AN ABSTRACT OF THE THESIS OF

Martin Šenkyřík for the degree of Doctor of Philosophy in Mathematics presented on April 25, 1995. Title: A Topological Approach to Dry Friction and Nonlinear Beams

Abstract approved: Redacted for Privacy

Ronald B. Guenther

Topological results are applied to boundary value problems modeling nonlinear beams and dry friction. A classical continuation theorem is used to prove existence results for nonlinear beams. Unified proofs, where possible, are given for all the physically relevant boundary conditions. Integration techniques and various integral inequalities are used to prove uniqueness results. Since the equation modeling dry friction exhibits discontinuities in the spatial variable the classical definition of a solution cannot be used; therefore, Filippov's definition of a solution is employed. This definition reformulates the original problem as a differential inclusion. A topological result for set-valued maps is used to prove an existence theorem for periodic solutions of a certain differential inclusion and it is applied to the original problem. Other known results for differential inclusions are also applied to the original equation and to other boundary value problems with discontinuities in the spatial variable.

©Copyright by Martin Šenkyřík

April 25, 1995

All rights reserved

A Topological Approach to Dry Friction and Nonlinear Beams

by

Martin Šenkyřík

A Thesis

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed April 25, 1995
Commencement June 1995

Doctor of Philosophy thesis of Martin Šenkyřík presented on April 25, 1995

APPROVED:

Redacted for Privacy

Major Professor, representing Mathematics

Redacted for Privacy

Head of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Redacted for Privacy

Martin Šenkyřík, Author

ACKNOWLEDGEMENT

First I would like to thank my wife, Jitka, and my sons, Adam and Jiri, for their support, help and understanding during my work on this thesis and throughout my mathematical career. It was a big challenge and it took a lot of courage to come to Corvallis without any knowledge of English or American culture.

During my stay in Corvallis I had much help from the faculty and staff of the Mathematics Department. Most importantly I would like to thank my advisor, Professor Ronald Guenther. During the time we worked together I had the opportunity to appreciate his professional, mathematical and interdisciplinary knowledge as well as his personal qualities. Whenever I needed any kind of help, Professor Guenther was always ready to help me. He will have a special place in my memory of Corvallis and Oregon State University. Many thanks to my committee members, Professors John Dilles, David Finch, John Lee, James Richman, and Enrique Thomann for serving on my committee and for their valuable consultations and comments on my work. Thanks to the office staff in the Mathematics Department, Carolyn Brumley, Jim Eddington, Susan Ellinwood, Ferne Simendinger and Donna Templeton. They were of great help to me and I always felt very comfortable asking them for assistance. I would also like to thank the Mathematics Department for providing financial support during my studies at Oregon State University.

I would never have come to Corvallis, I would never have completed my degree, and I would never have written this thesis if I had not received excellent mathematical training at my home institution, Palacky University in the Czech Republic. This outstanding school and its great teachers gave me the necessary background to succeed here at Oregon State University. I would also like to express my gratitude and respect to a very special person, Professor Irena Rachunkova of Palacky University, who was my first advisor. Her personality, hard work, and enthusiasm inspired me to become a research mathematician.

Finally, I would like to thank my friend, Professor Dalibor Froncek of Silesian University in the Czech Republic, for directing me during my high school years towards a career in mathematics. Without Dalibor I would never have majored in mathematics and have never become a mathematician.

TABLE OF CONTENTS

LIST OF FIGURES

DEDICATION

To my wife Jitka.

PREFACE

In 1976 Professor Andrzej Granas provided a new approach for establishing fixed points of certain mappings in his paper "Sur La méthode de continuité de Poincaré" [11]. This method is now known as Topological Transversality and has proven to be very powerful in various areas of nonlinear analysis. Topological Transversality was used in many papers of Professors Granas, Guenther and Lee to prove existence results for initial value and boundary value problems for nonlinear ordinary differential equations and for integral equations. In 1988 in the paper "Some existence results for the differential inclusions $y^{(k)} \in F(x, y, ..., y^{(k-1)})$, $y \in \mathcal{B}$," [12] Professors Granas, Guenther and Lee applied Topological Transversality to differential inclusions and proved some existence results for initial value and boundary value problems for nonlinear differential inclusions.

In his 1964 paper "Differential equations with discontinuous right-hand side" [6] Professor A.F. Filippov defined solutions of the differential equation

$$x' = f(t, x), \ \ x \in \mathbb{R}^n,$$

where f is discontinuous in both variables, as solutions to the differential inclusion

$$x'(t) \in \bigcap_{\delta>0} \ \bigcap_{\mu(N)=0} konv f(t, U(x(t), \delta) - N),$$

where $U(x, \delta) = \{y : |x-y| < \delta\}$ and $konvY$ is the closed convex hull of Y. Filippov's definition has a realistic physical meaning. I illustrate this in an example in Chapter I.

In this thesis we apply Topological Transversality and Filippov's theory to fourth order boundary value problems modeling nonlinear beams and to the dry friction equation. We show how powerful and natural each of these methods is by itself and how they become even more powerful when combined and used together.

We begin in Chapter I by introducing some notation. We recall some definitions and theorems from Topological Transversality and define Filippov's solutions to differential equations.

Chapter II deals with nonlinear beams, which are modeled by fourth order boundary value problems. Topological Transversality is applied to obtain existence results for all the physically relevant combinations of the three typical end conditions: clamped end, simply supported end, free end. All the existence theorems in this chapter enable us to establish the existence of solutions directly from the differential equation. Uniqueness is also studied by using Wirtinger-type integral inequalities.

The results of Chapter III are motivated by the dry friction equation, in particular by the existence of periodic solutions for this equation. Since the dry friction equation exhibits discontinuities in the spatial variable we cannot use the classical definition or Carathéodory's definition and we use Filippov's definition. Using the results given in Professor Kurzweil's book, "Ordinary differential equations" (Translation from the Czech edition by Michael Basch) [20], we show that under a reasonable hypothesis on the discontinuous right hand side of the equation

$$x' = f(t,x), \;\; x \in \mathbb{R}^n,$$

the multivalued convexification of f from Filippov's definition fits in the framework of the above mentioned paper [12] of Professors Granas, Guenther and Lee. The results of this paper are then applied to obtain existence results for the periodic and Dirichlet boundary value problems for Filippov's solutions. However, the existence result for periodic solutions involves a monotonicity condition which is applicable only to the dry friction equation in its least general form

$$u'' + bu' + cu + k\text{sgn}u' = e(t),$$

where $b, c, k \in \mathbb{R}, b, c, k > 0$ and e(t) is a measurable 1-periodic function. However, our goal is to obtain an existence result for a more general dry friction equation

$$u'' + b(u') + c(u) + k\text{sgn}u' = e(t),$$

or

$$u'' + u'd(u) + c(u) + k\text{sgn}u' = e(t),$$

where $k \in \mathbb{R}, k > 0$, b,c and d are nonlinear functions, and e is a measurable 1-periodic function. For this we prove an existence theorem with no monotonicity restrictions for the second order periodic boundary value problem for the differential inclusion

$$x'' \in F(t, x, x'), \;\; x(0) = x(1), \; x'(0) = x'(1),$$

where F is a set-valued function. This theorem is applicable to the more general dry friction equation. Some numerical computations are also presented to illustrate the results of Chapter III.

A TOPOLOGICAL APPROACH TO DRY FRICTION AND NONLINEAR BEAMS

1. INTRODUCTION

1.1. NOTATION

Throughout this thesis we try to use the most standard notation. In the case of some less frequently used notation we include an explanation in the section where it appears. Therefore, this section is only for reference purposes.

The following standard notation is used: $\mathbb{R} = (-\infty, \infty)$ denotes the real line; the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$ and its norm by $|x|$. $C(I)$ is the Banach space of continuous functions (with values in a Euclidean space) on the interval $I = [a, b]$. We define $|u|_0 = sup\{|u(t)|; t \in I\}$. $C^m(I)$ is the Banach space of functions whose m-derivatives are continuous on I with the norm

$$|u|_m = max\{|u|_0, |u'|_0, \ldots, |u^m|_0\}.$$

By $AC(I)$ we mean the set of absolutely continuous functions on I. For $1 \leq p < \infty$, $L^p(I)$ denotes the Banach space of p-th power, Lebesgue integrable functions with the norm

$$||u||_p = (\int_I |u|^p)^{\frac{1}{p}}.$$

$L^\infty(I)$ is the Banach space of essentially bounded, measurable functions together with the essential supremum norm, which is the limit of the L^p-norm as $p \to \infty$. The closed convex hull of Y is denoted by $konvY$. Let U be a subset of a metric space M, then ∂U and $cl(U)$ denote, respectively, the boundary and the closure of U in M. If X is a subset of a Banach space, then $Kv(X)$ is the family of all compact, convex, nonempty subsets of X. Suppose E and N are subsets of a finite dimensional Banach space, μ is a Lebesgue measure and $\phi(t,x)$ is a scalar valued function, then

$$\operatorname{ess}\max_{x\in E}\phi(t,x)=\inf_{\mu(N)=0}\ \sup_{x\in E-N}\phi(t,x).$$

The essential upper bound of the function $\phi(t,x)$ at the point x 8 is defined by

$$M_x\{\phi(t,x)\}=\lim_{\delta\to 0}\ \operatorname{ess}\max_{y\in U(x,\delta)}\phi(t,y),$$

where $U(x,\delta)=\{y:|x-y|<\delta\}$. Analogously the essential lower bound of the function $\phi(t,x)$ at the point x is denoted by $m_x\{\phi(t,x)\}$. For further discussion of L^p spaces, Lebesgue measure and Lebesgue integration see [7].

Definition 1.1.1 (Upper Semicontinuous Set Valued Function) *Let $H \subset \mathbb{R}^n$, $F:H\to 2^{\mathbb{R}^m}$. The function F is called upper semicontinuous at a point $y\in H$, if for every open set $V\subset\mathbb{R}^m$ such that $F(y)\subset V$ there exists $\delta>0$ so that $F(z)\subset V$ for z such that $|z-y|<\delta$ and $z\in H$. The function F is called upper semicontinuous if it is upper semicontinuous at every point $y\in H$.*

Definition 1.1.2 (Measurable Set Valued Function) *Let the set $H\subset\mathbb{R}^n$ be measurable, $F:H\to 2^{\mathbb{R}^m}$. The function F is called measurable if the set $\{y\in H;\ F(y)\cap E\neq\emptyset\}$ is measurable for every closed set $E\subset\mathbb{R}^m$.*

For other equivalent definitions of a measurable set valued function see [3].

1.2. TOPOLOGICAL TRANSVERSALITY

In this section we describe the topological principles that are further used. For proofs and details see [5] and [11]. Throughout this section we assume that E is a normed linear space, $K\subset E$ is a convex set and $U\subset K$ is an open set in K.

Definition 1.2.1 *Suppose X is a metric space and $F : X \to K$ is a continuous map. F is compact if F(X) is contained in a compact subset of K. F is completely continuous if it maps bounded subsets in X into compact subsets of K.*

Definition 1.2.2 *A compact map $F : cl(U) \to K$ is called admissible if it is fixed point free on $\partial(U)$. The set of all such maps will be denoted by $K_{\partial U}(cl(U), K)$.*

A map $F \in K_{\partial U}(cl(U), K)$ is essential if every compact map which agrees with F on ∂U has a fixed point in U. Otherwise F is inessential.

A homotopy $(H_\lambda : X \to K)$, $0 \le \lambda \le 1$ is said to be compact provided the map $H : X \times [0,1] \to K$ given by $H(x, \lambda) = H_\lambda(x)$ for $(x, \lambda) \in X \times [0,1]$ is compact.

Two maps $F, G \in K_{\partial U}(cl(U), K)$ are homotopic if there is a compact homotopy $H_t : cl(U) \to K$ for which $F = H_0$, $G = H_1$ and H_λ is admissible for each $\lambda \in [0,1]$.

Since every essential map agrees with itself on ∂U it has a fixed point. Now we formulate the Schauder fixed point theorem. For a detailed discussion see [5].

Theorem 1.2.3 *Suppose E is a normed linear space, $K \subset E$ is a convex set and $F : K \to K$ is a compact map. Then F has a fixed point in K.*

The following theorem follows from the Schauder fixed point theorem.

Theorem 1.2.4 *Let $p_0 \in U$ be fixed. Then the constant map sending each point of $cl(U)$ to p_0 is essential in $K_{\partial U}(cl(U), K)$.*

Proof. Let $G : cl(U) \to K$ be a compact map with $G = p_0$ on ∂U. Define

$$H(x) = \begin{cases} G(x), & \text{for } x \in cl(U) \\ p_0, & \text{for } x \in K - cl(U). \end{cases}$$

Since $H : K \to K$ is compact we get by the Schauder fixed point theorem that there exists an $x_0 \in K$ such that $H(x_0) = x_0$. From the definition of H and from the fact that $F = G$ on $\partial(U)$ it follows that $x_0 \in U$ and $x_0 = H(x_0) = G(x_0)$. Thus G has a fixed point and F is essential.

We now state the Topological Transversality Theorem without a proof. The proof is given in [5].

Theorem 1.2.5 (Topological Transversality) *Let F and G be homotopic maps in $K_{\partial U}(cl(U), K)$. Then F is essential if and only if G is essential.*

An immediate consequence of the previous two theorems is the following nonlinear alternative, which will be further used in Chapters 2 and 3.

Theorem 1.2.6 (Nonlinear Alternative) *Let $N : cl(U) \to K$ be a compact map, $p_0 \in U$, and $H_\lambda(u) = H(u, \lambda) : cl(U) \times [0,1] \to K$ a compact map with $H_1 = N$ and H_0 the constant map to p_0. Then either*

(1) N has a fixed point in $cl(U)$; or

(2) there exists a $\lambda \in (0,1)$ such that H_λ has a fixed point in ∂U.

Proof. Assume (1) fails so that N is fixed point free in $cl(U)$ and $N \in K_{\partial U}(cl(U), K)$. The map $H : cl(U) \times [0,1] \to K$ is compact. If in addition, H_λ were in $K_{\partial U}(cl(U), K)$ for each $\lambda \in [0,1]$, then N would be homotopic to a constant map and hence be essential by Theorems 1.2.4 and 1.2.5. Then N would have a fixed point, and this is a contradiction. Hence, if (1) fails there must be a $\lambda \in [0,1]$ such that $H_\lambda \notin K_{\partial U}(cl(U), K)$. Since H_λ is compact, this means that H_λ is not fixed point free on $\partial(U)$; that is, (2) holds for some $\lambda \in [0,1]$. $\lambda \neq 1$ because (1) fails and $\lambda \neq 0$ because $p_0 \notin \partial U$.

