AN ABSTRACT OF THE THESIS OF

Brian L. Kruchoski for the degree of Doctor of Philosophy in Civil Engineering presented on August 10, 1992.

Title: Identification of Structural Parameters and Hydrodynamic Effects for Forced and Free Vibration

Abstract approved: Redacted for Privacy

Dr. John W. Leonard

Statistically-based estimation techniques are presented in this study. These techniques incorporate structural test data to improve finite element models used for dynamic analysis.

Methods are developed to identify optimum values of the parameters of finite element models describing structures. The parameters which may be identified are : element area, mass density, and moment of inertia; lumped mass and stiffness; and the Rayleigh damping coefficients. A technique is described for incorporating hydrodynamic effects on small bodies by identifying equivalent structure mass, stiffness, and damping properties. Procedures are presented for both the free vibration problem and for forced response in the time domain.

The equations for parameter identification are formulated in terms of measured response, calculated response, the prior estimate of the parameters, and a weighting

matrix. The form of the weighting matrix is presented for three identification schemes : Least Squares, Weighted Least Squares, and Bayesian. The weighting matrix is shown to be a function of a sensitivity matrix relating structural response to the parameters of the finite element model. Sensitivities for the forced vibration problem are derived from the Wilson Theta equations, and are presented for the free vibration problem.

The algorithm used for parameter identification is presented, and its implementation in a computer program is described.

Numerical examples are included to demonstrate the solution technique and the validity and capability of the identification method. All three estimation schemes are found to provide efficient and reliable parameter identification for many modeling situations.

IDENTIFICATION OF STRUCTURAL PARAMETERS AND HYDRODYNAMIC EFFECTS FOR FORCED AND FREE VIBRATION

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed August 10, 1992

Commencement June 1993

APPROVED:

Redacted for Privacy

Professor of Civil Engineering in charge of major

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Date thesis is presented August 10, 1992

Typed by researcher for Brian L. Kruchoski

ACKNOWLEDGEMENT

The author wishes to express his sincere gratitude and appreciation to his advisor, Dr. John W. Leonard, Professor of Civil and Ocean Engineering, for his guidance and support throughout this work. The author would also like to thank his wife, Dr. Martha E. Kruchoski : without her encouragement, understanding, and many sacrifices, this work would not have been possible.

A grant provided by the Oregon State Computer Center is also gratefully acknowledged.

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CHAPTER I

INTRODUCTION

The proper design of many land-based and sea-based structures requires the ability to accurately predict their response to dynamic disturbances. These disturbances may originate from such sources as ground motions due to earthquakes, fluctuating wind loads, vibrating machinery, or wave-induced hydrodynamic effects. Often, a mathematical model of the structure is generated and is solved by computer to predict the response to a variety of expected loadings.

For certain complex or critical structures, scale models are constructed and subjected to laboratory tests. The results of these experiments are then used to validate and improve the mathematical model so that it may be used with greater confidence as a response-predictor (1)*. When systems are to be mass produced (e.g., vehicle structures and their components), or when several nearly identical

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*Numbers in parentheses denote entries in Bibliography.

structures will be needed throughout a project (e.g., transmission towers, offshore rigs for oil exploration), prototypes may be built. Results from tests of these structures can be used to provide data for validating the mathematical model; the model can then be used with greater confidence to evaluate and refine the preliminary design.

The objective of this study is to investigate statistically-based estimation techniques which use structural test data to improve finite element models. These models may be analyzed either for free vibrations or for forced response in the time domain. Techniques are developed to identify optimum mass, stiffness, and damping matrices describing land-based and sea-based structures.

Mathematical Modeling of Structures

Essential to the accurate analysis of a structure subjected to dynamic loads is a proper formulation of the mathematical model. For a linearly elastic structure, this description is based on the mass, elastic properties (stiffness), and energy-loss mechanism (damping) of the structure, as well as on the external sources of excitation (applied forces or support motion) (2). If a finite element formulation is used, the equations for the dynamic response of the structure can be written as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{P(t)\} \qquad (1.1)$$

where $[M]$ = structural mass matrix of size n x n

$[C]$ = structural mass matrix, n x n

$[K]$ = structural stiffness matrix, n x n

$\{P(t)\}$ = vector of time-dependent external loads ,n x 1

$\{x\},\{\dot{x}\},\{\ddot{x}\}$ = vector of nodal displacements, velocities, and accelerations, n x 1

and n is the number of displacement degrees-of-freedom (d.o.f.) used to discretize the structure (3).

The accuracy with which Eq. (1.1) predicts the response of the structure is directly dependent on the form and composition of the property matrices; i.e. the mass, damping, and stiffness matrices. The individual terms of these matrices are, in general, functions of basic structural parameters, such as element area, moment of inertia, and mass density. In an analysis, these quantities are typically treated as deterministic values, which in turn yield deterministic property matrices. However, in this study they are treated as random variables which are capable of assuming any value. Thus, property matrices based on these parameters are also random, or nondeterministic.

The randomness of the structural parameters arises due to the number, type, and severity of assumptions and idealizations that must be made by the analyst when constructing the finite element model (1). In the traditional approach to modeling, the analyst uses experience and judgment to establish "best" estimates of all the parameters, thus fixing these random variables at unchanging (deterministic) values. These values are then used to construct the structure property matrices [M], [C], and [K], and a solution to Eq. (1.1) is sought.

In order to assess the adequacy of the structural model, the solution obtained is often compared to a known response. This response is usually in the form of response measurements taken on a model or prototype structure subjected to known disturbances. In cases where the predicted response is not in satisfactory agreement with the measured response, the analyst revises the "best" estimates of the structural parameters, reformulates the property matrices, and finds a new solution to Eq. (1.1). The response from this solution is compared with the measured response and the iterative process is repeated until, according to some criteria, satisfactory agreement is obtained between predicted and measured response. A model adjusted in this manner may then be used with greater confidence as a predictor for the response of the structure under other forms of loading.

The iterative, trial-and-error technique described above illustrates the inherent random nature of the structural parameters. They can, and often do, assume various numerical values in an analysis, until a final set which gives a "best" fit to the "true" solution is found (4). Although this set of best estimates for the parameters may be sought by trial-and-error techniques, they may also be found in a more efficient, systematic, and consistent fashion through techniques of system identification (described later in this chapter).

The next two sections contain discussions of typical assumptions made when constructing finite element models of structures. Such assumptions give rise to the need to identify, in an iterative fashion, better estimates of structural parameters. Following these sections is a discussion of the methodology of system identification.

Structures *in Vacuo*

In the finite element method, the infinite degrees-of-freedom (d.o.f.) structure is modeled as an assemblage of elements connected at nodes, with a finite number of nodal d.o.f. In the discretization, the structural properties (mass, stiffness, damping) are concentrated at the nodes, as are the applied disturbances (loads and prescribed displacements) (5). In determining the equivalent nodal properties of the structure, many approximations and idealizations must be made. These result in a primary source of error in the mathematical model. Some common difficulties in structural idealization will now be discussed.

The mass and stiffness matrices for a structure are usually formed at the element level and then assembled into global structure matrices. For a beam element, for example, the mass and stiffness are well-defined functions of length, area, mass density, moments of inertia, and Young's modulus. Frequently, the values of these parameters are not well established, especially in structures exhibiting irregular or complex geometry. For instance, in the case of tapered members, "equivalent" prismatic values for the cross-sectional area and the moments of inertia are commonly used, even though the area and inertias vary along the length of the member (6). Such approximations introduce errors of indeterminate magnitude into the mass and stiffness matrices of the element.

At the global level, various idealizations are made in order to reduce the complexity of the model. Structural joints are usually modeled as either perfectly rigid or perfectly flexible, conditions which are never physically realized. As an alternative to this approach, it is possible to model a connection as partially constrained by specifying a lumped, or concentrated, stiffness in order to simulate the actual amount of joint rigidity. The use of concentrated stiffness may also be appropriate when haunched or flared members meet at a point. In such a situation, a concentrated mass might also be added at the joint in order to account for mass effects not included at the element level. Concentrated nodal masses are also used to account for various structural appurtenances and attachments. Such modifications to the global mass and stiffness matrices provide additional sources of modeling error.

The damping matrix shown in Eq. (1.1) is used to account for the total loss of energy in the system. Since the mechanisms of energy loss are not well-defined or fully understood for most structures, it is usually not possible to account for damping effects at the element level (2). Also, structural damping effects can come from several sources, such as material damping, joint friction, and nonstructural elements (such as shear walls) (3). For these reasons, the damping matrices for most structures are specified at the global level and are meant to incorporate all important effects as equivalent linear viscous damping.

Since the damping matrix cannot be formulated as an explicit function of the structural parameters, it is often convenient to use a special form of damping known as Rayleigh damping, or proportional damping (3). A Rayleigh damping matrix is assembled from a linear combination of the global mass and stiffness matrices according to

$$[C] = \alpha[M] + \beta[K] \tag{1.2}$$

where α and β are proportionality factors (the Rayleigh damping coefficients). Thus, the content of the structure damping matrix is dependent on the assumptions made on mass and stiffness effects as well as on the choice of α and β.

