$$

and

$$B = 8T - 1 + \tfrac{1}{2}(1 + \tfrac{1}{2}d)T^{-1} + O(T^{-2}),$$

so

$$\gamma = 1 - \tfrac{1}{16}T^{-1} + O(T^{-3}).$$

A similar calculation may be made for λ as given by (4.2d) with the result that the right hand side of (4.4) may be computed and yields

$$\varepsilon C = \frac{1}{32\sqrt{2}C_s T^2} + O(T^{-3}).$$

Reversion of the formal series gives $T(\varepsilon) = O(\varepsilon^{-1/2})$ as $\varepsilon \to 0+$. Since a solution is guaranteed to exist on $[0, T(\varepsilon)]$, it follows that $T_*(\varepsilon) \geq T(\varepsilon)$. *Q.E.D.*

Chapter 5

Analytical Results for Forced Waves in One Dimension

We now highlight a few analytical results available for the nonlinear wave equation, (1.7), when modeling surface waves propagating in a long, one-dimensional wavetank. It shall be assumed the disturbances are due to a wave generator located at $x = 0$, and that the waves are being driven onto still water of constant depth. There are two main results presented in this chapter, the first one being something of a curiosity, the second perhaps useful in the design of experiments to test the validity of the nonlinear wave model.

Perturbation Solution. Suppose that we rescale (1.7) by replacing u with Av, where A is a dimensionless quantity representing the (dimensionless) amplitude of the wave generator output. We then obtain the system

$$\begin{aligned} v_{tt} &= ((1 + Av)v_x)_x, \\ v(0, x) &= 0, \\ v_t(0, x) &= 0, \\ v(t, 0) &= g(t), \end{aligned}$$

where the viscous damping has been neglected. In the case of small amplitudes, we can look for a solution by means of perturbation expansion.

Suppose then that

$$v = v^0 + Av^1 + A^2v^2 + \cdots.$$

Then it may be verified that the two lowest terms are

$$v^0(t,x) = g(t-x)$$

and

$$v^1(t,x) = \tfrac{1}{2}xg'(t-x)g(t-x).$$

It might be expected that examination of the higher terms is tedious and this is indeed the case, yet an interesting trend develops in the representation of these solutions: it *appears* that

$$v^k(t,x) = \mathcal{P}_k(x\partial/\partial t)[g(t-x)]^{k+1}$$

where $\mathcal{P}_k$ is a polynomial of degree k, and as evidence we have

$$\begin{aligned}
v^0(t,x) &= g(t-x),\\
v^1(t,x) &= \tfrac{1}{4}(x\partial/\partial t)[g(t-x)]^2,\\
v^2(t,x) &= \{\tfrac{1}{24}(x\partial/\partial t)^2 - \tfrac{1}{8}(x\partial/\partial t)\}[g(t-x)]^3,\\
v^3(t,x) &= \{\tfrac{1}{192}(x\partial/\partial t)^3 - \tfrac{3}{64}(x\partial/\partial t)^2 + \tfrac{5}{64}(x\partial/\partial t)\}[g(t-x)]^4.
\end{aligned}$$

The associated polynomials up to fourth order are

$$\begin{aligned}
\mathcal{P}_0(z) &= 1,\\
\mathcal{P}_1(z) &= \tfrac{1}{4}z,\\
\mathcal{P}_2(z) &= \tfrac{1}{24}z^2 - \tfrac{1}{8}z,\\
\mathcal{P}_3(z) &= \tfrac{1}{192}z^3 - \tfrac{3}{64}z^2 + \tfrac{5}{64}z,\\
\mathcal{P}_4(z) &= \tfrac{1}{1920}z^4 - \tfrac{3}{320}z^3 + \tfrac{29}{640}z^2 - \tfrac{7}{128}z.
\end{aligned}$$

Whether or not this solution form continues for higher terms and is true in general has not been established.

