RATIONAL APPROXIMATIONS FOR FUNCTIONS OF A COMPLEX VARIABLE OBTAINED BY USE OF THE DARBOUX FORMULA

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THEORETICAL DEVELOPMENT

In a recent paper Squire (9, p. 94-108) obtained some rational approximations for functions satisfying the differential equation,

$$(1) \qquad y'(x) = f(x)\, y(x) \, ,$$

of type

$$(2) \qquad y(x) = y(a) \frac{1 + \sum_{n=o}^{N} A_n^N R_{n+1}(a)(x-a)^{n+1}}{1 + \sum_{n=o}^{N} (-1)^{n+1} A_n^N R_{n+1}(x)(x-a)^{n+1}} \, .$$

With this he obtained certain rational representations for e^x, namely

$$e^x \sim \frac{2+x}{2-x} \, ,$$

$$(3) \qquad e^x \sim \frac{12 + 6x + x^2}{12 - 6x + x^2} \, ,$$

$$e^x \sim \frac{120 + 60x + 12x^2 + x^3}{120 - 60x + 12x^2 - x^3} \, .$$

Squire mentions that these formulas were first obtained by

Hummel and Seebeck (3, p. 243-247) using a generalization of Taylor's expansion. In addition, he states that more recently Lanczos (5, p. 379-476) obtained them by a quadrature technique.

A survey article by Southard (8, p. 1-13) tabulates eighteen of the known methods for obtaining rational functions as approximations for functions of a real variable, while making most particular reference to a formula by Darboux, discussed by Whittaker and Watson (11, p. 125):

$$\phi^{(n)}(0)[f(z)-f(a)] = \sum_{m=1}^{n} (-1)^{m-1}(z-a)^{m} \left[\phi^{(n-m)}(1)f^{(m)}(z) - \phi^{(n-m)}(0)f^{(m)}(a)\right] \tag{4}$$

$$+ (-1)^{n}(z-a)^{n+1}\int_0^1 \phi(t)\, f^{(n+1)}(a+t[z-a])\, dt \; .$$

The class of functions for which the Darboux formula will give a rational approximation, he states, are those functions which satisfy the differential equation

$$f'(z) = R_1(z)\, f(z) + R_2(z) \; , \tag{5}$$

where $R_1(z)$ and $R_2(z)$ are rational functions of z. At the close of his paper Southard outlines some of the open questions with respect to application of Darboux' formula and the potential usefulness of that formula for obtaining rational approximations for complex z. In particular, for the class of functions satisfying the differential equation (1) he specifically suggests experimenting with the nth degree polynomial

$$\phi_n(t) = t^p (t-1)^{n-p}, \tag{6}$$

with p an arbitrary integral parameter, $p \geq 0$. This thesis deals with just such experimentation.

By arbitrary choice of n and p formulas can be obtained which are comparable to those obtained by Lanczos and Squire. For example, if $p = 5$, $n = 6$,

$$e^z \sim \frac{720 + 120z}{720 - 600z + 240z^2 - 60z^3 + 10z^4 - z^5}, \tag{7}$$

which for $z = 1$ yields $e \sim 2.718447$, and, if $p = 5$, $n = 7$,

$$e^z \sim \frac{2520 + 720z + 60z^2}{2520 - 1800z + 60z^2 - 120z^3 + 15z^4 - z^5}, \tag{8}$$

which yields $e \sim 2.718287$.

Insofar as comparison is being made it is interesting to note that in (6) the combination $p = 3$, $n = 6$, by Darboux' formula yields the last formula in (3), which yields $e \sim 2.7183$.

The importance of the variation of results stemming from the wide latitude of choice for n and p is that the flexibility thereby afforded will enable one to tailor the approximating function to fit a nonlinear case. It goes without saying that for various combinations of n and p, as well as for $n = p$, we obtain results for the well-documented linear case similar to those obtained by Southard, Hummel and Seebeck, Lanczos, Lotkin (6, p. 29-34), Squire, and Milne (7, p. 537-542).

