SEMIGRAPHICAL DESIGN OF SERVO-MECHANISMS
USING INVERSE ROOT LOCUS

by

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A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

June 1962
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Date thesis is presented May 16, 1962

Typed by Ramona Nestell
ACKNOWLEDGEMENT

I am particularly indebted to Dr. and Mrs. James M. Unosawa and family, who have made it possible for me to study in the United States and who have greatly encouraged me during my course of study.

I am also deeply grateful to Professor Louis N. Stone, Head of the Electrical Engineering Department, Oregon State University, who has been my instructor for the last two years and patiently helped me with many suggestions and ideas.

I would also like to express my deepest appreciation to my friends here in the United States and especially to Mr. Donald M. Takeuchi who has patiently read my thesis and corrected my English.

Without the help of these people, this thesis could never have been completed.
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I. INTRODUCTION

A number of approaches have been taken to the synthesis problem. The historically older approaches are represented by the frequency response methods (7, p. 246-362) of Nquist, Bode, and Nichols. Today the root-locus methods (9, 19, 21, 26) of Evans are widely used for the synthesis problem. Walter (28) has shown that lead networks can be selected by simple angular measurement to cause a root-locus to pass through any single location of the closed-loop transfer function. Guillemin (26) suggests a method which requires that (i) Compensation poles lie on the negative real axis, (ii) Plant poles are canceled by compensation zeros, except for a pole at the origin, and (iii) Compensation poles are determined graphically in order for some measure of control to be retained over the final results. Zaborszyk (29) suggests an iteration process, alternating between direct and inverse root-locus plots with small changes in the open-loop transfer function or the closed-loop transfer function for each step. At the end, a compromise may be reached which provides the poles of the fixed element, \( G_1(s) \), in the open-loop, and a closed-loop transfer function, \( G_0(s) \), which meets specifications on transient response, bandwidth, etc. Hausenbauer and Lago (14) present the synthesis problem for third-order systems. Design curves are given relating constant velocity, \( K_V \), bandwidth, and transient performance to pole and zero locations. Open-loop poles thus determined are real and do not necessarily coincide with prescribed poles which are
cancelled by compensation zeros. Aseltine (1) introduced an inverse root-locus method that graphically determines the open-loop pole locations from a given closed-loop pole-zero configuration. Aseltine (2) also presents the synthesis problem by using the inverse root-locus method which is based on the fact that the gain on an inverse root-locus plot must have the same value at all required open-loop poles. Aseltine's method (2) requires the poles of the open-loop transfer function to be real and the locations of the open-loop poles are calculated by the magnitude equation of the coincident equations and equation (6).

In this thesis, a semi-graphical design method using inverse root-locus is presented for determination of the compensation networks when the fixed open-loop poles and zero, specified closed-loop dominant poles, and a specified velocity constant, $K_v$, are given.

The following topics are discussed:

1. The system is restricted to a type 1 which is compensated by a simple lead, or lag network and tachometer transfer function.

2. The relationships between the velocity constant, $K_v$, and the pole-zero configuration.

3. The relation between the closed-loop pole-zero configuration and the approximate transient response.


5. The relation between the angle which makes the inverse root-locus pass through the selected point and the location of the dominant poles by phase-angle loci and equation (22).
6. Cascade compensation using the inverse root-locus method in the system which has the following features: (1) The dominant poles \((\omega_n, \gamma)\) of the closed-loop system are specified. (2) The fixed element \(G_1(s)\) of the open-loop system is specified. (3) The velocity constant \(K_v\) is specified.

7. The feedback compensation using the inverse root-locus method.

The method described here does not need pole cancellation, therefore a simple compensation can be used.

The locations of the pole-zero of the open- and closed-loop system in this method are calculated by using equations (5), (6), (7), (31), and the aid of \(s\)-plane, phase-angle loci \(K_o\) and equation (22).
II. REVIEW OF THE BASIC THEORY

1. The Relationship between the Velocity Constant \( K_v \) and the Poles and Zeros

The Type I system is characterized by the location of a pole at the origin. The over-all system function is governed by two general factors: (1) the fixed components and (2) the performance specifications, or the compensation network.

The relation between the velocity constant, \( K_v \), and the poles and zeros of the open- and closed-loop transfer function will be determined for a unity feedback system.

Let us consider the system of Figure 1.

![Simple single-loop feedback control system.](image)

The assumptions are that, (1) no multiple poles exist, (2) the order of the denominator of the closed-loop transfer function is larger than that of the numerator, and (3) the open- and closed-loop transfer functions are in the following factored forms:

\[
G(s) = G_1(s) G_2(s) = K \frac{(s + z_2)(s + z_3) \cdots (s + z_m)}{s(s + p_2)(s + p_3) \cdots (s + p_n)}
\]  

(1)

\[
G_c(s) = \frac{C(s)}{R(s)} = K_c \frac{\prod_{i=1}^{m} (s + x_i)}{\prod_{i=1}^{n} (s + \beta_i)}
\]  

(2)
where $S$ = a complex variable

$$q_j (i = 1, 2, \ldots, n) = \text{poles of closed-loop transfer function } G_0(s)$$

$$p_i (i = 1, 2, \ldots, n) = \text{poles of open-loop transfer function } G(s)$$

$$Z_j (i = 1, 2, \ldots, m) = \text{zeros of open- and closed-loop transfer function.}$$

The zeros of the open-loop transfer function are identical with the zeros of the closed-loop transfer function.

If the transfer function of the fixed component is

$$G_1(s) = \frac{K_1 (S + Z_1)(S + Z_2) \cdots (S + Z_m)}{S(S + P_1)(S + P_2) \cdots (S + P_n)}$$  \hspace{1cm} (3)$$

then, the transfer function of the compensation network is

$$G_2(s) = \frac{K}{K_1} \frac{(S + Z_1)(S + Z_2) \cdots (S + Z_m)}{(S + P_1)(S + P_2) \cdots (S + P_n)}$$  \hspace{1cm} (4)$$

In the complex plane, the poles of the closed-loop transfer function are marked by $\mathbb{E}$, the poles of the open-loop transfer function are marked by $x$, and the zeros of the open- and closed-transfer function by $0$.

The velocity constant, $K_v$, is defined by the following equations:

The velocity constant, $K_v$, can be related to the poles and zeros of the open-loop transfer function, $G(s)$, by the relation

$$K_v = \lim_{s \to 0} sG(s)$$

By Guillemin (26, p. 282), the relation between the velocity constant,
and the poles and zeros of the closed-loop transfer function is determined by:

\[
\frac{1}{K_v} = \sum_{i=1}^{n} \frac{1}{q_i} - \sum_{i=1}^{m} \frac{1}{Z_i} \tag{6}
\]

The velocity constant, \(K_v\), can be also related to the poles of the open- and closed-loop transfer function by the relation:

\[
K_v = \frac{\varphi_1 \varphi_2 \cdots \varphi_n}{p_2 p_3 \cdots p_n} \tag{7}
\]

Equations (6) and (7) represent the correlation between the velocity constant, \(K_v\), and the system response characteristics.

The correlation of the velocity constant and the system response characteristics is of basic importance in the servo synthesis.

2. The Closed Loop Pole-Zero Configuration and Transient Response

The transient response of the system is determined by the pole-zero configuration of the closed-loop system on the S-plane. The principal characteristics of the transient response of the system are fixed by one pair of conjugate complex poles; the dominant poles \((\omega_n, \gamma)\), \(-q_1\) and \(-q_2\) in Figure 2. The desired response, as shown in Figure 3, implies the important assumption that a pair of dominant complex poles exists.

The system characteristics are determined by the two dominant poles \((\omega_n, \gamma)\), \(-q_1\) and \(-q_2\), and any zeros in the significant part of the plane (16, 26).

The typical pole-zero configuration, as shown in Figure 2, requires either one of the following (9, 16, 17, 26):
Figure 2. Pole-zero configuration for the desired response.

Figure 3. The desired transient response to unit step function.
1. The other poles must be far away to the left of the dominant poles so that the transient due to these poles dies out very rapidly.

2. Any other poles near the imaginary axis must be near a zero since the magnitude of the transient term due to that pole is very small.

The approximate peak overshoot, $M_p$, and time to peak, $T_p$, are defined by (8, 27):

\[
T_p = \frac{1}{\omega_n \sqrt{1 - \gamma^2}} \left[ \pi \sum_{i=1}^{m} \frac{\left| \frac{g_i - Z_i}{Z_i} \right|}{2} + \sum_{i=3}^{n} \frac{\left| \frac{Z_i - g_i}{|g_i - Z_i|} \right|}{2} \right] \tag{8}
\]

\[
M_p = \frac{m}{n} \prod_{i=1}^{m} \left| \frac{g_i - Z_i}{Z_i} \right| \prod_{i=3}^{n} \left| \frac{Z_i - g_i}{|g_i - Z_i|} \right| e^{-\omega_n \gamma T_p} \tag{9}
\]

where \( \sum_{i=1}^{m} \frac{1}{Z_i - g_i} \) is the sum of the angles from the zeros to the dominant complex pole, \( \sum_{i=3}^{n} \frac{1}{Z_i - g_i} \) is the sum of the angles from the poles to the dominant complex pole, \( \prod_{i=1}^{m} \left| \frac{g_i - Z_i}{Z_i} \right| \) is the product of the ratios of the absolute distances between the zeros and the dominant complex pole and the values of the zeros, \( \prod_{i=3}^{n} \left| \frac{Z_i - g_i}{|g_i - Z_i|} \right| \) is the product of the ratios of the poles to the absolute distances between the poles and the dominant complex pole.

