AN ABSTRACT OF THE THESIS OF

Samantha A. Smee for the degree of Honors Baccalaureate of Science in Mathematics presented on May 26, 2010. Title: Applying Kuhn's Theory to the Development of Mathematics.

Abstract Approved:

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Sharyn Clough

Presented in this paper is an analysis of the body of scholarly work that attempts to apply Thomas Kuhn's theory of scientific revolutions to mathematics. These applications vary on several levels, from the authors' interpretation of Kuhn's original work to the definitions used for terms such as paradigm, revolution, and mathematics. The number of these works combined with their variety makes it difficult to grasp the whole of the literature. This paper is an attempt to consolidate and streamline this information in order to facilitate further research on the subject. This analysis makes it clear that Kuhn's theory can be successfully applied to mathematics in a way that provides an accurate view of the growth of mathematical knowledge.

Key Words: Thomas Kuhn, philosophy of science, mathematics, paradigms, revolutions

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Applying Kuhn's Theories to the Development of Mathematics

by

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A PROJECT

submitted to

Oregon State University
University Honors College

in partial fulfillment
of the requirements
for the degree of

Honors Baccalaureate of Science in Mathematics (Honors Associate)

Presented May 26, 2010

Commencement June 2010

Honors Baccalaureate of Science in Mathematics project of Samantha A. Smee presented May 26, 2010.

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I understand that my project will become part of the permanent collection of Oregon State University, University Honors College. My signature below authorizes the release of my project to any reader upon request.

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1. Introduction

Thomas Kuhn published *The Structure of Scientific Revolutions* in 1962. The book was in part an effort to determine systematically the nature of sociological influences on the work of scientists. To accomplish this, Kuhn analyzes the development of science from a historical perspective, focusing on the change from a geocentric view of the solar system (in this case the Ptolemaic system) to a heliocentric one (also known as the Copernican system). He determines that the methods used to create both theories were essentially scientific; however he argues that many other factors went into the reasoning that scientists used as they changed from one view to the other. The change was not just about the traditionally recognized virtues of the accuracy, simplicity, scope, fruitfulness or consistency of either theory; there were also political and religious motivations to consider. Kuhn carefully considers the historical context of the development of particular scientific theories, acknowledging that scientists are humans who are influenced in their choice of scientific theory by their broader social environments. However, he incorporates the explanatory role of this broader social influence without thereby portraying scientists as irrational.[1] His view of the social nature of scientific knowledge enabled Kuhn to account for changes in scientific theories without necessarily labeling the discarded ideas and theories as irrelevant or unscientific.

A side effect of Kuhn's theory about the role of social influences in theory choice is that it shows how science does not have to grow by straightforward accumulation of knowledge; i.e. by only building on existing theories. In the case of astronomy, the heliocentric view was developed in part by individuals who did not hold as tightly to certain religious beliefs as

[1] Early commentators on Kuhn's work inaccurately read this aspect of his theory as portraying all theory-choice in science as a product of 'mob psychology' (Lakatos, 1974).

those people who preferred a geocentric view and, as such, Copernicans were open to ideas that Ptolemaic followers were not. Because of the influence of their non-scientific beliefs, the Copernicans developed a system that did not build upon the Ptolemaic system. Instead, the new system replaced the old entirely.

As this example clearly demonstrates, Kuhn does not believe that scientific knowledge accumulates in any straightforward fashion where newer theories must necessarily build on old ones. Instead, he thinks that scientific knowledge progresses in two distinct phases. These phases are determined with reference to a set of laws and assumptions, agreed upon both consciously and unconsciously by scientists, which Kuhn refers to as a ‘paradigm’. Work that is contained within the paradigm is called ‘normal science’, while work that challenges a paradigm is called ‘revolutionary science’. Kuhn believes that science cycles through alternating phases of normal and revolutionary science. In the former, the traditionally recognized criteria of accuracy, scope, simplicity, fruitfulness, and consistency are employed to evaluate research results. During the latter, these criteria do not suffice for making decisions between competing paradigms, and extra-scientific concerns are often considered.

Since the publication of Kuhn’s book in 1962, many attempts have been made to see whether his theory can provide similar insights into the structure of other disciplines. Of particular interest to me is the application to mathematics. The growth of mathematics is often assumed to be strictly cumulative, meaning that it grows only by building on previous theories.[2] However, I believe that it is more complex than that. Do changes of the non-cumulative type, characterized by replacement of an old theory by a new one, as described in the astronomy debate happen within mathematics? Like scientists, mathematicians are human and are capable of being influenced by the social worlds they live in, as well as their views about their discipline. These broader influences impact how scientists choose between competing paradigms; since mathematicians have access to the same tools, there is potential for them to make similar choices. Exploring the implications of Kuhn's work for mathematics may give mathematicians a way to classify the changes that occur in mathematics, as well as a way to gain insight into the effects these social influences have on the discipline as a whole. Once we gain a greater understanding of how mathematics develops, we may be able to apply this knowledge to problems in mathematics education.

