AN ABSTRACT OF THE THESIS OF

PAUL JAMES CHOBAN (Name) for the M.S. (Degree) in Mathematics (Major)

Date thesis is presented May 14, 1964

Title AN INTRODUCTION TO SCATORS

Redacted for Privacy

Abstract approved ____________ (Major professor)

The indicated sum of a real scalar and a real or imaginary vector is called a scator. Either the scalar part or the vector part may be null. Scators generalize the complex variable to n-space. The algebra of scators is not generally associative under multiplication but the commutative and distributive laws are satisfied. A division algebra exists provided that division by a pure vector (except in the one dimensional case) is excluded. The basic laws of exponents are true excepting those for which the associative law does not hold. Exponential, trigonometric, hyperbolic and logarithmic functions are defined. All scators admit to an exponential representation and if the vector part is imaginary there exists a scator analog to DeMoivre's Theorem.

The derivative of a scator with respect to a scator is defined in terms of differentials. The analogs to the Cauchy-Riemann equations of complex variables are derived.

AN INTRODUCTION TO SCATORS

by

PAUL JAMES CHOBAN

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1964

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented May 14, 1964

Typed by Nancy Kerley

ACKNOWLEDGEMENTS

To Professor Hostetter and my wife in that order.

TABLE OF CONTENTS

	Page
INTRODUCTION	1
ALGEBRA OF SCATORS	3
TRANSCENDENTAL FUNCTIONS OF SCATORS	16
CALCULUS OF SCATORS	20
BIBLIOGRAPHY	25

AN INTRODUCTION TO SCATORS

INTRODUCTION

The beginning of modern vector analysis is usually traced to Grassmann's _Ausdehnungslehre_ published in 1844 and Hamilton's _Quaternions_ published the same year. About forty years later, in 1881 and 1884, Gibbs printed a privately circulated pamphlet _Elements of Vector Analysis_ (3, p. 17-90), that presents the subject in very nearly the same form that it appears today. In this pamphlet Gibbs defined transcendental functions of dyadics but not transcendental functions of vectors or higher order odd tensors.

The search for a useful definition of e^V where V is a vector led to the consideration of a new quantity which shall be called a scator.

Definition. A scator is the indicated sum of a scalar and a vector

$$\alpha = a + A = A + a$$

where a is a real scalar and A is a real or imaginary vector. If A is real the scator will be called real, and if A is imaginary, i.e., $A = iV$, where V is real, the scator will be said to be complex. Either a or A may be null.

Throughout this work lower case Latin letters will indicate elements of the scalar field, capital Latin letters will indicate elements of the vector field, and lower case Greek letters will indicate scators.

ALGEBRA OF SCATORS

Definition. Two scators are equal if and only if their scalar parts are equal and their vector parts are equal.

Thus, if $\alpha = a + A$ and $\beta = b + B$, then $\alpha = \beta$ if and only if $a = b$ and $A = B$.

Definition. The sum of two scators α and β, indicated by $\alpha + \beta$, is the algebraic sum of their scalar parts and their vector parts.

Thus, if $\alpha = a + A$ and $\beta = b + B$, then

$$\alpha + \beta = (a + b) + (A + B).$$

It follows that scators form an abelian group under addition. The identity element is the null scator $o + O$. The inverse of $\alpha = a + A$ is $-\alpha = -(a + A)$.

Definition. A scator is complete if and only if both parts are non-null. Otherwise a scator is incomplete. The subscript notation α_s and α_v will refer to the scalar and vector parts of α, respectively.

To define scator multiplication we introduce a new symbol, $\circ$, and indicate the product of two scators α and β as $\alpha \circ \beta$.

Definition. Let $\alpha = a + A$ and $\beta = b + B$, then

$$\alpha \circ \beta = (a + A) \circ (b + B)$$

$$= a \circ b + a \circ B + A \circ b + A \circ B,$$

where by definition

$$a \circ b = ab,$$

$$a \circ B = B \circ a = aB,$$

$A \circ B = B \circ A = A \cdot B$ where $A \cdot B$ is the inner product of the vectors A and B. Then

$$\alpha \circ \beta = (ab + A \cdot B) + (bA + aB).$$

It follows that scators are closed and commutative under multiplication. The identity element is the scator $1 + O$.

