AN ABSTRACT OF THE THESIS OF

MOON UP PARK for the MASTER OF SCIENCE

in MATHEMATICS presented on June 25, 1970

Title: GENERALIZED MEASURES OF DEFORMATION-RATES IN TORSIONAL FLOWS OF VISCOELASTIC FLUIDS BETWEEN TWO INFINITE PARALLEL PLANES

Abstract approved: Redacted for Privacy

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An introduction to the generalized measures of deformation-rates involving not only velocity gradients but also acceleration gradients is given and constitutive equations using these measures have been discussed. These new constitutive equations are then used to study torsional flows of viscoelastic fluids between two infinite parallel planes. In order to assess the advantage of this theory over the existing theories, a brief review of the existing nonlinear theories of continuous media has been made. In this review, it has been pointed out that the existing theories involve a number of terms in powers and products of the ordinary measures of strain or strain-rate and several unknown response functions of invariants of kinematic matrices. This is owing to the fact that the order

of the measures of strain and strain-rates have not been fixed and their generalized measures have not been used in the formulation of constitutive equations.

In the present investigation we have, by fixing a priori the orders of the measures of the deformation-rates mentioned above, developed the concept of the generalized measures and been able to obtain a suitable constitutive equation for viscoelastic fluids. The orders of the measures are so chosen that the resulting constitutive equation describes pseudoplastic fluids, for which the apparent coefficient of viscosity decreases with the increase in rate of shear. These new constitutive equations have been found to contain only four terms in the deformation-rate tensors and four rheological constants, and no unknown functions of the invariants.

The constitutive equation obtained thus is applied to torsional flows of a viscoelastic fluid between two infinite parallel planes. The velocity components and pressure have been expanded in power series of a small parameter, and first- and second-order approximations to the velocity components have been obtained. The normal stress differences, velocity profiles, apparent coefficient of viscosity and their behavior depending on the rheological constants have been investigated. The phenomena of reversed flows have also been discussed. Besides the freedom to choose the order of the measures, the rheological parameter entering the constitutive equations can also be suitably varied so as to correlate the theory with experiments. It is found that the behavior of a viscoelastic fluid depends on the sign as well as the magnitude of the rheological constants. Thus, the constitutive equations based on the concept of combined generalized measures predict adequately qualitative as well as quantitative information on the behavior of viscoelastic fluids and eliminate the need for assuming unknown response coefficients.

GENERALIZED MEASURES OF DEFORMATION-RATES IN TORSIONAL FLOWS OF VISCOELASTIC FLUIDS BETWEEN TWO INFINITE PARALLEL PLANES

by

Moon Up Park

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

June 1971

APPROVED:

Redacted for Privacy

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Date thesis is presented June 25, 1970

Typed for Moon Up Park by Eileen Ash

ACKNOWLEDGEMENTS

I would like to extend my sincere appreciation to Professor M. N. L. Narasimhan for his kind suggestion and guidance given me during the course of this investigation. It was from his inspired teaching and from the many informal lectures we had last summer that I acquired my interest in rheology.

I would also like to thank Professor B. R. Seth whose ideas have stimulated my own work in this field.

I am thankful to Professor A. T. Lonseth, Department of Mathematics, Oregon State University, for arranging partial financial support for the study made possible by the Atomic Energy Commission under grant number AT(45-1) - 1947.

My thanks are due to Professor D. S. Carter for going through the manuscript of my thesis.

I am also grateful to Mrs. Eileen Ash for typing the final draft of this thesis.

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NOTATION

The following is a list of symbols used in this analysis.

Symbols	Meaning
a_i	i_{th} vector component of acceleration
a_r, a_θ, a_z	physical components of acceleration in cylindrical coordinates
B	second deformation-rate matrix
I_B, II_B, III_B	first, second and third invariants of second deformation-rate matrix
b_{ij}	second deformation-rate tensor
B*	generalized second deformation-rate matrix
b^*_{ij}	generalized second deformation-rate tensor
D	first deformation-rate matrix
I_D, II_D, III_D	first, second and third invariants of first deformation-rate matrix
d_{ij}	first deformation-rate tensor
D*	generalized first deformation-rate matrix
d^*_{ij}	generalized first deformation-rate tensor
g_{ij}	metric tensor
I	identity matrix
k, k'	dimension correcting constants
n, n'	measure indices
p	pressure of fluid

Symbols	Meaning
q, q'	irreversibility indices
r, θ, z	cylindrical coordinates
R	Reynolds number
T	stress matrix
t_{ij}	stress tensor
t_{rr}, $t_{r\theta}$, t_{zz}, etc.	physical components of stress in cylindrical coordinates
t	time
v_i	i_{th} vector component of velocity
u, v, w	physical components of velocity in cylindrical coordinates
δ_{ij}	Kronecker delta
ε	dimensionless small parameter
η	dimension correcting constant
λ	frequency of oscillation
μ	Newtonian coefficient of viscosity
μ_a	apparent coefficient of viscosity
Ω	amplitude of oscillation
ρ	density of fluid
σ_1, σ_2	normal stress differences
τ	nondimensional time
,	partial differentiation
;	covariant differentiation

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GENERALIZED MEASURES OF DEFORMATION-RATES IN TORSIONAL FLOWS OF VISCOELASTIC FLUIDS BETWEEN TWO INFINTE PARALLEL PLANES

CHAPTER 1

INTRODUCTION

1.1 Preliminary Remarks

In the classical theory of fluid mechanics, the constitutive equations relate the stress tensor to the strain or strain-rate tensors linearly. The Newtonian viscosity depends only on pressure and temperature and is independent of the rate of shear.

Non-Newtonian fluids are those for which the above mentioned linear relation does not hold since these fluids exhibit certain phenomena such as variable viscosity depending on the rates of shear in the fluids, normal stress effects etc. and they may even depend on the previous history of deformation. Examples of such fluids are high polymer solutions, pastes, paints, colloidal solutions, condensed milk, etc. and these occur in everyday life as well as in industry. In order to study, for instance, the behavior of viscosity of such fluids, we define the apparent coefficient of viscosity as the ratio of shear stress to shear rate. The fluids for which this coefficient decreases with increasing shear-rate are known as pseudoplastic fluids and those for which it increases as dilatant fluids.

Internal constitution of the materials is responsible for these differences in behavior. In order to explain, therefore, the response of a material to the applied forces, we need to set up a relation, depending on the internal constitution of the material. This leads to the formulation of constitutive equations of materials which are relations between the stress and the deformation (strain) or motion (strain-rate). It is, therefore, important to study the internal constitution of these materials and to construct suitable constitutive equations for these materials.

A number of nonlinear constitutive equations have been proposed by various workers. The concept of stress is well defined but the measure of strain used is flexible as it should be. In the classical theory when the displacements are finite, the use of a linear measure of strain or strain-rate does not lead to a satisfactory solution of problems. The present trend to explain experimental results involving finite deformations is based on the use of a linear strain measure even though we know from experiments that the strain is nonlinear in character. Thus the order of the measure of strain is not fixed in the classical theory of constitutive equations and consequently they have become unnecessarily complicated and involve unknown response coefficients.

Thus, it becomes necessary to use generalized measures with their orders fixed, a priori, instead of the ordinary measure, in order to provide a satisfactory scientific basis for explaining viscoelastic behavior of real materials. Further, we shall find later in our investigation that in order to explain a variety of viscoelastic and viscoplastic phenomena, it will be necessary to combine generalized measures of different orders and construct accordingly suitable constitutive equations. Thus one can explain pseudoplastic, dilatant and many other types of phenomena in real fluids with the help of combined generalized measures of the rates of deformation

1.2 Object of the Present Study

In order to avoid any further complexity of the stress-strain relations, and at the same time to explain the phenomena arising out of finite deformations in the case of solidsand non-Newtonian behavior in the case of fluids, Seth (1964) introduced the generalized measure concept into continuum mechanics. He also suggested the generalized measure of deformation-rate to be used in fluid mechanics.

Narasimhan and Sra (1969) have found that in certain viscoelastic flows, the mere use of the generalized measure of rate of deformation involving velocity gradients would predict two of the normal stresses to be always equal, which is contrary to experiments (Truesdell and Noll, 1965). Hence they have suggested that for viscoelastic fluids, in addition to the generalized measures of deformation-rate involving velocity gradients, those of another deformation-rate involving acceleration gradients should also be used. They proposed the following constitutive equation

$$T = - pI + 2\,\mu\, D^* + 4\,\eta\, B^* , \qquad (1.2.1)$$

where D^* = generalized first deformation-rate matrix,

B^* = generalized second deformation-rate matrix,

p = isotropic pressure,

I = identity matrix,

μ,η = dimension correcting constants.

It is the object of the present investigation to study torsional flows of viscoelastic fluids characterized by the above constitutive equation between two infinite parallel planes, one of which performs torsional ocillations while the other is at rest.

1.3 Basic Assumptions

The following assumptions will be made in the analysis of the flow problems:

a. the flow is isothermal,

b. the fluid is homogeneous, isotropic and incompressible.

1.4 Plan of the Present Investigation

We have divided our work into four chapters. Chapter 2 is devoted to the generalized measures of deformation and rates of deformation to be used in the constitutive equations and a review of the nonlinear theories of continuum mechanics using ordinary measures. The constitutive equations involving the generalized measures of deformation-rates are set up for incompressible isotropic fluids. As an illustration, we have so fixed the orders of the measures of the deformation-rates and combined them suitably that the fluids obtained are found to be pseudoplastic.

In chapter 3 we apply the concept of generalized measures to torsional flows of viscoelastic fluids and show that this new powerful approach provides a sound scientific basis for constructing constitutive equations and enables one to explain non-Newtonian effects on real fluids adequately. The influence of generalized measures of rates of deformation on the velocity profiles has been determined and shown graphically.

Chapter 4 contains the summary and conclusion.

CHAPTER 2

THE CONSTITUTIVE EQUATIONS OF VISCOELASTIC MATERIALS

2.1 Preliminary Remarks

The fundamental conservation laws of the theory of continuous media are valid for all materials irrespective of their constitution. In order to take account of the nature of different materials, we must therefore find additional equations identifying the basic characteristics of the body with respect to the response sought. In the theory of continuous media this is done by introducing models appropriate to the particular class of phenomena under scrutiny.

In section 2 we cite some of the limitations of the classical theory of fluid dynamics; for example, its failure to explain the normal stress effects, variable viscosity of fluids, stress relaxation, etc. In section 3 we discuss various nonlinear theories that have emerged in an attempt to find suitable mathematical models which could explain the non-Newtonian behavior of fluids, and draw special attention to the fact that these constitutive equations are very complicated and involve many unknown response functions.

In section 4 we discuss the constitutive equations involving generalized measures. In section 5 we set up a suitable constitutive equation by combining three different orders of generalized measures of rates of deformation.