The Darboux formula (4) holds if $f(z)$ is analytic at all points on the straight line from a to z and if $\phi(t)$ is any polynomial of degree n. Requiring that $n > p$ (Southard treated the case $n = p$) we obtain several special formulas. For $p = 1$,

$$f(z) \sim f(a) + \frac{1}{n} f'(z)\,(z-a) + \sum_{m=1}^{n-1} \frac{n-m}{n} \frac{f^{(m)}(a)}{m!} (z-a)^m . \tag{9}$$

It is at once evident that as $n \to \infty$ the right hand side reduces to the Taylor series representation of $f(z)$, which converges for $|z-a| < R$. The term $\frac{1}{n} f'(z)(z-a)$ may be approximated by $\frac{1}{n} \frac{f(z) - f(a)}{z - a} (z-a)$, so (9) may be written as

$$f(z) \sim f(a) + f'(a)(z-a) + \sum_{m=2}^{n-1} \frac{n-m}{n-1} \frac{f^{(m)}(a)}{m!} (z-a)^m . \tag{10}$$

A form of the approximation free of $f'(z)$ results if (9) is rewritten as

$$(11)\quad f'(z) \sim n\,\frac{f(z)-f(a)}{z-a} + \sum_{m=1}^{n-1} \frac{(m-n)\,f^{(m)}(a)}{m!}(z-a)^{m-1} .$$

The functions are analytic; integration from a to z yields

$$f(z) \sim f(a) + n\int_a^z \frac{f(s)-f(a)}{s-a}\,ds$$

$$(12)\qquad + \sum_{m=1}^{n-1} \frac{m-n}{m}\,\frac{f^{(m)}(a)}{m!}(z-a)^m$$

$$= n\int_a^z \frac{f(s)-f(a)}{s-a}\,ds - n\sum_{m=1}^{n-1} \frac{f^{(m)}(a)}{m\,m!}(z-a)^m$$

$$+ \sum_{m=o}^{n-1} \frac{f^{(m)}(a)}{m!}(z-a)^m .$$

The last term is the nth partial sum of the Taylor series expansion of $f(z)$. As $n \rightarrow \infty$ the Taylor series becomes an exact representation of $f(z)$, hence the limit of the sum of the other terms must be zero. That is,

$$(13)\quad \lim_{n\rightarrow\infty} n\left[\int_a^z \frac{f(s)-f(a)}{s-a}\,ds - \sum_{m=1}^{n-1} \frac{f^{(m)}(a)}{m\,m!}(z-a)^m\right] = 0,$$

or

$$(14)\quad \int_a^z \frac{f(s)-f(a)}{s-a}\,ds = \sum_{m=1}^{\infty} \frac{f^{(m)}(a)}{m\,m!}(z-a)^m .$$

This series also converges for $|z-a| < R$. Combining (14) and (12) we get

$$f(z) \sim n \sum_{m=1}^{\infty} \frac{f^{(m)}(a)}{m\, m!}(z-a)^m - n \sum_{m=1}^{n-1} \frac{f^{(m)}(a)}{m\, m!}(z-a)^m \tag{15}$$
$$+ \sum_{m=o}^{n-1} \frac{f^{(m)}(a)}{m!}(z-a)^m ,$$

or

$$f(z) \sim n \sum_{m=n}^{\infty} \frac{f^{(m)}(a)}{m\, m!}(z-a)^m + \sum_{m=o}^{n-1} \frac{f^{(m)}(a)}{m!}(z-a)^m . \tag{16}$$

That is, the first term on the right replaces the terms of the Taylor series expansion from $m = n$ and beyond, saying in effect that

$$\lim_{n\to\infty} n \sum_{m=n}^{\infty} \frac{f^{(m)}(a)}{m\, m!}(z-a)^m = \lim_{n\to\infty} \sum_{m=n}^{\infty} \frac{f^{(m)}(a)}{m!}(z-a)^m , \tag{17}$$

which is easily verified. We note that (14) represents the definite integral of the Taylor series expansion of $f(z)$.