The idea of scators may be extended to quantities of the form

$$f + F + \Phi$$

where Φ is a Gibbs dyadic. Such a quantity will be called a scator of the second order. A scator of the n^{th} order is defined as

$$f + F + \Phi_2 + \ldots + \Phi_n$$

where Φ_k is a Gibbs polyadic of the k^{th} order. By definition

$$\alpha \circ (f + F + \Phi_2 + \ldots + \Phi_n) = \alpha \circ f + \alpha \circ F + \alpha \circ \Phi_2 + \ldots + \alpha \circ \Phi_n,$$

where

$$\Phi_k = \sum_{j=1}^{m} B_j^1 B_j^2 B_j^3 \ldots B_j^k ,$$

and

$$\alpha \circ \Phi_k = (\alpha \circ \sum_{j=1}^{m} B_j^1) \; B_j^2 B_j^3 \ldots B_j^k , \qquad m \geqslant 1 .$$

Note that scator multiplication is not generally associative, for if $\alpha = a + A$, $\beta = b + B$, and $\gamma = g + G$, then

$$(\alpha \circ \beta) \circ \gamma = [(a + A) \circ (b + B)] \circ (g + G)$$

$$= [(ab + A \cdot B) + (bA + aB)] \circ (g + G).$$

$$(\alpha \circ \beta) \circ \gamma = (abg + gA \cdot B + bA \cdot G + aB \cdot G) + (abG + A \cdot BG + bgA + agB), \tag{2.1}$$

while

$$\alpha \circ (\beta \circ \gamma) = (a + A) \circ [(b + B) \circ (g + G)]$$

$$= (a + A) \circ [(bg + B \cdot G) + (gB + bG)] .$$

$$\alpha \circ (\beta \circ \gamma) = (abg + gA \cdot B + bA \cdot G + aB \cdot G) + (abG + AB \cdot G + bgA + agB). \tag{2.2}$$

Subtracting equation (2. 2) from (2. 1) gives

$$(\alpha \circ \beta) \circ \gamma - \alpha \circ (\beta \circ \gamma) = A \cdot BG - AB \cdot G,$$

$$(\alpha \circ \beta) \circ \gamma - \alpha \circ (\beta \circ \gamma) = B \cdot (AG - GA) . \tag{2.3}$$

__Theorem 1.__ The necessary and sufficient condition that the product $\alpha \circ \beta \circ \gamma$ be associative is that β_v be orthogonal to α_v and γ_v or that α_v and γ_v be linearly dependent.

The proof is immediate since the right side of (2. 3) vanishes under the conditions of the hypothesis and conversely.

__Definition.__ The scator conjugate of $\alpha = a + A$ is $\bar{\alpha} = a - A$.

Thus $\alpha \circ \bar{\alpha}$ is always a scalar. Using this definition the following properties of conjugates are easily verified.

$$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta},$$

$$\overline{\alpha \circ \beta} = \bar{\alpha} \circ \bar{\beta},$$

$$\bar{\bar{\alpha}} = \alpha .$$

<u>Definition.</u> The absolute value of a complex scator $\alpha = a + Ai$, A real, is the scalar $|\alpha| = \sqrt{(\alpha \circ \bar{\alpha})} = \sqrt{a^2 + A \cdot A}$.

Hence for complex scators it follows immediately that

(2.4) $$\alpha \circ \bar{\alpha} = |\alpha|^2 .$$

<u>Definition.</u> The absolute value of the real scator $\alpha = a + A$ is $|\alpha| = \sqrt{a^2 + A \cdot A}$.

<u>Theorem 2.</u> If α and β are complex scators then $|\alpha \circ \beta| \leqslant |\alpha| |\beta|$.

Proof: Let $\alpha = a + Ai$ and $\beta = b + Bi$. Then

$$\begin{aligned} |\alpha \circ \beta|^2 &= |(a + Ai) \circ (b + Bi)|^2 = |(ab - A \cdot B) + (aB + bA)i|^2 \\ &= (A \cdot B)^2 + a^2b^2 + a^2 B \cdot B + b^2 A \cdot A + (B \cdot B)(A \cdot A) - (B \cdot B)(A \cdot A) \\ &= (a^2 + A \cdot A)(b^2 + B \cdot B) + (A \cdot B)^2 - (B \cdot B)(A \cdot A) \\ &= |\alpha|^2 |\beta|^2 + (A \cdot B)^2 - (B \cdot B)(A \cdot A), \end{aligned}$$

and with the aid of Schwarz's inequality

(2.5) $$|\alpha \circ \beta|^2 - |\alpha|^2 |\beta|^2 = (A \cdot B)^2 - |B|^2 |A|^2 \leqslant 0 ,$$

from which it follows that

$$|\alpha \circ \beta|^2 \leqslant |\alpha|^2 |\beta|^2,$$

and

$$|\alpha \circ \beta| \leqslant |\alpha|\,|\beta| \,.$$

From (2.5) it is evident that equality holds in this theorem when α_v and β_v are linearly dependent.