1.3. FILIPPOV SOLUTIONS

In this section we define Filippov solutions for systems of ordinary differential equations. For details and further properties see [6].

Many existence results for ordinary differential equations with discontinuities only in the time variable were proved by using Carathéodory's definition of a solution. Filippov's definition of a solution is more general than that of Carathéodory and includes it as a special case. The reader can find a complete theory of Filippov solutions, analogous to the classical or Carathéodory's theory in [6]. Consider an initial value or a boundary value problem for the system of differential equations

$$x' = f(t, x), \; x \in \mathbb{R}^n, \tag{1.3.1}$$

where f is $\mathbb{R}^n$ valued and $f = (f_1, \ldots, f_n)$. We assume f to be measurable, but make no continuity requirements on the function f. Based on the idea that the values of f on $N \subset \mathbb{R}^n$ with $\mu(N) = 0$, where μ is a Lebesgue measure, should play no role, Filippov defined solutions of (1.3.1) as solutions to the differential inclusion constructed as a convexification of f with respect to $x \in \mathbb{R}^n$ in the following way:

$$x'(t) \in \bigcap_{\delta>0} \bigcap_{\mu N=0} konv f(t, U(x(t), \delta) - N),$$

for almost every t, where $U(x, \delta) = \{y : |x - y| < \delta\}$ and $konv Y$ is the closed convex hull of Y. This definition is illustrated in the example at the end of this section.

Definition 1.3.1 *Let x be absolutely continuous on the interval [0,1]. If x satisfies*

$$x'(t) \in \bigcap_{\delta>0} \bigcap_{\mu N=0} konv f(t, U(x(t), \delta) - N) = K\{f(t, x)\} = k_t(x)$$

for almost every $t \in (0, 1)$, we say x is a solution of (1.3.1).

In the next definition, which is equivalent to Definition 1.3.1, we use the notation introduced in Section 1.1 on p.2. The proof of the equivalence of Definition 1.3.1 and the following Definition can be found in [6], p.203.

Definition 1.3.2 *Let x be absolutely continuous on the interval [0,1] and let $x \in \mathbb{R}^n$. If*

$$m_i(t,x) = m_x\{f_i(t,x_1,...,x_n\} \leq x_i' \leq M_x\{f_i(t,x_1,...,x_n)\} = M_i(t,x)$$

for i = 1,2,...,n and for almost every $t \in (0,1)$, then x is a solution of (1.3.1).

Remark 1 Let

$$k_t(x) = (k_t^1(x),...,k_t^n(x))$$

then

$$k_t^i \subset [m_i(t,x), M_i(t,x)]$$

for almost every $t \in (0,1)$. Or if we define $\mathcal{M}_i(x,t) = \max\{|m_i(t,x)|, |M_i(t,x)|\}$, then

$$k_t^i(x) \subset [-\mathcal{M}_i(t,x), \mathcal{M}_i(t,x)]$$

for almost every $t \in (0,1)$. Using vector notation and the symbol $cl(Y)$ for the closure of Y we get

$$k_t(x) \subset cl(U(0, |\mathcal{M}(x,t)|), \tag{1.3.2}$$

that we write

$$|k_t(x)| \leq |\mathcal{M}(x,t)| \tag{1.3.3}$$

for almost every $t \in (0,1)$, where $\mathcal{M}(x,t) = (\mathcal{M}_1(x,t),...,\mathcal{M}_n(x,t))$.

Example Consider the unforced damped oscillator with dry friction, which is modeled by the equation

$$u'' + 2u' + 3u + \mathrm{sgn} u' = 0 \tag{1.3.4}$$

together with the initial conditions

$$u(0) = u_0,\ u'(0) = u'_0.$$

Based on physical intuition we would expect that after some time no motion occurs. We show that Filippov's solution has the same property.

The equation can be written as a system

$$\begin{aligned} u &= x_1, \\ x_1' &= x_2, \\ x_2' &= -2x_2 - 3x_1 - \mathrm{sgn} x_2, \end{aligned}$$

or using Filippov's definition

$$x_1' \in \{x_2\},$$
$$x_2' \in \begin{cases} \{-2x_2 - 3x_1 - \mathrm{sgn} x_2\}, & \text{when } x_2 \neq 0 \\ \{(-3x_1 - 1,\ -3x_1 + 1)\}, & \text{when } x_2 = 0. \end{cases}$$

Because of the damping both x_1 and x_2 exhibit an exponential decay property. $x_1 \to -1/3$, $x_2 \to 0$ if we use the formula for the upper half-plane and $x_1 \to 1/3$, $x_2 \to 0$ if we use the formula for the lower half-plane. Since x_2 is alternating at some finite time $t = t_0$ we get

$$|x_1(t_0)| < 1/3,\ \ x_2(t_0) = 0.$$

In a sufficiently small neighborhood of t_0 we have

$$x_2' < 0, \text{ for } x_2 > 0,$$

$$x_2' > 0, \text{ for } x_2 < 0,$$

$$x_2' \in \{(-3x_1 - 1, \ -3x_1 + 1)\}, \text{ for } x_2 = 0.$$

So for $t > t_0$ we have $x_2' = 0$, $x_2 = 0$, $x_1' = 0$ thus x_1 is constant and since $x_1 = u$, u is constant; i.e. the particle has stopped at that position at the time $t = t_0 \doteq 5.9$.

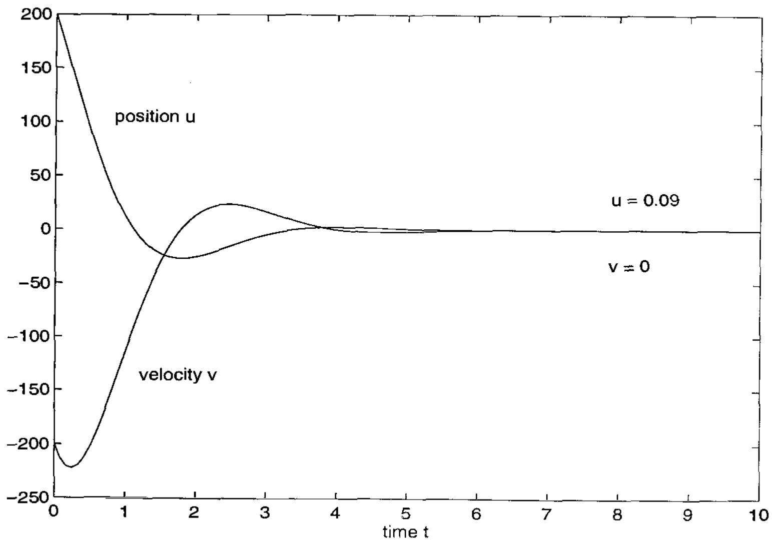

Figure 1. Graph of a solution of the unforced dry friction equation (1.3.3)

2. FOURTH ORDER BOUNDARY VALUE PROBLEMS AND NONLINEAR BEAMS

2.1. INTRODUCTION

The study of boundary value problems for ordinary differential equations has become very popular in the last few years. However, not too many results are available for fourth order and higher order problems, which is surprising in contrast with some of their important applications. An example of a phenomenon governed by a fourth order partial differential equation is the small transverse displacement of an elastic beam, see [15] p.195, where the following linear equation is derived:

$$u_{tt} + \frac{EI}{A} u_{xxxx} = -ku_t - \rho g + f(x,t),$$

t is the time variable, x is the longitudinal variable; E is Young's modulus; I is the moment of inertia; A is the cross section area; ρ is the density; g is the acceleration due to gravity; k is the damping coefficient and f is an external force.

In the remainder of this chapter we deal with the static (time independent case), we use t as a longitudinal variable rather than as a time variable and we "reserve" x for other purposes. In the static case the time derivatives vanish, and assuming the length of the beam is one, we get an ordinary differential equation, which can be written in the general nonlinear form

$$u^{(iv)} = f(t, u, u', u'', u'''),\ 0 \le t \le 1, \tag{2.1.1}$$

where now t is the longitudinal variable.

This equation together with various boundary conditions is studied in this chapter. We assume $f \in C([0,1] \times \mathbb{R}^4)$ and we consider continuous (classical) solutions, i.e. $u \in C^4([0,1])$ such that (2.1.1) is satisfied for all $t \in [0,1]$. All the results given in this paper can be generalized for Carathéodory solutions, using the same technique as in [27] and [23].

The three typical end conditions for the beam are

$$\begin{aligned} u(a) = u'(a) = 0, &\quad \text{end clamped at a,} \\ u(a) = u''(a) = 0, &\quad \text{end simply supported at a,} \\ u''(a) = u'''(a) = 0, &\quad \text{end free at a,} \end{aligned}$$

where $a = 0$ or $a = 1$.

The boundary conditions studied in this chapter are the physically relevant pairs of these end conditions, CC, CF, SS and CS, where C, S and F refer to a clamped, a simply supported and a free end, respectively. For example by CF we mean

$$u(0) = u'(0) = u''(1) = u'''(1) = 0.$$

By BC we mean any of these boundary conditions. The relation $u \in BC$ means that u satisfies BC. A beam with one end clamped and one end free is called a cantilever beam.

In Section 2.3 we prove an existence theorem under no sign or monotonicity restrictions and under the requirement, that there exists a bounded set, where f is bounded and where the bound of f depends linearly on the bound of the set. This theorem applies with some modifications to all of our boundary conditions and for $BC = CF$ it can be used to establish the existence of nonpositive solutions. Another theorem which applies to all of the boundary conditions is proved under a Bernstein - Nagumo like growth condition on f in u''', with some additional growth requirements on the lower derivatives. The additional requirements can be replaced by certain sign conditions in the case of both ends simply supported. Some modifications for the various boundary conditions are required in order to obtain the sharpest possible results. This section is concluded by two theorems with no growth restrictions on f, unfortunately, these are applicable only to special boundary conditions. One applies to $BC = CF$, and the second one, which is an extension of the results of Rodriguez and Tineo [26] for the second order Dirichlet problem, applies to $BC = SS$. This theorem can be used to establish the existence of nonpositive solutions.

Earlier work [21] establishes existence results for (2.1.1), BC under a Bernstein - Nagumo like growth condition on f in u''' and under a monotonicity condition. Further results are obtained by replacing the Bernstein-Nagumo like conditions by essentially different quadratic growth rate and integral inequality restrictions.

Existence results for (2.1.1), BC appear in [14] when f splits in a special way and grows sublinearly in u, u', u'', u'''.

In Section 2.4 we obtain a uniqueness result for $BC = CF$ under a Lipschitz condition on f, with a restriction on the Lipschitz constants. These restrictions can be weakened for each of the other boundary conditions. In the case of f depending only on x and u we get a uniqueness theorem under the assumption that f is nonincreasing in u. When $BC = SS$ and f depends only on t and u'' we can assume that f is nondecreasing in u''. The last uniqueness theorem is a generalization of this result in the sense that it allows f to depend on u'''.

It is very easy to verify the assumptions of each theorem for a given right-hand side of the equation. This is illustrated in the examples.

2.2. THE NONLINEAR ALTERNATIVE FOR BEAMS

First we prove a general existence theorem for nonlinear beams. For details and related results see [13], [14], [16], [21] and [10]. This theorem will be the main tool to prove existence results for (2.1.1), BC.

Theorem 2.2.1 (The Nonlinear Alternative for Beams) *Assume* $D \subset C^4[0,1]$ *is open and bounded,* $0 \in D$, $g(t,x,y,z,w,\lambda) \in C^0([0,1] \times \mathbb{R}^4 \times [0,1])$ *and* $g(t,x,y,z,w,1) = f(t,x,y,z,w)$. *Then either*

(1) (2.1.1), BC has a solution $u \in cl(D)$ *(closure of D)*

or

(2) there exists a $\lambda \in (0,1)$, *such that*

$$u^{(iv)} = \lambda g(t,u,u',u'',u''',\lambda), \tag{2.2.1}$$

BC has a solution $u_\lambda \in \partial(D)$ *(boundary of* D*).*

Proof. We define $D_{BC} = \{u \in D : u \in BC\}$

$$N_\lambda : C^3[0,1] \to C^0[0,1],\ \ 0 \le \lambda \le 1,$$

by

$$(N_\lambda u)(t) = \lambda g(t,u(t),u'(t),u''(t),u'''(t),\lambda).$$

N_λ is a continuous map. Let $j : C^4_{BC} \to C^3_0[0,1]$, where $C^3_0[0,1] = \{u : u \in C^3[0,1] and u(0) = 0\}$, $C^4_{BC} = \{u \in C^4[0,1] : u \in BC\}$, be the natural embedding defined by $ju = u$. From Arzela-Ascoli Theorem it follows that the map j is completely continuous. Further we define

$$L : C^4_{BC} \to C_0[0,1],$$

by

$$Lu(t) = u^{(iv)}(t).$$

It is easy to check by a direct calculation that L is invertible. Then

$$H_\lambda = L^{-1} N_\lambda j$$

defines a homotopy $H_\lambda : cl(D_{BC}) \to C^4_{BC}$. It is clear that the fixed points of H_λ are precisely the solutions of 2.2.1, BC. By our assumption H_λ is fixed point free on $\partial(D_{BC})$ for $\lambda \in (0,1)$. The complete continuity of j, the invertibility of L and the continuity of N_λ imply that the homotopy H_λ is compact. Since $0 \in D$ and H_0 is the zero map, the theorem now follows from Theorem 1.2.6 where we take $p_0 = 0$.

2.3. EXISTENCE RESULTS

Remark All of the boundary conditions CC, CF, SS, CS imply that u , u', u'', u''' have at least one zero point in $[0, 1]$.

