Supplement to

TORSION OF SANDWICH PANELS OF TRAPEZOIDAL, TRIANGULAR, AND RECTANGULAR CROSS SECTIONS

Derivation of Differential Equation and Its Application to Rectangular Panels with Loads Applied at Corners

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Supplement to

TORSION OF SANDWICH PANELS OF TRAPEZOIDAL, TRIANGULAR, AND RECTANGULAR CROSS SECTIONS[1]

Derivation of Differential Equation and its Application to Rectangular Panels with Loads Applied at Corners

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Introduction

Forest Products Laboratory Report No. 1871 (1)[3] presents two mathematical analyses of the torsion of rectangular sandwich plates. In one analysis the Saint Venant theory is used, although it does not satisfy the detail boundary conditions in regard to the applied load. In the other, a more rigorous treatment is used that satisfies all boundary conditions. In Report No. 1874 (2) the derivation of a system of suitable differential stress strain relations is carried out by means of the variational theorem of complementary energy in conjunction with Lagrangian multipliers. A system of differential equations was obtained. These equations (which can be applied to bending or twisting of sandwich panels) are then applied to the torsion of sandwich panels of trapezoidal, triangular, and rectangular cross sections by using the Saint Venant torsion in their solutions. The formula for the torsional stiffness of a sandwich panel of rectangular cross section so obtained agrees with the infinite series solution given in the Report No. 1871 (1).

The purposes of the present report are as follows

(1) To obtain from the differential stress strain relations and equations of equilibrium derived in the Report No. 1874 (2) a six order partial differential equation, corresponding to the role of the equation $\nabla^4 w = \frac{p}{D}$ in the thin solid plate theory (3), that governs

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[2]Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[3]Underlined numbers in parentheses refer to Literature Cited at the end of the text.

the small deflection of sandwich panels under bending or twisting. The problem of bending or twisting of sandwich panels thus reduced to the integration of this governing differential equation of deflection.

(2) By applying the governing differential equation and equations for stresses to solve the problem presented in Report No. 1871 (1) that is the torsion of rectangular sandwich panel having the torque applied by forces concentrated at the corners of the panel. The result which satisfies all boundary conditions shows that the expressions of homogeneous solution remain essentially the same and the series of particular solution converge more rapidly than those of the rigorous treatment presented in Report No. 1871 (1).

Notation

x, y, z	rectangular coordinates (fig. 1).
a, b	half length and width of sandwich.
h	half thickness of core.
t	thickness of facings.
E, ν	Young's modulus of elasticity and Poisson's ratio of the facings.
G	$\frac{E}{2(1+\nu)}$, shear modulus of the facings.
G_{xz}, G_{yz}	shear modulii of the core.
w	deflection of the panel in the z direction, Lagrangian multiplier.
β, γ	Lagrangian multipliers.
σ_x, σ_y, τ	stresses in facings.
$\tau_{xz}\ \tau_{yz}$	stresses in core.
p	load per unit area.
D_x	$\frac{G_{xz}}{Gth(1+\frac{t}{2h})^2}$
D_y	$\frac{G_{yz}}{Gth(1+\frac{t}{2h})^2}$
D	$\frac{1}{4Gth^2(1+\frac{t}{2h})^2}$
D_1	$(1-\nu)D_x + 2D_y$

D_2	$2D_x + (1 - \nu) D_y$
α_m	$\frac{(2m + 1) \pi}{2a}$
β_n	$\frac{(2n + 1) \pi}{2b}$
γ_m	$\left(\frac{D_x}{D_y} + \frac{D_x}{\alpha_m^2}\right)^{\frac{1}{2}}$
δ_n	$\left(\frac{D_y}{D_x} + \frac{D_y}{\beta_n^2}\right)^{\frac{1}{2}}$
P	resultant force applied at a corner.
P_1	$\frac{4P}{ab}$
c	$\frac{G_{yz}}{G_{xz}}$
A_m, B_m, C_m, D_m K_m, A_n, B_n, F_n, H_n, L_n	parameters.
T	applied torque.
θ	angle of twist per unit length in radians.

Derivation of Differential Equations for Deflection and Stresses

By setting $\alpha = 0$, equations (6), (7), (8), (10), (11), (12), (13), and (14) of Report No. 1874 (2) are reduced respectively to the following equations:

$$\tau_{xz} = t\left(1 + \frac{t}{2h}\right)\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y}\right) \tag{1}$$

$$\tau_{yz} = t\left(1 + \frac{t}{2h}\right)\left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x}\right) \tag{2}$$

$$\frac{\partial \tau_{xz}}{\partial x} = - \frac{\partial \tau_{yz}}{\partial y} - \frac{p}{2h} \tag{3}$$

$$\frac{\sigma_x - \nu \sigma_y}{hE\left(1 + \frac{t}{2h}\right)} = \frac{\partial \beta}{\partial x} \tag{4}$$

$$\frac{\sigma_y - \nu\sigma_x}{hE(1 + \frac{t}{2h})} = \frac{\partial\gamma}{\partial y} \tag{5}$$

$$\tau = Gh(1 + \frac{t}{2h})(\frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial x}) \tag{6}$$

$$\beta = \frac{\tau_{xz}}{G_{xz}} - \frac{\partial w}{\partial x} \tag{7}$$

$$\gamma = \frac{\tau_{yz}}{G_{yz}} - \frac{\partial w}{\partial y} \tag{8}$$

To find the physical interpretation of the Lagrangian multiplier $\underline{w}$ we notice that $\underline{p}$ under the double integral $\iint$ wpdxdy (which is a term contained in the energy expression I, equation (9) of Report No. 1874 (2)) represents the applied load intensity. We conclude that the term $\iint$ wpdxdy represents the virtual work and $\underline{w}$, the Lagrangian multiplier, is actually the deflection of the surface of sandwich panels.