2.2 Limitations of the Classical Theory of Continuous Media

In the classical theory of continuous media, the constitutive equation of incompressible viscous fluids is

$$T = - pI + 2\mu D, \tag{2.2.1}$$

with

$$I_D = 0, \tag{2.2.2}$$

where T = stress matrix,

D = deformation-rate matrix,

I = identity matrix,

I_D = first invariant of D,

p = fluid pressure,

μ = coefficient of viscosity.

The equation (2.2.1) is linear in D and viscosity is a function of temperature. Fluids whose behavior is governed by (2.2.1) are known as incompressible Newtonian fluids.

It was found by experiments that (2.2.1) of classical theory of continuous media cannot furnish explanations for any of the phenomena such as normal stress effects, variable viscosity, viscoelasticity, viscoplasticity, pseudoplasticity, stress relaxation, time-dependent effects, etc. which are exhibited by real fluids. Furthermore in the classical theory of Newtonian fluids rectilinear flows are possible in a cylinder of any cross section. But, for non-Newtonian fluids, it was discovered that such flows cannot be maintained in non-circular tubes without the application of an appropriate body-force distribution in addition to a uniform pressure gradient along the tube. Such flows are known as secondary flows (Ericksen, 1960).

2.3 Nonlinear Theories of Continuous Media

Since the classical theory of continuous media fails to explain many non-Newtonian phenomena, we need to set up suitable mathematical models which can explain these phenomena. A number of nonlinear theories have been proposed by various workers.

Reiner-Rivlin Theory (1945). According to this theory the constitutive equation for incompressible, isotropic viscous fluids is

$$T = -pI + \alpha_1 D + \alpha_2 D^2, \qquad (2.3.1)$$

where α_1 and α_2 are functions of the second and the third invariants of $\dot{D}$. This theory appears to be mathematically simpler than other theories, but the response coefficients α_1 and α_2 which are functions of the invariants of the first deformation-rate tensor are unknown and cannot be specified explicitly. This theory always predicts the existence of two equal normal stresses in certain viscometric flows, but experiments contradict such a prediction when the rate of shear becomes appreciably large.

Rivlin-Ericksen Theory (1955). Rivlin and Ericksen assumed that the stress at a point x and at time t is a function of the gradients, in the spatial system, of velocity, acceleration, second acceleration and higher accelerations at the point x, measured at time t. This assumption led to the formulation of the constitutive equation

$$T = \alpha_0 I + \sum_{p=1}^{N} \alpha_p (\Pi_p + \Pi_p^*), \tag{2.3.2}$$

for incompressible, isotropic fluids, where α's are unknown functions of the second and the third invariants of kinematic matrices, and Π_p and Π_p^* are certain matrix products formed from the kinematic matrices and its transpose respectively. This theory is successful in obtaining normal stresses which need not be equal, but the constitutive equation has been made very complicated by the introduction of several higher order kinematic matrices and unknown functions of their invariants.

Green and Rivlin Theory (1957). This theory is a further generalization of Rivlin and Ericksen's theory. In this theory the stress t_{ij} depends on the complete deformation history of the material and is assumed to be a functional of $\dfrac{\partial x^p(\tau)}{\partial X^q}$ over the range $-\infty < \tau \leq t$:

$$t_{ij} = F_{ij} \left(\frac{\partial x^p(\tau)}{\partial X^q} \right)_{\tau = -\infty}^{t} , \qquad (2.3.3)$$

where x^p and X^q refer to the deformed and undeformed states respectively. The remarks made earlier apply to this theory as well.

Oldroyd Theory (1951). Oldroyd proposed the following constitutive equation

$$(1 + \lambda_1 \frac{\delta}{\delta t})\, t_{ij}^{(e)} - 2k_1 (d_{im}\, t_j^{(e)m} + d_{jm}\, t_i^{(e)m})$$

$$= 2\mu (1 + \lambda_2 \frac{\delta}{\delta t})\, d_{ij} - 8\mu k_2 d_{im} d_j^m , \qquad (2.3.4)$$

where

$$\frac{\delta\, t_{ij}^{(e)}}{\delta t} = \frac{\partial\, t_{ij}^{(e)}}{\partial t} + t_{ij;m}^{(e)}\, v^m + t_{mj}\, v^m_{;i} + t_{im}\, v^m_{;j} \ ,$$

$t_{ij}^{(e)}$ = deviatoric part of the stress tensor,

λ_1, λ_2 = relaxation times,

and k_1 and k_2 are arbitrary scalar constants. Oldroyd (1958) introduced another generalization by using the Jaumann derivative instead of convective derivatives.

Noll's Theory (1958). Noll assumed that the stress in an incompressible fluid at time t depends, to within a hydrostatic pressure, on the history of motion, (in particular, the past history of the relative deformation gradient) up to time t. His constitutive equation has thus the form

$$T = -\,pI + \mathop{F}_{s=0}^{\infty}(G(s)), \tag{2.3.5}$$

where F is the constitutive functional and $G(s)$ is the history of the relative deformation gradient.

The solution of any problem in this theory depends on the experimental determination of the three material functions, that is, the viscosity and the two normal stress functions.

The ever-increasing complexity of the constitutive equations of continuous media and their ad hoc generalizations aimed at obtaining simple results have been criticized by Seth. He (1964, 1966) observed that the constitutive equations have to be complicated so long as we use ordinary measures of strain (or strain rate) in their formulation instead of the generalized measures.

2.4 Constitutive Equations Involving Generalized Measures

In order to avoid bringing unnecessary complications into the stress-strain relations, and at the same time to eliminate unknown response coefficients and to predict results fairly compatible with experimental investigations, Seth has strongly felt the need to construct generalized measures of deformation which should reduce to the known ones in special cases.

The ordinary measures of deformation-rate are

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \,, \qquad (2.4.1)$$

and

$$b_{ij} = \tfrac{1}{2}(a_{i,j} + a_{j,i} + 2\, v_{m,i}\, v^{m}{}_{,j}) \,, \qquad (2.4.2)$$

where the symbol $,_i$ denotes partial differentiation with respect to the spatial coordinate x^i.

Narasimhan and Sra (1969) generalized the ordinary measures of deformation-rates as follows:

$$d^*_{ij} = \frac{k}{m^q n^q}\left[\delta_{ij} - (\delta_{ij} - 2md_{ij})^{\frac{n}{2}}\right]^q , \tag{2.4.3}$$

$$b^*_{ij} = \frac{k'}{m'^{q'} n'^{q'}}\left[\delta_{ij} - (\delta_{ij} - 2m'b_{ij})^{\frac{n'}{2}}\right]^{q'} , \tag{2.4.4}$$

where m, m', k, and k' are dimension correcting constants and n, n' are measure indices and q, q' are irreversibility indices of generalized measures. The measures d^*_{ij} and b^*_{ij} are the generalized measures of deformation-rates and for n, $n'=2$, q, $q'=1$ and k, $k'=1$, these generalized measures reduce to the ordinary one d_{ij} and b_{ij} respectively.

Narasimhan and Sra (1969) proposed a new constitutive equation for incompressible and isotropic fluids of the following form:

$$T = -pI + 2\mu D^* + 4\eta B^* , \tag{2.4.5}$$

where $B^* = \|b^*_{ij}\|$ and $D^* = \|d^*_{ij}\|$ in matrix forms and η is the dimension correcting constant. Substituting the expressions for D^* and B^* from (2.4.3) and (2.4.4) into (2.4.5) we obtain, for incompressible isotropic fluids

$$T = - pI + \alpha_1 D + \alpha_2 D^2 + \beta_1 B + \beta_2 B^2 , \qquad (2.4.6)$$

where, for specific values of n, q, n', q', the coefficients α_1, α_2 are known functions of the invariants of D, and β_1, β_2 are known functions of the invariants of B with finite number of terms in each case.

It is obvious that whatever the positive integral values of n, q, n', q' the deviatoric part of the stress matrix can never contain more than four terms. This has a clear advantage over the general Rivlin- Ericksen constitutive equation or even its simplest form. In all other nonlinear theories the order of the measures of deformation-rates have not been fixed, and one does not know in these theories how to choose the rheological coefficients, since they are, in general, infinite series of the invariants of the kinematic matrices. On the other hand, by first generalizing the ordinary measures and then fixing the orders of the generalized measures, the nonlinearity has been condensed, essentially into two terms, viz. $2\mu D^*$ and $4\eta B^*$, and the rheological coefficients α_1, α_2, β_1 and β_2 occuring in (2.4.6) are also known explicitly.

A further advantage of the generalized measures is that they help to avoid the unnecessary introduction of a number of response functions.

Narasimhan and Sra (1969), by fixing the orders of the generalized measures appropriately, have discussed the dilatant fluid behavior. In our investigation we propose to consider combinations of generalized measures of deformation-rates in order to explain a variety of viscoelastic and viscoplastic phenomena. In particular we consider pseudoplastic behavior of materials for which the apparent coefficient of viscosity decreases with increasing rate of shear.

2.5 Combination of Generalized Measures of Different Orders

A variety of irreversible phenomena such as creep, fatigue, pseudoplasticity and dilatancy etc. can be successfully explained by using a combination of generalized deformation-rate measures rather than just one set of generalized measures. For illustration, we discuss here the pseudoplastic behavior of materials. This can be accomplished by first choosing three different sets of orders of measures (n_r, q_r, n_r', q_r'), where r takes values 1, 2 and 3 such that

$$(n_1, q_1, n_1', q_1') = (2, 1, 0, 0),$$

$$(n_2, q_2, n_2', q_2') = (2, 2, 0, 0), \qquad (2.5.1)$$

$$(n_3, q_3, n_3', q_3') = (2, 3, 2, 1).$$

Now from (2.4.3) and (2.4.4) we obtain

$$D_1{}^* = k_1 D, \quad D_2^* = k_2 D^2, \quad D_3^* = k_3 D^3,$$

$$B_1{}^* = 0 \ , \quad B_2{}^* = 0 \ , \quad B_3{}^* = k'B \ , \qquad (2.5.2)$$

where k's and k' are as before dimension correcting constants.

Since the generalized measures (2.4.3) and (2.4.4) are those of rates of deformation, we can combine them with their orders fixed above and obtain

$$D^* = k_1 D + k_2 D^2 + k_3 D^3,$$

and (2.5.3)

$$B^* = k'B.$$

We shall find later that these orders of measures chosen would predict pseudoplastic behavior of fluids.

Use of Cayley-Hamilton theorem now yields

$$D^* = k_1D + k_2D^2 + k_3(III_DI - II_DD + I_DD^2)$$
$$= k_3III_DI + (k_1 - k_3II_D)D + (k_2 + k_3I_D)D^2, \quad (2.5.4)$$

where I_d, II_D, III_D denote the first, second and third invariants of first deformation-rate matrix.