Theorem 2.3.1 *Let there exist* $m, M \in \mathbb{R}$, $m, M > 0$, *such that*

$$-m \leq f(t, x, y, z, w) \leq M \tag{2.3.1}$$

for

$$-M \leq w \leq m, \ \ -m \leq z \leq M, \ \ -M \leq y \leq m, \ \ -M \leq x \leq m. \tag{2.3.2}$$

Then (2.1.1), CF has a solution u satisfying

$$-M \leq u''' \leq m, \ \ -m \leq u'' \leq M, \ \ -M \leq u' \leq m, \ \ -M \leq u \leq m. \tag{2.3.3}$$

Proof. We apply the nonlinear alternative to the domain

$$D = \{u \in C^4([0,1]): \ -M < u < m, \ -M < u' < m, \ -m < u'' < M,$$

$-M < u''' < m\}$ and to the family of equations

$$u^{(iv)} = \lambda f(t, u, u', u'', u'''), \ CF.$$

Then $x \in \partial D$ if

$$-M \le x' \le m, \ -m \le x'' \le M, \ -M \le x''' \le m, \ \text{for} \ \ t \in [0,1] \ \text{and}$$

$$max\, x(t) = m \ \ \text{or} \ \ min\, x(t) = -M \ \ \text{on} \ \ [0,1],$$

or

$$-M \le x \le m, \ -m \le x'' \le M, \ -M \le x''' \le m, \ \text{for} \ \ t \in [0,1] \ \text{and}$$

$$max\, x'(t) = m \ \ \text{or} \ \ min\, x'(t) = -M \ \ \text{on} \ \ [0,1],$$

or

$$-M \le x \le m, \ -M \le x' \le m, \ -M \le x''' \le m, \ \text{for} \ \ t \in [0,1] \ \text{and}$$

$$max\, x''(t) = M \ \ \text{or} \ \ min\, x''(t) = -m \ \ \text{on} \ \ [0,1],$$

or

$$-M \le x \le m, \ -M \le x' \le m, \ m \le x'' \le M, \ \text{for} \ \ t \in [0,1] \ \text{and}$$

$$max\, x'''(t) = m \ \ \text{or} \ \ min\, x'''(t) = -M \ \text{on} \ [0,1].$$

From (2.3.1) it follows that any solution $u_\lambda \in cl(D)$ of

$$u^{(iv)} = \lambda g(t, u, u', u'', u''', \lambda),\ CF$$

for $\lambda \in (0,1)$ satisfies

$$-m < u_\lambda^{(iv)} < M.$$

Further CF implies $-M < u_\lambda''' < m$, $-m < u_\lambda'' < M$, $-M < u_\lambda' < m$ and $-M < u_\lambda < m$. Thus there is no solution u_λ on the boundary ∂D for $\lambda \in (0,1)$, the nonlinear alternative applies and this proves the theorem.

Example The following functions satisfy the hypotheses of Theorem 2.3.1 :

$$f = \frac{1}{3} z \cos y + \frac{1}{3} w \sin x \cos y \sin z \sin w + \frac{1}{3} tx \cos w,$$

with $m = M$, where $M > 0$, and M is arbitrary.

$$f = \frac{1}{2} w^4 + \frac{1}{2} \cos x,$$

with $m = M = 1$.

$$f = \frac{1}{2} e^{w^2 - 1} + \frac{1}{2} \cos x,$$

with $m = M = 1$.

Analogous theorems for the other boundary conditions can be stated. Depending on the locations of the zero points of the solution and its derivatives, conditions (2.3.2) and (2.3.3) need to be modified. The following lemma will be further used in the cases when the solution or some of its derivatives vanish at 0 and 1.

Lemma 2.3.2 *Let $x \in C^1[0,1]$, $x(0) = x(1) = 0$ and $-m \le x'(t) \le M$, where $m, M > 0$. Then*

$$|x(t)| < L, \quad \textit{for } t \in [0,1],$$

where $L = mM(m+M)^{-1}$. If $m = M$, then $L = M/2$.

Proof. From the equalities

$$x(t) = \int_0^t x'(s)ds \ ,$$

$$-x(t) = \int_t^1 x'(s)ds$$

it follows that

$$-mt \le x(t) \le Mt, \ \ M(1-t) \ge -x(t) \ge -m(1-t), \tag{2.3.4}$$

for $t \in [0,1]$. By a simple manipulation with (2.3.4) we get

$$-L \le x(t) \le L.$$

Since $x'(t)$ is continuous, two of the inequalities (2.3.4) are sharp, we get

$$-L < x(t) < L$$

and this completes the proof.

Remark For BC = SS, conditions (2.3.2) can be replaced by

$$-K \le w \le K, \ -\frac{K}{2} \le z \le \frac{K}{2}, \ -\frac{K}{2} \le y \le \frac{K}{2}, \ -\frac{K}{4} \le x \le \frac{K}{4},$$

where $K = max\{m, M\}$ and assertions (2.3.3) can be replaced similarly.

For BC = CS we have

$$-K \le w \le K, \ -K \le z \le K, \ -K \le y \le K, \ -\frac{K}{2} \le x \le \frac{K}{2},$$

and for BC = CC we have

$$-K \le w \le K, \ -K \le z \le K, \ -\frac{K}{2} \le y \le \frac{K}{2}, -\frac{K}{4} \le x \le \frac{K}{4}.$$

In both cases assertions (2.3.3) can be replaced by the assertions corresponding to the replacements of (2.3.2).

The examples given for CF also apply to SS, CS and CC.

Now we show that in Theorem 2.3.1 we can take $m = 0$. This will allow us to establish the existence of nonpositive solutions for the case $BC = CF$.

Lemma 2.3.3 *Let there exist $\epsilon, M \in \mathbb{R}$, and $\epsilon, M > 0$, such that*

$$-m \leq f(t,x,y,z,w) \leq M$$

for

$$-M \leq w \leq m, \ \ -m \leq z \leq M, \ \ -M \leq y \leq m, \ \ -M \leq x \leq m,$$

where $0 < m < \epsilon$. Then (2.1.1), CF has a solution u satisfying

$$-M \leq u''' \leq 0, \ \ 0 \leq u'' \leq M, \ \ -M \leq u' \leq 0, \ \ -M \leq u \leq 0.$$

Proof. The proof follows immediately from Theorem 2.3.1.

Theorem 2.3.4 *Suppose there exist $M \in \mathbb{R}$, $M > 0$, such that*

$$0 \leq f(t,x,y,z,w) \leq M$$

for

$$-M \leq w \leq 0, \ \ 0 \leq z \leq M, \ \ -M \leq y \leq 0, \ \ -M \leq x \leq 0.$$

Then (2.1.1), CF has a solution u satisfying

$$-M \leq u''' \leq 0, \ \ 0 \leq u'' \leq M, \ \ -M \leq u' \leq 0, \ \ -M \leq u \leq 0.$$

Proof. For each $n \in \mathbb{N}$ there exists $\delta_n \in \mathbb{R}$, $0 < \delta_n < M$ such that

$$-\delta_n \leq 0 \leq f(t,x,y,z,w) + \frac{1}{n} \leq M + \frac{2}{n}$$

for

$$-M \leq w \leq \delta_n, \ \ \delta_n \leq z \leq M, \ \ -M \leq y \leq \delta_n, \ \ -M \leq x \leq \delta_n.$$

From Lemma 2.3.3 it follows that

$$u_n^{(iv)} = f(t, u_n, u_n', u_n'', u_n''') + \frac{1}{n}, \quad CF \tag{2.3.5}$$

has a solution u_n and

$$-M - \frac{2}{n} \le u_n''' \le 0,\ \ 0 \le u_n'' \le M + \frac{2}{n},\ \ -M - \frac{2}{n} \le u_n' \le 0,\ \ -M - \frac{2}{n} \le u_n \le 0,$$

for all $n \in \mathbb{N}$.

The sequences $(u_n)_{n=1}^{\infty}$, $(u_n')_{n=1}^{\infty}$ and $(u_n'')_{n=1}^{\infty}$ are uniformly bounded and equicontinuous on $[0,1]$. The sequence $(u_n''')_{n=1}^{\infty}$ is uniformly bounded on [0,1]. From (2.3.5) we get that $(u_n^{iv})_{n=1}^{\infty}$ is uniformly bounded, so $(u_n''')_{n=1}^{\infty}$ is equicontinuous. Without loss of generality we may assume by using Arzela - Ascoli lemma that all four sequences are uniformly convergent on $[0,1]$. From (2.3.5) it follows that the function

$$u(t) = \lim_{n\to\infty} u_n(t) \text{ for } t \in [0,1]$$

is a solution of (2.1.1), CF and that it satisfies the assertions of the theorem.

Lemma 2.3.5 *Suppose $p, q, r_1, r_2 \in \mathbb{R}$, $r_1, r_2 \ge 0$, $r_1 + r_2 > 0$, $p, q \ge 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $g \in C([0,1] \times \mathbb{R}^2)$, $h \in L^q(r_1, r_2)$, $\omega \in C([0,\infty))$ is a positive function and*

$$\int_0^{\infty} \frac{ds}{\omega(s)} = \infty.$$

Then there exists an $r^* \in (1,\infty)$ such that for any function $u \in C^4([0,1])$ satisfying BC,

$$-r_1 \le u''(t) \le r_2, \tag{2.3.6}$$

and

$$|u^{(iv)}| \le \omega(|u'''|) g^{1/p}(t, u, u') h(u'')(1 + |u'''|)^{1/q} \tag{2.3.7}$$

for $|u'''(t)| \geq 1$, *the estimate*

$$|u'''(t)| \leq r^*, \tag{2.3.8}$$

holds on $[0,1]$.

Proof. The inequalities (2.3.6) and BC imply, that $|u| \leq max\{r_1,\ r_2\}$ and $|u'| \leq max\{r_1,\ r_2\}$. We define

$$g_0 = \sup\{|g(t,u,u')| :\ u \text{ satisfies BC and (2.3.6),}\}$$

$$k_0 = 2 \parallel g_0^{1/p} \parallel_{L^p(0,1)} \parallel h \parallel_{L^q(-r_1,r_2)}$$

and

$$\Omega(x) = \int_0^x \frac{ds}{\omega(|s|)}, \text{ for } x \in \mathbb{R}.$$

Then it follows that, $g_0(t) \in L^1(0,1)$, Ω is an odd function, $\Omega(\mathbb{R}) = \mathbb{R}$ and there exists an inverse mapping Ω^{-1}. Let $u \in C^4([0,1])$ satisfy (2.3.6), (2.3.7) and BC, then there exists $t_3 \in [0,1]$ so that $u'''(t_3) = 0$.

Suppose $t_3 \neq 1$ and that there exists $t_g \in (t_3, 1]$ such that

$$|u'''(t_g)| > r^*, \tag{2.3.9}$$

where

$$r^* = \Omega^{-1}(\Omega(1) + k_0).$$

Let $[a_1, b_1] \subset [t_3, 1]$ be the maximal interval such that $|u'''(t)| \geq 1$ for $t \in [a_1, b_1]$ and $t_g \in [a_1, b_1]$. Let $t_m \in (a_1, b_1]$ be such that $|u'''(t_m)| = \rho_1 = max\ \{u'''(t) :\ a_1 \leq t_m \leq b_1\}$. Then by (2.3.7) and the Hölder inequality

$$\int_{a_1}^{t_m} \frac{u^{(iv)}(t)}{\omega(|u'''(t)|)} dt \leq \int_0^1 g^{1/p}(t,u,u')h(u'')(1+|u'''|)^{1/q} dt$$

$$\leq k_0 = 2 \parallel g_0^{1/p} \parallel_{L^p(0,1)} \parallel h \parallel_{L^q(-r_1,r_2)} = k_0.$$

In the case $u'''(t) \geq 1$ on $[a_1, t_m]$ we get $\Omega(\rho_1) - \Omega(1) \leq k_0$, $\rho_1 \leq r^*$ and this is a contradiction. In the case $u'''(t) \leq -1$ we get $-\Omega(-\rho_1) + \Omega(-1) \leq k_0$ and further $-\rho_1 \geq \Omega^{-1}(-k_0 + \Omega(-1)) = \Omega^{-1}(-k_0 - \Omega(1)) = -r^*$. This is a contradiction. Similarly using the fact that

$$\int_{a_1}^{t_m} \frac{-u^{(iv)}(t)}{\omega(|u'''(t)|)} dt \leq k_0,$$

we can get a contradiction for $t_3 \neq 0$, $t_g \in [0, t_3)$ and this completes the proof.

Theorem 2.3.6 *Suppose g, ω, h, r_1, r_2, p,q and r^* are the same as in Lemma 2.3.5 and*

$$|f(t,x,y,z,w)| \leq \omega(|w|)g^{1/p}(t,x,y)h(z)(1+|w|)^{1/q} \tag{2.3.10}$$

for $|w| \geq 1$, $-r_1 \leq z \leq r_2$, $|y| \leq max\{r_1, r_2\}$, $|x| \leq max\{r_1, r_2\}$. Further suppose that

$$r^* < min\{r_1, r_2\}. \tag{2.3.11}$$

Then (2.1.1), BC has a solution u such that $|u^{(i)}| \leq r^*$ for i = 0,1,2,3.

Proof. We apply the nonlinear alternative to the domain $D = \{x \in C^4([0,1]) : |x'''| < r^* + 1, -r_1 < x'' \leq r_2, |x'| < max\{r_1, r_2\}, |x| < max\{r_1, r_2\}\}$ and to the family of equations

$$u^{(iv)} = \lambda f(t, u, u', u'', u'''), \;\; BC.$$

From Lemma 2.3.5 it follows that $|u'''| < r^*$ for $\lambda \in (0,1)$. From BC and (2.3.11) it follows that $-r_1 < u'' < r_2$, $|u'| < max\{r_1, r_2\}$, $|u| < max\{r_1, r_2\}$ for $\lambda \in (0,1)$. Thus the nonlinear alternative applies.

Remark We treated all the boundary conditions together and so we used those inequalities that apply to all of these boundary conditions. However, if the solution or the derivative vanishes at both endpoints, we can use Lemma 2.3.2 to get better inequalities.

Remark The growth condition (2.3.10) can be replaced by a one sided growth condition in the case BC = CF, because the number t_3 from the proof of Lemma 2.3.5 can be equal to 1. Lemma 2.3.7 shows that for BC = SS, the condition (2.3.11) can be replaced by (2.3.12) to obtain bounds on u'', u' and u.

Lemma 2.3.7 *Let u be a solution of*

$$u^{(iv)} = \lambda f(t, u, u', u'', u'''), \;\; SS,$$

for $\lambda \in (0, 1)$. Let r_1, r_2 be the constants from Lemma 2.3.5 Further let u satisfy (2.3.6),

$$f(t, x, y, -r_1, 0) < 0 \text{ and } f(t, x, y, r_2, 0) > 0, \tag{2.3.12}$$

for $|y| \leq max\{r_1, r_2\}$, $|x| \leq max\{r_1, r_2\}/2$, then

$$-r_1 < u''' < r_2.$$

Proof. Let $t_1 \in (0, 1)$ and $u''(t_1) = r_2$. Then $u'''(t_1) = 0$, $u^{(iv)}(y_1) \leq 0$ and this is a contradiction. Similarly we get a contradiction for $u''(t_1) = -r_1$ and this proves the lemma.