Theorem 3. If α and β are complex scators then $|\alpha + \beta| \leqslant |\alpha| + |\beta|$.

Proof. From (2.4) and the properties of scator conjugates it follows that

$$|\alpha + \beta|^2 = (\alpha + \beta) \circ (\overline{\alpha + \beta}) = (\alpha + \beta) \circ (\overline{\alpha} + \overline{\beta}) \,,$$

(2.6) $$|\alpha + \beta|^2 = \alpha \circ \overline{\alpha} + \alpha \circ \overline{\beta} + \overline{\alpha} \circ \beta + \beta \circ \overline{\beta}.$$

But since the conjugate of $\alpha \circ \overline{\beta}$ is $\overline{\alpha} \circ \beta$, from Theorem 2

$$\alpha \circ \overline{\beta} + \overline{\alpha} \circ \beta \leqslant 2|\alpha \circ \overline{\beta}| \leqslant 2|\alpha|\,|\overline{\beta}| = 2|\alpha|\,|\beta|.$$

If this inequality is used in (2.6) then

$$|\alpha + \beta|^2 \leqslant |\alpha|^2 + 2|\alpha|\,|\beta| + |\beta|^2 \,,$$

and

$$|\alpha + \beta| \leqslant |\alpha| + |\beta|.$$

The result can be extended to n complex scators by mathematical induction. It is apparent that the absolute value of a complex scator satisfies the requirements of a norm.

Definition. The quotient α / β of two scators α and β is that scator δ such that $\alpha = \beta \circ \delta$.

Theorem 4. If $\beta_s \neq o$ and $b^2 - B \cdot B \neq o$, there exists a unique quotient $\delta = \alpha / \beta$.

Proof: If $\alpha = a + A$ and $\beta = b + B$, let $\delta = d + D$. Then by the definition of a quotient

$$a + A = (b + B) \circ (d + D)$$

$$= (bd + B \cdot D) + (dB + bD),$$

and from equality of scators

$$a = bd + B \cdot D, \tag{2.7}$$

$$A = dB + bD. \tag{2.8}$$

Taking the inner product of (2.8) with B and solving for $B \cdot D$ gives

$$B \cdot D = \frac{A \cdot B - dB \cdot B}{b}, \qquad (b \neq o).$$

Substituting the right member of this equation into (2.7) and simplifying yields

$$ab = b^2 d + A \cdot B - dB \cdot B,$$

from which it follows that

$$d = \frac{ab - A \cdot B}{b^2 - B \cdot B}, \qquad (b^2 - B \cdot B \neq o).$$

Substituting the right member of this equation into (2.8) and simplifying yields

$$A(b^2 - B \cdot B) = (ab - A \cdot B)\, B + b\, (b^2 - B \cdot B)\, D.$$

Solving for D gives

$$D = \frac{(b^2 - B \cdot B)\, A - (ab - A \cdot B)\, B}{b(b^2 - B \cdot B)}.$$

Hence, if there exists a quotient it is

(2.9) $$\delta = d + D = \frac{a + A}{b + B} = \frac{(ab - A\cdot B)}{b^2 - B\cdot B} + \frac{(b^2 - B\cdot B)A - (ab - A\cdot B)B}{b(b^2 - B\cdot B)},$$

(2.9a) $$\frac{\alpha}{\beta} = \frac{a + A}{b + B} = \frac{(ab - A\cdot B)(b - B) + (b^2 - B\cdot B)A}{b(b^2 - B\cdot B)}, \quad b \neq o, \quad b^2 - B\cdot B \neq o.$$

To show that α is a quotient, from (2.9a) it follows that

$$\delta \circ \beta = \frac{(ab - A\cdot B)(b - B) \circ (b + B)}{b(b^2 - B\cdot B)} + \frac{(b^2 - B\cdot B)A \circ (b + B)}{b(b^2 - B\cdot B)}$$

$$= \frac{ab - A\cdot B + bA + A\cdot B}{b} = a + A.$$

To show that δ is unique suppose there exists another quotient $\gamma = g + G$. Then $\beta \circ \delta = \beta \circ \gamma$ implies

$$(b + B) \circ (d + D) = (b + B) \circ (g + G),$$

and from equality of scators

(2.10) $$bd + B\cdot D = bg + B\cdot G,$$

(2.11) $$dB + bD = gB + bG.$$

Solving (2.11) for G yields

(2.12) $$G = \frac{bD + (d - g)B}{b},$$

and solving (2.10) for g gives

(2.13) $$g = \frac{bd + B\cdot D - B\cdot G}{b}.$$

Substituting (2.12) into (2.13) and simplifying gives

$$g = \frac{b^2 d - dB\cdot B + gB\cdot B}{b^2},$$

and

$$(b^2 - B\cdot B)g = (b^2 - B\cdot B)d,$$

from which it follows that $g = d$. Using this result in (2. 12) it becomes immediately obvious that $G = D$ and hence the quotient δ is unique.

