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(Major professor)

The primary purpose of this thesis is to offer a geometric interpretation of substitutable, independent, and complementary commodities in consumption. The definitions of these concepts have been controversial. In this thesis, Hicks' definitions are adopted. As these definitions are dependent on properties of the substitution effect term in the Slutsky equation, this equation is derived and the theory leading up to the derivation is summarized. The equivalence of Hick's verbal and mathematical definitions is established in the case of three commodities. Using these definitions both two - and three-dimensional diagrams of substitutes, independents, and complements are given. Hayek's interpretation of Hicks' verbal definitions is reviewed and supplemented with the writer's interpretation. Finally an example of a utility function which exhibits substitutable and independent commodities is given.

A GEOMETRIC INTERPRETATION OF SUBSTITUTABLE, INDEPENDENT, AND COMPLEMENTARY COMMODITIES IN CONSUMPTION

by

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A GEOMETRIC INTERPRETATION OF SUBSTITUTABLE, INDEPENDENT, AND COMPLEMENTARY COMMODITIES IN CONSUMPTION

CHAPTER I

INTRODUCTION

Substitutable, independent, and complementary commodities in consumption are the central topics of this thesis. Hence, a natural point of departure would be an explanation or definition of these concepts in relation to demand theory. However, it is precisely at this point that the first obstacle is confronted. By reviewing the economic literature it becomes apparent that there is no unanimity among economists in their interpretation of these concepts. This point was crucial with respect to the origin of this thesis.

The writer, while attending a Mathematical Economics course taught by Professor A. N. Halter at Oregon State University, was asked to find a utility function that exhibited complementary commodities. The definition of complementarity employed was that given in Microeconomic Theory by Henderson and Quandt [5, p. 29], which was the text for the course. After a long and fruitless search for a utility function that illustrated complementarity, the writer turned to the literature in order to gain further insight about the problem. The results of reviewing the literature were inconsequential with regard to the immediate problem of finding the desired utility function but did reveal the controversy regarding the concepts of related goods.

A Brief Review of Substitutes, Independents, and Complements

Most of the relevant articles encountered, which dealt with related goods, were concerned with either one of two central themes: (1) a new set of proposed definitions for substitutes, independents, and complements [8, 12, 15], and/or (2) various arguments pertaining to some existing set of definitions [3, 4, 9, 10, 17, 18, 20]. Nearly everyone has rejected the set of definitions used by Fisher, Pareto, Edgeworth, and Marshall. Though their definitions no doubt had great intuitive appeal, they depended upon the cardinal measurability of utility. Assuming utility to be ordinally measurable suggested that new definitions were needed for substitutes, independents, and complements. Some economists responded (and are still responding) by offering their respective proposals, while others assumed the role of critics. An example of the situation can be seen in Ichimura's [8] "A Critical Note on the Definition of Related Goods." In this article he offers a new set of definitions and in passing notes that there are basically five others already in existence.

The concepts of substitutable, independent, and complementary commodities have been defined by three different methods: mathematical, verbal, and diagrammatic. While it would be desirable to have agreement among these three types of definitions, usually this has not been the case. For example, one writer's mathematical definition may not correspond at all to another's verbal definition, or even yet another's diagrammatic definition. But the state of affairs is even worse than this. Examples may be found where single writers have inconsistently utilized two or more of the above methods, e.g., a particular individual's mathematical definition may not correspond to his own verbal definition. At any rate, one can perceive that the state of affairs regarding related commodities is both unsettled and confusing.

The Interest in Complementarity

In reviewing the large amount of material which has been written about related commodities, one can discern another predominance. This is the attention given to complementary commodities. Substitutes seem to be viewed as a sort of counterpart to complements with little mention of independent commodities. The reason for economists' preoccupation with the notion of complementarity is not altogether apparent. A partial explanation might be that the precise intuitive notion of complementarity is somewhat vague and therefore harder to define. On the other hand, the term substitutes and its derivatives are very common, even inherent, to the subject of economics. Hence, an illusion of explicitness is probably imparted when these terms are used, and therefore, economists' notion of substitutes is ostensibly definite. To illustrate this, consider the economic term "the marginal rate of substitution," for which the term "complements" does not enjoy a comparable parallel. The marginal rate of substitution of Q_1 for Q_2 is interpreted verbally to be the rate at which a consumer would be willing to substitute Q_1 for Q_2 in order to maintain a given level of utility. Here the use of the term "substitute" is unequivocally clear, and would seemingly imply that Q_1 and Q_2 are substitutes. However, this may not be the case at all.

Objectives of This Thesis

"Slutsky's Equation," called the "Fundamental Equation of Value Theory" by Hicks, and its ramifications provide the bases for many contemporary economic investigations. The most commonly accepted definition of substitutes, independents, and complements is related to this equation via the substitution effect of a price change, which is a basic component of the equation [8, p. 179]. It is within this context that this thesis is set. The general objective is to present in one place in an organized and concise fashion the mathematics leading up to the substitution effect term and its verbal and geometric interpretations. The specific objectives are:

1. To derive the Slutsky equation from a few basic assumptions regarding an "economic" consumer.

2. To present the definitions of substitutable, independent, and complementary commodities in a mathematical form as given by Hicks [6, p. 311].

3. To show the correspondence between Hicks' verbal [6, p. 44] and mathematical definitions [6, p. 311].

4. To provide, for the first time, a three-dimensional geometric interpretation of the substitution term from which the relationships among commodities emanate.

5. To provide an original two-dimensional interpretation of substitutes, independents, and complements when three commodities are involved.

6. To show the correspondence between Hayek's [4] two-dimensional diagrammatic interpretation of Hicks' verbal definition and the proposed three-dimensional geometric interpretation.

7. To provide an example of a utility function which exhibits independent and substitutable commodities.

In Chapters II and III objectives one, two and three are accomplished. The geometric interpretation of substitutable, independent, and complementary commodities is given in Chapter IV. First, the three-dimensional interpretation of the substitution effect term is given. Second, the two-dimensional interpretation of the relationships among commodities is given. Third, Hayek's two-dimensional interpretation of Hicks' verbal definitions is given. Fourth, Hayek's diagrams are supplemented by the insight gained from the three-dimensional interpretation. Finally, the example of a utility function is presented.

A summary and the conclusions of the analysis make up Chapter V.

CHAPTER II

MATHEMATICAL MODEL OF CONSUMPTION AND DEMAND

Most economic theories may be represented in terms of mathematical models. The purpose of the present chapter is to construct a mathematical model of an "economic" consumer. A customary point of departure is to assume the consumer has a preference field or preference ordering regarding the commodities available to him. Another point of departure is to assume the consumer has a demand function for every commodity available to him. Both of these approaches do not imply the existence of a utility function unless the integrability condition is fulfilled. However, Wold [23, p. 62-63, p. 90-93] points out that unless the integrability condition is fulfilled, each approach is inconsistent, and when it is fulfilled the two approaches become equivalent. Hence, the point of departure in this thesis will be from the consumer's utility function.

The Utility Function

It is assumed that the consumer possesses a utility function [2, p. 13]. It is further assumed that there are a given number of commodities available to the consumer, and his utility function is defined for some fixed number of these commodities. The consumer's utility function is in the following form:

$$U = f(q_1, q_2, \ldots, q_n), \tag{2-1}$$

which may be thought of as a real-valued mapping from $P \subset R^n$ to R, where P denotes the domain of the function, which is the set of ordered n-tuples, each coordinate of which is non-negative, and R is the range of the function, which is the set of all real numbers [2, p. 59]. Hence, each coordinate value, q_i, where $i = 1, 2, \ldots, n$, must be greater than or equal to zero, which means the consumer cannot consume a negative amount of any commodity Q_i, as this is meaningless in the present context. Each q_i is a variable and represents some number of units of commodity Q_i. Every commodity is assumed to have a well-defined unit of measure. In terms of vector notation (2-1) may be written as

$$U = f(\mathbf{q}), \tag{2-2}$$

where it is understood that $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, and $\mathbf{q}$ may be called a <u>budget alternative</u> or simply a <u>budget</u>. U is designated as the level of utility.

It is further assumed that the consumer's utility function is globally continuous on its domain of definition and that it possesses derivatives [1] of at least the first two orders, which are also globally continuous. Global continuity means the functions are continuous on their domain P, or at every point in P [2, p. 160].

A further restriction is placed on the consumer's utility function by assuming that each first partial derivative of the function, f_i, for $i = 1, 2, \ldots, n$ is positive. This derivative is called the <u>marginal</u> <u>utility</u> of the i^{th} commodity, and expresses the rate of change in the level of utility with respect to the i^{th} variable. [2] Assuming each first partial derivative to be positive is tantamount to saying the consumer's level of utility increases as he takes more of any commodity Q_i. In other words, the consumer would rather have more of any commodity Q_i, than have less or an equal amount of it. [3] Hence, if (2-1) is evaluated at specific or fixed values for

---

[1] Derivatives are expressed in traditional notation, e.g., the derivative of $U = f(\mathbf{q})$ is denoted $dU = f_1 dq_1 + f_2 dq_2 + \cdots + f_n dq_n$, and the first partial derivative of U with respect to the i^{th} variable is denoted $\frac{\partial f(\mathbf{q})}{\partial q_i}$ or simply f_i. The "cross" partial derivatives are denoted $\frac{\partial^2 f(\mathbf{q})}{\partial q_i \partial q_j} = f_{ij}$, $i \neq j$. When $i = j$ it is called the second-order partial derivative.

[2] The term marginal utility does not imply that utility is cardinally measurable.

[3] It is assumed that the consumer never reaches the point of satiation or beyond for any commodity in a given time period. In reality the consumer could reach the point at which he would want no more of some commodity. At this point the commodity becomes a discommodity, and it is assumed in this thesis that no commodity ever becomes a discommodity.

each q_i one obtains

(2-3) $$U_{\mathcal{g}^0} = f(q_1^0, q_2^0, \ldots, q_n^0).$$ [4]

Now if any one of the coordinate values of $\mathcal{g}$, say the k^{th}, is increased to a higher fixed value and all other coordinate values remain as specified in (2-3), one obtains by re-evaluating the utility function

(2-4) $$U_{\mathcal{g}^1} = f(q_1^1, q_2^1, \ldots, q_k^1, \ldots, q_n^1), \quad \text{for} \quad \begin{array}{l} q_1^0 = q_1^1 \\ q_2^0 = q_2^1 \\ \ldots\ldots \\ q_{k-1}^0 = q_{k-1}^1 \\ q_k^0 < q_k^1 \\ q_{k+1}^0 = q_{k+1}^1 \\ \ldots\ldots \\ q_n^0 = q_n^1 \end{array},$$

where $U_{\mathcal{g}^1} > U_{\mathcal{g}^0}$. In this sense it can be said that the consumer's

[4] A specific or fixed value for some variable is denoted by a superscript e.g., q_i^0, q_i^1, etc. Likewise when a specific value is being employed for each coordinate of $\mathcal{g}$ one could write $\mathcal{g}^0 = (q_1^0, q_2^0, \ldots, q_n^0)$. $U_{\mathcal{g}^1}$ is a real number which is obtained by evaluating the utility function at $\mathcal{g}^1$.

utility function is strictly increasing with respect to each variable in its domain.

Indifference Surfaces

The loci of all budgets which satisfy

$$U_{q^0} = f(q), \tag{2-5}$$

are said to form an $n-1$ dimensional indifference surface in $P \subset R^n$. The system of all such surfaces fills the entire budget space [23, p. 83]. For any budget which satisfies (2-5), the consumer will be indifferent as to which particular budget he possesses, assuming no budget constraint. For the present, no restrictions are placed on these surfaces other than those which must logically follow from the heretofore stated assumptions concerning the consumer's utility function. For example, $\frac{\partial q_j}{\partial q_i} = -\frac{f_i}{f_j}$ which is negative since f_i and f_j are always positive. The negative of this partial derivative is called the rate of commodity substitution or more traditionally the marginal rate of substitution and is always positive for any two commodities.

According to the notation above, another $n-1$ dimensional indifference surface consists of the loci of all points which satisfy

$U_{\mathscr{g}1} = f(\mathscr{g})$, where $U_{\mathscr{g}0} < U_{\mathscr{g}1}$. Geometrically this means the loci of points satisfying $U_{\mathscr{g}0}$ lie between the loci of points satisfying $U_{\mathscr{g}1}$ and the origin. That is, the indifference surface corresponding to $U_{\mathscr{g}1}$ lies above the indifference surface for $U_{\mathscr{g}0}$.

Time Period

The consumer's utility function is required to remain fixed during the period under consideration, i. e., the consumer's desires or wants do not change during the period. Perhaps the easiest way to view this is to assume that the consumer receives a fixed amount of income at the beginning of each time period. Upon receiving his income the consumer goes to the market and spends all of his income on those commodities which are in the domain of his utility function. From the time he purchases the commodities up to the time he receives his next fixed amount of income, he consumes part or all of the commodities he has purchased. When he receives his next fixed amount of income, that is, at the beginning of the next time period, his utility function may have changed or taken a different form. In fact, it may not even be defined for the same variables or the same number of variables. However, this is of no importance, as in this thesis only one time period is considered, and hence only the fixed utility function corresponding to this time period. Since the consumer makes his choices among the Q_i at the beginning of the period, it is really expected utility that he considers.

At this point it is perhaps of worth to reflect upon the monumental work of Eugen E. Slutsky [19] , as his article "On the Theory of the Budget of the Consumer" is, as Henderson and Quandt [5, p. 41] state: "The article upon which the modern mathematical theory of consumer behavior is based."

Slutsky's first two hypotheses regarding the consumer's utility function are concerned with: (1) the continuity of the function and its first two derivatives, and (2) the constancy of the function during the time period considered. His third hypothesis states [19, p. 30]:

> (3) The hypothesis that the increment of utility obtained in passing from one combination of goods to another does not depend upon the mode of passage. In mathematical language, this leads to the condition:

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} = \frac{\partial^2 U}{\partial x_2 \partial x_1} .$$

This assumption has not been included in this thesis, as the abovementioned assumption concerning the global continuity of the utility function and its first two derivatives implies Slutsky's third hypothesis [2, p. 243]. Whether Slutsky noticed that his first hypothesis implied his third is of no importance, as his third hypothesis was worth noting, assuming his readers would not make the deduction.

The Budget Constraint

It was previously noted that the consumer receives a fixed amount of income each time period. One might consider that income is a variable over time. However, since only one specific time period will be considered, income may, for the present, be regarded as being a fixed value, which is again denoted by a superscript, e.g., I^0. Hence, for the time period under consideration, the consumer will be limited as to the amounts of each commodity he may purchase, as he has only a fixed amount of income to spend. This, of course, presupposes that the prices of all commodities are positive, i.e., $p_i > 0$, for $i = 1, 2, \ldots, n$. In addition to assuming prices are positive, it will be assumed for the present that they are fixed values throughout the time period.

The individual consumer is regarded as having no effect on prices. For any commodity he desires to purchase, he must pay the same price per unit, no matter how many units he purchases, i.e., the consumer receives no quantity discounts. Therefore the consumer will allocate his income to the n commodities in the domain of his utility function subject to the following restriction:

$$(2\text{-}6) \qquad I^0 = p_1^0 q_1 + p_2^0 q_2 + \cdots + p_n^0 q_n ,$$

where income and prices are fixed, and the q_i are variables in the present discussion. It is understood that each price is in terms of price per unit. The type and size of the unit corresponds to the well defined unit in which each quantity is measured.

Equation (2-6) is called the consumer's budget constraint and may be satisfied by an infinite number of budgets (regarded as an $n-1$ dimensional plane in $P \subset R^n$). Geometrically it is a constraint in the sense that the consumer is capable of purchasing any budget which lies between the budget constraint and the origin, that is, any budget which satisfies

$$(2\text{-}7) \qquad I^0 > p_1^0 q_1 + p_2^0 q_2 + \cdots + p_n^0 q_n \,.$$

However, because the consumer is required to spend all of his income on the commodities available to him, he must purchase some budget which satisfies (2-6). The budget constraint may be considered as an implicit function and is written

$$(2\text{-}8) \qquad g(\mathbf{q}) = 0 \,.$$

At this point it will be useful to make some observations regarding the budget constraint in two and three dimensions. First, let it be assumed that there are only two variables in the domain of the consumer's utility function so that his budget constraint takes the following form:

(2-9) $$I^0 = p_1^0 q_1 + p_2^0 q_2 \ .$$

It is now desired to draw a graph of this equation. By rewriting (2-9) one obtains

(2-10) $$q_2 = \frac{I^0}{p_2^0} - \frac{p_1^0}{p_2^0} q_1 ,$$

where q_2 is now considered to be an explicit function of q_1, i.e., $q_2 = h(q_1)$. Upon differentiating this function with respect to q_1, one obtains

(2-11) $$h'(q_1) = -\frac{p_1^0}{p_2^0} < 0 \ .$$

Hence, it is easy to see that the graph of (2-9) will resemble Figure 2-1. In particular the graph of this budget constraint will be a straight line which has a constant negative slope equal to $-\frac{p_1^0}{p_2^0}$ and intersects the Q_1 axis at $\frac{I^0}{p_1^0}$. Every point lying on this line represents a different budget the consumer could purchase by spending all of his fixed income at the prevailing fixed prices. And of course he could purchase any budget that lies between the budget constraint and the origin. However, at the prevailing prices this means he would not be spending all of his fixed income.