The expansion of a meromorphic function in an infinite series of rational functions, as discussed in Copson (1, p. 147) or Titchmarsh (10, p. 110), can be extended in the following way:

(i) let $f(z)$ be a meromorphic function having poles of first order, $z_1, z_2, \cdots, z_n$, with residues $b_1, b_2, \cdots, b_n$;

(ii) order the poles by their moduli,

$$0 \le |z_1| \le |z_2| \le \cdots\cdots\cdots \le |z_n| ; \tag{18}$$

finally, form the integral

$$(19) \qquad I = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)(w-z)}\, dw\ ,$$

where C is a circle of radius R enclosing all singularities. The residues are quickly evaluated, so

$$(20) \qquad I = \frac{f(z) - f(a)}{z - a} + \sum_{k=1}^{n} \frac{b_k}{(z_k - z)(z_k - a)}\ .$$

As $R \to \infty$ the number of poles could become arbitrarily large; if $f(w)$ is bounded $\lim_{R \to \infty} I = 0$, so

$$(21) \qquad \frac{f(z) - f(a)}{z - a} = - \sum_{k=1}^{\infty} \frac{b_k}{(z_k - z)(z_k - a)}$$

or

$$(22) \qquad f(z) = f(a) - \sum_{k=1}^{\infty} b_k \left[\frac{1}{z_k - z} - \frac{1}{z_k - a}\right].$$

The number of poles and the number of terms of the series might well be finite, providing a useful way of representing an analytic function exactly in terms of rational functions. Even if the number of poles were quite large there might be suitable approximations to $f(z)$ by retaining relatively few residues.

Equation (22) suggests another way of attacking the integral of (14). If $f(z)$ satisfies the given conditions

$$(23)\qquad n\int_a^z \frac{f(s)-f(a)}{s-a}\,ds = n\sum_{k=1}^{\infty}\int_a^z \frac{b_k}{(s-z_k)(z_k-a)}\,ds$$

$$= n\log\prod_{k=1}^{\infty}\left[\frac{z-z_k}{a-z_k}\right]^{\frac{b_k}{z_k-a}} .$$

From (11),

$$f(z) \sim n\log\prod_{k=1}^{\infty}\left[\frac{z-z_k}{a-z_k}\right]^{\frac{b_k}{z_k-a}} - n\sum_{m=1}^{n-1}\frac{f^{(m)}(a)}{m\,m!}(z-a)^m$$

$$(24)\qquad + \sum_{m=1}^{n-1}\frac{f^{(m)}(a)}{m!}(z-a)^m ,$$

and, formally at least,

$$(25)\qquad \log\prod_{k=1}^{\infty}\left[\frac{z-z_k}{a-z_k}\right]^{\frac{b_k}{z_k-a}} \sim \sum_{m=1}^{n-1}\frac{f^{(m)}(a)}{m\,m!}(z-a)^m .$$

Finally, as $n\to\infty$,

$$(26)\qquad \prod_{k=1}^{\infty}\left[\frac{z-z_k}{a-z_k}\right]^{\frac{b_k}{z_k-a}} = \exp\sum_{m=1}^{\infty}\frac{f^{(m)}(a)}{m\,m!}(z-a)^m ,$$

a by-product formula in which the rather special function on the right is expressed as a product.