Note that if the denominator is a complete complex scator a quotient always exists.

If $\beta_s = o$ then there exists a unique quotient if and only if α_v and β_v are linearly dependent and β_v is one dimensional.

As a corollary to Theorem 4 we have the following.

If $\beta_s \neq o$ and $b^2 - B \cdot B \neq o$, there exists a unique inverse $1/\beta$ such that $\beta \circ \frac{1}{\beta} = 1$.

Proof: From (2. 9) it follows directly that

$$1/\beta = \frac{1}{b + B} = \frac{b - B}{b^2 - B \cdot B} \quad . \tag{2. 14}$$

Equation (2. 14) suggests that the inverse can be obtained simply by multiplying the numerator and denominator of $1/\beta$ by $b - B$ and indeed it is true that

$$\frac{1}{\beta} = \frac{1 \circ (b - B)}{(b + B) \circ (b - B)} \quad .$$

In general however $\frac{\alpha}{\beta} \neq \frac{\alpha \circ \gamma}{\beta \circ \gamma}$.

<u>Theorem 5.</u> If β_s, $\gamma_s \neq o$ and $b^2 - B \cdot B$, $g^2 - G \cdot G \neq o$, then $\frac{\alpha}{\beta} = \frac{\alpha \circ \gamma}{\beta \circ \gamma}$ if and only if $(\frac{\alpha}{\beta} \circ \beta) \circ \gamma = \frac{\alpha}{\beta} \circ (\beta \circ \gamma)$.

The proof is a direct consequence of the definition of quotient.

Another consequence of the non-associativity of scator multiplication is the following.

Theorem 6. If $\beta_s \neq o$ and $b^2 - B\cdot B \neq o$, then $\frac{\alpha}{\beta} = \alpha \circ \frac{1}{\beta}$ if and only if α_v and β_v are linearly dependent.

Proof: Let $\alpha = a + A$ and $\beta = b + B$. Then the quotient α/β is given by

$$\frac{\alpha}{\beta} = \frac{a + A}{b + B} = \frac{(ab - A\cdot B)}{b^2 - B\cdot B} + \frac{(b^2 - B\cdot B)A - (ab - A\cdot B)B}{b(b^2 - B\cdot B)} \quad . \tag{2.9}$$

With the aid of (2. 14),

$$\alpha \circ \frac{1}{\beta} = \frac{ab - A\cdot B}{b^2 - B\cdot B} + \frac{b^2A - abB}{b(b^2 - B\cdot B)} \quad . \tag{2.15}$$

If $\alpha/\beta = \alpha \circ \frac{1}{\beta}$ the vector parts as well as the scalar parts of (2. 9) and (2. 15) must be equal.

Hence

$$b^2A - (B\cdot B)A - abB + (A\cdot B)B = b^2A - abB,$$

from which it follows that

$$(A\cdot B)B = (B\cdot B)A \quad .$$

Therefore A and B must be linearly dependent. Conversely if $A = kB$, the vector part of (2. 9) becomes

$$\frac{b^2kB - B\cdot B\,kB - abB + kB\cdot B\,B}{b\,(b^2 - B\cdot B)} = \frac{b^2kB - abB}{b(b^2 - B\cdot B)} = \frac{b^2A - abB}{b(b^2 - B\cdot B)} \quad ,$$

which is the vector part of (2. 15). Hence $\frac{\alpha}{\beta} = \alpha \circ \frac{1}{\beta}$.

In a similar manner it can be proved that if $\beta_s \neq o$ and

$b^2 - B \cdot B \neq o$, then $\frac{\alpha \circ \gamma}{\beta} = \frac{\alpha}{\beta} \circ \gamma$ if and only if β_v and γ_v are linearly dependent.

The equation $\delta \circ \beta = \alpha$ can also be solved for δ by multiplying $\delta \circ \beta$ by the inverse of β and using (2.3) to give a result identical to that of (2.9). The process is demonstrated below.

