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Vortex stretching and criticality for the three-dimensional Navier-Stokes equations

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A mathematical evidence—in a statistically significant sense—of a *geometric scenario* leading to *criticality* of the Navier-Stokes problem is presented. [http://dx.doi.org/10.1063/1.4752170]

Dedicated to Professor Peter Constantin on the occasion of his 60th birthday, with admiration.

I. PROLOGUE

Three-dimensional (3D) Navier-Stokes equations (NSE)—describing a flow of 3D incompressible viscous fluid—read

$$u_t + (u \cdot \nabla)u = -\nabla p + \triangle u,$$

supplemented with the incompressibility condition $\operatorname{div} u = 0$, where u is the velocity of the fluid and p is the pressure (here, the viscosity is set to 1); taking the curl yields the vorticity formulation,

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \triangle \omega,$$

where $\omega = \operatorname{curl} u$ is the vorticity of the fluid.

It is well known that both globally[2,3] and uniformly-locally (with suitable spatial decay at infinity)[29] finite energy data generate global-in-time weak (distributional) solutions to the 3D Navier-Stokes equations (NSE), satisfying global and local energy inequality, respectively. Despite much effort—since the pioneering work of Leray[2] in 1930's—the question of whether weak solutions may exhibit finite-time singularities remains an open problem. It is known that the set of all possible singularities is small—the one-dimensional (parabolic) Hausdorff measure of the singular set in $\Omega \times (0, T)$ is zero for any $T > 0$;[6] here, Ω is a global spatial domain.

There are various regularity criteria preventing the finite-time formation of singularities, mainly expressed either as a local or a global condition on a weak solution over a spatiotemporal domain, or as a condition on a regular solution approaching a potential singular time T^*. The conditions are given as boundedness of a suitable spatiotemporal norm, the common trait being that the norm is scaling-invariant (critical) with respect to the natural scaling in the Navier-Stokes model. In contrast, the *a priori* bounded quantities are all subcritical; moreover, there is a *scaling gap* between a regularity criterion in view and the corresponding *a priori* bounded quantity. As an illustration, here are two classical examples—for the velocity formulation—in L^p and the space of bounded mean oscillations (*BMO*) spaces. The regularity criteria are boundedness in $L^\infty_t L^3_x$[30] and $L^2_t BMO_x$,[23] and the corresponding *a priori* bounded quantities are $L^\infty_t L^2_x$[2,3] and $L^1_t L^\infty_x$,[4] respectively. These are manifestations of *supercriticality* of the Navier-Stokes problem.

A rigorous study of *geometric depletion of the nonlinearity* in the 3D NSE (as well as in the 3D Euler equations) was pioneered by Constantin, and it was based on a singular integral representation for α—the stretching factor in the evolution of the vorticity magnitude $|\omega|$ depleted by *local coherence of the vorticity direction*—"the story of alpha and omega."[15] This is fundamental as there is ample evidence—both numerical[5,7,12–14] and theoretical[17,18,21,34]—that regions of intense vorticity tend to self-organize in *coherent vortex structures*, most notably, quasi one-dimensional vortex filaments, displaying strong local coherence of the vorticity direction.

The mechanism of the geometric depletion of the nonlinearity was subsequently exploited in Ref. 16 to show that as long as the regions of intense vorticity exhibit local Lipschitz-coherence of the vorticity direction, no finite-time blow up can occur, and later in Ref. 28 where the Lipschitz-coherence was replaced by the $\frac{1}{2}$-Hölder coherence. A spatiotemporal localization of the $\frac{1}{2}$-Hölder coherence regularity criterion was performed in Refs. 32 and 35, and independently in Ref. 33. The aforementioned regularity criteria are all pointwise coherence conditions; hence, necessarily supercritical with respect to the NSE scaling. A local, scaling-invariant (critical) criterion over a parabolic cylinder below a potential singular point (x_0, t_0),

$$\int_{t_0-(2R)^2}^{t_0} \int_{B(x_0,2R)} |\omega(x,t)|^2 \rho^2_{\frac{1}{2},2R}(x,t)\, dx\, dt < \infty,$$