Although Kuhn rarely mentions mathematics in *Structure*, when he does, it is most often in conjunction with various sciences. From Kuhn's words alone, most would infer that he was content to consider mathematics to be a type of science. However, the consensus in the mathematical community appears to be the opposite. Mathematics may be related to science, and is certainly a useful tool for scientists, but most mathematicians believe that mathematics and science are not the same. However, even if mathematicians are right about this, the question remains whether mathematics differs enough from science that Kuhn's theories regarding science are not applicable to mathematics. This is a question that must be answered before proceeding to determine whether and/or how to apply Kuhn to mathematics.

[2] (Crowe, 1992)

The nature of mathematics is not all that must be addressed. Kuhn's use of terms such as 'revolution' and 'paradigm' must also be examined since these are open to interpretation. Much of the debate over the applicability of Kuhn to mathematics can be reduced to a debate of definitions about these key concepts. Changes in each definition affect the question whether and/or how Kuhn's work can be applied to mathematics. Once the definitions have been determined, using Kuhn's theories to classify the growth of mathematical knowledge becomes a simpler task. My review of the literature shows that, when these terms are properly defined, it does indeed appear that there can be, and have been non-cumulative changes in mathematics.

1.1 Thomas Kuhn's Theory of Scientific Revolutions

1.1.1 Paradigms and Normal Science

Clearly, paradigms are a key component of Kuhn's description of science and scientific change. Kuhn defines a paradigm as the collection of basic assumptions that are agreed upon, often unconsciously or implicitly, by the members within a discipline.[3] These assumptions can include scientific laws, observations, and theories. Though the scientists are not necessarily aware of it, they accept the defining elements of the paradigm as true, and as such do not spend time continuously questioning these elements. In the example of the Ptolemaic and Copernican theories, the paradigms of each would consist of different assumptions about moving bodies, the makeup of space and matter, as well as what would count as exemplars of these phenomena.

[3] See Chapter II in (Kuhn, 1962).

Normal science is the process of using a paradigm as a collection of reference points for discovering knowledge about the world. Kuhn describes this process as 'puzzle-solving'.[4] The term refers to how paradigms, while constraining the kinds of questions that are asked, do not supply all the answers within themselves. There are still unanswered questions about the world that scientists can address within the context of a paradigm, in order to develop their understanding of the world. These questions are puzzles within a paradigm. Puzzles can take many forms, like finding a more precise measurement of a physical constant, or testing the extension of an already known theory.[5]

1.1.2 Anomalies, Crises, and Revolutions

Normal science is done entirely within the context of an existing paradigm. As such, it is generally assumed by scientists that the results of their puzzle solving will be consistent with the elements that make up that paradigm. However, that is not always the case. Kuhn calls these contrary instances 'anomalies'.[6] An anomaly can be caused by many different factors, like human or experimental error. These problems can be identified and corrected. In cases when an explanation cannot be found, a single anomaly can often be explained away as a special case.

However, if unexplainable anomalies begin to accumulate, scientists typically begin to suspect a fault in the current paradigm. Anomalies bring these problems to light, and cause researchers to have doubts about the assumptions they are working with. This uncertainty regarding their paradigm causes some scientists to abandon normal science in favor of

[4] See Chapter IV in (Kuhn, 1962).
[5] (Kuhn, 1970, p. 15)
[6] (Kuhn, 1962, p. 52)

looking for solutions or a new paradigm. At this point, the discipline is in 'crisis'.[7] During a time of crisis, all manner of solutions are developed to fix the paradigm. Sometimes a paradigm can be salvaged, while other times it is replaced. When the old paradigm is replaced with a new one in response to crisis, a scientific revolution has occurred. Once the community has become acclimatized to the new paradigm, normal science begins again.

Shifting between paradigms is not an easy process. Often, if standard criteria of a good theory, such as accuracy and scope, cannot be used to compare the competing paradigms, then scientists may turn to social factors to help in the decision making process. This can be seen in the astronomy example, where religious and political influences played a greater role than scientific factors in the change from the Ptolemaic to the Copernican system. It is in this way that Kuhn's theory accounts for and explains the role of sociological influences in scientific practice.

1.2 Problems Arising Within Kuhn's Theory

The most important problem with Kuhn's theory, as mentioned above, is that of definitions. For example, one critic claimed to find the word 'paradigm' used in over twenty different ways in Kuhn's book.[8] Though that result may be slightly exaggerated, there is no doubt that Kuhn's definition is rather ambiguous. The proper scale of a paradigm and its relationship to a scientific community are not always clear. Similarly with the elements in a paradigm. Kuhn is not completely clear whether all parts of the paradigm must be directly related to the scientific theories in question. Is it possible for other ideas, such as shared values and ideals

[7] (Kuhn, 1962, p. 69)
[8] (Gillies, 1992, p. 270)

to be part of a paradigm? If so, this would open up another way that paradigms might change. For example, Kuhn spends a lot of time discussing how paradigms affect and are affected by the education of scientists.[9] Older science textbooks often teach the laws of the current paradigm without describing the revolutionary science that caused the paradigm to begin with. This reinforces the conception of science as growing through straightforward accumulation. However, since the publication of Kuhn's work, more awareness of the history of science has changed the way some textbooks are written. Newer textbooks tend to place more emphasis on the historical aspects of science and, as such, these books portray science as progressing in a more Kuhnian fashion. This is certainly an important change in the way we look at science, but is it a paradigmatic change? From Kuhn's ambiguous arguments in *Structure* it is impossible to tell.