Lifespan of Forced Waves. An application of (1.7) of particular interest is that of forced waves in a wave tank where, as shown in Fig. 2, a wave generator drives waves onto initially still water of constant depth. If damping mechanisms due to bottom and sidewall drag are negligible, linear theory predicts wave motion of permanent form, $u(t,x) = f(t - x/c)$, where c is the wave speed and f is represents the output of the wave generator. The present nonlinear model provides an alternative to the linear theory, and wave tank experimentation can be used to check the validity of this theory. It is necessary, though, to develop some sort of criteria for judging this validity. A prominent feature of the nonlinear model is that, in contrast with the results of the linear theory, "classical solutions" in one spatial dimension exist for only finite time, and in fact, numerical experiments indicate that this is due to the tendency of the waves to shoal and break. This distinction can be used to test (1.7) as a realistic model of wave propagation. Of course the validity of (1.7) as a wave model is dubious at, and near, the point of breaking, so we cannot expect to predict when *real* waves break; however, it is reasonable to suspect that the wave experiences significant modification in shape as this point is neared. For this reason, it is important to estimate the time at which breakdown of the classical solution occurs, for this provides a parameter for gauging the time required for nonlinear behavior to develop. In the following, an expression for this breakdown time, t_b, is found. For the special case of a still-water depth, H_0, and a sinusoidal driver, $f(t) = AH_0 \sin(2\pi t/T)$, it is found that

$$\frac{t_b}{T} = \frac{1}{\pi A}$$

for $A > 0$ and

$$\frac{t_b}{T} \approx \frac{1}{2} - \frac{1}{\pi A}$$

when $A \to 0-$. Thus, for example, if a wave generator with 2 s period drives waves of 0.1 m amplitude onto 2 m of still water, then $A = 0.05$ and the classical solution ceases to exist at about 19 s, a distance of 84 m from the generator.

The appropriate scaled expressions for the nonlinear model are

$$
\begin{aligned}
u_{tt} &= ((1+u)u_x)_x, \\
u(0,x) &= \varphi(x), \\
u_t(0,x) &= \psi(x), \\
u(t,0) &= f(t),
\end{aligned}
\tag{5.1}
$$

where f describes the output of the wave generator, and φ and ψ specify the initial conditions of the water. Later we shall take $\varphi = \psi = 0$, but for the moment we formulate the general problem.

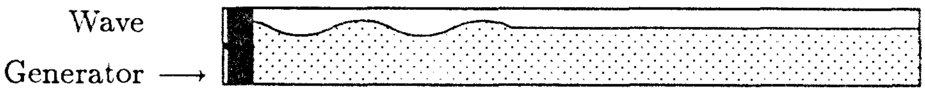

Fig. 2. Schematic of Wave Tank with Wave Generator.

The forced wave problem may be recast as a first order system:

$$
\begin{aligned}
u_t + v_x &= 0, \\
v_t + (1+u)u_x &= 0, \\
u(0,x) &= u_0(x) = \varphi(x), \\
v(0,x) &= v_0(x) = -\int_0^x \psi(\xi)\, d\xi, \\
u(t,0) &= f(t),
\end{aligned}
\tag{5.2}
$$

and is amenable to treatment by the hodograph transformation [2, 16]. Under this transformation, we compute the solution along characteristics as well as the characteristics themselves. The crossing of characteristics signals the breakdown of the classical solution. so we shall be interested primarily in computing the characteristics and their earliest crossing.

The transformation begins by introducing new coordinates α and β which parameterize the characteristics. If we write (5.2) in matrix form as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1+u & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0$$

then a curve $(\hat{x}(s), \hat{t}(s))$ is characteristic if

$$\hat{x}_s = \lambda \hat{t}_s$$

where λ is an eigenvalue of the square matrix. Since there are two eigenvalues, in this case $\pm\sqrt{1+u}$, we describe the characteristics in terms of the parameters α and β by

$$\hat{x}_\alpha = \hat{t}_\alpha \sqrt{1+\hat{u}}, \tag{5.3a}$$

$$\hat{x}_\beta = -\hat{t}_\beta \sqrt{1+\hat{u}}, \tag{5.3b}$$

where we set $\hat{u}(\alpha, \beta) = u(\hat{t}(\alpha, \beta), \hat{x}(\alpha, \beta))$ and $\hat{v}(\alpha, \beta) = v(\hat{t}(\alpha, \beta), \hat{x}(\alpha, \beta))$. Clearly,