Solving equations (4) and (5) for $\underline{\sigma_x}$ and $\underline{\sigma_y}$, gives

$$\sigma_x = \frac{Eh(1 + \frac{t}{2h})}{(1 - \nu^2)}(\frac{\partial\beta}{\partial x} + \nu\frac{\partial\gamma}{\partial y}) \tag{9}$$

$$\sigma_y = \frac{Eh(1 + \frac{t}{2h})}{(1 - \nu^2)}(\frac{\partial\gamma}{\partial y} + \nu\frac{\partial\beta}{\partial x}) \tag{10}$$

Substituting these expressions and equation (6) in equations (1) and (2) and carrying out the differentiations with respect to $\underline{x}$ and $\underline{y}$, we obtain

$$\tau_{xz} = th(1 + \frac{t}{2h})^2 G(\frac{2}{1 - \nu}\frac{\partial^2\beta}{\partial x^2} + \frac{\partial^2\beta}{\partial y^2} + \frac{1 + \nu}{1 - \nu}\frac{\partial^2\gamma}{\partial x\partial y}) \tag{11}$$

$$\tau_{yz} = th(1 + \frac{t}{2h})^2 G(\frac{2}{1 - \nu}\frac{\partial^2\gamma}{\partial y^2} + \frac{\partial^2\gamma}{\partial x^2} + \frac{1 + \nu}{1 - \nu}\frac{\partial^2\beta}{\partial x\partial y}) \tag{12}$$

By substituting for $\underline{\beta}$ and $\underline{\gamma}$ their expressions (7) and (8) into equations (11) and (12) the following equations are found

$$\frac{G_{yz}(1 - \nu)}{Gth(1 + \frac{t}{2h})^2}\tau_{xz} = 2\frac{G_{yz}}{G_{xz}}\frac{\partial^2\tau_{xz}}{\partial x^2} + (1 - \nu)\frac{G_{yz}}{G_{xz}}\frac{\partial^2\tau_{xz}}{\partial y^2}$$

$$+ (1 + \nu)\frac{\partial^2\tau_{yz}}{\partial x\,\partial y} - 2G_{yz}\frac{\partial}{\partial x}\nabla^2 w \tag{13}$$

$$\frac{G_{yz}(1-\nu)}{Gth(1+\frac{t}{2h})^2}\tau_{yz} = 2\frac{\partial^2\tau_{yz}}{\partial y^2} + (1-\nu)\frac{\partial^2\tau_{yz}}{\partial x^2} + (1+\nu)\frac{G_{yz}}{G_{xz}}\frac{\partial^2\tau_{xz}}{\partial y\,\partial x}$$

$$-\ 2G_{yz}\frac{\partial}{\partial y}\nabla^2 w \tag{14}$$

where $\underline{\nabla^2}$ is the Laplacian operator.

Differentiating equation (13) with respect to $\underline{x}$ and using equation (3), we obtain

$$(1+\nu-2\frac{G_{yz}}{G_{xz}})\frac{\partial^3\tau_{yz}}{\partial x^2\partial y} - (1-\nu)\frac{G_{yz}}{G_{xz}}\frac{\partial^3\tau_{yz}}{\partial y^3} + \frac{G_{yz}(1-\nu)}{Gth(1+\frac{t}{2h})^2}\frac{\partial\tau_{yz}}{\partial y} = 2G_{yz}\frac{\partial^2}{\partial x^2}\nabla^2 w$$

$$+\frac{G_{yz}}{G_{xz}h}\left[\frac{\partial^2 p}{\partial x^2} + \frac{(1-\nu)}{2}\frac{\partial^2 p}{\partial y^2}\right] - \frac{G_{yz}(1-\nu)}{2Gth^2(1+\frac{t}{2h})^2}\,p \tag{15}$$

Substituting for $\frac{\partial\tau_{xz}}{\partial x}$ its expression (3) in equation (14) gives

$$\left[2-(1+\nu)\frac{G_{yz}}{G_{xz}}\right]\frac{\partial^2\tau_{yz}}{\partial y^2} + (1-\nu)\frac{\partial^2\tau_{yz}}{\partial x^2} - \frac{G_{yz}(1-\nu)}{Gth(1+\frac{t}{2h})^2}\tau_{yz} = 2G_{yz}\frac{\partial}{\partial y}\nabla^2 w$$

$$+\frac{G_{yz}(1+\nu)}{2G_{xy}h}\frac{\partial p}{\partial y} \tag{16}$$

Differentiating equation (16) twice with respect to $\underline{x}$ and equation (15) once with respect to $\underline{y}$, then subtracting one from the other, we obtain the following differential equation for the shearing stress $\underline{\tau_{yz}}$.

$$D_x\frac{\partial^4\tau_{yz}}{\partial x^4} + (D_x+D_y)\frac{\partial^4\tau_{yz}}{\partial x^2\partial y^2} + D_y\frac{\partial^4\tau_{yz}}{\partial y^4} - D_xD_y\nabla^2\tau_{yz} = \frac{D_y}{2h}\frac{\partial}{\partial y}(D_x p - \nabla^2 p) \tag{17}$$

where

$$D_x = \frac{G_{xz}}{Gth(1+\frac{t}{2h})^2} \qquad D_y = \frac{G_{yz}}{Gth(1+\frac{t}{2h})^2}$$

Equation (17) can also be written as

$$D_x\nabla^4\tau_{yz} - (D_x - D_y)\frac{\partial^2}{\partial y^2}\nabla^2\tau_{yz} - D_xD_y\nabla^2\tau_{yz} = \frac{D_y}{2h}\frac{\partial}{\partial y}(D_x p - \nabla^2 p) \tag{18}$$