From equations (8) and (9) we can conclude that (8, 9):


2. Additional closed poles increase $T_p$.

3. $T_p$ is inversely proportional to the undamped natural frequency $\omega_n$ of the dominant poles.

4. As $T_p$ decreases, $M_p$ increases.
5. As the damping ratio $\gamma$ decreases, $M_p$ increases.

6. At times, the choice of $T_p$ and $M_p$ requires a compromise since $T_p$ or $M_p$ is not always desirable.

To avoid the calculation of $T_p$ and $M_p$, the charts (11, p. 122-127) which express the correlation between the pole-zero configuration and time response might be utilized.
III. INVERSE ROOT LOCUS

1. The Inverse Root-Locus Method

The root-locus method, as suggested by Evans, is used for the determination of the closed loop poles of a negative feedback system from the location of the poles and zeros of the open-loop system. The inverse root-locus method, as suggested by Aseltine, involves the determination of the open-loop poles from a knowledge of the closed loop poles. The definition of the inverse root loci is that the inverse root loci are plots of the variations of the poles of the open loop system function with changes in the closed loop gain.

The closed loop transfer function is defined by

\[ G_c(s) = \frac{C}{R(s)} = \frac{G(s)}{1 + G(s)} \]  \hspace{1cm} (10)

Once the closed loop transfer function is established by specifying its poles and zeros, the open-loop transfer function is determined by the inversion of equation (10) as

\[ G(s) = \frac{G_c(s)}{1 - G_c(s)} \]  \hspace{1cm} (11)

The inverse root-locus method is used to determine the positions of the roots of

\[ 1 - G_c(s) = 0 \]  \hspace{1cm} (12)

The gain condition for the inverse root-locus is determined by

\[ |G_c(s)| = 1 \]  \hspace{1cm} (13)
The angle condition is

$$\angle G_c(s) = n\pi$$  \hspace{1cm} (14)

where \(n\) is even.

Equations (13) and (14) are equivalent to the following two equations if they are expressed as equation (2)

$$K_c \prod_{i=1}^{m} \frac{|s+z_i|}{|s+p_i|} = 1$$  \hspace{1cm} (15)

$$\sum_{i=1}^{m} \frac{1}{s+z_i} - \sum_{i=1}^{n} \frac{1}{s+p_i} = n\pi \hspace{1cm} (n \text{ even})$$  \hspace{1cm} (16)

Equation (14), or equation (16), describes an inverse root-locus on which the sum of the angles is 0 degrees or 360 degrees. The gain and phase conditions are called the coincident condition for the inverse root locus.

The rules and relationships listed below permit accurate construction of certain parts of the inverse root-loci and indicate the approximate location of the other parts (9, 26, 29).

1. The locus is symmetrical with respect to the real axis.
2. The loci starting points are at the closed-loop poles and the loci ending points are at the closed-loop zeros or at infinity.
3. The number of branches going to infinity equals \(n - m\),

\[
\begin{align*}
    n &= \text{number of the closed-loop poles} \\
    m &= \text{number of the closed-loop zeros} \\
    \text{and if } n > m.
\end{align*}
\]

4. The branches going out to infinity will approach asymptotes
which make the following angles with the real axis

$$\theta_k = \frac{360^\circ}{n-m}$$  \hspace{1cm} (17)

where \( k = 0, 1, 2, \ldots, (n - m - 1) \) and \( m < n \).

5. There is always at least one branch \((k = 0)\) going to infinity. This branch is the section of real axis lying to the right of the extreme right real pole or zero of the closed-loop transfer function, \( G_c(s) \). (This point is different from the rules of root loci.)

6. All asymptotes meet at one point on the real axis, i.e., at the centroid.

$$\bar{c}_c = \frac{\sum_{i=1}^{n} \text{Re}(p_i) - \sum_{i=1}^{m} \text{Re}(z_i)}{n-m}$$  \hspace{1cm} (18)

7. When \( n-m > 2 \) \((26, p. 302)\)

$$\sum_{i=1}^{n} \frac{p_i}{\bar{c}_c} = \sum_{i=1}^{n} \frac{p_i}{L_i}$$  \hspace{1cm} (19)

8. If the total number of the closed-loop poles and zeros to the right of the \( S \) point on the real axis is even, then this point lies on the locus.

Examples: Real axis loci of the inverse root-locus.

![Real axis loci of the inverse root-locus](image)

Note: There is always one branch going to a positive infinity \((\text{Rule #5})\).
9. Consider the case where the loci have branches on the real axis between two closed-loop poles. There must be a point at which the loci break away from the real axis and enter into the complex region of the $S$-plane in order to approach zeros or the point of infinity.

Example:

\[ W(s) = \frac{1}{\prod_{i=1}^{m} (s + q_i)} = K \]

The breakaway point satisfies the magnitude condition

\[ s = \sigma + j\omega \]

Thus by taking the derivative of $W(s)$ and setting it equal to zero, the point $\sigma$ can be determined.

\[ \frac{dW}{d\sigma} = \frac{1}{\prod_{i=1}^{m} (\sigma + q_i)} = 0 \]

The point $\sigma$ is the root of this equation.

10. In calculating the breakaway point, the magnitude condition for inverse root-locus can be rewritten in the following form.

11. Similarly, a break-in point must be present when the locus exists between two zeros on the real axis.
Examples:

12. The break-in point can be calculated in the same way as the breakaway point.

13. It is also possible to have breakaway and break-in points between a pole and a zero on real axis.

Example:

14. The initial angle of the locus at a complex pole is the sum of the angles of all vectors connecting all poles and zeros to the complex poles in question. Angles of vectors from zeros are taken positive and those from poles are taken negative.

15. The initial angle of the locus at a complex zero is equal to the sum of the angles of all vectors connecting all poles and zeros to the complex poles in question. Angles of vectors from zeros are taken negative and those from poles are taken positive.

16. At real single poles or zeros the locus starts along the real axis.
17. At real double poles or zeros the two branches of the locus start either along the real axis or perpendicular to it.

18. At very low closed-loop gain the open-loop poles almost coincide with the closed-poles and with increasing gain they gradually shift to the closed-loop zeros or to infinity. The number of closed- and open-loop poles is the same with unity feedback (Figure 1 where $m < n$).

Applications of these rules and relationships to facilitate construction of the inverse root-loci is shown in Figure 4.
Figure 4. (a) Inverse root loci for

\[ G_0(s) = \frac{K_0}{s + a} \]

Figure 4. (b) Inverse root loci for

\[ G_0(s) = \frac{K_0}{(s + a)(s + b)} \quad a < b \]
Figure 4. (c) Inverse root loci for

\[ G_c(s) = \frac{K_c (s + b)}{(s + a) (s + c)} \quad \text{for} \quad a < b < c \]

Figure 4. (d) Inverse root loci for

\[ G_c(s) = \frac{K_c}{s^2 + 2 \tilde{\omega}_n \omega_n + \omega_n^2} \]
Figure 4. (e) Inverse root loci for

$$G_c(s) = \frac{K_c(s + c)}{s^2 + 2 \gamma \omega_n + \omega_n^2} \quad c > \Re(s)$$

Figure 4. (f) Inverse root loci for

$$G_c(s) = \frac{K_c(s + a)}{s^2 + 2 \gamma \omega_n + \omega_n^2} \quad a < \Re(s)$$
Figure 4. (g) Inverse root loci for

\[ G_c(s) = \frac{K_c}{(s^2 + 2 \sum \omega_n + \omega_n^2) (s + a)} \quad a < R_e(q_1) \]
Figure 14. (h) Inverse root loci for $G_0(s) = \frac{K(s + b)}{(s^2 + 2\gamma \omega_n s + \omega_n^2)(s + a)}$

\[ a > b \quad a = R_0(q_1) \]
The Angle $\Phi_0$ and the Phase-Angle Loci $\Phi_0$

The locus of roots of the denominator of equation (11) can be drawn by using the inverse root-locus method. For a given value of gain, a set of root on the loci will be obtained $(2, 2h)$.

In simple cases, such as in problems in which the poles and zeros of the closed-loop transfer function are on the real axis, the construction of the inverse root-locus is done by using the rules previously described. In more complicated cases, where the closed-loop transfer function has complex conjugate poles and in systems where the approximate locations of the inverse root-loci are not apparent, the construction of the inverse root-loci is done by using the phase-angle loci $(8)$, with the angle $\Phi_0$ equal to $n\pi 60 \ (n = 0, 1, 2, ....)$ degrees.

The design must force the roots to lie at the selected points $s_o$ and $\Phi_o$ which are complex poles of the open-loop transfer function. In general, an infinite number of pole-zero combinations of the closed-loop transfer function can force the inverse root-locus to pass through the selected point $s_o$ $(19, 20, 26)$. It is assumed that a single pole and zero of the closed-loop transfer function makes the inverse root-locus pass through the selected point $s_o$. When there are no restrictions on the locations of the compensator pole and zero, and no gain restriction, a root may be located at any point in the $s$-plane by use of a single-pole, single zero compensator $(20)$.