Since we shall deal with incompressible fluids, we have

$$I_D = 0, \quad (2.5.5)$$

and equations (2.5.3) and (2.5.4) together with (2.5.5) yield the constitutive equation

$$T = (-p + 2\mu k_3III_D)I + 2\mu(k_1 - k_2II_D)D$$
$$+ 2\mu k_2D^2 + 4\eta k'B, \quad (2.5.6)$$

or

$$T = \alpha_0I + \alpha_1 D + \alpha_2D^2 + \beta_1 B, \quad (2.5.7)$$

where

$$\alpha_0 = -p + 2\mu k_3III_D,$$

$$\alpha_1 = 2\mu(k_1 - k_2 II_D) ,$$

$$\alpha_2 = 2\mu k_2 ,$$

$$\alpha_3 = 4\eta k'.$$

CHAPTER 3

APPLICATION OF THE NEW THEORY OF CONSTITUTIVE EQUATIONS TO TORSIONAL FLOWS OF VISCOELASTIC FLUIDS BETWEEN TWO INFINITE PARALLEL PLANES

3.1 Preliminary Remarks

In section 2 we mention the basic equations governing continuous media, viz. the equation of continuity and the equations of motion which are expressed in cylindrical co-ordinates. In section 3, 4 and 5 using the constitutive equation developed in the previous chapter, the problem of the flow of an incompressible viscoelastic fluid due to torsional oscillations of an infinite plane when the fluid is bounded by another stationary parallel plane has been formulated and solved by expanding the velocity components and the pressure in powers of small amplitude of oscillation of the plane. First- and second-order approximations to the velocity, stress and deformation-rates have been obtained.

In section 6 and 7 asymptotic solutions for large values of R and small values of R have been obtained respectively. In section 8 the stress and deformation-rates have been expanded in powers of small parameter and the first- and second-order solutions have been obtained for large values of R as well as for small values of R. Section 9 deals with the discussion of the results.

3.2 Basic Equations of Continuous Media

The principles of conservation of mass and conservation of linear momentum lead to the following two equations respectively.

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + (\rho\ v^i)_{;i} = 0\ , \tag{3.2.1}$$

and the equations of motion are

$$\rho\ a^i = t^{ij}_{;j} + \rho\ f^i\ , \tag{3.2.2}$$

where v^i = i_{th} vector component of velocity,

a^i = i_{th} vector component of acceleration,

ρ = density of fluid,

t^{ij} = stress tensor,

f^i = i_{th} vector component of body force per unit mass,

$;j$ = covariant differentiation with respect to spatial coordinates x^j.

We shall assume the density ρ to be constant and there is no body force acting on the fluid. Consequently the equations of continuity and of motion reduce to the following forms:

$$v^{i}{}_{;i} = 0, \tag{3.2.3}$$

$$\rho a^{i} = t^{ij}{}_{;j}. \tag{3.2.4}$$

The equations of continuity and of motion in the cylindrical coordinates are

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{3.2.5}$$

and

$$\begin{aligned}
\rho a_r &= \frac{1}{r}\frac{\partial(rt_{rr})}{\partial r} + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial \theta} + \frac{\partial t_{zr}}{\partial z} - \frac{t_{\theta\theta}}{r}, \\
\rho a_\theta &= \frac{1}{r}\frac{\partial(rt_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial \theta} + \frac{\partial t_{\theta z}}{\partial z} + \frac{t_{r\theta}}{r}, \\
\rho a_z &= \frac{1}{r}\frac{\partial(rt_{zr})}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta z}}{\partial \theta} + \frac{\partial t_{zz}}{\partial z},
\end{aligned} \tag{3.2.6}$$

where

v_r, v_θ, v_z = physical components of velocity ,

a_r, a_θ, a_z = physical components of acceleration ,

t_{rr}, t_{rz}, etc. = physical components of stress tensor.

3.3 Formulation of the Problem

We consider an oscillatory flow between two infinite parallel planes a distance z_o apart. We assume that one of the planes at z_o is stationary and the other is performing torsional oscillations about its own axis with an angular velocity $\Omega \cos \lambda T$ where λ denotes the frequency of oscillation and Ω the amplitude of angular velocity. The space between the planes is occupied by an incompressible non-Newtonian fluid of density ρ.

Equations of Motion

The equations of motion in cylindrical coordinates for axisymmetric flow with azimuthal variation neglected are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = \frac{\partial t_{rr}}{\partial r} + \frac{\partial t_{rz}}{\partial z} + \frac{t_{rr} - t_{\theta\theta}}{r} ,$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) = \frac{\partial t_{r\theta}}{\partial r} + \frac{\partial t_{\theta z}}{\partial z} + 2\frac{t_{r\theta}}{r} , \tag{3.3.1}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = \frac{\partial t_{rz}}{\partial r} + \frac{\partial t_{zz}}{\partial z} + \frac{t_{rz}}{r} ,$$

and the equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 , \tag{3.3.2}$$

where u , v , and w are velocity components in the direction of r , θ and z respectively.

Boundary Conditions

The boundary conditions are

$$\begin{aligned} &u = 0, \quad v = r\Omega\cos\lambda t , \quad w = 0 , \quad \text{at } z = 0 , \\ &u = 0, \quad v = 0 , \quad w = 0 , \quad \text{at } z = z_o , \end{aligned} \tag{3.3.3}$$

3.4 Method of Solution

We assume the solution of the equations of motion in the forms;

$$\begin{aligned} u &= r\Omega F'(y, \tau) , \\ v &= r\Omega G(y, \tau) , \\ w &= -2z_o\Omega F(y, \tau) , \\ p &= 2\mu k_1\Omega\{ - p_1(y, \tau) + p_2(y, \tau)\frac{r^2}{z_o^2} \} , \end{aligned} \tag{3.4.1}$$

where $y = z/z_o$, $\tau = \lambda t$ and a prime denotes differentiation with respect to y and p_1 and p_2 are functions of y and τ. The equation of continuity is satisfied.

The stress tensor in physical components is given by

$$\begin{aligned} t_{rr} = &- p + 2\Omega(\mu k_1 F' + 2\eta k'\lambda\frac{\partial F'}{\partial\tau}) \\ &+ 2\Omega^2\{\mu k_2(F'^2 + \frac{r^2}{4z_o^2}F''^2) + 4\eta k'(F'^2 - FF'')\} \\ &+ 2\Omega^3\mu k_3 F'^3 , \end{aligned}$$

$$t_{\theta\theta} = -p + 2\Omega(\mu k_1 F' + 2\eta k'\lambda\frac{\partial F'}{\partial\tau})$$
$$+ 2\Omega^2\{\mu k_2(F'^2 + \frac{r^2}{4z_o^2}G'^2) + 4\eta k'(F'^2 - FF'')\}$$
$$+ 2\Omega^3\mu k_3 F'^3 ,$$

$$t_{zz} = -p - 2\Omega(\mu k_1 F' + 4\eta k'\lambda\frac{\partial F'}{\partial\tau})$$
$$+ 2\Omega^2\mu k_2\{4F'^2 + \frac{r^2}{4z_o^2}(F''^2 + G'^2)\}$$
$$+ 4\Omega^2\eta k'\{4(FF'' + 2F'^2) + \frac{r^2}{z_o^2}(F''^2 + G'^2)\}$$
$$- 2\Omega^3\mu k_3\{8F'^3 + \frac{3r^2}{4z_o^2}F'(F''^2 + G'^2)\} ,$$

$$t_{r\theta} = \tfrac{1}{2}\mu k_2(\frac{r\Omega}{z_o})^2 F''G' , \tag{3.4.2}$$

$$t_{rz} = \frac{r}{z_o}\Big[\Omega(\mu k_1 F'' + 2\eta k'\lambda\frac{\partial F''}{\partial\tau})$$
$$-\Omega^2\{\mu k_2 F'F'' - 4\eta k'(FF'' - FF''')\}$$
$$+ \Omega^3\mu k_3 F''\{3F'^2 + \frac{r^2}{4z_o^2}(F''^2 + G'^2)\}\Big],$$

$$t_{\theta z} = \frac{r}{z_o}\Big[\Omega(\mu k_1 G' + 2\eta k'\lambda\frac{\partial G'}{\partial\tau})$$
$$- \Omega^2\{\mu k_2 F'G' - 4\eta k'(F'G' - FG'')\}$$
$$+ \Omega^3\mu k_3 G'\{3F'^2 + \frac{r^2}{4z_o^2}(F''^2 + G'^2)\}\Big] .$$

It may be noted that the normal stresses are found to be not equal to one another as should be the case according to viscometric experiments. Thus as mentioned before, in order to explain viscoėlastic behavior of materials it is necessary to use generalized measures of the deformation-rates.

Equations of Motion in Dimensionless Form

The equations of motion (3.3.1) reduce to the dimensionless form:

$$R\left\{\frac{\partial F'}{\partial \tau} + \varepsilon(F'^2 - 2FF'' - G^2)\right\} = -4p_2 + F''' + \alpha\frac{\partial F''}{\partial \tau}$$
$$+ \varepsilon\{2\alpha(F''^2 - FF'''') + \beta(F''^2 - G'^2 - 2F'F'''))\}$$
$$+ \gamma\varepsilon^2\{3F'(2F''^2 + F'F''')$$
$$+ \frac{r^2}{4z_o^2}(3F''^2F''' + 2G'G''F'' + G'^2F''')\}, \qquad (3.4.3)$$

$$R\left\{\frac{\partial G}{\partial \tau} + 2\varepsilon(F'G - FG')\right\} = G'' + \alpha\frac{\partial G''}{\partial \tau} + 2\varepsilon\{\alpha(F''G' - FG''')$$
$$- \beta(F''G' + F'G'')\}$$
$$+ \gamma\varepsilon^2\{3F'(F'G'' + 2F''G')$$
$$+ \frac{r^2}{4z_o^2}(3G'^2G''' + 2F''F'''G' + G''F''^2)\}, \qquad (3.4.4)$$

$$R(-2\frac{\partial F}{\partial \tau} + 4\varepsilon FF') = 2p'_1 - 2p'_2\frac{r^2}{z_o^2} - 2F'' - 2\alpha\frac{\partial F''}{\partial \tau}$$

$$+ 2\varepsilon\{2\alpha(11F'F'' + FF''') + 14\beta F'F''$$

$$+ \frac{r^2}{z_0^2}(\beta + 2\alpha)(F''F''' + G'G'')\}$$

$$(3.4.5)$$

$$+ \gamma\varepsilon^2 \frac{r^2}{z_o^2}\{3F'F''(F''' + G') + \tfrac{1}{4}(F''^3 + F''G'^2)\} ,$$

where

$$\varepsilon = \frac{\Omega}{\lambda} \qquad R = \frac{\rho\lambda z_o^2}{\mu k_1} ,$$

$$\alpha = \frac{2\eta\lambda k'}{\mu k_1} , \qquad \beta = \frac{1}{2}\frac{k_2\lambda}{k_1} , \qquad (3.4.6)$$

$$\gamma = \frac{k_3\lambda^2}{k_1} .$$

Boundary Conditions

The boundary conditions (3.3.3) now become

$$\begin{array}{llll} F = 0, & F' = 0, & G = \cos\tau, & \text{at } y = 0, \\ F = 0, & F' = 0, & G = 0, & \text{at } y = 1. \end{array} \qquad (3.4.7)$$