For the more general case, $1 < p < n$, Darboux' formula (4) may be written out as

$$f(z) \sim f(a) + \sum_{m=1}^{n-p} (-1)^{m-1} \frac{\binom{p}{m}}{(n)_m} f^{(m)}(z)\,(z-a)^m$$

(27)

$$+ \sum_{m=1}^{n-p} \frac{(n-p)_m}{(n)_m} \frac{f^{(m)}(a)}{m!} (z-a)^m ,$$

where

(28) $$(n)_m = n(n-1)\cdots\cdots(n-m+1), \quad m \geq 1.$$

If, in (27), the $f^{(m)}(z)$ is replaced by $\dfrac{f^{(m-1)}(z) - f^{(m-1)}(a)}{z-a}$

an inductive process quickly yields

(29) $$f^{(m)}(z) = \frac{f(z) - \sum_{k=o}^{m-1} f^{(k)}(a)\,(z-a)^k}{(z-a)^m} ,$$

hence the Darboux formula goes as

$$f(z) \sim f(a) + \sum_{m=1}^{n-p} (-1)^{m-1} \frac{\binom{p}{m}}{(n)_m} \left[f(z) - \sum_{k=o}^{m-1} f^{(k)}(a)\,(z-a)^k \right]$$

(30)

$$+ \sum_{m=1}^{n-p} \frac{(n-p)_m}{(n)_m} \frac{f^{(m)}(a)}{m!} (z-a)^m .$$

This rather clumsy result could probably be reduced in form but it should be sufficient to write down some formulas which result for special values of p. For the case $p = 1$, already discussed, (30) takes the form

$$(31)\quad f(z) \sim f(a) + \frac{f(z) - f(a)}{z - a} + \sum_{m=1}^{n-1} \frac{n-m}{n} \frac{f^{(m)}(a)}{m!} (z-a)^m ,$$

which reduces at once to

$$(32)\qquad f(z) \sim f(a) + \sum_{m=1}^{n-1} \frac{n-m}{n-1} \frac{f^{(m)}(a)}{m!} (z-a)^m ,$$

precisely the form of (10).

For $p = 2$,

$$f(z) \sim f(a) + \frac{f'(a)(z-a)}{n^2 - 3n + 3} + \sum_{m=1}^{n-2} \frac{(n-m)(n-m-1)}{n^2 - 3n + 3} \frac{f^{(m)}(a)}{m!} (z-a)^m$$

(33)

$$= f(a) + f'(a)(z-a) + \sum_{m=2}^{n-2} \frac{(n-m)(n-m-1)}{n^2 - 3n + 3} \frac{f^{(m)}(a)}{m!} (z-a)^m .$$

For $p = 3$,

$$f(z) \sim f(a) + f'(a)(z-a) \frac{3n - 7}{n^3 - 6n^2 + 14n - 13} - \frac{f''(a)\,(z-a)^2}{n^3 - 6n^2 + 14n - 13}$$

(34)

$$+ \sum_{m=1}^{n-3} \frac{(n-m)(n-m-1)(n-m-2)}{n^3 - 6n^2 + 14n - 13} \frac{f^{(m)}(a)}{m!} (z-a)^m$$

$$= f(a) + f'(a)(z-a) + \frac{n^3 - 9n^2 + 26n - 26}{n^3 - 6n^2 + 14n - 13} \frac{f''(a)}{2!} (z-a)^2$$

$$+ \sum_{m=3}^{n-3} \frac{(n-m)(n-m-1)(n-m-2)}{n^3 - 6n^2 + 14n - 13} \frac{f^{(m)}(a)}{m!} (z-a)^m .$$

Other formulas of similar nature have been developed but not included in the present work. A section on applications of these ideas appears to be more interesting than additional formulations.

APPLICATIONS

Some applications of the theory for the simple differential equation

$$(35) \qquad y'(z) - y(z) = 0\,, \qquad y(0) = 1,$$

have already been indicated for n = p, the set of rational fractions in (3) published by Squire. Some variations of these approximations for $p \neq n$ are obtained quickly from the formulations above; if a = 0, n = 6 in (9), the case p = 1, it follows at once that

$$(36) \qquad y = 1 + \frac{y+5}{6} z + \frac{z^2}{3} + \frac{z^3}{12} + \frac{z^4}{72} + \frac{z^5}{720}$$

or

$$(37) \qquad e^z \sim \frac{720 + 600z + 240z^2 + 60z^3 + 10z^4 + z^5}{120(6 - z)}\,.$$

This rational approximation function yields $e \sim 2.718332$. The formulas of (7) and (8) are obtained by choosing a = 0, p = 5, n = 6 and 7 in the formula of (27).