$$\begin{aligned} \hat{u}_\alpha &= u_t \hat{t}_\alpha + u_x \hat{x}_\alpha, \\ \hat{u}_\beta &= u_t \hat{t}_\beta + u_x \hat{x}_\beta, \\ \hat{v}_\alpha &= v_t \hat{t}_\alpha + v_x \hat{x}_\alpha, \\ \hat{v}_\beta &= v_t \hat{t}_\beta + v_x \hat{x}_\beta, \end{aligned}$$

so we can solve for u_t, u_x, v_t, and v_x in terms of $\hat{u}_\alpha$, $\hat{u}_\beta$, $\hat{v}_\alpha$, and $\hat{v}_\beta$. Doing this and substituting these expressions into (5.2) leads to the relations

$$\begin{aligned} -\hat{u}_\alpha \sqrt{1+\hat{u}} + \hat{v}_\alpha &= 0, \\ \hat{u}_\beta \sqrt{1+\hat{u}} + \hat{v}_\beta &= 0. \end{aligned} \tag{5.4}$$

Equations (5.3) and (5.4), the hodograph transformation of the original system, are to be solved now. From (5.4) we have upon integration

$$-\frac{2}{3}(1+\hat{u})^{3/2} + \hat{v} = K_1(\beta), \tag{5.5a}$$

$$\frac{2}{3}(1+\hat{u})^{3/2} + \hat{v} = K_2(\alpha) \tag{5.5b}$$

where K_1 and K_2 are "constants" of integration. Let $\tau > 0$ represent a moment of time. As shown in Fig. 3a we associate with the point $(0, \tau)$ two characteristic curves, C^1 and C^2. The curve C^1 eminates from a point on the x-axis and is the characteristic generated by varying β with α held fixed, and passes through $(0, \tau)$. The curve C^2 is the characteristic eminating from $(0, \tau)$, generated by letting α vary while holding β fixed. Finally, for $(x, t) \in C^2$, let C^3 be the characteristic shown, intersecting C_2 at that point.

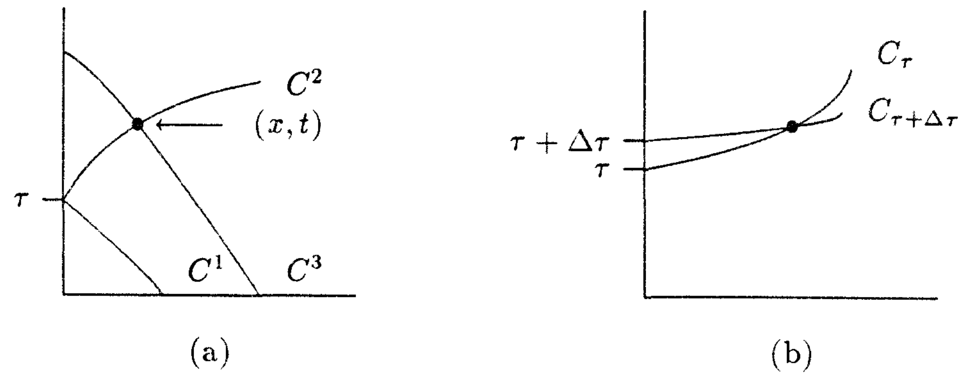

Fig. 3. Characteristic Curves in the (x, t)-Plane.

We now treat the special case of disturbances being driven into a region initially at rest, i.e., $u_0 = v_0 = 0$. Since u and v are initially zero, it follows from (5.5b) that $K_2 = 2/3$ so that along C^1 and C^3 we have

$$\frac{2}{3}(1 + \hat{u})^{3/2} + \hat{v} = \frac{2}{3},$$

which is to say that

(5.6a) $$\frac{2}{3}(1 + u(t, x))^{3/2} + v(t, x) = \frac{2}{3}$$

and

(5.6b) $$\frac{2}{3}(1 + f(\tau))^{3/2} + v(\tau, 0) = \frac{2}{3}.$$

Approaching (x, t) along C^2 we have from (5.5a)

$$-\frac{2}{3}(1+u(t,x))^{3/2}+v(t,x)=-\frac{2}{3}(1+f(\tau))^{3/2}+v(\tau,0). \tag{5.6c}$$