The values of α and β are usually selected on the basis of previous modeling experience or results of vibration tests.

Structures in Fluid

In addition to the idealizations discussed in the previous section, additional assumptions must be made in the analysis of structures subjected to hydrodynamic loads. These assumptions involve the inclusion of effects of fluid mass and fluid damping in the structural model.

In the modeling of wave-induced forces on structures, the applied nodal loads specified in Eq. (1.1) are often expressed by means of Morison's equation (7,8,9) :

$$\{P(t)\} = \rho C_M[\bar{V}]\{\dot{u}\} - \rho(C_M-1)[\bar{V}]\{\ddot{x}\} + \\ + (0.5\rho C_D[\bar{A}])\{|\dot{u}-\dot{x}|\}\{\dot{u}-\dot{x}\}^T \tag{1.3}$$

where $\{u\}$, $\{\dot{u}\}$, $\{\ddot{u}\}$ = known fluid kinematics, n x 1

$\{x\}$, $\{\dot{x}\}$, $\{\ddot{x}\}$ = unknown structure kinematics, n x 1

ρ = fluid density

$[\bar{A}]$, $[\bar{V}]$ = projected area and displaced volume of the structure, n x n (diagonal)

C_M, C_D = empirical mass and damping coefficients for calculation of fluid forces.

In general, the coefficients C_M and C_D depend on several real fluid effects, as well as on characteristics of the structure; e.g., roughness and proximity of members (9). The determination of these coefficients is difficult and not well-defined, and much attention has been devoted to this topic in the literature (9,10,11).

The right hand side of Eq. (1.3) contains a nonlinear term which is dependent on the relative velocity between the fluid and the structure. This nonlinearity can cause great difficulty in the analysis (9,11). One common practice is to linearize this drag-related term (9) and consider the total effect to consist of two components : one component acting as a driving force related to the fluid velocity, and one acting as fluid-induced damping of the structure. Using this approach, one may linearize Eq. (1.3) and combine it with Eq. (1.1) to give

$$([M] + \rho(C_M-1)[\bar{V}])\{\ddot{x}\} + ([C] + 0.5\rho C_D\Psi[\bar{A}])\{\dot{x}\} + + [K]\{x\} = \rho C_M[\bar{V}]\{\ddot{u}\} + (0.5\rho C_D[\bar{A}])\{|\dot{u}|\}\{\dot{u}\}^T \qquad (1.4)$$

where Ψ is a linearization factor related either to the root mean square value of the relative velocity or the time average of the fluid velocity (9).

The first matrix in Eq. (1.4) may be regarded as the effective mass of the structure, while the second matrix represents the effective damping. These matrices include both structure and fluid effects.

The effective mass matrix shown in Eq. (1.4) must account for the added mass effect of the fluid (C_M-1), fluid trapped inside structural elements, and the mass of any excrescences, as well as the *in vacuo* mass of the structure (9,12). The fluid-related terms are typically treated as additional masses concentrated at the structural nodes. Appropriate values are difficult to establish.

The effective damping matrix in Eq. (1.4) cannot be assembled with a large degree of confidence. Besides the limitations described in the previous section, the appearance of the drag coefficient C_D and the linearization factor Ψ lend further uncertainty to the damping matrix. Additional difficulties may arise during the solution of Eq. (1.4) since the fluid damping term alters the assumed proportional form of the damping matrix (9).

System Identification Method

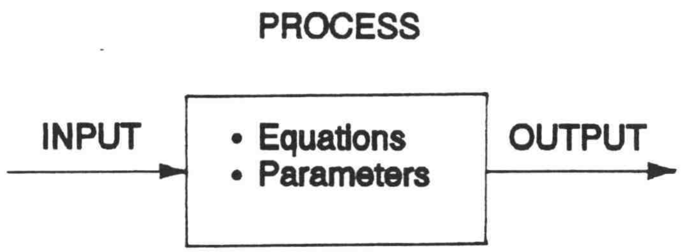
A principal goal in the reduction of a complex structure to a mathematical model is to be able to use the model, with confidence, to predict the response to various prescribed disturbances. Applications might include the determination of the mode shapes and frequencies of free vibration, the calculation of the history of displacements of a structure during an earthquake, or the prediction of the response to wave-induced forces on a structure submerged in the ocean. Frequently, data from actual tests of the structure, or of a scale model, exist and these data are used by the analyst to aid in refining the mathematical model.

The application of methods of system identification as they apply to the mathematical modeling of structures will now be addressed.

The mathematical description of a structure usually consists of three components : the input to the mathematical model; the series of equations which constitute the model; and the output from the model. The input typically consists of a given disturbance; e.g., applied loads and prescribed displacements. The equations serving as the model must be specified both with respect to form (e.g., a set of second-order linear differential equations) and with respect to parameters (e.g., the coefficients of the descriptive equations). Equation (1.1) is an example of form and parameters together defining a model. In a dynamic analysis, the output from the mathematical model of a structure typically consists of either time-dependent kinematics or of the frequencies and mode shapes of free vibration.

The three components detailed above - the input, the equations, and the output - may also be described as the input, the process, and the response of the (structural) system being modeled (see Fig. 1).

In many engineering applications, the input and the process are known (or assumed known) and the response of the system is sought. The solution is obtained in a relatively straightforward manner and is usually unique. In some applications the process and its response may be known, and



DESIGN PROBLEM	• GIVEN	:	Input & Process
	• DETERMINE	:	Output
INVERSE PROBLEM	• GIVEN	:	Input & Output
	• DETERMINE	:	Process

Figure 1. Structural Model as Input, Process, and Output

it is desired to determine the input which gave rise to the known response. In other instances, the input and response of a system or process may be known, and a mathematical description of the process is desired.

The latter two problems are referred to as "inverse problems" (13,14) and do not generally possess unique solutions. However, techniques of system identification allow one to determine a solution which "best fits" the known conditions (15,16,17). In particular, we are concerned with the last modeling situation in which it is required to determine some or all of the parameters of a known process, given the form of the process, the input into the process, and the output.

Consider a test structure instrumented to measure some structural response of interest. This response, as previously discussed, could be mode shapes, frequencies, or kinematics. The measured response can be thought of as an observable random vector of measurements, $\{Y\}$, polluted by a zero-mean, unobservable, random error vector, $\{\Gamma\}$. The response measurements are a function of the parameters of the system which may be denoted as the random vector $\{Z\}$. The function relating the response and the parameters is the known process and may be represented by a known mapping matrix, $[H]$ (18). Thus, this system may be expressed as

$$\{Y\} = [H]\{Z\} + \{\Gamma\}. \tag{1.5}$$

Having measured {Y} and knowing [H], one may solve for the parameter vector {Z} such that the response predicted by the mathematical model best fits the measured response according to some criteria (4).

Different assumptions regarding the statistical properties of the parameter vector {Z} and the zero-mean random error vector {Γ} result in three different identification procedures (18,19) : Least Squares, Weighted Least Squares, and Bayesian. Similarities and differences among the three methods will be addressed in Chapter III. However, following is a brief description of how these three methods allow for the estimation of the parameter vector.

Suppose a mathematical model is generated for the instrumented structure under consideration. Prior to the testing of the structure, the analyst makes a best (prior) estimate of the structural parameters, denoted as {r_p}. The analysis is performed, yielding a calculated (predicted) structural response based on the prior estimates of the parameters. Let this response be denoted as {$Q(r_p)$}.

The structure is now tested and the measured response {Q_m} becomes available. Assuming that there is an unacceptable difference between the predicted response and the measured response, this difference may be used to refine the prior estimate of the parameters. This new, better estimate of the parameters is given for all three estimation techniques by

$$\{\hat{r}\} = \{r_p\} + [W]\ (\{Q_m\} - \{Q(r_p)\}) \qquad (1.6)$$

where [W] is a weighting matrix which possesses a different form for Least Squares, Weighted Least Squares, and Bayesian identification (19).

The weighting matrix, [W], will be functionally dependent on the mapping matrix, [H], appearing in Eq. (1.5). This can be demonstrated by expressing the solution to the mathematical model as a Taylor Series expansion about the prior parameter estimate (4) to obtain

$$(\{Q(r)\} - \{Q(r_p)\}) = [T](\{r\} - \{r_p\}) + \{\epsilon\} \qquad (1.7)$$

where [T] is a sensitivity matrix relating the change in structural response {Q} to changes in the parameters {r}, and $\{\epsilon\}$ is an error vector. Comparing the form of this equation to that given in Eq. (1.5), one sees that the sensitivity matrix is the mapping matrix of the process. Furthermore, linear estimation theory shows (18) that the weighting matrix [W] is a function of the mapping matrix [H] (or equivalently, [T]), and of the statistical properties of {Z}, {Y}, and $\{\Gamma\}$. The form of this function varies for each of the three identification procedures.