Differentiating equation (16) with respect to $\underline{y}$ and adding equation (15), gives

$$4hD(D_x - D_y)\frac{\partial}{\partial y} \nabla^2\tau_{yz} = D_xD_y\nabla^4 w + DD_y[2\nabla^2 p - (1 - \nu)D_x p] \tag{19}$$

where

$$D = \frac{1}{4Gth^2(1 + \frac{t}{2h})^2}$$

Differentiating equation (18) with respect to y and applying equation (19), we obtain

$$\nabla^6 w - (1 - \frac{D_y}{D_x})\frac{\partial^2}{\partial y^2}\nabla^4 w - D_y\nabla^4 w = D[-\frac{2}{D_x}\nabla^4 p + (1 - \nu + 2\frac{D_y}{D_x})\nabla^2 p +$$

$$(1+\nu)(1 - \frac{D_y}{D_x})\frac{\partial^2 p}{\partial y^2} - (1 - \nu)D_y p] \tag{20}$$

In the same manner as the derivation of equation (18), (19), and (20), or by considering the condition of symmetry, we obtain

$$D_y\nabla^4\tau_{xz} - (D_y - D_x)\frac{\partial^2}{\partial x^2}\nabla^2\tau_{xz} - D_xD_y\nabla^2\tau_{xz} = \frac{D_x}{2h}\frac{\partial}{\partial x}(D_y p - \nabla^2 p) \tag{21}$$

$$4hD(D_y - D_x)\frac{\partial}{\partial x}\nabla^2\tau_{xz} = D_xD_y\nabla^4 w + DD_x[2\nabla^2 p - (1 - \nu)D_y p] \tag{22}$$

$$\nabla^6 w - (1 - \frac{D_x}{D_y})\frac{\partial^2}{\partial x^2}\nabla^4 w - D_x\nabla^4 w = D[\frac{-2}{D_y}\nabla^4 p + (1 - \nu + 2\frac{D_x}{D_y})\nabla^2 p$$

$$(1+\nu)(1 - \frac{D_x}{D_y})\frac{\partial^2 p}{\partial x^2} - (1 - \nu)D_x p] \tag{23}$$

Subtracting equation (23) from equation (20), gives

$$(D_xD_y - D_x\frac{\partial^2}{\partial x^2} - D_y\frac{\partial^2}{\partial y^2})\nabla^4 w = D[2\nabla^4 p - D_1\frac{\partial^2 p}{\partial x^2} - D_2\frac{\partial^2 p}{\partial y^2} + (1 - \nu)D_xD_y p] \tag{24}$$

where

$$D_1 = (1 - \nu)D_x + 2D_y \qquad D_2 = 2D_x + (1 - \nu)D_y$$

It is seen that the problem of bending of rectangular sandwich panel by a lateral load p reduces to the integration of equation (24). The shearing stresses τ_{yz} and τ_{xz} can now be determined from equation (18) or (21) and equation (3).

Once w, τ_{yz} and τ_{xz} are obtained, the remaining five quantities β, γ, τ, σ_x, and σ_y can be readily found from equations (7), (8), (6), (9), and (10) by differentiation. It is of interest to note that equation (24) reduces to the differential equation of the sandwich plate given by Reissner as equation (70) in reference (4) if G_{xz} is assumed to be equal to G_{yz}. When $G_{xz} = G_{yz} = \infty$ equation (24) reverts to the known form of this equation for the homogeneous plate.

Torsion of Sandwich Panel of Rectangular Cross Section having the Torque Applied by Forces Concentrated at the Corners of the Panel (fig. 1)

The Loading

For the purpose of integrating equation (24) for the deflection of a rectangular sandwich panel by the loading shown in figure 1 we express the load intensity p in the form of a double trigonometric series:

$$p = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} A_{mn} \sin\frac{(2m + 1)\pi x}{2a} \sin\frac{(2n + 1)\pi y}{2b} \tag{a}$$

To calculate any particular coefficient $A_{m'n'}$ of this series for a given load distribution, that is, for a given p, we multiply both sides of equation (a) by $\sin\frac{(2n' + 1)\pi y}{2b} dy$ and integrate from 0 to b. Observing that

$$\int_{-b}^{b} \sin\frac{(2n + 1)\pi y}{2b} \sin\frac{(2n' + 1)\pi y}{2b} dy = 0 \qquad \text{when } n \neq n'$$

$$\int_{-b}^{b} \sin\frac{(2n + 1)\pi y}{2b} \sin\frac{(2n' + 1)\pi y}{2b} dy = b \qquad \text{when } n = n'$$

we find in this way

$$\int_{-b}^{b} p \sin\frac{(2n' + 1)\pi y}{2b} dy = b \sum_{m=o}^{\infty} A_{mn'} \sin\frac{(2m + 1)\pi x}{2a} \tag{b}$$

Multiplying both sides of equation (b) by $\sin\frac{(2m' + 1)\pi x}{2a}$ dx and integrating from 0 to a, we obtain

$$\int_{-a}^{a} \int_{-b}^{b} p \sin\frac{(2m' + 1)\pi x}{2a} \sin\frac{(2n' + 1)\pi y}{2b} dxdy = abA_{m'n'}$$

from which

$$A_{m'n'} = \frac{1}{ab} \int_{-a}^{a} \int_{-b}^{b} p \sin\frac{(2m' + 1)\pi x}{2a} \sin\frac{(2n' + 1)\pi y}{2b} dxdy \tag{c}$$

In the case of the four concentrated loads applied as shown in figure 1 equation (c) is integrated over four very small areas at the corners of the panel. Equation (c) becomes

$$A_{m'n'} = \frac{4}{ab} \sin\frac{(2m' + 1)\pi}{2} \sin\frac{(2n' + 1)\pi}{2} \int_{a-\delta}^{a} \int_{b-\delta}^{b} p d_x d_y$$

where δ can be made as small as desired. It is evident that the value of the double integral is equal to the concentrated load P or

$$A_{mn} = \frac{4P}{ab} \sin\frac{(2m + 1)\pi}{2} \sin\frac{(2n + 1)\pi}{2} = \frac{4P}{ab}(-1)^{m+n}$$