Solving (5.6) for $u(t, x)$, we obtain

$$u(t,x)=f(\tau). \tag{5.6a}$$

Thus $\hat{u}$ is constant along C^2 so by integrating (5.3a) along C^2, we find that x and t are related by

$$x=\sqrt{1+f(\tau)}(t-\tau). \tag{5.7b}$$

Notice that if the point (x, t) is specified, we can solve (5.7b) for τ and use (5.7a) to compute u. To investigate the breakdown of classical solutions of this problem, we use (5.7b) to find the time at which the α-characteristics cross. Suppose that C_τ and $C_{\tau+\Delta\tau}$ are the characteristic curves eminating from the t-axis as shown in Fig. 3b where we imagine $\Delta\tau > 0$ being as small as we please. If these two curves intersect, a classical solution ceases to exist. From (5.7b), this is when

$$\sqrt{1+f(\tau)}(t-\tau)=\sqrt{1+f(\tau+\Delta\tau)}(t-\tau-\Delta\tau)$$

or in terms of a derivative with respect to τ,

$$\frac{d}{d\tau}\left(\sqrt{1+f(\tau)}(t-\tau)\right)=0.$$

The computation shows that

$$t_b(\tau)=\tau+2\frac{1+f(\tau)}{f'(\tau)}$$

is the time at which neighboring characteristics cross, provided the second term on the right in nonnegative. With this expression we can find the time at which breakdown of the classical solution occurs by looking for the minimum of $t_b(\tau)$ as we allow τ to vary over $[0, \infty)$, excluding those intervals in which $(1 + f(\tau))/f'(\tau) < 0$. In general this may be a messy problem, yet the physically important case of sinusoidal driving, $f(\tau) = A\sin(2\pi\tau/T)$, lends itself neatly to analysis. A straightforward computation verifies that, when $A > 0$, the earliest crossing of characteristics corresponds to $\tau = 0$, so we have

$$\frac{t_b}{T} = \frac{1}{\pi A}. \tag{5.8}$$

In terms of "dimensional" quantities, $A = A_0/H_0$ where A_0 is the amplitude of the driven wave out of the wave generator, and H_0 is the depth of the initially still water, so it follows in this case that the breakdown time is a function of the "relative amplitude" and the period of the wave generator output.

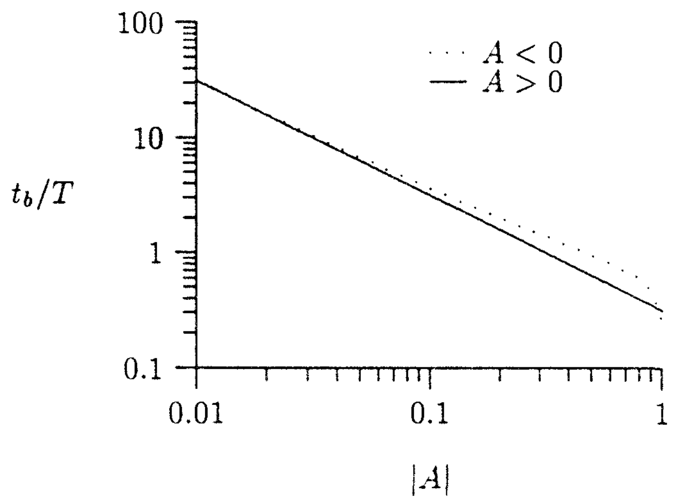

Fig. 4. Breakdown Times for Sinuoidal Drivers.

When $A < 0$, the analysis is more complicated, and a simple formula like (5.8) is not available. The results, however, are plotted in Fig. 4 along with those for the case $A > 0$, and it may be shown that

$$\frac{t_b}{T} \sim \frac{1}{2} - \frac{1}{\pi A}$$

as $A \to 0-$. As shown in the figure, the behavior of t_b/T becomes interesting as $A \to -1$. A bit of calculation shows

$$\frac{t_b}{T} \sim \frac{1}{4} + \frac{2}{\pi}\sqrt{1+A}\left(1 + \tfrac{29}{48}(1+A) + \tfrac{1363}{2560}(1+A)^2 + O(1+A)^3\right)$$

as $A \to -1$. Notice that (5.8) gives, in contrast to the result of Chapter 4, the fact that when $A > 0$, the lifespan for the driven wave (in one space dimension) behaves like $T_*(A) = O(A^{-1})$ whereas the pure Cauchy problem only has the weaker result $T_*(\varepsilon) \geq T(\varepsilon) = O(\varepsilon^{-1/2})$.