The complete process for parameter identification can now be seen as an iterative one. The analyst assumes prior estimates of the structural parameters, $\{r_p\}$, and predicts the response of the structure, $\{Q(r_p)\}$. After determining the sensitivity of the response to the parameters (expressible as [T]), a weighting matrix is formed. This weighting matrix, [W], is used to calculate a better estimate, $\{\hat{r}\}$, of the structural parameters. This new estimate is based on $[\{Q_m\}-\{Q(r_p)\}]$, the difference between measured and predicted response. The process is repeated until this difference may be regarded as negligible or until there is no significant difference between $\{\hat{r}\}$ and $\{r_p\}$. This process is depicted in Fig. 2.

Scope of Work

In this study, statistically-based estimation techniques are presented. These techniques use structural test data to improve finite element models for dynamic analysis. Methods are developed to identify optimum values of the parameters of finite element models describing land-based and sea-based structures. The parameters to be identified are those contributing to the terms of the structural mass, damping, and stiffness matrices. This study is limited to models which exhibit linear structural behavior, and to parameters which appear as linear terms in the mass, damping, and stiffness matrices. Identification for the product of two or more parameters is not considered.

To date, there have been no studies reported on the identification of structural parameters for forced vibration using standard finite element modeling techniques and calculated kinematics. Also, there have been no studies on

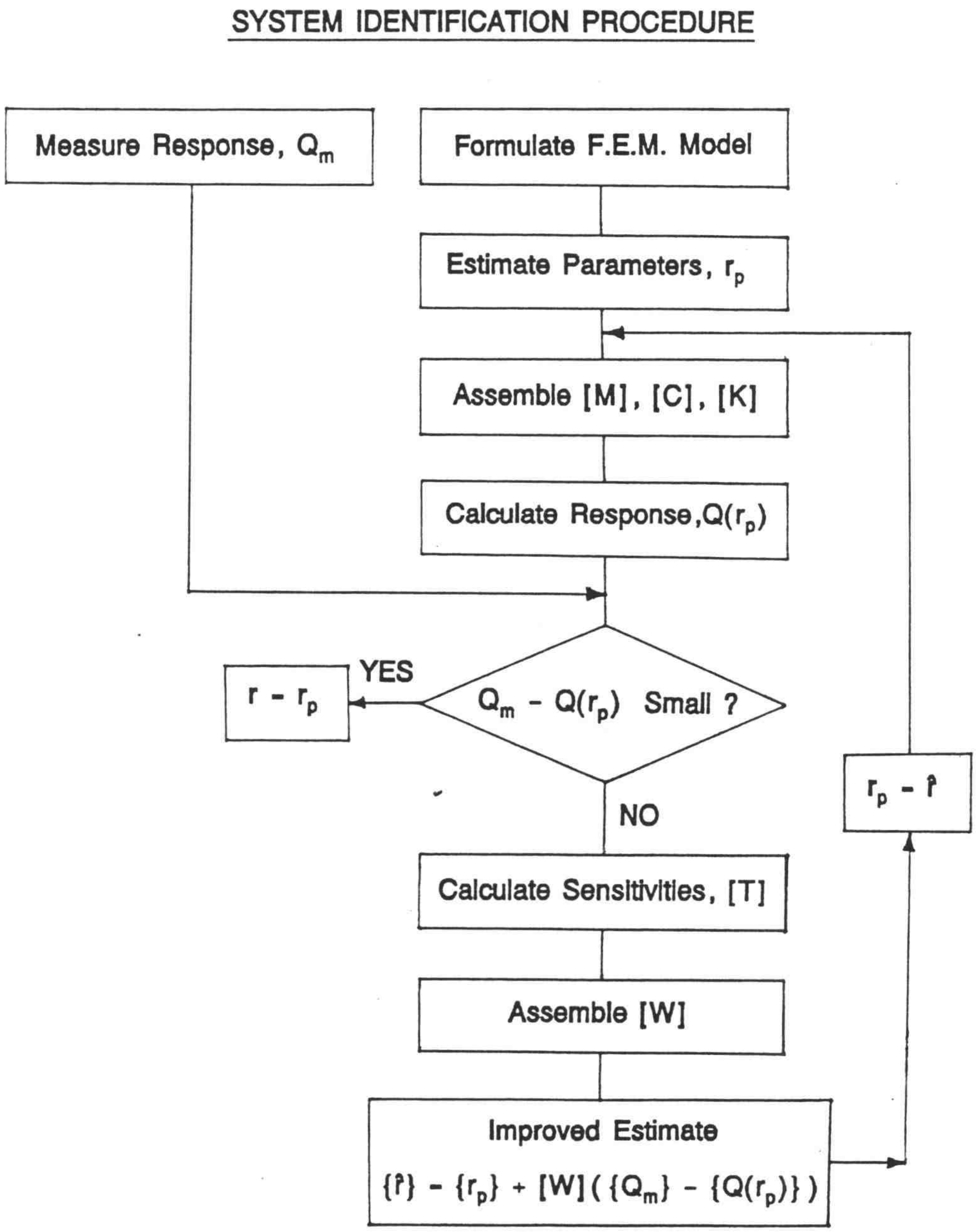


Figure 2. Parameter Identification Flowchart

the identification of concentrated mass and stiffness or element mass density, or on the identification of structural parameters to account for uncertainties in the response to hydrodynamic loading.

Included in this study are the effects of element mass, area, and moment of inertia; the effects of concentrated mass and stiffness; the effects of the Rayleigh damping coefficients; and the effects of hydrodynamic loading on the mass and damping characteristics of the structure. Techniques are presented for both the free vibration problem and for forced vibration in the time domain.

An overview of previous research relating to the system identification of structural parameters is presented in Chapter II. In Chapter III, the basic estimation equations are formulated, and the forms of the weighting matrix corresponding to the three estimation schemes are presented. The mapping matrix is decomposed into the product of two submatrices, and the forms of these matrices are given for use with free vibration and forced vibration models. Chapter IV contains a discussion of the algorithm used in the identification process and its implementation in the computer program which was developed. Numerical examples are included in Chapter V to demonstrate the validity and capabilities of the identification procedures detailed in this study. A summary of the work presented, as well as conclusions and possible extensions, are contained in Chapter VI.

CHAPTER II

PREVIOUS RESEARCH

Introduction

In Chapter I, the application of system identification techniques to mathematical modeling was described. It was seen that identification methods can be used to refine a mathematical model of a structure by providing better estimates of the parameters of the model. Much of the available literature in system identification addresses the problem of process identification with emphasis towards the design of control systems. Surveys of this literature are contained in papers by Bekey (13) and Astrom and Eykhoff (20). However, the present study is concerned with the estimation of model parameters. Therefore, only works which address this problem will be mentioned here. Particular attention will be given to references which discuss the estimation of structural parameters.

Much of the research in system identification has been centered in the aerospace industry, with applications to shock and vibration analysis of aerospace structures. Collins, et al. (21) presented one of the earliest literature surveys in 1970. The authors classified the existing research into two major categories - techniques applicable in the frequency domain and techniques applicable in the time domain - with further categorization according to type of model, type of estimation technique, and other criteria. This survey was updated by Hart and Yao (14) in 1977, with emphasis given to techniques suitable for structural dynamics. Other early state-of-the-art papers include those by Flannelly and Berman (22) and Bowles and Straeter (23). Flannelly and Berman addressed basic problems of system identification for aerospace structures, while Bowles and Straeter reviewed some computational considerations for various schemes in use at the time.

An overview of existing system identification methodology was presented in 1988 by the ASCE Task Committee on Methods for Identification of Large Structures in Space (24). This paper classified the available techniques according to three criteria : 1) model and parameters, 2) measurement data, and 3) estimation algorithm. Special attention was given to methods which are suitable for large, flexible space structures.

Numerous techniques and applications in system identification are presented in textbooks by Eykhoff (15), Ljung (16), and Soderstrom and Stoica (17). The emphasis in these works is on process identification and control systems; applications to structures are not discussed.

Methods for the identification or estimation of structural parameters may be grouped by many categories : parametric or nonparametric, frequency domain or time domain, single degree-of-freedom (d.o.f.) or multiple d.o.f., etc. Hart and Yao (14) grouped the methods into two categories : those which require prior estimates of the parameters and those which do not. This approach will be continued in the present study, beginning with methods which do not require prior estimates.

Many of the methods presented in the literature do not use mass, stiffness, or damping matrices in modeling structures. In some of those which do use these matrices, the original form or size of the matrices must be changed in order to perform the identification. In such cases, the physical significance of the mass and stiffness matrices is lost. Methods in which the full order, standard form, and physical significance of these matrices is preserved will be specially noted as such.

No Prior Estimates of Parameters

Gersch (25) and Gersch, et al. (26) developed a procedure for determining the period and damping of multiple d.o.f. structures. The method produces maximum likelihood estimates using time series with autoregressive moving average (ARMA).

A scheme for identifying the parameters of a nonlinear, single d.o.f. model was presented by DiStefano and Rath (27). The method is based on the minimization of a quadratic loss function, and requires the measured acceleration, velocity, displacement, and forcing function in the identification process.

DiStefano and Pena-Pardo (28) investigated the response of a linearly-damped viscous model and a nonlinear stiffness model. The method they developed uses an invariant imbedding filter operating on data from experiments conducted on seismic tables.