Let us consider the angle contributed at a point $s_o$ in the $s$-plane by a pair of complex poles, $s^*$ and $\Phi^*$, of the closed-loop
transfer function shown in Figure 5.

The complex plane is subdivided into three regions (I), (II) and (III) as shown in Figure 6.

The roots $s_o$ and $\bar{s}_o$ which are complex conjugate poles of the open-loop transfer function are located at any point within one of the three regions in the $s$-plane. The type of compensation required to reshape a root-locus through a selected point is normally designated by the area in which the selected point lies (20).

The angle which a single pole and zero of closed-loop transfer function must provide to make $s_o$ lie on the inverse root-locus is equal to the angle designated in Figure 5 as the angle $\theta_o$ and is determined as follows.

1. If the complex roots, $s_o$ and $\bar{s}_o$, are in the region (I) as shown in Figure 7, the phase angle at a point $s_o$ is

$$\alpha + \beta = 360^\circ + \theta_o.$$

2. If the complex roots, $s_o$ and $\bar{s}_o$, are in the region (II) as shown in Figure 8, the phase-angle at a point $s_o$ is

$$\alpha + \beta = 360^\circ - \theta_o.$$

3. If the complex roots, $s_o$ and $\bar{s}_o$, are in the region (III) as shown in Figure 9, the phase-angle at a point $s_o$ is

$$\alpha + \beta = \theta_o.$$
Figure 5. s-plane plot for angle condition

Figure 6. Regions in s-plane
Figure 7. $s$-plane plot for angle condition 1

$\alpha + \beta - \Phi_0 = 360^\circ$

Figure 8. $s$-plane plot for angle condition 2

$\alpha + \beta + \Phi_0 = 360^\circ$
Figure 9. \( s \)-plane plot for angle condition 3

The phase-angle at a root \( s_0 \) is summarized below:

\[
\alpha + \beta = 360^\circ + \Xi_0 \quad \text{Region (1)}
\]
\[
= 360^\circ - \Xi_0 \quad \text{(II) (22)}
\]
\[
= \Xi_0 \quad \text{(III)}
\]

The phase-angle at a root \( s_0 \) must be supplemented by the contribution of another pole and zero so that \( s_0 \) lies on the inverse root-locus. In other words, the phase-angle at a root \( s_0 \) will be forced to be \( n360^\circ \) (\( n = 0,1,2,\ldots \)).

The angle \( \Xi_0 \) is determined by the position of \( s_0 \) corresponding to the position \( s^* \) in the \( s \)-plane. To determine the angle \( \Xi_0 \) quickly,
the phase-angle loci for two complex poles of the closed loop transfer function in the s-plane as shown in Figure 10 is utilized.

If the angle $\beta_0$ is known, the phase-angle $(\alpha + \beta)$ at a selected root $s_0$ is obtained from equation (22) which shows that this phase-angle $(\alpha + \beta)$ will be greater than or less than $360^\circ$ or equal to $\beta_0$.

The phase-angle $(\alpha + \beta)$ at $s_0$ can also be obtained by the following method:

1. The angle $\alpha$ of $s^*$ to $s_0$ and the angle $\beta$ of $\bar{s}^*$ to $s_0$ are measured in the s-plane.

2. To obtain the phase-angle $(\alpha + \beta)$ at $s_0$, the angles $\alpha$ and $\beta$ are summed.

Consider $s^*$ as the dominant complex root $(\omega_n, \gamma)$ and a selected root $s_0$ in region (I) as shown in Figures 5 and 7.

The following conclusions are drawn from equation (22) in order that $s_0$ be forced to lie on the inverse root locus.

1. The phase-angle $(\alpha + \beta)$ at $s_0$ from the dominant complex roots $(\omega_n, \gamma)$, $s^*$ and $\bar{s}^*$, is $360 + \beta_0$ as shown in Figure 7.

2. Thus, the phase-angle at $s_0$ from the poles and zeros of the closed-loop transfer function, except the dominant complex poles, is $n360^\circ - \beta_0$, where $n$ is an integer.

3. When $n = 0$, then the phase-angle at $s_0$ is $-\beta_0$.

Equation (22) is useful in judging the angle condition for the inverse root-locus. The angle $\beta_0$ is obtained from Figure 10 or in the s-plane.

A pole and zero may be placed at any location which satisfies
Figure 10. Phase-angle loci $H_0$ for two conjugate complex poles.

$s^*$ = dominant pole of the closed-loop system

$s_0$ = complex conjugate pole of the open-loop system
or selected point in $s$-plane
the requirement for the angle $\theta_0$. The simplest approach is to select one of the points (perhaps the zero) arbitrarily, then lay off the angle $\theta_0$ with a spiral to locate the other point (pole). The numerical demonstration in this case is shown as follows.

**Example 1.** The closed-loop transfer function is assumed to have one pair of conjugate complex poles, one real pole, and one finite zero:

$$\frac{C(s)}{R(s)} = \frac{K_c(s + z_2)}{(s^2 + 2\gamma \omega_n s + \omega_n^2)(s + p_3)}$$

The following specifications are considered:

$$\gamma = 0.5$$
$$\omega_n = 10 \text{ rad/s}$$
$$z_2 = 8$$

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_c(s + 8)}{(s^2 + 10s + 100)(s + p_3)}$$

The inverse root-locus has to pass through a selected root $s_0$; $s_0$ is selected as $s_0 = -4 + j4$.

The location of a pole-$q_3$ of the closed-loop transfer function will be determined as follows.

The selected root, $s_0 = -4 + j4$, corresponds to a point (0.115, 0.462) in Figure 10 where it shows the phase-angle at a point $s_0$ is $\theta = 7.5^\circ$.

The phase-angle at a selected root $s_0$ is also obtained on the $s$-plane by using the graphical method and equation (22).

The location of the selected root $s_0$ is in region (I). The phase-angle ($\alpha + \beta$) at a root $s_0$ from the dominant poles, $s^*$
Figure 11. Inverse root loci for $\frac{C(s)}{R} =$

\[
\frac{K_a(s + 8)}{(s^2 + 10s + 100)(s + 9.2)}
\]
and \(360^\circ + 75^\circ\).

To pass the inverse root-locus through the root \(s_0\), pole-\(q_3\) and zero-\(z_1\) of the closed-loop transfer function have to provide the angle requirement, since the phase-angle on the inverse root-locus is equal to \(n360^\circ\) \((n = 0, 1, 2, \ldots)\).

In order for the inverse root-locus to pass through \(s_0\), the angle formed by pole \(q_3\) and zero \(z_1\) must be \(-7.5^\circ\).

To get this angle, the location of pole \(q_3\) has to be the left of the zero \(z_1\).

The location of pole \(q_3\) is determined graphically as shown in Figure 11, and it is \(-q_3 = -9.2\).

The closed-loop transfer function in which the inverse root-locus passes through root \(s_0\) becomes

\[
\frac{C(s)}{R(s)} = \frac{K_c (s+8)}{(s^2 + 10s + 100)(s + 9.2)}
\]

The inverse root-locus for this example is also shown in Figure 11.

**Example 2.** The closed-loop transfer function considered is

\[
\frac{C(s)}{R(s)} = \frac{K_c (s+3)}{(s+1+j1)(s+1-j1)(s + 2.76)}
\]

The pole-zero configuration for \(\frac{C(s)}{R(s)}\) is shown in Figure 12. The location of zero \(-3\) is on the left of pole -2.76 on the real axis. The angle \(\varphi_0\) formed by a zero and a pole on the real axis to a root \(s_0\) on the complex inverse root-locus from a dominant pole is expressed.

\[
\varphi_0 = -\varphi_z + \varphi_p = +\varphi.
\]
Figure 12. Inverse root loci for

\[ \frac{C(s)}{R(s)} = \frac{K_o(s + 3)}{(s + 1 + j)(s + 1 - j)(s + 2.76)} \]
The angle $\Theta_0$ formed by a zero and a pole is positive. From equation (22), it is obvious that the complex inverse root-loci from dominant poles is in region (II) as shown in Figure 12. Figure 12 also shows the inverse root-locus for this example. It is clear that equation (22) can be used to find the approximate location of the complex inverse root-locus in the regions and the order of the location of a pole and a zero on the real axis.
IV. DESIGN OF SERVO-MECHANISMS USING INVERSE ROOT LOCUS

1. Cascade Compensation

In the design of feedback-control systems, the ultimate goal is the synthesis of a system to perform according to prescribed specifications. Synthesis is the determination of the required components from the statement of the system requirement. In other words, synthesis is determined in the complex plane by obtaining the open-loop system transfer function from the desired closed-loop input and output time functions.

The desired transient response is approximately determined by the location of the dominant complex poles of the closed loop transfer function (8, 9, 11, 16, 17, 26).

The desired steady-state value is given by the velocity constant, $K_v$.

The desired response will be discussed in the following procedures.

(1) The desired transient response is specified by the magnitudes of the peak overshoot, $M_p$, and time to peak, $T_p$.

(2) The location of the dominant complex poles of the closed-loop system which satisfy specification (1) is determined.

(3) The locations of the poles and zeros, except the dominant poles, of the closed-loop system are determined to satisfy the specifications for the velocity constant, $K_v$, and coincident condition.
The magnitudes of $M_p$ and $T_p$ are checked from equations (8) and (9), or by using the charts (11, p. 122-127).