Example The hypotheses of Theorem 2.3.6 are satisfied for

$$f = \frac{1}{20}((zw^2 \cos(x^5 t) + \frac{1}{100}),$$

with $r_1 = -2$ and $r_2 = 2$. Since $r^* > 1$ we must take $min\{|r_1|,\ |r_2|\} > 1$. Further we have

$$|f| \leq \frac{1}{20}(z + \frac{1}{100})(1 + |w|)^2.$$

Now $\omega(|w|) = 1 + |\omega|,\ g = 1,\ h = \frac{1}{20}(z + \frac{1}{100}),\ p = \infty,\ q = 1,$

$$k_0 = \frac{1}{10}\int_{-2}^{2}(z + \frac{1}{100})dz = 0.404\ ,$$

$$\Omega(x) = \int_0^x \frac{ds}{1 + |s|} = ln(|x| + 1)$$

and

$$\Omega^{-1}(y) = e^y - 1 \text{ for } x \geq 0 \text{ and } \Omega(1) = \ln 2.$$

So

$$r^* = \Omega^{-1}(ln2 + 0.404) = e^{\ln 2 + 0.404} - 1 = 1.996 < 2 = min\{|r_1|,\ |r_2|\}.$$

The proof of Theorem 2.3.6 can be modified by defining

$$r^* = \Omega^{-1}(\Omega(a) + k_0),$$

where $a > 0$, so $r^* < a$ and we get no restrictions on the $min\{|r_1|,\ |r_2|\}$. For a special choice of boundary conditions we can use a sharper version of Theorem 2.3.6 as described in the first remark following Theorem 2.3.6.

Now we prove an existence result with no growth restrictions for BC = CF.

Lemma 2.3.8 *Suppose* $g \in C([0,1] \times \mathbb{R}^4)$ *and that there exist* $c_1, c_2 \in \mathbb{R},\ c_1, c_2 > 0$ *such that*

$$g(t, x, y, z, -c_1) < 0 \textit{ and } \quad g(t, x, y, z, c_2) > 0 \tag{2.3.13}$$

for $t \in [0,1]$, $(x,y,z) \in [-c_2, c_1]^3$. *Let* u *be a solution of*

$$u^{(iv)} = g(t,u,u',u'',u'''),\ \ CF \tag{2.3.14}$$

such that

$$-c_1 \le u''' \le c_2,\ \ -c_2 \le u^i \le c_1 \ \ for\ i = 0,1,2. \tag{2.3.15}$$

Then

$$-c_1 < u''' < c_2,\ \ -c_2 < u^i < c_1 \ \ for\ i = 0,1,2. \tag{2.3.16}$$

Proof. Let $t_0 \in (0,1)$ and $u'''(t_0) = c_2$, then $u^{(iv)}(t_0) = 0$ but from (2.3.13) it follows that $u^{(iv)}(t_0) > 0$ and this is a contradiction. If $u'''(0) = c_2$, then $u^{(iv)}(0) \le 0$ and from (2.3.13) we get a contradiction, so $u''' < c_2$ on $[0,1]$. Similarly we can prove that $u''' > c_1$ on $[0,1]$. The sharp inequalities for the lower derivatives and for the solution follow from CF.

Lemma 2.3.9 *Let* $c_1, c_2 \in \mathbb{R}$, $c_1, c_2 > 0$ *and*

$$f(t,x,y,z,-c_1) \le 0, \quad f(t,x,y,z,c_2) \ge 0 \tag{2.3.17}$$

for $t \in [0,1]$, $(x,y,z) \in [-c_2, c_1]^3$. *Then the function*

$$g(t,x,y,z,w,\lambda) = \lambda[f(t,x,y,z,w) + (1-\lambda)w]$$

satisfies (2.3.13) for $\lambda \in (0,1)$, $t \in [0,1]$, $(x,y,z) \in [-c_2, c_1]^3$. *For* $\lambda = 1$ *we have*

$$g(t,x,y,z,w,1) = f(t,x,y,z,w).$$

Proof. The lemma follows immediately from the definition of the function g.

Theorem 2.3.10 *Suppose f satisfies (2.3.17), then (2.1.1), CF has a solution that satisfies (2.3.15).*

Proof. We apply the nonlinear alternative to the domain $D = \{x \in C^4([0,1]) : -c_1 < x''' < c_2, \ -c_2 < x^{(i)} < c_1 \text{ for } i = 0,1,2\}$ and the family of equations

$$u^{(iv)} = \lambda[f(t,x,y,z,w,) + (1-\lambda)w], \ CF. \tag{2.3.18}$$

From Lemmas 2.3.8 and 2.3.9 it follows, that there are no solutions of (2.3.18) on the boundary of D for $\lambda \in (0,1)$.

Example The hypotheses of Theorem 2.3.10 are satisfied for

$$f = a(t)[x^k y^l z^m] + w^{2n+1} e^{w^2} + 1,$$

where $a(t) \in C([0,1])$ is and k,l,m,n are nonnegative integers.

Now we prove an existence result with no growth restrictions for BC = SS.

Lemma 2.3.11 *Suppose $g \in C([0,1] \times \mathbb{R}^4)$ and that there exist $\epsilon, c_1, c_2 \in \mathbb{R}$, $\epsilon, c_1, c_2 > 0$ such that*

$$g(t,x,y,z,0) < 0 \tag{2.3.19}$$

for $t \in [0,1]$, $x \in (-L/2, L/2)$, $y \in (-L, L)$ and $z \in [-\epsilon, 0]$, where $L = c_1 c_2 (c_1 + c_2)^{-1}$. Further let u be a solution of

$$u^{(iv)} = g(t,u,u',u'',u'''), \ SS, \tag{2.3.20}$$

such that $u''(t) \geq -\epsilon$, $-c_1 \leq u'''(t) \leq c_2$ for $t \in [0,1]$. Then $u''(t) > 0$ for $t \in (0,1)$ and $u'''(1) < 0 < u'''(0)$.

Proof. Let u be as above and $u''(t_0) = 0$, where $t_0 \in [0,1)$. If $u'''(t_0) = 0$, then $u^{(iv)}(t_0) < 0$. Thus, under the assumption that $u'''(t_0) \leq 0$ there exists $t_1 \in (t_0, 1)$ such that $u''(t_1) < 0$, $min\{u''(t) : t \in [t_0, 1)\} = u''(t_1)$ and $u'''(t_1) = 0$. Further from (2.3.19) we get $u^{iv}(t_1) < 0$ and this is a contradiction, which proves that $u'''(t_0) > 0$ if $t_0 \in [0,1)$ and $u''(t_0) = 0$. Since $u'''(0) > 0$, there exists $t_2 \in (0,1]$ such that $u''(t_2) = 0$, and $u''(t) > 0$ for $t \in (0, t_2)$. By the part of the proof above we have $t_2 = 1$. If $u'''(1) = 0$, then by (2.3.19) $u^{iv}(1) < 0$. This contradicts the fact that $u''(t) > 0$ for $t \in (0,1)$ and proves the lemma.

Lemma 2.3.12 *Suppose* $g \in C([0,1] \times \mathbb{R}^4)$ *and that there exist* $c_1, c_2 \in \mathbb{R}$, $c_1, c_2 > 0$ *such that*

$$g(t,x,y,z,-c_1) > 0, \quad g(t,x,y,z,c_2) > 0 \tag{2.3.21}$$

for $t \in [0,1]$, $x \in (-L/2, L/2)$, $y \in (-L, L)$, $z \in [0, L)$, *where* $L = c_1 c_2 (c_1 + c_2)^{-1}$. *Further let* u *be a solution of (2.3.20)* $u''(t) > 0$ *for* $t \in (0,1)$, $u'''(1) < 0 < u'''(0)$ *and* $-c_1 \leq u'''(t) \leq c_2$ *for* $t \in [0,1]$. *Then*

$$-c_1 < u'''(t) < c_2$$

for $t \in [0,1]$.

Proof. Suppose that $u'''(t_1) = c_2$, where $t_1 \in [0,1]$, then $t_1 < 1$ and from (2.3.19) it follows that $u^{(iv)}(t_1) > 0$. This is a contradiction, so we get $u'''(t) < c_2$ for $t \in [0,1]$. Using the same argument we get $u'''(t) > -c_1$ for $t \in [0,1]$ and this completes the proof of the lemma.

Theorem 2.3.13 *Let there exist* $c_1, c_2 \in \mathbb{R}$, $c_1, c_2 > 0$ *such that*

$$f(t,x,y,z,-c_1) \geq 0, \quad f(t,x,y,z,c_2) \geq 0 \tag{2.3.22}$$

for $t \in [0,1]$, $x \in (-L/2, L/2)$, $y \in (-L, L)$, $z \in [0, L)$, *where* $L = c_1c_2(c_1+c_2)^{-1}$. *Further let*

$$f(t,x,y,0,0) \leq 0 \tag{2.3.23}$$

for $t \in [0,1]$, $x \in (-L/2, L/2)$ *and* $y \in (-L, L)$. *Then (2.1.1), SS has a solution* u *satisfying* $u \in (-L/2, 0]$, $u' \in (-L, L)$, $u'' \in [0, L)$ *and* $u''' \in [-c_1, c_2]$.

Proof. By the Tietze-Urysohn lemma there exists a continuous function $h : \mathbb{R}^2 \to [-1,1]$ such that $h(0,0) = -1$ and $h(z,-c_1) = h(z,c_2) = 1$ for $z \in [0, L]$. We define

$$g_n(t,x,y,z,w) = f(t,x,y,z,w) + h(z,w)/n \text{ for } n \in \mathbb{N}.$$

Then

$$g_n(t,x,y,0,0) \leq -1/n < 0$$

for $t \in [0,1]$, $x \in (-L/2, L/2)$, $y \in (-L, L)$, $n \in \mathbb{N}$ and

$$g_n(t,x,y,z,-c_1) \geq 1/n > 0,$$

$$g_n(t,x,y,z,c_2) \geq 1/n > 0$$

for $t \in [0,1]$, $x \in (-L/2, L/2)$, $y \in (-L, L)$, $z \in [0, L)$, $n \in \mathbb{N}$. We define

$$D_n = \{x \in C^4([0,1]) : -L/2 < x < L/2, -L < x' < L, -\epsilon_n < x'' < L, -c_1 < x''' < c_2\},$$

where $1 > \epsilon_n > 0$ is such that

$$g_n(t,x,y,z,0) < 0$$

for $t \in [0,1]$, $x \in (-L/2, L/2)$, $y \in (-L, L)$ and $z \in [-\epsilon_n, 0]$. From Lemma 2.3.11 and Lemma 2.3.12 it follows that the boundary value problem

$$u^{(iv)} = \lambda g_n(t, u, u', u'', u'''), \ \ SS$$

has no solutions on the boundary of D_n for $\lambda > 0$. By the nonlinear alternative the boundary value problem

$$u^{(iv)} = g_n(t, u, u', u'', u'''), \ \ SS \tag{2.3.24}$$

has a solution $u_n \in cl(U_n)$ (closure of U_n). The sequences $(u_n)_{n=1}^{\infty}$, $(u_n')_{n=1}^{\infty}$ and $(u_n'')_{n=1}^{\infty}$ are uniformly bounded and equicontinuous on $[0, 1]$. The sequence $(u_n''')_{n=1}^{\infty}$ is uniformly bounded on [0,1]. From (2.3.24) we get that $(u_n^{iv})_{n=1}^{\infty}$ is uniformly bounded, so $(u_n''')_{n=1}^{\infty}$ is equicontinuous. Without loss of generality we may assume by using Arzela-Ascoli theorem that all four sequences are uniformly convergent on $[0, 1]$. From (2.3.24) and the definition of the function g_n it follows that the function

$$u(t) = \lim_{n\to\infty} u_n(t) \text{ for } t \in [0, 1]$$

is a solution of (2.1.1), SS and that it satisfies the assertions of the theorem.

Example The hypotheses of Theorem 2.3.13 are satisfied for

$$f = a(t)[x^k y^l z^m] + w^{2n} e^{w^2} - 1,$$

where $a(t) \in C([0, 1])$ and k,l,m,n are nonnegative integers or

$$f = tx^5 y^3 z^4 + w^{14} - 1.$$

2.4. UNIQUENESS RESULTS

First we recall two lemmas. The proofs of these lemmas can be found in [2] and [19]. Lemma 2.4.2 is often referred to as Wirtinger's Inequality.

Lemma 2.4.1 *Let $u \in AC(0,1)$, $u' \in L^2(0,1)$ and $u(t_0) = 0$, where $0 \le t_0 \le 1$. Then*

$$\int_0^1 u^2(t)dt \le (2/\pi)^2 \int_0^1 (u'(t))^2 dt.$$

Lemma 2.4.2 *Let $u \in AC(0,1)$, $u' \in L^2(0,1)$ and $u(0) = u(1) = 0$, where $0 \le t_0 \le 1$. Then*

$$\int_0^1 u^2(t)dt \le (1/\pi)^2 \int_0^1 (u'(t))^2 dt.$$

Theorem 2.4.3 *Suppose there exist positive constants α, β, γ and δ such that*

$$\frac{16\alpha}{\pi^4} + \frac{8\beta}{\pi^3} + \frac{4\gamma}{\pi^2} + \frac{2\delta}{\pi} < 1. \tag{2.4.1}$$

Further, suppose

$$|f(t,x_1,y_1,z_1,w_1) - f(t,x_2,y_2,z_2,w_2)| \le \alpha|x_1 - x_2| + \beta|y_1 - y_2| + \gamma|z_1 - z_2| + \delta|w_1 - w_2| \tag{2.4.2}$$

for any $t \in [0,1]$, $x_i, y_i, z_i, w_i \in \mathbb{R}$, $i = 1,2$. Then (2.1.1), CF has at most one solution.

Proof. Let u_1 and u_2 be solutions of (2.1.1), CF. If $v = u_1 - u_2$, then

$$v(0) = v'(0) = v''(1) = v'''(1) = 0. \tag{2.4.3}$$

We define

$$\rho = (\int_0^1 (v^{(iv)}(t))^2 dt)^{1/2}.$$

From Lemma 2.4.1 we get the following inequalities:

$$(\int_0^1 (v'''(t))^2 dt)^{1/2} \le 2\rho/\pi,$$

$$(\int_0^1 (v''(t))^2 dt)^{1/2} \leq 4\rho/\pi^2,$$

$$(\int_0^1 (v'(t))^2 dt)^{1/2} \leq 8\rho/\pi^3,$$

$$(\int_0^1 (v(t))^2 dt)^{1/2} \leq 16\rho/\pi^4.$$

From (2.4.2) it follows that

$$\rho \leq \rho(\frac{16\alpha}{\pi^4} + \frac{8\beta}{\pi^3} + \frac{4\gamma}{\pi^2} + \frac{2\delta}{\pi}).$$

From (2.4.1) and (2.4.3) we get $\rho = 0$, $v^{(iv)} = 0$, $v''' = 0$, $v'' = 0$, $v' = 0$, $v = 0$. and this proves the uniqueness.