Other approaches which do not require prior estimates of parameters include those by Torkamani and Hart (29,30), Berman and Hannelly (31), and Berman, et al. (32). The papers by Torkamani and Hart detail a method for calculating the impulse response function of single and multiple d.o.f. systems. The method involves the discretization of Duhamel's integral and uses data from records of ground motions during earthquakes. The latter two works presented methods for identifying parameters of a linear, discrete model using modal data. Methods of computing the effects of changes in mass and stiffness on natural frequencies and normal modes were studied. Masri, et al. (33,34,35) proposed a method for the nonparametric identification of chainlike systems which are nonlinear and have multiple degrees-of-freedom. Time histories of the excitation and response are needed, and the magnitude of all masses must be known.

A method which uses the free-decay response-time function of a structure has been presented by Ibrahim and Mikulcik (36), Ibrahim (37), and Ibrahim and Pappa (38). In this method (Ibrahim Time Domain Technique), two response matrices were formed from a special oversized model and were used to calculate eigenvalues and eigenvectors. The damping factors were then extracted from this information. Modal confidence factors were used to distinguish between computational modes and structural modes.

System identification techniques were applied by Kaplan, et al. (39) to determine the hydrodynamic coefficients in Morison's equation from measured forces on an offshore structure. A generalization of linear Kalman filtering was used to convert an assumed two-point boundary-value problem to an initial-value problem, resulting in a sequential estimator. The coupled, first-order differential equations which were generated are nonlinear and time-varying.

A method was proposed by Batill and Hollkamp (40) to predict the transient response of multiple d.o.f. structures subjected to arbitrary input. An ARMA model was created from the linear, discrete-time transfer function. The identification process requires a two-stage, iterative technique and an overspecified model, and uses free vibration response or impulse response data.

Prior Estimates of Parameters

In a previously cited study of damped systems, DiStefano and Rath (27) also detailed an identification scheme in which the nonlinear structural model was reduced to a system of first-order equations using invariant imbedding. A loss function is then minimized in a least squares fashion, and the problem solved using a filtering method and a Gauss-Newton method. Experimental errors were quantified by inclusion of a vector of observation errors.

Ibanez (41) presented a method for obtaining the pseudo-inverse of a matrix, thus allowing for the weighted least squares solution of ill-conditioned linear systems. Such systems result when there is an inequality between the number of degrees-of-freedom used to model a structure and the number of response measurements available.

Beliváeau (42,43) addressed the estimation of mass, stiffness, and damping matrices using natural frequencies, mode shapes amplitudes and phase angles, and damping constants. Eigenvalue and eigenvector sensitivities are used in the procedure and an optimization is performed on an objective function based on least squares, maximum likelihood, or Bayesian inference. Beliváeau (44) has also presented a scheme for identifying mass, stiffness, and damping matrices. This method uses least squares criteria and acceleration records with a system of equations similar to the equations of motion.

Identification of damping coefficients was the subject of a study by Caravani and Thomson (45). The identification algorithm involves the recursive minimization of a loss function based on weighted least squares, and operates on the equations of motion in the frequency domain.

Caravani, et al. (46) used displacement, velocity, and acceleration data to identify stiffness and damping effects. Two algorithms were presented : one based on a least squares approach and one based on weighted least squares. Both methods require the stiffness and damping matrices to be of tridiagonal form.

Beliveau and Favillier (47) developed a method for estimating stiffness and damping parameters of buildings based on acceleration measurements at each story. The analysis was done in the frequency domain, and sinusoidal loads and displacements were assumed. An objective function based on Bayesian inference was minimized using a modified Newton-Raphson scheme.

Shinozuka, et al. (48) transformed the equations of motion into state equations at discrete times, and then into ARMA models. The method of instrumental variables and the method of limited information likelihood were then used in the identification process. The coefficient matrices of the original equations of motion were recovered from the parameter estimates of the ARMA model.

Capecchi and Vestroni (49) replaced a finite element model for a structure by an analytical polynomial function in terms of the structural parameters. A Bayesian approach was used with a classical Newton method to iteratively solve for the optimum values of the structural parameters. The method was applied to a four-story frame consisting of beam, truss, and plate elements. Pseudo-experimental mode shapes and frequencies were used in the identification process.

Lee and Yun (50) presented a method for the estimation of the model equations and parameters for linear, multiple d.o.f. structures. The equations of motion were transformed into a state-space equation, and then into a stochastic ARMA (ARMAX) model. A sequential prediction error method was used to estimate the parameters of the ARMAX model; the system parameters were then recovered from the ARMAX parameters. The technique was applied to a two d.o.f. and a three d.o.f. shear building, using the floor accelerations as measured data.

Studies in which the full order, standard form, and physical significance of the mass and stiffness matrices are maintained include those by Collins, et al. (4), Hart and Torkamani (19), Leonard and Warren (1), Leonard, et al. (51), and Leonard and Khouri (52).

Bayesian techniques for mass and stiffness characteristics of a finite element model have been presented by Collins, et al. (4). Initial estimates of element parameters are updated using measured mode shapes and natural frequencies in conjunction with eigenvalue and eigenvector sensitivities. The method allows for the inclusion of both modeling error and measurement error.

Hart and Torkamani (19) summarized least squares, weighted least squares, and Bayesian techniques for use with *a priori* estimates of structural parameters. Numerical examples were presented which demonstrate the effects of various assumptions regarding measurement and modeling errors. Only free vibration was considered.

Leonard and Warren (1) incorporated the Bayesian techniques presented by Collins, et al. (4) into a finite element program, EASE2 (53). The program was used for free vibration estimation to iteratively modify section properties in a model of a construction crane. The crane was modeled with beam elements, and test data consisted of the frequencies and mode shapes of the first two modes. Leonard, et al. (51) and Leonard and Khouri (52) extended the capabilities of the program to include plate and shell elements.

CHAPTER III

FORMULATION OF PROBLEM

Introduction

In this chapter, linear estimation theory is presented in general form. The theory for Bayesian estimation is presented first, followed by Weighted Least Squares and Least Squares estimation theory.

The estimation equations required for the identification of structural parameters are then formulated. These equations are shown to be a function of the sensitivity matrix, [T], relating the change in structure response to the change in the parameters of the finite element model. This matrix is expressed as a product of two submatrices, [T] = [B][A]. Forms of [T] are presented for both free and forced vibration. Equations for calculating the terms in submatrix [B] are presented for free vibration identification. For forced vibration, equations are derived which allow the terms of [B] to be calculated. Forms of the sensitivity submatrix [A] are detailed for the identification of element, lumped, and damping parameters.

Linear Estimation Theory

Basic linear estimation theory is presented in this section. This theory allows a vector of random variables to be estimated from observed values of another random vector.

For the structural identification problem, improved estimates of the structural parameters are sought; these estimates use the measured response of an instrumented structure. In the next section of this chapter, system identification for a structure is shown to be an application of linear estimation theory.

Assume the output from some known process is related to the parameters of the process according to the linear model

$$\{Y\} = [H]\{Z\} + \{\Gamma\} \tag{3.1}$$

Let np = number of parameters, and nd = number of available data (measurements). The terms in the above equation are defined as

$\{Y\}$ = Observable random vector, size nd x 1

$[H]$ = Known mapping matrix, nd x np

$\{Z\}$ = Random parameter vector, np x 1

$\{\Gamma\}$ = Unobservable random error vector, nd x 1

The rank of [H] is np $\leq$ nd (18).

Use the notation E(*) to indicate the expected value of a random variable; important statistical properties of $\{Z\}$ and $\{\Gamma\}$ are

$$E(\{Z\}) = \{\mu_Z\},\ np \times 1 \tag{3.2a}$$

$$E[(\{Z\}-\{\mu_Z\})(\{Z\}-\{\mu_Z\})^T] = [V_{ZZ}],\ np \times np \tag{3.2b}$$

$$E(\{\Gamma\}) = \{0\},\ nd \times 1 \tag{3.3a}$$

$$E(\{\Gamma\}\{\Gamma\}^T) = [V_{\Gamma\Gamma}],\ nd \times nd \tag{3.3b}$$

$$E(\{\Gamma\}\{Z\}^T) = \{0\}, \text{ nd x np} \tag{3.4}$$

The zero value of the last term arises from the assumption that the experimental error is independent of the parameters. The expected value of {Z} is $\{\mu_Z\}$, and is assumed known for Bayesian estimation (54); it is assumed unknown for Least Squares and Weighted Least Squares estimation (18). The matrices $[V_{ZZ}]$ and $[V_{\Gamma\Gamma}]$ are defined, respectively, as the variances of the parameter vector and the variance of the error vector; it is assumed that their values are known.

For the remainder of this section, matrix notation will be dropped for clarity of the presentation. However, it should be remembered that all terms are either vectors or arrays.