Equating the terms independent of r/z_o and the coefficient of $(r/z_o)^2$ on both sides of (3.4.5) we get two equations. Integrating the equation arising out of the terms independent of r/z_o we get p_1, while integrating the second one we get

$$p_2 = \tfrac{1}{2}\,\varepsilon(2\alpha + \beta)(F''^2 + G'^2) + \phi(\tau) + p_3 \,, \tag{3.4.8}$$

where

$$p_3 = \gamma\varepsilon^2 \int \{3F'F''(F''' + G') + \tfrac{1}{2} F''(F''^2 + G'^2)\}\, dy.$$

With this expression for p_2 the equation of motion (3.4.3) becomes

$$\begin{aligned} R\{\frac{\partial F'}{\partial\tau} + \varepsilon(F'^2 - 2FF'' - G^2)\} = & -4(\phi + p_3) + \alpha\frac{\partial F''}{\partial\tau} \\ & - \varepsilon\{2\alpha(F''^2 + 2G'^2 + FF'''') \\ & + \beta(F''^2 + 3G'^2 + 2F'F''')\} \\ & + \gamma\varepsilon^2\{3F'(2F''^2 + F'F''') \\ & + \frac{r^2}{4z_o^2}(3F''^2F''' + 2G'G''F'' + G'^2F''')\} . \end{aligned} \tag{3.4.9}$$

3.5 Solutions of the Equations of Motion in Power Series of ε

The equations of motion (3.4.9) have nonlinear terms associated with the small parameter ε, and hence the method of Poincare can be applied (Courant and Hilbert, 1962). Following this method we assume that a solution can be found by expanding the function $F(y,\tau)$ and $G(y,\tau)$ in ascending powers of the small parameter ε. We further assume that the parameter ε is sufficiently small so that the series expansions converge fast enough and the first two terms give good accuracy. On substituting the series

$$\begin{aligned} F &= F_o + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots, \\ G &= G_o + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots, \\ \phi &= \phi_o + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \ldots, \end{aligned} \tag{3.5.1}$$

into (3.4.3) and (3.4.9) and equating the coefficients of like powers of ε we obtain the following system of linear partial differential equations:

$$R\frac{\partial F_o'}{\partial \tau} = F_o''' + \alpha\frac{\partial F_o'''}{\partial \tau} - 4\phi_o \,, \tag{3.5.2}$$

$$R\frac{\partial G_o}{\partial \tau} = G_o'' + \alpha\frac{\partial G_o''}{\partial \tau} \,, \tag{3.5.3}$$

and

$$R\{\frac{\partial F_1'}{\partial \tau} + (F_o'^2 - 2F_oF_o'' - G_o^2)\}$$

$$= F_1''' - 4\phi_1 - \beta\{F_o''^2 + 2F_o'F_o''' + G_o'^2\}$$

$$+ \alpha\{\frac{\partial F_1'''}{\partial \tau} - 2(F_oF_o'''' + F_o''^2 + G_o'^2)\} , \qquad (3.5.4)$$

$$R\{\frac{\partial G_1}{\partial \tau} + 2(F_o' G_o' - F_oG_o')\} = G_1'' - 2\beta(G_o'F_o'' + F_o'G_o'')$$

$$+ \alpha\{\frac{\partial G_1''}{\partial \tau} + 2(F_0''G_o' - F_oG_o''')\} , \qquad (3.5.5)$$

The boundary conditions to be satisfied are

$$F_m = 0, \quad F_m' = 0, \quad G_o = \cos\tau, \quad G_{m+1} = 0, \quad \text{at } y = 0,$$

$$(3.5.6)$$

$$F_m = 0, \quad F_m' = 0, \quad G_m = 0, \quad \text{at } y = 1,$$

for $m = 0, 1, 2, \ldots$

<u>First-order Solutions</u>

The solution of equation (3.5.2) and (3.5.3) satisfying the boundary conditions (3.5.6) are

$$F_o(y,\tau) = 0,$$

$$G_o(y,\tau) = \mathrm{Re}\left[\frac{e^{i\tau}\sinh\{A(1-y)\}}{\sinh A}\right]$$

$$= \psi_1(y)\cos\tau + \psi_2(y)\sin\tau, \qquad (3.5.7)$$

$$\phi_o(\tau) = 0,$$

where

$$A = a + ib,$$

$$a = \left[\frac{R\{(1 + \alpha^2)^{\frac{1}{2}} + \alpha\}}{2(1 + \alpha^2)}\right]^{\frac{1}{2}}, \quad b = \left[\frac{R\{(1 + \alpha^2)^{\frac{1}{2}} - \alpha\}}{2(1 + \alpha^2)}\right]^{\frac{1}{2}}, \tag{3.5.8}$$

$$\psi_1(y) = \frac{\cosh\{a(y-2)\}\cos by - \cosh ay\ \cos\{b(y-2)\}}{\cosh 2a - \cos 2b},$$

$$\psi_2(y) = \frac{\sinh\{a(y-2)\}\sin by - \sinh ay\ \sin\{b(y-2)\}}{\cosh 2a - \cos 2b}, \tag{3.5.9}$$

and the symbol Re denotes the real part of the functions.

Second-order Solution

The solutions of (3.5.4) and (3.5.5) can be so chosen as to consist of a steady and an unsteady part as follows:

$$F_1(y,\tau) = f(y) + h(y)e^{2i\tau}, \tag{3.5.10}$$

$$\phi_1(\tau) = K + Me^{2i\tau}, \tag{3.5.11}$$

with the understanding that only the real parts of the complex quantities have any physical meaning. Here K and M are constants.

From (3.5.5) we obtain

$$G_1(y,\tau) = 0. \tag{3.5.12}$$

Substituting the expressions (3.5.10) and (3.5.11) into (3.5.4) and (3.5.5), and equating the coefficients of $e^{2i\lambda t}$ and terms independent of it, we obtain two equations for $f(y)$ and $h(y)$:

$$f''' = a_1 \cosh\{2a(y-1)\} + a_2 \cosh\{2b(y-1)\} - 4K , \tag{3.5.13}$$

$$2iRh' - (1 + 2i\alpha)h''' = A_1[\cosh\{2A(y-1)\} - 1] - 4M , \tag{3.5.14}$$

where

$$\begin{aligned} a_1 &= \frac{R\{4\alpha + 3\beta - (1 + \alpha^2)^{\frac{1}{2}}\}}{2(1 + \alpha^2)^{\frac{1}{2}}(\cosh 2a - \cos 2b)} , \\ a_2 &= \frac{R\{4\alpha + 3\beta + (1 + \alpha^2)^{\frac{1}{2}}\}}{2(1 + \alpha^2)^{\frac{1}{2}}(\cosh 2a - \cos 2b)} , \end{aligned} \tag{3.5.15}$$

$$A_1 = \frac{R - A(4\alpha + 3\beta)}{2(\cosh 2A - 1)} .$$

Solving (3.5.13) we obtain the function $f(y)$ and the constant K:

$$8f(y) = a_2 b^{-3}\sin\{2b(1-y)\} - a_1 a^{-3}\sinh\{2a(1-y)\}$$
$$+ (a_2 b^{-3}\sin 2b - a_1 a^{-3}\sinh 2a)(3y^2 - 2y^3 - 1)$$
$$+ 2(a_2 b^{-2}\cos 2b - a_1 a^{-2}\cosh 2a)y(1-y)^2 \quad (3.5.16)$$
$$+ 2(a_2 b^{-2} - a_1 a^{-2})y^2(1-y) ,$$

$$8K = 3a_2 b^{-3}(b + b\cos 2b - \sin 2b)$$
$$- 3a_1 a^{-3}(a + a\cosh 2a - \sinh 2a). \quad (3.5.17)$$

Solving (3.5.14) we obtain the function $h(y)$ and constant M;

$$h(y) = C_o + C_1 e^{By} + C_2 e^{-By} + C_3 y + C_4 \sinh\{2A(y-1)\} , \quad (3.5.18)$$

$$M = \frac{R}{2}\left[\frac{(4\alpha + 3\beta) - (1 - 3\alpha^2 - 3\alpha\beta)}{4(1 + \alpha^2)(\cosh 2A - 1)} - iC_3\right] \quad (3.5.19)$$

where

$$A = a + ib , \qquad B = c + id ,$$

$$c = \left[\frac{R\{2\alpha + (1 + 4\alpha^2)^{\frac{1}{2}}\}}{(1 + 4\alpha^2)}\right]^{\frac{1}{2}}, \quad (3.5.20)$$

$$d = \left[\frac{R\{(1 + 4\alpha^2)^{\frac{1}{2}} - 2\alpha\}}{(1 + 4\alpha^2)}\right]^{\frac{1}{2}},$$

and C's are known functions of a, b, c and d.

Velocity field

The transverse velocity is

$$v = r\Omega\{\psi_1(y)\cos\tau + \psi_2(y)\sin\tau\} + 0(\varepsilon^2), \tag{3.5.21}$$

where the last term can be neglected, and ψ_1 and ψ_2 are given by (3.5.8) and (3.5.9).

We may separate the radial and axial velocity components u and w into a steady part, denoted by u_s and w_s, and a fluctuating part, denoted by u_f and w_f respectively;

$$u_s = \frac{r\Omega^2}{4\lambda}\Big[- a_2b^{-2}\cos\{2b(1-y)\} - a_1a^{-2}\cosh\{2a(1-y)\}$$
$$+ 3(a_2b^{-3}\sin 2b - a_1a^{-3}\sinh 2a)y(1-y)$$
$$+ (a_2b^{-2}\cos 2b - a_1a^{-2}\cosh 2a)(1-y)(1-3y)$$
$$+ (a_2b^{-2} - a_1a^{-2})(3y-2)y\Big], \tag{3.5.22}$$

$$w_s = -\frac{z_o\Omega^2}{4\lambda}\Big[a_2b^{-3}\sin\{2b(1-y)\} - a_1a^{-3}\sinh\{2a(1-y)\}$$

$$+ (a_2 b^{-3} \sin 2b - a_1 a^{-3} \sinh 2a)(3y^2 - 2y^3 - 1)$$

$$+ 2(a_2 b^{-2} \cos 2b - a_1 a^{-2} \cosh 2a)(1-y)^2 y$$

$$+ 2(a_2 b^{-2} - a_1 a^{-2})(1-y)y^2 \Bigg] , \qquad (3.5.23)$$

and

$$u_f = \frac{r\Omega^2}{\lambda} \mathrm{Re} \Bigg[\{C_3 + C_1 \, Be^{By} - C_2 Be^{-By}$$
$$+ 2AC_4 \cosh(2Ay - 2A)\} e^{2i\tau} \Bigg] , \qquad (3.5.24)$$

$$w_f = - \frac{2z_o \Omega^2}{\lambda} \mathrm{Re} \Bigg[\{C_o + C_1 e^{By} + C_2 e^{-By} + C_3 y$$
$$+ C_4 \sinh(2Ay - 2A)\} e^{2i\tau} \Bigg] . \qquad (3.5.25)$$

Thus the above analysis completes the solution for the velocity field and the solution for the pressure field can also be obtained from the equation (3.4.1).