There are also problems with Kuhn's definition of 'revolution'. While it is clear that a revolution in science means that a great change has occurred, Kuhn's writing in *Structure* makes it difficult to determine how large a change must be to count as a revolution. If the required changes are too large, revolutions would be an extremely rare occurrence; perhaps so rare that it would not be possible to find a pattern in their nature. On the other hand, if the required changes are relatively small, then it could be argued that revolutions are occurring all the time, a situation that would rob the concept of its usefulness. In addition, there are disputes among readers of Kuhn about how much of a previous paradigm may remain after a revolution.[10]

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[9] See Chapter XI in (Kuhn, 1962).
[10] For examples of this in the context of mathematics, see the works of Michael Crowe and Joseph Dauben.

This point is especially important in the case of mathematical revolutions, because of the tendency of previously proved theories to still be held true, even when a considerable change has occurred in mathematics. Some mathematicians and philosophers of mathematics might take this as support for the claim that mathematics grows by straightforward accretion, but it is my opinion that while mathematics can proceed cumulatively it also has periods of non-cumulative growth and even revolution.

2. Mathematics

2.1 Conceptions of Mathematics

Mathematics clearly has very close ties to science, but, as mentioned above, mathematics is not typically regarded as a science. Commonly cited differences between the two are the objects under investigation, and the methods used. In science, it is clear that it is the world around us that is being studied. For example, physics is the study of motion; botany is the study of plants, etc. In mathematics, it is not quite so simple. The objects under investigation in mathematics can be numbers, shapes, or logical relationships. Though we can see examples of these things in the world, their ideal forms, which mathematicians study, are mental constructs. Additionally, mathematics uses deductive methods as opposed to the inductive methods most often used by scientists. Mathematicians and philosophers of mathematics often take these differences to mean that mathematics and science are fundamentally different, and as such must change in different ways.

One attribute often associated with mathematics is that of certainty. The results of mathematical research are commonly thought to be certain, whereas the results of science are fallible. Once a mathematical theorem is proven, many people believe it is true forever. Clearly that is not the case with scientific theories, which is one of the reasons that revolutions occur in science. However, I argue that this is not the case with mathematical theories either.

Due to the use of deductive reasoning in mathematics, theorems can have deductive certainty, but this does not mean that theorems proved in this way are always true. All that deductive certainty entails is validity, i.e. *if* the premises of a proof are true, *then* the conclusion must follow. Since there is no guarantee that any particular premises will always be held true, the truth of the conclusion of a proven theorem is not guaranteed either.

The absence of such guarantees can be shown throughout the history of mathematics. The ancient Greeks began by assuming that incommensurable magnitudes could not exist, i.e. they assumed all numbers could be represented as a ratio of two whole numbers. However, upon the discovery that these magnitudes did exist, the theories that were derived from this assumption no longer held true. Because they were no longer useful to ancient mathematicians, the false theories were discarded. New theories were developed that assumed incommensurable magnitudes—and these new theories did not depend on the previous, now inaccurate theories. The practice of mathematics allows for theories to be discarded, if the premises are later shown to be false. In fact it is just this practice that explains how completely new mathematical theories are introduced. These new theories are examples of non-cumulative growth, as they do not build on the accepted truths of prior theories. Contrary to popular belief, and as the deductive nature of certainty in mathematics makes clear, there are at least a few historical instances of non-cumulative changes in mathematics. Whether these changes constitute *revolutions* in math is the question to which I now turn.

2.1.1 Michael Crowe's Mathematics

Michael Crowe is a professor emeritus in the history and philosophy of science at University of Notre Dame, with some training in mathematics. His 1975 article, *Ten 'laws' concerning* *patterns of change in the history of mathematics*, was one of the first works to address applying Kuhn's theories to mathematics. The last of these ten laws states that revolutions never occur in mathematics. According to Crowe, the most important aspect of his final law is the preposition 'in'. No revolutions ever occur *in* mathematics, but they can occur outside of mathematics, e.g. in mathematical nomenclature, symbolism, methodology, and metamathematics.[11] This implies that Crowe does not consider the latter elements to be part of mathematics proper. However, Crowe never defends this stipulation of what counts as mathematics proper. To exacerbate the problem, not only does he not defend his definition of mathematics proper, but the definition itself remains largely implied. Crowe never gives an example of something that *is* a part of mathematics. The features that, for Crowe, are not a part of mathematics—mathematical nomenclature, symbolism, methodology, and metamathematics—are, he acknowledges, susceptible to revolutionary change.[12] However, I argue that some of the features that Crowe designates as 'non-mathematics', and capable of revolutionary change, should actually be seen as integral to mathematics proper.