The expected value (mean) and variance of the output vector Y can be expressed in terms of the mean and variance of the parameter vector. By definition,

$$\mu_Y = E(Y) \tag{3.5}$$

and it follows from Eqs. (3.1), (3.2a), and (3.3a) that

$$\mu_Z = E(HZ+\Gamma) = H\mu_Z \tag{3.6}$$

The variance V_{YY} is defined as

$$V_{YY} = E[(Y-\mu_Y)(Y-\mu_Y)^T] \tag{3.7}$$

which becomes, upon substitution of Eqs. (3.1) and (3.6) and expansion of the product :

$$V_{YY} = H\ E[(Z-\mu_Z)(Z-\mu_Z)^T]H^T + E(\Gamma)E(Z-\mu_Z)^T H^T + + H\ E[(Z-\mu_Z)]E(\Gamma^T) + E(\Gamma\Gamma^T) \quad (3.8)$$

Substituting the relationships given by Eqs. (3.2b) and (3.3a), one obtains

$$V_{YY} = HV_{ZZ}H^T + V_{\Gamma\Gamma} \quad (3.9)$$

In a similar fashion, the following statistical properties can be demonstrated :

$$V_{ZY} = E[(Z-\mu_Z)(Y-\mu_Y)^T] = V_{ZZ}H^T \quad (3.10a)$$

$$V_{YZ} = V_{ZY}{}^T = HV_{ZZ}{}^T \quad (3.10b)$$

$$E(ZZ^T) = V_{ZZ} + \mu_Z\mu_Z{}^T \quad (3.10c)$$

$$E(YY^T) = V_{YY} + \mu_Y\mu_Y{}^T \quad (3.10d)$$

$$E(ZY^T) = V_{ZY} + \mu_Z\mu_Y{}^T \quad (3.10e)$$

$$E(YZ^T) = V_{YZ} + \mu_Y\mu_Z{}^T \quad (3.10f)$$

All three estimation schemes - Bayes, Weighted Least Squares, and Least Squares - are based on a class of linear estimators, $\hat{Z}$, of the form

$$\hat{Z} = G + DY \quad (3.11)$$

where

G = vector of real numbers, np x 1

D = matrix of real numbers, np x nd

To select the best, linear, unbiased estimator (55,56) from the class, G and D are selected so that

$$E(\hat{Z}-Z) = 0 \tag{3.12}$$

and

$$\Pi = E[(\hat{Z}-Z)(\hat{Z}-Z)^T] \quad \text{is minimized} \tag{3.13}$$

In the above, Π is the squared-error loss function (57) and is an nd x nd matrix. Equation (3.12) can be alternately expressed as

$$E(\hat{Z}) = E(Z) = \mu_Z \tag{3.14}$$

The loss function is minimized when

$$\delta\Pi = 0 \tag{3.15}$$

where $\delta\Pi$ indicates the first variation of Π.

Bayesian Estimation. In Bayesian estimation, the expected value of the parameter vector, μ_Z, is assumed to be known. In order to determine the forms of G and D in Eq. (3.11), the requirements of Eqs. (3.13) and (3.14) must be met.

The loss function defined in Eq. (3.13) can be expanded by substituting (3.11) and performing the indicated matrix operations, to obtain

$$\begin{aligned}\Pi = GG^T + GE(Y^T)D^T - GE(Z^T) + DE(Y)G^T + GE(YY^T)D^T + \\ - DE(YZ^T) - E(Z)G^T - E(ZY^T)D^T + E(ZZ^T)\end{aligned} \tag{3.16}$$

If the relationships given by Eq. (3.10) are substituted into this equation, the loss function becomes

$$\Pi = GG^T + G\mu_Y{}^T D^T - G\mu_Z{}^T + D\mu_Y G^T +$$
$$+ D(V_{YY}+\mu_Y\mu_Y{}^T)D^T - D(V_{YZ}+\mu_Y\mu_Z{}^T) - \mu_Z G^T +$$
$$- (V_{ZY}+\mu_Z\mu_Y{}^T)D^T + (V_{ZZ}+\mu_Z\mu_Z{}^T) \qquad (3.17)$$

Equation (3.15) indicates that the minimization of Π is accomplished by setting its first variation equal to zero (the null matrix). Since the loss function Π is a function of both G and D, the variation can be written as

$$\delta\Pi = \delta_G\Pi + \delta_D\Pi = 0 \qquad (3.18)$$

where $\delta_G\Pi$ and $\delta_D\Pi$ indicate the variation of Π with respect to G and D.

It can be shown (18) that the operation specified in the above equation yields the following :

$$\delta_G\Pi = 0 : G + D\mu_Y - \mu_Z = 0 \qquad (3.19a)$$

$$\delta_D\Pi = 0 : G\mu_Y{}^T + D(V_{YY}+\mu_Y\mu_Y{}^T) - (V_{ZY}+\mu_Z\mu_Y{}^T) = 0 \qquad (3.19b)$$

The desired form of G is found from the first of these two expressions :

$$G = \mu_Z - D\mu_Y \qquad (3.20)$$

It should be noted that the requirement specified in Eq. (3.12), $E(\hat{Z}-Z) = 0$, leads to the same result; therefore, this criterion is automatically satisfied. Also note that G and μ_Z are not mathematically independent. This is a feature exhibited by Bayesian estimation, but not by the other estimation schemes.

The second term of the variation equation, given by Eq. (3.19b), can be rearranged as

$$(G+D\mu_Y-\mu_Z)\mu_Y{}^T + DV_{YY} - V_{ZY} = 0 \qquad (3.21)$$

The expression within the parentheses is equal to zero, by virtue of Eq. (3.20). Thus, it is required that

$$DV_{YY} - V_{ZY} = 0 \qquad (3.22)$$

from which

$$D = V_{ZY}V_{YY}{}^{-1} \qquad (3.23)$$

Substitution of V_{ZY} and V_{YY} from Eq. (3.10) yields the desired form of D :

$$D = (V_{ZZ}H^T)(HV_{ZZ}H^T + V_{\Gamma\Gamma})^{-1} \qquad (3.24)$$

With G and D now fully determined, the final form of the linear estimator is obtained by substitution of Eqs. (3.20) and (3.24) into Eq. (3.11), giving

$$\hat{Z} = \mu_Z + V_{ZZ}H^T(HV_{ZZ}H^T + V_{\Gamma\Gamma})^{-1}(Y-\mu_Y) \qquad (3.25)$$

In the next section of this chapter it is shown that

$$\mu_Z = 0 \qquad (3.26a)$$

$$\mu_Y = 0 \qquad (3.26b)$$

for the structural identification problem. In that instance, $G = \mu_Z-D\mu_Y = 0$, and the estimator becomes

$$\hat{Z} = V_{ZZ}H^T(HV_{ZZ}H^T + V_{\Gamma\Gamma})^{-1} Y \tag{3.27}$$

Weighted Least Squares Estimation. Recall that the desired form of the linear estimator is given by

$$\hat{Z} = G + DY \tag{3.11}$$

In Least Squares and Weighted Least Squares estimation, it is assumed that μ_Z, the expected value of the parameter vector, is unknown and is independent of G (18).

The first criterion that must be satisfied by the estimator is

$$E(\hat{Z}-Z) = E(G+DY-Z) = 0 \tag{3.28}$$

After substitution of $Y = HZ+\Gamma$, and expansion and simplification of the terms, it can be demonstrated that this criterion requires

$$G + (DH-I)\mu_Z = 0 \tag{3.29}$$

where I is the identity matrix. Since G and μ_Z are assumed to be mathematically independent, this equation is satisfied if

$$G = 0 \tag{3.30a}$$

$$DH = I \tag{3.30b}$$

The second criterion to be satisfied requires that

$$\delta\Pi = \delta_G\Pi + \delta_D\Pi = 0 \tag{3.18}$$

where

$$\Pi = E[(\hat{Z}-Z)(\hat{Z}-Z)^T]$$
$$= E[(G+DY-Z)(G+DY-Z)^T] \qquad (3.31)$$

This loss function is subject to the constraints of Eqs. (3.30). It can be shown (18) that expansion and simplification of Eq. (3.31), subject to Eqs. (3.30), leads to

$$\Pi = DV_{\Gamma\Gamma}D^T + (I-DH)\nu + \nu^T(I-H^TD^T) \qquad (3.32)$$

where ν is a Lagrangian multiplier used to incorporate the constraint of Eq. (3.30b).

It can be demonstrated (18) that minimizing Π according to Eq. (3.18) requires

$$DV_{\Gamma\Gamma} - \nu^TH^T = 0 \qquad (3.33)$$

Solve for D

$$D = \nu^TH^TV_{\Gamma\Gamma}^{-1} \qquad (3.34a)$$

and post-multiply both sides of the equation by H to obtain

$$DH = \nu^TH^TV_{\Gamma\Gamma}^{-1}H \qquad (3.34b)$$

Since DH = I from Eq. (3.30b), this becomes

$$I = \nu^TH^TV_{\Gamma\Gamma}H \qquad (3.35)$$

For the identity to hold, the Lagrangian multiplier must assume the form

$$\nu^{T} = (H^{T}V_{\Gamma\Gamma}^{-1}H)^{-1} \quad (3.36)$$

When this result is substituted into Eq. (3.34), the final form of D is established as

$$D = (H^{T}V_{\Gamma\Gamma}^{-1}H)H^{T}V_{\Gamma\Gamma}^{-1} \quad (3.37)$$

With this D, and with G = 0 from Eq. (3.30a), the desired estimator is obtained from Eq. (3.11) :

$$\hat{Z} = (H^{T}V_{\Gamma\Gamma}^{-1}H)^{-1}\ H^{T}V_{\Gamma\Gamma}^{-1}\ Y \quad (3.38)$$

Least Squares Estimation. The Least Squares form of the estimator $\hat{Z}$ can be obtained directly from the result for Weighted Least Squares.

