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NORMAN FRED LINDQUIST for the DOCTOR OF PHILOSOPHY
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Title: REPRESENTATIONS OF CENTRAL CONVEX BODIES
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William J. Firey

Some centrally symmetric convex polytopes can be represented
as finite vector sums of line segments; for a three dimensional cen-
tral polyhedron, this property is equivalent to the polyhedron having
faces which are centrally symmetric. The dissertation treats pro-
properties of centrally symmetric convex bodies in n-dimensional Eu-
clidean space and their relation to arbitrary (not necessary finite)
vector sums of line segments.

Specifically, it is shown that a convex polytope is a finite vec-
tor sum of segments if and only if its faces of some fixed dimension
r, r greater than or equal 2, are finite vector sums of segments.
Further, if a sequence of convex bodies which are finite vector sums
of segments converge to a convex body, then that limit body is a pro-
jection body, i.e. a convex body whose support function has repre-
sentation

\[ H(u) = \int_{S^{n-1}} |(u, v)| S(K, d\omega(v)) . \]
Here the Radon-Stieltjes integral is taken with respect to the surface area function of some central convex body \( \overline{K} \) and is integrated over the unit spherical surface \( S^{n-1} \). The projection bodies form a closed subset of the set of all convex bodies.

It is known that the support function of a sufficiently smooth, centrally symmetric convex body has representation

\[
H(u) = \int_{S^{n-1}} |(u, v)| h(v) d\omega(v). \quad (A)
\]

Here \( h \) is an even function defined on the unit spherical surface \( S^{n-1} \) and \( d\omega(v) \) is the surface element on \( S^{n-1} \) at \( v \). A necessary and sufficient condition on a function \( h \), defined on \( S^{n-1} \), is found for (A) to be the support function of a central convex body. Furthermore, every support function of a centrally symmetric convex body has a representation of the form

\[
H(u) = \int_{S^{n-1}} |(u, v)| \mu(d\omega(v)) \quad (B)
\]

where \( \mu \) is a finite signed measure defined over the Borel sets on \( S^{n-1} \). This is done by showing that every central body is the difference of two projection bodies in the sense that to every central body we may add vectorally a projection body so as to obtain a projection
body. A partial answer is given to the problem of characterizing those signed measures defined over the Borel sets on $S^{n-1}$ for which (B) is a representation of the support function of a central body.

The paper concludes with some generalizations of some formulas of Blaschke [Kreis und Kugel, p. 156] for the volume, surface area and total mean curvature of a centrally symmetric convex body.
Representations of Central Convex Bodies

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Head of Department of Mathematics

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REPRESENTATIONS OF CENTRAL CONVEX BODIES

I. INTRODUCTION

It is known that some centrally symmetric convex polytopes can be represented as finite vector sums of line segments. In fact, for a three dimensional central polyhedron, this property is equivalent to the polyhedron having centrally symmetric faces. In the following we investigate properties of centrally symmetric convex bodies and their relation to arbitrary (not necessary finite) vector sums of line segments.

In this initial chapter we state the concepts from the theory of convex bodies which will be necessary for the understanding of this work. For further results and proofs see the monograph of T. Bonnesen and W. Fenchel [5].

Our work is set in n-dimensional Euclidean space $\mathbb{E}^n (n \geq 2)$. If $u$ is an element of $\mathbb{E}^n$, then we will call $u$ either a point or a vector. The origin will be denoted by $0$. The inner product of two points $u, v$ in $\mathbb{E}^n$ will be written $(u, v)$ and the length $\|u\|$ of a vector $u$ is $\sqrt{(u, u)}$. We shall use $S^{n-1}$ to denote the unit spherical surface in $\mathbb{E}^n$, that is

$$S^{n-1} = \{ u \in \mathbb{E}^n : \|u\| = 1 \}.$$
If \( u \) is in \( S^{n-1} \) and if \( c \) is any real number, then the set 
\[ \Pi = \{x \in \mathbb{R}^n : (x, u) = c\} \]
will be called a hyperplane with normal \( u \). The hyperplane \( \Pi \) determines two closed half spaces

\[ L_1 = \{x : (x, u) > c\} \quad \text{and} \quad L_2 = \{x : (x, u) < c\}. \]

More generally, if \( u_1, \ldots, u_{n-p}, p < n \) are linearly independent vectors in \( S^{n-1} \) and if \( c_1, \ldots, c_{n-p} \) are real numbers, then the intersection of the corresponding \((n-p)\) hyperplanes will be called a \( p \)-plane. In the case that \( p = 1 \) we will speak of a line and in the case \( p = 2 \) we will speak of a plane. The terms hyperplane and \((n-1)\)-plane are then synonymous. By definition an \( n \)-plane coincides with the whole space. If all the \( c_i \) are zero, then the resulting \( p \)-plane will be termed a \( p \)-dimensional subspace and is known to be isomorphic to \( p \)-dimensional Euclidean space \( \mathbb{E}^p \).

A convex body \( K \) is a compact convex subset of \( \mathbb{E}^n \). If \( K \) fails to contain interior points, we will say that \( K \) is degenerate. Whether or not \( K \) is degenerate, there is a \( p \)-dimensional subspace of \( \mathbb{E}^n \) in which \( K \) or a translate of \( K \) may be considered non-degenerate. For this reason we shall speak of \( K \) as being \( p \)-dimensional.

A hyperplane \( \Pi \) is said to be a support plane of the convex body \( K \) if \( K \) is contained in one of the closed half-spaces
determined by \( \Pi \) and if \( \Pi \cap K \) is not empty. If \( \Pi \) is the support plane \( \{x: (x, u) = c\} \) and if \( K \) is contained in the half-space \( \{x: (x, u) \leq c\} \), then \( \Pi \) will be termed the support plane with outer normal \( u \). Then, for any non-zero vector \( u \), there will be a unique support plane with \( u/\|u\| \) as its outer normal.

Let \( K \) be a convex body in \( \mathbb{E}^n \) and let \( H \) be the real valued function defined over \( \mathbb{E}^n \) by the equation

\[
H(u) = \max_{x \in K} (x, u).
\]

This function \( H \) is called the support function of \( K \). For \( u \) in \( S^{n-1} \), \( \{x: (x, u) = H(u)\} \) is the support plane of \( K \) with outer normal \( u \). \( H \) is known to possess the following three properties:

(a) \( H(0) = 0 \),

(b) \( H(\lambda u) = \lambda H(u), \ \lambda > 0 \) \[\text{[positive homogeneity]}\],

(c) \( H(u+v) \leq H(u) + H(v) \) \[\text{[sub-additivity]}\].

(1b) and (1c) imply that \( H \) is a convex function. Additionally, if the origin is an element of our convex body \( K \), then the support function of \( K \) is known to be non-negative.

The support function of the point \( a \), considered as a convex body, has representation \( \langle u, a \rangle \) and the support function of the line segment \( s(a) \) joining \( a \) to \( -a \) has a representation \( \|\langle u, a \rangle\| \).

A function \( G \) defined over \( \mathbb{E}^n \) which satisfies condition (1)
is known to be the support function of a unique convex body. This body is the set

\[ \{ x : (x, u) \leq G(u) \text{ for all } u \in E^n \} . \]

We also know that \( 0 \) is in \( K \) if and only if the support function of \( K \) is non-negative over \( E^n \).

If \( \lambda \) is a real number, then for any set \( R \subseteq E^n \) we denote by \( \lambda R \) the set of all points \( \lambda x \), where \( x \) is in \( R \). If \( Q \) is another subset of \( E^n \), then \( R + Q \) will denote the set of points \( x + y \), where \( x \) is in \( R \) and \( y \) is in \( Q \). In particular, if \( Q \) is the point \( a \), then \( R + a \) originates from \( R \) by the translation \( x' = x + a \).

If \( K_1, \ldots, K_r \) are convex bodies in \( E^n \) and if \( \lambda_1, \ldots, \lambda_r \) are non-negative numbers, then

\[ K = \lambda_1 K_1 + \ldots + \lambda_r K_r \quad (2) \]

is also a convex body in \( E^n \). Thus, if \( a \) is a point in \( E^n \) and if \( K \) is a convex body, \( K' = K + a \) is also a convex body in \( E^n \). \( K' \) originates from \( K \) by the translation \( x' = x + a \). In what follows we will be interested in properties of convex bodies which are invariant under translations; in fact, we will often assume that our body is translated so that it contains the origin.

The support function \( H \) of the convex body \( K \) defined by (2)
has the representation

\[ H(u) = \lambda_1 H_1(u) + \ldots + \lambda_r H_r(u), \]

where \( H_i \) is the support function of \( K_i \). If \( H \) is the support function of the body \( K \), the support function of the convex body \( K + a \) has representation \( H(u) + (u, a) \).