Hence we find

$$p = \frac{4P}{ab} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (-1)^{m+n} \sin \alpha_m x \sin \beta_n y \tag{25}$$

where

$$\alpha_m = \frac{(2m + 1)\pi}{2} \qquad \beta_n = \frac{(2n + 1)\pi}{2b}$$

The Particular Solution

For the loading shown in figure 1 the deflection w is an odd function of x and y. With this restriction we take the following expression as the particular solution for deflection

$$w = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} w_{mn} h \sin \alpha_m x \sin \beta_n y \tag{26}$$

in which the constant w_{mn} must be chosen so as to satisfy equation (24). Substituting expression (26) and (25) into equation (24), we find

$$w_{mn} = \frac{DP_1(-1)^{m+n}[2(\alpha_m^2 + \beta_n^2)^2 + D_1\alpha_m^2 + D_2\beta_n^2 + (1 - \nu)D_xD_y]}{h(\alpha_m^2 + \beta_n^2)^2 (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \tag{27}$$

where

$$P_1 = \frac{4P}{ab}$$

Taking the particular solution of equation (18) for τ_{yz} which must be an odd function of x and an even function of y as

$$\frac{\tau_{yz}}{G_{yz}} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} A_{mn} \beta_n h \sin \alpha_m x \cos \beta_n y \tag{28}$$

and substituting this expression with equation (25) into equation (18), we obtain

$$A_{mn} = \frac{2DP_1 (-1)^{m+n} (\alpha_m^2 + \beta_n^2 + D_x)}{h (\alpha_m^2 + \beta_n^2) (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \tag{29}$$

Using equations (26), (27), (28), and (29) the remaining six particular solutions of τ_{xz}, β, γ, τ, σ_x, and σ_y can readily be obtained from equations (3), (7), (8), (6), (9), and (10).

These particular solutions are:

$$\frac{\tau_{xz}}{G_{xz}} = 2DP_1 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n} \alpha_m (\alpha_m^2 + \beta_n^2 + D_y)}{(\alpha_m^2 + \beta_n^2)(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \cos \alpha_m x \sin \beta_n y \tag{30}$$

$$\beta = DP_1 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n} \alpha_m [D_x(\nu - 1)(\alpha_m^2 + \beta_n^2 + D_y) - (D_x - D_y)(1 + \nu)\beta_n^2]}{(\alpha_m^2 + \beta_n^2)^2 (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \cos \alpha_m x \sin \beta_n y \tag{31}$$

$$\gamma = DP_1 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n} \beta_n [D_y(\nu - 1)(\alpha_m^2 + \beta_n^2 + D_x) + (D_x - D_y)(1 + \nu)\alpha_m^2]}{(\alpha_m^2 + \beta_n^2)^2 (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \sin \alpha_m x \cos \beta_n y \tag{32}$$

$$\frac{\tau}{G} = 2DP_1 h\left(1 + \frac{t}{2h}\right) \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n} \alpha_m\beta_n [(D_x + D_y)(\alpha_m^2 + \beta_n^2) - (D_1\alpha_m^2 + D_2\beta_n^2) - (1 - \nu)D_xD_y]}{(\alpha_m^2 + \beta_n^2)^2 (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \cos \alpha_m x \cos \beta_n y \tag{33}$$

$$\frac{\sigma_x}{E} = DP_1 h \frac{\left(1 + \frac{t}{2h}\right)}{(1 + \nu)} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n}[(\alpha_m^2 + \nu\beta_n^2)(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y) + 2\alpha_m^2\beta_n^2(D_x - D_y)]}{(\alpha_m^2 + \beta_n^2)^2 (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \sin \alpha_m x \sin \beta_n y \tag{34}$$

$$\frac{\sigma_y}{E} = DP_1 h \frac{\left(1 + \frac{t}{2h}\right)}{(1 + \nu)} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n}[(\nu\alpha_m^2 + \beta_n^2)(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y) + 2\alpha_m^2\beta_n^2(D_y - D_x)]}{(\alpha_m^2 + \beta_n^2)^2 (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \sin \alpha_m x \sin \beta_n y \tag{35}$$

It is seen that the above series converge more rapidly than those given in the Report No. 1871 (1).

The Homogeneous Solutions

In order that all the boundary conditions can be satisfied solutions other than the particular solution must be found. This is accomplished by setting the left side of equation (24) equal to zero, (p = 0). A suitable general integral of this equation is

$$w = h \sum_{m=o}^{\infty} \left[\frac{C_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} + \frac{K_m \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b}\right] \sin \alpha_m x$$

$$+ h \sum_{n=o}^{\infty} \left[\frac{F_n \sinh \beta_n x + H_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} + \frac{L_n \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a}\right] \sin \beta_n y \qquad (36)$$

where

$$\gamma_m = \left(\frac{D_x}{D_y} + \frac{D_x}{\alpha_m^2}\right)^{\frac{1}{2}} \qquad \delta_n = \left(\frac{D_y}{D_x} + \frac{D_y}{\beta_n^2}\right)^{\frac{1}{2}} \qquad (37)$$

and C_m, D_m, K_m, F_n, H_n, and L_n are arbitrary constants to be determined later from the boundary conditions. The expression (36) is considered the homogeneous solution of w because it does not contribute to the loading.

It is noticed here that the expressions (36) and (37) are similar to those found in the previous Report No. 1871 (1).

In view of the equations (36) and (16) the homogeneous solution of equation (18) for τ_{yz} is:

$$\frac{\tau_{yz}}{G_{yz}} = h \sum_{m=o}^{\infty} \alpha_m \left[A_m \frac{\cosh \alpha_m y}{\cosh \alpha_m b} + B_m \frac{\cosh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b}\right] \sin \alpha_m x + h \sum_{n=o}^{\infty} \beta_n \left[A_n \frac{\sinh \beta_n x}{\cosh \beta_n a}\right.$$

$$\left. + B_n \frac{\sinh \delta_n \beta_n x}{\cosh \delta_n \beta_n a}\right] \cos \beta_n y \qquad (38)$$

Substituting equation (38) and (36) into equation (16) and using equations (37), we obtain

$$A_m = \frac{4D_m}{2 - c[(1 - \nu)\gamma_m^2 + 1 + \nu]} \qquad (39)$$

$$B_m = \frac{(\gamma_m^2 - 1) K_m}{\gamma_m (1 - c)} \qquad (40)$$

$$A_n = \frac{-4H_n}{[(1 - \nu)\delta_n^2 - 2c + 1 + \nu]} \qquad (41)$$

$$B_n = \frac{-(\delta_n^2 - 1) L_n}{\delta_n (1 - c)} \tag{42}$$

By using the expressions (36) and (38) we obtain the remaining six homogeneous solutions of $\underline{\tau_{xz}}$, $\underline{\beta}$, $\underline{\gamma}$, $\underline{\tau}$, $\underline{\sigma_x}$, and $\underline{\sigma_y}$ by means of equations (3), (7), (8), (6), (9), and (10).