Remark Theorem 2.4.3 holds for any of the other three boundary conditions. Condition (2.4.1) can be weakened by using Lemma 2.4.2 instead of Lemma 2.4.1, when the solution or some of its derivatives vanish at both endpoints. So for $BC = SS$ we have

$$\frac{4\alpha}{\pi^4} + \frac{4\beta}{\pi^3} + \frac{2\gamma}{\pi^2} + \frac{2\delta}{\pi} < 1.$$

If $BC = CC$, then

$$\frac{4\alpha}{\pi^4} + \frac{4\beta}{\pi^3} + \frac{4\gamma}{\pi^2} + \frac{2\delta}{\pi} < 1$$

and for $BC = CS$ we have

$$\frac{8\alpha}{\pi^4} + \frac{8\beta}{\pi^3} + \frac{4\gamma}{\pi^2} + \frac{2\delta}{\pi} < 1.$$

Example The hypotheses of Theorem 2.4.3 are satisfied for

$$f = 50t^2 + \alpha \sin x + \beta \cos y + \gamma z + \delta w,$$

where

$$\frac{16\alpha}{\pi^4}+\frac{8\beta}{\pi^3}+\frac{4\gamma}{\pi^2}+\frac{2\delta}{\pi}<1.$$

For the other boundary conditions we can apply the previous remark and the inequality can be weakened.

Theorem 2.4.4 *Suppose $f \in C([0,1]\times \mathbb{R}^2)$ and*

$$(f(t,x_1,y_1)-f(t,x_2,y_2))(x_1-x_2)\leq \frac{\pi^2}{4}(y_1-y_2)^2 \tag{2.4.4}$$

for any $t\in[0,1],\ x_i,y_i\in\mathbb{R},\ i=1,2$. Then the boundary value problem

$$u^{(iv)}=f(t,u,u'),\ \ BC \tag{2.4.5}$$

has at most one solution.

Proof. Let u_1 and u_2 be solutions to (2.4.5). If $v=u_1-u_2$, then v satisfies BC and

$$-v^{(iv)}+f(t,u_1,u_1')-f(t,u_2,u_2')=0.$$

Multiplication of the above equation by v, integration from 0 to 1 and Lemma 2.4.1 yield:

$$0=-\int_0^1 v^{(iv)}v\,dt+\int_0^1(f(t,u_1,u_1')-f(t,u_2,u_2'))(u_1-u_2)dt$$

$$\leq -\int_0^1 v^{(iv)}v\,dt+\frac{\pi^2}{4}\int_0^1(v')^2dt=\int_0^1 v'''v'dt+\frac{\pi^2}{4}\int_0^1(v')^2dt$$

$$=\int_0^1(-(v'')^2+\frac{\pi^2}{4}(v')^2)dt\leq 0.$$

Using Fourier series (see [2] and [19]), we get

$$v' = A\sin(\frac{\pi}{2}(t+t_0)),$$

where $t_0 \in [0,1]$ and $v'(t_0) = 0$. BC implies that $v'(t) = 0$ and $v(t) = 0$ for $t \in [0,1]$ and this proves the theorem.

Example Assume f(t,x,y) satisfies the hypotheses of the mean value theorem. For $y_1 = y_2$ condition (2.4.4) becomes

$$(f(t,x_1,y_1) - f(t,x_2,y_1))(x_1 - x_2) \leq 0$$

and this is true if $f_x(t,x,y) \leq 0$ for any $(t,x,y) \in [0,1] \times \mathbb{R}^2$. By the mean value theorem for $y_1 \neq y_2$ condition (2.4.4) is equivalent to

$$f_x(t,x_0,y_0)\frac{(x_1-x_2)^2}{(y_1-y_2)^2} + f_y(t,x_0,y_0)\frac{x_1-x_2}{y_1-y_2} - \frac{\pi^2}{4} \leq 0,$$

where $x_0 \in [x_1,x_2]$, $y_0 \in [y_1,y_2]$. This is satisfied if $f_x(t,x,y) < 0$ and

$$(f_y(t,x,y))^2 + \pi^2 f_x(t,x,y) \leq 0,$$

for any $(t,x,y) \in [0,1] \times \mathbb{R}^2$. For $f = -x + \sin y$ we have $(\cos y)^2 + \pi^2(-1) \leq 0$, thus by the above argument the function satisfies the hypotheses of Theorem 2.4.4.

Theorem 2.4.5 *Suppose $f \in C([0,1] \times \mathbb{R})$ and*

$$(f(t,z_1) - f(t,z_2))(z_1 - z_2) \geq 0 \tag{2.4.6}$$

for any $t \in [0,1]$, $z_i \in \mathbb{R}$, $i = 1,2$. Then the boundary value problem

$$u^{(iv)} = f(t,u''), \ \ SS \tag{2.4.7}$$

has at most one solution.

Proof. Let u_1 and u_2 be solutions to (2.4.7). If $v = u_1 - u_2$, then v satisfies SS and

$$-v^{(iv)} + f(t, u_1'') - f(t, u_2'') = 0.$$

Multiplying the above equation by v'' and integrating from 0 to 1, we get

$$0 = -\int_0^1 v^{(iv)} v'' dt + \int_0^1 (f(t, u_1'') - f(t, u_2''))(u_1'' - u_2'') dt$$

$$\geq -\int_0^1 v^{(iv)} v'' dt = \int_0^1 (v''')^2 dt \geq 0.$$

Thus, $v''' = 0$, $v'' = 0$, $v' = 0$, $v = 0$ and this proves the theorem.

Example The hypotheses of Theorem 2.4.5 are satisfied for

$$f = t^2 z^m + z^n, \text{ where m and n are odd, positive integers.}$$

Remark In Theorem 2.4.4 we could let f depend formally on all the variables, but nothing would be gained, since condition (2.4.4) requires f to depend only on t, x and y. Similarly for Theorem 2.4.5 condition (2.4.6) requires f to depend only on t and z.

Remark The following theorem shows that condition (2.4.6) in Theorem 2.4.5 can be weakened, but we obtain a weaker result.

Theorem 2.4.6 *Suppose* $f \in C([0,1] \times \mathbb{R})$ *and*

$$(f(t, z_1) - f(t, z_2))(z_1 - z_2) \geq -(z_1 - z_2)^2 \pi^2 \qquad (2.4.8)$$

for any $t \in [0,1]$, $z_i \in \mathbb{R}$, $i = 1, 2$. *Then any two solutions of the boundary value problem (2.4.7) differ by* $A \sin \pi x$, *where* $A \in \mathbb{R}$.

Proof. Let u_1 and u_2 be solutions of (2.4.7). If $v = u_1 - u_2$, then similarly as in the proofs of Theorem 2.4.4 and Theorem 2.4.5 we obtain

$$\int_0^1 ((v''')^2 - \pi^2 (v'')^2) dt = 0.$$

Using Fourier series and the Parseval equality, we get $v = A \sin \pi x$ (see [2] and [19]) and this proves the theorem.

Example The hypotheses of Theorem 2.4.6 are satisfied for

$$f = t\pi^2 \cos z.$$

Theorem 2.4.7 *Suppose* $h \in C[0,1]$ *and*

$$f(t, z_1, w_1) - f(t, z_2, w_2) + h(t)|w_1 - w_2| > 0 \tag{2.4.9}$$

for any $t \in [0,1]$, $z_i, w_i \in \mathbb{R}$, $i = 1, 2$ *and* $z_1 > z_2$. *Then the boundary value problem*

$$u^{(iv)} = f(t, u'', u'''), \ \ SS \tag{2.4.10}$$

has at most one solution.

Proof. Let u_1 and u_2 be solutions to (2.4.10) and set $v = u_1 - u_2$. Then v satisfies SS. We assume $v''(t) > 0$ on $(\alpha, \beta) \subset (0,1)$. Further from (2.4.9) we get

$$v^{(iv)} + h(t)|v'''| > 0$$

and

$$((\exp(\int_0^t h(s) \mathrm{sgn}(v'''(s)) ds) v'''(t))' > 0. \tag{2.4.11}$$

Suppose there existed a $t_0 \in (0,1)$ such that $v''(t_0 > 0$ and let (α, β) be the maximal interval containing t_0 on which $v''(t) > 0$. Then $v''(\alpha) = v''(\beta) = 0$, $v'''(\alpha) \geq 0$,

$v'''(\beta) \leq 0$ and there exists a $t_1 \in (\alpha, \beta)$ such that $v'''(t_1) = 0$. By integrating (2.4.11) from α to t_1 and from t_1 to β, we get $-v'''(\alpha) > 0$, $v'''(\beta) > 0$ and this is a contradiction. So $v''(t) \leq 0$ on $[0, 1]$.

If $v''(t) < 0$ on $(\alpha, \beta) \subset (0, 1)$, then $-v''(t) > 0$ on $(\alpha, \beta) \subset (0, 1)$ and the same argument as above implies $-v''(t) \leq 0$ on $[0, 1]$. Thus, $v''(t) = 0$ on $[0, 1]$. SS implies that $v(t) = 0$ on $[0, 1]$ and this completes the proof of the theorem.

Example The hypotheses of Theorem 2.4.7 are satisfied for

$$f = t^2 z^m + z^n + tw, \text{ where m and n are odd, positive integers.}$$

Remark In Theorem 2.4.7 we could let f depend formally on all the variables, but just like in the case of Theorems 2.4.4, 2.4.5 and 2.4.6, nothing would be gained.

3. BOUNDARY VALUE PROBLEMS WITH DISCONTINUITIES IN THE SPATIAL VARIABLE AND DRY FRICTION

3.1. INTRODUCTION

The results presented in this chapter were motivated by the dry friction equation, in particular by the question of the existence of periodic solutions for this equation. The dry friction phenomenon, which is also known as the stick-slip phenomenon, occurs very frequently in many technical problems. First we prove an existence result for the periodic boundary value problem (PBVP):

$$x' = f(t,x),\ x \in \mathbb{R}^n \tag{3.1.1}$$

$$x(0) = x(1) \tag{3.1.2}$$

where $f = (f_1, ..., f_n)$, $x = (x_1, ..., x_n)$. Further we prove an existence theorem for the second order scalar periodic boundary value problem:

$$u'' = f(t,u,u'),\ u \in \mathbb{R} \tag{3.1.3}$$

$$u(0) = u(1),\ \ u'(0) = u'(1). \tag{3.1.4}$$

Applications to the dry friction equation are also presented. In addition we also prove an existence theorem for the second order Dirichlet problem.

In this chapter we assume f to be measurable, but make no continuity requirements on the function f. We use Filippov's definition of a solution, which is defined in Chapter I, Section 1.3. For further reference and details see [6].

A standard approach to boundary value problems with discontinuities in the spatial variable is to solve the problem on each side of the discontinuity separately and then try to match these solutions. A totally different approach is used here. Using Filippov's theory, we reformulate the BVP's as differential inclusions and then apply the existence theorems proved in Section 3.3 and in [12] to obtain existence results for the PBVP and for the Dirichlet Problem.

3.2. THE NONLINEAR ALTERNATIVES FOR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS

In this section we state two versions of the nonlinear alternative for differential inclusions. The proof is given in [12] and it is based on a set-valued analogue of a classical result of Leray and Schauder, which had been proved in [5]. First we introduce some notation and give the necessary definitions.

Let $k \geq 0$ and $[k] = \{0, 1, \ldots, k\}$. Given a bounded set $A \subset (\mathbb{R}^n, |\ |)$, let $|A| = sup\{|a| : a \in A\}$. Let $C_0^k = \{u \in C^k[0,1] : u(0) = 0\}$. For each $i \in [k-1]$ let $g_i : C_{k-1}([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ be a continuous linear form such that there exists $\tilde{g}_i$, $\tilde{g}_i : C_{k-1}([0,1], \mathbb{R}) \to \mathbb{R}$, and $g(\phi(x)v) = \tilde{g}(\phi(x))v$ for each real-valued function $\phi \in C^{k-1}$ and $v \in \mathbb{R}^n$. Define

$$\mathcal{B} = \{u \in C^{k-1} : g_i(u) = 0,\ i \in [k-1])\},\ and\ C^n_{\mathcal{B}} = \{u \in C^n : u \in \mathcal{B}\}.$$

We define a linear operator $\Lambda : C^k_{\mathcal{B}} \to C$ by $\Lambda y = y^{(k)}$. If X is a subset of a Banach space E, then $Kv(X)$ is the family of all compact, convex, nonempty subsets of X.

Definition 3.2.1 *Let $p \geq 1$. A set valued function $F : [0,1] \times \mathbb{R}^{kn} \to Kv(\mathbb{R}^n)$ is an L^p-Carathéodory function provided:*

(a) the map $x \to F(t,x)$ is upper semicontinuous for all $t \in [0,1]$;

(b) the map $t \to F(t,x)$ *is measurable for all* $x \in \mathbb{R}^{kn}$*;*

(c) for each $r > 0$ *there exists* $h_r \in L^p[0,1]$ *such that* $|x| \le r$ *implies* $|F(t,x)| \le h_r(t)$ *for almost all* $t \in [0,1]$.

Theorem 3.2.2 (Nonlinear alternative for regular problems) *Consider an* L^p*-Carathéodory function* F *and the family of problems*

$$x^{(k)} \in \lambda[F(t,x,x',\ldots,x^{(k-1)})], \;\; x \in \mathcal{B}, \tag{3.2.1}$$

for $\lambda \in [0,1]$.

(A) *Let* U *be a bounded, open set in* $C_{\mathcal{B}}^{k-1}$ *with* $0 \in U$*. Then either*

$$x \in F(t,x,x',\ldots x^{(k-1)}), \;\; x \in \mathcal{B}, \tag{3.2.2}$$

has a solution in cl(U)*, the closure of* U*, or (3.2.1) has a solution on* ∂U*, the boundary of* U*, for some* $\lambda \in (0,1)$.

(B) *Suppose there exists a constant* M *such that for any* $\lambda \in (0,1)$ *and any solution* x *to (3.2.1) we have* $|x|_{k-1} \le M$*. Then (3.2.2) has a solution.*

Proof. See [12].

Theorem 3.2.3 (Nonlinear alternative for problems in resonance) *Consider an* L^p*-Carathéodory function* F *and the family of problems*

$$x^{(k)} + \alpha x \in \lambda[F(t,x,x',\ldots,x^{(k-1)}) + \alpha x], \;\; x \in \mathcal{B}, \tag{3.2.3}$$

for $\lambda \in [0,1]$*, where* $\alpha \neq 0$ *is fixed and is not an eigenvalue of* Λ.