'Mathematical nomenclature' simply refers to the names we give to mathematical objects, such as numbers. As times change and languages evolve, the names for mathematical objects may certainly change as well. This may appear to be a trivial part of mathematical knowledge, but when we consider the integral role that abstract entities play in mathematics, we can see its importance. For example, numbers are one of the foundational objects of mathematics, and they are completely abstract. Without names for numbers it would be difficult to use them. Since numbers are the objects upon which the earliest mathematics was

[11] (Crowe, 1975, p. 19)
[12] (Crowe, 1975, p. 19)

based, it is difficult to imagine mathematics progressing far without mathematical nomenclature to help us use them efficiently. A similar argument can be made about symbolism in mathematics. 'Mathematical symbolism' refers to the use of symbols to represent mathematical objects. Just like nomenclature, symbolism allows us to efficiently represent abstract concepts within mathematics. One example of a change in mathematical symbolism is the change from using words to represent unknown quantities to using letters. Both mathematical nomenclature and symbolism are extremely important tools for the use of mathematics. They are perhaps so important that one could argue that mathematics would not have developed into its modern form without them.

Mathematical 'methodology' and 'metamathematics' are very closely connected concepts. Methodology refers to the rules and methods that govern a discipline. In mathematics, this specifically refers to the types of logical methods that are used to make mathematical advances. In contrast, metamathematics is the term used to denote the study of the structure and methods of mathematics, as well as beliefs regarding the nature of mathematics. Both of these aspects of mathematics have changed over time. The development of logical, especially deductive, methods used in mathematics by the ancient Greeks was a methodological change from previous practice, as was the introduction of the rigorous methods of calculus promoted by Cauchy.[13] Metamathematical changes include the acceptance of incommensurable line segments and the changes in mathematical thinking caused by Gödel's incompleteness theorems.[14] Mathematical methodology and metamathematics have given mathematics its very structure. Is it possible for them to be

[13] (Dauben, 1992, p.73)
[14] (Dauben, 1984, p.64)

completely separate from mathematics proper? I think not, as they are clearly intertwined with mathematics at a very basic level, and changes in them serve to shape the progression of mathematics.

To review, Crowe admits that non-cumulative, and even revolutionary changes can and have occurred in the above areas, but he insists that these areas are not a part of mathematics proper. I think this is an error on his part, and I believe the above discussion demonstrates the ways that these areas are integral to mathematics. Because he gives no defense of his definitions and examples of elements within mathematics, Crowe's view of mathematics seems unnecessarily and arbitrarily restrictive. And as I argue below, Crowe's picture of mathematics is inconsistent with the way most mathematicians and philosophers of mathematics see the discipline.

2.1.2 Other views of mathematics

Michael Crowe is not the only one to weigh in on the question of revolutions in mathematics. Many other philosophers and mathematicians have written responses to his 1975 article. Each respondent has had a slightly different view of mathematics, which correspondingly changes their view of the question of revolutions in mathematics.

Caroline Dunmore, a philosopher of mathematics with some training in mathematics, believes that mathematics consists of two parts.[15] The first part consists of mathematical objects, like notation, terminology, definitions, and theorems. The second piece is the views

[15] (Dunmore, 1992, p. 211)

and methods that constitute metamathematics. These two pieces are distinct, according to Dunmore, but, she argues, considered together they *become* mathematics.

In contrast to this position is the mathematics proposed by Joseph Dauben, a professor who studies the history of mathematics at the City University of New York. His view of mathematics combines Dunmore's objects and metamathematics into an inseparable unit.[16]

It is important to consider these different ideas of what mathematics is, because these ideas can affect the outcome of applying Kuhn's theory to mathematics. The views that each philosopher has regarding mathematics influence their approaches to the task of applying Kuhn. As we will see in later sections, each of these definitions of mathematics will correspond to different definitions of paradigms and revolutions.

2.2 Mathematics vs. Views of Mathematics

The main question driving all of the different conceptions of mathematics reviewed above is whether to include meta-level views of mathematics and other abstract entities as part of mathematics proper. The various answers to this question impact the arguments about revolutions in mathematics because typically more examples of revolutions in mathematics can be found when these pieces are included. This may be because it is easier for mathematicians to reinterpret existing theories in terms of their new views of mathematics than to discard the older theories entirely.

[16] (Dauben, 1984)

3. Paradigms in Mathematics

Just as paradigms are an important part of Kuhn's arguments about the structure of science, so too are they important in the analysis of mathematics. An analysis of what would count as a paradigm in mathematics would not only help in understanding the mathematical equivalent of normal science, but would also help in determining what may count as revolutionary mathematics.

3.1 Formalizing Paradigms

In his 1999 paper, *On Classification of Scientific Revolutions,* Ladislav Kvasz, a Slovakian philosopher and mathematician, tries to determine the nature of scientific revolutions, so that any possible implications for mathematics might be more easily identified. To do this, Kvasz reformulates Kuhn's rather ambiguous concept of a paradigm into a much more rigorous definition.