3.6 Asymptotic Solutions for Large Values of the Reynolds Number

For large values of R the functions $G_o(y,\tau)$, $f(y)$ and $h(y)$ become

$$G_o(y,\tau) = e^{-ay} \cos(by - \tau) , \qquad (3.6.1)$$

$$f(y) = a_3\{e^{-2ay} + 2ay(1-y)^2 - (1 + 2y)(1 - y)^2\} \, , \tag{3.6.2}$$

$$h(y) = \mathrm{Re}\,\frac{C_5}{2AB}\left[2Ae^{-By} - Be^{-2Ay} + \frac{2A - B}{B}\{B(y - 1) - 1 - e^{-B(1 - y)} + e^{-By}\}\right] \tag{3.6.3}$$

where

$$a_3 = \frac{(1 + \alpha^2)\{(1 + \alpha^2)^{\frac{1}{2}} - (4\alpha + 3\beta)\}}{8a(1 + \alpha^2)^{\frac{1}{2}}\{\alpha + (1 + \alpha^2)^{\frac{1}{2}}\}} \, , \tag{3.6.4}$$

$$C_5 = \frac{(6\alpha + 3\beta) + i(1 - 9\alpha^2 - 9\alpha\beta)}{4(1 + 9\alpha^2)} \, , \tag{3.6.5}$$

α and β are given by (3.4.6).

Consequently, the transverse velocity (3.5.21) for large values of R can be written as

$$v = r\Omega e^{-ay}\cos(by - \tau) \, , \tag{3.6.6}$$

where

$$a = \frac{z_o}{2}\left(\frac{\rho}{\eta k'}\right)^{\frac{1}{2}} a^* \, ,$$

$$b = \frac{z_o}{2} \left(\frac{\rho}{\eta k'}\right)^{\frac{1}{2}} b^* ,$$

$$\nu = \frac{\mu}{\rho} ,$$

$$a^* = \left\{\frac{\alpha(1 + \alpha^2)^{\frac{1}{2}} + \alpha^2}{1 + \alpha^2}\right\}^{\frac{1}{2}}$$

$$b^* = \left\{\frac{\alpha(1 + \alpha^2)^{\frac{1}{2}} - \alpha^2}{(1 + \alpha^2)}\right\}^{\frac{1}{2}}. \tag{3.6.7}$$

Similarly, for large values of R the steady parts of the radial and axial velocities are

$$u_s = \frac{2r\Omega^2 a_3}{\lambda} \{3y(1 - y) + a(1 - y)(1 - 3y) - ae^{-2ay}\} , \tag{3.6.8}$$

$$w_s = - \frac{2z_o\Omega^2 a_3}{\lambda} \{2ay(1-y)^2 - (1+2y)(1-y)^2 + e^{-2ay}\} , \tag{3.6.9}$$

and the unsteady parts of the radial and axial velocities are

$$u_f = \mathrm{Re}\, \frac{rC_5\Omega^2}{\lambda}\left[e^{-2Ay} - e^{-By} + \frac{2A-B}{2AB}\{1-e^{-By}-e^{-B(1-y)}\}\right], \tag{3.6.10}$$

$$w_f = - \mathrm{Re} \frac{z_o C_5 \Omega^2}{\lambda AB} \Big[2Ae^{-By} - Be^{-2Ay}$$

$$+ \frac{2A-B}{B} \{B(y-1) - 1 - e^{-B(1-y)} + e^{-By}\} \Big] , \qquad (3.6.11)$$

where A and B are given by (3.5.20).

3.7 Solutions for Small Values of the Reynolds Number

For small values of R the functions $G_o(y,\tau)$, $f(y)$ and $h(y)$ become

$$G_o(y,\tau) = (1-y)\{(1+\psi_3)\cos\tau - \psi_4 \sin\tau\} + O(R^3) , \qquad (3.7.1)$$

$$f(y) = y^2(1-y)^2\{a_4(3-y) + a_5(10-10y-5y^2-y^3)\} + O(R^3) , \qquad (3.7.2)$$

$$h(y) = \mathrm{Re} \Big[C_6 + C_7 y + C_8 y^2 + C_9 y^3 + C_{10} y^4$$

$$+ 2AC_4(y-1)\{1 + \tfrac{2}{3} A^2 (y-1)^2\} + C_4 \, O(R^2) \Big] , \qquad (3.7.3)$$

where

$$\psi_3(y) = \frac{Ry(y-2)}{6(1+\alpha^2)} \left\{ \alpha + \frac{R(\alpha^2-1)(3y^2-6y-4)}{60(1+\alpha^2)} \right\} ,$$

$$\psi_4(y) = \frac{Ry(y-2)}{6(1+\alpha^2)} \left\{ 1 + \frac{R\alpha(3y^2-6y-4)}{30(1+\alpha^2)} \right\} ,$$

$$a_4 = \frac{(1+5\alpha^2+3\alpha\beta)(1+\alpha^2)^{\frac{1}{2}}(a^4+b^4) - (5\alpha+3\beta)(1+\alpha^2)(a^4-b^4)}{40\{3(a^2-b^2) + (a^4+b^4)\}},$$

$$b_4 = \frac{(1+5\alpha^2+3\alpha\beta)(1+\alpha^2)^{\frac{1}{2}}(a^6+b^6) - (5\alpha+3\beta)(1+\alpha^2)(a^6-b^6)}{840\{3(a^2-b^2) + (a^4+b^4)\}},$$

and C's are known functions of A and B.

Consequently, the transverse velocity (3.5.21) can now be written

$$v = r\Omega(1 - y)\{(1 + \psi_3)\cos\tau - \psi_4\sin\tau\}. \qquad (3.7.4)$$

The steady parts of the radial and axial velocities are

$$\begin{aligned} u_s = {} & 2r^2\Omega^2\lambda^{-1}y(1 - y)\{a_4(6 - 15y + 5y^2) \\ & + 2a_5(20 - 70y + 7y^2 - 35y^3 + 7y^4)\} \end{aligned} \qquad (3.7.5)$$

$$\begin{aligned} w_s = {} & -2z_o\Omega^2\lambda^{-1}y^2(1 - y)^2\{a_4(3 - y) \\ & + a_5(10 - 10y - 5y^2 - y^3)\}, \end{aligned} \qquad (3.7.6)$$

and the unsteady parts of the radial and axial velocitites are

$$u_f = r\Omega^2\lambda^{-1}\mathrm{Re}\left[\{C_7 + 2C_8y + 3C_9y^2 + 4C_{10}y^3 + 2A^2C_4(1 + 2A^2(y-1)^2\}\,e^{2i\lambda t}\right], \tag{3.7.7}$$

$$w_f = -2z_o\Omega^2\lambda^{-1}\ \mathrm{Re}\left[\{C_6 + C_7y + C_8y^2 + C_9y^3 + C_{10}y^4 + 2AC_4(y-1)(1 + \tfrac{2}{3}A^2(y-1)^2)\}\,e^{2i\lambda t}\right]. \tag{3.7.8}$$

3.8 Discussion of the Results

The first-order approximation is equivalent to neglecting the convective terms compared with the time rate of change of velocity components in the equations of motion. This is valid provided the amplitude of oscillation is sufficiently small.

Transverse Velocity. The first-order solution corresponds to the unsteady shear layer for the transverse velocity. For large values of R the transverse velocity is

$$v = r\Omega e^{-ay}\cos(by - \tau)\ , \tag{3.8.1}$$

and is oscillatory with an amplitude decreasing exponentially with distance from the oscillating plane, and a phase which progressively diminishes with increasing distance from the oscillating plane.

The critical distance, over which the amplitude falls off by a factor of e, is z_o/a. The phase difference between the two planes is given by the constant b.

Attenuation coefficient of shear wave a/z_o is given by

$$\left(\frac{a}{z_o}\right)^2 = \frac{\rho a^{*2}}{4\eta k'} , \tag{3.8.2}$$

where

$$a^{*2} = \frac{\alpha^2 + \alpha(1 + \alpha^2)^{\frac{1}{2}}}{1 + \alpha^2} . \tag{3.8.3}$$

In order to investigate the behavior of the attenuation coefficient we study its variation with α. Differentiating a^{*2} with respect to α and solving the resulting expression after setting it to zero, we obtain $\alpha = -1/\sqrt{3}$ and $\alpha \to \infty$. It is found that the attenuation coefficient reaches a maximum for $\alpha = -1/\sqrt{3}$ and $\alpha \to \infty$. This behavior of the attenuation coefficient is also given in figure (3.1).

We now consider the behavior of transverse velocity for the cases $k' \to 0$, $k' > 0$ and $k' < 0$.

<u>Case 1 When $k' \to 0$.</u> When the constant k' approaches zero, the constants a and b become

$$a = b = z_o\left(\frac{\lambda}{2k_1\nu}\right) = \left(\frac{R}{2}\right)^{\frac{1}{2}}, \tag{3.8.4}$$

and the transverse velocity becomes

$$v = r\, e^{-(R/2)^{\frac{1}{2}} y} \cos\{(R/2)^{\frac{1}{2}} y - \tau\}, \tag{3.8.5}$$

which for fixed $r\Omega$ depends only on R and λ.

Case 2 When $k' > 0$. In this case the constant a and b become frequency-dependent and a^{*2} is a monotonic increasing function which approaches a limit 2 as α tends to infinity.

Case 3 When $k' < 0$. The constant a and b become frequency-depenpent. In this case the constant a^{*2} at first increases with frequency, but when $\alpha = -1/\sqrt{3}$, it attains its maximum value of $\frac{1}{4}$, after which it drops to zero. That is, an oscillation of very high frequency is propagated without much attenuation. The critical frequency at which the maximum damping occurs is given by

$$\lambda_c = \frac{\lambda}{\sqrt{3}\alpha}, \tag{3.8.6}$$

and the ratio $\lambda/\lambda_c = \sqrt{3}\,\alpha$ depends only on α.

The corresponding Reynolds number at this critical frequency is

$$R_c = \frac{R}{\sqrt{3}\alpha} \quad , \tag{3.8.7}$$

and the ratio $R/R_c = \sqrt{3}\alpha$ depends only on α. The ratio

$$\frac{a^{*2}}{\lambda_c} = \frac{\sqrt{3}\ R}{8\lambda} = \frac{\sqrt{3}\ R_c}{8\lambda_c} \tag{3.8.8}$$

is a constant for all negative values of k'.