Let \( H \) be a support function of a convex body. Then for any two points in \( \mathbb{E}^n \) we set

\[ H'(u; v) = \lim_{h \to 0^+} \frac{[H(u+hv)-H(u)]}{h}. \]

\( H'(u; v) \) is the directional derivative of \( H \) at \( u \) in the direction \( v \) and is known to exist for all \( u \) and \( v \) in \( \mathbb{E}^n \). As a function of \( v \), it satisfies the conditions required of a support function. If \( K \) is the body with support function \( H \) and if \( u \) is in \( S^{n-1} \) then \( H'(u; v) \) is a representation of the support function of the intersection of \( K \) with its support plane with outer normal \( u \). If \( K \) is the sum of several convex bodies, then the directional derivative of the support function of \( K \) is the sum of the directional derivatives of the support functions of the summands of \( K \).

On the space of all convex bodies in \( \mathbb{E}^n \) we may introduce the following metric: Let \( Z = \{u; \|u\| \leq 1\} \) denote the unit ball in \( \mathbb{E}^n \). Then, let \( K \) and \( K' \) be two convex bodies in our space. Since
\[ p(K, K') = \max_{\mathbf{u} \in S^{n-1}} |H(u) - H'(u)| \]

where \( H \) and \( H' \) are the support functions of \( K \) and \( K' \) respectively. It is known that \( p(K, K') \) satisfies the conditions required of a metric. The topology on the space of all convex bodies is the topology induced by this metric. In particular, a sequence of convex bodies \( K_i \) is said to converge to the convex body \( K \) if \( \{p(K, K_i)\} \) has limit zero. In view of the equivalent definition of the metric we have the following: A sequence of convex bodies \( K_i \) with support functions \( H_i \) converges to a convex body \( K \) with support function \( H \) if and only if the sequence \( \{H_i\} \) of support functions converges uniformly on \( S^{n-1} \) to \( H \).

Fundamental to the study of convex bodies is Blaschke's selection theorem: If \( \{K_i\} \) is an infinite sequence of convex bodies each of which is contained in the same sphere, then this sequence contains a subsequence which converges to a convex body.
A convex body $\mathbf{K}$ is centrally symmetric (or central) with center $t$ if every chord of $\mathbf{K}$ which passes through $t$ is bisected by $t$. If $H$ is the support function of $\mathbf{K}$, then $\mathbf{K}$ is centrally symmetric with center $t$ if and only if

$$H(u) - H(-u) = (2t, u).$$

(4)

For the most part we will be concerned with central bodies having center at the origin, and in this case we have $H(u) = H(-u)$. It is easy to show that, if $\{\mathbf{K}_i\}$ is a sequence of central convex bodies which converge to a convex body $\mathbf{K}$, then the limit body $\mathbf{K}$ will also be centrally symmetric.

If we project $\mathbf{K}$ orthogonally onto a $p$-dimensional subspace $E^p$, then we obtain a convex body $\overline{\mathbf{K}}$ in $E^p$. The support function of $\overline{\mathbf{K}}$ is the restriction to $E^p$ of the support function of $\mathbf{K}$. From this, one sees that the projection of a central convex body is again a central convex body.

The $n$-dimensional volume or measure of a convex body $\mathbf{K}$ will be denoted by $V(\mathbf{K})$. In particular the volume of the unit ball in $E^n$ will be denoted by $\mathcal{L}_n$. The volume functional is known to be positively homogeneous of degree $n$, that is, for $\lambda > 0$,

$$V(\lambda \mathbf{K}) = \lambda^n V(\mathbf{K});$$

and it is known to depend continuously on its argument, that is $\{V(\mathbf{K}_i)\}$ converges to $V(\mathbf{K})$ when $\{\mathbf{K}_i\}$ converges to $\mathbf{K}$. 
Now let $K = \lambda_1 K_1 + \ldots + \lambda_r K_r$ where the weights $\lambda_i$ are arbitrary non-negative numbers. It is known that $V(K)$ is a homogeneous polynomial of degree $n$ in the weights $\lambda_i$, to wit

$$V(K) = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \ldots \sum_{i_n=1}^{r} V(K_{i_1}, K_{i_2}, \ldots, K_{i_n}) \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_n}$$

(5)

The coefficients $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ of this polynomial are the mixed volumes of the bodies $K_{i_1}, K_{i_2}, \ldots, K_{i_n}$ and are defined so as to be symmetric in their arguments. Generally, in studying mixed volumes, it suffices to look at $V(K_1, \ldots, K_n)$, where the $K_i$ are not necessarily distinct. The mixed volume $V(K_1, \ldots, K_n)$ is continuous in each argument and reduces to $V(K)$ when all of the arguments equal $K$. We will also need the following property of mixed volumes: $V(K_1, \ldots, K_n) = 0$ if and only if there do not exist line segments $s_i \subseteq K_i$ with direction $v_i$ for which the $n$ vectors $v_i$ are linearly independent.

Next we treat some measure theoretic concepts which will be needed. For proofs and ideas related to surface area functions the reader is referred to the paper of W. Fenchel and B. Jessen [8] or to the monograph of H. Busemann [6].

Let $\mathcal{B}$ denote the field of Borel sets on the unit spherical surface $S^{n-1}$. Let $\Phi$ be a finite-valued, non-negative measure
defined on \( B \). Then, for continuous functions \( G \) defined over \( S^{n-1} \) the Radon-Stieltjes integral of \( G \) with respect to \( \Phi \), taken over \( \omega \in B \), exists and will be denoted by

\[
\int_{\omega} G(u)\Phi(d\omega(u)).
\]

It is shown in [8] that if \( K \) is a non-degenerate convex body, then there is a unique non-negative finite measure \( S(K, \cdot) \) defined over \( B \) with the property that for each convex body \( \overline{K} \) with support function \( \overline{H} \)

\[
nV(\overline{K}, K, \ldots, K) = \int_{S^{n-1}} \overline{H}(u)S(K, d\omega(u)).
\]  

This set function is called the surface area function of the body \( K \). Alternately, \( S(K, \omega) \) can be defined as the \((n-1)\)-dimensional volume of the set of all those boundary points of \( K \) at which there is at least one support plane with outer normal in \( \omega \).

Furthermore, if \( \{K_i\} \) is a sequence of convex bodies which converges to the body \( K \), then the associated sequence of surface area functions \( \{S(K_i, \cdot)\} \) converges to the set function \( \Phi \) satisfying

\[
nV(\overline{K}, K, \ldots, K) = \int_{S^{n-1}} \overline{H}(u)\Phi(d\omega(u)).
\]
for each convex body $\overline{K}$. Thus, $S(K,\cdot) = \Phi$.

It is known that $S(K,\cdot)$ satisfies

$$S(K,\omega) \geq 0 \text{ over } B,$$  \hspace{1cm} (7)

with strict inequality for open hemispheres, and

$$\int_{S^{n-1}} (v, u)S(K, d\omega(u)) = 0 \text{ for all } v \in S^{n-1}. \hspace{1cm} (8)$$

Condition (8) amounts to saying that the mass distribution over $S^{n-1}$, such that $S(K,\omega)$ is the mass on $\omega$, has its centroid at the center of $S^{n-1}$. W. Fenchel and B. Jessen [8] and A. D. Alexandrov [2] and in a restricted sense Minkowski [5, §60] have shown that any totally-additive set function defined over $B$ and satisfying conditions (7) and (8) is the surface area function of a convex body which is unique to within a translation.

If $K$ is centrally symmetric, then

$$S(K,\omega) = S(K,\omega')$$

where $\omega'$ denotes the antipodal set on $S^{n-1}$ to $\omega$. In this case, condition (8) is automatically satisfied. Thus, each non-negative measure $\Phi$ defined on $B$ which satisfies $\Phi(\omega) = \Phi(\omega')$ for all $\omega$ in $B$ and is positive over open hemispheres is the surface area
function of a convex body which is unique to within a translation. If $K'$ is the reflection of $K$ in the origin, then

$$S(K', \omega) = S(K, \omega') = \phi(\omega') = \phi(\omega) = S(K, \omega).$$

Thus, to within a translation, $K = K'$ and so is central.
II. CONVEX POLYTOPES

By a convex polytope \( P \) we will mean any bounded subset of \( \mathbb{E}^n \) which can be represented as the intersection of a finite number of closed half-spaces. If \( P \) has a center of symmetry, then we will call \( P \) a central polytope. Let \( P \) be an \( n \)-dimensional convex polytope and consider a support plane \( \Pi \) of \( P \) with outer normal \( \mathbf{v} \). If \( \Pi \cap P \) contains more than a single point, then we will call this intersection the face of \( P \) with outer normal \( \mathbf{v} \) and will denote it by \( F(\mathbf{v}) \). \( F(\mathbf{v}) \) will be a convex polytope of dimension at least one and at most \( n-1 \).

If \( H \) is the support function of \( P \), then the value of the support function of \( F(\mathbf{v}) \) at \( \mathbf{u} \) will be the directional derivative of \( H \) at \( \mathbf{v} \) with respect to \( \mathbf{u} \) \([5, \text{p. 26}]\), that is, \( H'(\mathbf{v}; \mathbf{u}) \) is the support function of \( F(\mathbf{v}) \).