Kvasz starts with something he refers to as an "epistemic framework" of a theory, a concept roughly analogous to Kuhn's paradigm.[17] Kvasz divides the epistemic framework of a theory into three parts: the formal, the conceptual, and the evidential.[18] They serve to provide for those working within the framework the same kind of information that a paradigm provides to scientists. Each piece of the framework brings different information. The formal frame is the formal structure of the theory. It encompasses the theory's symbolic language and descriptions of the theory in that language. The conceptual frame consists of the semantic structure, explanation, and interpretation of the theory. The last piece is the evidential

[17] (Kvasz, 1999, p. 208)
[18] (Kvasz, 1999, p. 211-212)

frame, which contains the perceptive structure of the theory. When taken together these pieces form Kvasz's equivalent of a paradigm.

To illustrate each of these aspects of the framework, Kvasz uses the scientific example of the epistemic framework of Newtonian mechanics. The formal frame of Newtonian mechanics is the symbols, such as m for mass, and also relationships between the symbols, like $F = m \cdot a$, and the description of motion as second-order dynamics, which enhances the meaning of the symbols. The conceptual frame includes the fundamental quantities, e.g. mass and volume, derived quantities, as well as the legitimate explanations and questions allowed in Newtonian mechanics. For example, in Newtonian mechanics we can explain freefall with gravitational force, but it is inappropriate to ask *why* gravity works, because Newtonian mechanics cannot explain that. The evidential frame of Newtonian mechanics is what allows us to perceive mechanics on earth and in space as a unified theory, in contrast to Aristotelian mechanics, for example, which treats the two areas separately. This Newtonian example helps us to more clearly see the nature of an epistemic framework as it applies in science. Kvasz concludes that, conceived of as epistemic frameworks, mathematics has had a number of paradigms and that these have shifted over time.[19]

Euclidean geometry can provide us with a mathematical example of an epistemic framework, which supports Kvasz' claim that the concept of an epistemic framework can be applied to mathematics well as science. The formal frame is made up of the symbols of Euclidean geometry, e.g. A, B, π, $\parallel$, (here the points A and B, pi, and parallel lines), and the description of Euclidean geometry—a constructive geometry where the parallel postulate holds. The

[19] (Kvasz, 1999, Chap. 2)

conceptual frame would contain the fundamental objects, such as lines, planes, and angles, as well as proven theorems and unproven questions (restricted by the parallel postulate). Finally, the modern evidential frame of Euclidean geometry is the perception of Euclidean geometry as one of many different types of geometries.

As can be seen from the above analysis, Kvasz's approach to mathematical paradigms is based primarily on determining the exact nature of the elements within a paradigm. This results in a very formal, yet universal, notion of a paradigm and all of its parts, which can be applied to both mathematics and science.

3.2 Other Interpretations of Mathematical Paradigms

While Kvasz has certainly presented one of the more formal interpretations of Kuhn's concept of a paradigm, and discussed how it may apply to mathematics, his interpretation is not the only one available. In this section I examine the work of Joseph Dauben and Leo Corry who present alternative views regarding paradigms in mathematics.

3.2.1 Cultural Paradigms

In his work on mathematical paradigms, Joseph Dauben examines the impact cultural differences have on the work of mathematicians, specifically in the cultures of ancient Greece and China. He studies the differences in the reactions of these cultures to their separate discoveries of incommensurable magnitudes. The different paradigms in which ancient Greek and Chinese mathematicians worked caused them to have very different responses to this discovery. For the Greeks, incommensurable magnitudes caused problems within their paradigm. However, the Chinese paradigm was already equipped to deal with these magnitudes, so it was not adversely affected.

The ancient Greek paradigm that Dauben refers to is the mathematical practices of the Pythagoreans. The Pythagoreans had an interesting view of the mathematical world. For the Pythagoreans, 'number was the measure of all things', meaning that everything could be assigned some ratio of whole numbers.[20] Essentially, the Pythagoreans understood the world as being wholly rational. Unfortunately, these views—which Dauben considers to be something analogous to a 'Pythagorean paradigm'—caused trouble for the Pythagoreans when it came to computing certain geometric entities. For example, consider a right triangle with two legs of length 1. Using the Pythagorean Theorem, the remaining side of the triangle can be computed to be $\sqrt{2}$. However, $\sqrt{2}$ is irrational and cannot be expressed as a ratio, so any Pythagoreans attempting to compute this would be unpleasantly surprised; unable to comprehend such a number. Thus, it is easy to see why the discovery of irrational numbers caused such uproar among the Pythagoreans. The existence of irrational numbers went against the Pythagorean paradigm, inciting great changes in Greek mathematics. As we shall see, however, the same situation produced a different outcome in the context of Chinese mathematics.