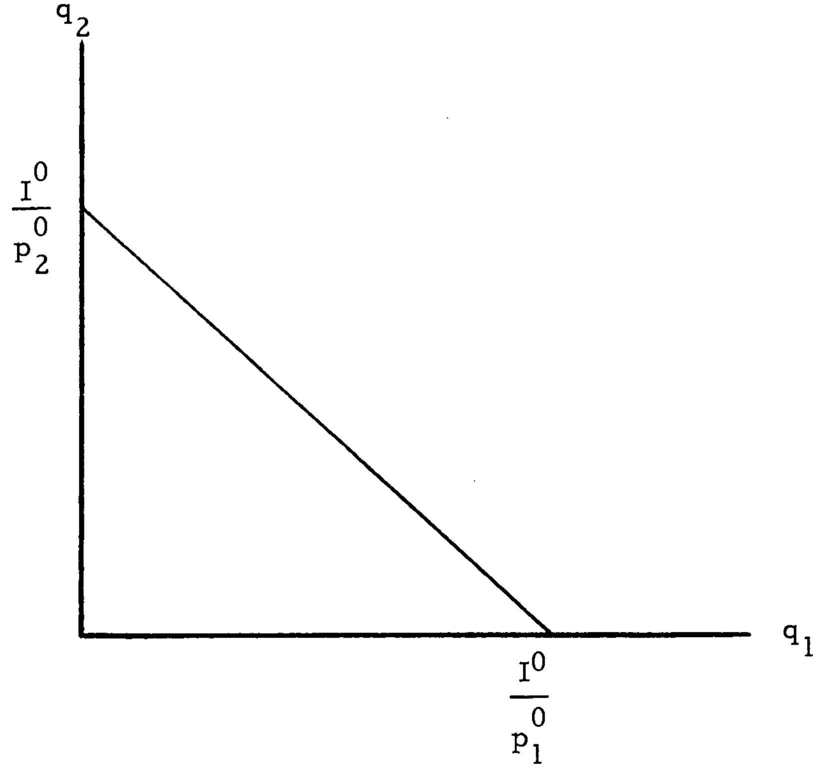

Figure 2-1. Budget Constraint in Two Dimensions.

Using the same argument, it can be seen that if the domain of the consumer's utility function is defined for three variables, his budget constraint is represented by some triangular shaped plane in three dimensions. (See Figure 2-2.) Hence, each

(2-12) $$\frac{\partial q_i}{\partial q_j} = - \frac{p_j^0}{p_i^0} < 0, \quad \text{for all} \quad i = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3,$$

where $i \neq j$.

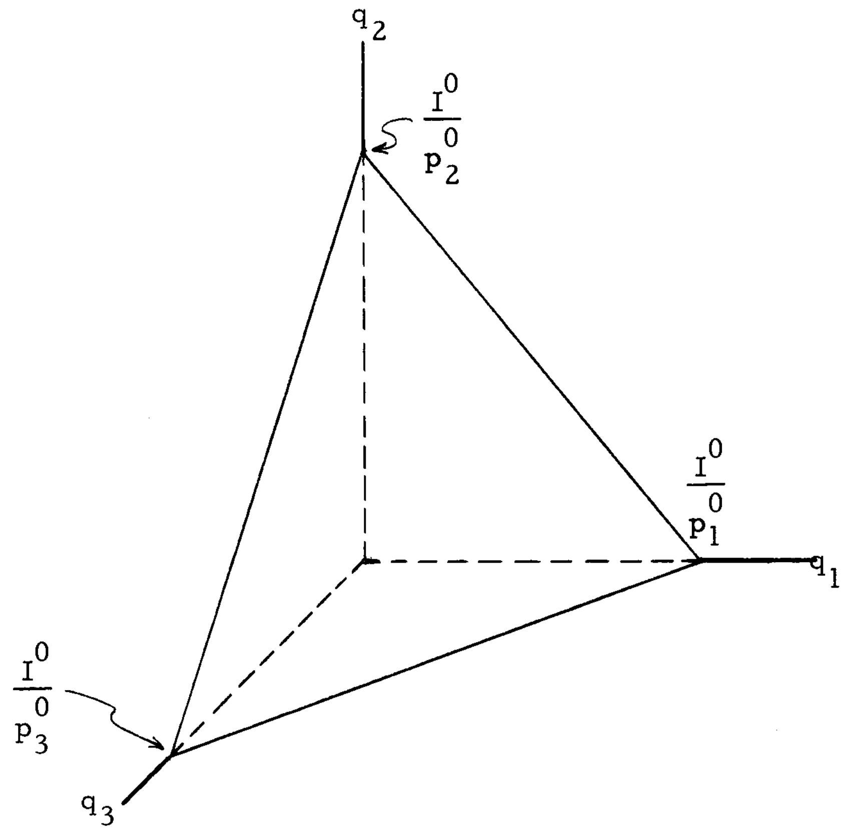

Figure 2-2. Budget Constraint in Three Dimensions.

Consumer Behavior

It has already been stated that the consumer spends all of his given income and that his utility function is not altered in any way in the time period under consideration. To carry out the analysis contained in this thesis, it must be additionally assumed that the consumer obeys the following three laws:[4]

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[4] Some texts give postulates or axioms of consumer rationality. However, these postulates are concerned with the consumer's preference field or ordering and are basic to the establishment of a utility function. As the existence of a utility function is assumed from the outset in this thesis, the writer does not deem it necessary to state

(1) The consumer can evaluate his utility function for any arbitrary budget. This implies that he has complete knowledge concerning his function. The resulting value is called the <u>utility of the budget</u> and is designated $U_{\mathcal{g}^k}$.

(2) The consumer is always willing to forego possession of any arbitrary budget, $\mathcal{g}^k$, and possess instead some arbitrary budget, $\mathcal{g}^m$, if and only if $U_{\mathcal{g}^k} < U_{\mathcal{g}^m}$.[5] If and only if two arbitrary budgets have equal utilities, i.e., $U_{\mathcal{g}^k} = U_{\mathcal{g}^m}$, is the consumer <u>indifferent</u> with respect to willingness to possess either budget.

(3) The consumer purchases that budget, of all budgets he can obtain, which maximizes the value of his utility function, provided such a budget exists. This budget is called the <u>optimal budget</u> and is designated by $\mathcal{g}^*$.

---

such postulates as the postulate of transitivity. The writer feels that these postulates are embodied in the "laws of consumer behavior" and the assumptions concerning the consumer's utility function.

[5] The one and only reason the phrase "willing and able" is not used here is that the consumer may not be able to purchase budget $\mathcal{g}^m$ because of his income limitation.

The Utility Index

The various utilities of budgets form a continuous ordinal index. The index is ordinal in the following sense. For two arbitrary budgets, assume $U_{\mathscr{g}^0} = 5$ and $U_{\mathscr{g}^1} = 10$. It cannot be said that the utility of $\mathscr{g}^1$ is twice the utility of $\mathscr{g}^0$.[6] All that can be said is that the consumer is willing to forego possession of $\mathscr{g}^0$ in order to possess $\mathscr{g}^1$. No statement or conjecture is made regarding what utility means to the consumer. It cannot be said that utility is synonymous to joy, pleasure, pain, happiness, or any other descriptive noun. Utility is independent of any psychological, physiological, philosophical, or other connotations [19, p. 27-28]. In the words of Henderson and Quandt [5, p. 6]:

> All information pertaining to the satisfaction that the consumer derives from various quantities of commodities is contained in his utility function.
>
> The concept of utility and its maximization are void of any sensuous connotation.

The term "satisfaction" could be replaced with "amount of utility," in order to preclude any connotations about utility. About the only concrete statement that can be made is that the level of utility

---

6 If this statement could be made the utility index would be called a cardinal index.

increases (decreases) if any one or any group of quantities in a given budget increases (decreases). If some quantities increase and others decrease in a given budget to form a new arbitrary budget, it cannot be said a priori which budget yields the greatest utility, if in fact either does. In this case the consumer can tell if one budget has greater utility than the other only after he evaluates his utility function for the two budgets.

As the level of utility is an ordinal index, it is not necessary that the consumer's utility function be unique. In general, any increasing monotonic transformation[7] of the consumer's utility function may be used in place of the given utility function, as it will preserve the order of the utility index, and an ordinal index is an index which exhibits only order. Conditions for a constrained maximum of the consumer's utility function (which will be dealt with later) are invariant when any increasing monotonic transformation of the utility function is used in place of the given utility function [5, p. 17-20].

Constrained Maximization of the Consumer's Utility Function

Having specified the framework of a mathematical model of consumer behavior, it is now desired to examine consumer behavior

---

[7] A function $F(U)$ is an increasing monotonic transformation of $U = f(\mathscr{g})$ if $F(U_{\mathscr{g}^0}) < F(U_{\mathscr{g}^1})$ whenever $U_{\mathscr{g}^1} < U_{\mathscr{g}^0}$.

more closely within the model and establish any conditions which must be fulfilled. The consumer wants to purchase the budget which yields the greatest utility and simultaneously satisfies his budget constraint, i. e., he wants to fulfill law number three. Hence the consumer faces a type of constrained maximization problem, which may be handled mathematically.

Graphic Representation of a Constrained Maximum

To gain elementary understanding of a constrained maximum it is perhaps useful to refer to the traditional graphic representation, where there are only two commodities in the domain of the consumer's utility function. This is given in Figure 2-3.

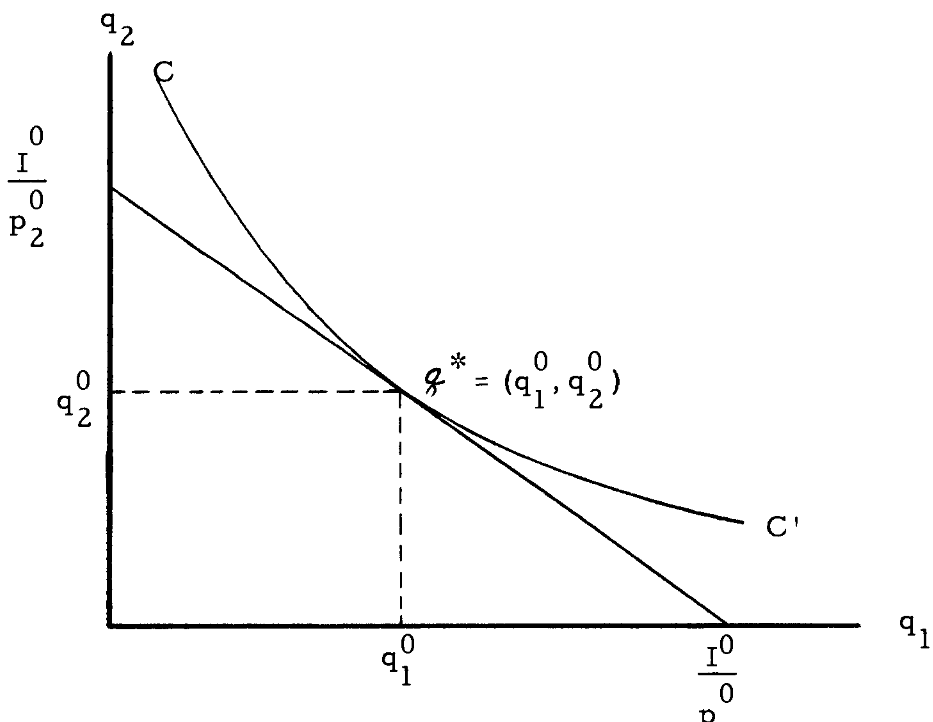

Figure 2-3. Graphic Representation of a Constrained Maximum.

CC' represents a section of a traditional n-1 dimensional indifference surface, which is tangent to the consumer's budget constraint. The student of economics learns that the consumer purchases the optimal budget represented by $\mathcal{g}^*$, as this budget will yield the greatest utility at the prevailing fixed prices and income.

In order to locate $\mathcal{g}^*$ in our mathematical model, necessary and sufficient conditions for a constrained maximum must be fulfilled. In loose verbal terms, fulfillment of the necessary conditions locates the point of tangency between the budget constraint and the indifference surface, provided no corner solutions exist [5, p. 15-16]. Fulfillment of the sufficient conditions insures that the indifference surface lies above the budget constraint. See Figure 2-4 where not every point of the indifference curve lies above the tangency at $\mathcal{g}^1$. As will be shown in the next section, this type of indifference curve is inadmissible.

Necessary Conditions for a Constrained Maximum

Lagrange's Method [2, p. 266-268] establishes the necessary conditions for attaining a constrained maximum in the type of problem the consumer faces. The mechanics of this method consist of equating each first partial derivative of the utility function to a real number λ, which is called the Lagrange multiplier, times the corresponding first partial derivative of the budget constraint and solving these n

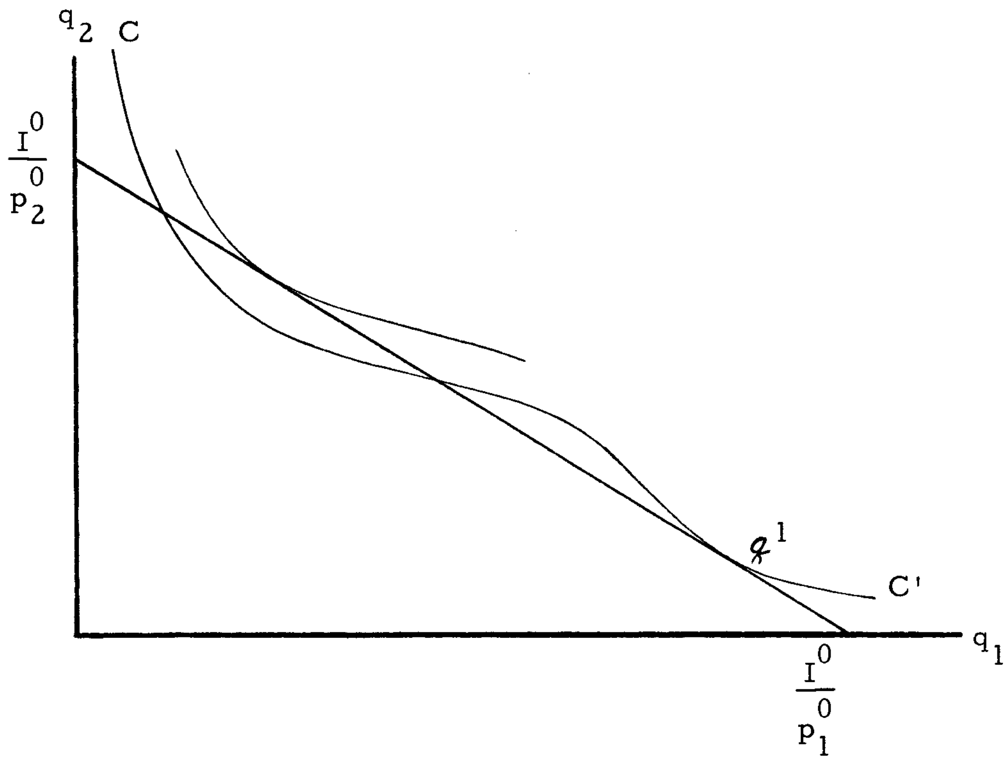

Figure 2-4. An Inadmissible Indifference Curve with Valid Budget Constraint.

equations together with the budget constraint for q and λ. This may be expressed as the following system of $n+1$ equations:

$$
\begin{aligned}
f_1 &= \lambda g_1 \\
f_2 &= \lambda g_2 \\
&\cdots\cdots \\
f_n &= \lambda g_n \\
g(q) &= 0,
\end{aligned}
\tag{2-13}
$$

which are to be solved simultaneously for q and λ. From equation (2-8) it follows that

$$g_i = p_i^0, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{2-14}$$

Equations (2-13) may be rewritten to form the following system in which λ has been eliminated:

$$\begin{aligned} & p_1^0 q_1 + p_2^0 q_2 + \cdots + p_n^0 q_n = I^0 , \\ & \frac{f_i}{f_j} = \frac{p_i^0}{p_j^0}, \quad \text{for all} \quad i \neq j = 1, 2, \ldots, n. \end{aligned} \tag{2-15}$$

It can be seen in general that there are $n-1$ independent rates of commodity substitution. Equations (2-15) are to be solved for any budgets, $\mathscr{g}$, that satisfy them. However, it will be seen that there must be a unique $\mathscr{g}$ which satisfies (2-15) if demand functions are to exist, and this budget is the proposed candidate for the optimal budget.