Chapter 6
Numerical Results

Forced Waves in One Dimension. To exhibit the behavior of the solution for the nonlinear wave equation $\eta_{tt} = g\mathrm{div}\,((H_0 + \eta)\nabla\eta)$, computations were made for the case of one and two spatial dimensions. Fig. 5 shows the progression in one dimension of a wave being forced onto a wave tank initially at rest. Sinusoidal waves of 0.4 m (peak-to-peak) and 4 s period were driven onto still water of depth $H_0 = 1.5$ m. With the surface waves progressing from left to right, we see the shape of the sinusoid beginning to alter at 6 s; at 8 s the leading edge is becoming steep while the troughs are more pronounced, and this trend continues at 10 s. The usual linear wave equation predicts waves of permenant form, so a perfect sinusoid would propagate unchanged in the linear theory. Further experiments indicate the presence of numerical problems shortly after 12 s. Indeed, from the results of the previous chapter on the lifespan of classical solutions for forced waves, we see from (5.8) and Fig. 4 that with $A = 0.2/1.5$, the solution will break down at $t = 9.55$ s. The fact that "numerical breakdown" of the solution occurs after this is probably due to two things: (i) there are smoothing effects of the discretization used which prevent breaking at the wavefront, and (ii) it is difficult to say precisely when the onset of numerical problems occurs as the degradation is gradual.

The Initial Value Problem in Two Dimensions. Fig. 6 & 7 show the results of calculations in two spatial dimensions for the special case $\eta(0, x) =$

0, $\eta_t(0,x) = \psi(x)$ where ψ is supported compactly on R^2 and selected so as to introduce two identical disturbances. As only qualitative behavior of the solution will be discussed, the precise form of ψ and the choice of H_0 are unimportant. Fig. 6 displays the formation and interaction of the surface waves caused by the initial disturbance ψ as described by the nonlinear wave equation

$$\eta_{tt} = g((H_0 + \eta)\eta_x)_x + g((H_0 + \eta)\eta_y)_y,$$

whereas for comparison, Fig. 7 shows the results obtained when the linear equation

$$\eta_{tt} = g(H_0\eta_x)_x + g(H_0\eta_y)_y$$

is used. The nonlinear waves tend, in all cases, to be of lesser amplitude than those resulting from the linear model, and indeed, are steeper as is evident at $t = 12$ and 16. Notice also that the superposition of the waves in the linear model at $t = 16$ is markedly different from the interaction predicted by the nonlinear model as is evidenced by the height and separation of the peaks.

The nonlinear waves tend to increase in steepness but the eventual breakdown of the solutions is not quite clear. Loosely speaking, we have the effects of nonlinearity tending to steepen the wave near the wavefront, while on the other hand, we expect a certain amount of dispersion to take place as the disturbance continues to evolve. If this dispersion reduces the amplitude of the wave quick enough, then perhaps the nonlinearity will become negligible at which point the wave progresses like a linear wave without breakdown. This is of course speculative, but the point is that existence of global solutions for small enough initial data *in two space dimensions* has not yet been disproved. The results of Li & Chen [20] however, suggest the possibility that such a solution does not exist. When, however, the damping term, $\alpha\eta_t$, is added, there is a glimmer of hope, yet no analytical results. What is needed to prove the nonexistence of a global classical solution is an example showing breakdown of these solutions occuring no matter how small the initial data are chosen; perhaps by looking at radial solutions and using techniques based upon the characteristics of the equation this example may be produced.

Solution by Finite Differences. The calculations were made using centered finite difference approximations for the time and space derivatives with special attention paid to preserving the divergence form of the equation. Thus, for example, in one spatial dimension, the discretizations

$$\eta_{tt}(k\Delta t, m\Delta x) \approx (\Delta t)^{-2}\{\eta_m^{k+1} - 2\eta_m^k + \eta_m^{k-1}\}$$

and

$$((H_0+\eta)\eta_x)_x(k\Delta t, m\Delta x) \approx$$
$$\frac{1}{\Delta x}\left\{(H_0 + \frac{\eta_{m+1}^k + \eta_m^k}{2})\frac{\eta_{m+1}^k - \eta_m^k}{\Delta x} - (H_0 + \frac{\eta_m^k + \eta_{m-1}^k}{2})\frac{\eta_m^k - \eta_{m-1}^k}{\Delta x}\right\}$$

were used. To maintain stability, a "practical" version of the Courant-Friedrichs-Lewy condition for the linear (constant propagation speed) wave equation was used, presumably to good effect. In its standard form, this condition requires Δt selected so that