For his knowledge of Chinese mathematics, Dauben refers to the ancient Chinese mathematics text, *Jiu Zhang Suan Shu* (*Nine Chapters on the Mathematical Art). Nine Chapters* is one of the oldest Chinese mathematical texts. It is often considered to be the Chinese equivalent of Euclid's *Elements* as it was a fundamental text in mathematics for the

[20] (Dauben, 1995, p. 127)

ancient Chinese. Chapter four of *Nine Chapters*, *Shao Guang*, discusses the extraction of square roots. This is described using the geometric process of finding the length of the side of a square that has a given area. Chinese mathematicians developed an algorithm to compute these roots, and if the given area was a perfect square, such as 16 or 36, the algorithm terminated in a finite number of steps. However, if the area was not a perfect square, then the steps could continue indefinitely. In these cases, if the Chinese did not find a solution within a certain number of steps, they ceased working, saying that the number was incomputable. Unlike Greek mathematicians however, the Chinese were not troubled by this result, likely, Dauben argues, because of the differences in these cultural paradigms. Nothing in the Chinese paradigm prevented Chinese mathematician from accepting irrational numbers as they were, whereas the Pythagorean paradigm rejected them outright. Though the Chinese were interested in proofs, they did not use the axiomatic methods that the Greeks were so well known for. Instead, ancient Chinese mathematics was primarily algorithmic in nature. In addition, the Chinese did not hold to the Pythagorean ideal of the world as rational in measure. Thus there was no contradiction to the structure of Chinese mathematics, or to their mathematical world-view. It is for these reasons that Chinese mathematicians, while recognizing that irrational numbers were different from rational ones, did not have need for a change in their paradigm at this discovery.

Though Dauben's look at mathematical paradigms is not as thorough as Kvasz's, it definitely shows how cultural influences can be interpreted in a paradigmatic fashion. This is an especially interesting viewpoint when considering ancient mathematics, as the elements in an ancient paradigm, such as theorems, constants, etc., are not as likely to be as cut and dry as they are in the present.

3.2.2 Structural Paradigms

Leo Corry, a philosopher of science at Tel-Aviv, has also analyzed Kuhn's notion of paradigm in relation to mathematics. He thinks that, though Kuhn's idea of a paradigm is rather ambiguous, we can assert some fundamental qualities that all paradigms must have*. First, a paradigm must be something that differs from an individual discovery or theory. Secondly, a paradigm must be able to influence the development of theories. With this classification of a paradigm in mind, Corry turns to an example regarding the structure of modern algebra.[21]

Before the rise of modern, structural algebra, algebra was considered to be the study of algebraic forms and polynomial equations, especially the problem of equation solvability. However, the publication of the earliest modern algebra textbooks brought forth a new emphasis on algebra as the study of algebraic structures.[22] This emphasis is what Corry wishes to consider a paradigm. Though there are no theorems explicitly stating that algebra is the study of structures, the changes to this view of the discipline clearly shaped our modern theories of algebra. In addition, Corry notes that one unique feature of these types of mathematical paradigms is that there is no logical reason why a mathematician cannot work within both the nineteenth-century algebra paradigm, and the modern algebra paradigm

[21] (Corry, 1995, p. 185)
[22] For example, Corry cites *Moderne Algebra* [1930] by B. L. van der Waerden as the first text on algebraic structures. (Corry, 1995, p. 186)

simultaneously. This is an interesting point, because there is much discussion about whether scientists are capable of such work during a shift in scientific paradigm.

All three of these different pictures of mathematical paradigms, from Kvasz to Corry, place emphasis on different aspects of Kuhn's original concept. Kvasz emphasizes a paradigm as a network of interconnected theories and concept, while Dauben focuses on the cultural aspects that may be included in a paradigm. Finally, Corry considers as aspects of a paradigm those underlying assumptions mathematicians have about the structure of their particular discipline.

4. ***Revolutions in Mathematics***

The word ‘revolution’, in the sense that Thomas Kuhn uses it, means a great change within a discipline—a specific type of upheaval. However, as I have noted, Kuhn is ambiguous as to how great a change must be in order to constitute a revolution and how much of a previous incarnation of a discipline should remain post-revolution. These issues have been the cause of a great deal of conflict in the discussion of revolutions in mathematics. In this section I discuss the definitions of revolution that appear most often in the literature on this topic, as well as the way that the various definitions of mathematics and paradigms affect these discussions about revolutions.

4.1 Strict Revolutions

In this section I will discuss a certain type of revolutions that I call ‘strict revolutions’. Strict revolutions are characterized by the fact that these revolutions completely discard previous ideas and/or theories associated with them.

4.1.1 Michael Crowe

Recall that Michael Crowe claims as his tenth law that there are no revolutions in mathematics. I have argued that, due to Crowe’s views about the nature of mathematics, it is difficult to determine how useful any of his conclusions are. As discussed above, Michael Crowe’s views of mathematics are unclear, because he does not provide specific examples of the elements within mathematics proper.

However, Crowe does provide a clear explanation of what he means by revolution. Unfortunately, it is an overly restrictive definition. It requires that “some previously existing entity must be overthrown and irrevocably discarded”.[23] The conditions of this type of revolution are difficult to satisfy because they are so severe. Given his limited view of mathematics, it is unsurprising that Crowe concludes that there are no revolutions in mathematics. Further, if we remove nomenclature, symbolism, methodology, and metamathematics from within mathematics—as Crowe stipulates that we should— it then seems that there is little left in mathematics, but scattered numerical concepts and computations. This formulation of mathematics does not allow for an in-depth study of the development of mathematics. As it is this developmental type of study that typically motivates the application of Kuhn’s methods to mathematics in the first place, it is easy to see where many authors have found fault with this Crowe’s work. While being one of the first to address the mathematical revolution question in terms of Kuhn, Crowe is too restrictive in his definition of revolution to adequately capture the aspects of mathematics that make such questions interesting to philosophically-minded mathematicians and historians and philosophers of mathematics.