If one considers the regular octahedron in \( \mathbb{E}^3 \), then one sees that the faces of a central polytope need not be central. On the other hand it is known \([1; 7, \text{p. 28}; 12, \text{p. 47}]\) that in \( \mathbb{E}^3 \) central symmetry of the 2-dimensional faces of a convex polytope \( P \) implies that \( P \) possesses central symmetry. Alexandrov \([1]\) gives a proof of this theorem and states that in an analogous fashion one may prove that central symmetry of the \((n-1)\)-dimensional faces of an \( n \)-dimensional convex polytope \( P \) implies that \( P \) is central. A proof of
this generalization has recently appeared [16]. We will give in this
chapter some additional information as to the character of convex
polytopes which have faces which are centrally symmetric.

**THEOREM 1.** If all of the faces of dimension greater than or equal
two of a convex polytope \( P \) of dimension \( n \) possess central sym-
metry, then so does \( P \).

**PROOF:** If \( n = 3 \), this is Alexandrov's Theorem and is true. Let
\( P \) be a \( k \)-dimensional convex polytope and let \( H \) be its support
function. Since the face \( F(v) \) with outer normal \( v \) is central, its
support function \( H'(v; \cdot) \) satisfies by (4)

\[
H'(v; u) - H'(v; -u) = (t(v), u),
\]

(9)

where \( t(v) \) is a translation vector. Let us project \( P \) onto a 3-
dimensional subspace \( E^3 \). This projection is then a 3-dimensional
convex polytope \( \hat{P} \) with support function \( \hat{H} \), the restriction of \( H \)
to \( E^3 \). We can then calculate the support functions of the faces
\( \hat{F}(v) \) of \( \hat{P} \), where \( v \) is a unit vector in \( E^3 \). Thus, for \( u \)
in \( E^3 \) and \( v \) a unit vector in \( E^3 \), we have
\[
\hat{H}'(v;u) = \lim_{t \to 0^+} \frac{[\hat{H}(v+tu) - \hat{H}(v)]}{t} \\
= \lim_{t \to 0^+} \frac{[H(v+tu) - H(v)]}{t} \\
= H'(v;u).
\]

Hence, \( \hat{H}'(v;\cdot) \) is the restriction of \( H'(v;\cdot) \) to the subspace \( E^3 \).

Then by (9) and (10) for \( u \) and the unit vector \( v \) in \( E^3 \)

\[
\hat{H}'(v;u) - \hat{H}'(v;-u) = H'(v;u) - H'(v;-u) = (t(v), u).
\]

This shows that the face \( \hat{F}(v) \) of \( \hat{P} \) with other normal \( v \) is central. Since \( v \) was arbitrary, we have that all polygonal faces of \( \hat{P} \) are central, and so by Alexandrov's Theorem \( \hat{P} \) is central.

Since we projected \( P \) onto an arbitrary three-dimensional subspace, all projections of \( P \) onto three dimensional subspaces are centrally symmetric. This implies that \( P \) is central [5, p. 124].

By repeatedly applying Theorem 1 to the faces of \( P \) one proves the following corollary:

COROLLARY. If all of the two dimensional faces of a convex polytope \( P \) are centrally symmetric, then \( P \) is centrally symmetric.

The support function of the line segment joining the points \( a \) and \(-a\) has the representation \( H(u) = |(u, a)| \). We find it useful to calculate \( H'(w;u) \).
LEMMA 1. If $H(u) = |(a, u)|$, then

$$H'(w;u) = \begin{cases} 
(a, u) & \text{if } (a, w) > 0, \\
|(a, u)| & \text{if } (a, w) = 0, \\
(a, u) & \text{if } (a, w) < 0.
\end{cases} \quad (11)$$

PROOF. To compute $H'(w;u)$ we compute the limit of the quotient

$$\frac{|(a, w + tu)| - |(a, w)|}{t} \quad (12)$$

as $t$ goes to zero through positive values. If $(a, w) > 0$, then for $t$ sufficiently small $(a, w + tu) > 0$ and so

$$|(a, w + tu)| - |(a, w)| = t(a, w).$$

Thus (12) becomes $(a, u)$. If $(a, w) = 0$, (12) becomes $t|(a, u)|/t$ or simply $|(a, u)|$. Finally, if $(a, w) < 0$ and $t$ sufficiently small $(a, w + tu) < 0$ and (12) becomes $-(a, u)$. As $t$ tends to zero, we have (11).

If we now form the vector sum of $r$ segments $s(a_1), \ldots, s(a_r)$, we obtain a convex polytope $P$ whose support function $H$ has representation

$$H(u) = \sum_{i=1}^{r} |(a_i, u)|.$$
Evidently, \( H(-u) = H(u) \) and so \( P \) has the origin as a center of symmetry. If \( P \) is a convex polytope whose support function \( H \) has representation

\[
H(u) = \sum_{i=1}^{r} |(u, a_i)| + (t, u), \quad (13)
\]

then \( P \) will be called a finite sum of segments. If a polytope is a finite sum of segments, we see from (13) that \( P \) has \( t \) as a center of symmetry.

It is interesting to note that in two dimensional Euclidean space a convex polygon \( P \) is a finite sum of segments if and only if \( P \) is central. In fact, if \( P \) has the origin as a center of symmetry and if \( s(a_i) \) are the line segments through the origin which are congruent to the edges of \( P \), then \( P \) is the vector sum of the \( s(a_i) \).

[For a detailed proof see 14, p. 5].

Ogard [14, p. 10] gives an improvement of Alexandrov's Theorem stated earlier. He explicitly shows that the polygonal faces of a three dimensional polytope are central if and only if the polytope is a finite sum of segments. This theorem also lends itself to dimensional generalization. We do this in two steps.

THEOREM 2. If a convex polytope \( P \) is a finite sum of segments, then so are the faces of \( P \).
PROOF. Since $P$ is centrally symmetric, we may assume that it has the origin as a center of symmetry. Thus the support function $H$ of $P$ has representation

$$H(u) = \sum_{i=1}^{r} |(a_i, u)|.$$ 

The support function of the intersection of $P$ and its support plane with outer normal $v$ has, by (11), the representation

$$H'(v; u) = \sum_{I(v)} |(a_i, u)| + (\sum_{II(v)} a_i - \sum_{III(v)} a_i, u).$$

Here $I(v)$ indicates that we are to sum over those indices $i$ for which $(a_i, v) = 0$, $II(v)$ indicates that we sum over those $i$ for which $(a_i, v) > 0$ and $III(v)$ is to indicate that we sum over those $i$ for which $(a_i, v) < 0$. Now, if the intersection of $P$ and its support plane with outer normal $v$ contains at least one line segment, the index set $I(v)$ will not be empty. Thus, the face $F(v)$ of $P$ with outer normal $v$ has a support function of the form (13) and is a finite sum of segments.

Next we prove the converse of Theorem 2.

THEOREM 3. If all of the faces of a convex polytope $P$ are finite
sums of segments, then $P$ is a finite sum of segments.

**PROOF.** First we note that the faces of $P$ are centrally symmetric since they are finite sums of segments. By Theorem 1, $P$ is centrally symmetric, and we can assume then that $P$ has its center of symmetry at the origin.

Let $H$ be the support function of $P$. Since $F(v)$, the face of $P$ with outer normal $v$, is the sum of segments, its support function $H'(v;\cdot)$ has representation

$$H'(v;u) = \sum_{i=1}^{m(v)} |(a_i(v), u)| + (t(v), u),$$

where the $a_i(v)$ are orthogonal to $v$ and $t(v)$ is a translation vector. Let $A$ be the set of all $a_i(v)$ obtained by allowing $v$ to range over $S^{n-1}$. Since there is a finite number of faces of $P$, $A$ will be a finite set. We then write $A = \{a_1, \ldots, a_m\}$.

Let $a_k$ be in $A$ and let $\hat{P}$ be the orthogonal projection of $P$ onto an $(n-1)$-dimensional subspace orthogonal to $a_k$. Shephard has shown [16, p. 1209] that the vertices of $\hat{P}$ are the images under this projection of edges of $P$ parallel to $a_k$. Choose $v$ orthogonal to $a_k$. $F(v)$ is the intersection of $P$ with a support plane $\Pi(v)$ with outer normal $v$ and, since $v$ is orthogonal to $a_k$, $\Pi(v)$ is a support plane of $\hat{P}$. Because $\hat{P}$ is a polyhedron,
albeit degenerate, \( \hat{P} \cap \Pi(v) \) will contain a vertex of \( \hat{P} \) and, hence, \( F(v) \) will contain an edge parallel to \( a_k \). Thus, the set of all \( a_i(v) \), for some fixed \( v \), contains all elements of \( A \) orthogonal to \( v \).

We now define a function \( G \) by

\[
G(u) = \sum_{i=1}^{m} |(a_i, u)|. 
\]

\( G \) is the support function of a centrally symmetric convex polyhedron \( Q \). By Lemma 1 the support function of the intersection of \( Q \) with its support plane with outer normal \( v \) has the representation

\[
G'(v; u) = \sum_{*} |(a_i, u)| + (T(v), u). 
\]

Here \( * \) means that the sum is to be taken over those \( i \) for which \( (a_i, v) = 0 \) and \( T(v) \) is a translation vector.