where

$$\rho_{\gamma,r}(x,t) = \sup_{y\in B(x,r),\, y\neq x} \frac{\left|\sin\varphi\Big(\eta(x,t), \eta(y,t)\Big)\right|}{|x-y|^{\gamma}}$$

is a γ-Hölder measure of coherence of the vorticity direction η at the point (x, t), was presented in Ref. 36. On the other hand, a corresponding (subcritical) *a priori* bound had been previously obtained in Ref. 11,

$$\int_0^T \int_{\mathbb{R}^3} |\omega(x,t)||\nabla\eta(x,t)|^2\, dx\, dt \le \frac{1}{2}\nu^{-2} \int_{\mathbb{R}^3} |u_0(x)|^2\, dx,$$

where ν is the viscosity.

A different geometric approach to the study of possible singularity formation in 2D and 3D incompressible flows was developed by Cordoba and Fefferman;[24,25,27] in particular, non-existence of "tube collapse singularities" in 3D incompressible inviscid flows was shown in Ref. 25, and non-existence of a more general class of "squirt singularities" in incompressible flows—including the flows described by the 3D NSE—was presented in Ref. 31.

The purpose of this Article is to present a mathematical evidence—in a statistically significant sense—of a geometric scenario leading to the *criticality* of the Navier-Stokes problem. More precisely, utilizing the *ensemble averaging process* introduced in our recent study of turbulent cascades in *physical scales* of 3D incompressible fluid flows,[38–41] we show that the ensemble-averaged vortex stretching term is positive across a range of scales extending from a square root of a Kraichnan-type micro-scale to the macro-scale. Combining this with the *a priori* bound on the decrease of the distribution function of the vorticity obtained by Constantin in Ref. 11—as well as with the general mechanism of creation and dynamics of vortex filaments in turbulent flows (cf. Ref. 17)—indicates a geometric scenario in which the region of intense vorticity (defined as the region in which the vorticity magnitude—near a possible singular time—exceeds a fraction of the L^∞-norm) comprises of macro-scale-long vortex filaments with the diameters of the cross-sections scaling like $\dfrac{1}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$. This is exactly the *scale* of *local one-dimensional sparseness* of the region of intense vorticity needed to prevent a formation of a finite-time singularity.[42]

II. GEOMETRIC MEASURE-TYPE REGULARITY CRITERION

In this section, we briefly recall a *geometric measure-type* regularity criterion for solutions to the 3D NSE obtained recently by one of the authors; for details, see Ref. 42.

Definition 2.1: Let x_0 be a point in $\mathbb{R}^3$, $r > 0$, S an open subset of $\mathbb{R}^3$ and δ in $(0, 1)$.

The set S is linearly δ-sparse around x_0 at scale r in weak sense *if there exists a unit vector d in S^2 such that*

$$\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \le \delta.$$

For $M > 0$, denote by $\Omega_t(M)$ the vorticity super-level set at time t; more precisely,

$$\Omega_t(M) = \{x \in \mathbb{R}^3 : |\omega(x,t)| > M\}.$$

The vorticity version of the local one-dimensional (linear) sparseness regularity criterion is as follows.

Theorem 2.1: *Suppose that a solution u is regular on an interval* $(0, T^*)$.

Fix δ *in* $(0, 1)$, *and let* $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$ *and* $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$. *Assume that there exists* $\epsilon > 0$ *such that for any t in* $(T^* - \epsilon, T^*)$, *either*

(i) $t + \dfrac{1}{d_0^2 \|\omega(t)\|_\infty} \geq T^*$ *(d_0 is an absolute constant appearing in the local-in-time analytic smoothing in L^∞; cf. Ref. 42), or*

(ii) there exists $s = s(t)$ *in* $\left[t + \frac{1}{4d_0^2\|\omega(t)\|_\infty}, t + \frac{1}{d_0^2\|\omega(t)\|_\infty}\right]$ *such that for any spatial point* x_0, *there exists a scale* r, $0 < r \leq \frac{1}{2d_0^2\|\omega(t)\|_\infty^{\frac{1}{2}}}$, *with the property that the super-level set* $\Omega_s(M)$ *is linearly* δ*-sparse around* x_0 *at scale r in weak sense; here,* $M = M(\delta) = \frac{1}{d_0^\alpha}\|\omega(t)\|_\infty$.