The Complete Solution

From the foregoing analysis the complete solutions may be written as follows:

$$\frac{w}{h} = \frac{DP_1}{h} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n}[2(\alpha_m^2 + \beta_n^2) + D_1\alpha_m^2 + D_2\beta_n^2 + (1-\nu)D_xD_y]}{(\alpha_m^2 + \beta_n^2)^2(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \sin \alpha_m x \sin \beta_n y$$

$$+ \sum_{m=o}^{\infty} \left[\frac{C_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} + \frac{K_m \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b}\right] \sin \alpha_m x$$

$$+ \sum_{n=o}^{\infty} \left[\frac{F_n \sinh \beta_n x + H_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} + \frac{L_n \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a}\right] \sin \beta_n y \tag{43}$$

$$\frac{\tau_{yz}}{G_{yz}} = 2DP_1 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n} \beta_n(\alpha_m^2 + \beta_n^2 + D_x)}{(\alpha_m^2 + \beta_n^2)(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \sin \alpha_m x \cos \beta_n y$$

$$+ h \sum_{m=o}^{\infty} \alpha_m \left(A_m \frac{\cosh \alpha_m y}{\cosh \alpha_m b} + B_m \frac{\cosh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b}\right) \sin \alpha_m x$$

$$+ h \sum_{n=o}^{\infty} \beta_n \left(A_n \frac{\sinh \beta_n x}{\cosh \beta_n a} + B_n \frac{\sinh \delta_n \beta_n x}{\cosh \delta_n \beta_n a}\right) \cos \beta_n y \tag{44}$$

$$\frac{\tau_{xz}}{G_{xz}} = 2DP_1 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n} \alpha_m(\alpha_m^2 + \beta_n^2 + D_y)}{(\alpha_m^2 + \beta_n^2)(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \cos \alpha_m x \sin \beta_n y$$

$$+ ch \sum_{m=o}^{\infty} \alpha_m \left(A_m \frac{\sinh \alpha_m y}{\cosh \alpha_m b} + B_m \frac{\sinh \gamma_m \alpha_m y}{\cosh \gamma_m \alpha_m b}\right) \cos \alpha_m x$$

$$+ ch \sum_{n=o}^{\infty} \beta_n \left(A_n \frac{\cosh \beta_n x}{\cosh \beta_n a} + B_n \frac{\cosh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a}\right) \sin \beta_n y \tag{45}$$

$$\beta = - DP_1 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n}\alpha_m[D_x(1-\nu)(\alpha_m^2 + \beta_n^2 + D_y) + (D_x - D_y)(1+\nu)\beta_n^2]}{(\alpha_m^2 + \beta_n^2)^2(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \cos \alpha_m x \sin \beta_n y$$

$$+ h \sum_{m=o}^{\infty} \alpha_m \left[\frac{(cA_m - C_m) \sinh \alpha_m y - D_m\alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + \frac{(cB_m\gamma_m - K_m) \sinh \gamma_m\alpha_m y}{\gamma_m \cosh \gamma_m\alpha_m b} \right] \cos \alpha_m x + h \sum_{n=o}^{\infty} \beta_n \left[\frac{(cA_n - F_n - H_n)\cosh \beta_n x - H_n\beta_n x \sinh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left. + \frac{(cB_n - L_n\delta_n) \cosh \delta_n\beta_n x}{\delta_n \cosh \delta_n\beta_n a} \right] \sin \beta_n y \tag{46}$$

$$\gamma = - DP_1 \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n}\beta_n[D_y(1-\nu)(\alpha_m^2 + \beta_n^2 + D_x) - (D_x - D_y)(1+\nu)\alpha_m^2]}{(\alpha_m^2 + \beta_n^2)^2(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \sin \alpha_m x \cos \beta_n y$$

$$+ h \sum_{m=o}^{\infty} \alpha_m \left[\frac{(A_m - C_m - D_m) \cosh \alpha_m y - D_m\alpha_m y \sinh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + \frac{(B_m - K_m\gamma_m) \cosh \gamma_m\alpha_m y}{\gamma_m \cosh \gamma_m\alpha_m b} \right] \sin \alpha_m x + h \sum_{n=o}^{\infty} \beta_n \left[\frac{(A_n - F_n) \sinh \beta_n x - H_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left. + \frac{(B_n\delta_n - L_n) \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a} \right] \cos \beta_n y \tag{47}$$

$$\frac{\tau}{G} = (1 + \frac{t}{2h}) \left\{ -2DP_1 h \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{(-1)^{m+n}\alpha_m\beta_n[D_1\alpha_m^2 + D_2\beta_n^2]}{(\alpha_m^2 + \beta_n^2)^2(D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \right.$$

$$- \frac{(D_x + D_y)(\alpha_m^2 + \beta_n^2) - (1-\nu)D_xD_y}{(\alpha_m^2 + \beta_n^2)^2 (D_x\alpha_m^2 + D_y\beta_n^2 + D_xD_y)} \cos \alpha_m x \cos \beta_n y$$