(A) *Let* U *be a bounded, open set in* $C_{\mathcal{B}}^{k-1}$ *with* $0 \in U$*. Then either (3.2.2) has a solution in cl*(U)*, the closure of* U*, or (3.2.3) has a solution on* ∂U*, the boundary of* U*, for some* $\lambda \in (0,1)$.

(B) *Suppose there exists a constant* M *such that for any* $\lambda \in (0,1)$ *and any solution* x *to (3.2.3) we have* $|x|_{k-1} \le M$*. Then (3.2.2) has a solution.*

Proof. See [12].

3.3. EXISTENCE THEOREMS FOR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS

In this section we recall two theorems on differential inclusions from [12] and we prove an existence theorem for periodic solutions to a second order scalar differential inclusion. The Nonlinear Alternatives from Section 3.2 are used to prove these theorems.

Theorem 3.3.1 (Periodic problem for inclusions) *Let $F(t,x) : [0,1] \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ be L^p-Carathéodory and consider the problem*

$$x' \in F(t,x), \;\; x(0) = x(1), \tag{3.3.1}$$

where F satisfies

$$x \cdot F(t,x) \leq 0^* \text{ for all } |x| \geq r, \text{ for some fixed } r > 0. \tag{3.3.2}$$

Then (3.3.1) has a solution.

Proof. See [12].

Theorem 3.3.2 (Periodic problem for second order scalar inclusions) *Let $F(t,x) : [0,1] \times \mathbb{R}^2 \to Kv(\mathbb{R})$ be an L^p-Carathéodory function. Further let*

$$|F(t,y,z) - \alpha y - z g(y)| < C, \tag{3.3.3}$$

for all $y, z \in \mathbb{R}$, where $\alpha, C \in \mathbb{R}, \;\; C > 0, \;\; 0 < |\alpha| < \pi^2$ and $g \in C^0(\mathbb{R})$. Then

$$u'' \in F(t,u,u'), \;\; u(0) = u(1), \; u'(0) = u'(1), \tag{3.3.4}$$

has at least one solution.

Proof. The proof is an application of the nonlinear alternative for problems in resonance, Theorem 3.2.3. We need to establish *a priori* bounds independent of λ for the $C^1[0,1]$ norm of the solutions to

$$u'' - \alpha u \in \lambda[F(t,u,u') - \alpha u], \;\; u(0) = u(1), \; u'(0) = u'(1), \tag{3.3.5}$$

where $\lambda \in (0,1)$, and $\alpha \neq 0$ is fixed. Since $|\alpha| < \pi^2$ we can see that $-\alpha$ is not an eigenvalue of the associated homogeneous problem. Let u be a solution of the family (3.3.5), where $0 < \lambda < 1$. Then

$$u'' = \lambda(w - \alpha u) + \alpha u, \;\; u(0) = u(1), \; u'(0) = u'(1), \tag{3.3.6}$$

where $w(t) \in F(t,u,u')$ for almost every $t \in (0,1)$ and $w(t)$ is measurable. The existence of the function w is a consequence of Kuratowski-Ryll-Nardzewski theorem†; see [1] for Kuratowski-Ryll-Nardzewski theorem. We integrate (3.3.6) from 0 to 1 and use the identity $\int_0^1 u'g(u)dt = 0$ to get

$$0 = \alpha\int_0^1 udt + \lambda\int_0^1 (w - \alpha u)dt = \alpha\int_0^1 udt + \lambda\int_0^1 (w - \alpha u - u'g(u)))dt.$$

This implies that

$$|\int_0^1 udt| \leq \frac{\lambda C}{|\alpha|} \leq \frac{C}{|\alpha|},$$

and for some $t_0 \in (0,1)$ we have

$$|u(t_0)| = |\int_0^1 udt| \leq \frac{C}{|\alpha|}.$$

Further, by the Hölder inequality we have

$$|u(t)| \leq |u(t_0)| + \int_{t_0}^t |u'(s)|ds \leq \frac{C}{|\alpha|} + (|\int_0^1 |u'(s)|^2 ds)^{\frac{1}{2}} \tag{3.3.7}$$

---

†This argument is frequently used in the theory of differential inclusions; see [9], [8], [12] and [17].

for $t \in [0,1]$. Now we find an estimate for the L_2 norm of u'. We multiply (3.3.6) by u and integrate from 0 to 1 to get

$$-\int_0^1 (u')^2 dt = \alpha \int_0^1 u^2 dt + \lambda \int_0^1 (wu - \alpha u^2 - g(u)u'u)dt. \tag{3.3.8}$$

Here we integrated by parts on the left hand side of the equation and on the right hand side of the equation we used the identity

$$\int_0^1 g(u)u'udt = u(1)G(u(1)) - u(0)G(u(0)) - \int_0^1 G(u)u'dt = 0,$$

where $\frac{dG(u)}{du} = g$. From (3.3.7), (3.3.8) and from the Wirtinger inequality for periodic functions, we obtain

$$\int_0^1 |u'|^2 dt \leq |\alpha| \int_0^1 u^2 dt + \lambda C \int_0^1 |u| dt \leq \frac{|\alpha|}{\pi^2} \int_0^1 |u'|^2 dt + \lambda \frac{C^2}{|\alpha|} + \lambda C (\int_0^1 |u'|^2 dt)^{\frac{1}{2}}. \tag{3.3.9}$$

If $\|u'\|_{L_2(0,1)} < 1$, then we have the desired estimate, so we assume $\|u'\|_{L_2(0,1)} \geq 1$. After a simple manipulation with (3.3.9) we get

$$\|u'\|_{L_2(0,1)} \leq K,$$

where

$$K = \frac{C(C + |\alpha|)\pi^2}{|\alpha|(\pi^2 - |\alpha|)} > 0.$$

Here we used the assumption $|\alpha| < \pi^2$. From (3.3.7) it follows that

$$|u(t)| \leq \frac{C}{|\alpha|} + K, \text{ for } t \in [0,1].$$

Finally, we need an estimate on u'. We multiply (3.3.6) by u'' and integrate from 0 to 1 to get

$$\int_0^1 (u'')^2 dt = \int_0^1 (\alpha u u'' + \lambda(w u'' - \alpha u u''))dt = -\alpha \int_0^1 (u')^2 dt + \lambda \int_0^1 (wu'' - \alpha u u'')dt. \tag{3.3.10}$$

From (3.3.3), (3.3.10) and the Hölder inequality, it follows that

$$\int_0^1 (u'')^2 dt \leq |\alpha| \int_0^1 |u'|^2 dt + \lambda C \int_0^1 |u''| dt + \lambda \int_0^1 u'' u' g(u) dt \leq$$

$$\alpha K^2 + \lambda C \int_0^1 |u''| dt + \lambda M \int_0^1 |u'' u'| dt \leq \alpha K^2 + C\Big(\int_0^1 |u''|^2 dt\Big)^{\frac{1}{2}} + KM\Big(\int_0^1 |u''|^2 dt\Big)^{\frac{1}{2}},$$

where $M = max\{g(u) : |u| \leq \frac{C}{|\alpha|} + K\}$. If $\|u''\|_{L_2(0,1)} < 1$, then we have the desired estimate, so we assume $\|u''\|_{L_2(0,1)} \geq 1$ to obtain

$$\|u''\|^2_{L_2(0,1)} \leq (\alpha K^2 + C + KM)\|u''\|_{L_2(0,1)}$$

and

$$\|u''\|_{L_2(0,1)} \leq L,$$

where $L = \alpha K + C + KM$. Since there exists $t_1 \in (0,1)$ such that $u'(t_1) = 0$, the fundamental theorem of calculus and Hölder inequality yield

$$|u'(t)| \leq \int_{t_1}^t u''(s) ds \leq \|u''\|_{L_2(0,1)} \leq L \text{ for } t \in [0,1],$$

and the required *a priori* bounds are established. This proves the theorem.

Theorem 3.3.3 (Dirichlet Problem for Inclusions) *Let*

$$F : [0,1] \times \mathbb{R}^{2n} \to Kv(\mathbb{R}^n)$$

be L^p-Carathéodory and consider the problem

$$x'' \in F(t, x, x'), \;\; x(0) = x(1) = 0, \tag{3.3.11}$$

with F(t,x,x') = G(t,x,x') + H(t,x,x'). Further let $z = (z_1, z_2)$, $z_1, z_2 \in \mathbb{R}^n$, and

(a) $z_1 \cdot G(t,z) \geq 0$ for all $(t,z) \in [0,1] \times \mathbb{R}^{2n}$,

(b) $|G(t,z)| \leq c(t,z_1)|z_2|^2 + d(t,z_1)$ where $c(t,z_1)$, $d(t,z_1)$ are bounded on bounded sets,

(c) $|H(t,z)| \leq L(|z_1|^\alpha + |z_2|^\beta)$ for some $\alpha \in [0,1]$, $\beta \in [0,1)$, $L \in \mathbb{R}$ and $L \geq 0$. Then the system (3.3.11) has at least one solution.

Proof. See [12].

3.4. EXISTENCE THEOREMS FOR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DISCONTINUITIES IN THE SPATIAL VARIABLE

In this section we apply the results of the previous section to boundary value problems with discontinuities in the spatial variable. By solutions, we mean the Filippov solutions defined in Section 1.3.

Theorem 3.4.1 (Periodic problem for Filippov solutions) *Suppose f(t,x) is measurable in* $[0,1] \times \mathbb{R}^n$ *and suppose further that for any bounded, closed domain* $D \subset [0,1] \times \mathbb{R}^n$*, there exists an integrable function B(t), which may depend on D, such that*

$$|f(t,x)| \leq B(t) \tag{3.4.1}$$

almost everywhere in D. Moreover, assume that for all $(t_0, x_0) \in [0,1] \times \mathbb{R}^n$*, there exist* δ_1*,* $\delta_2 > 0$ *and a function C(t):* $[t_0 - \delta_1, t_0 + \delta_1] \to \mathbb{R}$ *such that*

$$|f(t,x)| \leq C(t) \tag{3.4.2}$$

for $(t,x) \in [t_0 - \delta_1, t_0 + \delta_1] \times cl(U(x_0, \delta_2)) = cl(U(t_0, x_0, \delta_1, \delta_2))$ *(clY is the closure of Y) , where at the endpoints, we consider appropriate one-sided neighborhoods. Finally, suppose there is an* $r > 0$ *such that for every* x*,* $|x| > r$ *there exists an* $\epsilon > 0$*,* $\epsilon \leq |x| - r$ *such that*

$$x \cdot f(t,y) \leq 0 \tag{3.4.3}$$

for $y \in U(x,\epsilon) - N$*, where* $\mu(N) = 0$*. Then the periodic boundary value problem*

$$x' = f(t,x), \; x \in \mathbb{R}^n \tag{3.4.4}$$

$$x(0) = x(1) \tag{3.4.5}$$

has a solution.

Proof. To establish the result we apply Theorem 3.3.1. First we prove that $k_t(x)$ is upper semicontinuous for all $t \in$ [0,1] as a function of x. We can write

$$k_t(x) = \bigcap_{n=1,\dots} \bigcap_{\mu N=0} konvf(t, U(x, 1/n) - N).$$

We define an open set

$$\Omega(k_t(x), \epsilon) = \{y \in \mathbb{R}^n : d(y, k_t(x)) < \epsilon\}.$$

Further we define

$$K_n = \{\mathbb{R}^n - \Omega(k_t(x), \epsilon)\} \cap \bigcap_{\mu N=0} konvf(t, U(x, 1/n) - N)$$

for $n \in \mathbb{N}$. From (3.4.2) it follows that there exists an $N_0 \in \mathbb{N}$ such that for $n > N_0$, K_n is a compact set. Since $K_n \subset K_{n-1}$, there exists an $n_0 \in \mathbb{N}$ such that $K_{n_0} = \emptyset$ otherwise

$$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$$

and this is a contradiction. If $z \in U(x, 1/2n_0)$ then $U(z, 1/2n_0) \subset U(x, 1/n_0)$ so

$$k_t(z) \subset \bigcap_{\mu N=0} konvf(t, U(z, \frac{1}{2n_0}) - N) \subset \bigcap_{\mu N=0} konvf(t, U(x, \frac{1}{n_0}) - N) \subset \Omega(k_t(x), \epsilon).$$

The above inclusion implies that $k_t(x)$ is upper semicontinuous for all $t \in [0, 1]$ as a function of x.

From the results given in [20] (see Remark 2) it follows that $k_t(x)$ is measurable for all $x \in \mathbb{R}^n$ as a function of t.

Now we show that for each $r > 0$, there exists an integrable function $h_r(t)$ such that if $|z| \leq r$, then $|k_t(z)| \leq h_r(t)$ for almost all $t \in (0, 1)$. Since $D = [0, 1] \times [-r, r]$ is bounded we can apply (1.3.3) and (3.4.1) to get

$$|k_t(z)| \leq |\mathcal{M}(z, t)| \leq h_r(t)$$

for almost every $t \in (0,1)$, where $h_r(t)$ is integrable. In order to apply Theorem 3.3.1 it remains to prove

$$x \cdot k_t(x) \leq 0$$

for all $|y| \geq r$ and for some fixed $r > 0$. From (3.4.3) it follows that there exists $\epsilon > 0$ such that

$$x \cdot f(t, U(x,\epsilon) - N) \leq 0,$$

where $\mu(N) = 0$ and this means that $f(t, U(x,\epsilon) - N)$ is a subset of the half plane $P = \{y : x \cdot y \leq 0\}$ so

$$konv f(t, U(x,\epsilon) - N) \subset P$$

and

$$k_t(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} konv f(t, U(x,\delta) - N) \subset P.$$

The last inclusion is equivalent to $x \cdot k_t(x) \leq 0$. Now the theorem follows from Theorem 3.3.1.

Remark 2 Observe that the upper semicontinuity in x for almost all $t \in [0,1]$ for $k_t(x)$ would follow from the weaker assumption: for all $x_0 \in \mathbb{R}^n$ and almost every $t_0 \in [0,1]$, there exists $\delta > 0$ and $C \geq 0$ such that $|f(t_0, x)| \leq C$ for $|x - x_0| < \delta$. However, to apply Theorem 3.3.1 and Theorem 18.6.3, [20] the stronger assumption of the theorem is used to guarantee upper semicontinuity of $k_t(x)$ in x for all $t \in [0,1]$ and the measurability of $k_t(x)$ in t for all $x \in \mathbb{R}^n$. See also Remark 3.