If the magnitudes of $M_p$ and $T_p$ of the response are within the specifications of procedure (1), the system response is acceptable. If one of them exceeds the specifications of procedure (1), the procedures from (2) to (4) will be repeated until the specifications for $M_p$ and $T_p$ are satisfied.

The coincident condition is derived from the fact (1, 2): In an inverse root-locus diagram, the gain, $K_c$, will have the same value at each open-loop pole.

An inverse root-locus is necessary that will pass through the poles of the fixed element $G_1(s)$ of the open-loop transfer function as a root locus passes through the poles of the closed-loop transfer function.

The poles and zeros of the open-loop transfer function which are not in the transfer function of the fixed element $G_1(s)$ have to be formed by the compensation network $G_2(s)$ in cascade as shown in Figure 1.

In this thesis, the location of the dominant poles ($\omega_n$, $\gamma$), the fixed element $G_1(s)$ and the velocity constant, $K_v$, are specified to simplify the procedures for the determination of the practical compensation network $G_2(s)$. If the velocity constant, $K_v$, is not specified, $K_v$ will be calculated from the pole-zero configuration of the closed-loop transfer function.

With the addition of a dipole in the closed loop transfer function on the negative real axis as shown in Figure 13, the velocity constant,
$K_v$, is given by the expression

$$
\frac{1}{K_v} = \frac{1}{K_{vo}} + \frac{1}{g d} - \frac{1}{Z_d}
\]

$$

$$
= \frac{1}{K_{vo}} - \frac{\delta}{d^2 - \delta^2/4}
\]

![Diagram of Dipole Configuration](image)

**Figure 13. Dipole Configuration**

$K_{vo}$ is determined by the dominant poles $(\omega_n, \gamma)$ and the fixed element $G_1(s)$ which are specified. The term $\delta^2/4$ can be neglected, because the distance $\delta$ between the dipole is very small compared to the distance $d$. The expression for the approximate $K_v$ becomes

$$
\frac{1}{K_v} = \frac{1}{K_{vo}} - \frac{\delta}{d^2}
\]

(23)

It is obvious that the approximate $K_v$ can be calculated. The addition of the dipole modifies the step-function response only slightly (26, p. 43), but the modified response is within the specifications.

The design of a system which meets certain performance specifications will be considered in the following procedures.

(a) $N$ is considered as the total number of fixed closed-loop poles (5, p. 185).
Consider that \( R \) poles and \( s \) zeros exist in the compensation network \( G_2(s) \).

If the velocity constant, \( K_v \), or the gain, \( K \), of the open-loop transfer function is not specified,

\[
R + s \geq N - 1 \quad (24)
\]

If the velocity constant, \( K_v \), is specified,

\[
R + s \geq N. \quad (25)
\]

If the dominant poles of the closed-loop transfer function and the velocity constant, \( K_v \), are specified,

\[
N \text{ is equal to } 2. \quad N = 2. \text{ Then } R + s \geq 2. \quad (26)
\]

It is assumed that the compensation network has one pole and one zero as \( R = s = 1 \).

The compensation network can be realized by a simple lead, or lag circuit.

It is assumed that the fixed element, \( G_1(s) \), has \( r \) poles. The poles of the open-loop transfer function, \( G(s) \), are \( r + R \) and the poles of the closed-loop transfer function are also \( r + R \). Therefore, \( r + R - N \) poles of the closed-loop transfer function have to be determined. For example, it is assumed that there are 3 poles of the fixed element, \( G_1(s) \), and 2 poles of the fixed closed-loop transfer function,

\[
r = 3 \quad N = 2.
\]

The velocity constant, \( K_v \), is specified and \( R \) poles and \( s \) zeros of the compensation network, \( G_2(s) \), are assumed to be \( R = 1 \) and \( s = 1 \), respectively. Then, \( 3 + 1 - 2 = 2 \) poles of the closed-loop transfer
function have to be determined.

The dominant poles of the closed-loop transfer function and the fixed element, $G_1(s)$, of the open-loop transfer function are specified in this thesis. If the compensation network is assumed in the form of a simple lead or lag network as one pole and one zero, then it is possible to determine the order of the closed-loop transfer function from equation (26).

(b) It is obvious that the phase-angle loci, $\mathbb{L}_o$, in Figure 10 is very useful to determine the approximate location of the complex inverse root-locus from the dominant poles of the closed-loop transfer function. If the approximate location of the complex inverse root locus from the dominant poles is given in some region of the s-plane, then the locations of the desired pole and zero of the closed-loop transfer function on the real axis are determined from the phase-angle loci, $\mathbb{L}_o$, in Figure 10 and equation (22).

(c) The location of the dominant poles, and other poles and zeros, of the closed-loop transfer function can be determined in such a way that the inverse root-locus must pass through the poles of the fixed element, $G_1(s)$, of the open-loop transfer function, simultaneously.

This statement can be expressed by the coincident condition equations.

It is considered:

(1) The poles of the fixed element, $G_1(s)$, are $-p_1, -p_2, -p_3, \ldots, -p_n$.

(2) The dominant poles are $-q_1$ and $-q_2$, or $(\omega_n, \gamma)$. 
(3) The other poles of the closed-loop transfer function are \(-q_3, -q_4, \ldots, -q_n\).

(4) The zeros of the open- and closed-loop transfer function are \(-z_1, -z_2, \ldots, -z_m\).

The coincident conditions are expressed as follows:

\[ K_c = \frac{\omega_n^2 q_3 \cdots q_n}{z_1 z_2 \cdots z_m} \]
\[ = \frac{(z_1 - P_1)(z_2 - P_2) \cdots (z_n - P_n)}{(z_1 - P_1)(z_2 - P_2) \cdots (z_m - P_m)} \]  \hspace{1cm} (27)

\[ \sum_{i=1}^{m} \frac{1}{z_i - P_k} - \sum_{j=1}^{n} \frac{1}{P_i - P_k} = n360^\circ \] \hspace{1cm} (28)

(d) The relations between the velocity constant, \(K_v\), and poles and zeros of the system are expressed by equations (5), (6) and (7). In these equations, if the dominant poles, \(-q_1\) and \(-q_2\), of the closed-loop transfer functions are expressed as \((\omega_n, \gamma)\), then equations (6) and (7) can be expressed as follows:

\[ \frac{1}{K_v} = \frac{2\gamma}{\omega_n} + \frac{1}{\frac{1}{\beta^3}} + \frac{1}{\frac{1}{\beta^4} - \frac{1}{z_1} - \frac{1}{z_2} \cdots - \frac{1}{z_m}} \] \hspace{1cm} (29)

\[ K_v = \frac{\omega_n^2 q_3 \cdots q_n}{P_2 P_3 \cdots P_n} \] \hspace{1cm} (30)

(e) In most cases of the feedback system, the order \(n\) of the denominator and the order \(m\) of the numerator of the system transfer function is expressed as

\[ n - m \geq 2 \]

Then, the sum of the open-loop poles is identical with the sum
of the closed-loop poles (26, p. 302) such that

\[ 2 \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i = P_2 + P_3 + \cdots + P_n \]  

(31)

The poles \( P_{r+1}, P_{r+2}, \ldots, P_n \) and the zeros \( z_{u+1}, \ldots, z_m \) of the compensation network, and the poles \( q_1, q_2, \ldots, q_n \) of the closed-loop transfer function are determined by equations (29), (30) and (31), since the dominant poles \( (\omega_n, \gamma) \), the velocity constant, \( k_v \), the poles \( P_1(=0), P_2, \ldots, P_r \) and zeros \( z_1, z_2, \ldots, z_u \) of the fixed element \( G_1(s) \) are specified. Fortunately, in the practical design problem, the number of the equations is greater than the number of the unknown values. The choice of equations are free.

Usually, the fixed element, \( G_1(s) \), have a simple form in the servo-mechanisms system.

It is assumed that the poles of \( G_1(s) \) are on the real axis. The approximate location of the inverse root-locus and the number of the poles and zeros of the open- and closed-loop system can be determined by using procedures (a) and (b).

If the velocity constant, \( k_v \), is specified, the locations of the poles and zeros of the open- and closed-loop system can be determined by equations (6) and (7). The coincident condition equations (27) and (28) can be used to check the designed system. This is explained later in the general case and illustrated in Example 3.

It is assumed that the fixed element, \( G_1(s) \), has a pair of complex conjugate poles which are located to the left side of the dominant poles of the closed-loop transfer function. The
approximate location of the inverse root-locus and the number of the poles and zeros of the open- and closed-loop system can be also determined by using procedures (a) and (b).

The locations of the poles and zeros of the open-loop transfer function can be determined by the coincident condition equation (27) expressed as trigonometric function, since the locations of the poles and zeros of the closed-loop transfer function are determined by the phase-angle from the complex conjugate poles of the open-loop transfer function. This is also explained in the general case and illustrated in Example 4.

In designing the system, a simple design procedure and a simple, economical, and reliable compensation are preferable. At times, a simple design may be obtained but difficulty may arise in getting a realistic compensator network.

(i). Case 1--The fixed element, $G_1(s)$, has two real poles, one at origin and the other on the negative real axis.