$$+ h^2 \sum_{m=o}^{\infty} \alpha_m^2 \left[\frac{[(1+c)A_m - 2(C_m + D_m)] \cosh \alpha_m y - 2D_m\alpha_m y \sinh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + \frac{[(c\gamma_m^2 + 1)B_m - 2K_m\gamma_m] \cosh \gamma_m\alpha_m y}{\gamma_m \cosh \gamma_m\alpha_m b} \right] \cos \alpha_m x + h^2 \sum_{n=o}^{\infty} \beta_n^2 \left[\frac{[(1+c)A_n - 2(F_n + H_n)] \cosh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left. \left. - \frac{2H_n\beta_n x \sinh \beta_n x}{\cosh \beta_n a} + \frac{[(c + \delta_n^2)B_n - 2L_n\delta_n] \cosh \delta_n\beta_n x}{\delta_n \cosh \delta_n\beta_n a} \right] \cos \beta_n y \right\} \tag{48}$$

$$
\frac{\sigma_x}{E} = \frac{1+\frac{t}{2h}}{1+\nu}\Bigg\{ DP_1 h \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} \frac{(-1)^{m+n}[(\alpha_m^2+\nu\beta_n^2)(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y)+2\alpha_m^2\beta_n^2(D_x-D_y)]}{(\alpha_m^2+\beta_n^2)^2\,(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y)} \sin\alpha_m x \sin\beta_n y
$$

$$
-h^2\sum_{m=o}^{\infty}\alpha_m^2\left[\frac{(\frac{c-\nu}{1-\nu}A_m - C_m + \frac{2\nu}{1-\nu}D_m)\sinh\alpha_m y - D_m\alpha_m y\cosh\alpha_m y}{\cosh\alpha_m b} + \frac{(\frac{c-\nu}{1-\nu}\gamma_m B_m - \frac{1-\nu\gamma_m^2}{1-\nu}K_m)\sinh\gamma_m\alpha_m y}{\gamma_m\cosh\gamma_m\alpha_m b}\right]\sin\alpha_m x
$$

$$
+h^2\sum_{n=o}^{\infty}\beta_n^2\left[\frac{(\frac{c-\nu}{1-\nu}A_n - F_n - \frac{2}{1-\nu}H_n)\sinh\beta_n x - H_n\beta_n x\cosh\beta_n x}{\cosh\beta_n a} + \frac{(\frac{c-\nu}{1-\nu}\delta_n B_n - \frac{\delta_n^2-\nu}{1-\nu}L_n)\sinh\delta_n\beta_n x}{\delta_n\cosh\delta_n\beta_n a}\right]\sin\beta_n y\Bigg\} \tag{49}
$$

$$
\frac{\sigma_y}{E} = \frac{1+\frac{t}{2h}}{1+\nu}\Bigg\{ DP_1 h \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} \frac{(-1)^{m+n}[(\nu\alpha_m^2+\beta_n^2)(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y)+2\alpha_m^2\beta_n^2(D_y-D_x)]}{(\alpha_m^2+\beta_n^2)^2(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y)} \sin\alpha_m x \sin\beta_n y
$$

$$
+h^2\sum_{m=o}^{\infty}\alpha_m^2\left[\frac{(\frac{1-\nu c}{1-\nu}A_m - C_m - \frac{2}{1-\nu}D_m)\sinh\alpha_m y - D_m\alpha_m y\cosh\alpha_m y}{\cosh\alpha_m b} + \frac{(\frac{1-\nu c}{1-\nu}\gamma_m B_m - \frac{\gamma_m^2-\nu}{1-\nu}K_m)\sinh\gamma_m\alpha_m y}{\gamma_m\cosh\gamma_m\alpha_m b}\right]\sin\alpha_m x
$$

$$
-h^2\sum_{n=o}^{\infty}\beta_n^2\left[\frac{(\frac{1-\nu c}{1-\nu}A_n - F_n + \frac{2\nu}{1-\nu}H_n)\sinh\beta_n x - H_n\beta_n x\cosh\beta_n x}{\cosh\beta_n a} + \frac{(\frac{1-\nu c}{1-\nu}\delta_n B_n - \frac{1-\nu\delta_n^2}{1-\nu}L_n)\sinh\delta_n\beta_n x}{\delta_n\cosh\delta_n\beta_n a}\right]\sin\beta_n y\Bigg\} \tag{50}
$$

Determinations of Six Parameters C_m, D_m, K_m, F_n, H_n, and L_n in the Expression of Deflection w from the Six Boundary Conditions

As seen from the expressions of the complete solutions of w, τ_{yz}, τ_{xz}, β, γ, τ, σ_x, and σ_y given in the preceeding section, the problem of torsion of rectangular sandwich panel has been reduced to finding the six arbitrary constants C_m, D_m, K_m, F_n, H_n, and L_n. These six constants can be evaluated from the following six boundary conditions of the sandwich panel:

(1) The requirement $\tau_{yz} = 0$ at $y = \pm b$ gives

$$B_m = -\gamma_m A_m \tag{51}$$

(2) The requirement $\tau_{xz} = 0$ at $x = \pm a$ gives

$$B_n = -\delta_n A_n \tag{52}$$

(3) The requirement $\tau = 0$ at $y = \pm b$ gives

$$\frac{1+c}{2} A_m - C_m - D_m(1 + \alpha_m b \tanh \alpha_m b) + \frac{B_m(c\gamma_m^2 + 1)}{2\gamma_m} - K_m = 0 \tag{53}$$

(4) The requirement $\tau = 0$ at $x = \pm a$ gives

$$\frac{1+c}{2} A_n - F_n - H_n(1 + \beta_n a \tanh \beta_n a) + \frac{B_n(\delta_n^2 + c)}{2\delta_n} - L_n = 0 \tag{54}$$

(5) By means of the Fourier sine transform[4] of equation (49) and (50) it can be shown that the requirement, $\sigma_x = 0$ at $x = \pm a$ gives

---

[4]Both sides of equation (49) are multiplied by $\sin \beta_n y$ and integrated from 0 to b. Both sides of equation (50) are multiplied by $\sin \alpha_m x$ and integrated from 0 to a. The following integrals are needed for these operations.