If $\alpha = a + A$, $\beta = b + B$ where $b \neq o$ and $b^2 - B \cdot B \neq o$, $\delta = d + D$ and $\delta \circ \beta = \alpha$, then from (2.3) we can write

$$(\delta \circ \beta) \circ \frac{1}{\beta} = \delta \circ (\beta \circ \frac{1}{\beta}) + \frac{B \cdot (BD - DB)}{b^2 - B \cdot B} = \alpha \circ \frac{1}{\beta} ,$$

and with the aid of (2.15)

$$(2.16) \qquad \delta = d + D = \frac{ab - A \cdot B}{b^2 - B \cdot B} + \frac{bA - aB}{b^2 - B \cdot B} - \frac{B \cdot (BD - DB)}{b^2 - B \cdot B} .$$

The scalar part of δ in this equation is identical to the scalar part of (2.9). The vector parts are also identical, for from (2.16) it follows that

$$b^2 D - (B \cdot B)D = bA - aB - (B \cdot B)D + (B \cdot D)B ,$$

and on simplifying and introducing the dyadic idemfactor I

$$(2.17) \qquad Db^2 - D \cdot BB = D \cdot (b^2 I - BB) = bA - aB .$$

Substituting $B = \sqrt{(B \cdot B)}\, E_1$ where E_1 is a unit vector in the direction of B of an orthonormal set of n vectors $E_1, E_2, \ldots, E_n$, one can write

$$b^2 I - BB = b^2 \sum_{j=1}^{n} E_j E_j - B \cdot B\, E_1 E_1$$

$$= (b^2 - B \cdot B) E_1 E_1 + b^2 \sum_{j=1}^{n} E_j E_j \ .$$

If $B \cdot B \neq o$ its reciprocal is

(2.18) $$(b^2 I - BB)^{-1} = \frac{1}{b^2} \sum_{j=1}^{n} E_j E_j - \frac{E_1 E_1}{B \cdot B}$$

$$= \frac{E_1 E_1}{b^2 - B \cdot B} + \frac{1}{b^2} \sum_{j=2}^{n} E_j E_j \ .$$

Postmultiplying (2.17) by $(b^2 I - BB)^{-1}$ and simplifying gives

(2.19) $$D = (bA - aB) \cdot (b^2 I - BB)^{-1} \ .$$

Now

(2.20) $$bA - aB = b \sum_{j=1}^{n} a_j E_j - a\sqrt{(B \cdot B)}\ E_1 \ .$$

With the aid of (2.18) and (2.20) we can write (2.19) as

$$D = \frac{b a_1 E_1}{b^2 - B \cdot B} + \frac{1}{b} \sum_{j=2}^{n} a_j E_j - \frac{aB}{b^2 - B \cdot B}$$

$$= \frac{b^2 a_1 E_1 + (b^2 - B \cdot B) \sum_{j=2}^{n} a_j E_j - abB - B \cdot B a_1 E_1 + B \cdot B a_1 E_1}{b(b^2 - B \cdot B)}$$

$$= \frac{(b^2 - B \cdot B)A - (ab + A \cdot B)B}{b(b^2 - B \cdot B)}, \text{ since } B \cdot B a_1 E_1 = (A \cdot B)B,$$

which is identical to the vector part of (2. 9).

Definition. If n is a positive integer and α is any scator, then

$$\alpha^n = \alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha \text{ to n factors.}$$

From the properties of scator algebra and the definition of powers the scalar laws of exponents are true with the following restrictions. For $n > 1$,

$$\alpha^n \circ \beta^n = (\alpha \circ \beta)^n$$

if and only if α_v and β_v are linearly dependent; and

$$\left(\frac{\alpha}{\beta}\right)^n = \frac{\alpha^n}{\beta^n}$$

if β_v is null.

Powers of scators can be extended to include all integers by defining

$$a^{o} = 1, \quad a \text{ non-null},$$

$$a^{-n} = \frac{1}{a^{n}} \quad , \quad a_{v} \text{ non-null}, \ a^{2} - |A|^{2} \neq o.$$

TRANSCENDENTAL FUNCTIONS OF SCATORS

Definition. If k is any real or imaginary scalar and α any scator then

(3.1) $$e^{k\alpha} = 1 + k\alpha + \frac{k^2\alpha^2}{2!} + \frac{k^3\alpha^3}{3!} + \ldots$$

(3.2) $$\cosh k\alpha = 1 + \frac{k^2\alpha^2}{2!} + \frac{k^4\alpha^4}{4!} + \ldots$$

(3.3) $$\sinh k\alpha = k\alpha + \frac{k^3\alpha^3}{3!} + \frac{k^5\alpha^5}{5!} + \ldots$$

(3.4) $$\cos k\alpha = 1 - \frac{k^2\alpha^2}{2!} + \frac{k^4\alpha^4}{4!} - \frac{k^6\alpha^6}{6!} + \ldots$$