Then, there exists $\gamma > 0$ *such that* ω *is in* $L^\infty\Big((T^* - \epsilon, T^* + \gamma); L^\infty\Big)$, *i.e.,* T^* *is not a singular time*.

The proof is based on a very intimate interplay between the diffusion in the model—represented by the local-in-time analytic smoothing in L^∞—and the geometric properties of the harmonic measure (via the *harmonic measure majorization principle*).

The analyticity estimate on solutions needed is a vorticity version of the estimate given in Ref. 37; this was based on a general method for estimating uniform radius of spatial analyticity in L^p-spaces introduced in Ref. 19, which was in turn inspired by the (analytic) Gevrey-class method presented in Ref. 10 (see also Ref. 20).

The key geometric harmonic measure estimate used in the proof is a generalization of the classical Beurling's problem,[1] conjectured in Ref. 8 and solved by Solynin in Ref. 22 (a symmetric version of the problem was previously resolved in Ref. 9); for more details, see Ref. 42.

Remark 2.1: A rudimentary version of Theorem 2.1 was previously obtained in Ref. 26. The condition needed in Ref. 26 is a much stronger condition; essentially, a requirement of a local existence of a sparse *coordinate projection*. In contrast, all that is needed here is a local sparseness of an one-dimensional *trace* in a *very weak sense*.

III. THE REGION OF INTENSE VORTICITY

There is strong numerical evidence that the regions of high vorticity organize in coherent vortex structures[5,7,12–14] and in particular, in elongated vortex filaments (tubes). In addition, an in-depth analysis of creation and dynamics of vortex tubes in 3D turbulent incompressible flows was presented in Ref. 17 (see also Refs. 18, 21, and 34).

Consider a flow near the first (possible) singular time T^*, and define *the region of intense vorticity* at time $t < T^*$ to be the region in which the vorticity magnitude exceeds a fraction of $\|\omega(t)\|_\infty$; keeping the notation from Sec. II, this corresponds to the set $\Omega_t\Big(\frac{1}{c_1}\|\omega(t)\|_\infty\Big)$, for some $c_1 > 1$.

Denote a suitable *macro-scale* associated with the flow by R_0. The picture painted by numerical simulations indicates that the region of intense vorticity comprises—in a statistically significant sense—of vortex filaments with the lengths comparable to R_0.

Let us for a moment accept this as a probable geometric blow up scenario. The length scale associated with the diameters of the cross-sections can then be estimated *indirectly*, by estimating the rate of the decrease of the total volume of the region of intense vorticity $\Omega_t\Big(\frac{1}{c_1}\|\omega(t)\|_\infty\Big)$.

Taking the initial vorticity to be a finite Radon measure, Constantin showed[11] that the L^1-norm of the vorticity is *a priori* bounded over *any* finite time-interval; a desired estimate on the total volume of the region of intense vorticity follows simply from the Tchebyshev inequality,

$$\mathrm{Vol}\left(\Omega_t\left(\frac{1}{c_1}\|\omega(t)\|_\infty\right)\right) \le \frac{c_2}{\|\omega(t)\|_\infty} \ (c_2 > 1).$$

This implies the decrease of the diameters of the cross-section of at least $\frac{c_3}{\|\omega(t)\|_\infty^{\frac{1}{2}}}(c_3 > 1)$, which is exactly the scale of *local one-dimensional sparseness* of the region of intense vorticity needed to prevent the formation of singularities presented in Theorem 2.1. In other words, the Navier-Stokes problem in this scenario becomes *critical*.

A key step in justifying this scenario is providing a *mathematical evidence* of persistence—in a statistically significant sense—of the R_0-long vortex filaments (at this point, the evidence is purely numerical). A term responsible for the creation of vortex filaments is the *vortex-stretching term*,

$$(\omega \cdot \nabla)u \cdot \omega = S\omega \cdot \omega,$$

where S is the strain matrix. One way to identify the range of (longitudinal) scales at which the dynamics of creation and persistence of vortex filaments takes place is to identify the *range of scales of positivity* of $S\omega \cdot \omega$. In Sec. IV, we will show that the range of positivity of $S\omega \cdot \omega$—in a statistically significant sense—extends from a power of a Kraichnan-type micro-scale to the macro-scale R_0. It is worth pointing out that the argument is *dynamic*—based on ensemble averaging local dynamics described by the full 3D Navier-Stokes system.