$$\int \sinh qx \sin sx \, dx = \frac{1}{q^2+s^2}(q \cosh qx \sin sx - s \sinh qx \cos sx)$$

$$\int x \cosh qx \sin sx \, dx = \frac{qx}{q^2+s^2} \sinh qx \sin sx - \frac{sx}{q^2+s^2} \cosh qx \cos sx - \frac{q^2-s^2}{(q^2+s^2)^2} \cosh qx \sin sx + \frac{2qs}{(q^2+s^2)^2} \sinh qx \cos sx$$

$$DP_1h \sum_{m=o}^{\infty} \frac{(-1)^n[(\alpha_m^2+\nu\beta_n^2)(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y) + 2\alpha_m^2\beta_n^2(D_x-D_y)]}{(\alpha_m^2+\beta_n^2)^2(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y)} = \frac{2}{b}\sum_{m=o}^{\infty}(-1)^{m+n}$$

$$(\alpha_m h)^2 \left[\frac{\alpha_m(\frac{c-\nu}{1-\nu}A_m - C_m + \frac{2\nu}{1-\nu}D_m)}{\alpha_m^2+\beta_n^2} - \frac{D_m\alpha_m}{\alpha_m^2+\beta_n^2}\left(\alpha_m b \tanh \alpha_m b - \frac{\alpha_m^2-\beta_n^2}{\alpha_m^2+\beta_n^2}\right)\right.$$

$$\left. + \frac{\alpha_m(\frac{c-\nu}{1-\nu}\gamma_m B_m - \frac{1-\nu\gamma_m^2}{1-\nu}K_m)}{\gamma_m^2\alpha_m^2+\beta_n^2}\right] - (\beta_n h)^2 \left\{\left(\frac{c-\nu}{1-\nu}A_n - F_n - \frac{2}{1-\nu}H_n\right)\tanh \beta_n a - H_n\beta_n a\right.$$

$$\left. + \left[\frac{c-\nu}{1-\nu}B_n - \frac{\delta_n^2-\nu}{(1-\nu)\delta_n}L_n\right]\tanh \delta_n\beta_n a\right\} \qquad (55)$$

and

(6) The requirement, $\sigma_y = 0$ at $y = \pm b$ gives

$$DP_1h \sum_{n=o}^{\infty} (-1)^m \frac{[(\nu\alpha_m^2+\beta_n^2)(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y)+2\alpha_m^2\beta_n^2(D_y-D_x)]}{(\alpha_m^2+\beta_n^2)^2(D_x\alpha_m^2+D_y\beta_n^2+D_xD_y)} = -(\alpha_m h)^2$$

$$\left\{\left(\frac{1-\nu c}{1-\nu}A_m - C_m - \frac{2}{1-\nu}D_m\right)\tanh \alpha_m b - D_m\alpha_m b + \left[\frac{1-\nu c}{1-\nu}B_m - \frac{\gamma_m^2-\nu}{(1-\nu)\gamma_m}K_m\right]\tanh \gamma_m\alpha_m b\right\}$$

$$+ \frac{2}{a}\sum_{n=o}^{\infty}(-1)^{m+n}(\beta_n h)^2 \left[\frac{\beta_n(\frac{1-\nu c}{1-\nu}A_n - F_n + \frac{2\nu}{1-\nu}H_n)}{\alpha_m^2+\beta_n^2} - \frac{H_n\beta_n}{\alpha_m^2+\beta_n^2}\left(\beta_n a \tanh \beta_n a + \frac{\alpha_m^2-\beta_n^2}{\alpha_m^2+\beta_n^2}\right)\right.$$

$$\left. + \frac{\beta_n(\frac{1-\nu c}{1-\nu}\delta_n B_n + \frac{\nu\delta_n^2-1}{1-\nu}L_n)}{\alpha_m^2+\delta_n^2\beta_n^2}\right] \qquad (56)$$

Solving equations (53) and (54) by using equations (39), (40), (41), (42), (51), and (52), gives

$$D_m = \frac{(\gamma_m-1)\left\{c[1+\nu+\gamma_m^2(1-\nu)]-2\right\}C_m}{2(c\gamma_m^4-2\gamma_m^2+c)-(\gamma_m^2-1)(1+\alpha_m b \tanh \alpha_m b)\left\{c[1+\nu+\gamma_m^2(1-\nu)]-2\right\}} \qquad (57)$$

$$H_n = \frac{(\delta_n^2-1)[(1-\nu)\delta_n^2-2c+1+\nu]F_n}{2(\delta_n^4-2c\delta_n^2+1)-(\delta_n^2-1)(1+\beta_n a \tanh \beta_n a)[(1-\nu)\delta_n^2-2c+1+\nu]} \qquad (58)$$

Substituting expressions (57) and (58) into expressions (51), (52), and using equations (39), (40), (41), (42), gives

$$K_m = \frac{4\gamma_m^2(1-c)\,C_m}{2(c\gamma_m^4-2\gamma_m^2+c)-(\gamma_m^2-1)(1+\alpha_m b \tanh \alpha_m b)\left\{c[1+\nu+\gamma_m^2(1-\nu)] - 2\right\}} \tag{59}$$

$$L_n = \frac{-4\delta_n^2(1-c)F_n}{2(\delta_n^4-2c\delta_n^2+1)-(\delta_n^2-1)(1+\beta_n a \tanh \beta_n a)[(1-\nu)\delta_n^2 + 2c+1+\nu]} \tag{60}$$

Substituting equations (57) and (58) into equations (39 and (41) gives

$$A_m = \frac{-4(\gamma_m^2-1)\,C_m}{2(c\gamma_m^4-2\gamma_m^2+c)-(\gamma_m^2-1)(1+\alpha_m b \tanh \alpha_m b)\left\{c[1+\nu+\gamma_m^2(1-\nu)] - 2\right\}} \tag{61}$$

$$A_n = \frac{-4(\delta_n^2-1)F_n}{2(\delta_n^4-2c\delta_n^2+1) - (\delta_n^2-1)(1+\beta_n a \tanh \beta_n a)[(1-\nu)\delta_n^2 -2c+1+\nu]} \tag{62}$$

Equations (51), (52), (57), (58), (59), (60), (61), and (62), show that the constants $\underline{D_m}$, $\underline{H_n}$, $\underline{K_m}$, $\underline{L_n}$, $\underline{A_m}$, $\underline{A_n}$, $\underline{B_m}$, $\underline{B_n}$ can be expressed in terms of two arbitrary constants $\underline{C_m}$ and $\underline{F_n}$.