(3.5) $$\sin k\alpha = k\alpha - \frac{k^3\alpha^3}{3!} + \frac{k^5\alpha^5}{5!} - \frac{k^7\alpha^7}{7!} + \ldots$$

From the first three definitions above it follows that

$$e^{\alpha} = \cosh\alpha + \sinh\alpha \quad , \quad e^{-\alpha} = \cosh\alpha - \sinh\alpha,$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \quad , \quad e^{-i\alpha} = \cos\alpha - i\sin\alpha .$$

Thus

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2} \quad , \quad \sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2} \quad ,$$

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} , \quad \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} .$$

Other properties of exponential functions that can be easily demonstrated are:

(3.6) $$e^{a + A} = e^{a} \circ e^{A} = e^{a}e^{A}$$

(3.7) $$e^{k_1\alpha} \circ e^{k_2\alpha} = e^{(k_1 + k_2)\alpha}$$

(3.8) $e^{\alpha} \circ e^{\beta} = e^{\alpha + \beta}$ if and only if α_v, β_v are linearly dependent.

<u>Theorem 7a.</u> If $\alpha = a + Ai$ there exist scalars r and t such that $\alpha = re^{iA_1 t}$ where A_1 is a unit vector in the direction of A, $r = |\alpha|$, and $t = \tan^{-1} \frac{|A|}{a}$.

Proof: Let

$$a = r \cos A_1 t = r \cos t, \quad A = r \sin A_1 t = rA_1 \sin t \ .$$

Then

$$(a + Ai) \circ (a - Ai) = a^2 + A^2 = |\alpha|^2 = r^2 \rightarrow r = |\alpha| \ .$$

And

$$\frac{A}{a} = \frac{rA_1 \sin t}{r \cos t} \rightarrow \frac{|A|}{a} = \tan t \rightarrow t = \tan^{-1} \frac{|A|}{a} \ .$$

Hence

$$\alpha = a + Ai = r(\cos A_1 t + i \sin A_1 t)$$

$$= re^{iA_1 t}$$

(3.9) $$\alpha = a + Ai = |\alpha| \, e^{iA_1 t}, \qquad t = \tan^{-1} \frac{|A|}{a} \ .$$

We call (3.9) the polar form of α.

<u>Theorem 7b.</u> If $\alpha = a + A$ and A is real, α may be expressed in the form

$$a + A = \sqrt{a^2 - A^2} \quad e^{A_1 t}, \qquad t = \tanh^{-1} \frac{|A|}{a}, \quad |a| > |A| \ ,$$

or

$$a + A = \sqrt{A^2 - a^2} \ A_1 \circ e^{A_1 t}, \quad t = \coth^{-1} \frac{|A|}{a}, \quad |a| < |A| \ .$$

The proof of the first form follows directly as in Theorem 7a by using the hyperbolic functions. To prove the second make the substitution

$$a = rA_1 \circ \sinh A_1 t \quad , \qquad A = rA_1 \cosh A_1 t.$$

The function $e^{iA_1 t}$ is periodic with a period of $2\pi k A_1 i$ where k is any integer, since

$$e^{iA_1(t+2\pi k)} = \cos A_1(t + 2\pi k) + i \sin A_1(t + 2\pi k)$$
$$= \cos A_1 t + i \sin A_1 t = e^{iA_1 t} \ .$$

Thus it is possible to use (3.9) to determine powers and extract roots of scators.

With the aid of (3.8) it follows that for α complex and n a positive integer,

$$\text{(3.10)} \qquad \alpha^n = (|\alpha| \, e^{iA_1 t})^n = |\alpha|^n \, e^{iA_1 nt} \ .$$

In particular, if $|\alpha| = 1$ then (3.10) becomes the scator analog of De Moivre's Theorem. In view of (2.9) and the definition of negative powers, equation (3.10) holds true for negative exponents as well.

Similarly, if $\alpha = \beta^n$, n a positive integer, then

$$\beta = \alpha^{\frac{1}{n}} = (|\alpha| \, e^{iA_1 t})^{\frac{1}{n}} = |\alpha|^{\frac{1}{n}} \, e^{iA_1 \frac{(t+2\pi k)}{n}},$$

k any integer.

<u>Definition.</u> If $\alpha = e^{\beta}$ then $\log \alpha = \beta$.

<u>Theorem 8.</u> If $\alpha = a + A$ is non-null and α is complex or α is real and $|a| > |A|$, then there exists a scator β such that $\alpha = e^{\beta}$.