In Least Squares estimation, it is assumed that the variance matrix for the measurement error is of the form

$$V_{\Gamma\Gamma} = v^{2}I \quad (3.39)$$

where v^2 is the square of the standard deviation of the measurement error. With this variance, Eq. (3.38) becomes

$$\hat{Z}= [H^{T}(\sigma^{2}I)^{-1}H]^{-1}H^{T}(\sigma^{2}I)^{-1}Y \quad (3.40)$$

which simplifies to the desired form

$$\hat{Z} = (H^{T}H)^{-1}\ H^{T}\ Y \quad (3.41)$$

An examination of Eqs. (3.27), (3.38), and (3.41) shows that, for all three estimation schemes, the estimator for the parameters can be cast in the form

$$\hat{Z} = DY \tag{3.42}$$

Formulation of the Estimation Equations for Structural Identification

Suppose a mathematical (finite element) model is generated for an instrumented test structure. Prior to testing, the analyst makes a best (prior) estimate of the values of the structural parameters, denoted by $\{r_p\}$. The analysis is performed, yielding a predicted structural response $\{Q(r_p)\}$ based on the prior estimate of the parameters. An improved estimate of the response $\{Q(r)\}$ can be obtained by expressing the solution to the mathematical model of the structure as a Taylor Series expansion about the prior estimate of the parameters.

Consider a continuous, differentiable function $f(\tau)$. The Taylor Series expansion of this function about a nearby point $\bar{\tau}$ is given by

$$f(\tau) = f(\bar{\tau}) + \frac{\partial f}{\partial \tau}(\tau - \bar{\tau}) \tag{3.43}$$

where the partial derivative of the function is evaluated at the point $\bar{\tau}$ (58). For the structural identification problem, the function under consideration is the response of the structure, $\{Q\}$, which is a function of the structural parameters, $\{r\}$. Therefore, expanding this response in a

Taylor Series expansion about the (nearby) prior estimate of the parameters, $\{r_p\}$, yields

$$\{Q(r)\} = \{Q(r_p)\} + \left[\frac{\partial Q}{\partial r}\right](\{r\}-\{r_p\}) \tag{3.44}$$

where the derivative term is evaluated at $\{r_p\}$. Define a sensitivity matrix, [T], as

$$[T] = \left[\frac{\partial Q}{\partial r}\right] \tag{3.45}$$

which relates the change in structural response to the change in the structural parameters (4). This definition allows Eq. (3.44) to be written as

$$\{Q(r)\} - \{Q(r_p)\} = [T](\{r\}-\{r_p\} \tag{3.46}$$

After the structure is tested, the measured response $\{Q_m\}$ becomes available, and can be considered as the "true" response to be matched by the finite element model. The measured data will be polluted by an error vector $\{\epsilon\}$, assumed to be zero mean with a variance of $[V_{\epsilon\epsilon}]$; that is,

$$E(\{\epsilon\}) = \{0\} \quad : nd \times 1 \tag{3.47a}$$

$$E(\{\epsilon\}\{\epsilon\}^T) = [V_{\epsilon\epsilon}] \quad : nd \times nd \tag{3.47b}$$

Thus, when the experimental response is used, Eq. (3.46) becomes

$$(\{Q_m\} - \{Q(r_p)\}) = [T](\{r\} - \{r_p\}) + \{\epsilon\} \tag{3.48}$$

This equation is of the same form as the standard linear estimation equation, given by

$$\{Y\} = [H]\{Z\} + \{\Gamma\} \tag{3.1}$$

where

$$\{Y\} = \{Q_m\} - \{Q(r_p)\} \tag{3.49a}$$

$$[H] = [T] \tag{3.49b}$$

$$\{Z\} = \{r\} - \{r_p\} \tag{3.49c}$$

$$\{\Gamma\} = \{\epsilon\} \tag{3.49d}$$

Recall that the estimator for {Z} was assumed to be of the form

$$\{\hat{Z}\} = \{G\} + [D]\{Y\} \tag{3.11}$$

It was shown that {G} = {0} for all three estimation schemes, so that the estimator can be written as

$$\{\hat{Z}\} = [D]\{Y\} \tag{3.42}$$

Let [W] represent [D] for the structural identification problem, and note that

$$\{\hat{Z}\} = \{\hat{r}\} - \{r_p\} \tag{3.50}$$

Substituting these terms into Eq. (3.42), along with {Y} from Eq. (3.49a), one obtains

$$(\{\hat{r}\} - \{r_p\}) = [W](\{Q_m\} - \{Q(r_p)\}) \qquad (3.51)$$

If this equation is rearranged, the improved estimate of the structural parameters, $\{\hat{r}\}$, is found to be

$$\{\hat{r}\} = \{r_p\} + [W](\{Q_m\} - \{Q(r_p)\}) \qquad (3.52)$$

In this equation, [W] is referred to as the weighting matrix, and is given by the three forms of [D] derived in the previous section of this chapter. In those three expressions, the following substitutions are made based on the correspondence of terms :

$$[H] = [T] \qquad (3.53a)$$

$$[V_{zz}] = [V_{rr}] \qquad (3.53b)$$

$$[V_{\Gamma\Gamma}] = [V_{\epsilon\epsilon}] \qquad (3.53c)$$

Therefore, the weighting matrix for the structural identification problem is given for the three estimation schemes by

<u>Bayes</u>

$$[W] = [V_{rr}][T]^T([T][V_{rr}][T]^T + [V_{\epsilon\epsilon}])^{-1} \qquad (3.54a)$$

<u>Weighted Least Squares</u>

$$[W] = ([T]^T[V_{\epsilon\epsilon}]^{-1}[T])^{-1}[T]^T[V_{\epsilon\epsilon}]^{-1} \qquad (3.54b)$$

<u>Least Squares</u>

$$[W] = ([T]^T[T])^{-1}[T]^T \qquad (3.54c)$$

The matrices $[V_{rr}]$ and $[V_{\epsilon\epsilon}]$ are specified by the analyst, and reflect the uncertainty in the parameters and data, respectively.

In the previous section of this chapter, the Bayesian estimator for {Z} was reduced to the form $\{\hat{Z}\} = [D]\{Y\}$ because it was anticipated that

$$\{\mu_Z\} = \{0\} \tag{3.26a}$$

$$\{\mu_Y\} = \{0\} \tag{3.26b}$$

for the structural identification problem. This can now be readily seen from Eq. (3.49), since

$$\{\mu_Z\} = E(\{r\}-\{r_p\}) = \{r_p\} - \{r_p\} = \{0\} \tag{3.55a}$$

and

$$\{\mu_Y\} = E(\{Q_m\}-\{Q(r_p)\}) = \{Q_m\} - \{Q_m\} = \{0\} \tag{3.55b}$$

In this section, it was shown that the improved estimate of the structural parameters is given by

$$\{\hat{r}\} = \{r_p\} + [W](\{Q_m\} - \{Q(r_p)\}) \tag{3.52}$$

where the weighting matrix [W] is a function of the sensitivity matrix [T]. The following section considers the sensitivity matrix in more detail.

<u>Forms of the Sensitivity Matrix, [T]</u>

As defined in Eq. (3.45), the sensitivity matrix [T] relates the change in structural response to the change in the parameters as

$$[T] = \left[\frac{\partial Q}{\partial r}\right] \tag{3.45}$$

Recall that Q represents the calculated response of the structure. For free vibration analysis, Q comprises the squared natural frequencies and the mode shapes. For forced vibration, Q comprises the time-history kinematics; i.e., the displacements, velocities, and accelerations at the nodal degrees-of-freedom. For both types of analysis, the response will be shown to be an explicit function of the mass and stiffness matrices, [M] and [K]. (For forced vibration, the response will also be dependent on the damping matrix, [C]; however, this matrix will be formed from a linear combination of [M] and [K]). The mass and stiffness matrices, in turn, are explicit functions of the structural parameters (e.g., element areas). Thus, the chain rule of differentiation may be used with Eq. (3.45) to express the sensitivity matrix as

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \left[\left[\frac{\partial Q}{\partial M}\right]\left[\frac{\partial M}{\partial r}\right] + \left[\frac{\partial Q}{\partial K}\right]\left[\frac{\partial K}{\partial r}\right]\right] \qquad (3.56)$$

For computational ease and efficiency, [T] may be expressed as the product of two submatrices (1,4) as follows :

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \left[\left[\frac{\partial Q}{\partial M}\right]\ \left[\frac{\partial Q}{\partial K}\right]\right] \begin{bmatrix} \left[\frac{\partial M}{\partial r}\right] \\ \left[\frac{\partial K}{\partial r}\right] \end{bmatrix} \qquad (3.57)$$

Defining

$$[B] = \left[\left[\frac{\partial Q}{\partial M} \right] \left[\frac{\partial Q}{\partial K} \right] \right] \qquad (3.58a)$$

$$[A] = \left[\begin{array}{c} \left[\frac{\partial M}{\partial r} \right] \\ \left[\frac{\partial K}{\partial r} \right] \end{array} \right] \qquad (3.58b)$$

the sensitivity matrix can be written as

$$[T] = [B][A] \qquad (3.59)$$

The exact forms of [B] and [A] depend on the type of response (i.e., free vibration or forced vibration) and the type of parameters to be estimated.