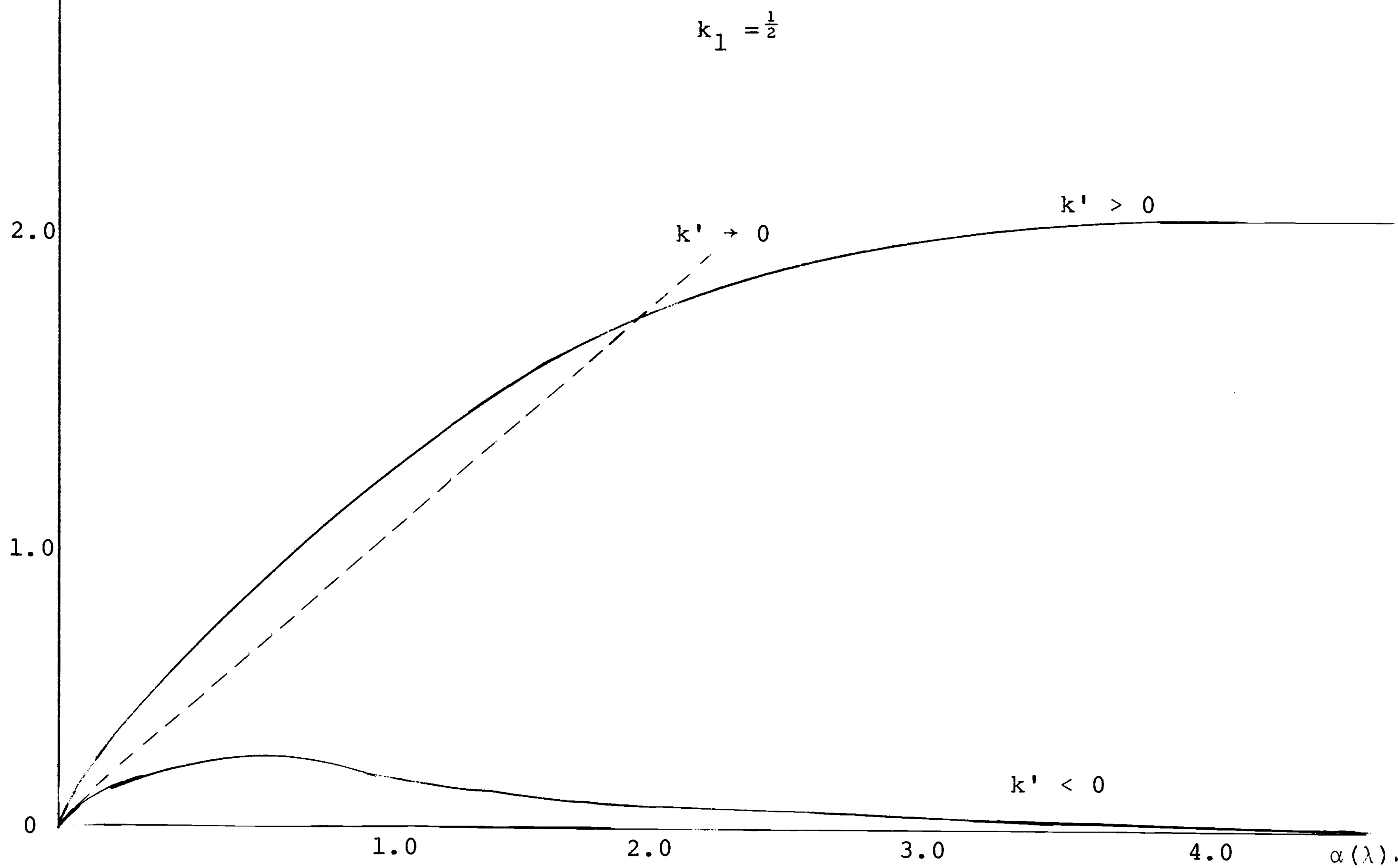
From the figure (3.1) it is interesting to note that the damping for $k' \to 0$ is found to be more rapid than for $k' > 0$ for all α except in a narrow range given by

$$\frac{2a^2}{R} = \frac{\alpha + (1 + \alpha^2)^{\frac{1}{2}}}{1 + \alpha^2} < 1 \ , \tag{3.8.9}$$

that is for α approximately less than 1.6. Again the damping for $k' \to 0$ is found to be more rapid than for $k' < 0$.

Figure 3.1 Attenuation Coefficient of Shear Wave

$k_1 = \frac{1}{2}$



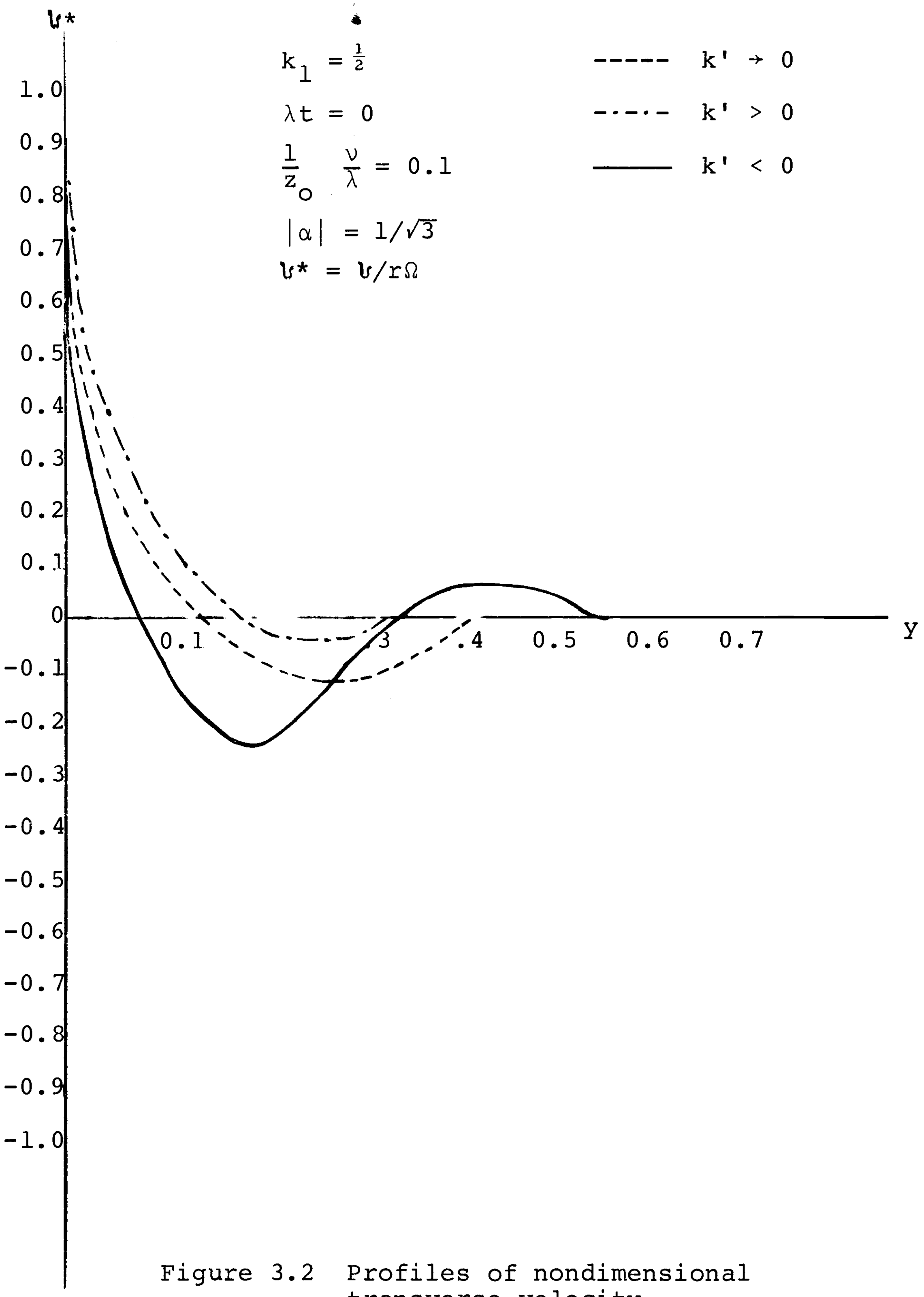


Figure 3.2 Profiles of nondimensional transverse velocity

This shows the nature of the dependence of the damping on the measure index k'. This is essentially due to the viscoelastic nature of the fluid. The measure index k_2 does not play any role in this respect. Numerical values for the damping of the shear wave and for the distance between nodes have been plotted against the dimensionless variables in figure (3.2).

For small values of R the fluid has transverse velocity for all y. The profile assumes the form of a polynomial in y, and the fluid acquires phase lag with respect to the oscillating plane. In the limit $R \to 0$, the transverse velocity is linear in y and vanishes near the stationary plane.

Radial and Axial Velocities. The second-order solution yields the radial and axial velocities composed of steady and unsteady components with frequency twice that of the oscillating plane.

The centrifugal and shearing forces, acting near the oscillating plane gives rise to steady components of radial and axial velocities. The centrifugal force causes fluid to be thrown radially outwards and consequently fluid is drawn inwards along the axis of oscillation towards the oscillating plane. The fluid thrown radially outwards must be balanced by fluid sucked radially inwards. To maintain this inward flow, a radial pressure gradient is induced.

For large R , boundary layers are found to be formed near the planes and the steady components of radial and axial velocities consist of three terms, one of which decreases exponentially with y and hence its influence diminishes outside the boundary layer. Also near an oscillating plane the steady radial and axial velocities increase linearly with the distance from the plane. Near a stationary plane the steady radial velocity increases linearly with the distance from the plane while the steady axial velocity diminishes.

The steady radial and axial velocities in non-dimensional form, denoted by u_s^* and w_s^* , are

$$u_s^* = (1 - y)\{(1 - 3y) + 3a^{-1}y\} - e^{-2ay} , \qquad (3.8.10)$$

$$w_s^* = (1 - y)^2\{a^{-1}(1 + 2y) - 2y\} - a^{-1}e^{-2ay} , \qquad (3.8.11)$$

The points at which the steady radial and axial velocity components attain their maxima can be obtained by differentiating and setting the resulting expression to zero;

$$y_r = \frac{4a - 3}{6(a - 1)} , \qquad y_a = \frac{a}{3(a - 1)} , \qquad (3.8.12)$$

respectively.

We now consider the behavior of steady radial and axial velocities for the cases $k' \to 0$, $k' > 0$ and $k' < 0$.

Case 1 When $k' \to 0$. As frequency approaches infinity, y_r and y_a approach 2/3 and 1/3 respectively; that is, at infinite frequency the steady axial velocity has its maximum at 1/3 whereas the steady radial velocity has its maximum at 2/3 and minimum at 1/3. For all finite frequency, y_r and y_a are greater than 2/3 and 1/3 respectively. The magnitude of steady radial and axial velocity components increases as frequency increases and attains maximum values at infinite frequency.