Proof: By Theorems 7a and 7b we can write α in the form

$$\alpha = a + A = re^{A_1 t}$$

where r is real and positive. Writing $r = e^{b}$ it follows that

$$\alpha = e^{b}e^{A_1 t} = e^{b + A_1 t} = e^{\beta} \quad .$$

Real logarithms exist for real scators provided that the magnitude of the scalar part is greater than the magnitude of the vector part.

CALCULUS OF SCATORS

Definition. A neighborhood of a scator ζ_o is the set of all scators ζ such that for any positive number d,

$$|\zeta - \zeta_o| < d.$$

Definition. Let the single valued function $\phi(\zeta)$ be defined on some neighborhood of ζ except possibly at ζ_o. Then η is the limit of $\phi(\zeta)$ as ζ approaches ζ_o and we write

$$\lim_{\zeta \to \zeta_o} \phi(\zeta) = \eta$$

if for every $e > o$ there exists a $d > o$ such that whenever $o < |\zeta - \zeta_o| < d$, then $|\phi(\zeta) - \eta| < e$. If there is no such η, the limit of $\phi(\zeta)$ at ζ_o does not exist.

Definition. Let the function $\phi(\zeta)$ be defined on some neighborhood of ζ_o. Then $\phi(\zeta)$ is continuous at ζ_o if

$$\lim_{\zeta \to \zeta_o} \phi(\zeta) = \phi(\zeta_o).$$

A function is continuous on a neighborhood if it is continuous at every scator on the neighborhood.

Definition.[1] A scator infinitesimal is a scator variable which approaches the null scator.

Thus a scator is an infinitesimal if and only if its scalar and vector parts approach nullity.

Definition. Let $\phi(\zeta)$ be a function of the independent variable ζ. If ζ_o is a particular value of ζ let the difference

$$\Delta\phi(\zeta_o) = \phi(\zeta) - \phi(\zeta_o).$$

Then the function $\phi(\zeta)$ will be said to be differentiable at ζ_o if one can write

$$\Delta\phi(\zeta_o) = \Delta\zeta \circ \Gamma + \Delta\zeta \circ \epsilon \tag{4.1}$$

where ϵ is a scator infinitesimal, that is, $\lim_{\Delta\zeta \to O} \epsilon = O$, and Γ is a scator whose order is at most one higher than that of $\phi(\zeta)$. Γ will be called the derivative at ζ_o of $\phi(\zeta)$ with respect to ζ and we write

$$\Gamma = \phi'(\zeta_o) = \frac{d\phi(\zeta_o)}{d\zeta} . \tag{4.2}$$

A detailed study of higher order scators like Γ is beyond the scope of this work.

---

[1]The definitions of scator infinitesimal, differentiability, and differential, follow closely those given by I. M. Hostetter in a privately circulated manuscript, Corvallis, Oregon.

Definition. If the infinitesimal $\Delta\eta$ is the change in a scator η we denote the principal part by $d\eta$ and call it the differential of η.

Hence from (4. 1) and (4. 2) it follows that

$$d\phi(\zeta_o) = d\zeta \circ \phi'(\zeta_o). \tag{4.3}$$

If ζ is in turn a function of a scalar parameter t, $\zeta = \zeta(t)$, $\phi(\zeta_o) = \phi(\zeta(t_o))$. If $\zeta(t)$ is differentiable at t_o then

$$\Delta\zeta(t_o) = \Delta t \circ \zeta'(t_o) + \Delta t \circ \eta$$

where η is a scator infinitesimal. Substituting this into (4. 1) for $\Delta\zeta$ gives

$$\Delta\phi\,\zeta(t_o) = [\,\Delta t \circ \zeta'(t_o) + \Delta t \circ \eta] \circ [\,\phi'(\zeta_o) + \epsilon\,]\ .$$

The principal part of $\Delta\phi\,\zeta(t_o)$ is

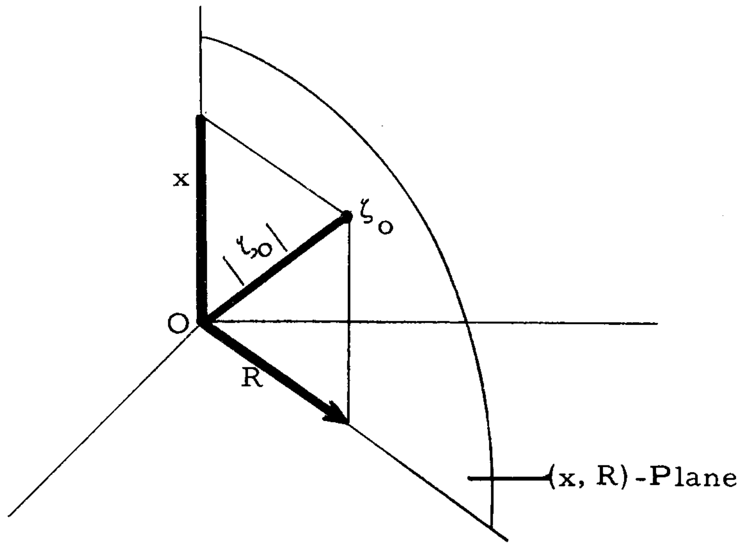
$$d\phi(t_o) = [\,dt \circ \zeta'(t_o)] \circ \phi'(\zeta_o)$$