IV. A DYNAMIC ESTIMATE ON THE VORTEX-STRETCHING TERM ACROSS A RANGE OF SCALES

We begin by recalling the concept of *ensemble averaging* with respect to (K_1, K_2)-*covers at scale R*, introduced in our work on existence and locality of turbulent cascades in *physical scales* of 3D incompressible flows[38–41] (for more details, see, e.g., Ref. 41).

Let $R_0 > 0$, and assume (for convenience) that the macro-scale domain of interest is the ball $B(0, R_0)$, $B(0, 2R_0)$ contained in the global spatial domain Ω. Consider a locally integrable physical density of interest f, and let $0 < R \le R_0$; the time interval of interest is $(0, T)$.

In what follows, we utilize refined spatiotemporal cut-off functions $\phi = \phi_{x_0,R,T} = \psi\,\eta$, where $\eta = \eta_T(t) \in C^\infty(0, T)$ and $\psi = \psi_{x_0,R}(x) \in \mathcal{D}(B(x_0, 2R))$ satisfying

$$0 \le \eta \le 1, \quad \eta = 0 \text{ on } (0, T/3), \quad \eta = 1 \text{ on } (2T/3, T), \quad \frac{|\eta'|}{\eta^{\rho_1}} \le \frac{C_0}{T}, \tag{4.1}$$

and

$$0 \le \psi \le 1, \quad \psi = 1 \text{ on } B(x_0, R), \quad \frac{|\nabla\psi|}{\psi^{\rho_2}} \le \frac{C_0}{R}, \quad \frac{|\triangle\psi|}{\psi^{2\rho_2-1}} \le \frac{C_0}{R^2}, \tag{4.2}$$

for some $\frac{1}{2} < \rho_1, \rho_2 < 1$. In particular, $\phi_0 = \psi_0\eta$, where ψ_0 is the spatial cut-off (as above) corresponding to $x_0 = 0$ and $R = R_0$.

For x_0 near the boundary of the macro-scale domain, $S(0, R_0)$, we assume additional conditions,

$$0 \le \psi \le \psi_0, \tag{4.3}$$

and, if $B(x_0, R) \not\subset B(0, R_0)$, then $\psi \in \mathcal{D}(B(0, 2R_0))$ with $\psi = 1$ on $B(x_0, R) \cap B(0, R_0)$ satisfying, in addition to (4.2), the following:

$$\psi = \psi_0 \text{ on the part of the cone centered at zero and passing through}$$
$$S(0, R_0) \cap B(x_0, R) \text{ between } S(0, R_0) \text{ and } S(0, 2R_0), \tag{4.4}$$

and

$$\begin{aligned}&\psi = 0 \text{ on } B(0, R_0) \setminus B(x_0, 2R) \text{ and outside the part of the cone}\\ &\text{centered at zero and passing through } S(0, R_0) \cap B(x_0, 2R) \qquad (4.5)\\ &\text{between } S(0, R_0) \text{ and } S(0, 2R_0).\end{aligned}$$

A *physical scale R* is realized via suitable ensemble averaging of the localized quantities with respect to "(K_1, K_2)-covers" at scale R.

Let K_1 and K_2 be two positive integers, and $0 < R \le R_0$. A cover $\{B(x_i, R)\}_{i=1}^n$ of the macro-scale domain $B(0, R_0)$ is a (K_1, K_2)-*cover at scale R* if

$$\left(\frac{R_0}{R}\right)^3 \le n \le K_1 \left(\frac{R_0}{R}\right)^3,$$

and any point x in $B(0, R_0)$ is covered by at most K_2 balls $B(x_i, 2R)$. The parameters K_1 and K_2 represent the maximal *global* and *local multiplicities*, respectively. Considering the time-averaged, per unit mass—spatially localized to the cover elements $B(x_i, R)$—local quantities $\hat{f}_{x_i,R}$,

$$\hat{f}_{x_i,R} = \frac{1}{T}\int_0^T \frac{1}{R^3}\int_{B(x_i,2R)} f(x,t)\phi^{\delta}_{x_i,R,T}(x,t)\,dx\,dt,$$