Since the \( a_i \), which are orthogonal to \( v \) are precisely those which we have called \( a_i(v) \), we have

\[
\sum_{*} |(a_i, u)| = \sum_{i=1}^{m(v)} |(a_i(v), u)|. 
\]

Thus, we have
\[ G'(v; u) = \sum_{i=1}^{m(v)} |(a_i(v), u)| + (T(v), u) \]

A comparison of (14) and (15) shows that the faces of \( P \) and \( Q \) with parallel outer normals have identical support functions and so are congruent. But this implies that \( P \) and \( Q \) are identical \[5, p. 116\]. Since \( Q \) is by construction a finite sum of line segments, so then is \( P \).

Repeated application of Theorem 2 and Theorem 3 gives the following corollary.

**COROLLARY 1.** If all faces of \( P \) of some fixed dimension \( r, 2 \leq r \leq n-1 \) are finite sums of segments, then \( P \) is a finite sum of segments.

The observation that for two dimensional faces of a convex polytope \( P \) central symmetry is equivalent to being a finite vector sum of segments, yields a second corollary.

**COROLLARY 2.** If \( P \) is a convex polytope whose two dimensional faces are centrally symmetric, then \( P \) is a finite vector sum of segments and conversely.
III. PROJECTION BODIES

Let \( \Phi \) be a finite-valued, non-negative measure defined over the field of Borel sets \( B \) on the unit spherical surface \( S^{n-1} \).

Then the Radon-Stieltjes integral

\[
H(u) = \int_{S^{n-1}} |(u,v)| \Phi(d\omega(v))
\]

exists since \( |(u,v)| \) is a continuous function of \( v \) defined over \( S^{n-1} \). As a function of \( u \) in \( E^n \), (16) obviously satisfies conditions (1a) and (1b) of a support function. Since

\[
|(u,w,v)| \leq |(u,v)| + |(w,v)|
\]

and since \( \Phi \) is non-negative, (16) also satisfies condition (1c). Thus, (16) is a representation of the support function of a convex body. Evidently, this body has the origin as a center of symmetry.

If \( \Phi \) is such that it has the value \( \lambda_i > 0 \) at the one point sets \( \{v_i\}, i = 1, \ldots, r \), in \( B \) and vanishes for all \( \omega \) in \( B \) which are disjoint from \( \{v_1, \ldots, v_r\} \), then (16) takes the form

\[
H(u) = \sum_{i=1}^{r} |(u,\lambda_i v_i)|
\]

In this case we obtain the support function of a convex polytope which is a finite sum of segments. For this reason, the convex body whose
support function has representation given by (16) will be termed a sum of segments. As before, we will also use this term for any convex body which can be translated so that its support function is of the type (16).

If we consider the surface area function $S(\overline{K}, \cdot)$ of a convex body $\overline{K}$, then by (16) we have

$$nV(\overline{K}_o, \overline{K}, \ldots, \overline{K}) = \int_{S^{n-1}} H_o(u)S(\overline{K}, d\omega(u)),$$  \hspace{1cm} (17)

for any convex body $\overline{K}_o$ with support function $H_o$. In particular, we have for the surface area $S(\overline{K})$ of $\overline{K}$

$$S(\overline{K}) = nV(\overline{Z}, \overline{K}, \ldots, \overline{K}) = \int_{S^{n-1}} S(\overline{K}, d\omega(u)) = S(\overline{K}, S^{n-1}) .$$

If in (17) we choose $\overline{K}_o$ to be the line segment $s(v)/2$ where $v$ is in $S^{n-1}$, then we obtain

$$nV(\frac{1}{2}s(v), \overline{K}, \ldots, \overline{K}) = \frac{1}{2} \int_{S^{n-1}} |(u,v)|S(\overline{K}, d\omega(u)) .$$  \hspace{1cm} (18)

This is the brightness of $\overline{K}$ in the direction of $v$, that is the $(n-1)$-dimensional volume of the orthogonal projection of the convex body $\overline{K}$ onto the $(n-1)$-dimensional subspace with normal $v$ [5,
It is easily seen that the right hand side of (18), considered as a function of \( v \), can be extended to be positively homogeneous of degree one without changing its form. In so doing, one obtains a representation of the support function of a convex body \( K \). The descriptive name of projection body has been given to this body [5, p. 45]. Since \( S(K, v)/2 \) is a finite, non-negative measure over \( B \), a comparison of (16) and (18) shows that this projection body is a sum of segments and, thus, is centrally symmetric.

Firey [10, p. 18] has shown that the converse is also true, to wit if \( K \) is a non-degenerate sum of segments, then its support function, restricted to the unit spherical surface \( S^{n-1} \), is the brightness function of an essentially unique centrally symmetric convex body \( \overline{K} \). We will use the terms projection body and sum of segments interchangeably.

Our next theorem has to do with approximating convex bodies which are sums of segments by convex polytopes which are finite sums of segments.

**THEOREM 4.** If \( K \) is a sum of segments, then \( K \) can be arbitrarily well approximated by convex polytopes which are finite sums of segments.

**PROOF.** Let \( H \), with representation
be the support function of $K$. It is sufficient to show that (19) can be approximated uniformly over $S^{n-1}$ by a finite sum. Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{\Phi(S^{n-1})}$. We partition $S^{n-1}$ into disjoint Borel sets $\omega_1, \ldots, \omega_r$ in such a way that the chordal diameters are all less than $\delta$. Then, for arbitrary but fixed $v_i$ in $\omega_i$ and for unit vectors $u$, we have

$$H(u) = \int_{S^{n-1}} |(u, v)\Phi(d\omega(v))|$$

(19)

$$\left| \int_{S^{n-1}} |(u, v)\Phi(d\omega(v))| - \sum_{i=1}^{r} |(u, v_i)\Phi(\omega_i)| \right|$$

$$\leq \sum_{i=1}^{r} \int_{\omega_i} \left| |(u, v)| - |(u, v_i)| \right| \Phi(d\omega(v))$$

$$\leq \sum_{i=1}^{r} \int_{\omega_i} \left| |(u, v - v_i)| \right| \Phi(d\omega(v))$$

$$\leq \sum_{i=1}^{r} \int_{\omega_i} v - v_i \right| \Phi(d\omega(v))$$

$$\leq \delta \sum_{i=1}^{r} \int_{\omega_i} \Phi(d\omega(v)) = \delta \Phi(S^{n-1}) = \epsilon.$$
Thus $K$ is approximated by

$$P_{\varepsilon} = \sum_{i=1}^{r} s(v_i)\Phi(\omega_i)$$

in the sense that $\rho(P_{\varepsilon}, K) < \varepsilon$, where the metric is that defined in Chapter 1.

We note in passing that the choice of the unit vectors $v_i$ in $\omega_i$ is arbitrary. Thus the approximating polytope in Theorem 4 is also dependent on the choice of these vectors.

Let us look more closely at the representation

$$H(u) = \int_{S^{n-1}} |(u, v)|\Phi(d\omega(v))$$

of the support function of a convex body $K$. Without loss of generality, we may suppose that

$$\Phi(\omega) = \Phi(\omega')$$

where $\omega'$ is the antipode of $\omega$ on $S^{n-1}$. If this were not so, then we could replace $\Phi$ by the arithmetic mean of its values at antipodal sets without altering the value of the integral in (20).

Let $D(v)$ be an open hemisphere of $S^{n-1}$ with pole $v$ and let $\partial D(v)$ denote its equatorial boundary. If, for some $v$ in
and so $\Phi(D(-v))$ vanish, then

$$\Phi(\omega) = \Phi(\omega \cap \partial D(v))$$

and from (15) $H(v) = H(-v) = 0$. Thus, the support plane with outer normal $v$ would coincide with the support plane with outer normal $-v$. This would imply that the body $K$ was a subset of a hyperplane with outer normal $v$ and so would be degenerate. If (20) is to be the support function of a non-degenerate convex body, then $\Phi$ may be considered as the surface area function of a centrally symmetric convex body [see p. 10].

If $K$ is degenerate, then for some unit vector $w$ we have

$$\Phi(D(w)) = \Phi(D(-w)) = 0.$$ 

If we now let $S^{n-2}$ denote the common equatorial boundary of $D(w)$ and $D(-w)$, we have for $\Phi$

$$\Phi(\omega) = \Phi(\omega \cap S^{n-1})$$

Then considering $\Phi$ as a set function defined over the Borel sets on $S^{n-2}$ we may write for $u$ in $E^{n-1}$, the subspace of $E^n$ orthogonal $w$,

$$H(u) = \int_{S^{n-2}} |(u, v)| \Phi(d\omega(v)).$$
K can then be considered as a convex body in $E^{n-1}$. If $K$ is not degenerate when considered as a convex body in $E^{n-1}$, then $\Phi$ can be considered as the surface area function of an $(n-1)$-dimensional central convex body $\overline{K}$. In this case we say that $K$ is an $(n-1)$-dimensional projection body.

We define an $(n-p)$-dimensional projection body $(2 \leq p \leq n-1)$ in a like manner. When we wish to emphasize the dimensionality of $K$ we will explicitly do so. Otherwise, we will simply speak of $K$ as a projection body.