$$= dt\,[\,\zeta'(t_o) \circ \phi'(\zeta_o)]\ ,$$

and

$$\frac{d\phi(t_o)}{dt} = \zeta'(t_o) \circ \phi'(\zeta_o)\ . \tag{4.4}$$

The value of the derivative in (4. 3) may depend upon the value of $d\zeta$, the direction in scator space. Let $\zeta = x + R$, $\phi(\zeta) = u(x, R) + V(x, R)$. We seek the conditions upon the functions u and V such that the value of the derivative is independent of the choice of dx and dR. This implies that the derivative at ζ_o on the (x, R)-plane (see sketch) is the same for all directions on that plane.



If the partial derivatives u_x, V_x, u_R, V_R are continuous at ζ_o then by the mean value theorem for scalar and for vector functions

$$\Delta u = (u_x + e_1)\ \Delta x + \Delta R \cdot (u_R + E_1),$$
$$\Delta V = (V_x + E_2)\ \Delta x + \Delta R \cdot (V_R + \Phi),$$

where e_1 is a scalar infinitesimal, E_1 and E_2 are vector infinitesimals, and Φ is a dyadic infinitesimal. The principle part of $\Delta\phi = \Delta u + \Delta V$ is, therefore,

(4.5) $d\phi(\zeta_o) = (u_x + V_x)\ dx + dR \cdot (u_R + V_R) = (dx + dR) \circ \phi'(\zeta_o)$.

Let $R = rR_1$, where R_1 is a unit vector either real or imaginary and $r = |R|$. Then for $dx = o$, that is, if ζ approaches ζ_o along a direction parallel to R,

$$dR = R_1\ dr$$

for all dR in the (x, R)-plane. And (4.5) becomes

(4.6) $$R_1 \cdot (u_R + V_R) = R_1 \circ (\phi'(\zeta_o)).$$

Similarly, if $dR = O$,

(4.7) $$u_x + V_x = \phi'(\zeta_o).$$

Since it is desired that $\phi'(\zeta_o)$ be the same in (4.6) and (4.7) we substitute the latter into the former with the result that

$$R_1 \cdot u_R + R_1 \cdot V_R = R_1 \circ u_x + R_1 \circ V_x.$$

Equating the scalar parts and vector parts gives

(4.8) $$R_1 \cdot u_R = R_1 \cdot V_x, \quad R_1 \cdot V_R = R_1 u_x$$

as the conditions which must be satisfied by u and V if the derivative is to be unique in the (x, R)-plane. These are the scator analogs of the Cauchy-Riemann equations.

As an example of a function which satisfies (4.8) consider

$$\phi(\zeta) = \zeta^2 = (x + R)^2 = x^2 + R \cdot R + 2xR.$$

Since $u = x^2 + R \cdot R$, and $V = 2xR$, the partial derivatives are given by

$$u_R = 2R, \qquad V_x = 2R,$$

$$u_x = 2x, \qquad V_R = 2xI,$$

where I is the dyadic idemfactor. Now

$$R_1 \cdot u_R = R_1 \cdot 2R = R_1 \cdot V_x,$$

$$R_1 \cdot V_R = R_1 \cdot 2xI = R_1 2x = R_1 u_x$$

as is required by equation (4.8).

BIBLIOGRAPHY

1. Brand, Louis. Vector and tensor analysis. New York, Wiley, 1947. 437 p.

2. Franklin, Philip. Functions of complex variables. Englewood Cliffs, Prentice-Hall, 1958. 246 p.

3. Gibbs, J. Willard. The scientific papers of J. Willard Gibbs. Vol. 2, New York, Dover, 1961. 284 p.

4. Hostetter, I. M. An extension of Gibbs vector analysis to n-space. Journal of Mathematics and Physics 15:191-204. 1936.

5. Weatherburn, C. E. Elementary vector analysis. London, Bell, 1921. 184 p.