Sufficient Conditions for a Constrained Maximum

Generally speaking, sufficient conditions for a constrained maximum are more difficult to derive than necessary conditions. However, they have been derived for the type of constrained maximization problem the consumer faces.

An indifference surface is said to be convex, or convex to the origin, at an arbitrary point $\mathscr{q}$ on the surface if the surface lies above the tangent plane at $\mathscr{q}$ everywhere in the vicinity of $\mathscr{q}$. The following sufficient conditions for a constrained maximum establish the convexity of the indifference surface at $\mathscr{q}$. If

(2-16) $$(-1)^k D^k(\mathscr{q}) > 0, \quad \text{for all} \quad k = 2, 3, \ldots, n$$

is satisfied for some $\mathscr{q}^0$, where

(2-17) $$D^k(\mathscr{q}) = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_k \\ f_1 & f_{11} & f_{12} & \cdots & f_{1k} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_k & f_{k1} & f_{k2} & \cdots & f_{kk} \end{vmatrix},$$

then the indifference surface is said to be convex at this value of $\mathscr{q}^0$.

A traditional illustration of the foregoing is given by Figure 2-3. Solution of the necessary conditions, (2-15), for some hypothetical utility function involving only two commodities yields the coordinate values for $\mathscr{q}^* = (q_1^0, q_2^0)$. By inserting $\mathscr{q}^*$ into (2-16) it would be seen that the inequality holds. This implies that the indifference curve is convex to the origin at $\mathscr{q}^*$ as shown.

Demand Functions

When equations (2-15) are solved for $\mathfrak{q}^*$, each coordinate value q_i^0, depends in general on $p_1^0, p_2^0, \ldots, p_n^0$ and I^0. Suppose for the moment that the solution to (2-15) is unique for any fixed set of prices and income. This means that each coordinate value of $\mathfrak{q}^*$ is unique and in general

$$q_i = \phi_i(p_1, p_2, \ldots, p_n, I), \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{2-17}$$

Now $\phi_i(p_1, p_2, \ldots, p_n, I)$ is called the demand function for commodity Q_i, where p_i and I are variables.

If the solution to (2-15) is not unique, then more than one value for some q_i can be obtained from one fixed set of prices and income, and these particular ϕ_i's are therefore not functions. Not having demand functions would be devastating to traditional economic theory.

Recall that (2-16) does not say that the indifference surface is convex at all budgets lying on it, unless it holds for all budgets on the surface. However, if (2-16) holds for all $\mathfrak{q}$, then (2-15) will have a unique solution for $\mathfrak{q}^*$. In summary it may be said that if (2-16) holds for any $\mathfrak{q}$, then the solution to (2-15) will be unique and further each commodity, Q_i, will have a well-defined demand function.

CHAPTER III

MATHEMATICAL ANALYSIS OF THE CONSUMPTION AND DEMAND MODEL

Given the model and analytic results in Chapter II, it is the purpose of the present chapter to conduct a further mathematical analysis on the model. The so-called "Slutsky Equation" or the "Fundamental Equation of Value Theory" shall be derived and then a subsequent analysis made on the income and substitution terms of this equation. From an analysis of the substitution term a discussion of substitutable, independent, and complementary goods shall emanate.

Derivation of the Slutsky Equation

Let the necessary conditions for the constrained maximum (2-13) be written in the form

$$
\begin{aligned}
p_1q_1 + p_2q_2 + \cdots + p_rq_r + \cdots + p_nq_n &= I \\
-\lambda p_1 + f_1 &= 0 \\
-\lambda p_2 + f_2 &= 0 \\
\cdots & \\
-\lambda p_r + f_r &= 0 \\
\cdots & \\
-\lambda p_n + f_n &= 0 ,
\end{aligned}
\tag{3-1}
$$

where now each p_i will be allowed to vary. Differentiating partially with respect to p_r one obtains

$$
\begin{aligned}
&p_1 \frac{\partial q_1}{\partial p_r} + p_2 \frac{\partial q_2}{\partial p_r} + \cdots + p_r \frac{\partial q_r}{\partial p_r} + q_r + \cdots + p_n \frac{\partial q_n}{\partial p_r} = 0 \\
&-p_1 \frac{\partial \lambda}{\partial p_r} + f_{11} \frac{\partial q_1}{\partial p_r} + f_{12} \frac{\partial q_2}{\partial p_r} + \cdots + f_{1r} \frac{\partial q_r}{\partial p_r} + \cdots + f_{1n} \frac{\partial q_n}{\partial p_r} = 0 \\
(3\text{-}2) \quad &\cdots\cdots\cdots\cdots\cdots\cdots \\
&-(\lambda + p_r \frac{\partial \lambda}{\partial p_r}) + f_{r1} \frac{\partial q_1}{\partial p_r} + f_{r2} \frac{\partial q_2}{\partial p_r} + \cdots + f_{rr} \frac{\partial q_r}{\partial p_r} + \cdots + f_{rn} \frac{\partial q_n}{\partial p_r} = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&-p_n \frac{\partial \lambda}{\partial p_r} + f_{n1} \frac{\partial q_1}{\partial p_r} + f_{n2} \frac{\partial q_2}{\partial p_r} + \cdots + f_{nr} \frac{\partial q_r}{\partial p_r} + \cdots + f_{nn} \frac{\partial q_n}{\partial p_r} = 0 .
\end{aligned}
$$

This may be rewritten in matrix notation as

$$
(3\text{-}3) \quad
\begin{bmatrix}
0 & p_1 & p_2 & \cdots & p_n \\
-p_1 & f_{11} & f_{12} & \cdots & f_{1n} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
-p_r & f_{r1} & f_{r2} & \cdots & f_{rn} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
-p_n & f_{n1} & f_{n2} & \cdots & f_{nn}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \lambda}{\partial p_r} \\
\frac{\partial q_1}{\partial p_r} \\
\cdots \\
\frac{\partial q_r}{\partial p_r} \\
\cdots \\
\frac{\partial q_n}{\partial p_r}
\end{bmatrix}
=
\begin{bmatrix}
-q_r \\
0 \\
\cdots \\
\lambda \\
\cdots \\
0
\end{bmatrix} .
$$

By using Cramer's rule one finds in general

$$\text{(3-4)} \quad \frac{\partial q_s}{\partial p_r} = \frac{\begin{vmatrix} 0 & p_1 & \cdots & p_{s-1} & -q_r & p_{s+1} & \cdots & p_n \\ -p_1 & f_{11} & \cdots & f_{1,s-1} & 0 & f_{1,s+1} & \cdots & f_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_r & f_{r1} & \cdots & f_{r,s-1} & \lambda & f_{r,s+1} & \cdots & f_{rn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_n & f_{n1} & \cdots & f_{n,s-1} & 0 & f_{n,s+1} & \cdots & f_{nn} \end{vmatrix}}{\begin{vmatrix} 0 & p_1 & p_2 & \cdots & p_n \\ -p_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_r & f_{r1} & f_{r2} & \cdots & f_{rn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}}$$

for r and arbitrary s. Using the fact from (2-13) and (2-14) that

$$\text{(3-5)} \qquad f_i = \lambda p_i, \quad i = 1, 2, \ldots, n,$$

one may rewrite the denominator in (3-4) as

$$\text{(3-6)} \qquad \begin{vmatrix} 0 & \frac{f_1}{\lambda} & \frac{f_2}{\lambda} & \cdots & \frac{f_n}{\lambda} \\ -\frac{f_1}{\lambda} & f_{11} & f_{12} & \cdots & f_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -\frac{f_r}{\lambda} & f_{r1} & f_{r2} & \cdots & f_{rn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -\frac{f_n}{\lambda} & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} = -\frac{1}{\lambda^2} D,$$

where

$$\text{(3-7)} \qquad D = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_s & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1s} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2s} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_r & f_{r1} & f_{r2} & \cdots & f_{rs} & \cdots & f_{rn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n & f_{n1} & f_{n2} & \cdots & f_{ns} & \cdots & f_{nn} \end{vmatrix}.$$

It should be noted that D is the same as $D^k(g)$ given in (2-17) when k is equal to n.

Expanding the determinant in the numerator of (3-4) by co-factors of the elements in column $s+1$, one obtains

$$(3\text{-}8)\quad \begin{vmatrix} 0 & p_1 & \cdots & p_{s-1} & -q_r & p_{s+1} & \cdots & p_n \\ -p_1 & f_{11} & \cdots & f_{1,s-1} & 0 & f_{1,s+1} & \cdots & f_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_r & f_{r1} & \cdots & f_{r,s-1} & \lambda & f_{r,s+1} & \cdots & f_{rn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_n & f_{n1} & \cdots & f_{n,s-1} & 0 & f_{n,s+1} & \cdots & f_{nn} \end{vmatrix} = -q_r(-\frac{1}{\lambda})D_s + \lambda(-\frac{1}{\lambda^2})D_{rs},$$

where D_{rs} is the cofactor of element f_{rs} of D, and D_s is the cofactor of the first row and column s+1 of D, or element f_s of D.

Hence,

$$(3\text{-}9)\qquad \frac{\partial q_s}{\partial p_r} = -\frac{q_r \lambda D_s}{D} + \frac{\lambda D_{rs}}{D}, \quad \text{for} \quad r, s = 1, 2, \ldots, n.$$

As they stand, equations (3-9) are difficult to interpret verbally. The first and second terms on the right of (3-9) are known traditionally as the income and substitution terms respectively. Further analysis, in which the determinant notation used in (3-9) shall be changed to partial derivative notation, will justify the use of this nomenclature.

Income Effect Term

To change the first term on the right hand side of (3-9) to partial derivative notation, consider differentiating equations (3-1) with respect to I. Upon doing so the following system is obtained:

$$
\begin{aligned}
& p_1 \frac{\partial q_1}{\partial I} + p_2 \frac{\partial q_2}{\partial I} + \cdots + p_n \frac{\partial q_n}{\partial I} = 1 \\
& -p_1 \frac{\partial \lambda}{\partial I} + f_{11} \frac{\partial q_1}{\partial I} + f_{12} \frac{\partial q_2}{\partial I} + \cdots + f_{1n} \frac{\partial q_n}{\partial I} = 0 \\
(3\text{-}10) \quad & -p_2 \frac{\partial \lambda}{\partial I} + f_{21} \frac{\partial q_1}{\partial I} + f_{22} \frac{\partial q_2}{\partial I} + \cdots + f_{2n} \frac{\partial q_n}{\partial I} = 0 \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
& -p_n \frac{\partial \lambda}{\partial I} + f_{n1} \frac{\partial q_1}{\partial I} + f_{n2} \frac{\partial q_2}{\partial I} + \cdots + f_{nn} \frac{\partial q_n}{\partial I} = 0
\end{aligned}
$$

Equations (3-10) can be rewritten in matrix notation as

$$
(3\text{-}11) \quad \begin{bmatrix} 0 & p_1 & p_2 & \cdots & p_n \\ -p_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ -p_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda}{\partial I} \\ \frac{\partial q_1}{\partial I} \\ \frac{\partial q_2}{\partial I} \\ \vdots \\ \frac{\partial q_n}{\partial I} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .
$$

Again using Cramer's rule, one finds for arbitrary s

$$(3\text{-}12)\quad \frac{\partial q_s}{\partial I} = \frac{\begin{vmatrix} 0 & p_1 & p_2 & \cdots & p_{s-1} & 1 & p_{s+1} & \cdots & p_n \\ -p_1 & f_{11} & f_{12} & \cdots & f_{1,s-1} & 0 & f_{1,s+1} & \cdots & f_{1n} \\ -p_2 & f_{21} & f_{22} & \cdots & f_{2,s-1} & 0 & f_{2,s+1} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_n & f_{n1} & f_{n2} & \cdots & f_{n,s-1} & 0 & f_{n,s+1} & \cdots & f_{nn} \end{vmatrix}}{\begin{vmatrix} 0 & p_1 & p_2 & \cdots & p_n \\ -p_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ -p_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}}, \quad s = 1, 2, \cdots, n.$$

According to (3-6) the denominator of the term on the right may be written as $-\frac{1}{\lambda^2}D$. After substituting $p_i = \frac{f_i}{\lambda}$ and expanding the determinant in the numerator by cofactors of column $s+1$, one obtains

$$(3\text{-}13)\qquad \begin{vmatrix} 0 & \frac{f_1}{\lambda} & \frac{f_2}{\lambda} & \cdots & \frac{f_{s-1}}{\lambda} & 1 & \frac{f_{s+1}}{\lambda} & \cdots & \frac{f_n}{\lambda} \\ -\frac{f_1}{\lambda} & f_{11} & f_{12} & \cdots & f_{1,s-1} & 0 & f_{1,s+1} & \cdots & f_{1n} \\ -\frac{f_2}{\lambda} & f_{21} & f_{22} & \cdots & f_{2,s-1} & 0 & f_{2,s+1} & \cdots & f_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\frac{f_n}{\lambda} & f_{n1} & f_{n2} & \cdots & f_{n,s-1} & 0 & f_{n,s+1} & \cdots & f_{nn} \end{vmatrix} = -\frac{1}{\lambda} D_s$$

where D_s is the cofactor of element f_s of D.

Hence, one obtains in general

$$(3\text{-}14)\qquad \frac{\partial q_s}{\partial I} = \lambda \frac{D_s}{D}, \quad \text{for} \quad s = 1, 2, \ldots, n .$$

Substituting the above result into (3-9) yields

$$(3\text{-}15)\qquad \frac{\partial q_s}{\partial p_r} = -q_r \frac{\partial q_s}{\partial I} + \lambda \frac{D_{rs}}{D}, \quad \text{for} \quad r, s = 1, 2, \ldots, n.$$

The preceding analysis is not original with this writer and follows closely the method outlined by Hicks [6, p. 307-309]. Equations (3-15) were originally due to Slutsky [19] in 1915. Overlooked in the literature, this result was again obtained independently by Hicks and Allen [7] in 1934 and termed the "Fundamental Equation of Value Theory" by Hicks [6, p. 309].