Remark 3 The hypotheses of Theorem 3.4.1 and the results given in [20] imply that $k_t(x)$ is measurable for all $x \in \mathbb{R}^n$ as a function of t. In particular this follows from Theorem 18.6.3, p. 351, where under the assumption (3.4.2) it is shown that a certain multifunction E constructed from f is a Scorza Dragoni function[‡] (Definition 18.5.3). In Appendix 18.8. and 18.9. it is shown that $E = k_t(x)$. From Theorem A18.5.11 it follows that any Scorza Dragoni function is measurable as a function of t for all $x \in \mathbb{R}^n$.

However, in most applications we deal with functions $f(t,x)$ which exhibit only jump type discontinuities, as in the case of the dry friction equation. Thus, we could consider $f(t,x)$ to be such that $k_t(x)$ is measurable for all $x \in \mathbb{R}^n$ as a function of t, which is usually an easily verifiable condition. If we were to make this additional assumption , we could avoid referring to the technical proofs in [20].

Remark 4 Condition (3.4.3) becomes

$$x \cdot f(t,x) \leq 0$$

at all points where f is continuous. In a neighborhood of a point of discontinuity, x, condition (3.4.3) requires the function f to take values within the half plane $P = \{y : x \cdot y \leq 0\}$. We can neglect sets of measure zero.

Definition 3.4.2 *We say $u = x_1$ is a Filippov solution of*

$$u'' = f(t,u,u'),\ \ u(0) = u(1),\ u'(0) = u'(1), \tag{3.4.6}$$

if (x_1, x_2) is a Filippov solution of

$$x_1' = x_2,$$

[‡]Let the set $G \subset \mathbb{R} \times \mathbb{R}^n$ be open and $E : G \to Kv(\mathbb{R}^n)$. We say that E is a Scorza Dragoni function if for every $\epsilon > 0$ there exists a measurable set $A_\epsilon \subset \mathbb{R}$ such that $\mu(\mathbb{R} - A_\epsilon) < \epsilon$ and F restricted to $G \cap (A_\epsilon \times \mathbb{R}^n)$ is upper semicontinuous.

$$x_2' = f(t, x_1, x_2),$$

$$x_1(0) = x_1(1),\ \ x_2(0) = x_2(1).$$

Theorem 3.4.3 (Second order scalar periodic problem for Filippov solutions) *Suppose f(t,y,z) is measurable in* $[0,1] \times \mathbb{R}^2$ *and suppose further that for any bounded, closed domain* $D \subset [0,1] \times \mathbb{R}^n$, *there exists an integrable function, B(t), which can depend on D, such that*

$$|f(t, y, z)| \leq B(t) \tag{3.4.7}$$

almost everywhere in D. Moreover, assume that for all $(\bar{t}, \bar{y}, \bar{z}) \in [0,1] \times \mathbb{R}^2$ *there exist* $\delta_1,\ \delta_2 > 0$ *and a function C(t):* $[\bar{t} - \delta_1, \bar{t} + \delta_1] \to \mathbb{R}$ *such that*

$$|f(t, y, z)| \leq C(t) \tag{3.4.8}$$

for $((t, y, z) \in [\bar{t} - \delta_1, \bar{t} + \delta_1] \times cl(U((\bar{y}, \bar{z}), \delta_2))$ *(clY is the closure of Y) , where at the endpoints, we consider appropriate one-sided neighborhoods. Finally, suppose that for each* $(y, z) \in \mathbb{R}^2$ *there exists an* $\epsilon > 0$ *such that*

$$\sup_{(x,y)\in\mathbb{R}^2} |f(t, \tilde{y}, \tilde{z}) - \alpha y - zg(y)| < C, \tag{3.4.9}$$

for $(\tilde{y}, \tilde{z}) \in U((y, z), \epsilon) - N$, *where* $\mu(N) = 0$ *and* α, C *and* g *are like in Theorem 3.3.2. Then the periodic boundary value problem*

$$u'' = f(t, u, u'),\ \ u(0) = u(1),\ u'(0) = u'(1), \tag{3.4.10}$$

where $u \in \mathbb{R}$, *has a solution.*

Proof. To establish the result we apply Theorem 3.3.2. By the same argument as in the proof of Theorem 3.4.1 we get that

$$K\{f(t, x_1, x_2)\} = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} konv f(t, U((x_1, x_2), \delta) - N),$$

where f is a function satisfying (3.4.7), (3.4.8) is an L^p-Carathéodory function. Let $(h = col(x_2, f(t, x_1, x_2))$, where $col(x_2, f(t, x_1, x_2)$ is the column vector with components x_2 and $f(t, x_1, x_2)$, then

$$K\{h(t, x_1, x_2)\} = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} konvh(t, U((x_1, x_2), \delta) - N) = col(x_2, K\{f(t, x_1, x_2\}).$$

So (3.4.10) is equivalent to

$$u'' \in K\{f(t, u, u')\}, \ u(0) = u(1), \ u'(0) = u'(1).$$

From (3.4.9) it follows that $K\{f(t, x, y)\}$ satisfies (3.3.3), so Theorem 3.3.2 applies and this completes the proof.

Theorem 3.4.4 (Dirichlet Problem for Filippov solutions) *Let $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be measurable in $[0,1] \times \mathbb{R}^{2n}$ and suppose for any bounded closed domain $D \subset [0,1] \times \mathbb{R}^{2n}$, there exists an integrable function $B(t)$, which can depend on D, such that*

$$|f(t,x)| \leq B(t)$$

almost everywhere in D. For all $(t_0, x_0) \in [0,1] \times \mathbb{R}^{2n}$ there exist $\delta_1, \delta_2 > 0$ and a function $C : [t_0 - \delta_1, t_0 + \delta_1] \to \mathbb{R}$ such that

$$|f(t,x)| \leq C(t)$$

for $(t,x) \in [t_0 - \delta_1, t_0 + \delta_1] \times cl(U(x_0, \delta_2)) = cl(U(t_0, x_0, \delta_1, \delta_2))$.

Further let $z = (z_1, z_2)$, $z_1, z_2 \in \mathbb{R}^n$,

$$f(t,z) = g(t,z) + h(t,z),$$

where

(a) $z_1 \cdot g(t,y) \geq 0$ *for all* $(t,z) \in [0,1] \times \mathbb{R}^{2n}$ *and* $y \in U(z,\epsilon) - N$, $\mu(N) = 0$ *for some* $\epsilon > 0$,

(b) $|g(t,z)| \leq c(t,z_1)|z_2|^2 + d(t,z_1)$ *where* $c(t,z_1)$, $d(t,z_1)$ *are bounded on bounded sets,*

(c) $|h(t,z)| \leq L(|z_1|^\alpha + |z_2|^\beta)$ *for some* $\alpha \in [0,1]$, $\beta \in [0,1)$ *and* $L \geq 0$.

Then the boundary value problem

$$x'' = f(t,x,x'), \;\; x \in \mathbb{R}^n$$

$$x(0) = x(1) = 0$$

has at least one solution.

Proof. The proof is the same as that of Theorem 3.4.1 and Theorem 3.4.3 but instead of using Theorem 3.3.1 and Theorem 3.3.2 we use Theorem 3.3.3 and we define solutions analogously as in Definition 3.4.2.

3.5. APPLICATIONS TO THE DRY FRICTION EQUATION

Application of Theorem 3.4.1

Theorem 3.4.1 is applicable to the dry friction equation

$$u'' + bu' + cu + k\mathrm{sgn}u' = e(t), \tag{3.5.1}$$

where $b, c, k \in \mathbb{R}$, $b, c, k > 0$, e(t) is a measurable 1-periodic function with

$$\sup_{t \in \mathbf{R}} |e(t)| = M.$$

(1) If $u = x_1$, we can rewrite this equation as a system in the following way:

$$x_1' = x_2 - x_1,$$

$$x_2' = (1-b)(x_2 - x_1) - cx_1 - k\text{sgn}(x_2 - x_1) + e(t).$$

Condition (3.4.3) then becomes

$$-(x_1^2 + (b-1)x_2^2 + (c-b)x_1x_2) + |x_2|(k+M) < 0$$

for all x_1, x_2 such that $x_1^2 + x_2^2 > r^2$ for some fixed $r \in \mathbb{R}$. By completing the square we get

$$-(x_1 + \frac{c-b}{2}x_2)^2 - (b - 1 - \frac{(c-b)^2}{4})x_2^2 + |x_2|(k+M) < 0. \tag{3.5.2}$$

If $b > 1$ and $c \in (b - 2\sqrt{b-1}, b + 2\sqrt{b-1})$ then there exists $r \in \mathbb{R}$, which depends on b, c, k and M, such that (3.5.2) is satisfied for $x_1^2 + x_2^2 > r^2$ and by Theorem 1 there exists a 1-periodic solution. Moreover, the proof of Theorem 5 [12] gives the *apriori* estimate $x_1^2 + x_2^2 < r^2$ so we get

$$u^2 + (u' + u)^2 < r^2.$$

If $k < M$, then there does not exist a constant solution.

(2) By the above discussion there exists a 1-periodic solution of the equation

$$u'' + 2u' + 3u + \text{sgn}u' = 2\sin 2\pi t. \tag{3.5.3}$$

(3.5.2) becomes

$$-(x_1 + \frac{x_2}{2})^2 - \frac{3}{4}x_2^2 + 3|x_2| < 0.$$

This inequality holds for example if $x_1^2 + x_2^2 > 25$, so we get the bound

$$u^2 + (u' + u)^2 < 25$$

and since $k < M$ we have no constant solutions.

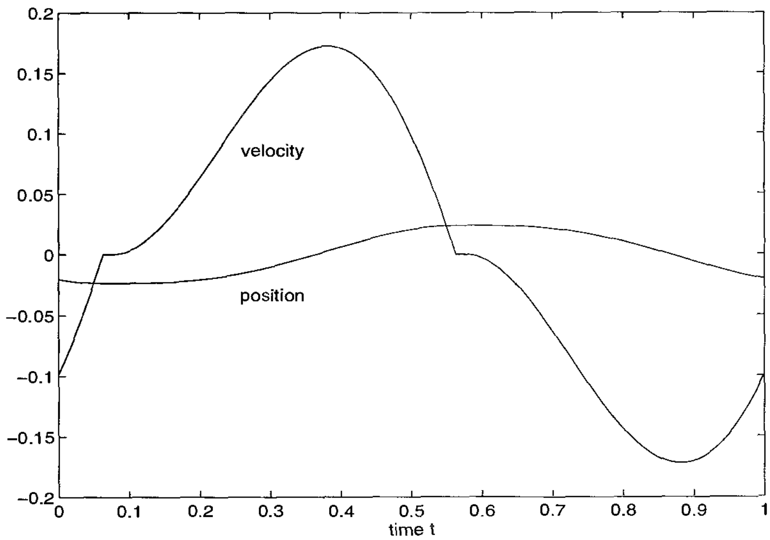

Figure 2. Graph of a periodic solution of the forced dry friction equation (3.5.3)

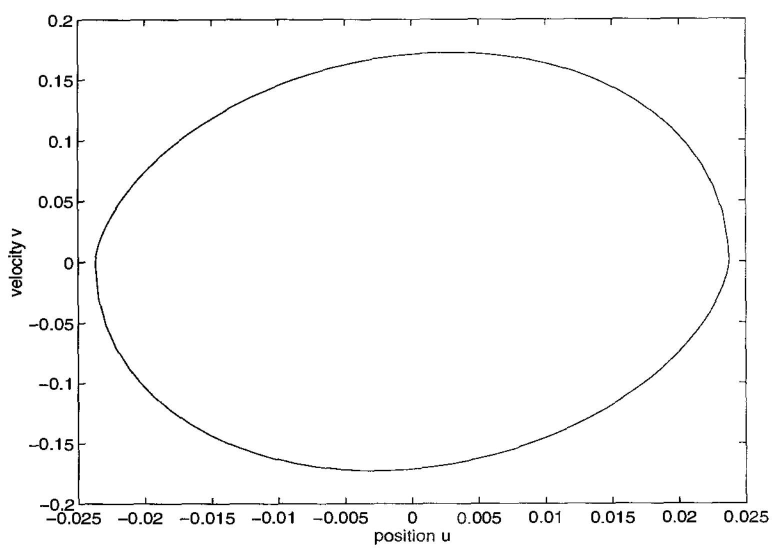

Figure 3. Phase diagram of the periodic solution of (3.5.3) from Figure 2

(3) An example of the unforced damped oscillator with dry friction is given in Chapter I, Section 1.3.

Application of Theorem 3.4.3

Theorem 3.4.3 is applicable to a more general dry friction equation with the following nonlinearities:

$$u'' + b(u') + c(u) + k\mathrm{sgn} u' = e(t), \tag{3.5.4}$$

or

$$u'' + d(u)u' + c(u) + k\mathrm{sgn} u' = e(t), \tag{3.5.5}$$

where $k \in \mathbb{R}, k > 0$, b,c and g are nonlinear functions, e is a measurable 1-periodic function.

(1) Consider the equation

$$u'' + b(u') + c(u) + k\mathrm{sgn} u' = e(t),$$

where b, c are continuous functions and $e(t)$ is a 1-periodic measurable function with

$$\sup_{t\in\mathbb{R}} |e(t)| = M.$$

Condition (3.4.9) becomes:

$$|-b(z) - c(y) \pm k\mathrm{sgn} z + e(t) - \alpha y - zg(y)| \leq$$

$$|-b(z) - zg(y)| + |-c(y) - \alpha y| + |e(t) + k| \leq C.$$

If there exist $\beta, \gamma, C_1, C_2 \in \mathbb{R}$, $0 < |\gamma| < \pi^2$, such that

$$|b(z) - \beta z| \leq C_1, \tag{3.5.6}$$

$$|c(y) - \gamma y| \leq C_2,$$

for all $y, z \in \mathbb{R}$, then we can take $C = C_1 + C_2 + M + k$ and condition (3.4.9) is satisfied. We considered $g(y) = -\beta$, $\alpha = -\gamma$. Thus, our theorem applies for example to the equation

$$u'' + 2u' - u'e^{-(u')^2} + 3u - ue^{-(u)^2} + \mathrm{sgn}u' = 2\sin 2\pi t. \tag{3.5.7}$$

The graph and the phase diagram of equation (3.5.7) is in Figures 4 and 5. We can see that the amplitude has increased compared to equation (3.5.3). This is due to the decreased damping and decreased stiffness of the spring. We can observe similar phenomena also in examples (2) and (3) that follow.