It has not been pointed out by the various authors cited that certain recursion methods simplify and greatly extend the range of this type of approximation. The Darboux formula for $n = p$ has been studied extensively by Dyche (2, p. 1-18). More general formulas have been obtained but let the special case $n = 2$ be considered in the study below:

$$(38)\qquad f(z) = f(a) + \frac{z-a}{2}[f'(a) + f'(z)] + \frac{(z-a)^2}{12}[f''(a) - f''(z)] .$$

The solution of the differential equation

$$(39)\qquad y'(z) - y(z) = 0 , \qquad y(a) = b ,$$

is obtained easily from (38) since $y'' = y' = y$. That is,

$$(40)\qquad y = b\,\frac{1 + \frac{z-a}{2} + \frac{(z-a)^2}{12}}{1 - \frac{z-a}{2} + \frac{(z-a)^2}{12}} ,$$

which, for $a = 0$, $b = 1$, is the form of the second approximation in (3). But note that a kind of analytic continuation is at once possible if the value of y at the first point, say $z = 1$, is used to evaluate y at $z = 2$, then repeating the process. Equation (40) then takes the form of a recursion formula (or difference equation), namely

$$(41) \qquad y_{k+1} = y_k \frac{1 + \frac{z-a}{2} + \frac{(z-a)^2}{12}}{1 - \frac{z-a}{2} + \frac{(z-a)^2}{12}} , \qquad y_o = 1 .$$

The solution is at once

$$(42) \qquad y_k = \left\{ \frac{1 + \frac{z-a}{2} + \frac{(z-a)^2}{12}}{1 - \frac{z-a}{2} + \frac{(z-a)^2}{12}} \right\}^k$$

for general values of z-a. If, in particular, z-a = 1,

$$(43) \qquad y_k = \left(\frac{19}{7}\right)^k = (2.71428)^k ,$$

and a special case is $y_{10} = 21{,}700$, which compares well with the exact value $e^{10} = 22{,}076$. The case n = 3, as studied by Lanczos, would yield

$$(44) \qquad y_k = \left(\frac{193}{71}\right)^k$$

which would give an approximation to e^{100} correct to three significant figures. Few approximation techniques are valid over such a wide range.

Second order differential equations with polynomial coefficients are still of considerable practical interest. Let the next example be the linear Bessel differential equation of zero order,

$$(45) \quad xy''(x) + y'(x) + xy(x) = 0, \; y(0) = 1, \; y'(0) = 0 .$$

Let the independent variable be the real number x and, for convenience, let $y'(x) = z(x)$. The formula of (38) must be applied twice. From the differential equation,

$$
(46) \qquad
\begin{aligned}
y'(x) &= z, & z'(x) &= -\frac{z + xy}{x}, \\
y''(x) &= -\frac{z + xy}{x}, & z''(x) &= \frac{xy + (2-x^2)z}{x^2},
\end{aligned}
$$

so, if

$$
(47) \qquad y(a) = b, \qquad z(a) = c,
$$

the two Darboux representations have the form

$$
(48) \qquad
\begin{aligned}
y &= b + \frac{x-a}{2}(c+z) + \frac{(x-a)^2}{12}\left[-\frac{c+ab}{a} + \frac{z+xy}{x}\right], \\
z &= c + \frac{x-a}{2}\left[-\frac{c+ab}{a} - \frac{z+xy}{x}\right] + \frac{(x-a)^2}{12}\left[\frac{ab+(2-a^2)c}{a^2} - \frac{xy+(2-x^2)z}{x^2}\right]
\end{aligned}
$$