**THEOREM 5.** If $K$ is a non-degenerate convex body and if $\{K_i\}$ is a sequence of $n$-dimensional projection bodies which converge to $K$, then $K$ is an $n$-dimensional projection body.

**PROOF.** We assume that the origin is a common center of symmetry of the $K_i$ and an interior point of $K$. Since the $K_i$ are $n$-dimensional projection bodies, their support functions $H_i$ have representation

$$H_i(u) = \int_{S^{n-1}} |(u, v)| S(\overline{K}_i, d\omega(v)) , \quad (21)$$

where the $\overline{K}_i$ are centrally symmetric convex bodies. Let $H$ be the support function of $K$ and so the sequence $\{H_i\}$ converges uniformly over $S^{n-1}$ to $H$. $K$ is contained in a sphere of radius...
M and K contains a sphere of radius m > 0; thus we have

\[ m \|u\| < H(u) < M \|u\| \]  

(22)

for all u in \( \mathbb{R}^n \). Now because \( \{H_i\} \) converges uniformly over \( S^{n-1} \) to H, (22) must be true for all but a finite number of the \( H_i \). Without loss of generality, we then assume that we have omitted this finite set of \( H_i \) and re-indexed so that

\[ m \|u\| < H_i(u) < M \|u\| \]  

(23)

is valid for all \( i \) and u.

Next, we show that the central bodies \( \overline{K}_1', \overline{K}_2', \ldots \) appearing in the right hand side of (21) are all contained in a common sphere. Let \( \sigma_i \) denote one-half the minimum over \( S^{n-1} \) of (21), \( \overline{D}_i \) the maximum over \( S^{n-1} \) of \( \overline{H}_i(u) + \overline{H}_i(-u) \) (where \( \overline{H}_i \) is the support function of \( \overline{K}_i \) ), and \( \xi_p \) the p-volume of the p-dimensional unit ball. From (23) and Cauchy's surface area formula [5, p. 48] we have

\[ S(\overline{K}_i) = \frac{1}{\xi_{n-1}} \int_{S^{n-1}} H_i(u)S(Z, d\omega(u)) \]

\[ \leq \frac{M}{\xi_{n-1}} \int_{S^{n-1}} S(Z, d\omega(u)) = \frac{nM\xi_n}{\xi_{n-1}} . \]  

(24)

Minkowski's inequality [5, p. 109] and (24) yield
\[ V(K_i) \leq \xi_n \frac{[S(K_i)/n \xi_n]}{n^{n-1}} \]

\[ \leq \xi_n [M/\xi_{n-1}]^{n/n-1}. \]

By (23) we note that \( \bar{\sigma}_i > \frac{1}{2} m \) and consequently from an inequality of Firey [9] conclude that

\[ D_i = \frac{nV(K_i)}{\bar{\sigma}_i} \leq \frac{2n \xi_n}{m} [M/\xi_{n-1}]^{n/n-1}. \]

We denote the right hand side of this inequality by \( R \) and observe that it is independent of \( i \). Whereby, we have that all of the \( K_i \) are contained in a sphere of radius \( R \).

Since all the \( K_i \) are contained in a common sphere, we may apply Blaschke's selection theorem to conclude that \( \{K_i\} \) contains a subsequence \( \{K_{i(j)}\} \) which converges to a convex body \( K \).

An alternate way of writing (21) [5, p. 45] is

\[ H_{i(j)}(u) = nV(s(u), \bar{K}_{i(j)}, \ldots, \bar{K}_{i(j)}). \] (25)

By the continuity of the mixed volumes, the right side of (25) converges to \( nV(s(u), K, \ldots, \bar{K}) \) as \( \{K_{i(j)}\} \) converges to \( K \). At the same time we have \( \{\bar{H}_{i(j)}\} \) converging to \( \bar{H} \), and so

\[ H(u) = nV(s(u), \bar{K}, \ldots, \bar{K}). \] (26)
But, since \( \{S(\tilde{K}_{i(j)}, \omega)\} \) converges to \( S(\tilde{K}, \omega) \) [p. 9], the right hand side of (26) is twice the brightness function of \( \tilde{K} \), that is

\[
H(u) = \int_{S^{n-1}} \|(u, v)\| S(\tilde{K}, d\omega(v))
\]

Thus, \( K \) is an n-dimensional projection body.

To take care of the case where \( K \) is degenerate, we need a preliminary result.

**THEOREM 6.** The orthogonal projection of an n-dimensional projection body onto a p-dimensional subspace is a p-dimensional projection body.

**PROOF.** Let \( K \) be an n-dimensional projection body. Its support function \( H \) has representation

\[
H(u) = \int_{S^{n-1}} \|(u, v)\| S(\tilde{K}, d\omega(v))
\]

As in the proof of Theorem 4, there are polytopes \( P_i \) with support functions \( H_i \) with representations

\[
H_i(u) = \sum_{j(i)} \|(u, v_{j(i)})\| S(\tilde{K}, \omega_{j(i)})
\]
for which

$$\max_{u \in S^{n-1}} \left| H(u) - H_1(u) \right| < \frac{1}{i}.$$ 

Thus, the sequence \( \{P_i\} \) converges to \( K \).

Now we project \( K \) and \( P_i \) orthogonally onto a p-dimensional subspace \( E^p \). The support function of \( \hat{P}_i \), the projection of \( P_i \), has, for \( u \) in \( E^p \), representation

$$\hat{H}_i(u) = \sum_{j(i)} |(u, v_{j(i)})| S(K, \omega_{j(i)})$$

$$= \sum_{j(i)} |(u, \hat{v}_{j(i)})| S(K, \omega_{j(i)}),$$

where \( \hat{v}_{j(i)} \) is the projection of \( v_{j(i)} \) onto \( E^p \). Thus, \( \hat{P}_i \) is a finite sum of segments in \( E^p \) and so is a projection body.

The support function of \( \hat{K} \), the projection of \( K \), is the restriction of \( H \) to the subspace \( E^p \). If \( \hat{H} \) is the support function of \( \hat{K} \), then we have

$$\max_{u \in S^{p-1}} \left| \hat{H}(u) - \hat{H}_i(u) \right| = \max_{u \in S^{p-1}} \left| H(u) - H_1(u) \right|$$

$$\leq \max_{u \in S^{n-1}} \left| H(u) - H_1(u) \right| < \frac{1}{i}.$$
Thus, $\{\mathcal{H}_i\}$ converges uniformly over $S^{p-1}$ to $\mathcal{H}$ and, consequently, $\{\mathcal{P}_i\}$ converges to $\mathcal{K}$. Now, an application of Theorem 5 yields that $\mathcal{K}$ is a projection body.

By combining Theorem 5 and Theorem 6 we have the following theorem.

**THEOREM 7.** If $K$ is the limit of a sequence of projection bodies, then $K$ is a projection body.

**PROOF.** If $K$ is non-degenerate, this is Theorem 5. If $K$ is degenerate, then it must lie in some $p$-dimensional plane, and, by a suitable translation of $K$, we may assume that this $p$-plane is a $p$-dimensional subspace. In this subspace, $K$ may be considered as a non-degenerate convex body. We now project each member of the sequence of projection bodies, which converges to $K$, onto this subspace containing $K$. We thus form by Theorem 6 a sequence of projection bodies in this subspace which converge to $K$. Now an appeal to Theorem 5 shows that $K$ is also a projection body.

For any convex body $K$ in $E^n$ and any class $\mathcal{K}$ of convex sets in $E^n$, $K$ is said to be approximable by the class $\mathcal{K}$ if there exists a sequence of sets $\{K_i\}$, such that, as $i$ tends to infinity $K_i$ tends to $K$, where each set $K_i$ is a vector sum of a finite number of sets of the class $\mathcal{K}$, [15, p. 9]. An approximation
problem is to determine necessary and sufficient conditions for \( K \) to be approximable by a class \( K \).

Theorem 4 and Theorem 7 then provide the solution of an approximation problem: Let \( K \) be the class of all line segments in \( E^n \). Then \( K \) is approximable by \( K \) if and only if \( K \) is a projection body. This solves a question alluded to by Shephard [15, p.9].
It is known [4, p. 154; 5, p. 29] that if $K$ is a smooth (e.g. with analytical support function), centrally symmetric convex body with center at the origin, then its support function $H$ has representation

$$H(u) = \int_{S^{n-1}} |(u, v)| h(v) d\omega(v). \tag{27}$$

Here, $h$ is an even function defined on $S^{n-1}$ and $d\omega(v)$ is the surface element on $S^{n-1}$ at $v$. If $h$ is positive, then (27) represents the support function of a projection body. On the other hand, if $h$ is always negative, then $H$ is not a convex function and so (27) does not represent the support function of a convex body. Generally, $h$ will fluctuate in sign and we seek conditions on a function $h$ defined over $S^{n-1}$ to guarantee that (27) is the representation of the support function of a central body.

First, one might ask whether all central convex bodies have a representation which is analogous to (27). The affirmative answer is given by the next theorem.