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{c}$$

where c is the speed of the propagating wave. Since in the nonlinear case the characteristics propagate with speed $[g(H_0+\eta)]^{1/2}$, we select in our implementation

$$\frac{\Delta t}{\Delta x} \leq [g(H_0+\eta_0)]^{-1/2}$$

where η_0 is the largest value of η in the current solution step. Thus Δt may be modified as necessary in the computation although in the cases examined it was chosen sufficiently small that no such modifications were required.

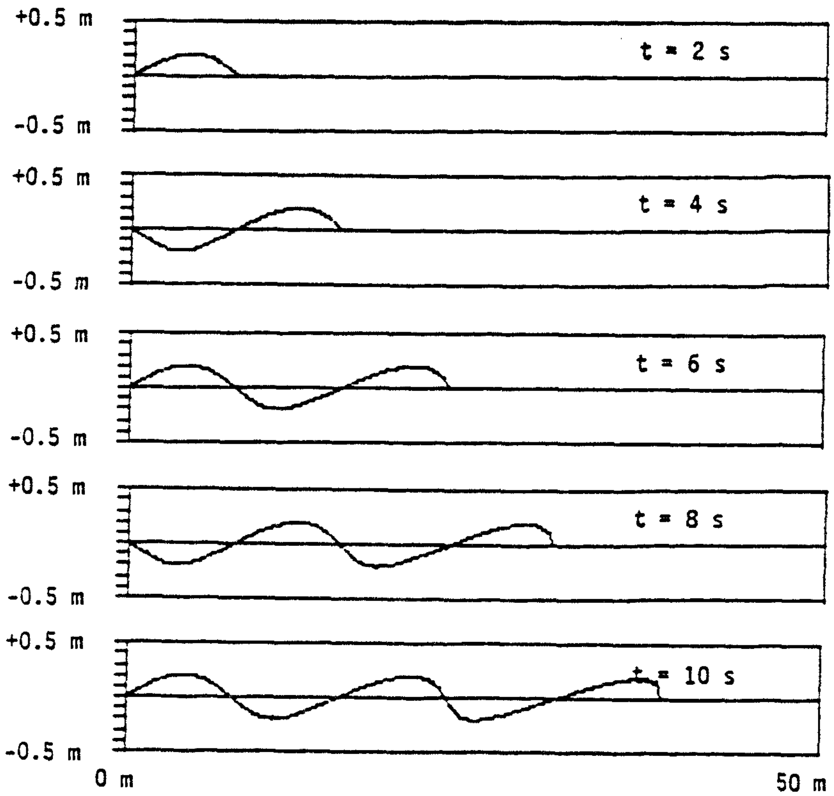

Fig. 5. Example of a Forced Wave.