and the recursion notation leads to

$$
(49) \qquad
\begin{aligned}
&(y_{k+1} - y_k)\left[1 - \frac{(x-a)^2}{12}\right] - z_{k+1}\frac{x-a}{2}\left[1 + \frac{x-a}{6x}\right] - z_k\frac{x-a}{2}\left[1 - \frac{x-a}{6a}\right] \\
&\qquad = 0, \\
&\frac{x-a}{2}\left[y_{k+1}\left(1 + \frac{x-a}{6x}\right) + y_k\left(1 - \frac{x-a}{6a}\right)\right] \\
&\quad + z_{k+1}\left[1 + \frac{x-a}{2x} + \frac{(x-a)^2(2-x^2)}{12x^2}\right] - z_k\left[1 - \frac{x-a}{2a} + \frac{(x-a)^2(2-a^2)}{12a^2}\right] \\
&\qquad = 0.
\end{aligned}
$$

Elimination of z_{k+1} yields

$$y_{k+1}\left\{\frac{1-\dfrac{(x-a)^2}{12}}{\dfrac{x-a}{2}\left(1+\dfrac{x-a}{6x}\right)}+\frac{\dfrac{x-a}{2}\left(1+\dfrac{x-a}{6x}\right)}{1+\dfrac{x-a}{2x}+\dfrac{(x-a)^2(2-x^2)}{12x^2}}\right\}$$

(50)

$$=y_k\left\{\frac{1-\dfrac{(x-a)^2}{12}}{\dfrac{x-a}{2}\left(1+\dfrac{x-a}{6x}\right)}-\frac{\dfrac{x-a}{2}\left(1+\dfrac{x-a}{6a}\right)}{1+\dfrac{x-a}{2x}+\dfrac{(x-a)^2(2-x^2)}{12x^2}}\right\}$$

$$+z_k\left\{\frac{1-\dfrac{x-a}{6a}}{1+\dfrac{x-a}{6x}}+\frac{1-\dfrac{x-a}{2a}+\dfrac{(x-a)^2(2-a^2)}{12a^2}}{1+\dfrac{x-a}{2x}+\dfrac{(x-a)^2(2-a^2)}{12x^2}}\right\}.$$

Similarly,

$$z_{k+1}\left\{\frac{1+\dfrac{x-a}{2x}+\dfrac{(x-a)^2(2-x^2)}{12x^2}}{\dfrac{x-a}{2}\left(1+\dfrac{x-a}{6x}\right)}+\frac{\dfrac{x-a}{2}\left(1+\dfrac{x-a}{6x}\right)}{1-\dfrac{(x-a)^2}{12}}\right\}$$

(51)

$$=-y_k\left\{1+\frac{1-\dfrac{x-a}{6a}}{1+\dfrac{x-a}{6x}}\right\}+z_k\left\{\frac{1-\dfrac{x-a}{2a}+\dfrac{(x-a)^2(2-a^2)}{12a^2}}{1+\dfrac{x-a}{2x}\quad\dfrac{(x-a)^2(2-x^2)}{12x^2}}-\frac{\dfrac{x-a}{2}\left(1-\dfrac{x-a}{6a}\right)}{1-\dfrac{(x-a)^2}{12}}\right\}.$$

Although clumsy in appearance note that the values of $y(x)$ and $z(x)$ are given for integer multiples of $x-a$ in terms of the values of $y(x)$ and $z(x)$ at the preceding point. Such an operation lends itself well to digital equipment. Because of indeterminant forms the case $a = 0$ must be worked out separately,

$$(52)\quad y(x) = \frac{(80-x^2)(6-x^2)}{2(240+17x^2+x^4)}, \quad z(x) = \frac{-x(240-13x^2)}{2(240+17x^2+x^4)},$$

$$x \geq 0 .$$

Then $y(1) = \frac{395}{516} = 0.7655$ and $z(1) = -\frac{227}{516} = -0.4399$,

which are off three and two places each in the last figure. If, again, $z-a = 1$, then