It is interesting to note that the steady radial and axial velocity components vanish when the combined measure parameter $\beta = 1/3$ and reverse directions when $\beta < 1/3$.

Case 2 When $k' > 0$ or $k' < 0$. In this case y_r and y_a depend on frequency λ as well as measure index k'. The magnitude of axial velocity is smallest when $\alpha = 0$ $(k' \to 0)$. For $\alpha = 0.5$ $(k' > 0)$ it increases slightly, but when $\alpha = -0.5$ $(k' < 0)$ the increment of magnitude of axial velocity is about twice that of the case $\alpha = 0.5$.

The influence of α , involving the viscoelastic effects, on steady radial velocity is found to be similar to that on axial velocity. The steady radial and axial velocity components vanish when

$\beta = \frac{1}{3} \left\{ (1 + \alpha^2)^{\frac{1}{2}} - 4\alpha \right\}$ and reverse direction when

$$\beta < \frac{1}{3} \left\{ (1 + \alpha^2)^{\frac{1}{2}} - 4\alpha \right\}.$$

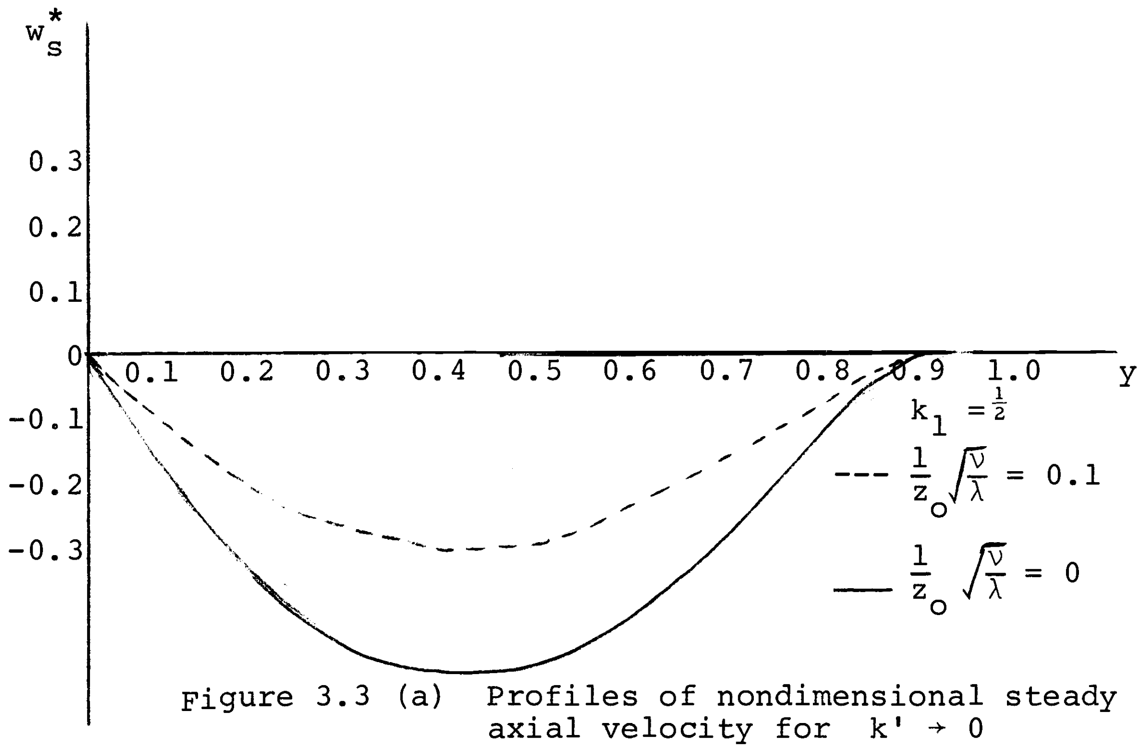


Figure 3.3 (a) Profiles of nondimensional steady axial velocity for $k' \to 0$

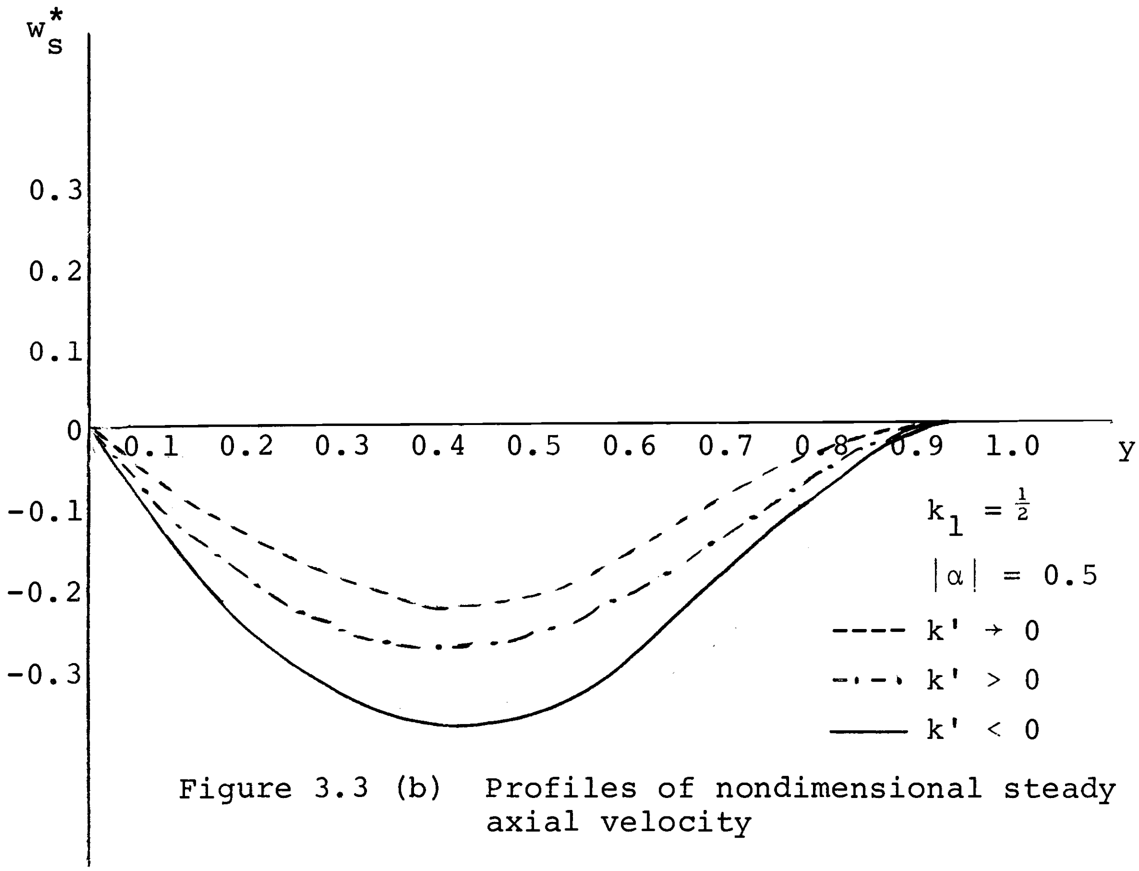


Figure 3.3 (b) Profiles of nondimensional steady axial velocity

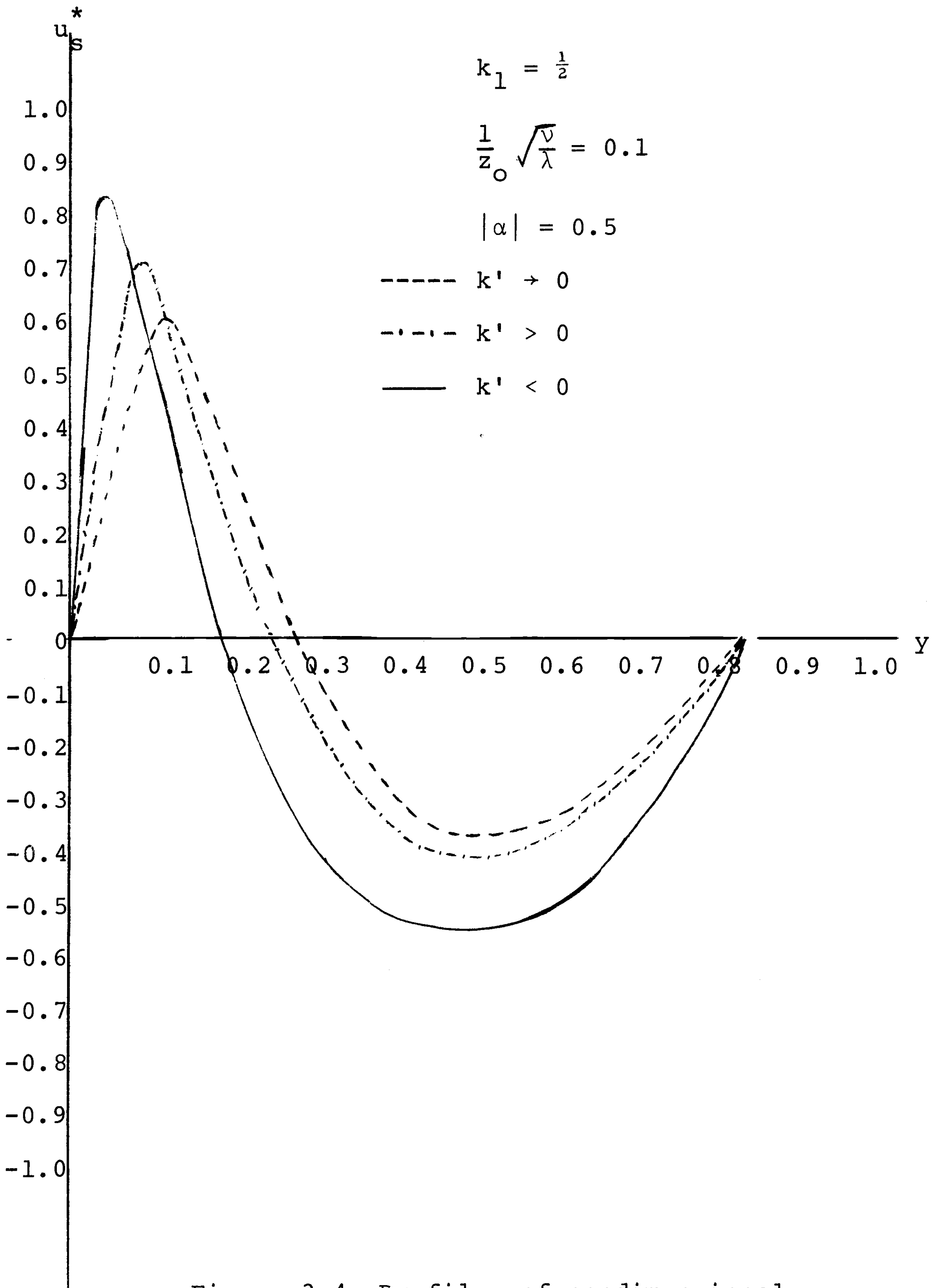


Figure 3.4 Profiles of nondimensional steady radial velocity

Normal Stress Differences. By expanding the deformation-rate and stress tensors in powers of ε and taking the first-order terms, we obtain the non-vanishing first deformation-rate, denoted by δ ,

$$\delta = d_{\theta z}^{(0)} = \frac{r\Omega G_o'}{2z_o} , \tag{3.8.13}$$

and the first-order normal stresses

$$\begin{aligned} t_{rr}^{(0)} &= - p , \\ t_{\theta\theta}^{(0)} &= - p + 2\mu k_2 \delta^2 , \\ t_{zz}^{(0)} &= - p + 2\mu k_2 \delta^2 + 64\eta k' z_o^2 r^{-2} \delta^2 . \end{aligned} \tag{3.8.14}$$