4.1.2 Caroline Dunmore

Like Crowe, Caroline Dunmore uses strict revolutions in her analysis. However, because of her conception of mathematics, her theory offers much more interesting results than Crowe’s. As noted above, Dunmore thinks of mathematics in two parts; an object part, and a metamathematical part. Because she includes the metamathematical aspects of mathematics, her analysis yields actual revolutions. These ‘meta-level’ revolutions, as Dunmore calls

[23] (Crowe, 1975, p. 19)

them, represent strict revolutionary changes in metamathematics. However, Dunmore asserts that no revolutions occur on the object level of mathematics. The example of the shift from Euclidean to non-Euclidian geometries can be interpreted as a meta-level revolution.[24]

According to Dunmore, when non-Euclidean geometries were developed there was a revolution in the ways geometry was conceived; meta-mathematically speaking, there was a change from thinking of the single Euclidean geometry as the only possible geometry to the notion that Euclidean geometry was consistent with many other non-Euclidean geometries. Because the first idea is incompatible with the latter one, it was discarded, causing a strict revolution. On the other hand, Euclidean geometry is still used; so on the object level there was no revolution. I argue, however, that there are numerous other ways to interpret the geometry example that reveal revolutions even in the object level.

4.2 Conceptual Revolutions

This section concerns what I will call ‘conceptual revolutions’. Unlike strict revolutions, conceptual revolutions leave previous concepts in place, albeit often in a diminished capacity.

4.2.1 Joseph Dauben

Joseph Dauben, in a 1984 response to Crowe, offers another definition of revolution. He felt that Crowe’s definition was too restrictive to be useful to philosophers and historians of mathematics. Dauben’s work is focused on ‘conceptual revolutions’, a view of revolutions that allows for the possibility of revolutionary changes in views regarding mathematics, and

[24] (Dunmore, 1992, p.212)

changes within metamathematics. These revolutions do not require the complete elimination of previous entities the way strict revolutions do. For example, the creation of non-Euclidean geometry caused a change in mathematicians' conceptions of mathematics. Instead of holding to the previous belief that Euclidean geometry was the only possible geometry, they changed their views, allowing non-Euclidean geometries to become legitimate in mathematics. This is a good example of the type of revolution Dauben proposes, because non-Euclidean geometries did not eliminate Euclidean geometry. Instead, Euclidean geometry became one of many different geometries. This is the essence of Dauben's revolutions: Nothing is eliminated and the former concepts are retained, just with a significantly lower stature than they had previously.

4.2.2 Yuxin Zheng

Yuxin Zheng, a teacher of mathematics and professor of philosophy at Nanjing University, holds a view of mathematical revolutions that is a hybrid of the views of Dunmore and Dauben. Zheng believes that best explanations of the concepts 'mathematics' and 'revolution' will come from careful observation of the ordinary usages of the words.[25] It is his opinion that Caroline Dunmore presents an appropriate view of mathematics, and Joseph Dauben's view of revolutions is accurate. Using the example of Euclidean and non-Euclidean geometry, we can note that Zheng holds the view that there were revolutions on both the meta- and object-levels of mathematics. However, instead of strict revolutions, as Dunmore would define them, these revolutions were conceptual ones. Both the metamathematics and the objects of Euclidean geometry were left behind and incorporated into a new comprehensive theory of multiple geometries.

[25] (Zheng, 1992, p. 171)

4.3 Multiple Revolutions

This section discusses those theories that include multiple types of revolutions. Some of these notions of revolutions are the same as the ones discussed above, but new definitions are also introduced. The primary proponent of this type of analysis is Ladislav Kvasz, who presents three types of scientific revolutions, created by building off of his concept of an epistemic framework, introduced earlier. Kvasz's term for revolution is 'epistemic ruptures' because they are revolutions that come about via changes in the epistemic framework.

4.3.1 Epistemic Ruptures

Kvasz's identifies four types of rupture, three of which cause revolutionary changes. They are idealizations, re-presentations, objectivisations, and re-formulations.[26] Each of these is caused by different levels of rupturing in the framework. I will treat each of these types of epistemic ruptures in turn, along with mathematical and scientific examples of each type.

The first type of rupture, idealization, is the most drastic type of revolutionary change. It is a change in the way important concepts are idealized in a theory. For example, in science, Kvasz cites the Galilean rupture—the change from Aristotelian to Newtonian mechanics—as an idealization, because of the fundamental changes in how these two theories idealize motion. In mathematics, the Pythagorean rupture, the name which Kvasz uses to denote the change from computational Egyptian and Babylonian mathematics to deductive Greek mathematics, is an idealization because of the changes in these theories' idealization of

[26] (Kvasz, 1999, p. 220-222)

shapes. These types of ruptures cause a revolution that is a strict revolution, in that a former doctrine is replaced by a new one.