Substitution Effect Term

Now in order to change the second term on the right-hand side of (3-15) to partial derivative notation, consider differentiating the following system with respect to p_r for arbitrary r:

$$f(q_1, q_2, \ldots, q_n) = c, \quad \text{where} \quad c = \text{constant}$$

(3-16)
$$\begin{aligned} -\lambda p_1 + f_1 &= 0 \\ \cdot \quad \cdot \quad &\cdot \quad \cdot \\ -\lambda p_r + f_r &= 0 \\ \cdot \quad \cdot \quad &\cdot \quad \cdot \\ -\lambda p_n + f_n &= 0 \,. \end{aligned}$$

Equations (3-16) are identical to equations (3-1) with the exception that the budget constraint is replaced by the condition that utility is held at a constant level. Differentiating with respect to p_r yields

$$f_1 \frac{\partial q_1}{\partial p_r} + f_2 \frac{\partial q_2}{\partial p_r} + \cdots + f_n \frac{\partial q_n}{\partial p_r} = 0$$

$$-p_1 \frac{\partial \lambda}{\partial p_r} + f_{11} \frac{\partial q_1}{\partial p_r} + f_{22} \frac{\partial q_2}{\partial p_r} + \cdots + f_{1n} \frac{\partial q_n}{\partial p_r} = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

(3-17)

$$-(\lambda + p_r \frac{\partial \lambda}{\partial p_r}) + f_{r1} \frac{\partial q_1}{\partial p_r} + f_{r2} \frac{\partial q_2}{\partial p_r} + \cdots + f_{rn} \frac{\partial q_n}{\partial p_r} = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$-p_n \frac{\partial \lambda}{\partial p_r} + f_{n1} \frac{\partial q_1}{\partial p_r} + f_{n2} \frac{\partial q_2}{\partial p_r} + \cdots + f_{nn} \frac{\partial q_n}{\partial p_r} = 0.$$

The preceding system is written in matrix notation as

(3-18)

$$\begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ -p_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_r & f_{r1} & f_{r2} & \cdots & f_{rn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -p_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda}{\partial p_r} \\ \frac{\partial q_1}{\partial p_r} \\ \vdots \\ \frac{\partial q_r}{\partial p_r} \\ \vdots \\ \frac{\partial q_n}{\partial p_r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix},$$

Now solving for $\frac{\partial q_s}{\partial p_r}$ in a manner similar to the preceding, one obtains for arbitrary r and s

$$(3\text{-}19) \qquad \frac{\partial q_s}{\partial p_r} = \frac{\lambda D_{rs}}{D}, \quad r, s = 1, 2, \ldots, n .$$

Recall that the above result was obtained by holding utility constant. Therefore, (3-19) can be written in the following form:

$$(3\text{-}20) \qquad \frac{\lambda D_{rs}}{D} = \left(\frac{\partial q_s}{\partial p_r} \right)_{U = \text{constant}}, \quad r, s = 1, 2, \ldots, n.$$

Because D is the determinant of a real symmetric matrix, D_{rs} is equal to D_{sr}. It follows that the substitution effect is symmetric in the sense that

$$(3\text{-}21) \qquad \left(\frac{\partial q_s}{\partial p_r} \right)_{U = \text{constant}} = \left(\frac{\partial q_r}{\partial p_s} \right)_{U = \text{constant}}, \quad r, s = 1, 2, \ldots, n.$$

<u>Combined Income and Substitution Effects</u>

Using the results obtained above, equations (3-15) can be re-written. Since $\frac{\partial q_s}{\partial I}$ is the partial derivative of the quantity of commodity s with respect to income, it is tacitly assumed that prices are held constant by definition. Therefore, equations (3-14) can be rewritten

$$(3\text{-}22) \qquad \frac{\lambda D_s}{D} = \left(\frac{\partial q_s}{\partial I}\right)_{p_i = \text{constant}}, \quad s = 1, 2, \ldots, n.$$

Now the Slutsky equation (3-15) can be written as

$$(3\text{-}23) \qquad \frac{\partial q_s}{\partial p_r} = -q_r\left(\frac{\partial q_s}{\partial I}\right)_{p_i = \text{constant}} + \left(\frac{\partial q_s}{\partial p_r}\right)_{U = \text{constant}}, r, s = 1, 2, \ldots, n.$$

It should be more clear that the first term is the income term and the second is the substitution term. Next it shall be shown how these terms can be interpreted as the traditional income and substitution effects respectively.

Geometric Interpretation of Income and Substitution Effects

The present section deals with the traditional geometric interpretation of the income and substitution terms, which involves the indifference curves for two commodities. It should be pointed out at the outset that the Slutsky equation involves rates of change, whereas only discrete changes are shown diagrammatically. As a consequence, the substitution and income effects are discrete changes which are said to correspond to rates of change. The geometric result is a total discrete change that is said to correspond to a rate of change comprised of the two components of equation (3-23).

Assume that only two commodities, Q_1 and Q_2, are available to the consumer. In this case the consumer's budget constraint will resemble the budget constraint in Figure 2-1. As pointed out earlier, solution to the necessary conditions, (2-13), will locate the point of tangency between the budget constraint and the indifference curve. Assuming the sufficient conditions are satisfied, Figure 2-3 represents the given situation and in the following presentation it is always assumed that the sufficient conditions are satisfied.

Starting initially from $\mathcal{q}^*$ as illustrated in Figure 2-3, suppose the price of Q_1 decreases from p_1^0 to p_1^1. The effect of this price decrease on the budget constraint is a rotation about the point $\frac{I^0}{p_2^0}$ to the new position represented in Figure 3-1 by the line extending from $\frac{I^0}{p_2^0}$ to $\frac{I^0}{p_1^1}$. By solving the new necessary conditions, in which p_1^0 has been replaced by p_1^1, a new optimal budget, $\mathcal{q}^{**}$, is obtained. Of course, $\mathcal{q}^{**}$ represents a higher level of utility due to the fact that p_1 was decreased and the consumer can actually purchase more than before the price decrease. Therefore $\mathcal{q}^{**}$ is on an indifference curve in Figure 3-1 which is farther from the origin than CC'.

This total discrete change from $\mathcal{q}^*$ to $\mathcal{q}^{**}$, as shown in Figure 3-1, may be broken into two parts by drawing a line segment

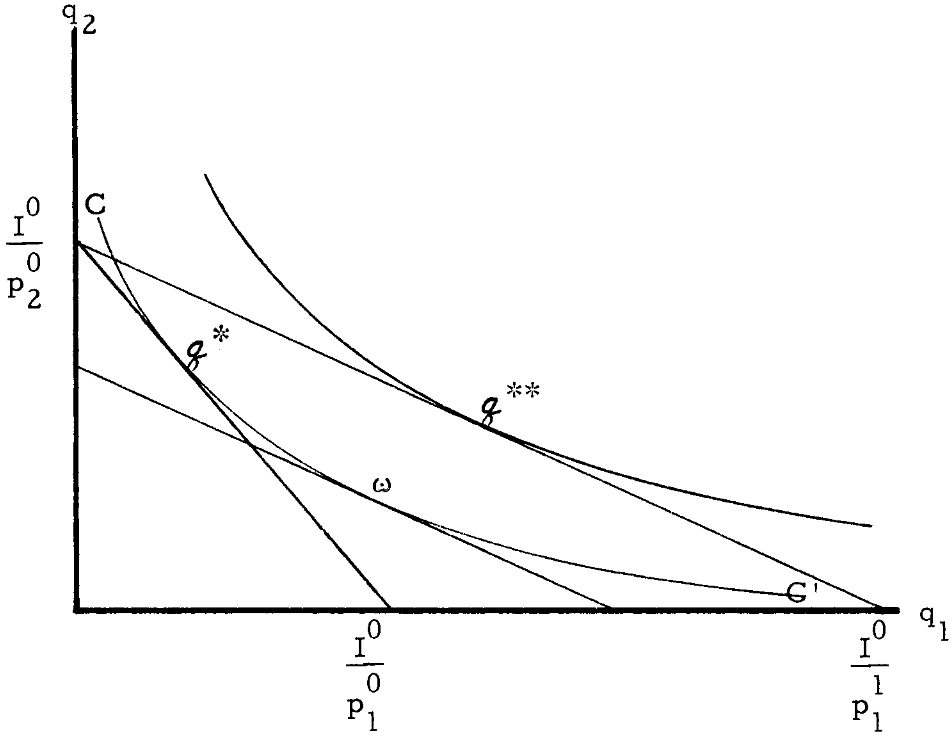

Figure 3-1. Geometric Interpretation of Price Change on the Optimal Budget.

parallel to the new budget constraint and tangent to the original indifference curve CC'. Economists use the term <u>real income</u> in describing this parallel translation. The consumer is said to experience no change in real income if he maintains his exact level of utility as before the price decrease.

Recall that the substitution effect is a rate of change when utility is held at a constant level, or alternatively when real income is held constant. Hence, the movement from g^* to ω corresponds to the substitution effect of a price decrease and is measured along the original indifference curve. On the other hand, the movement from ω to $\mathcal{g}^{**}$ is termed the income effect. In making this movement, which is exemplified by the parallel translation of the budget constraint, prices are held constant, and real income is effectively increased due to the fall in price of Q_1.

In general, when the price of commodity r decreases (increases) the consumer increases (decreases) his level of utility by means of the substitution and income effects of the price change. The substitution effect of the price change is merely a re-equating of the rates of commodity substitution holding real income constant. The income effect of a price change is the increase (decrease) in utility caused by moving from the result of the substitution effect to the new optimal budget and is due to the increase (decrease) in real income while prices are held constant. But the change in real income is actually due to the initial price change.[8]

It should be noted that although Slutsky originally deduced equation (3-15) he did not interpret it in the manner given in this thesis. He did not use the terms substitution effect, income effect, or real income. A later geometric depiction given by Schultz [18, p. 44] corresponding to Slutsky's verbal interpretation also differs from the preceding figures. Thus it would appear that there are two different interpretations of the Slutsky equation. However, that there

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[8] This is, no doubt, one reason why economics appears incomprehensible to university sophomores.

are two different ways to interpret the Slutsky equation should be of no concern. This seeming paradox is resolved by recalling that the Slutsky equation is in terms of rates of change and that there are always at least two ways of showing these on a diagram utilizing discrete changes.

Substitutes, Independents and Complements

The purpose of the present section is to define substitutable, independent, and complementary commodities. For brevity and on the basis of the definitions of substitutable, independent, and complementary commodities employed in this thesis, these three relationships shall henceforth be referred to collectively as the triad. Each individual relationship is then referred to as an element of the triad, of which there are three.[9] The elements of the triad have no unique definition which has been universally accepted, and their definitions have been subjected to many changes.

An Intuitive Notion of the Triad

Intuitively two goods might be said to be substitutes if they can replace each other to some degree in the consumer's budget. In this

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[9] In this context the elements of the triad are mutually exclusive. There are some writers who would take opposition with this. Pearce [15, p. 139] for instance does not seem to feel that the relationship between commodities should be exclusory.

intuitive everyday sense there are many examples of pairs of goods which might be thought of as substitutes for one another. One example of such a pair is sliced bread and dinner rolls. The consumer might not want all sliced bread or all dinner rolls in his budget for a given period. That is, he wants some combination of the two and he would be willing to substitute some rolls for some bread and vice versa. The crux of the argument here is that if the consumer takes more rolls in his budget he will take less bread and if he takes less rolls in his budget he will take more bread, i. e., he will substitute some of one for some of the other. The idea of one commodity being a replacement for the other is important here.

Intuitively if the quantity taken of one commodity does not depend on the quantity taken of another commodity the pair of commodities may be said to be independent commodities. Usually one would think of examples where the two goods are totally unrelated such as .22 caliber rifle cartridges and quarts of ice cream. Ordinarily one would not think of replacing ice cream with .22 cartridges at the dinner table, or on the other hand, loading his rifle with ice cream.

An intuitive example of complementary commodities might be weiners and hot dog buns, as this pair of goods tends to be consumed together. Complementary commodities then are those which tend to go together or be used together, and in this sense if the consumer increases his consumption of hot dog buns in his budget he will also increase his intake of wieners, as he will customarily want to use them together. Wieners complement the hot dog buns, i.e., the wieners supply a lack in the hot dog buns.

The description above of the elements of the triad correspond to some everyday sense of the terms. These descriptions concern only pairs of goods and make no mention of price, income, or utility level which are essential in the formal definition to be used in this thesis.

Historical Definitions of the Triad

There have been many attempts to formally define the elements of the triad which have added to the confusion generated by the intuitive definitions. In reveiwing utility theory, Stigler [20, p. 384-386] comments on historically the first formal definition of the triad given by Auspitz and Lieben in 1889. Their definition merely depended on the sign of the cross partial of the utility function, $\frac{\partial^2 U}{\partial q_i \partial q_j}$, where $i \neq j$. If the cross partial was positive, zero, or negative the commodities Q_i and Q_j were complements, independents, or substitutes respectively. This simple definition although incorporated in the works of Edgeworth, Marshall, Fisher and Pareto was later discarded as it was seen to be dependent on the measurability of utility. In addition, Stigler mentions another definition which was given by Johnson and also a version which has been offered by Hicks and Allen.

In 1950 Ichimura [8] offered a new set of definitions for the triad. He summarized the current situation by stating that five other definitions had been previously advanced. The five definitions he listed were those given by: (1) Edgeworth and Pareto, (2) Allen and Hicks, (3) Slutsky and Hicks, (4) Hicks' literary definition, and (5) Lange. The fact that the definition of the triad is a yet debated subject can be seen by browsing current economic literature, where as late as 1962 Lerner [12] proposed a new set of definitions.

From the foregoing discussion emerges the question as to which definition of the triad is to be used in this thesis. The formal definition to be used herein is the one given by Hicks [6, p. 311] in his mathematical appendix, as this definition seems to be the most widely accepted.

Hicks' Mathematical Definition

The definition given by Hicks [6, p. 311] depends upon the sign of the substitution term in equations (3-20).[10] Following Hicks

10 It is interesting to note that Slutsky made no mention of the elements of the triad when explaining his equation. In fact as previously noted, he interpreted the entire equation quite differently from Hicks and Allen [1].

Q_r and Q_s are said to be complements or substitutes as $\left(\frac{\partial q_s}{\partial p_r}\right)_{U = \text{constant}}$ is negative or positive. If $\left(\frac{\partial q_s}{\partial p_r}\right)_{U = \text{constant}}$ is zero, Q_r and Q_s are said to be independent of one another.

A verbal description of the triad can be derived from the mathematical definition as follows. Starting from some optimal budget $\mathcal{q}^*$, suppose the price of Q_r increases while all other prices remain constant. After re-equating his rates of commodity substitution with the appropriate price ratios, the consumer will be worse off than before the price increase, i. e., he will be at a lower level of utility than before the price increase. However, after the price increase suppose that the consumer is given an increment of income. The size of this increment is such that its expenditure along with the original amount of income yields a level of utility identical to the original level. That is, the boost in income is just enough to create a tangency between the original indifference surface and the altered budget constraint, and the consumer is said to experience no change in real income. It can be proved that at this new point of tangency the quantity of Q_r will be less than the quantity of Q_r at the initial optimal budget $\mathcal{q}^*$.[11] If at this new point of tangency on the original indifference surface, the quantities taken of any

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[11] This comes from the fact that $\left(\frac{\partial q_r}{\partial p_r}\right)_{U = \text{constant}} < 0$ or alternately that $\frac{\lambda D_{rr}}{D} < 0$, c.f., Hicks [6, p. 310].

other commodities have fallen (risen, not changed) from the amount taken at $\mathcal{g}^*$ these commodities are said to be complementary (substitutable, independent)with Q_r.

To summarize, if the price of Q_r increases and simultaneously the consumer receives an increase in income such that his real income remains constant, those commodities which he now purchases less (more, the same) of are called complements (substitutes, independents).

Hicks' verbal definition and its relation to his mathematical definition. Hicks [6, p. 44] gave a verbal definition of substitutes and complements unlike the verbal explanation given above.[12] His verbal definition has to do with changes in the rates of commodity substitution as movements are made in a particular direction on a given indifference surface. Hicks specifies that Q_3 is a substitute (complement) for Q_2 if the rate of commodity substitution of Q_3 for Q_1 is diminished (increased) when Q_2 is substituted for Q_1 in such a way as to leave the consumer no better off than before.[13]

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[12] Mosak [13, p. 23n] states that Hicks' verbal and mathematical definitions are at variance, and Samuelson [16, p. 184] notes that Hicks [6, p. 44, p. 311] offers two or more definitions of complementarity which are not equivalent. In a later article Samuelson [17, p. 379n] seems to retract the criticism of Hicks, but to the writer's knowledge the issue for three commodities remained unsettled until now.

[13] Hicks used different notation and terminology in explaining substitutes and complements. Notably, he used the term "money" in

A cursory comparison of Hicks' verbal and mathematical definitions may not reveal the correspondence between them. The frame of reference for both definitions is along a single indifference surface. Notice that Hicks' verbal definition is in terms of three commodities. It will be demonstrated later that the consumer's utility function must be defined for at least three commodities if the phenomena of complements or independents are to be possible. In reference to three specific commodities, the mathematical definition is concerned with the direction of the movement of q_r when p_s changes, whereas the verbal definition is concerned with the direction of the movement of the rate of commodity substitution of Q_3 for Q_1 when Q_2 is substituted for Q_1 (the quantity of Q_3 being held constant). An investigation to show the concordance of the two definitions for three commodities may be of merit.