The fixed element, $G_1(s)$, of the open-loop transfer function is considered as

$$ G_1(s) = \frac{K_1}{s(s+a)} $$

where $a$ is a positive real value.

The dominant poles ($\omega_n$, $\gamma$) of the closed-loop transfer function and the velocity constant $K_v$ are specified.

Let's consider the case in which

$$ \left| \text{Re}(-\frac{\omega_n}{\gamma}) \right| \geq a $$
The real part of the dominant poles is greater than or equal to the real pole $\zeta$ of $G_1(s)$.

The specified pole-zero configuration of the system is shown in Figure 13.

![Figure 13. The specified pole-zero configuration of the system.](image)

In order to pass the inverse root-locus through the poles of the fixed element, $G_1(s)$, the location of the inverse root-locus from the dominant poles has to be in region (I) in Figure 6, since a break-in point of the complex inverse root locus is between poles $\zeta$ and $P_1(=0)$ of $G_1(s)$.

It is assumed that a pole and a zero of the closed-loop transfer function put the locus in region (I). The order of this pole-zero is determined from equation (22), such that the location of a closed-loop pole is on the left of a zero as shown in Figure 14. This case is illustrated in Example 1.
Figure 14. The pole-zero configuration of the system.
- $x$ -- open-loop poles.
- $x$ -- closed-loop poles.

The third pole $-P_3$ of the open-loop transfer function is on the left of the pole $-q_3$ of the closed-loop transfer function.

The relationship between $K_v$, and the pole and zero positions are derived from equations (6) and (7).

$$
\frac{1}{K_v} = \frac{2\gamma}{\omega_n} + \frac{1}{q_3} - \frac{1}{z_1}
$$  \hspace{1cm} (33)

$$
K_v = \frac{\omega_n^2 q_3}{\alpha P_3}
$$  \hspace{1cm} (34)

The difference of the order between the numerator $m$ and the denominator $n$ of the closed-loop transfer function is two. Then, the following equation is derived from equation (31).

$$
2 \gamma \omega_n + q_3 = \alpha + P_3
$$  \hspace{1cm} (35)

The locations of the closed-loop pole, $q_3$, and open-loop pole, $p_3$, are determined from equations (34) and (35). The location of
the zero, \(z_1\), is determined from equation (33) by substituting \(q_3\) into equation (33).

The numerical example is illustrated in Example 3.

**Example 3.** Synthesis is initiated by a determination of the specifications, including the transfer function of the fixed element of the system and the performance characteristics. The following specifications are considered:

1. The fixed element of the system is described by the transfer function

   \[
   G_1(s) = \frac{10}{s(s + 10)}
   \]

2. The performance characteristics are specified by the dominant poles of the closed-loop system, and the velocity constant:

   \[
   \omega_n = 25 \quad \text{rad/s}
   \]
   \[
   \zeta = 0.5
   \]
   \[
   K_v = 30 \quad /s
   \]

   The closed loop system function \(G(s)\) is chosen as

   \[
   \frac{C(s)}{R(s)} = \frac{K_c(s + z_1)}{(s^2 + 2\zeta \omega_n s + \omega_n^2)(s + \frac{\omega_n}{\zeta})}
   \]

   The open-loop system function has three poles and one zero. The zero of the open-loop system function is identical with the zero of the closed-loop system function.

   The real part of the dominant poles is greater than \(P_2 = 10\) as

   \[
   |R_e(-q_1)| \geq 10
   \]

   The complex inverse root-locus has to be in region (I) in
Figure 6 to pass through the poles $-p_1 = 0$ and $-p_2 = 10$ of the open-loop system. It is obvious that the zero $z_1$ is located at the right of the pole $q_3$ of the closed-loop system. The location of the pole $q_3$ of the closed-loop system and the pole $P_3$ of the open-loop system are determined from equations (34) and (35):

$$30 = \frac{(25)^2q_3}{10 P_3}$$

$$25 + q_3 = 10 + P_3.$$

Therefore, the locations of $P_3$ and $q_3$ are determined:

$$q_3 = 13.8$$

$$P_3 = 28.8$$

The location of the zero is determined from equation (33):

$$\frac{1}{30} = \frac{1}{25} + \frac{1}{13.8} - \frac{1}{z_1}$$

Therefore, $z_1 = 12.7$

The open-loop system function is

$$G_T(s) = G_{ii}(s) G_{t2}(s) = \frac{K_v(1 + s/z_1)}{s(1 + s/P_3)(1 + s/P_3)}$$

$$= \frac{P_3 P_3}{z_1} \frac{K_v (s + z_1)}{s(s + P_3)(s + P_3)}$$

$$= \frac{(10)(28.8)(30)(s + 12.7)}{(12.7)(s + 10)(s + 28.8)}$$

The transfer function of the compensation network is determined:

$$\frac{G_2(s)}{G_1(s)} = \frac{1}{G_{t2}(s)} \frac{G(s)}{G_{t2}(s)} = \frac{s(s + 10)}{10} \frac{(10)(28.8)}{12.7} \frac{30(s + 12.7)}{s(s + 10)(s + 28.8)}$$

$$= \frac{28.8}{12.7} \frac{30(s + 12.7)}{s(s + 28.8)}$$
Figure 15. Inverse root-locus for \( \frac{C(s)}{R(s)} = \frac{Kc(s+12.7)}{(s^2+2s+5)(s+38)} \) in Example 3.
The inverse root-locus for this example is shown in Figure 15.

In this example, the dominant poles give a good approximation of the system transient because of the dipole location of $-q_3$ and $-z_1$ (26, p. 43).

(ii) Case 2—The fixed element, $G_1(s)$, has three poles, a pair of complex conjugate poles and one at the origin.

It is assumed that the complex conjugate poles of the fixed element, $G_1(s)$, are not located close to the origin, since the effect of a simple cascade compensation is unsatisfactory (26, p. 338-339).

When the fixed element, $G_1(s)$, has a pair of complex conjugate poles close to the origin, a feedback compensation is usually used (26, p. 338-339). This is discussed later in this thesis.

It is assumed that the location of the complex conjugate poles $-p_2$ and $-p_3$ of $G_1(s)$ is on the left side of the dominant poles of the closed-loop transfer function as shown in Figure 16.

![Figure 16. Specified pole-zero configuration](image-url)
The angle subtended by open-loop poles \(-p_2\) and \(-p_3\) at a dominant pole \(-q_1\) is denoted by \(\xi_0\) as shown in Figure 16. The angle subtended by closed-loop dominant poles \(-q_1\) and \(-q_2\) at an open-loop pole \(-p_2\) is denoted by \(360^\circ - \xi_0\) as shown in Figure 16, since the complex inverse root-locus from the dominant poles is in region II as shown in Figure 6 and the angle is denoted by equation (22).

The angle which makes open-loop pole \(-p_2\) lie on the inverse root-locus is formed by other closed-loop pole-zero and it has to be \(n360^\circ + \xi_0\). If \(n = 0\), the angle formed at \(-p_2\) by other pole-zero becomes \(\xi_0\). It is assumed that the angle \(\xi + \xi_0\) at \(-p_2\) is formed by one closed-loop pole and one zero on the real axis which are denoted by \(-q_3\) and \(-z_1\), respectively. It is obvious that as a result of Example 2, the location of the closed-loop pole \(-q_3\) is to the right of the zero \(-z_1\), since the complex inverse root-locus from the dominant poles of the closed-loop transfer function is in region (II).

When the complex inverse root-locus from a dominant pole \(-q_1\) passes through the open-loop pole \(-p_2\), the inverse root-locus from the closed-loop pole \(-q_3\) has to pass through the open-loop pole \(-p_1\) \((= 0)\), simultaneously, to satisfy the coincident condition, as shown in Figure 17.
The compensation network in this case is formed as $K(s + z_1)$ which is an ideal derivative plus proportional compensator. The ideal derivative plus proportional compensation, $K(s + z_1)$, is not used, since it is difficult to construct and it requires much equipment, and it amplifies any spurious signals or noise (9, p. 276). Instead, a passive network $\frac{K(s + z_1)}{s + p_4}$ is used as a compensator.

The closed-loop transfer function is required by equation (26) to have an additional pole $-q_4$, since $N = 3$, $R = s = 1$, and $r = 3$. If the additional pole $-q_4$ is located well to the left in the $s$-plane as shown in Figure 17, the system performance is not significantly changed by the presence of additional pole $-q_4$, since the large magnitude means that this pole has little effect on either transient or frequency response (26, p. 41-43 and p. 306).
In this case, the magnitude condition for the inverse root-locus can be expressed as

\[
\frac{|z_1-p_1||z_2-p_2||z_3-p_3||z_4-p_4|}{|z_1-p_1|} = \frac{\omega_n^2 \theta_3 \theta_4}{\theta_1} = K_c \quad (36)
\]

Equation (36) can be rearranged as

\[
\frac{\theta_1}{|z_1-p_1|} \cdot \frac{|z_3-p_3|}{\theta_3} = \frac{\omega_n^2 \theta_4}{|z_1-p_1||z_2-p_2||z_4-p_4|} = K_c
\]

or

\[
\frac{\theta_1}{|z_1-p_1|} \cdot \frac{|z_3-p_3|}{\theta_3} = K_c \quad (37)
\]

\[
\frac{\omega_n^2 \theta_4}{|z_1-p_1||z_2-p_2||z_4-p_4|} = K_c \quad (38)
\]

Equation (37) can be explained in trigonometric form with the aid of Figure 18 (20, p. 272).