The first parts on the right side of equation (55) and (56) can be expressed in terms of $\underline{C_m}$ by means of equations (57), (59), (61), and (40).

The second parts on the right side of equations (55) and (56) can be expressed in terms of F_n by means of equations (58), (60), (62), and (42). Thus equations (55) and (56) may be solved for $\underline{C_m}$ and $\underline{F_n}$ in terms of the load $\underline{P}$ for as many values of $\underline{m}$ and $\underline{n}$ as desired. These values can then be substituted in equations (43), (44), (45), (48), (49), and (50) to obtain the deflections of and stress in the sandwich panel.

Determination of Torsional Rigidity $\frac{M}{\theta}$

The loads acting at the corners of the sandwich panel form a couple the magnitude of which is

$$T = 2Pb$$

The angle of twist per unit length is

$$\theta = \frac{w\big|_{x=a,\, y=b}}{ab}$$

The displacement $\underline{w}$ is given by equation (43).

Thus the torsional rigidity can be expressed as

$$\frac{T}{\theta} = \frac{2Pab^2}{w\big|_{x=a,\; y=b}} \tag{63}$$

Conclusion

The results of the foregoing analysis show that the series of the particular solution obtained by the present method converge more rapidly than those found in Report No. 1871 (1) and the series of homogeneous solution remain essentially the same. It is expected that the numerical results will be close to the results computed in Report No. 1871 (1).

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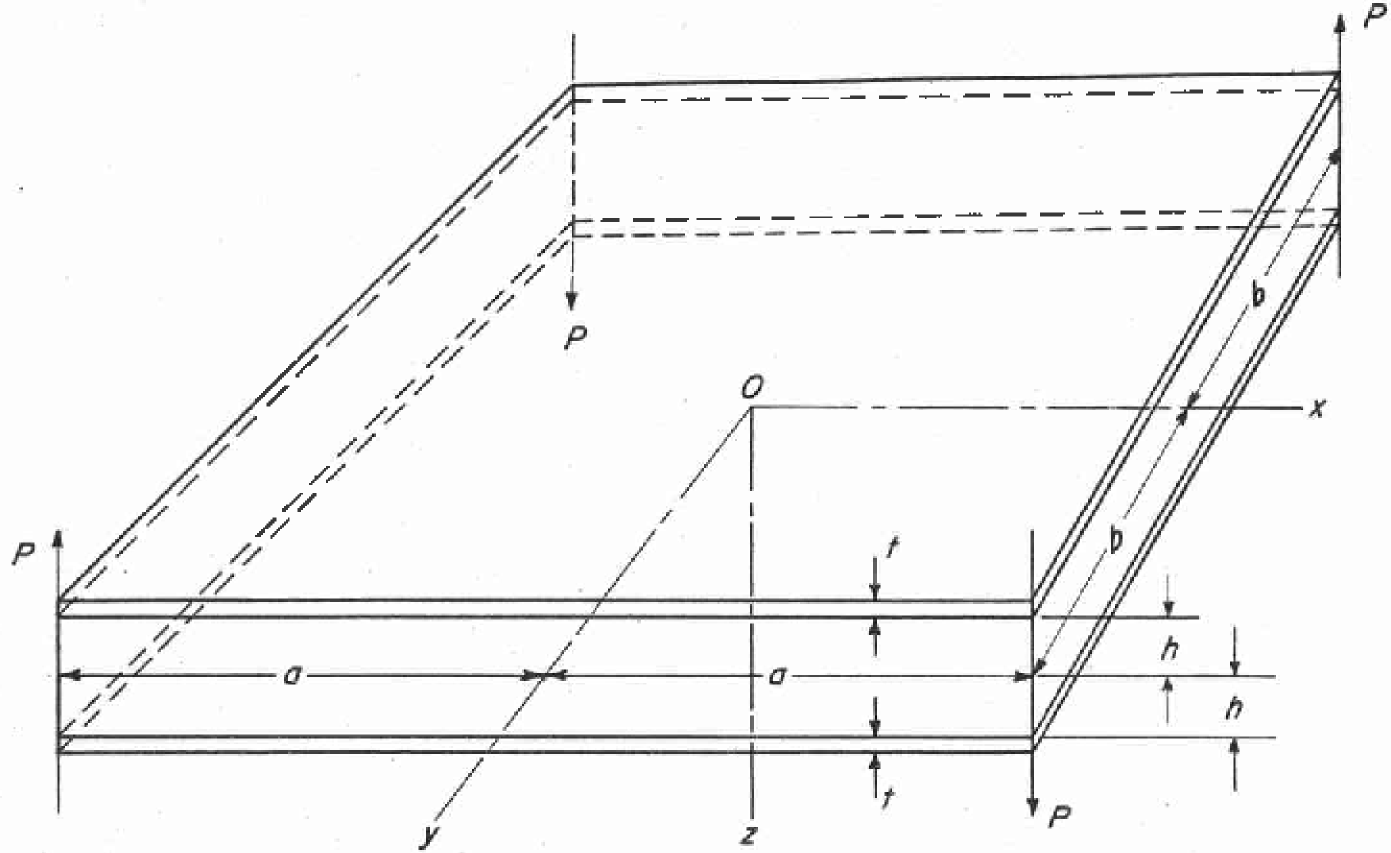

Figure 1. --Sketch of sandwich panel.