The second type of rupture is re-presentation. These ruptures create new areas in the field by re-creating the fundamental objects. There are many examples of these ruptures in mathematics. They include the Cartesian rupture, which represents the birth of analytic geometry and a recreating of curves, as well as the Leibnizean rupture—the birth of differential and integral calculus.

Objectivisations are the third kind of rupture, and the last of the revolutionary kinds. These ruptures are characterized by a change in the ontological status of the objects they affect. One example of this is the Lobachevskyean rupture, which is responsible for non-Euclidean geometry. This rupture did not change any of the mechanics of Euclidean geometry; however it did change the ontological status of geometry. Similarly with the Einsteinian rupture in physics. Both the objectivisation and re-presentation types of epistemic ruptures fall into the category of conceptual revolutions, like the idealizations discussed above.

The last type of rupture is re-formulations. This kind of rupture makes the smallest changes, which are not revolutions but rather direct extensions of a theory. An example of a mathematical re-formulation is the switch from using Roman numerals to using Arabic notation. Both kinds of numerals give the same results in calculations; however Arabic numerals are clearly more convenient to use.

All of these classifications of mathematical revolutions provide solutions to the ambiguities that are present in Kuhn’s work. However, I will argue in the next section that Kvasz’s solution of defining multiple types of scientific revolutions that may be applied to mathematics is likely to provide philosophers and mathematicians with the most accurate view of the development of mathematics.

5. Conclusion

From my review of the literature on the topic of applying the theories of Kuhn to the development of mathematics, I have established that Kuhn's theories can be applied to mathematics, in some very general sense, at least. I have presented numerous examples in support of my claim. However, having established that Kuhn's theories can be applied to mathematics, there are specific factors that influence the details of this application, and it is to these more specific factors that I now turn.

5.1 Factors Influencing the Application of Kuhn's Theory

The first, and perhaps most important of these factors, is the definition of mathematics that one subscribes to. Without a clear image of what mathematics is, Kuhn's theory cannot be applied in a way that is coherent. Second, is the issue of scale of paradigms and, by extension, revolutions. In cases where paradigms are too small in scale—say at the scale of individual theories—revolutions and paradigmatic changes occur too often for mathematical revolutions to be a useful concept to analyze. The opposite effect can happen if paradigms are conceived of on an overly large scale. In these cases, shifts rarely occur at all, which prohibits mathematicians from seeing any kind of useful pattern in the changes. With these important factors in mind, I next make a case for how best to conceive of the details of applying Kuhn's theories to mathematics.

5.2 The Ideal Application of Kuhn to Mathematics

It is my opinion that, in response to the first factor noted above, the most useful way to define mathematics is modeled by Joseph Dauben, namely, his view that mathematics consists not only of theories and concepts, but also of metamathematics and mathematicians' notions of how mathematics behaves. I believe that the most interesting effect of using Kuhn's theory in conjunction with mathematics is the insights it can give into these additional aspects of various mathematical disciplines. For example, in the often mentioned cases of the discovery of non-Euclidean geometries, I find the meta-mathematical conceptual leap from the mindset of a singular geometry (Euclidean) to thinking instead of multiple geometries (non-Euclidean) to be the most important aspect of this shift. Therefore, to me, metamathematical concepts are an essential part of any definition of mathematics.

With respect to defining paradigms, I agree with Leo Corry's idea that a paradigm cannot be an individual theory or result, and that it must be on such a scale that it is able to influence the development of new theories. However, I believe that there is an important historical aspect to what paradigms entail in mathematics. In Joseph Dauben's work, he emphasized the cultural paradigms of ancient mathematicians. In moving to more recent time periods, however, I believe that cultural differences have become less salient, due to the global nature of mathematics as a discipline. With the advent of modern communication systems, mathematicians are able to communicate with international colleagues in a way that was not possible previously. This has virtually eliminated the more parochial nature of mathematical work that existed before modern times. It is for this reasons that I believe mathematical paradigms are most accurately viewed as being dependent on historical context. In the case of primitive mathematics, paradigms may not even be an applicable concept. Moving to ancient mathematics, cultural influences have a great influence. And finally, in more modern times, paradigms are based more on structure and theorems, and less on cultural factors.

Finally, in considering the topic of revolutions, I think that, because it is evident that many different types of revolutions are possible in mathematics, the strategy of using multiple conceptions of revolutions (following the example of Ladislav Kvasz) will enable the most accurate and detailed representation of the effects that revolutions have on mathematics. Using such a classification of revolution in this kind of project will enable philosophers and mathematicians to examine which types of revolution happen most frequently—because they all occur—and determine the reasons for such a phenomenon. This kind of detailed approach will be most useful in creating an accurate description of mathematical growth.

To summarize, I have argued that, when attempting to apply Kuhn to mathematics, three key things should be included in the analysis. First, any conception of mathematics that is used should involve a metamathematical component. Second, it should be kept in mind that paradigms can be conceived as dependent on historical contexts. Lastly, multiple types of revolutions should be considered. Utilizing this framework provides the most fruitful application of Kuhn to mathematics. It gives both mathematicians and philosophers the most detailed description of the growth of mathematics, which in turn will allow this analysis to be successfully applied to other fields.

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