Figure 18. \(s\)-plane diagram for equation (39).
From Figure 18,

\[
\frac{z_1}{|z_1 - p_2|} = \frac{\sin \psi_1}{\sin \theta_f} \quad \frac{|q_3 - p_2|}{q_3} = \frac{\sin \theta_f}{\sin \psi_3}
\]

Therefore, equation (37) can be explained as

\[
\frac{\sin \psi_1}{\sin \psi_3} = K_c
\]

(39)

The angle relation in Figure 18 is

\[
\psi_1 - \psi_3 = \Xi_0 - \theta_f = \Xi_0'
\]

(40)

From equations (39) and (40)

\[
\tan \psi_1 = \frac{\sin \Xi_0'}{\cos \Xi_0' - \sqrt{K_c}}
\]

(41)

The locations of the closed-loop pole \(-q_3\) and the closed-loop zero \(-z_1\) are determined by plotting the angles \(\psi_1\) and \(\psi_3\) on the s-plane as denoted in Figure 18. The angles \(\psi_1\) and \(\psi_3\) are computed by equations (40) and (41), since \(K_c\), \(\Xi_0\) and \(\theta_f\) are given by the specifications. An open-loop pole \(-p_4\) corresponding to a closed-loop pole \(-q_4\) can be determined by equation (31) and its location is on the real axis between the closed-loop pole \(-q_4\) and zero \(-z_1\).
Example 4. The specifications considered are as follows:

1) The fixed element $G_1(s)$ of the system is described by the transfer function

$$G_1(s) = \frac{K_i}{s(s + 10 + j6)(s + 10 - j6)}$$

2) The closed-loop dominant poles are specified as

$$\omega_n = 10 \text{ rad/s}$$

$$\gamma = 0.5$$

3) The compensator network consists of one pole and one zero.

It is clear that the open-loop transfer function has four poles, therefore the closed-loop transfer function also has four poles. By equation (26), two poles of the closed-loop transfer function have to be determined. This can also be considered from the above specifications, since the number of open-loop poles and the number of closed-loop poles are identical.

The closed-loop transfer function can be considered as

$$\frac{C_R(s)}{R(s)} = \frac{K_c(s + z_1)}{(s^2 + 10s + 100)(s + q_3)(s + q_4)}$$

Consider $-q_4$ to be

$$-q_4 = -30$$

The location of the closed-loop pole $-q_4$ is well to the left in the $s$-plane as shown in Figure 19. The pole $-q_4$ does not change the system response, significantly (26, p. 111).

The value of $K_c$ can be determined by equation (38) and with the
aid of s-plane in Figure 19 as

\[ K_c = \frac{10^2 \times 30}{5.6 \times 15.4 \times 20.8} = 1.67 \]

Then

\[ \frac{1}{K_c} = 0.598 \]

The angle \( \theta_0 \) is determined from Figure 10, or s-plane of Figure 19 as

\[ \theta_0 \approx 42^\circ \]

The angle \( \theta_{f4} \) is determined from Figure 19 as

\[ \theta_{f4} \approx 16.5^\circ \]

The angle \( \theta_0' \) is determined by equation (40) as

\[ \theta_0' = \theta_0 - \theta_{f4} = 42^\circ - 16.5^\circ = 25.5^\circ \]

Since angle \( \theta_0' \) and \( K_c \) are determined, the angle \( \psi_1 \) can be determined by equation (41) as

\[ \tan \psi_1 = \frac{\sin(25.5^\circ)}{\cos(25.5^\circ) - 0.598} = 1.43 \]

\[ \psi_1 = \tan^{-1}(1.43) = 55^\circ \]

The angle \( \psi_3 \) can be determined by equation (40) as

\[ 55^\circ - \psi_3 = 25.5^\circ \]

then

\[ \psi_3 = 29.5^\circ \]

The locations of the closed-loop pole \(-q_3\) and zero \(-z_1\) are determined by plotting the angles \( \psi_1 \) and \( \psi_3 \) as shown in Figure 19.
The locations of pole $-q_3$ and zero $-z_1$ are

$$-q_3 = -6.6, \quad -z_1 = 9.6$$

The location of the open-loop pole $-p_4$ is determined by equation (31) as

$$10 + 6.6 + 30 = 20 + p_4$$

then

$$-p_4 = -26.6$$

The determination of all locations of the pole-zero of the open- and closed-loop transfer function are complete.

The velocity constant, $K_v$, is determined by equations (6), or (7), or (29), or (30) to get

$$K_v = 5.5 \quad /s$$

The closed-loop transfer function is

$$\frac{C}{R}(s) = \frac{1.67 (s + 9.6)}{(s^2 + 10s + 100) (s + 6.6) (s + 30)}$$

The open-loop transfer function is

$$G(s) = G_1(s)G_2(s) = \frac{K_1(s + 9.6)}{s(s^2 + 20s + 136) (s + 26.6)}$$

$K_1$ is determined as

$$K_v = \lim_{s \to 0} sG(s) = \frac{K_1(9.6)}{(136) (26.6)} = 5.5$$

then

$$K_1 = \frac{(136) (26.6) (5.5)}{(9.6)} \approx 2072$$
Figure 19. s-plane diagram for angle condition and inverse root-loci for

\[
\frac{C}{R}(s) = \frac{K_0(s + 9.6)}{(s^2 + 10s + 100) (s + 6.6) (s + 30)} \quad \text{in Example 4.}
\]
2. Feedback Compensation

The choice of a method of compensation generally depends upon the specific system involved, the available components, economic reasons, and the designer's experience and judgement. Feedback compensation is sometimes preferable to cascade compensation.

The block diagram of a system employing feedback compensation is shown in Figure 20. There is a unity feedback loop (major loop) besides the compensated loop (minor loop) in order to maintain a direct correspondence between the output and input.

![Block Diagram](image)

Figure 20. A system employing feedback compensation.

In general, the fixed element, $G_f(s)$, of the forward transfer function is expressed as

$$G_f(s) = \frac{K_1(s+Z_1)(s+Z_2) \cdots (s+Z_m)}{s(s+P_1)(s+P_2) \cdots (s+P_r)}$$ (4.2)
The transfer function $H_0(s)$ of the compensator (or the feedback path transfer function $H_c(s)$) is expressed as

$$H_c(s) = \frac{K_0(s + S_1)(s + S_2) \cdots (s + S_R)}{(s + S_{R+1})(s + S_{R+2}) \cdots (s + S_K)}$$  \hfill (43)

The transfer function $\frac{C_A(s)}{A(s)}$ for the minor loop of Figure 20 is expressed as

$$\frac{C_A(s)}{A(s)} = K_1 \frac{(s + S_1)(s + S_2) \cdots (s + S_R)(s + S_{R+1}) \cdots (s + S_K)}{s(s + P'_i)(s + P'_j) \cdots \cdots (s + P'_{j} + P'_{k} - 1)}$$  \hfill (44)

where

$$S_i = P_i$$
$$i = 1, 2, \ldots, r$$

$$P'_j = \text{the roots of } 1 + H_0(s)G_1(s) = 0$$

The order of the factor $s$ in the numerator of $H_0(s)$, equation (43), must be equal to or higher than the type of the forward transfer function $G_1(s)$ shown in Figure 20 (9, p. 334). The number of the poles of $G_1(s)$ is different from the number of the zeros of $H_0(s)$. It is obvious that the zeros of the transfer function $\frac{C_A(s)}{A(s)}$ for the minor loop are the product of the zeros of $G_1(s)$ and the poles of $H_0(s)$. The poles of $\frac{C_A(s)}{A(s)}$ are the roots of $1 + G_1(s) H_0(s) = 0$.

The compensation must be capable of stabilizing the system and of adjusting the transient response to meet specifications. Aside from such considerations as physical realizability and economics, this simply means that the compensator must be able to move the roots of the characteristic equation to suitable locations (25, p. 301).
The first, and simplest case of feedback compensation to be considered, is the use of a tachometer which has the transfer function

\[ H_c(s) = K_h s \]  \hspace{1cm} (45)

The fixed element \( G_1(s) \) is considered as

\[ G_1(s) = \frac{K_1}{s(s + a)} \]  \hspace{1cm} (46)

where \( a \) = positive number.

The roots of \( 1 + G_1(s) H_c(s) = 0 \) is obtained as

\[ -p_1^* = 0 \]
\[ -p_2^* = -(a + K_1K_h) \]

The pole \( -p_2^* \) of \( C_A(s) \) moves to the left of the pole \( -p_2 \) of \( G_1(s) \). The poles \( -p_1^* \) and \( -p_2^* \) of \( C_A(s) \) are located at the origin and \(-a-K_1K_h\), since the transfer function of \( C_A(s) \) is

\[ C_A(s) = \frac{K_1}{s(s + a + K_1K_h)} \]  \hspace{1cm} (47)

The movement of the pole \( -p_2^* \) of \( C_A(s) \) is shown in Figure 21.

Next, it is also considered that the fixed element \( G_1(s) \) is

\[ G_1(s) = \frac{K_1}{s(s + p_2)(s + p_3)} \]  \hspace{1cm} (48)

where the poles \( p_2 \) and \( p_3 \) are complex conjugate.