(for some $0 < \delta \le 1$), the *ensemble average* $\langle F\rangle_R$ is defined as

$$\langle F\rangle_R = \frac{1}{n}\sum_{i=1}^{n} \hat{f}_{x_i,R}\,.$$

The ensemble averages (with the fixed multiplicities K_1 and K_2) act as a "detector" of significant sign-fluctuations of the density in view. More precisely, if the density exhibits significant sign-fluctuations on the scales comparable or greater than R, the ensemble averages at scale R—with respect to all admissible (K_1, K_2)-covers—will respond by exhibiting a wide range of values, from positive through zero to negative. This can be seen by rearranging the cover elements to emphasize the positive and the negative parts of the density, respectively. The larger the multiplicities, the finer the detection. In contrast, if the ensemble averages at scale R—with respect to all admissible (K_2, K_2)-covers (again, with the fixed multiplicities)—are nearly independent on the particular choice of the cover, and say positive, this indicates that the density is essentially positive on the scales comparable or greater than R.

As expected, for a non-negative density f, all the averages are comparable to each other throughout the full range of scales R, $0 < R \le R_0$; in particular, they are all comparable to the simple average over the integral domain. More precisely,

$$\frac{1}{K_*}F_0 \le \langle F\rangle_R \le K_* F_0, \qquad (4.6)$$

for all $0 < R \le R_0$, where

$$F_0 = \frac{1}{T}\int \frac{1}{R_0^3}\int f(x,t)\phi_0^{\delta}(x,t)\,dx\,dt,$$

and $K_* = K_*(K_1, K_2) > 1$.

Consider now a global-in-time weak solution u (say, a global-in-time "local Leray solution" on $\mathbb{R}^3 \times (0, \infty)$ in the sense of Ref. 29), and let T be the first (possible) singular time.

A spatiotemporal localization of the evolution of the enstrophy was presented in Refs. 32 and 35. Considering a (K_1, K_2)-cover $\{B(x_i, R)\}_{i=1}^n$ at scale R, the following expression for the time-integrated $B(x_i, R)$-localized vortex-stretching terms transpires,

$$\int_0^t \int (\omega \cdot \nabla) u \cdot \phi_i \, \omega \, dx \, ds = \int \frac{1}{2} |\omega(x,t)|^2 \psi_i(x) \, dx + \int_0^t \int |\nabla \omega|^2 \phi_i \, dx \, ds$$
$$- \int_0^t \int \frac{1}{2} |\omega|^2 \big((\phi_i)_s + \triangle \phi_i \big) \, dx \, ds$$
$$- \int_0^t \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds, \tag{4.7}$$

for any t in $(2T/3, T)$, and $1 \le i \le n$.

Denoting the time-averaged local vortex-stretching terms per unit mass associated to the cover element $B(x_i, R)$ by $VST_{x_i,R,t}$,

$$VST_{x_i,R,t} = \frac{1}{t} \int_0^t \frac{1}{R^3} \int (\omega \cdot \nabla) u \cdot \phi_i \, \omega \, dx \, ds, \tag{4.8}$$

the main quantity of interest is the ensemble average of $\{VST_{x_i,R,t}\}_{i=1}^n$,

$$\langle VST \rangle_{R,t} = \frac{1}{n} \sum_{i=1}^n VST_{x_i,R,t}. \tag{4.9}$$

Before stating the theorem, let us introduce the key macro-scale quantities, E_0, P_0, and σ_0. Denote by $E_{0,\,t}$ time-averaged enstrophy per unit mass associated with the macro-scale domain $B(0, 2R_0) \times (0, t)$,

$$E_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{1/2} \, dx \, ds,$$

by $P_{0,\,t}$ a modified time-averaged palinstrophy per unit mass,

$$P_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, ds + \frac{1}{t} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x,t)|^2 \psi_0(x) \, dx,$$

(the modification is due to the shape of the temporal cut-off η), and by $\sigma_{0,\,t}$ a corresponding Kraichnan-type scale,

$$\sigma_{0,t} = \left(\frac{E_{0,t}}{P_{0,t}} \right)^{\frac{1}{2}}.$$

Until now, there was no connection between the spatial macro-scale R_0 and the global time scale T. At this point, it is convenient to assume $R_0 \le \sqrt{T}$ (in addition, without loss of generality, suppose that $T \le 1$); in the case $R_0 > \sqrt{T}$, the proof can be modified similarly to the calculations in Refs. 39 and 40.