Consider a utility function defined for three commodities, Q_1, Q_2, and Q_3. According to the mathematical definition, Q_3 and Q_2 are substitutes (independents, complements) if the substitution term, $\frac{\lambda D_{32}}{D}$, i.e., $\left(\frac{\partial q_3}{\partial p_2}\right)_{U = \text{constant}}$, is positive (zero, negative). The sign of the substitution term is determined solely by

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place of Q_1. Furthermore, as in his mathematical definition, he did not delineate independent commodities, but this takes merely a minor addition. If the rate of commodity substitution of Q_3 for Q_1 remains constant as Q_2 is substituted for Q_1 then Q_3 is independent of Q_2.

the sign of D_{32} as λ must be positive and D must be negative by the sufficient conditions for the constrained maximum. Upon expanding D_{32} one obtains

(3-24) $$D_{32} = f_1^2 f_{23} + f_3 f_2 f_{11} - f_1 f_3 f_{21} - f_1 f_2 f_{13} .$$

To examine the verbal definition one must take the partial derivative of the rate of commodity substitution of Q_3 for Q_1 with respect to Q_2. Upon doing so one obtains

(3-25) $$\frac{\partial(\frac{f_3}{f_1})}{\partial q_2} = \frac{\partial(\frac{f_3}{f_1})}{\partial q_1} \cdot \frac{\partial q_1}{\partial q_2} + \frac{\partial(\frac{f_3}{f_1})}{\partial q_2} \cdot \frac{\partial q_2}{\partial q_2} + \frac{\partial(\frac{f_3}{f_1})}{\partial q_3} \cdot \frac{\partial q_3}{\partial q_2} .$$

This equation may be rewritten as

(3-26) $$\left(\frac{\partial(\frac{f_3}{f_1})}{\partial q_2}\right)_{U = \text{constant}} = \frac{\partial(\frac{f_3}{f_1})}{\partial q_2} - \frac{\partial(\frac{f_3}{f_1})}{\partial q_1} \cdot \frac{f_2}{f_1} ,$$

since $\frac{\partial q_2}{\partial q_2} = 1$, $\frac{\partial q_1}{\partial q_2} = -\frac{f_2}{f_1}$, and $\frac{\partial q_3}{\partial q_2} = 0$ as q_3 is to remain constant and not vary as Q_2 is substituted for Q_1 along the given indifference surface. By evaluating (3-26), the following results:

(3-27) $$\left(\frac{\partial(\frac{f_3}{f_1})}{\partial q_2}\right)_{U = \text{constant}} = \frac{f_1^2 f_{32} - f_1 f_3 f_{12} - f_1 f_2 f_{31} + f_2 f_3 f_{11}}{f_1^3} .$$

Now this result may be written in more familiar terms as

$$(3\text{-}28) \qquad \left(\frac{\partial(\frac{f_3}{f_1})}{\partial q_2} \right)_{U = \text{constant}} = \frac{D_{32}}{f_1^3} \, ,$$

the sign of which depends solely on D_{32} since $f_1 > 0$.

The connection between the Hicksian mathematical and verbal definitions is now readily apparent. If Q_3 and Q_2 are to be substitutes (independents, complements) for one another D_{32} must be negative (zero, positive) according to the mathematical definition. But this in turn implies that the rate of commodity substitution of Q_3 for Q_1 is decreasing (constant, increasing) as Q_2 is substituted for Q_1 (the quantity of Q_3 held constant).

Limits on the Number of Different Relationships for a Given Utility Function

It was indicated earlier that a utility function defined for only two commodities could exhibit only substitutable commodities. This follows for two commodities since D_{12} is equal to $f_1 f_2$ which must be positive, and hence $\frac{\lambda D_{12}}{D}$, i.e., $\left(\frac{\partial q_1}{\partial p_2} \right)_{U = \text{constant}}$, must be positive as $D > 0$ and $\lambda > 0$. A generalization of the above is possible. It is impossible for all commodities in the domain of definition of a given utility function to be complements, independents, or a combination of both. The reason for this is clear. If the quantity of one commodity decreases the quantity of at least one other commodity must increase in order to keep the consumer at the original level of utility. Since it is impossible for all commodities to be complements with one another, it would be interesting to know the maximum number of complements possible if the consumer possesses a utility function defined for n commodities. Kamien [9] gives the answer to this question.

If there are n commodities in question, then there are $\frac{n!}{2!(n-2)!}$ different combinations of the n commodities taken two at a time, or equivalently, there are $\frac{n(n-1)}{2}$ different pairs of commodities. Of these $\frac{n(n-1)}{2}$ possible pairs, Kamien determines the minimum number which may be substitutes and the maximum number which may be complements. He shows that there must be at least $n-1$ positive substitution terms and there may be at most $\frac{(n-1)(n-2)}{2}$ negative substitution terms. Hence at least $n-1$ of all possible pairs must be substitutes and there can be at most $\frac{(n-1)(n-2)}{2}$ pairs of complements. Kamien goes on to state that if any of the pairs are independent, the minimum number of substitution pairs will be unaffected but the maximum number of complement pairs will be reduced.

Cross Elasticity of Demand

It is perhaps worthwhile to point out here that the sign of the substitution term is quite different from the sign of the cross elasticity of demand given by $\frac{p_r}{q_s} \cdot \frac{\partial q_s}{\partial p_r}$. Some writers suggest that the sign of the cross elasticity of demand for two commodities indicates whether they are complements, independents, or substitutes. Leftwich [11, p. 45] states:

> The relationship with which we are most concerned are those between complementary commodities.
>
> When commodities are substitutes for each other, the cross elasticity between them will be positive . . . Commodities which are complementary to each other have negative cross elasticities.

Leftwich seems to offer an additional definition of substitutes and complements on pages 74-75. Kuhlman and Thompson [10, p. 509] however have disproved this notion and stated:

> Using Hicks's definitions the above argument demonstrates whenever two goods are compared that they can be substitutes even though the cross elasticities are negative. Furthermore, it shows that two goods can be complements when the cross elasticities are positive.

The basic limitation of the cross elasticity of demand is it does not account for the income effect which must be eliminated before one can determine whether two commodities are substitutes, independents, or complements. Stonier and Hague [21, p. 85] perhaps use better nomenclature when they suggest that the sign of the cross elasticity of demand tells whether or not two goods are in <u>joint demand</u>.

CHAPTER IV

A GEOMETRIC INTERPRETATION OF SUBSTITUTES, INDEPENDENTS, AND COMPLEMENTS

The purpose of the present chapter is to illustrate geometrically the mathematical and verbal definitions of the triad given in Chapter III. As was previously noted, it is impossible for complements or independents to exist if only two commodities reside in the domain of the consumer's utility function. Since the consumer's utility function must be defined for at least three commodities if these elements of the triad are to be depicted, it would seem that the geometric diagrams should be given in at least three dimensions. Hicks [6, p. 45] realized the difficulties of diagrammatically illustrating complementarity and indicated that he preferred to avoid the geometric difficulties when he stated:

> But the problem of related goods cannot be treated on a two-dimensional indifference diagram. It needs three dimensions to represent the two related goods and money (the necessary background). This means that the theory is most conveniently represented either in algebra (an algebraic version will be given in the Appendix) or, as here, in ordinary words.

The cognitive geometric illustrations offered in this thesis follow Hicks' mathematical definitions and are concerned mainly with the location of tangency on the budget constraint. For this reason, and because instrumental indifference surfaces are very difficult to draw in three dimensions, the indifference surface has been omitted in all three-dimensional diagrams. However, for the sake of clarity, diagrams for two commodities showing a substitution effect will accompany the three-dimensional diagrams in order to show the corresponding shifts in the budget constraint. In general these diagrams can be thought of as slices through the three-dimensional diagrams. It will be seen by considering these sections that an original two-dimensional interpretation of the triad results.

Hicks' verbal definition furnishes some insight into the shape of the indifference surface. Despite Hicks' opinion concerning the geometric difficulties, this insight was convincingly utilized by Hayek [4] in two-dimensional diagrams. His diagrams illustrate the contours of a three-dimensional indifference surface for the different types of related goods. Diagrams similar to his will be constructed and supplemented by the three-dimensional budget constraint interpretation offered by the writer. Of course the graphic illustrations have the aforementioned limitation of depicting discrete changes instead of the appropriate rates of change.

Three-dimensional Interpretation of the Triad

Consider starting from an optimal budget $\mathcal{g}^* = (q_1^0, q_2^0, q_3^0)$, i.e., a point of tangency between the budget constraint and an indifference surface. This initial situation is depicted in Figure 4-1. Figures 4-1a and 4-2a can be thought of as slices parallel to the Q_1-Q_2 axis plane and through the optimal budget, $\mathcal{g}^*$, in Figures 4-1b and 4-2b respectively. These slices represent a constant level of Q_3, namely where the level of Q_3 is equal to q_3^0. The reader is reminded to imagine a convex indifference surface tangent to the budget constraint at $\mathcal{g}^*$ in the three-dimensional diagram.

Now assume that the price of Q_2 increases from p_2^0 to p_2^1. The effect of this price change on the budget constraint is given in Figure 4-2. In Figure 4-2a the new budget constraint resulting from the increase in p_2 is given by the line extending from $\frac{I^0}{p_2^1}$ to $\frac{I^0}{p_1^0}$ and in Figure 4-2b by the triangular plane with vertices at $\frac{I^0}{p_2^1}$, $\frac{I^0}{p_1^0}$ and $\frac{I^0}{p_3^0}$. It is not shown in either diagram but this new budget constraint would, of course, be tangent to some new and lower-level indifference surface if all previous assumptions regarding the utility function hold. This fact is not important at the present, as only the original indifference surface needs to be shown for illustration of the substitution effect of a price change.

Recall that the substitution effect is depicted by a jump from $\mathcal{g}^*$ to ω along the initial indifference surface in Figure 3-1. The position of ω is determined by a parallel translation of the new

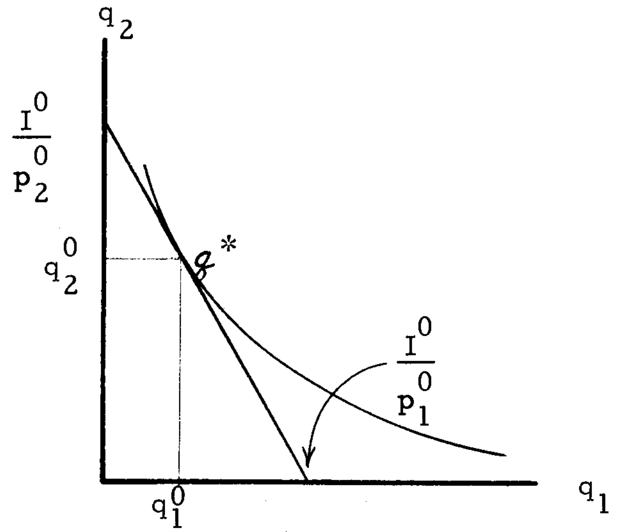

Figure 4-1a. Tangency Point Between Budget Constraint and Indifference Curve.

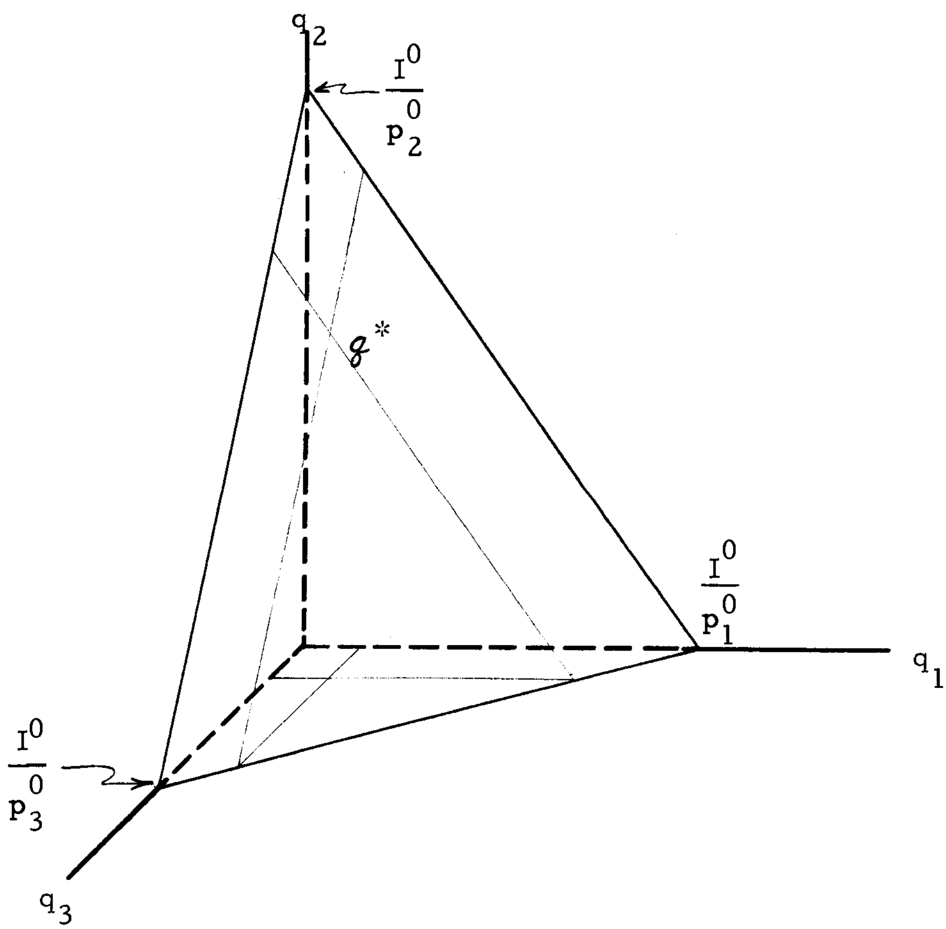

Figure 4-1b. Tangency Point Between Budget Constraint and Imagined Indifference Surface.

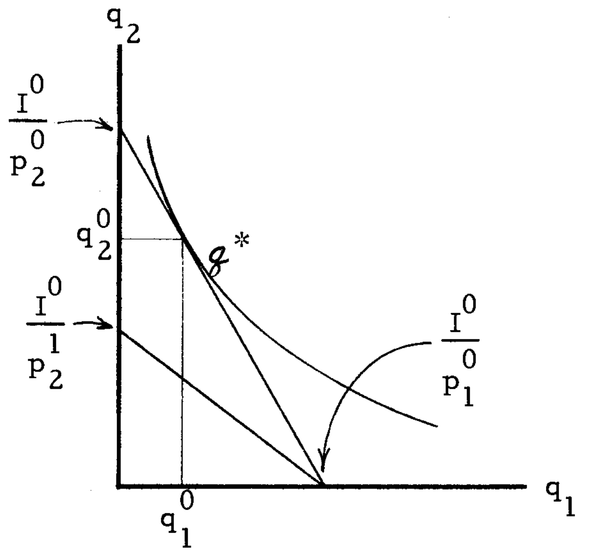

Figure 4-2a. Effect of a Price Change on the Budget Constraint in Two Dimensions.

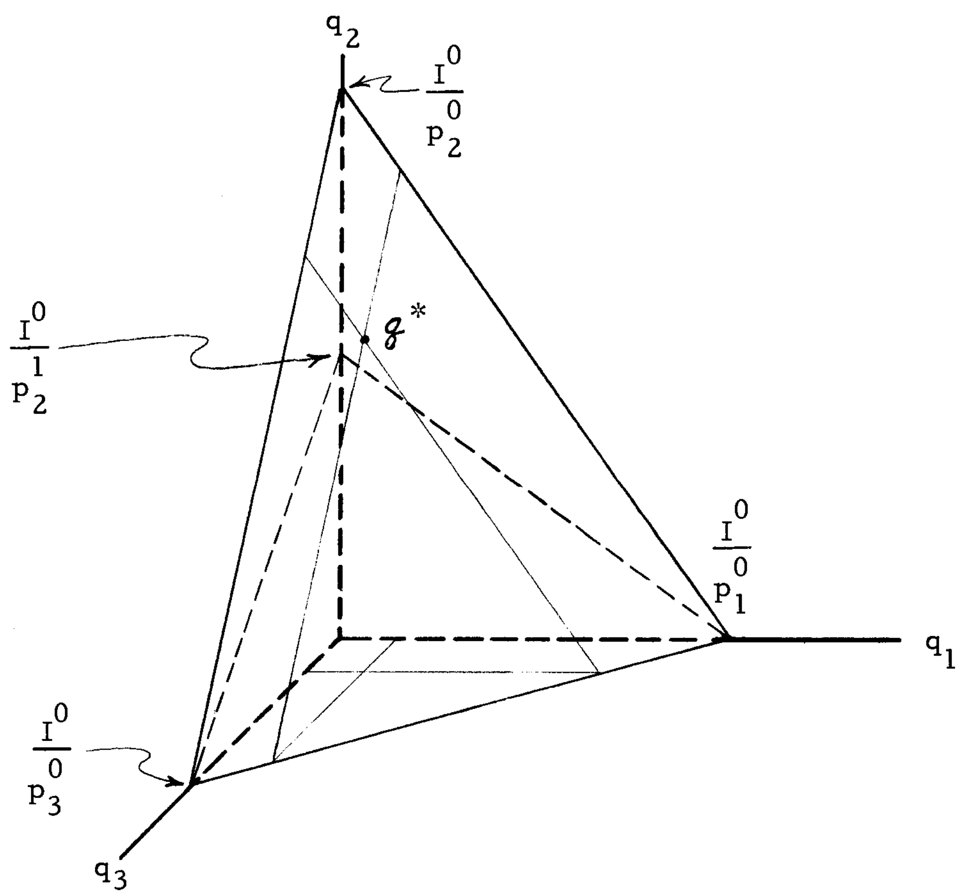

Figure 4-2b. Effect of a Price Change on the Budget Constraint in Three Dimensions.

budget constraint to a tangency with the initial indifference surface. This is known as holding real income constant by increasing actual income enough that the consumer remains at the same level of utility as before the price increase. This parallel translation of the new budget constraint is depicted in Figure 4-3.