**THEOREM 8.** If $K$ is a central convex body with center at the origin, then there is a finite signed measure $\mu$ defined over the Borel sets $B$ on $S^{n-1}$ for which the support function $H$ of $K$ has representation
\[ H(u) = \int_{S^{n-1}} |(u, v)| \mu(d\omega(v)) . \]

**PROOF.** Let \( \{K_i\} \) be a sequence of smooth, central bodies whose support functions \( H_i \) have representations

\[ H_i(u) = \int_{S^{n-1}} |(u, v)| h_i(v) d\omega(v) \]  \hspace{1cm} (28)

and which converge to \( K \) (smooth convex bodies are known to be dense in the space of all convex bodies [5, p. 35]). We define signed measures \( \mu_i \) over \( B \) by

\[ \mu_i(\omega) = \int_{\omega} h_i(v) d\omega(v) \, , \]

and then (28) becomes

\[ H_i(u) = \int_{S^{n-1}} |(u, v)| \mu_i(d\omega(v)) . \]  \hspace{1cm} (29)

Let \( \mu_i = \mu_i^+ - \mu_i^- \) by a Jordan decomposition of \( \mu_i \) where \( \mu_i^+ \) and \( \mu_i^- \) are non-negative measures defined over \( B \) [11, p. 123]. Thus (29) becomes
\[ H_i(u) = \int_{S^{n-1}} |(u, v)| \mu_i^+(d\omega(v)) - \int_{S^{n-1}} |(u, v)| \mu_i^-(d\omega(v)). \] (30)

From Chapter 3, we know that the two right hand integrals in (30) are themselves support functions of projection bodies \( K_i^+ \) and \( K_i^- \), that is the support function \( H_i^\pm \) of \( K_i^\pm \) has representation

\[ H_i^\pm(u) = \int_{S^{n-1}} |(u, v)| \mu_i^\pm(d\omega(v)). \]

\( K_i, K_i^+ \), and \( K_i^- \) all have the origin as a center of symmetry, and, consequently, their respective support functions are non-negative over \( E^n \). As in the proof of Theorem 5 we have

\[ 0 \leq H_i^-(u) \leq H_i^+(u) \leq H_i(u) \leq M \|u\| \]

for all \( i \). This implies that the two sequences \( \{K_i^+\} \) and \( \{K_i^-\} \) of projection bodies are uniformly bounded and hence by Blaschke's selection theorem contain subsequences which converge. By suitable omissions and relabeling, we can assume that these sequences themselves converge. By Theorem 7 the limits of these sequences are themselves projection bodies, which we denote by \( K^+ \) and \( K^- \) respectively. Then there are non-negative measures \( \mu^+ \) and \( \mu^- \) defined over \( B \) for which the support functions \( H^+ \) and \( H^- \)
respectively have representations

\[ H^+(u) = \int_{S^{n-1}} |(u, v)| \mu^+(d\omega(v)) \]

and

\[ H^-(u) = \int_{S^{n-1}} |(u, v)| \mu^-(d\omega(v)) \].

Next define a signed measure \( \mu \) over \( B \) by

\[ \mu(\omega) = \mu^+(\omega) - \mu^-(\omega) \].

The left hand side of (30) converges to \( H(u) \) and [see Chapter 2] the right hand side converges to

\[ \int_{S^{n-1}} |(u, v)| \mu^+(d\omega(v)) - \int_{S^{n-1}} |(u, v)| \mu^-(d\omega(v)) = \int_{S^{n-1}} |(u, v)| \mu(d\omega(v)) \].

Consequently,

\[ H(u) = \int_{S^{n-1}} |(u, v)| \mu(d\omega(v)) \],

which is our desired representation.

In the proof of Theorem 8 we have that the support function of a central body \( K \) can be written as the difference of the support functions of two projection bodies. Thus, any central convex body can be
written as the difference of two projection bodies in the sense that there is a projection body whose vector sum with the given central body is again a projection body. This decomposition is not unique.

Now, if $\mu$ is any finite signed measure defined over the Borel sets $B$ on $S^{n-1}$, one may form the integral

$$\int_{S^{n-1}} |(u, v)| \mu(d\omega(v))$$

and ask whether it is a representation of the support function of a central convex body. If $\mu$ is non-negative, then the answer is affirmative: The integral is the support function of a projection body. On the other hand, if $\mu$ is always non-positive, then the integral, as a function of $u$, is not convex and so the integral is not the support function of a convex body.

Historically, the question has been posed for smooth bodies [4; 5]. Specifically, conditions on a function $h$ defined over $S^{n-1}$ have been sought in order that

$$\int_{S^{n-1}} |(u, v)| h(v)d\omega(v)$$

be the support function of a smooth central body. Our next theorem will give an answer to this latter problem. First we need a lemma
upon which our solution is based.

LEMMA 2. In 2-dimensional Euclidean space the only centrally symmetric convex bodies are projection bodies.

PROOF. Let $K$ be a centrally symmetric convex body in $E^2$. Let $\{P_i\}$ be a sequence of central polygons which converge to $K$. We have noted [p. 16] that central polygons are finite vector sums of line segments and, hence, are projection bodies. Since the limit of a sequence of projection bodies is again a projection body [Theorem 7], $K$ is a projection body.

We may conclude directly from this lemma that in $E^2$ for (32) to represent the support function of a 2-dimensional central body it is necessary that $h$ be non-negative over $S^1$. In general, for 2-dimensional Euclidean space, (31) will represent the support function of a centrally symmetric convex body if and only if $\mu(\omega) \geq 0$ for all $\omega$ in $B$.

Let $M$ be an arbitrary 2-dimensional subspace of $E^n$ with $U$ its unit spherical surface; thus, $U = M \cap S^{n-1}$. For $\overline{v}$ in $U$ we define $L(\overline{v})$ to be the set of all vectors $v$ in $S^{n-1}$ which, when projected orthogonally onto $M$, have the same direction as $\overline{v}$, that is all $v$ in $S^{n-1}$ for which

$$(v, \overline{v}) > 0$$

(33)
and for all \( u \) in \( U \) orthogonal to \( \overline{v} \)

\[
(v - \overline{v}, u) = 0.
\]

In 3-dimensional Euclidean space \( L(\overline{v}) \) is simply a great semi-circle with \( \overline{v} \) in the middle and excluding both end points. Generally, if \( M \) is the orthogonal complement of \( M \), then the union of the closures of \( L(\overline{v}) \) and \( L(-\overline{v}) \) will be \( S^{n-1} \cap M \). Finally, denote by \( d\sigma(v) \) the \((n-2)\)-dimensional surface element at \( v \) on \( L(\overline{v}) \). We now state our next theorem.

**THEOREM 9.** In order that

\[
H(u) = \int_{S^{n-1}} |(u, v)| h(v) d\omega(v)
\]

be a representation of the support function \( H \) of a central convex body \( K \), it is necessary and sufficient that \( h \) satisfy

\[
\int_{L(\overline{v})} (v, \overline{v})^2 h(v) d\sigma(v) \geq 0
\]

for all 2-dimensional subspaces \( M \) of \( E^n \) and for all vectors \( \overline{v} \) in \( S^{n-1} \cap M \).

**PROOF.** It is known [5, p. 124] that \( K \) is centrally symmetric if
and only if all orthogonal projections of $K$ onto 2-dimensional subspaces $M$ of $E^n$ are central. If $H$ is the support function of $K$, then the restriction of $H$ to $M$ will be the support function of the projection of $K$ onto $M$. Thus $K$ will be centrally symmetric if and only if all restrictions of $H$ to 2-dimensional subspaces of $E^n$ are support functions of 2-dimensional central bodies, or, what is the same thing, support functions of 2-dimensional projection bodies.

Now let $M$ be an arbitrary 2-dimensional subspace of $E^n$ and let $U$ be the unit spherical surface in $M$, that is $U = S^{n-1} \cap M$. We propose to give an explicit description of the restriction of (32) to $M$. If $\mathbf{v}(0)$ is a fixed vector in $U$, then we may parameterize $U$ by the angular measure $\xi$ measured counterclockwise from $\mathbf{v}(0)$. We then denote vectors in $U$ by $\mathbf{v}(\xi)$ where $0 \leq \xi \leq 2\pi$ and write $L(\xi)$ for $L(\mathbf{v}(\xi))$.

Let $u$ be any vector in $M$ and let $v$ be a unit vector in $E^n$. If $v$ is orthogonal to all vectors $u$ in $M$, then $|(u, v)| = 0$. If $v$ is not orthogonal to all vectors in $M$, $v$ is in $L(\xi)$ for some unique $\xi$ and

$$v = (v, \mathbf{v}(\xi))\mathbf{v}(\xi) + \text{terms orthogonal to } M.$$  \hfill (34)

For $u$ in $M$ we have from (33) and (34)
\[ |(u, v)| = |(u, \overline{\nu}(\xi))|(v, \overline{\nu}(\xi)) \, . \] (35)

We continue to parameterize \( \mathbb{E}^n \). For \( v \) in \( S^{n-1} \) we consider \( v \) as an \( n \)-tuple \((v_1, \ldots, v_n)\) chosen so that if \( v \) is in \( M \), then \( v \) is of the form \((0, \ldots, 0, v_{n-1}, v_n)\). Next, we express the coordinates of \( v \) in hyperspherical coordinates [3, p. 233]:

\[
\begin{align*}
v_1 &= \cos \theta_1 \\
v_2 &= \sin \theta_1 \cos \theta_2 \\
v_3 &= \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
&\quad \vdots \\
v_{n-2} &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \cos \theta_{n-2} \\
v_{n-1} &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \sin \theta_{n-2} \cos \xi \\
v_n &= \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-3} \sin \theta_{n-2} \sin \xi
\end{align*}
\]

where \( 0 \leq \theta_i \leq \pi \), \( i = 1, \ldots, n-2 \), and \( 0 \leq \xi \leq 2\pi \). The vector which we have denoted by \( \overline{\nu}(\xi) \) has coordinates \( v_1 = 0, \ldots, v_{n-2} = 0, v_{n-1} = \cos \xi, v_n = \sin \xi \).