Theorem 4.1: *Let u be a global-in-time local Leray solution on* $\mathbb{R}^3 \times (0, \infty)$, *regular on* $(0, T)$. *Suppose that, for some* $t \in (2T/3, T)$,

$$C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} < R_0, \tag{4.10}$$

where $M_0 = \sup_t \int_{B(0,2R_0)} |u|^2 < \infty$, *and* $C > 1$ *a suitable constant depending only on the cover parameters.*

Then,

$$\frac{1}{C} P_{0,t} \le \langle VST \rangle_{R,t} \le C \, P_{0,t}, \tag{4.11}$$

for all R satisfying

$$C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} \le R \le R_0. \tag{4.12}$$

Remark 4.1: The macro-scale domain $B(0, R_0)$ is placed at the origin for convenience only; it can be placed anywhere in $\mathbb{R}^3$.

Proof: Recall that

$$\int_0^t \int (\omega \cdot \nabla) u \cdot \phi_i \, \omega \, dx \, ds = \int \frac{1}{2} |\omega(x, t)|^2 \psi_i(x) \, dx + \int_0^t \int |\nabla \omega|^2 \phi_i \, dx \, ds$$
$$- \int_0^t \int \frac{1}{2} |\omega|^2 \big((\phi_i)_s + \triangle \phi_i \big) \, dx \, ds$$
$$- \int_0^t \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds, \tag{4.13}$$

for any t in $(2T/3, T)$, and $1 \leq i \leq n$; the last two terms need to be estimated.

For the first term, the properties of the spatiotemporal cut-off function ϕ_i—setting $\rho_1 = \rho_2 = 3/4$—paired with the condition $t > \frac{2}{3} T \geq \frac{2}{3} R_0^2 \geq \frac{2}{3} R^2$ yield

$$\int_0^t \int \frac{1}{2} |\omega|^2 \big((\phi_i)_s + \triangle \phi_i \big) \, dx \, ds \leq C \frac{1}{R^2} \int_0^t \int |\omega|^2 \phi_i^{1/2} \, dx \, ds. \tag{4.14}$$

The second term—the localized transport term—will be estimated similarly as in Ref. 32; the powers of the cut-off function ϕ_i will be distributed somewhat differently leading to a bit more precise estimate.

Setting the cut-off parameters ρ_1 and ρ_2 to 7/8, the following sequence of bounds transpires.

$$\int_0^t \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds$$
$$\leq C \frac{1}{R} \int_0^t \int \Big(|\omega|^2 \phi_i \Big)^{3/4} |u| \, \Big(|\omega|^2 \phi_i^{1/2} \Big)^{1/4} \, dx \, ds$$
$$\leq C \frac{1}{R} \int_0^t \Big(\int |u|^{4/3} |\omega|^2 \phi_i \, dx \Big)^{3/4} \, \Big(\int |\omega|^2 \phi_i^{1/2} \, dx \Big)^{1/4} ds. \tag{4.15}$$

The first spatial integral is bounded as follows:

$$\int |u|^{4/3} |\omega|^2 \phi_i \, dx$$
$$\leq \Big(\sup_s \int_{B(x_i, 2R)} |u|^2 \, dx \Big)^{2/3} \, \Big(\int \big(|\omega| \phi_i^{1/2} \big)^6 \, dx \Big)^{1/3}$$
$$\leq C \, \Big(\sup_s \int_{B(x_i, 2R)} |u|^2 \, dx \Big)^{2/3} \, \Big(\int |\nabla \big(\phi_i^{1/2} \omega \big)|^2 \, dx \Big), \tag{4.16}$$

(the last line by the Sobolev Embedding Theorem).