Figure 4-3a shows two items which Figure 4-3b does not. The first is the new budget constraint as was given in Figure 4-2. Its retention in Figure 4-3b tends to make the diagram unwieldy, so it has been deleted. However, it is important to realize that the plane with vertices $\frac{I^1}{p_1^0}$, $\frac{I^1}{p_2^1}$, and $\frac{I^1}{p_3^0}$ is a parallel translation out from the origin of the new budget constraint which was depicted in Figure 4-2b. The new budget constraint is translated parallel until it becomes tangent to the initial indifference surface, and this tangency is the other item not shown in Figure 4-3b. The position of the tangency on the parallel translation will determine whether the commodities in question are substitutes, independents, or complements.

Refer now to Figure 4-4, which is similar to Figure 4-3 but with the omission of all extraneous items. It was previously pointed out that $\left(\frac{\partial q_r}{\partial p_r}\right)_{U = \text{constant}}$ must be negative, which indicates that the quantity of Q_r must fall if its price increases. Geometrically this means the tangency ω must occur below the initial level of Q_2. Hence the intersection of the original budget constraint and parallel

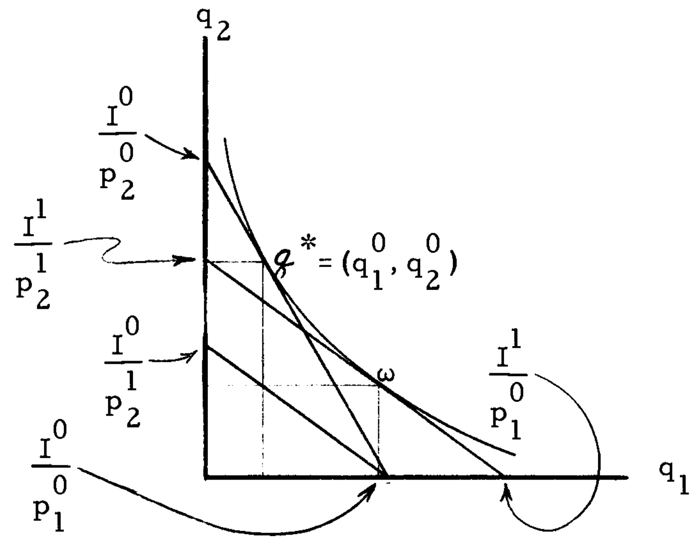

Figure 4-3a. Parallel Translation of the Budget Constraint in Two Dimensions.

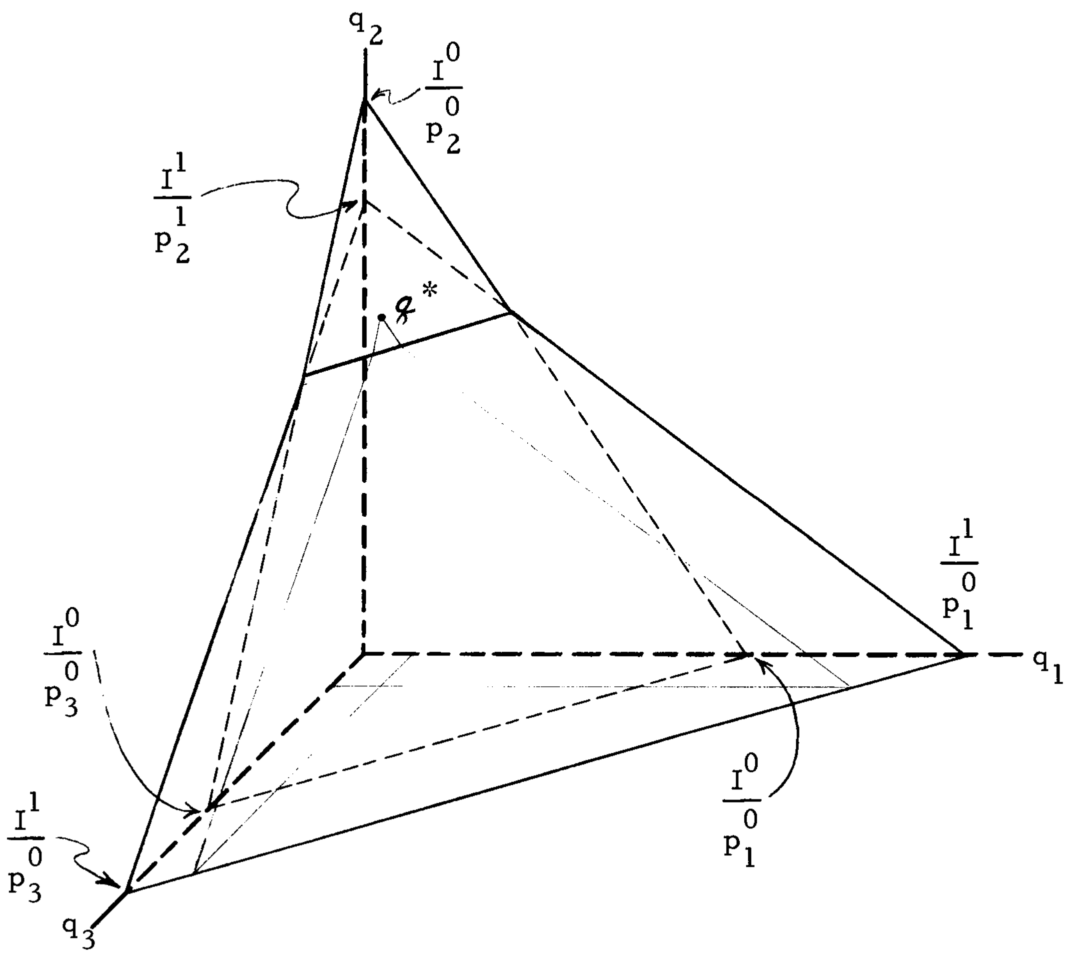

Figure 4-3b. Parallel Translation of the Budget Constraint in Three Dimensions.

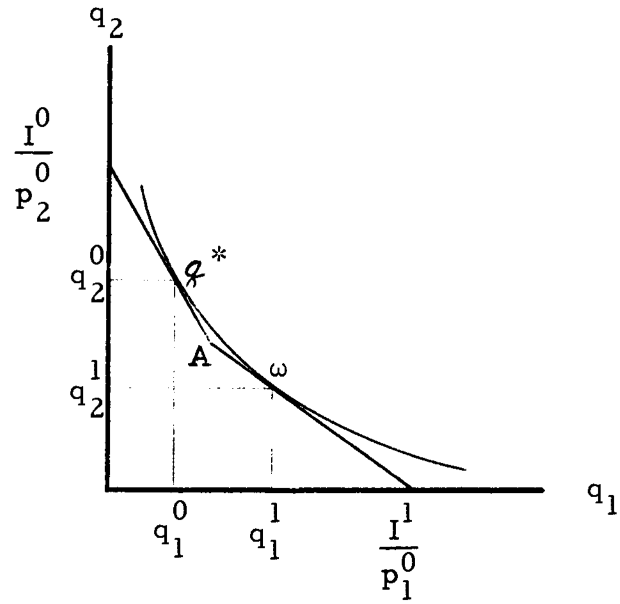

Figure 4-4a. Substitution Effect in Two Dimensions.

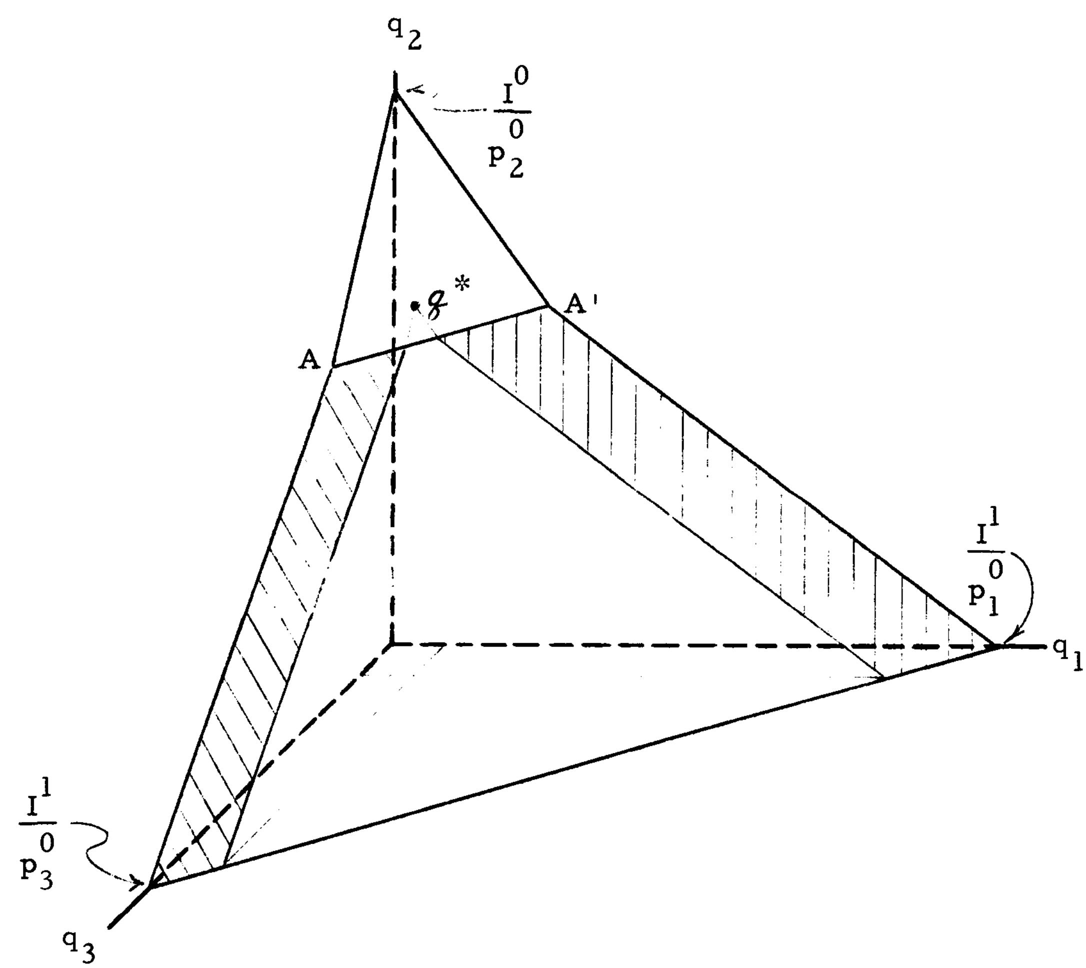

Figure 4-4b. Substitution Effect in Three Dimensions.

translation must occur below q_2^0. This intersection is given by A in Figure 4-4a and the line between A and A' in Figure 4-4b. As the quantity of Q_2 must decrease, the quantity of at least one of the other commodities must increase if the consumer is to remain at the same level of utility, i. e., on the original indifference surface. The plane with vertices A, A', $\frac{I^1}{p_1^0}$ and $\frac{I^1}{p_3^0}$, called the <u>relevant region</u>, is the region of the parallel translation which must be tangent to the initial indifference surface at some point. The uncross-hatched portion of the relevant region designates the area in which the quantities of both Q_1 and Q_3 are greater than q_1^0 and q_3^0 respectively. The area with the vertical cross-hatching denotes the area in which the quantity of Q_3 is less than q_3^0 and the quantity Q_1 is greater than q_1^0. Similarly, the diagonally cross-hatched area corresponds to a quantity of Q_3 greater than q_3^0 and a quantity of Q_1 less than q_1^0. The line separating the vertically cross-hatched area and the uncross-hatched area represents a quantity of Q_1 greater than q_1^0 and the quantity of Q_3 equal to q_3^0. Likewise, the line separating the diagonally cross-hatched and the uncross-hatched area represents a quantity of Q_3 greater than q_3^0 and the quantity of Q_1 equal to q_1^0. The indifference surface must be tangent to one of those portions of the relevant region described above. Table 1 is used to summarize the possible outcomes. Thus, the position of the

Table 1. Summary of Possible Outcomes of Substitution Effect for Three-dimensional Diagrams.

Point of Tangency, ω	Effect on Q_1	Effect on Q_3	Conclusion
1. Diagonally cross-hatched area	decrease	increase	Q_2 and Q_1 are complementary. Q_2 and Q_3 are substitutable.
2. Line between diagonally and uncross-hatched area	none	increase	Q_2 and Q_1 are independent. Q_2 and Q_3 are substitutable.
3. Uncross-hatched area	increase	increase	Q_2 and Q_1 are substitutable Q_2 and Q_3 are substitutable.
4. Line between vertically and uncross-hatched area	increase	none	Q_2 and Q_1 are substitutable. Q_2 and Q_3 are independent.
5. Vertically cross-hatched area	increase	decrease	Q_2 and Q_1 are substitutable. Q_2 and Q_3 are complementary.

original tangency between the original indifference surface and relevant region of the parallel translation of the new budget constraint determines whether commodities are substitutes, independents, or complements.

As indicated earlier, it is impossible for all commodities to be complements with each other. This is exemplified by the fact that in the relevant region no area corresponds to a simultaneous decrease in the quantities of Q_1 and Q_3 below q_1^0 and q_3^0 respectively, to accompany the mandatory decrease of Q_2 below q_2^0.

Two-dimensional Interpretation of the Triad

Disregarding the budget constraint for the moment, consider making a sequence of slices parallel to the Q_1-Q_2 axis plane at various levels of Q_3 and through a given indifference surface. If each of these slices are then projected orthogonally onto the Q_1-Q_2 axis plane a diagram similar to Figure 4-5 is the result. On each individual indifference curve the quantity of Q_3 is constant, and of course collectively, all indifference curves represent the same constant level of utility. Indifference curves farther from the origin indicate lower levels of Q_3 and correspond to slices made closer to the Q_1-Q_2 axis plane. Conversely, the level of Q_3 increases the closer the indifference curve is to the origin. This follows since the indifference surface is convex to the origin at every point.

With these concepts in mind consider Figure 4-2a and the events which it depicts. It represents the projection of a slice made

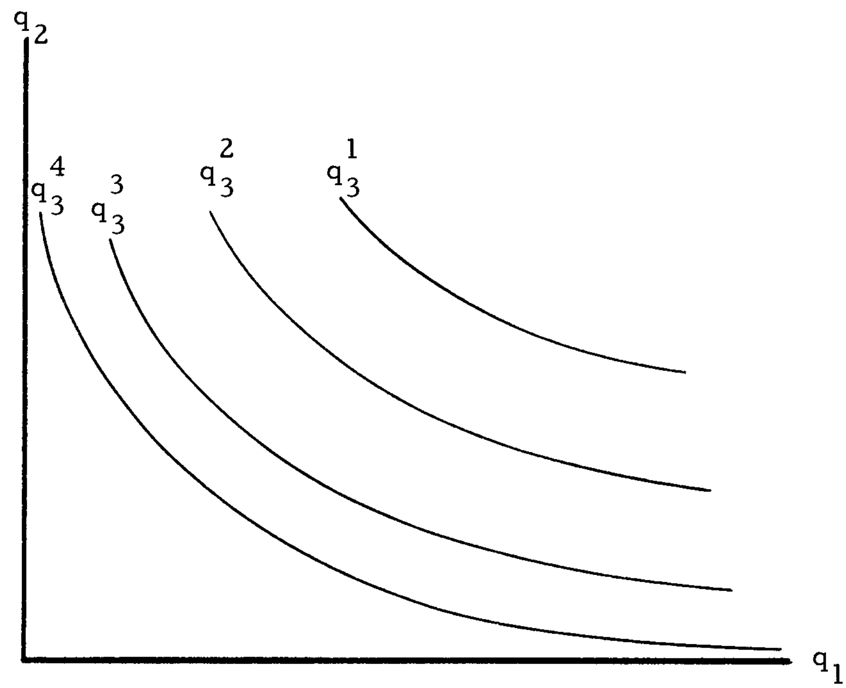

Figure 4-5. Orthogonal Projection of Indifference Surface Slices onto the Q_1-Q_2 Axis Plane.

through the indifference surface, the original budget constraint, and the new budget constraint, at q^*. The slice is projected orthogonally onto the Q_1-Q_2 axis plane and reflects a constant level of Q_3, q_3^0. The new budget constraint is to be translated parallel until it becomes tangent to the original indifference surface. The position of this tangency, ω, when projected orthogonally onto the Q_1-Q_2 axis plane will determine the relationships among the three commodities.