Free Vibration. Recall that the equations of motion for a structure are given by

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{P(t)\} \qquad (1.1)$$

For a free vibration analysis, Q represents the frequencies and mode shapes obtained from the solution to this equation in the absence of damping and applied loading; that is

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \qquad (3.60)$$

The solution for the displacement is assumed to be harmonic and of the form

$$\{x\} = \{\phi\}\sin(\omega t - \gamma) \tag{3.61}$$

where $\{\phi\}$ contains the amplitudes of motion (mode shapes) for the n degrees-of-freedom, ω is the natural frequency of vibration, and γ is a phase angle (59). Substitution of $\{x\}$ and its second time derivative into Eq. (3.60) yields, after rearranging,

$$\left([K] - \omega^2[M]\right)\{\phi\} = \{0\} \tag{3.62}$$

A nontrivial solution of this equation, which is a form of eigenproblem, requires that the determinant of the term in parentheses vanishes. Define

$$\lambda = \omega^2 \tag{3.63}$$

as the squared natural frequency; it is therefore required that

$$|\,[K] - \lambda[M]\,| = \{0\} \tag{3.64}$$

Note that Eq. (3.64) actually represents n simultaneous equations written in compact form, where n is the number of d.o.f. The solution will therefore yield n values of the squared natural frequency stored in the vector $\{\lambda\}$, and n sets of the vector $\{\phi\}$, the mode shapes. Thus, for a free vibration analysis, the complete response is given by

$$\{Q\} = \begin{bmatrix} \{\lambda\} \\ \{\phi\} \end{bmatrix} \tag{3.65}$$

and the sensitivity matrix is given by

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \begin{bmatrix} \left[\frac{\partial \lambda}{\partial r}\right] \\ \left[\frac{\partial \phi}{\partial r}\right] \end{bmatrix} \tag{3.66}$$

If the form given by Eq. (3.57) is used, the sensitivity matrix becomes

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \begin{bmatrix} \left[\frac{\partial \lambda}{\partial M}\right] & \left[\frac{\partial \lambda}{\partial K}\right] \\ \left[\frac{\partial \phi}{\partial M}\right] & \left[\frac{\partial \phi}{\partial K}\right] \end{bmatrix} \begin{bmatrix} \left[\frac{\partial M}{\partial r}\right] \\ \left[\frac{\partial K}{\partial r}\right] \end{bmatrix} \tag{3.67}$$

Comparing this with the form shown in Eq. (3.59), the sensitivity submatrix [B] for free vibration may be written as

$$[B] = \begin{bmatrix} \left[\frac{\partial \lambda}{\partial M}\right] & \left[\frac{\partial \lambda}{\partial K}\right] \\ \left[\frac{\partial \phi}{\partial M}\right] & \left[\frac{\partial \phi}{\partial K}\right] \end{bmatrix} \tag{3.68}$$

Submatrix [A] is given in Eq. (3.58b).

It should be noted that a full set of experimental data (Q_m) need not be measured; i.e., frequencies and mode shapes do not have to be measured in every mode and for every d.o.f. Define :

p = number of measured λ

q = number of modes for which ϕ is measured

s = number of d.o.f. at which ϕ is measured

nd = p+(q*s) = total number of measured data

np = number of parameters to be identified

n = number of d.o.f. in finite element model

The dimensions of the matrices are thus

$$[T] : nd \times np$$

$$[B] : nd \times 2n^2$$

$$[A] : 2n^2 \times np$$

Recall the equation that specifies the improved estimate of the structural parameters :

$$\{\hat{r}\} = \{r_p\} + [W](\{Q_m\} - \{Q(r_p)\}) \tag{3.52}$$

It can be seen that the dimensions of these matrices are

$$\{\hat{r}\}, \{r\} : np \times 1$$

$$Q_m, Q(r_p) : nd \times 1$$

$$[W] : np \times nd$$

The terms of submatrix [B] shown in Eq. (3.68) are arranged in the following order :

$$\left[\frac{\partial \lambda}{\partial M}\right] = \begin{bmatrix} \frac{\partial \lambda_1}{\partial M_{11}} & \frac{\partial \lambda_1}{\partial M_{12}} & \cdots & \frac{\partial \lambda_1}{\partial M_{nn}} \\ \frac{\partial \lambda_2}{\partial M_{11}} & \frac{\partial \lambda_2}{\partial M_{12}} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \lambda_{np}}{\partial M_{11}} & \frac{\partial \lambda_{np}}{\partial M_{12}} & \cdots & \frac{\partial \lambda_{np}}{\partial M_{nn}} \end{bmatrix} : \text{Size } np \times n^2 \tag{3.69a}$$

$$\left[\frac{\partial \lambda}{\partial K}\right] = \begin{bmatrix} \frac{\partial \lambda_1}{\partial K_{11}} & \frac{\partial \lambda_1}{\partial K_{12}} & \cdots & \frac{\partial \lambda_1}{\partial K_{nn}} \\ \frac{\partial \lambda_2}{\partial K_{11}} & \frac{\partial \lambda_2}{\partial K_{12}} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \lambda_{np}}{\partial K_{11}} & \frac{\partial \lambda_{np}}{\partial K_{12}} & \cdots & \frac{\partial \lambda_{np}}{\partial K_{nn}} \end{bmatrix} \quad : \text{ Size } np \times n^2 \qquad (3.69b)$$

$$
\left[\frac{\partial \phi}{\partial M}\right] = \begin{bmatrix}
\frac{\partial \phi_{11}}{\partial M_{11}} & \frac{\partial \phi_{11}}{\partial M_{12}} & \cdots & \frac{\partial \phi_{11}}{\partial M_{nn}} \\
\frac{\partial \phi_{21}}{\partial M_{11}} & \frac{\partial \phi_{21}}{\partial M_{12}} & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \phi_{s1}}{\partial M_{11}} & \frac{\partial \phi_{s1}}{\partial M_{12}} & \cdots & \frac{\partial \phi_{s1}}{\partial M_{nn}} \\
\hline
\frac{\partial \phi_{12}}{\partial M_{11}} & \frac{\partial \phi_{12}}{\partial M_{12}} & \cdots & \frac{\partial \phi_{12}}{\partial M_{nn}} \\
\frac{\partial \phi_{22}}{\partial M_{11}} & \frac{\partial \phi_{22}}{\partial M_{12}} & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \phi_{s2}}{\partial M_{11}} & \frac{\partial \phi_{s2}}{\partial M_{12}} & \cdots & \frac{\partial \phi_{s2}}{\partial M_{nn}} \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
\frac{\partial \phi_{1q}}{\partial M_{11}} & \frac{\partial \phi_{1q}}{\partial M_{12}} & \cdots & \frac{\partial \phi_{1q}}{\partial M_{nn}} \\
\frac{\partial \phi_{2q}}{\partial M_{11}} & \frac{\partial \phi_{2q}}{\partial M_{12}} & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \phi_{sq}}{\partial M_{11}} & \frac{\partial \phi_{sq}}{\partial M_{12}} & \cdots & \frac{\partial \phi_{sq}}{\partial M_{nn}}
\end{bmatrix} \quad : \text{ Size } (q*s) \times n^2 \qquad (3.70a)
$$

$$
\left[\frac{\partial\phi}{\partial K}\right] = \begin{bmatrix}
\frac{\partial\phi_{11}}{\partial K_{11}} & \frac{\partial\phi_{11}}{\partial K_{12}} & \cdots & \frac{\partial\phi_{11}}{\partial K_{nn}} \\
\frac{\partial\phi_{21}}{\partial K_{11}} & \frac{\partial\phi_{21}}{\partial K_{12}} & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial\phi_{s1}}{\partial K_{11}} & \frac{\partial\phi_{s1}}{\partial K_{12}} & \cdots & \frac{\partial\phi_{s1}}{\partial K_{nn}} \\
\hline
\frac{\partial\phi_{12}}{\partial K_{11}} & \frac{\partial\phi_{12}}{\partial K_{12}} & \cdots & \frac{\partial\phi_{12}}{\partial K_{nn}} \\
\frac{\partial\phi_{22}}{\partial K_{11}} & \frac{\partial\phi_{22}}{\partial K_{12}} & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial\phi_{s2}}{\partial K_{11}} & \frac{\partial\phi_{s2}}{\partial K_{12}} & \cdots & \frac{\partial\phi_{s2}}{\partial K_{nn}} \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
\frac{\partial\phi_{1q}}{\partial K_{11}} & \frac{\partial\phi_{1q}}{\partial K_{12}} & \cdots & \frac{\partial\phi_{1q}}{\partial K_{nn}} \\
\frac{\partial\phi_{2q}}{\partial K_{11}} & \frac{\partial\phi_{2q}}{\partial K_{12}} & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial\phi_{sq}}{\partial K_{11}} & \frac{\partial\phi_{sq}}{\partial K_{12}} & \cdots & \frac{\partial\phi_{sq}}{\partial K_{nn}}
\end{bmatrix} \quad : \text{ Size } (q*s) \times n^2 \qquad (3.70b)
$$

In the above, the first subscript on ϕ indicates the degree-of-freedom and the second subscript denotes the mode number.