(2) If we replace $b(x')$ by $x'd(x)$ then we need an estimate on

$$|-zd(y) - zg(y)|,$$

and this vanishes if we take $g = -d$. Thus, our theorem applies for example to the equation

$$u'' + 2(u^2 + 1)u' + 3u - ue^{-(u)^2} + \mathrm{sgn}u' = 2\sin 2\pi t. \tag{3.5.8}$$

(3) We can also consider a combination of the cases (1) and (2) of the form

$$u'' + b(u') + d(u)u' + c(u) + k\mathrm{sgn}u' = e(t),$$

where $b(z)$ satisfies (3.5.6). The function $g = -d - \beta$. So our theorem applies also to the equation

$$u'' + u'e^{-(u')^2} + 2(u^2 + 1)u' + 3u - ue^{-(u)^2} + \mathrm{sgn}u' = 2\sin 2\pi t. \tag{3.5.9}$$

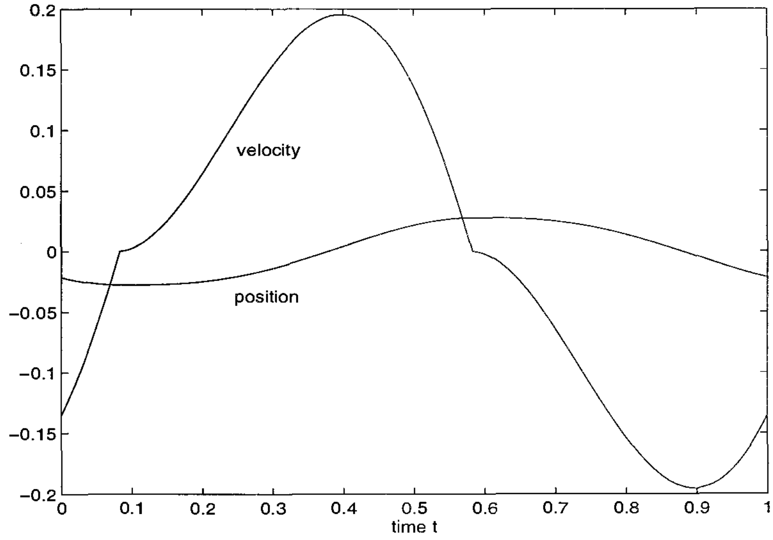

Figure 4. Graph of a periodic solution of the forced dry friction equation (3.5.7)

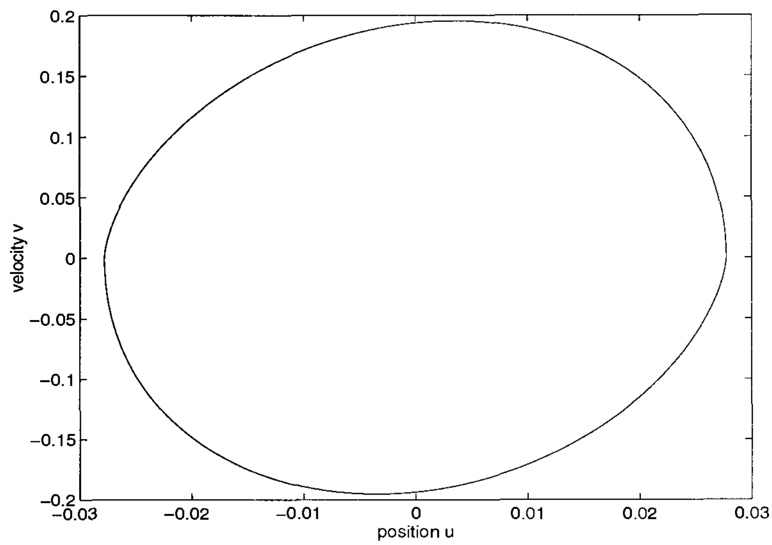

Figure 5. Phase diagram of the periodic solution of (3.5.7) from Figure 4

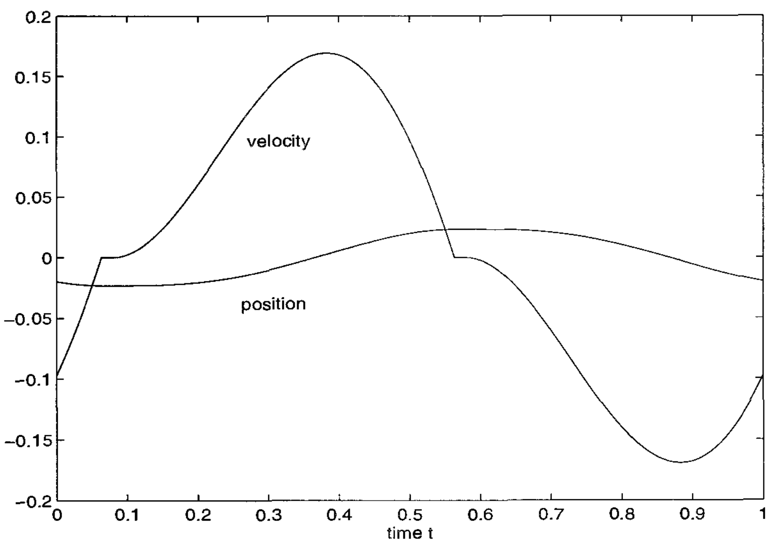

Figure 6. Graph of a periodic solution of the forced dry friction equation (3.5.8)

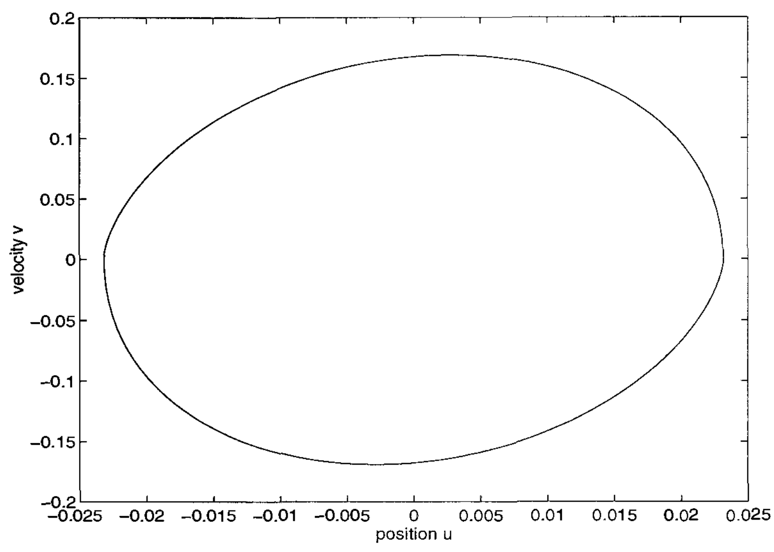

Figure 7. Phase diagram of the periodic solution of (3.5.8) from Figure 6

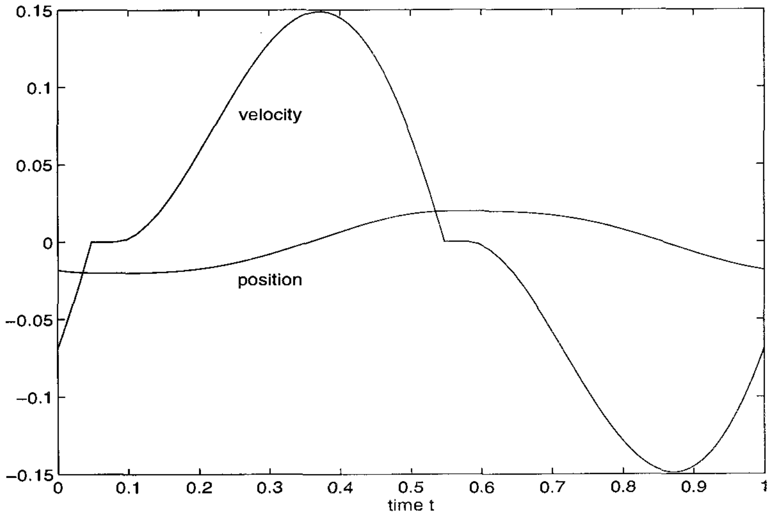

Figure 8. Graph of a periodic solution of the forced dry friction equation (3.5.9)

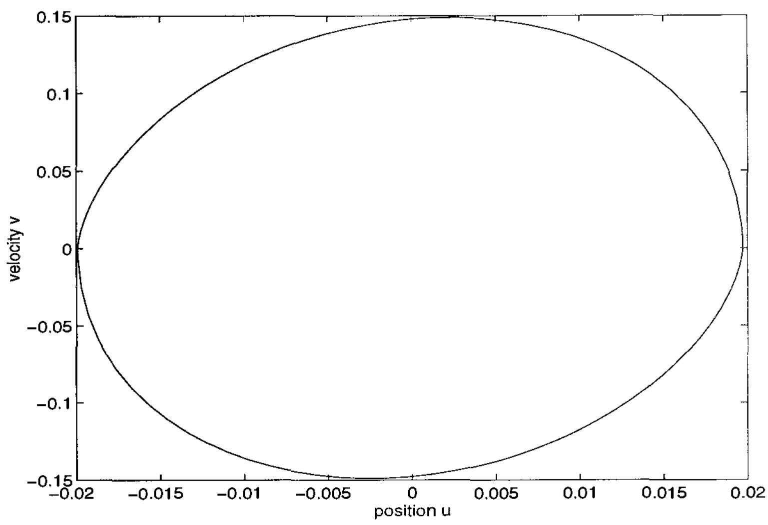

Figure 9. Phase diagram of the periodic solution of (3.5.9) from Figure 8

BIBLIOGRAPHY

[1] J.P. Aubin, A. Cellina, **Differential inclusions**, A series of comprehensive studies in mathematics, No. 264, Springer-Verlag, Berlin - Heidelberg - New York, 1984.

[2] E.F. Beckenbach and R. Bellman, **Inequalities**, Springer-Verlag, Berlin - Heidelberg - New York, 1965.

[3] C. Castaing and M. Valadier, **Convex analysis and measurable multifunctions**, Lect. notes in math., No.580, Springer-Verlag, Berlin - Heidelberg - New York, 1977.

[4] K. Deimling, **Multivalued differential equations**, De Gruyter series in nonlinear analysis and applications, 1, 1992

[5] J. Dugundji and A. Granas, **Fixed point theory**, Vol 1, Monographie Matematyczne, PNW, Warszaawa, 1982.

[6] A.F.Filippov, *Differential equations with discontinuous right-hand side*, Translat. Amer. Math. Soc. **42**, 199-231 (1964).

[7] G.B. Folland, **Real analysis**, John Wiley & Sons, New York - Chichester - Brisbane - Toronto - Singapore, 1984.

[8] M. Frigon, A. Granas, *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C.R. Acad. Sci. Paris, Serie 1, p. 819-822, (1990).

[9] M. Frigon, A. Granas, Z.E.A. Guennoun *Sur l'intervalle maximal d'existence de solutions pour des inclusions différentielles*, C.R. Acad. Sci. Paris, t. **306**, Serie I, p. 747-750, (1988).

[10] R.E. Gaines and J.L. Mawhin, **Coincidence degree and nonlinear differential equations**, Springer-Verlag, Berlin - Heidelberg - New York, 1977.

[11] A. Granas, *Sur La méthode de continuité de Poincaré*, Comptes Rendus Acad. Sci. Paris, **282**, 983-985, (1976).

[12] A. Granas, R.B. Guenther and J.W. Lee, *Some existence results for the differential inclusions*

$$y^{(k)} \in F(x, y, ..., y^{(k-1)}),\ y \in \mathcal{B},$$

C.R. Acad. Sci. Paris, t. **307**, Serie I , 391-396, (1988).

[13] A. Granas, R.B. Guenther and J.W. Lee, *Some general existence principles in the Carathéodory theory of nonlinear differential systems*, J. Math. Pures Appl. **70**, 153-196, (1991).

[14] A. Granas, R.B. Guenther and J.W. Lee, *Nonlinear boundary value problems for some classes of ordinary differential equations*, Rocky Mountain Journal of Mathematics **10**, 35-58, (1980).

[15] R.B. Guenther and J.W. Lee, **Partial differential equations of mathematical physics and integral equations**, Prentice-Hall, Englewood Cliffs, New Jersey, 1988.

[16] R.B. Guenther and J.W. Lee, *Some existence results for nonlinear integral equations via topological transversality*, Journal of Integral Equations and Applications, Volume **5**, Number 2, 195-209, (1993).

[17] R.B. Guenther, J.W. Lee and M. Šenkyřík, *The Filippov approach to boundary and initial value problems and applications*, **Boundary value problems for functional differential equations**, World Scientific Publishing, (in press).

[18] C.P. Gupta, **On a third-order three-point boundary value problem at resonance**, Diff. Int. Equations, **1**, Vol. 2, 1-12, (1989).

[19] G.H. Hardy, J.E. Littlewood and G. Pólya, **Inequalities**, Cambridge University Press, London - New York, 1952.

[20] J. Kurzweil, **Ordinary differential equations** (Translation from the Czech edition by Michael Basch), Elsevier, Amsterdam - Oxford - New York - Tokyo, 1986.

[21] J.L. Lee and D. O'Regan, *Boundary value problems for nonlinear fourth order equations with applications to nonlinear beams*, Utilitas Mathematica **44**, Winnipeg, Canada , 57-74, (1993).

[22] I. Rachunková, *Multiplicity results for four-point boundary value problems*, Nonlinear Anal., TMA **18**, 497-505, (1992).

[23] I. Rachunková, *On some three-point problems for third-order differential equations*, Mathematica Bohemica **117**, 98-110, (1992).

[24] I. Rachunková, *On the number of solutions of the Neumann problem for the ordinary second order differential equation*, Annales Mathematicae Silesianae **7**, Katowice, 79-87, (1993).

[25] I. Rachunková, *The first kind periodic solutions of differential equations of the second order*, Math. Slovaca **39**, 407-415, (1989).

[26] A. Rodriguez and A. Tineo, *Existence theorems for the Dirichlet problem without growth restrictions*, Journal of Mathematical Analysis and Applications **135**, 1-7, (1988).

[27] M. Šenkyřík, *An existence theorem for a third-order three-point boundary value problem without growth restrictions*, Mathematica Slovaca **42**, No. 4, 465 - 469, (1992).

[28] M. Šenkyřík, *Existence of multiple solutions for a third-order three-point regular boundary value problem*, Mathematica Bohemica, **119**, No. 2, 113-121, (1994).

[29] M. Šenkyřík, *Method of lower and upper solutions for a third-order three-point regular boundary value problem*, Acta Univ. Palack. Olomouc, Fac.-Rerum Natur. Math. **XXXI**, 60-70, (1992).

[30] M. Šenkyřík, *On a third-order three-point regular boundary value problem*, Acta Univ. Palack. Olomouc, Fac. Rerum Natur. Math. **XXX**, 75-86, (1991).

[31] M. Šenkyřík and R.B. Guenther, *Boundary value problems with discontinuities in the spatial variable*, Journal of Mathematical Analysis and Applications (in press).