$$(53)\quad \begin{aligned} y_{k+1} &= y_k \frac{11a(11x^2+6x+2) - x(6x+1)(6a-1)}{a(157x^2+78x+23)} \\ &+ z_k \frac{a(6a-1)(11x^2+6x+2) + x(6x+1)(11a^2-6a+2)}{a^2(157x^2+78x+23)} \end{aligned}$$

and

$$(54)\quad \begin{aligned} z_{k+1} &= -x\, y_k \frac{11(12ax-1)}{a(157x^2+78x+23)} \\ &+ x\, z_k \frac{11x(11a^2-6a+2) - a(6x+1)(6a-1)}{a^2(157x^2+78x+23)} . \end{aligned}$$

In particular, $y(2) = \frac{395(508) - 227(472)}{516(807)} = \frac{23{,}379}{104{,}103}$ $= 0.2246$ and $z(2) = -\frac{395(253) + 227(89)}{516(807)} = -\frac{60{,}069}{104{,}103}$ $= -0.5770$. The exact values are 0.2239 and -0.5767. The remarkable thing is that the recursion may be extended at least as far as $k = 10$, the last pair of entries being

$y(10) = -0.2467$ and $z(10) = -0.0276$. The exact values are -0.2459 and -0.0435 respectively.

Instead of recursion relationships the approximation might well be obtained as a formal closed expression. Consider the case

$$(55) \qquad y''(x) + y(x) = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

The pair of Darboux recursion formulas is at once

$$(56) \qquad \begin{aligned} &(y_{k+1} - y_k)\left[1 - \frac{(x-a)^2}{12}\right] - \frac{x-a}{2}(z_{k+1} + z_k) = 0, \\ &(y_{k+1} + y_k)\frac{x-a}{2} + (z_{k+1} - z_k)\left[1 - \frac{(x-a)^2}{12}\right] = 0, \\ &\qquad y_o = 1, \quad z_o = 0. \end{aligned}$$

For $z-a = 1$ the solution of the pair of difference equations

$$(57) \qquad \begin{aligned} &11(y_{k+1} - y_k) - 6(z_{k+1} + z_k) = 0, \\ &6(y_{k+1} + y_k) + 11(z_{k+1} - z_k) = 0, \\ &\qquad y_o = 1, \quad z_o = 0, \end{aligned}$$

may be quickly obtained by a Laplace transform procedure,

$$(58) \qquad y_k = \cos k\left(\cos^{-1}\frac{85}{157}\right), \qquad z_k = -\sin k\left(\sin^{-1}\frac{132}{157}\right).$$

Note that for all values of k

$$(59) \qquad y_k^2 + z_k^2 = 1 .$$

For $k = 1, 2$, and 3 the values of y_k are 0.5414, -0.4134, and -0.9894, while the exact values are 0.5403, -0.4162, and -0.9900. It is amusing to note that for any desired angle measured by a rational number (58) yields a Pythagorean triangle approximation, precisely the method of constructing a table of sinusoidal functions practiced by the Babylonians of 2000 B.C.

These results might well be compared with a paper by Kulikov (4, p. 1135-1143), who devised an approximation technique for the second order linear equation

$$(60) \qquad \alpha y''(x) + f(x)\, y'(x) + F(x)\, y(x) = 0 , \qquad \alpha = \text{constant}.$$

He specifically studied the Bessel equation (45) but achieved no such range of approximation as $0 < x < 10$. Also, his procedures seem to be quite special and not immediately applicable to differential equations of order higher than two.

Of course any approximation technique should be tested against nonlinear equations. Something was done along such lines by Lotkin, using a combination of the Darboux formula and iteration procedures. Perhaps the best field for future extensions of the present technique lies in the study of those iteration techniques which are primarily based on the usual Taylor series expansions. And the problem of estimation of error is virtually untouched.

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