In terms of our hyperspherical coordinates the surface element \( d\omega(v) \) on \( S^{n-1} \) becomes
If we define a function $g$ over $U$ by

$$d\omega(v) = (\sin \theta_1)^{n-2}(\sin \theta_2)^{n-3}\cdots(\sin \theta_{n-2})d\theta_1\cdots d\theta_{n-2}d\xi$$

$$= (v, \vec{v}(\xi))(\sin \theta_1)^{n-3}\cdots(\sin \theta_{n-3})d\theta_1\cdots d\theta_{n-2}d\xi. \quad (37)$$

Here we have used

$$(v, \vec{v}(\xi)) = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2}.$$ 

The set $L(\xi)$ is composed of those vectors with coordinates given by (36) where $\xi$ is fixed and $0 < \theta_i < \pi$. Thus

$$(\sin \theta_1)^{n-3}(\sin \theta_2)^{n-4}\cdots(\sin \theta_{n-3})d\theta_1\cdots d\theta_{n-2}$$

is the element of surface $d\sigma(v)$ at $v$ on $L(\xi)$. Then, for $v$ in $L(\xi)$, (37) becomes

$$d\omega(v) = (v, \vec{v}(\xi))d\sigma(v)d\xi. \quad (38)$$

Using (35) and (38) we see that $H$ restricted to $M$ has the representation

$$H(u)_{|u \in M} = \int_{L(\xi)}^{2\pi} \int_0^\infty |(u, \vec{v}(\xi))(v, \vec{v}(\xi))|^2 h(v)d\sigma(v)d\xi. \quad (39)$$

If we define a function $g$ over $U$ by
then (39) takes the form

$$g(\bar{v}(\xi)) = \int_{L(\xi)} (v, \bar{v}(\xi))^2 h(v) d\sigma(v),$$

(40)

$$H(u) = \int_0^{2\pi} |(u, \bar{v}(\xi))| g(\bar{v}(\xi)) d\xi.$$  

(41)

The restriction of $H$ to $M$ is to be the support function of a projection body. This will be the case if $g$ is non-negative over $U$. On the other hand, if $g$ is non-negative over $U$, (41) will be the support function of a central body in $M$. Hence, since $M$ was arbitrary,

$$H(u) = \int_{\mathbb{S}^{n-1}} |(u, \nu)| h(\nu) d\omega(\nu)$$

will be the representation of a support function of a central body if and only if (40) is non-negative for all vectors $\bar{v}(\xi)$ in $U$ and for all subspaces $M$ of $\mathbb{E}^n$.

Theorem 9 gives only a partial answer to the problem of characterizing the signed measures for which

$$H(u) = \int_{\mathbb{S}^{n-1}} |(u, \nu)| \mu(d\omega(\nu))$$

(42)
is a representation of the support function of a central body. Our
next goal is to extend the result of Theorem 9.

For any signed measure $\nu$ defined over the Borel sets $B$ on $S^{n-1}$ we may write

$$\nu = \nu_1 + \nu_2 + \nu_3$$

where $\nu_1$ is absolutely continuous with respect to Lebesgue measure on $B$, and $\nu_2$ and $\nu_3$ are singular with respect to Lebesgue measure [11, p. 182]. More exactly, $\nu_2$ is purely atomic, by which we mean there exists a countable set $C$ such that

$$\nu_2(S^{n-1} \setminus C) = 0,$$

and $\nu_3$ has the property that $\nu_3(\{u\}) = 0$ for every one point set $\{u\}$ in $B$.

In what follows, we give necessary and sufficient conditions on a signed measure, assumed absolutely continuous with respect to Lebesgue measure, in order that (42) be a representation of the support function of a central body. We also do the same for purely atomic signed measures. Unfortunately, we can not treat the case where a signed measure is singular and vanishes for each one point set in $B$.

First, we look at the absolutely continuous case. If $\nu$ is a finite signed measure defined over $B$ which is absolutely continuous with respect to Lebesgue measure, then the Radon-Nikodym Theorem [11, p. 128] states that there exists a finite valued integrable
function $h$ on $S^{n-1}$ for which

$$v(\omega) = \int_{\omega} h(v) d\omega(v).$$

**THEOREM 10.** In order that

$$H(u) = \int_{S^{n-1}} |(u, v)| v(d\omega(v)),$$

where $v$ is absolutely continuous with respect to Lebesgue measure, be a representation of the support function of a central body $K$, it is necessary and sufficient that

$$\int_{\text{L}(\overline{v})} (v, \overline{v})^2 h(v) d\sigma(v) \geq 0$$

for all subspaces $M$ of $E^n$ and for all vectors $\overline{v}$ in $S^{n-1} \cap M$. Here $h$ is the Radon-Nikodym derivative defined by

$$v(\omega) = \int_{\omega} h(v) d\omega(v).$$

**PROOF.** Using (44) we may write (43) as

$$H(u) = \int_{S^{n-1}} |(u, v)| h(v) d\omega(v).$$
Then the proof is identical with that of Theorem 9.

Next, let \( \nu \) be a finite signed measure which is purely atomic. If \( \nu \) has a non-zero value at only a finite number of points and if these values are positive, then we have the support function of a finite vector sum of segments.

**THEOREM 11.** Let \( \nu \) be a purely atomic, finite, signed measure defined over the Borel sets on \( S^{n-1} \). Let \( L(\nu) \) be as before and let \( \mu(\nu; \cdot) \) be the signed measure defined over the Borel sets on \( L(\nu) \) by \( \mu(\nu; \omega) = \nu(\omega) \). Then

\[
H(\nu) = \int_{S^{n-1}} |(u, \nu)| \nu(d\omega(\nu))
\]

will be a representation of the support function of a central convex body \( K \) if and only if

\[
\int_{L(\nu)} (v, \nu) \mu(\nu; d\omega(\nu)) \geq 0
\]

for all vectors \( \nu \) in \( S^{n-1} \cap M \) and for all 2-dimensional subspaces \( M \) of \( E^n \).

**PROOF.** As in the proof of Theorem 9, we project \( K \) orthogonally onto a 2-dimensional subspace \( M \). Then, for \( \nu \) in \( L(\nu) \), we
have

\[ |(u, v)| = |(u, \overline{v})|(v, \overline{v}) . \]

Now, the restriction of \( v \) to the Borel sets of \( L(\overline{v}) \) is either purely atomic or else has the value zero for each Borel set on \( L(\overline{v}) \).

We parameterize \( M \cap S^{n-1} \), as in the proof of Theorem 9, by the angular measure \( \xi \). Thus, we write \( \overline{v}(\xi) \) for \( \overline{v} \) and \( L(\overline{v}(\xi)) \) for \( L(\overline{v}) \). For \( u \) restricted to \( M \), \( H \) has the representation

\[ H(u) = \int_0^{2\pi} \int_{L(\overline{v}(\xi))} |(u, \overline{v}(\xi))|(v, \overline{v}(\xi))\mu(\overline{v}(\xi); d\omega(v))d\xi . \]

If we let \( g \) be defined over \( M \cap S^{n-1} \) by

\[ g(\overline{v}(\xi)) = \int_{L(\overline{v}(\xi))} (v, \overline{v}(\xi))\mu(\overline{v}(\xi); d\omega(v)) , \]

then we have

\[ H(u) = \int_0^{2\pi} |(u, \overline{v}(\xi))|g(\overline{v}(\xi))d\xi . \]

This will be the representation of a support function of a 2-dimensional central body if and only if

\[ g(\overline{v}(\xi)) = \int_{L(\overline{v}(\xi))} (v, \overline{v}(\xi))\mu(\overline{v}(\xi); d\omega(v)) \geq 0 \]

for \( 0 \leq \xi \leq 2\pi \). Since \( M \) was arbitrary, the theorem follows.
V. MEASURE NUMBERS

Blaschke [4, p. 156] has given some interesting formulas for the volume, surface area and total mean curvature of a 3-dimensional central convex body. These formulae, as given, are only valid for bodies whose support functions are sufficiently smooth. In this chapter, we generalize these formulae so as to be valid for all central bodies and for dimensions greater than or equal three. We will let \( \Omega \) denote the unit spherical surface \( S^{n-1} \) in \( E^n \) and use \( * \) in conjunction with summation formulae to signify that we are allowing the \( i_j \) to run independently from \( 1 \) to \( n \).