Observe Figure 4-6 where the pertinant areas to be considered have been delineated with cross-hatched borders. Since $\left(\frac{\partial q_2}{\partial p_2}\right)_{U = \text{constant}}$ must be negative it follows that ω must occur

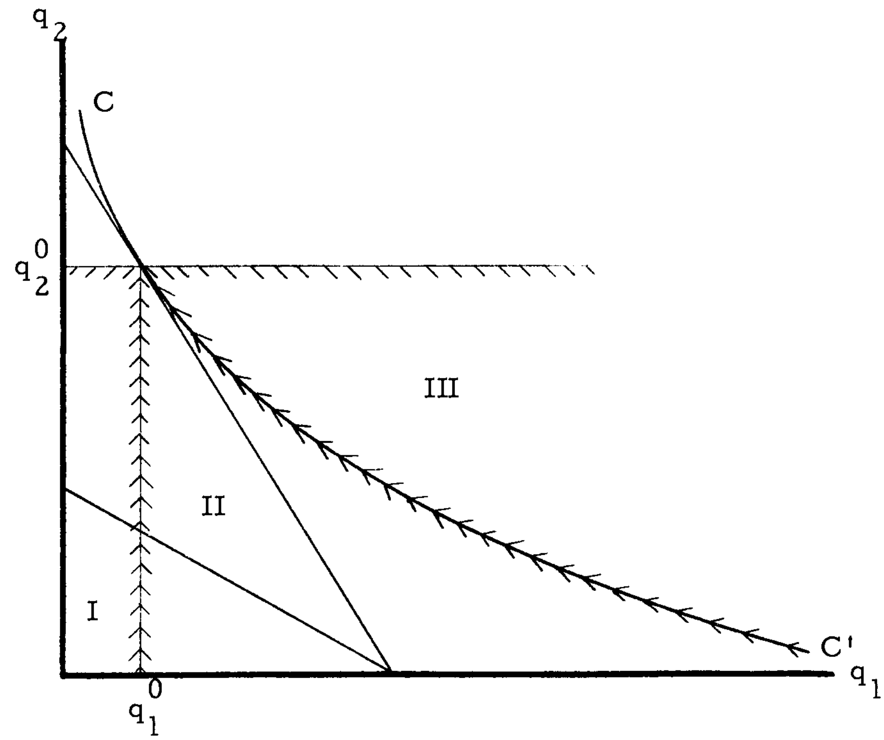

Figure 4-6. Geometric Interpretation of the Triad in Two Dimensions.

below the initial level of Q_2, q_2^0. If the projection of ω falls in area I, commodities Q_1 and Q_2 are complements, as in this area the respective levels of Q_1 and Q_2 are simultaneously less than their initial levels of q_1^0 and q_2^0. Furthermore, commodities Q_2 and Q_3 are substitutes, since the indifference curve on which ω lies when projected into this area is closer to the origin than the original indifference curve and depicts a level of Q_3 greater than q_3^0. As the increase in the level of Q_3 accompanies the manditory decrease in the level of Q_2, these two commodities are substitutes. Similarly if the projection of ω falls in area II, Q_2

is a substitute for both Q_1 and Q_3, as in this area the levels of both Q_1 and Q_3 are greater than q_1^0 and q_3^0 respectively. Area III depicts levels of Q_1 greater than q_1^0 and Q_3 less than q_3^0. If the projection of ω falls in this area it will imply that Q_2 and Q_1 are substitutes, whereas Q_2 and Q_3 are complements.

There are two cases yet to be examined. If the projection of ω falls on the line separating areas I and II, which represents the initial level of Q_1, commodities Q_2 and Q_3 will be substitutes and Q_2 and Q_1 will be independents. If ω lies on the original indifference curve, commodities Q_2 and Q_1 are substitutes, whereas Q_2 and Q_3 are independents. For three commodities Figures 4-3a and 4-4a illustrate this last situation. However if the consumer's utility function is defined for only two commodities, the substitution effect shown in Figures 4-3a and 4-4a illustrates Q_2 and Q_1 as being substitutes. Table 2 may be used to summarize these results.

Hayek's Interpretation of the Triad

Figures 4-7 and 4-8 are similar to those given by Hayek [4]. The system of indifference curves in both figures represent contours, i.e., slices made parallel to the Q_1-Q_3 axis plane, of a single indifference surface for three commodities projected orthogonally

Table 2. Summary of Possible Outcomes of Substitution Effect for Two-dimensional Diagrams.

Point of Tangency, ω	Effect on Q_1	Effect on Q_3	Conclusion
Area I	decrease	increase	Q_2 and Q_1 are complements. Q_2 and Q_3 are substitutes.
Line separating area I and II	none	increase	Q_2 and Q_1 are independents. Q_2 and Q_3 are substitutes.
Area II	increase	increase	Q_2 and Q_1 are substitutes. Q_2 and Q_3 are substitutes.
Original indifference curve	increase	none	Q_2 and Q_1 are substitutes. Q_2 and Q_3 are independents.
Area III	increase	decrease	Q_2 and Q_1 are substitutes. Q_2 and Q_3 are complements.

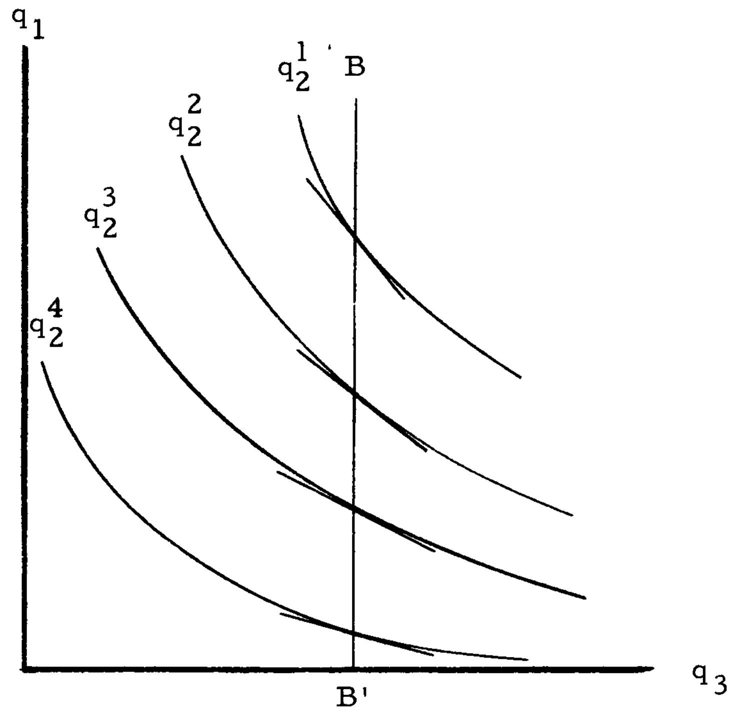

Figure 4-7. Hayek's Interpretation of Substitutes.

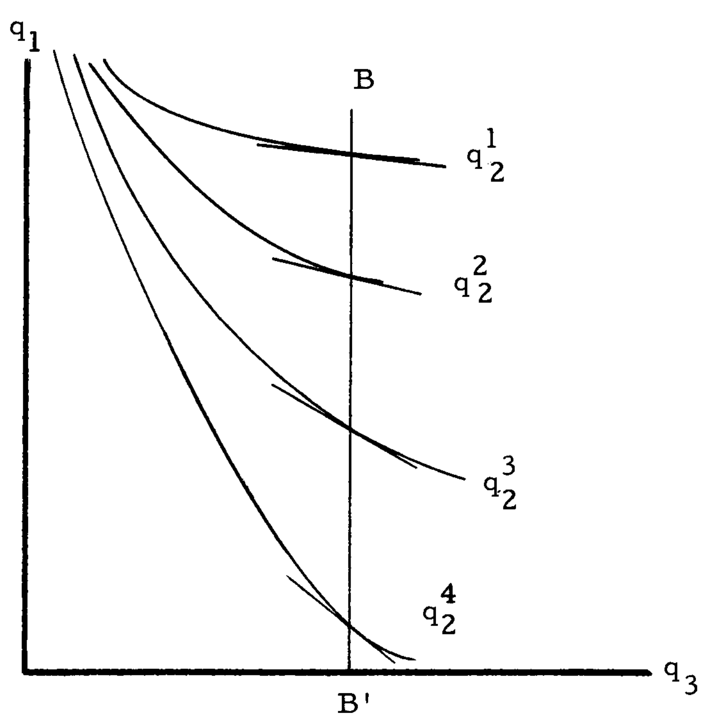

Figure 4-8. Hayek's Interpretation of Complements.

onto the Q_1-Q_3 axis plane. On each individual indifference curve the quantity of Q_2 is constant. Indifference curves farther from the origin indicate lower levels of Q_2, or conversely the level of Q_2 increases the closer the contour is to the origin.

In Hicks' verbal definition given in Chapter III the relationship between commodities Q_3 and Q_2 depends on the direction $\frac{f_3}{f_1}$ takes as Q_2 is substituted for Q_1 at a constant level of Q_3 and utility. A constant level of Q_3 is depicted by the line segment BB', and the system of indifference curves represent a single indifference surface or a constant level of utility. Movements along BB' towards B' indicate that the quantity of Q_2 is increasing and of course the quantity of Q_1 is decreasing, i. e., Q_2 is being substituted for Q_1. The short line segments drawn at the points of intersection between BB' and the system of indifference curves depict $\frac{\partial q_1}{\partial q_3}$. Now in Figure 4-7 $\frac{\partial q_1}{\partial q_3}$ increases for movements along BB' in the direction of B', i. e., the rate of commodity substitution of Q_3 for Q_1 decreases. Hence according to the Hicksian verbal definition Q_3 and Q_2 are substitutes.

In Figure 4-8 the rate of commodity substitution of Q_3 for Q_1 increases as Q_2 is substituted for Q_1 along BB', which means Q_3 and Q_2 are complements. A diagram similar to Figures 4-7 and 4-8 that illustrates Q_3 and Q_2 as independent commodities would have $\frac{\partial q_1}{\partial q_3}$ constant along BB'.

Hayek's Diagrams with Budget Constraint

Consider supplementing Figures 4-7 and 4-8 with the relevant region of the parallel translation. Figure 4-9 shows the projection of the relevant region in Figure 4-4b onto the Q_1-Q_3 axis plane. The result of this projection on the Q_1-Q_3 axis plane is given in Figure 4-10.

If indifference curves illustrating complementarity are combined with the budget constraint the result is a diagram like Figure 4-11. As in Figure 4-4b q^* designates the initial tangency between the indifference surface and the budget constraint. At q^*, $\frac{\partial q_1}{\partial q_3}$ equals $-\frac{p_3^0}{p_1^0}$ according to the dictates of the necessary conditions for a constrained maximum. This can be noted by observing that the short line segment which depicts $\frac{\partial q_1}{\partial q_3}$ is parallel to the line segment extending from $\frac{I^1}{p_3^0}$ to $\frac{I^1}{p_1^0}$. Also note that the line segment BB' falls upon the line which designates Q_3 and Q_2 as independent commodities.[14]

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14 The position of BB' is dependent upon the position of q^*. Hayek did not provide any rationale for the placement of BB' in his work. Since the triad has been defined in respect to price changes, the location of BB' is determined by movements away from q^*. Although Hicks' verbal definition is not in respect to price changes either, the above has shown that the verbal and mathematical definitions are equivalent.

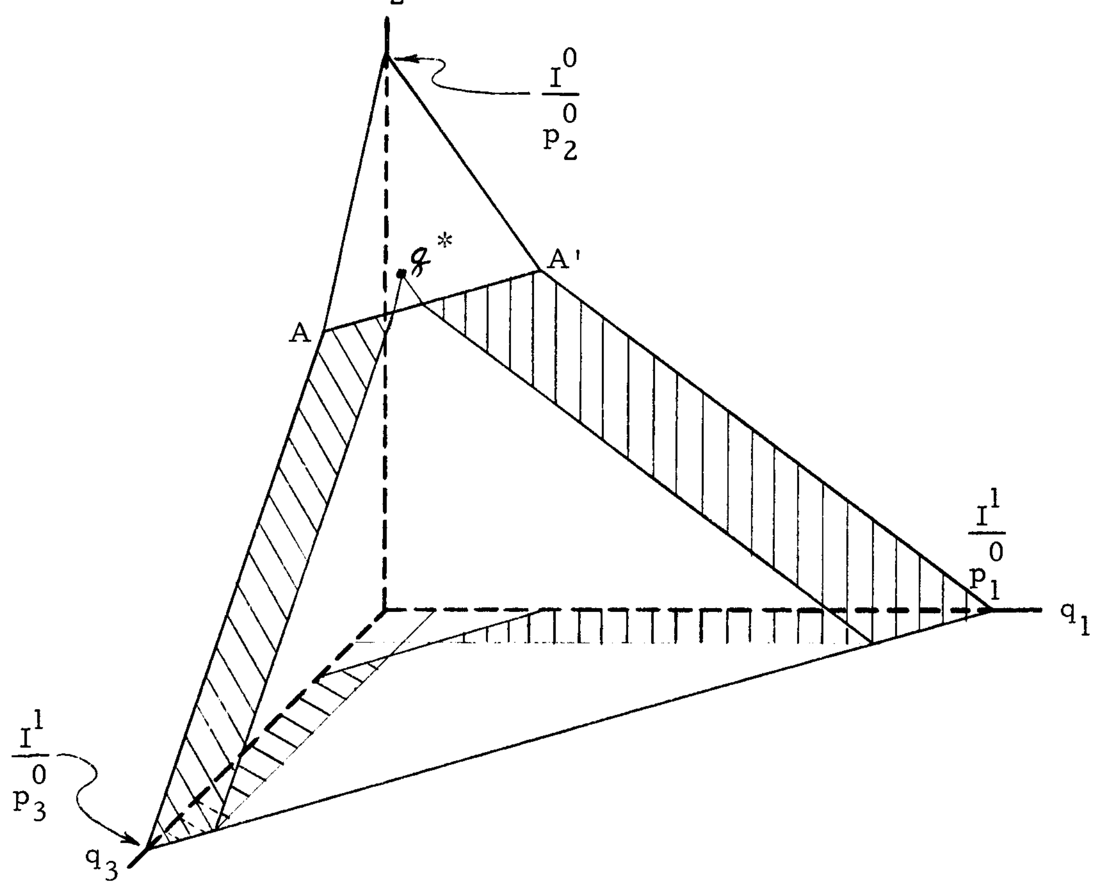

Figure 4-9. Orthogonal Projection of the Relevant Region onto the Q_1-Q_3 Axis Plane.

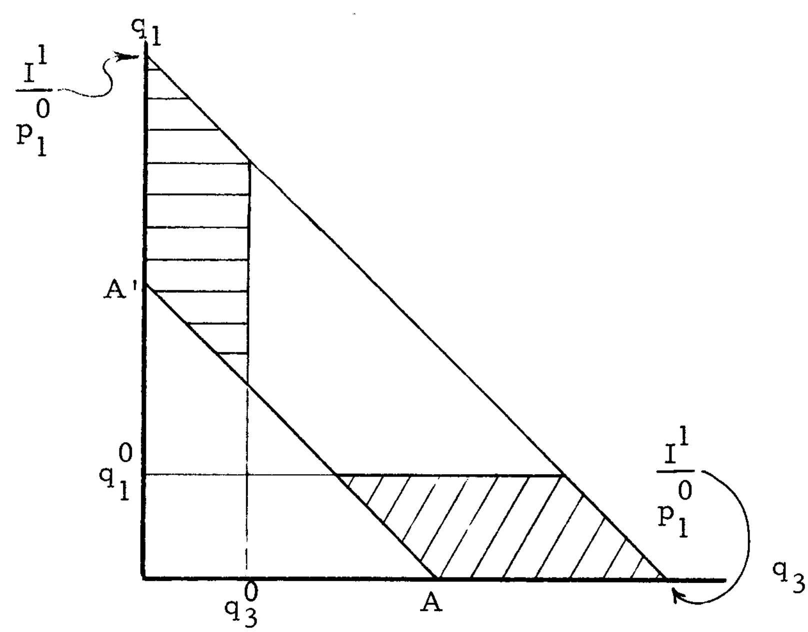

Figure 4-10. Relevant Region of Figure 4-9 in the Q_1-Q_3 Axis Plane.