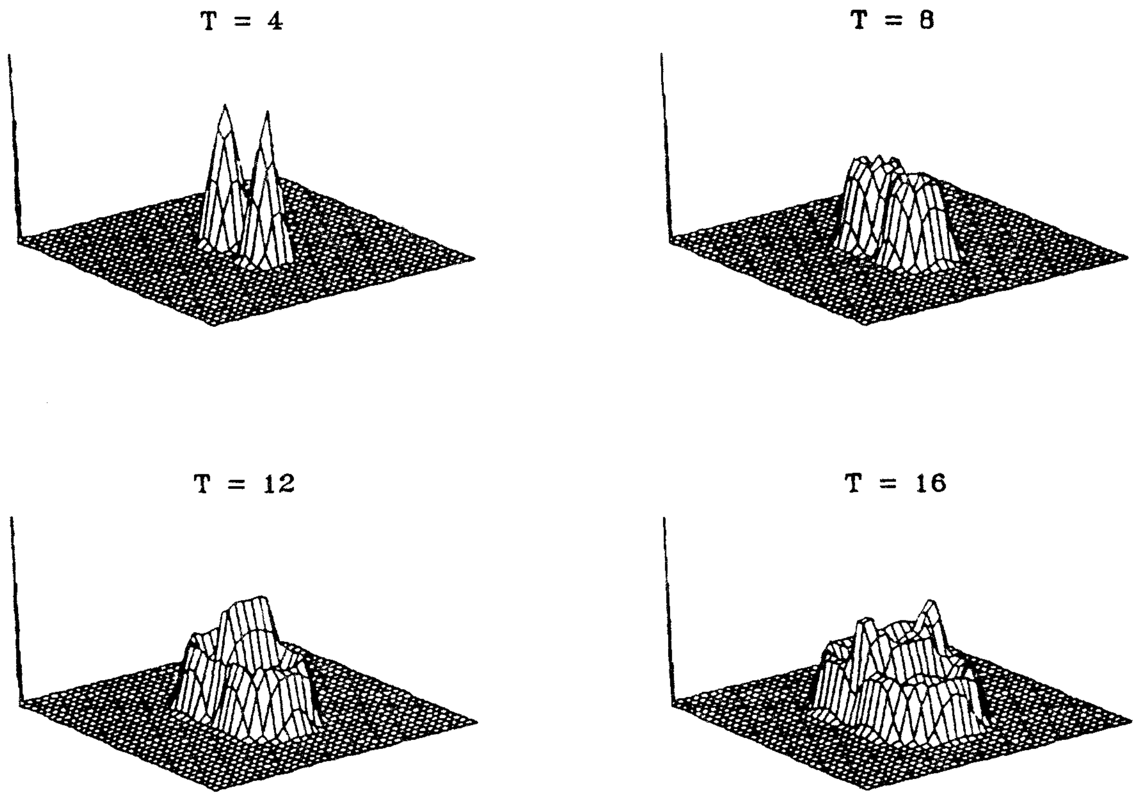

Fig. 6. Example of the Nonlinear Wave.

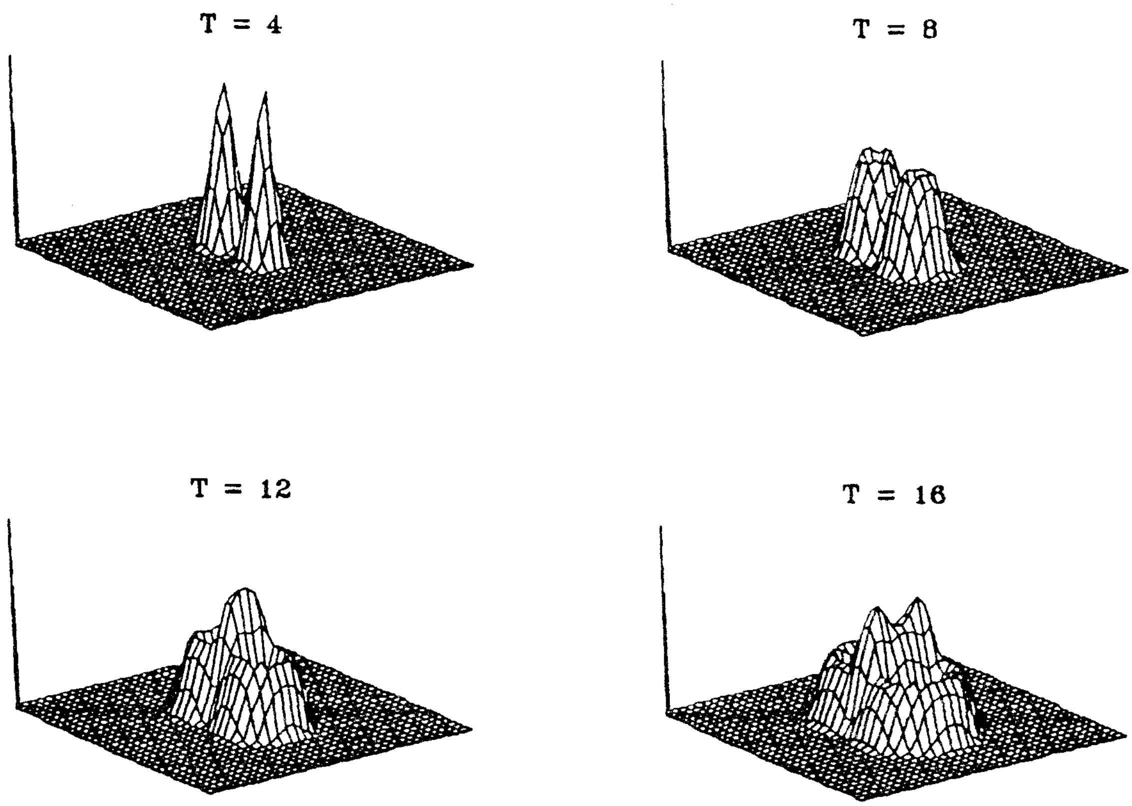

Fig. 7. Example of a Linear Wave.