The normal stress differences, denoted by σ_1 and σ_2 , are given by

$$\begin{aligned} \sigma_1 &= t_{\theta\theta}^{(0)} - t_{rr}^{(0)} = 2\mu k_2 \delta^2 , \\ \sigma_2 &= t_{zz}^{(0)} - t_{rr}^{(0)} = 2\mu k_2 \delta^2 + 64\eta k' z_o^2 r^{-2} \delta^2 . \end{aligned} \tag{3.8.15}$$

The normal stress differences are proportional to the square of transverse shear-rate. When k' approaches zero, σ_2 tends to σ_1 .

We consider the behavior of the normal stress differences for large values of R as well as for small values of R.

<u>Case 1 When R is large.</u> For large values of R, the first-order non-vanishing deformation-rate is given by

$$\delta = \delta_m \sin(\tau - by - \theta_o) , \tag{3.8.16}$$

where $\theta_o = \tan^{-1}(a/b)$ and $\delta_m = \frac{1}{2} z_o^{-1} r \, \Omega \, R^{\frac{1}{2}} (1 + \alpha^2)^{-\frac{1}{4}} e^{-ay}$.

The normal stress differences are

$$\begin{aligned} \sigma_1 &= \mu k_2 \delta_m^2 \, [1 - \cos\{2(\tau - by - \theta_o)\}] , \\ \sigma_2 &= \delta_m^2 \{\mu k_2 + 32\eta k' \frac{r^2}{z_o^2}\} [1 - \cos\{2(\tau - by - \theta_o)\}] . \end{aligned} \tag{3.8.17}$$

The normal stress differences oscillate with twice the frequency of the transverse velocity and deformation-rate, and decreases exponentially with y. A special feature of the flow is that a sinusoidal deformation-rate gives rise to sinusoidal stress differences which are in phase with the frequency of deformation-rate. When $r \to 0$, σ_1 approaches zero and

$$\sigma_2 \to 8\eta k' \Omega^2 R (1 + \alpha^2)^{-} \, e^{-2ay} [1 - \cos\{2(\tau - by - \theta_o)\}] . \tag{3.8.18}$$

This shows there is an axial flow along the axis of oscillation and its direction is determined by k'. When $r \to 0$ and $k' \to 0$, σ_2 approaches zero and there is no motion along the axis of oscillation.

<u>Case 2 When R is small.</u> For small values of R, the first-order non-vanishing first deformation-rate is given by

$$d_{\theta z}^{(0)} = \frac{\Omega r}{12 z_o (1 + \alpha^2)} \Bigg[-6(1 + \alpha^2)\cos\tau$$

$$+ R\{2(y - 1)^2 + y(y - 2)\} (\cos\tau - \alpha\sin\tau)$$

$$+ R^2 \left\{ \frac{6y(y - 1)^2(y - 2) + (3y^2 - 6y - 4)(3y^2 - 6y + 2)}{60(1 + \alpha^2)} \right\}$$

$$\{2\alpha \sin\tau + (1 - \alpha^2)\cos\tau\} \Bigg], \tag{3.8.19}$$

and the first-order normal stresses are given by

$$t_{rr}^{(0)} = - p ,$$

$$t_{\theta\theta}^{(0)} = - p + \mu k_2 \left(\frac{\Omega r}{2 z_o}\right)^2 \Bigg[1 + \cos 2\tau \tag{3.8.20}$$

$$- R \left\{ \frac{2(y - 1)^2 + y(y - 2)}{3(1 + \alpha^2)} \right\} (1 + \cos 2\tau - \sin 2\tau) \Bigg] ,$$

$$t_{zz}^{(0)} = - p - \left\{\mu k_2 \left(\frac{\Omega r}{2z_o}\right)^2 + 8\eta k'\right\}\left[1 - \cos 2\tau - \sin 2\tau\right.$$
$$\left. - R\left\{\frac{2(y-1)^2 + y(y-2)}{3(1+\alpha^2)}\right\}(1 - \cos 2\tau + \sin 2\tau)\right].$$

The normal stress differences are given by

$$\sigma_1 = \mu k_2 \left(\frac{\Omega r}{2z_o}\right)^2 \left[1 + \cos 2\tau\right.$$
$$\left. - R\left\{\frac{2(y-1)^2 + y(y-2)}{3(1+\alpha^2)}\right\}(1 + \cos 2\tau - \sin 2\tau)\right],$$

$$\sigma_2 = \left\{\mu k_2 \left(\frac{\Omega r}{2z_o}\right)^2 + 8\eta k'\right\}\left[\sin 2\tau + \cos 2\tau - 1\right. \qquad (3.8.21)$$
$$\left. + R\left\{\frac{2(y-1)^2 + y(y-2)}{3(1+\alpha^2)}\right\}(\sin 2\tau - \cos 2\tau + 1)\right].$$

For small values of R the normal stress differences assume the form of a polynomial in y.

Apparent Coefficient of Viscosity.

The transverse shear-rate is given by

$$t_{\theta z}^{(0)} = 2\mu\left(k_1\delta + k_3\delta^3 + \alpha k_1 \frac{\partial\delta}{\partial\tau}\right), \qquad (3.8.22)$$

and the apparent coefficient of viscosity, denoted by μ_a, is

$$\mu_a = 2\mu(k_1 + k_3\delta^2 + \alpha k_1\delta^{-1}\frac{\partial\delta}{\partial\tau})$$

$$= 2\mu\{k_1 + k_3\delta^2 + \alpha k_1\cot(\tau + \theta_1)\} , \qquad (3.8.23)$$

where $\theta_1 = \tan^{-1}(\psi'_1/\psi'_2)$.

Since the apparent coefficient of viscosity approaches the Newtonian viscosity coefficient as the shear-rate tends to zero, we have

$$\lim_{\delta \to 0}\mu_a = 2\mu\{k_1 + \alpha k_1\cot(\tau + \theta_1)\}$$
$$= \mu \qquad (3.8.24)$$

which leads to

$$k_1 + \alpha k_1\cot(\tau + \theta_1) = \tfrac{1}{2} . \qquad (3.8.25)$$

Using this result in (3.8.23), the apparent coefficient of viscosity may be written as

$$\mu_a = \mu(1 + 2k_3\delta^2) . \qquad (3.8.26)$$

Since the apparent coefficient of viscosity of a real pseudoplastic fluid is defined to be positive and decreases with increasing shear-rate, we should have

$$\frac{d\mu_a}{d\delta} = 4\mu k_3 \delta < 0 \; . \tag{3.8.27}$$

That is, the combined measure index k_3 should be negative, since $\mu > o$.

This completes the discussion of the behavior of pseudoplastic fluids based on the combined generalized measures of deformation-rates.

CHAPTER 4

SUMMARY AND DISCUSSIONS

The response of real materials to external forces is, in general, nonlinear in character. The classical theory which relates the stress to the strain or strain-rate linearly fails to explain these non-linear phenomena. The failure of the classical theories to explain the non-linear response of materials led to the search for more general theories.

In our present work, we have given a brief discussion of the various constitutive theories proposed by Reiner, Rivlin, Ericksen, Green, Oldroyd and Noll. All of these theories have been developed using ordinary measures of deformation or deformation-rate and have resulted in very complicated constitutive equations involving terms in powers and products of kinematic matrices and also a number of unknown response functions. The main source of all these difficulties is the use of ordinary measures of deformation or deformation-rate in the constitutive equations of non-linear materials. Any such restriction put on the strain measure will naturally result in straining the constitutive equation into complicated forms.

In order to avoid any further complexity of the stress-strain relations and at the same time to explain the phenomena arising out of finite deformations in the case of solids and non-Newtonian behavior in the case of fluids, Seth (1964) introduced the generalized measure concept into continuum mechanics. He also suggested the generalized measure of deformation-rate to be used in fluid mechanics.

Narasimhan and Sra (1969) have found that in certain viscometric flows of viscoelastic fluids, the mere use of the generalized measure of the rate of deformation involving velocity gradients predicts two of the normal stresses to be equal which is contrary to experiments. Hence they have suggested that in addition to the generalized measure of deformation-rate involving velocity gradients, that of a second deformation-rate involving acceleration gradients should be used. This is reasonable since the viscoelastic behavior depends not only on velocity gradients but also on acceleration gradients. Further there is no need to use higher order kinematic tensors of deformation-rates, since their generalized measures play an adequate role of predicting viscoelastic phenomena.

In the present thesis, we have developed the concept of combining generalized measures of deformation-rates of different orders, fixed a priori, in order to explain certain non-linear phenomena such as pseudoplasticity which cannot be explained by the mere use of generalized measures. A suitable constitutive equation based on the combined generalized measures of rates of deformation has been constructed for viscoelastic fluids. For illustration purposes, the orders of the measures have been so chosen that the resulting constitutive equation describes pseudo-plastic fluids. This constitutive equation has been applied to study the flow generated by torsional oscillations of an infinite plane in the presence of another parallel plane at rest and situated at a finite distance from the oscillating plane.

The constitutive equation obtained thus does not contain any unknown functions of the invariants of kinematic matrices and hence provides a great improvement over other theories of constitutive equations. Besides the freedom to choose the orders of the generalized measures of rates of deformation, one can vary the combinations of these measures so as to correlate the theory with experiments involving a variety of irreversible phenomena and thus this theory provides a lot of flexibility. Since the constitutive equations using combined generalized measures are much simpler and flexible than other theories, this approach has enabled us to discuss the results with greater clarity. After the orders of the combined generalized measures have been fixed, one needs to know only the values of the rheological constants, in order to obtain from our analysis concrete information on the behavior of any fluid.

Expressions for the velocity field, the stress-field, and the apparent coefficient of viscosity have been obtained and their behaviors have been adequately discussed. It is interesting to note that the behaviors of these quantities depend both on the sign and the magnitude of the rheological constants.

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