Combining the bounds (4.15) and (4.16) leads to

$$\int_0^t \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds$$
$$\leq C \frac{1}{R} \Big(\sup_s \int_{B(x_i, 2R)} |u|^2 \, dx \Big)^{1/2} \Big(\int_0^t \int |\nabla \big(\phi_i^{1/2} \omega \big)|^2 \, dx \, ds \Big)^{3/4} \Big(\int_0^t \int |\omega|^2 \phi_i^{1/2} \, dx \, ds \Big)^{1/4}$$
$$\leq \frac{1}{8} \int_0^t \int |\nabla \big(\phi_i^{1/2} \omega \big)|^2 \, dx \, ds + C \frac{\Big(\sup_s \int_{B(x_i, 2R)} |u|^2 \, dx \Big)^2}{R^2} \frac{1}{R^2} \int_0^t \int |\omega|^2 \phi_i^{1/2} \, dx \, ds. \tag{4.17}$$

Utilizing the commutator estimate (with $\rho_1 = \rho_2 = 3/4$)

$$\begin{aligned}\int |\nabla(\phi_i^{\frac{1}{2}}\omega)|^2\,dx \\ &\le 2\int |\nabla\omega|^2\phi_i\,dx + C\int\left(\frac{|\nabla\phi_i|}{\phi_i^{\frac{1}{2}}}\right)^2|\omega|^2\,dx \\ &\le 2\int |\nabla\omega|^2\phi_i\,dx + C\frac{1}{R^2}\int|\omega|^2\phi_i^{1/2}\,dx,\end{aligned} \tag{4.18}$$

in the first term of the above inequality yields the final bound for the localized transport term,

$$\begin{aligned}\int_0^t\int\frac{1}{2}|\omega|^2(u\cdot\nabla\phi_i)\,dx\,ds \\ &\le \frac{1}{4}\int_0^t\int|\nabla\omega|^2\phi_i\,dx\,ds + C\frac{1}{R^2}\int_0^t\int|\omega|^2\phi_i^{1/2}\,dx\,ds \\ &+ C\left(\frac{\sup_s\int_{B(x_i,2R)}|u|^2\,dx}{R}\right)^2\frac{1}{R^2}\int_0^t\int|\omega|^2\phi_i^{1/2}\,dx\,ds.\end{aligned} \tag{4.19}$$

Note that the factor

$$\frac{\sup_s\int_{B(x_i,2R)}|u|^2\,dx}{R}$$

is scaling-invariant, and—consequently—the bound (4.19) is dimensionally correct. However, for an arbitrary global-in-time local Leray solution, it is not *a priori* bounded (it is *a priori* bounded, e.g., assuming a uniform-in-time bound on the L^3-norm; this, however, automatically implies regularity[30]). The best one can do in general is to simply write $\sup_s\int_{B(x_i,2R)}|u|^2\,dx \le M_0$.

Taking this into account, and applying the bounds (4.14) and (4.19) in the expression (4.13)—describing the dynamics of the vortex-stretching term localized to the cover element $B(x_i, R)$—leads to

$$\int_0^t\int(\omega\cdot\nabla)u\cdot\phi_i\,\omega\,dx\,ds = \int\frac{1}{2}|\omega(x,t)|^2\psi_i(x)\,dx + \int_0^t\int|\nabla\omega|^2\phi_i\,dx\,ds + I_i, \tag{4.20}$$

where

$$|I_i| \le \frac{1}{4}\int_0^t\int|\nabla\omega|^2\phi_i\,dx\,ds + C\max\{M_0^2, R_0^{\frac{1}{2}}\}\frac{1}{R^4}\int_0^t\int|\omega|^2\phi_i^{1/2}\,dx\,ds.$$

Ensemble-averaging (4.20) and utilizing the inequality (4.6) several times implies that as long as

$$C\max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\}\,\sigma_{0,t}^{\frac{1}{2}} < R_0, \tag{4.21}$$

$$\frac{1}{C}\,P_{0,t} \le \langle VST\rangle_{R,t} \le C\,P_{0,t},$$

for all R satisfying

$$C\max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\}\,\sigma_{0,t}^{\frac{1}{2}} \le R \le R_0.$$

□

Remark 4.2: Suppose that T is the first (possible) singular time, and that the macro-scale domain contains some of the spatial singularities (at time T). This, paired with the assumption that u is a global-in-time local Leray solution implies

$$\lim_{t\to T^-}\sigma_{0,t} = 0;$$

hence, the condition (4.10) in the theorem is automatically satisfied for any t near the singular time T.

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