**Lemma 3.** If \( s_1, \ldots, s_n \) are line segments of length \( a_1, \ldots, a_n \) respectively, then the mixed volume of these line segments has representation

\[
V(s_1, \ldots, s_n) = a_1 \cdots a_n [v_1, \ldots, v_n]/n!. 
\]

Here \( v_i \) is the unit vector parallel to \( s_i \) and \( [v_1, \ldots, v_n] \) is the absolute value of the determinant whose successive columns are the vectors \( v_1, \ldots, v_n \).

**Proof.** We form the vector sum \( s_1 + \ldots + s_n \) and compute its volume using (5):
\[ V(s_1 + \ldots + s_n) = \sum_{*} V(s_1, \ldots, s_n) \]
\[ = n! V(s_1, \ldots, s_n). \quad (45) \]

This last equality is true because the mixed volumes of segments is zero if two of its arguments are identical [5, p. 41] and since the mixed volumes are symmetric in their arguments.

On the other hand, \( s_1 + \ldots + s_n \) is an \( n \)-dimensional parallel-opiped or has zero volume and so its volume is given by the absolute value of the determinant whose successive columns are the vectors \( s_1, \ldots, s_n \), that is \( [s_1, \ldots, s_n] \) [13, p. 90]. The volume of \( s_1 + \ldots + s_n \) is equal to the volume of \( a_1 v_1 + \ldots + a_n v_n \) and so

\[ V(s_1 + \ldots + s_n) = [a_1 v_1, \ldots, a_n v_n] = a_1 \ldots a_n [v_1, \ldots, v_n]. \quad (46) \]

(45) and (46) yield

\[ n! V(s_1, \ldots, s_n) = a_1 \ldots a_n [v_1, \ldots, v_n]. \]

Let \( K \) be the projection body whose support function \( H \) has representation

\[ H(u) = \int_{\Omega} |(u, v)| \mu(d\omega(v)). \]
Thus, \( \mu \) is a non-negative measure defined over the Borel sets on \( \Omega \). As in the proof of Theorem 4 for each \( \epsilon > 0 \) we may decompose \( \Omega \) into disjoint Borel sets \( \omega_1, \ldots, \omega_r \) for which

\[
\left| \mathcal{H}(u) - \sum_{i=1}^{r} \langle (u, v_i) \rangle \mu(\omega_i) \right| < \epsilon,
\]

where \( v_i \) is an arbitrary vector in \( \omega_i \). Then

\[
\mathcal{H}_\epsilon(u) = \sum_{i=1}^{r} \langle (u, v_i) \rangle \mu(\omega_i)
\]

is a representation of the support function of a convex polytope \( P_\epsilon \) which is a finite sum of segments, i.e., \( P_\epsilon = \sigma_1 + \ldots + \sigma_r \) where the \( \sigma_i \) are segments of length \( 2\mu(\omega_i) \) and parallel to \( v_i \). The volume of \( P_\epsilon \) can be computed using Lemma 3:

\[
V(P_\epsilon) = \sum_{*}^{n} \frac{2^n}{n!} [v_{11}, \ldots, v_{in}] \mu(\omega_{i1}) \ldots \mu(\omega_{in}).
\]

As \( \epsilon \) tends to zero, \( P_\epsilon \) converges to \( K \). The continuity of the volume functional shows that \( V(P_\epsilon) \) converges to \( V(K) \). Also we have
\[
\sum_{n!} \frac{2^n}{n!} [v_{i_1}, \ldots, v_{i_n}] \mu(\omega_{i_1}) \ldots \mu(\omega_{i_n})
\]

converging to

\[
\frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] \mu(d\omega(v_1)) \ldots \mu(d\omega(v_n)) ,
\]

where the multiple Radon-Stieltjes integration is carried out over the Cartesian product

\[
\Omega^n = \Omega \times \Omega \times \ldots \times \Omega .
\]

This yields our next theorem.

**THEOREM 12.** If \( K \) is a projection body whose support function has representation

\[
H(u) = \int_{\Omega} |(u, v)| \mu(d\omega(v)) ,
\]

then

\[
V(K) = \frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] \mu(d\omega(v_1)) \ldots \mu(d\omega(v_n)) .
\]

If \( K \) is an arbitrary central convex body, its support function has representation
\[ H(u) = \frac{1}{\Omega} \int_{\Omega} |(u, v)| \mu(d\omega(v)) , \]

where \( \mu \) is a signed measure defined over \( B \). Let \( \mu = \mu^+ - \mu^- \) be a Jordan decomposition of \( \mu \) into the difference of two non-negative measures. Then for \( \lambda \geq 1 \) the set function \( \mu + \lambda \mu^- \) is also a non-negative measure defined over \( B \) and, consequently,

\[ H_\lambda(u) = \frac{1}{\Omega} \int_{\Omega} |(u, v)| (\mu + \lambda \mu^-)(d\omega(v)) \]

is a representation of the support function of a projection body \( K_\lambda \).

By Theorem 12 the volume of \( K_\lambda \) is given by

\[ V(K_\lambda) = \frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] (\mu + \lambda \mu^-)(d\omega(v_1)) \ldots (\mu + \lambda \mu^-)(d\omega(v_n)) \]

\[ = \frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] \mu(d\omega(v_1)) \ldots \mu(d\omega(v_n)) \]

\[ + \lambda \frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] \mu(d\omega(v_1)) \ldots \mu(d\omega(v_{n-1})) \mu^-(d\omega(v_n)) \]

\[ + \ldots + \lambda^n \frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] \mu^-(d\omega(v_1)) \ldots \mu^-(d\omega(v_n)) . \]

We note that this is a polynomial of degree \( n \) in \( \lambda \).
Furthermore \( K_\lambda = K + \lambda K^\circ \) and so by (5)

\[
V(K_\lambda) = V(K) + \lambda n V(K, \ldots, K, K^\circ) + \ldots + \lambda^n V(K^\circ),
\]

which is also a polynomial of degree \( n \) in \( \lambda \). Since these two polynomials are equal for \( \lambda \geq 1 \), they must be equal for all values of \( \lambda \). In particular, for \( \lambda = 0 \), we have the following theorem.

**THEOREM 13.** If \( K \) is a centrally symmetric convex body whose support function has representation

\[
\int |(u, v)| \mu(d_\omega(v))
\]

then

\[
V(K) = \frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] \mu(d_\omega(v_1)) \ldots \mu(d_\omega(v_n)).
\]

By employing a procedure similar to that used in the proof of Theorem 13, we arrive at an expression for the mixed volumes of central convex bodies.

**THEOREM 14.** If \( K_1, \ldots, K_n \) are centrally symmetric convex bodies, then

\[
V(K_1, \ldots, K_n) = \int_{\Omega^n} [v_1, \ldots, v_n] \mu_1(d_\omega(v_1)) \ldots \mu_n(d_\omega(v_n)).
\]
where the support function $H_i$ of $K_i$ has representation

$$H_i(u) = \int |(u, v)| \mu_i(d\omega(v)) \, \Omega$$

**Proof.** Let $K = \lambda_1 K_1 + \ldots + \lambda_n K_n$. The support function $H$ of $K$ has representation

$$H(u) = \lambda_1 H_1(u) + \ldots + \lambda_n H_n(u) = \int_{S^{n-1}} |(u, v)| \mu(d\omega(v))$$

where $\mu = \lambda_1 \mu_1 + \ldots + \lambda_n \mu_n$. From Theorem 13

$$V(K) = \sum_{\ast} \lambda_1 \ldots \lambda_n \frac{2^n}{n!} \int_{\Omega^n} [v_1, \ldots, v_n] \mu_1 \ldots \mu_n (d\omega(v_1)) \ldots (d\omega(v_n)). \quad (47)$$

On the other hand,

$$V(K) = V(\lambda_1 K_1 + \ldots + \lambda_n K_n) = \sum_{\ast} \lambda_1 \ldots \lambda_n V(K_1, \ldots, K_n). \quad (48)$$

A comparison of the coefficient of the terms $\lambda_1 \lambda_2 \ldots \lambda_n$ in (47) and (48) yields the required result.

As an instance of Theorem 14 we have that the surface area $S(K)$ of $K$ is given by
\[ S(K) = \frac{2^n}{\zeta_{n-1}(n-1)!} \int_{\Omega^n} [v_1, \ldots, v_n] d\omega(v_1)\mu(d\omega(v_2)) \cdots \mu(d\omega(v_n)) , \]

where \( \zeta_{n-1} \) is the \((n-1)\)-dimensional volume of the unit ball in \((n-1)\)-dimensional space. The validity of this expression follows by noting that \( S(K) = nV(Z, K, \ldots, K) \), and that the support function of the unit ball \( Z \) in \( E^n \) has representation

\[ \frac{1}{2\zeta_{n-1}} \int_{\Omega} |(u, v)| d\omega(v) = 1 . \]

[5, p. 48].

Finally, if we restrict ourselves to \( E^3 \) and to smooth convex bodies, we obtain the formulas of Blaschke [4, p. 156] by considering \( V(K), nV(K, K, Z), \) and \( nV(K, Z, Z, \ldots) \) respectively.