The transfer function of the fixed element \( G_1(s) \) is rewritten
Figure 21. The movement of the pole $-p_2'$ of $\frac{C(s)}{A(s)}$.

Figure 22. The movement of the poles $-p_2'$ and $-p_3'$ of $\frac{C(s)}{A(s)}$. 

\[ -p_2' = -a - \frac{k_1 k_2}{a} \quad -p_2 = -a \]
The feedback compensation is considered as

\[ H_0(s) = K_h s \]  \hspace{1cm} (50)

The transfer function \( \frac{C_A(s)}{A} \) for the minor loop is

\[
\frac{C_A(s)}{A} = \frac{K_i}{S(S^2 + 2\alpha S + \alpha^2 + \beta^2 + K_1 K_h)}
\]

\[
= \frac{K_1}{S(S + \alpha - j\sqrt{\beta^2 + K_1 K_h})(S + \alpha + j\sqrt{\beta^2 + K_1 K_h})} \]  \hspace{1cm} (51)

From equation (51), it can be concluded that the poles \(-p_2'\) and \(-p_3'\) of \( \frac{C_A(s)}{A} \) move in parallel with the imaginary axis as shown in Figure 22.

The movement of the poles \(-p_2\) and \(-p_3\) to the location \(-p_2'\) and \(-p_3'\) makes the system to be less stable.

To avoid this effect, the following feedback compensation is used

\[ H_b(s) = \frac{K_h s(s + s_1)}{(s + s_2)} \]  \hspace{1cm} (52)

The transfer function \( H_b(s) \) of the compensator moves the poles \(-p_2\) and \(-p_3\) to suitable locations \(-p_2'\) and \(-p_3'\) in the s-plane away from the origin.

This is illustrated in Example 5.
Example 5. The following transfer functions are considered

\[ G_i(s) = \frac{K_i}{s(s+1+i)(s+1-i)} \]

\[ H_c(s) = \frac{K_R s(s+s_1)}{s+s_2} \]

The feedback compensator, \(H_c(s)\), moves the poles \(p_2\) and \(p_3\) of \(G_i(s)\) to the desired locations \(-p'_2\) and \(-p'_3\) which are

\[-p'_2 = -30 + j30\]

\[-p'_3 = -30 - j30\]

The desired location \(-p'_4\) is considered as

\[-p'_4 = -30\]

The minor loop transfer function \(\frac{C(s)}{A(s)}\) is

\[
\frac{C}{A}(s) = \frac{K_1(s+s_2)}{s(s+p'_2)(s+p'_3)(s+p'_4)}
\]

\[ = \frac{K_1(s+s_2)}{s(s+30-j30)(s+30+j30)(s+30)} \]

The pole-zero configuration for the transfer function \(\frac{C}{A}(s) H_c(s)\) is shown in Figure 23.

The coincident conditions with feedback \(H_c(s)\) for the inverse root-locus are

\[
\left| \frac{C}{A}(s) H_c(s) \right| = 1 \quad (53)
\]

\[
\angle \frac{C(s)}{A(s)} H_c(s) = \gamma \pi \quad (54)
\]

where \(n\) is an even number.
Figure 23. Pole-zero configuration for $\frac{G(s)}{A(s)}$ $H_0(s)$ in Example 5.
The transfer function \( \frac{C(s)}{A(s)} H_0(s) \) is

\[
\frac{C(s)}{A(s)} H_0(s) = \frac{K_1 K_2 (s + s_1)}{(s + p_2')(s + p_3')(s + p_4')}
\]

\[
= \frac{K_1 K_2 (s + s_1)}{(s + 30 - j30)(s + 30 + j30)(s + 30)}
\]

The location of a zero \( s_1 \) of \( \frac{C(s)}{A(s)} H_0(s) \) can be determined as illustrated in Example 1.

The angle \( \theta_0 \) subtended by the poles \(-p_2\) and \(-p_3\) at the pole \(-p_2'\) is \( \theta_0 \approx 20^\circ \).

The location of a zero \( s_1 \) is to the right of the pole \(-p_4'\), since the complex conjugate poles \(-p_2\) and \(-p_3\) of the fixed element \( G_1(s) \) are in region (I) of Figure 6, as illustrated in Example 1. It is shown in Figure 23.

The location of a zero \( s_1 \) is determined as

\[-s_1 = -14.9\]

The difference between the pole number and the zero number of the minor loop transfer function \( \frac{C(s)}{A(s)} \) is greater than 2 such that

\[n - m > 2\]

where

\[n = 4\]

\[m = 1\]

Equation (31) can be used to determine the location of the pole of the feedback compensator \( H_c(s) \) as

\[S_2 + P_a + P_3 = P_4' + P_3' + P_4\]

or

\[S_2 + 2 Re(P_2) = P_4' + 2 Re(P_4')\]
therefore
\[ s_2 + 2 = 30 + 60 \]
then
\[ -s_2 = -88 \]

The gain, \( K_1 K_h \), can be calculated by coincident equation (27),
or (53) with the aid of the s-plane of Figure 23 as
\[
K_1 K_h = \frac{|P_3 + P_3' \parallel P_4 + P_4'|}{|P_2 + s_1|} = 3680
\]

If \( K_1 = 1,000 \), the feedback gain \( K_h \) is
\[ K_h = 3.68 \]

In order to move the poles \(-p_2 \) and \(-p_3 \) of the fixed element
\( G_1(s) \) to the poles \(-p_2' \) and \(-p_3' \), the feedback compensator \( H_c(s) \) is

\[
H_c(s) = \frac{3.68 \ s(s + 14.9)}{(s + 88)}
\]

The block diagram for the system is shown in Figure 24. The
block diagram of Figure 24 can be reduced to a single equivalent
block as shown in Figure 25.

If \( G_2(s) \) is the cascade compensation, it is possible to design
the system of Figure 25 by using the cascade compensation method.
Figure 24. The block diagram for the system in Example 5.

Figure 25. Equivalent block diagram of Figure 24.
(1). Design of the Feedback Compensator

It is considered that the fixed element, $G_1(s)$, has two poles on the real axis, one on the origin.

A numerical example is illustrated below.

Example 6. The specifications considered are as follows:

(1) The fixed element, $G_1(s)$, is described by the transfer function

$$G_1(s) = \frac{1}{s(s + 1)}$$

(2) The dominant poles of the closed-loop transfer function are

$$\omega_n = 30 \text{ rad/s}$$

$$\gamma = 0.5$$

(3) The velocity constant $K_v$ are

(a) $K_v = 30 \text{ /s}$

(b) $K_v = 50 \text{ /s}$

The block diagram of the system is shown in Figure 26.

(a) The velocity constant, $K_v$, is specified as

$$K_v = 30 \text{ /s}$$

The tachometer transfer function is used as the feedback compensator such that

$$H_c(s) = K_h s$$

Equation (6), or (29) shows the following relation between the velocity constant $K_v$ and pole-zero configuration

$$\frac{1}{K_v} = \frac{2\gamma}{\omega_n}$$
then

\[ \frac{1}{30} = \frac{1}{30} \]

The velocity constant, \( k_v \), is specified by only dominant poles. The system response is governed by the dominant poles only, therefore the number of poles of the over-all transfer function is two.

For the minor loop, the minor loop transfer function is

\[ \frac{C}{A}(s) = \frac{1}{s(s + p')^2} = \frac{1}{s(s + 1 + k_a)} \]

The number of poles of \( \frac{C}{A}(s) \) is two.

The transfer function \( G_2(s) \) of Figure 26 does not have a pole because the pole number of the over-all system and the transfer function \( \frac{C}{A}(s) \) is the same.

The pole \( p'_2 \) of \( \frac{C}{A}(s) \) can be determined by equation (7), or (30)

\[ k_v = \frac{\omega_n^2}{p'_2} \]

then

\[ 30 = \frac{(30)^2}{p'_2} \]

\[ p'_2 = 30 \]

The pole \(-p'_1\) and \(-p'_2\) of \( \frac{C}{A}(s) \) are located at the origin and \(-30\).

The tachometer gain \( k_h \) can be obtained by the following equation

\[ p'_2 = p_2 + k_h = 1 + k_h \]

then \( k_h = 29 \)
Figure 26. Block diagram for the system in Example 6.

Figure 27. Block diagram for the system.
Figure 28. Equivalent block diagram of Figure 27.

Figure 29. Final block diagram of the system in Example 6 (a).
The block diagram of the system is shown in Figure 27. The minor loop is reduced as shown in Figure 28.

The following equation is derived from Figure 28.

\[
\frac{\mathcal{C}}{\mathcal{E}}(s) = \frac{K_2}{s(s + 30)}
\]

The gain \(K_2\) of the over-all system can be determined from

\[
K_v = \lim_{s \to 0} s\frac{\mathcal{C}}{\mathcal{E}}(s) = \frac{K_2}{30} \quad K_v = 30 \text{ }/s
\]

then

\[
K_2 = 900
\]

The final block diagram of the system is shown in Figure 29.