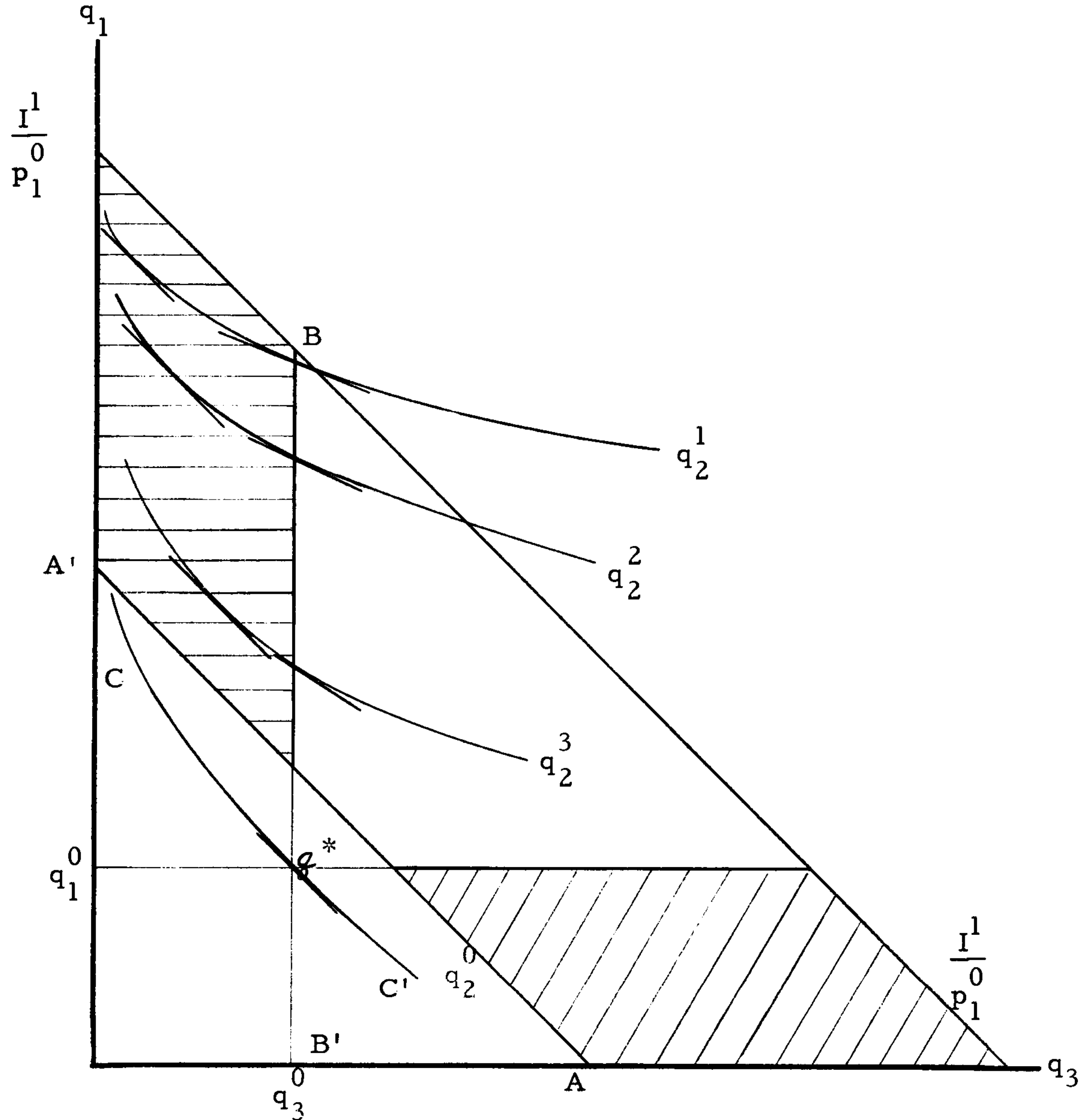

Figure 4-11. Relevant Region of Figure 4-10 Superimposed onto Hayek's Interpretation of Complements.

The next step is to determine the location of the tangency between the original indifference surface and the parallel translation of the new budget constraint. Here again, it is assumed that income has been incremented such that the parallel translation is tangent to the original indifference surface. Hence, the tangency must occur somewhere in the relevant region, just as in the three–dimensional diagrams. Since the parallel translation was the result of a price increase of commodity Q_2, while real income is held constant, neither p_1 nor p_3 have changed. This indicates that at the point of tangency, $\frac{\partial q_1}{\partial q_3}$ must again equal $-\frac{p_3^0}{p_1^0}$.

By observing Figure 4-11 it can be seen that in the relevant region $\frac{\partial q_1}{\partial q_3}$ equals $-\frac{p_3^0}{p_1^0}$ only in the horizontally cross-hatched area. Above $\mathcal{g}^*$ on line segment BB' $\frac{\partial q_1}{\partial q_3}$ is increasing. Furthermore, to the right of BB' and above $\mathcal{g}^*$, $\frac{\partial q_1}{\partial q_3}$ on any indifference curve is greater than $\frac{\partial q_1}{\partial q_3}$ evaluated at $\mathcal{g}^*$, i.e., all portions of the indifference curves lying to the right and above $\mathcal{g}^*$ are "flatter" than CC' at $\mathcal{g}^*$. This means, since $\frac{\partial q_1}{\partial q_3}$ has increased due to the increase in p_2 (Q_3 and Q_2 are complements), that $\frac{\partial q_1}{\partial q_3}$ is now equal to $-\frac{p_3^0}{p_1^0}$ somewhere to the left and above $\mathcal{g}^*$, and that the only possible area where this can occur is the horizontally cross-hatched area.

To indicate points where $\frac{\partial q_1}{\partial q_3}$ is equal to $-\frac{p_3^0}{p_1^0}$ short line segments have been drawn tangent to the indifference curves in the horizontally cross-hatched area. These points designate possible candidates for ω, the tangency between the original indifference surface and the parallel translation of the new budget constraint. Hence, the outcome of the two-dimensional analysis is the same as in the three-dimensional diagrams, and similar supplements to Hayek's diagrams which concur with the three-dimensional diagrams could be given to illustrate either substitutes or independents.

Example of Utility Function with Independent Commodities

Often a particular example helps clarify a theoretical argument. With this objective in mind the writer would like to illustrate a part of the foregoing analysis by examining a particular utility function. While many functions could be chosen for this purpose, in general it is not an easy task to find a function which will exhibit certain specific properties.[15] The function to be examined was chosen because it illustrates substitutable and independent commodities. It is defined for only three commodities so it will be possible to view the final analysis in terms of the preceding figures. The

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[15] The writer was not able to find a utility function defined for three commodities which would exhibit complementarity.

form of the particular function of interest is

(4-1) $$U = q_1 q_2 e^{q_3}, \qquad q_1, q_2, q_3 \geq 0 .$$

If it is not obvious the reader may verify that this function satisfies all assumptions specified in Chapter I.

Using Lagrange's Method the given utility function is maximized subject to the budget constraint for three commodities. Taking first partial derivatives of the utility function one obtains

(4-2) $$\begin{aligned} f_1 &= q_2 e^{q_3} \\ f_2 &= q_1 e^{q_3} \\ f_3 &= q_1 q_2 e^{q_3} . \end{aligned}$$

These first partials may now be used to solve system (3-1) written as

(4-3) $$\begin{aligned} p_1 q_1 + p_2 q_2 + p_3 q_3 &= I \\ -\lambda p_1 + q_2 e^{q_3} &= 0 \\ -\lambda p_2 + q_1 e^{q_3} &= 0 \\ -\lambda p_3 + q_1 q_2 e^{q_3} &= 0 , \end{aligned}$$

for q_1, q_2, and q_3. Upon solving one obtains the following solutions

$$(4\text{-}4) \qquad q_1 = \frac{p_3}{p_1}, \quad q_2 = \frac{p_3}{p_2}, \quad q_3 = \frac{I}{p_3} - 2,$$

which are the demand functions for Q_1, Q_2, and Q_3.

Now one must calculate the second-order and cross-partial derivatives of the utility function to verify that it satisfies the sufficient conditions for a constrained maximum. These derivatives are given as follows:

$$(4\text{-}5) \qquad \begin{aligned} f_{11} &= 0 \\ f_{22} &= 0 \\ f_{33} &= q_1 q_2 e^{q_3} \\ f_{12} &= e^{q_3} \\ f_{13} &= q_2 e^{q_3} \\ f_{23} &= q_1 e^{q_3} \end{aligned} \quad .$$

Using these values one may now determine if the required inequalities in (2-16) are satisfied. The first inequality is given as

$$(4\text{-}6) \qquad (-1)^2 D^2(\mathscr{q}) = 2q_1 q_2 e^{3q_3} > 0 .$$

The second inequality is

$$(4\text{-}7) \qquad (-1)^3 D^3(\mathscr{q}) = q_1^2 q_2^2 e^{4q_3} > 0.$$

Since these inequalities hold for any $\mathscr{q}$ which has positive coordinate values, the function is convex at every point and the sufficient conditions will be satisfied for any $\mathscr{q}^*$.

Now that it has been demonstrated that the utility function under consideration satisfied the necessary and sufficient conditions for a constrained maximum, the next task is to determine the relationship between commodities Q_1, Q_2, and Q_3, i. e., which elements of the triad they represent. For this one must calculate D_{12}, D_{13}, and D_{23}, which are easily determined to be

$$\begin{aligned} & D_{12} = 0 \\ (4\text{-}8) \qquad & D_{13} = -q_1^2 q_2 e^{3q_3} < 0 \\ & D_{23} = -q_1 q_2^2 e^{3q_3} < 0 . \end{aligned}$$

With the above results the sign of the substitution effect resulting from any particular price change is easy to determine. The substitution effects of particular interest are given as

$$\left(\frac{\partial q_1}{\partial p_2}\right)_{U\,=\,\text{constant}} = 0$$

(4-9)
$$\left(\frac{\partial q_1}{\partial p_3}\right)_{U\,=\,\text{constant}} = \frac{1}{p_1}$$

$$\left(\frac{\partial q_2}{\partial p_3}\right)_{U\,=\,\text{constant}} = \frac{1}{p_2} .$$

Since p_1 and p_2 must be positive, it follows from the mathematical definition that Q_1 and Q_2 are independent commodities with respect to each other while Q_3 is a substitute for both Q_1 and Q_2. The same results follow for the verbal definition since

$$\left(\frac{\partial\left(\frac{f_1}{f_3}\right)}{\partial q_2}\right)_{U\,=\,\text{constant}} = 0$$

(4-10)
$$\left(\frac{\partial\left(\frac{f_3}{f_2}\right)}{\partial q_1}\right)_{U\,=\,\text{constant}} = -\frac{q_2}{q_1} < 0$$

$$\left(\frac{\partial\left(\frac{f_2}{f_1}\right)}{\partial q_3}\right)_{U\,=\,\text{constant}} = -\frac{q_1}{q_2} < 0 .$$

Given the foregoing results it is an easy matter to find the budget which maximizes utility for any given income and set of prices. Suppose the given function is to be maximized when

$$
\text{(4-11)} \qquad
\begin{aligned}
p_1 &= \$2/\text{unit} \\
p_2 &= \$2/\text{unit} \\
p_3 &= \$4/\text{unit} \\
I &= \$20.
\end{aligned}
$$

Then the budget which maximizes the level of utility is found to be $\mathcal{g}^* = (2, 2, 3)$ and the budget yields a utility level of $4e^3$. Now suppose the price of Q_2 increases to \$4/unit. It is now desired to determine the amount of additional income this particular consumer would have to be given in order to keep him at the previous utility level of $4e^3$, under this new set of prices. Upon doing this the tangency, ω, between the parallel translation of the new budget constraint and the utility surface represented by

$$
\text{(4-12)} \qquad q_1 q_2 e^{q_3} = 4e^3
$$

can be determined. Once this tangency point is known it is a simple matter to determine the region in which ω lies.

Because the last three equations of (4-3) must hold in general for a point of tangency one can solve and determine that

$$q_1 = 2$$
$$q_2 = 1,$$

from which it follows that

$$e^{q_3} = 2e^3,$$

or,

$$q_3 = 3 + \ell n(2)$$
$$\doteq 3.69315 \; .$$

Hence,

$$\omega \doteq (2, 1, 3.69),$$

and it follows that this solution implies that

$$I = 2 \cdot 2 + 4 \cdot 1 + 4 \cdot 3.69315$$
$$\doteq 22.7726,$$

or the consumer's income would need to be increased by about $2.77 in order to keep him at the same level of utility.

Since q_1 did not change in response to the increase of p_2 it should be obvious that the new point of tangency lies on a line similar to the line in Figure 4-4b which separates the diagonally cross-hatched area and the uncross-hatched area. In terms of Figure 4-6, the orthogonal projection of ω lies on the line separating area I and II. To view the relationship between commodities Q_1 and Q_2 in terms of a figure like 4-11, which is designed to show the relationship between commodities Q_3 and Q_2, the axes must be relabeled with Q_1 on the horizontal axis and Q_3 on the vertical axis. Then the line segment BB' lies upon the line separating the uncross-hatched and diagonally cross-hatched areas. With this modification, one must look at $\frac{\partial q_3}{\partial q_1}$ at g^* which is equal to $-\frac{1}{2}$. At a constant level of utility if Q_2 is substituted for Q_3 there is no effect on the rate of commodity substitution of Q_1 for Q_3, i.e., $\left(\frac{\partial(\frac{f_1}{f_3})}{\partial q_2} \right)_{U = \text{constant}} = 0$. This means that all along the line segment BB', $\frac{\partial q_3}{\partial q_1}$ is constant and equal to $-\frac{1}{2}$. Furthermore, $-\frac{p_1^0}{p_3^0}$ is constant and equal to $-\frac{1}{2}$. Therefore, after the increase in p_2 from \$2/unit to \$4/unit and the increase in income of \$2.77, ω must occur on the line separating the diagonally cross-hatched and uncross-hatched areas.

Summary

A particular utility function, as given in (4-1) has been examined. According to the mathematical and verbal definitions used in this thesis, commodities Q_1 and Q_2 are independent, and Q_1 and Q_3 are substitutes for each other as are Q_2 and Q_3. This function has been examined in a particular example to illustrate the correspondence between (1) the mathematical and verbal definitions of the triad and (2) the three geometric diagrams illustrating independent commodities.

CHAPTER V

SUMMARY AND CONCLUSIONS

Summary

From a few basic assumptions regarding an "economic" consumer, the Slutsky equation was derived and discussed. The substitution effect term of the equation $\left(\frac{\partial q_r}{\partial p_s}\right)_{U = \text{constant}}$ was used to define the elements of the triad in a manner similar to Hicks. The commodities Q_r and Q_s are defined as substitutable, independent, or complementary according to whether the sign of the substitution effect term is positive, zero, or negative, respectively. It was shown mathematically, for the first time for three commodities, that Hicks' verbal definition corresponds to his mathematical definition. In the case of more than three commodities, this correspondence depends upon how one interprets Hicks' "money".

Using these definitions, a three-dimensional interpretation was proposed, which depicted the budget constraint in relation to an indifference surface. Moving away from an optimal budget due to a price change, it was shown that the position of the tangency, ω, on the relevant region of the parallel translation of the new budget constraint determines whether the commodities are substitutable, independent, or complementary. An original two-dimensional interpretation was derived which corresponded to the three-dimensional interpretation. The relationship among commodities was seen to depend on the position of ω when projected orthogonally onto the Q_1-Q_2 axis plane. Hayek's two-dimensional diagrams of Hicks' verbal definition were supplemented with a budget constraint in order to illustrate the correspondence between Hayek's diagrams and the three-dimensional diagrams of this thesis. The utility function $U = q_1 q_2 e^{q_3}$ was shown to exhibit Q_1 and Q_3, Q_2 and Q_3 as substitutes, and Q_1 and Q_2 as independents.

Conclusions

The following conclusion can be drawn from the analysis: Hicks' Mathematical Definition $\equiv$ Hicks' verbal definition for three commodities $\equiv$ Hayek's diagrams $\equiv$ writer's diagrams. This conclusion bears upon both teaching of economic theory and upon research applications. In teaching, the above equivalence means that students need not be confused by Hicks' verbal and mathematical definitions of the triad. Further, it gives introspection to the definitions of related goods, the possibility of which had been denied them by such an eminent economist as Stigler [20, p. 386]. In research, the equivalence gives insight into further work that can be carried out in demand analysis. It should provide a basis for specifying the functional form of demand functions particularly for commodities that are supposed complements. It also continues to challenge the researcher in finding a utility function that illustrates complementary commodities.

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