Epilogue

There are a number of partial differential equations which more or less successfully describe the evolution of shallow water waves, and in the present study we add one more. This new equation, which in a simple, special case may be written as $u_{tt} + \mu u_t = \operatorname{div}((1+u)\nabla u)$, is distinguished by the presence of a damping term based upon physical principles instead of the rather artificial damping terms appended to other shallow water models. Indeed, the system of shallow water equations,

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0, \\ u_t + uu_x + vu_y + gh_x &= 0, \\ v_t + uv_x + vv_y + gh_y &= 0, \end{aligned}$$

of interest to oceanographers and engineers presents a difficult system to solve numerically. Actual techniques of computation [**32**] generally involve replacing the right hand side of the last two equations with the dissipative terms $-\alpha u$ and $-\alpha v$. There is no physical significance for such terms despite the appeal of simplicity, and so it is unclear just what numerical results represent, especially when the nonlinearity of the equation is retained. Similarly, the integral representation of the solution for the wave model presented by Gallagher [**6**] fails even to converge as there is a pole on the axis of integration. This is "remedied" by introducing a damping factor ad hoc with the net effect of shifting the singularity off the real axis. It is not clear, though, what is being described—the ***dissipative*** flow of a perfect fluid in irrotational motion! The model presented in the present study is nonlinear and contains, by derivation, a viscous damping term and for this reason is felt to be a significant contribution to shallow water theory.

On the mathematical end of things, it is necessary to establish conditions for the well-posedness of problems arising in a physical setting, and this has been done for classical solutions of the Cauchy problem with initial data $u(0,x)$, $u_t(0,x)$, of compact support, sufficiently small in $H^4(R^2)\times H^3(R^2)$. The proof of existence of solutions is essentially a fixed point argument: The solution of the linear equation

$$u_{tt}+\mu u_t = \operatorname{div}\left((1+v)\nabla u\right)$$

(with appropriate initial data) is $u = \mathcal{G}(v)$; the solution of the nonlinear problem of interest then is a fixed point of $\mathcal{G}$. Sufficiently small initial data force $\mathcal{G}$ to be a contraction in a "low norm" space—say, H^1—and existence of a fixed point follows from an appeal to the Banach contraction mapping theorem. Further, we may use this low norm contraction with a "high norm" boundedness—say, boundedness in H^4—to establish contraction by $\mathcal{G}$ in the spaces naturally interpolated by the low- and high-norm spaces, and another appeal to the Banach theorem gives existence in the classical solution space, $C^2([0,T]\times R^2)$. A result on the dependence of the lifespan of the classical solution upon the size of the initial data is established and given in Chapter 4.

Naturally this work was performed with the hope of eventually validating the model experimentally. Results given in Chapter 5 perhaps will be useful in setting up such an experiment in a standard wave tank. At present, this validation has not been performed. Numerical computation of solutions of the nonlinear model are presented in Chapter 6. There was little difficulty in the finite difference scheme as long as dynamic selection of the time step was made in order to preserve stability of the iteration.

There are a few points worth drawing attention to. First, it would be very desirable to have in hand a *global* existence result for classical solutions. The results of Lax and John cited in the introduction suggest the eventual presence of wave breaking, but these are in one spatial dimension. In two dimensions, the naturally occuring situation, the viscous dissipation and inherent spatial dispersion of the waves may work to prevent this breaking, but this need to be investigated. Also, the effect of the damping term on the lifespan of solutions is surely of interest, but yet unknown. The model equation presented in Chapter 1 which includes the additional advective term (i.e., the term $\mathbf{U} \cdot \nabla h_t$ representing transport due to a strong mean current) is being used to investigate the stability of "edge waves" in the presence of a longshore current profile. Preliminary results are promising.

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