(b) The velocity constant, \(K_v\), is specified as

\[
K_v = 50 \text{ }/s
\]

The specified velocity constant, \(K_v\), is not satisfied by dominant poles only, since equation (6) or (29) shows that

\[
\frac{1}{K_v} = \frac{2}{\omega_n q_3} + \frac{1}{q_3} + \frac{1}{z_1}
\]

then

\[
\frac{1}{50} = \frac{1}{30 q_3} + \frac{1}{z_1}
\]

It is assumed that

(1) One pole, \(q_3\), and one zero, \(z_1\), of the closed-loop transfer function are added to satisfy the velocity constant specification

(2) One pole, \(q_3\), and one zero, \(z_1\), exist on the real axis

(3) They are expressed as dipole.
Equation (6), or (29) is expressed as

\[ \frac{1}{K_v} = \frac{2}{\bar{\omega}_n \omega} + \frac{1}{q} - \frac{1}{z_1} \]

then

\[ \frac{1}{50} = \frac{1}{30} + \frac{1}{q_3} - \frac{1}{z_1} \]

\[ \frac{1}{q_3} < \frac{1}{z_1} \quad (a) \]

The zero, \( z_1 \), is located to the right of the pole, \( q_3 \), on the real axis.

To satisfy equation (a), the following equation has to be satisfied

\[ \frac{1}{z_1} - \frac{1}{q_3} = \frac{1}{75} \quad (b) \]

It is assumed that the distance between the dipole is 0.5. The location of the zero, \( z_1 \), and the pole, \( q_3 \), of the closed-loop transfer function are determined by equation (b) as

\[ \frac{1}{z_1} - \frac{1}{z_1 + 0.5} = \frac{1}{75} \]

then

\[ z_1 = 5.87, \quad q_3 = 6.37 \]

The pole-zero configuration for the system is shown in Figure 30. Figure 30 also shows the inverse root-locus for the system. The block diagram of the system is shown in Figure 26.

The poles, \(-p_2^1\) and \(-p_3^1\), of \( G(s) \) can be determined by the coincident conditions (27) and (28), since when the inverse root-
Figure 30. Pole-zero configuration and the inverse root-loci for the system in Example 6 (b).
locus from the dominant poles passes through $-p_1 = -p_1' (= 0)$, the inverse root-locus passes through the poles, $-p_2'$ and $-p_3'$, of $\frac{C_A(s)}{E(s)}$ simultaneously.

In this thesis, the locations of the poles, $-p_2'$ and $-p_3'$, of $\frac{C_A(s)}{E(s)}$ are determined by equations (30) and (31).

\[
K_v = \frac{\omega_n q_3'}{p_2 p_3'}
\]

\[
2 \omega_n + q_3 = p_2' + p_3'
\]

then

\[
50 = \frac{(30)^2(6.37)}{p_2 p_3'}
\]

\[
30 + 6.37 = p_2' + p_3'
\]

Equations (c) and (d) are rearranged as

\[
p_2' p_3' = 114
\]

\[
p_2' + p_3' = 36.37
\]

therefore

\[
p_2' = 3.5, \quad p_3' = 32.9
\]

The block diagram for the system is shown in Figure 31. The block diagram with the minor loop for the system is shown in Figure 26.

The transfer function $\frac{C_A(s)}{A(s)}$ is expressed as

\[
\frac{C_A(s)}{A(s)} = \frac{1}{s(s + 1 + K_h)} = \frac{1}{s(s + p_2')}
\]

\[
= \frac{1}{s(s + 3.5)}
\]
**Figure 31.** Equivalent block diagram of the system in Example 6 (b).

**Figure 32.** Final block diagram of the system in Example 6 (b).
The feedback compensator $H_c(s) = K_h$ moves the pole $-p_2 = -1$ of the forward transfer function of the minor loop to $-p'_2 = -3.5$. The feedback compensator gain, $K_h$, can now be determined

$$p'_2 = p_2 + K_h$$

$$K_h = p'_2 - p_2 = 3.5 - 1 = 2.5$$

The forward transfer function $C_B(s)$ of the over-all system is expressed as (Figure 31)

$$\frac{C}{E}(s) = \frac{K_c(s + 5.87)}{s(s + 32.9)(s + 3.5)}$$

The velocity constant, $K_v$, is expressed as

$$K_v = \lim_{s \to 0} s\frac{C}{E}(s)$$

The gain, $K_2$, of the system can be determined as

$$50 = K_2 \frac{5.87}{(32.9)(3.5)}$$

then

$$K_2 = 980$$

The forward transfer function $C_B(s)$ is expressed as

$$\frac{C}{E}(s) = G_2(s) \frac{C}{A}(s)$$

Therefore, the transfer function $G_2(s)$ can be determined as

$$G_2(s) = \frac{C}{E}(s) \frac{C}{A}(s)$$

$$= \frac{K_c(s + 5.87)}{s(s + 32.9)(s + 3.5)} \frac{1}{s(s + 3.5)}$$

$$= \frac{980(s + 5.87)}{(s + 32.9)}$$
The final block diagram for the system which satisfy the specifications is shown in Figure 32.
V. CONCLUSIONS

The use of inverse root-locus for synthesis is attractive from the point of view of simplicity and amount of information which can be obtained quickly. It is also useful in determining open-loop characteristic from measured closed-loop response.

A semi-graphical method is presented for finding the compensation network, given fixed open-loop poles and zeros, and specified closed-loop dominant poles, and specified velocity constant $K_v$. For any number of specified closed-loop poles and zeros, the minimum total number of compensation poles and zero is given by equation (24) or (25). As presented, the method does not require that the poles of the fixed element, $G_1(s)$, be real. Phase-angle measurements at the fixed poles of the open-loop system and equation (22) are made to determine the type of compensation network required. In contrast with Aseltine's method (2), equations (5), (6), (7) and (31) are used because they are more analytic and simple.

It is also shown that a semi-graphical method using the inverse root-locus is useful in the feedback compensator.

The method used is one of the approaches to the synthesis problem and it is simple, compared with other approaches.


APPENDIX
I. Derivation of Equation (7).

Equation (7) will be derived as follows:

From Figure 1, the equation

$$E(s) = \frac{R(s)}{1 + G(s)}$$

is established. The equation may be rewritten

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

$$= \frac{C(s)}{R(s)} \frac{1}{G(s)} R(s)$$

then,

$$E(s) = \frac{S(s + P_2)(s + P_3) \cdots (s + P_n)}{(s + Q_1)(s + Q_2) \cdots (s + Q_n)} R(s)$$

Suppose $r(t)$ is a unit ramp function $r(t) = tu(t)$, then

$$R(s) = \frac{1}{s^2}$$

therefore

$$E(s) = \frac{(s + P_2)(s + P_3) \cdots (s + P_n)}{s(s + Q_1)(s + Q_2) \cdots (s + Q_n)}$$

If $E(s)$ possesses poles in the left-half plane only, the steady-state error is given by the final-value theorem,

$$e_{ss} = \lim_{s \to 0} SE(s) = \frac{P_2 P_3 \cdots P_n}{Q_1 Q_2 \cdots Q_n}$$

By definition, the steady-state error is

$$e_{ss} = \frac{1}{K_v}$$

Therefore, equation (7) is derived.

$$K_v = \frac{Q_1 Q_2 \cdots Q_n}{P_2 P_3 \cdots P_n}$$
II. Nomenclature

\[ C(s) = \text{controlled output, Figure 1} \]
\[ R(s) = \text{controlling input} \]
\[ E(s) = \text{error} \]
\[ G(s) = G_1(s)G_2(s) = \text{open-loop transfer function} \]
\[ G_1(s) = \text{transfer function of fixed element, or plant} \]
\[ G_2(s) = \text{transfer function of compensation network} \]
\[ G_c(s) = \frac{C}{R}(s) = \text{closed-loop transfer function} \]
\[ H_c(s) = \text{feedback transfer function} \]
\[ \frac{C}{A}(s) = \text{minor loop transfer function, Figure 20.} \]
\[ q_1(i = 1, 2, \ldots, n) = \text{poles of closed-loop transfer function} \]
\[ p_1(i = 1, 2, \ldots, r) = \text{poles of the fixed element transfer function, } G_1(s) \]
\[ p_1(i = r + 1, r + 2, \ldots, n) = \text{poles of the compensation network transfer function, } G_2(s) \]
\[ z_1(i = 1, 2, \ldots, u) = \text{zeros of the fixed element transfer function, } G_1(s) \]
\[ z_i(i = u + l, u + 2, \ldots, m) = \text{zeros of the compensation network transfer function, } G_2(s) \]
\[ s_1(i = 1, 2, \ldots, h) = \text{zeros of feedback transfer function, } H_c(s) \]
\[ s_1(i = h + 1, h + 2, \ldots, k) = \text{poles of feedback transfer function, } H_c(s) \]
\[ p_1^i(i = 1, 2, \ldots, r + k - h - 1) = \text{poles of minor loop transfer function, } \frac{G}{A}(s) \]

K = open-loop gain constant
$K_c$ = closed-loop gain constant

$K_v$ = velocity constant

$K_l$ = gain constant of the fixed element transfer function, $G_l(s)$

$K_h$ = gain constant of the feedback transfer function, $H_0(s)$

$\omega_n$ = undamped natural frequency

$\zeta$ = damping ratio

$M_p$ = peak overshoot with step function input

$T_p$ = time to peak with step function input

$t$ = time

$s = \sigma + j\omega$ = complex variable, or complex frequency