The eigenvalue (λ) and eigenvector (ϕ) derivatives in the above expressions are presented later in this chapter; the submatrix [A] is discussed in the last section.

<u>Forced Vibration.</u> For a forced vibration analysis, Q comprises the nodal displacements, velocities, and accelerations (at each time step) obtained from the solution of the equations of motion :

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{P(t)\} \tag{1.1}$$

Details of the technique used for the solution are presented later in this chapter. The solution can be represented as

$$\{Q\}=\left[\{x_*\}\ \{\dot{x}_*\}\ \{\ddot{x}_*\}\right]^T \tag{3.71}$$

where the subscript '*' indicates the solution obtained at the current time step.

For this response, the sensitivity matrix is given by

$$[T = \left[\frac{\partial Q}{\partial r}\right] = \begin{bmatrix} \left[\dfrac{\partial x_*}{\partial r}\right] \\ \left[\dfrac{\partial \dot{x}_*}{\partial r}\right] \\ \left[\dfrac{\partial \ddot{x}_*}{\partial r}\right] \end{bmatrix} \tag{3.72}$$

After application of the chain rule of differentiation, [T] can be written as

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \begin{bmatrix} \left[\frac{\partial x_*}{\partial M}\right] & \left[\frac{\partial x_*}{\partial C}\right] & \left[\frac{\partial x_*}{\partial K}\right] \\ \left[\frac{\partial \dot{x}_*}{\partial M}\right] & \left[\frac{\partial \dot{x}_*}{\partial C}\right] & \left[\frac{\partial \dot{x}_*}{\partial K}\right] \\ \left[\frac{\partial \ddot{x}_*}{\partial M}\right] & \left[\frac{\partial \ddot{x}_*}{\partial C}\right] & \left[\frac{\partial \ddot{x}_*}{\partial K}\right] \end{bmatrix} \begin{bmatrix} \left[\frac{\partial M}{\partial r}\right] \\ \left[\frac{\partial C}{\partial r}\right] \\ \left[\frac{\partial K}{\partial r}\right] \end{bmatrix} \tag{3.73}$$

The damping matrix [C] will not explicitly appear in the analysis; rather, it will be formed from the mass and stiffness matrices according to the form of Rayleigh damping :

$$[C] = \alpha[M] + \beta[K] \tag{1.2}$$

The solution for the kinematics (x_*, $\dot{x}_*$, $\ddot{x}_*$) will be found to be an explicit function of the mass and stiffness matrices and of the damping coefficients α and β. Thus, the sensitivity of the kinematics with respect to the damping matrix (e.g., $\partial x_*/\partial C$) will be accounted for when calculating $\partial x_* \partial M$, $\partial x_*/\partial K$, etc., since α and β will appear in the equations. This will be described later in this chapter.

Using Rayleigh damping therefore allows Eq. (3.73) to be collapsed to

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \begin{bmatrix} \left[\frac{\partial x_*}{\partial M}\right] & \left[\frac{\partial x_*}{\partial K}\right] \\ \left[\frac{\partial \dot{x}_*}{\partial M}\right] & \left[\frac{\partial \dot{x}_*}{\partial K}\right] \\ \left[\frac{\partial \ddot{x}_*}{\partial M}\right] & \left[\frac{\partial \ddot{x}_*}{\partial K}\right] \end{bmatrix} \begin{bmatrix} \left[\frac{\partial M}{\partial r}\right] \\ \left[\frac{\partial K}{\partial r}\right] \end{bmatrix} \tag{3.74}$$

Upon comparison of this with the form shown in Eq. (3.59), the sensitivity submatrix [B] for forced vibration may be written as

$$[B] = \begin{bmatrix} \left[\dfrac{\partial x_*}{\partial M}\right] & \left[\dfrac{\partial x_*}{\partial K}\right] \\ \left[\dfrac{\partial \dot{x}_*}{\partial M}\right] & \left[\dfrac{\partial \dot{x}_*}{\partial K}\right] \\ \left[\dfrac{\partial \ddot{x}_*}{\partial M}\right] & \left[\dfrac{\partial \ddot{x}_*}{\partial K}\right] \end{bmatrix} \tag{3.75}$$

Submatrix [A] corresponds to the last term in Eq. (3.74); it is the same as that given in Eq. (3.58b).

Recall that a full set of experimental data (Q_m) need not be measured. Define :

p = number of measured x

q = number of measured $\dot{x}$

s = number of measured $\ddot{x}$

nd = p+q+s = total number of measured data

np = number of parameters to be identified

n = number of d.o.f. in finite element model

With these definitions, the dimensions of the matrices are thus

[T] : nd x np

[B] : nd x $2n^2$

[A] : $2n^2$ x np

which are the same as in the identification problem for free vibration.

The terms of submatrix [B] shown in Eq. (3.75) are arranged in the following order :

$$\left[\frac{\partial x_*}{\partial M}\right] = \begin{bmatrix} \frac{\partial x_{*1}}{\partial M_{11}} & \frac{\partial x_{*1}}{\partial M_{12}} & \cdots & \frac{\partial x_{*1}}{\partial M_{nn}} \\ \frac{\partial x_{*2}}{\partial M_{11}} & \frac{\partial x_{*2}}{\partial M_{12}} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x_{*p}}{\partial M_{11}} & \frac{\partial x_{*p}}{\partial M_{12}} & \cdots & \frac{\partial x_{*p}}{\partial M_{nn}} \end{bmatrix} \quad : \text{ Size } p \times n^2 \tag{3.76a}$$

$$\left[\frac{\partial x}{\partial K}\right] = \begin{bmatrix} \frac{\partial x_{*1}}{\partial K_{11}} & \frac{\partial x_{*1}}{\partial K_{12}} & \cdots & \frac{\partial x_{*1}}{\partial K_{nn}} \\ \frac{\partial x_{*2}}{\partial K_{11}} & \frac{\partial x_{*2}}{\partial K_{12}} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x_{*p}}{\partial K_{11}} & \frac{\partial x_{*p}}{\partial K_{12}} & \cdots & \frac{\partial x_{*p}}{\partial K_{nn}} \end{bmatrix} \quad : \text{ Size } p \times n^2 \tag{3.76b}$$

$$\left[\frac{\partial \dot{x}_*}{\partial M}\right] = \text{Same order} \quad : \text{ Size } q \times n^2 \tag{3.76c}$$

$$\left[\frac{\partial \dot{x}_*}{\partial K}\right] = \text{Same order} \quad : \text{ Size } q \times n^2 \tag{3.76d}$$

$$\left[\frac{\partial \ddot{x}_*}{\partial M}\right] = \text{Same order} \quad : \text{ Size } s \times n^2 \tag{3.76e}$$

$$\left[\frac{\partial \ddot{x}_*}{\partial K}\right] = \text{Same order} \quad : \text{ Size } s \times n^2 \tag{3.76f}$$

The kinematic derivatives in the above expressions are derived later in this chapter. The submatrix [A] is discussed in the last section.

Equations (3.74) through (3.76) are applicable to the identification of all element-level parameters (e.g., element areas) and the structure-level parameters (concentrated masses and stiffnesses). Later in this chapter, it is shown that the kinematic derivatives are explicit functions of the structure mass and stiffness matrices which, in turn, are explicit functions of these parameters. Note that the damping matrix [C] was eliminated from Eq. (3.73) through the assumption of Rayleigh damping; thus, the term $[\partial C/\partial r]$ is not accounted for in the equations already presented. However, since damping is accounted for by using $[C] = \alpha[M] + \beta[K]$, the equations which give the solution for the kinematics x_*, $\dot{x}_*$, and $\ddot{x}_*$ will be explicit functions of α and β, as well as [M] and [K] (60). Thus, for identification of the Rayleigh damping coefficients, Eq. (3.72) can be expressed directly in terms of these two parameters as

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \begin{bmatrix} \left[\dfrac{\partial x_*}{\partial r}\right] \\ \left[\dfrac{\partial \dot{x}_*}{\partial r}\right] \\ \left[\dfrac{\partial \ddot{x}_*}{\partial r}\right] \end{bmatrix} = \begin{bmatrix} \left[\dfrac{\partial x_*}{\partial \alpha}\right] & \left[\dfrac{\partial x_*}{\partial \beta}\right] \\ \left[\dfrac{\partial \dot{x}_*}{\partial \alpha}\right] & \left[\dfrac{\partial \dot{x}_*}{\partial \beta}\right] \\ \left[\dfrac{\partial \ddot{x}_*}{\partial \alpha}\right] & \left[\dfrac{\partial \ddot{x}_*}{\partial \beta}\right] \end{bmatrix} \tag{3.77}$$

For the sensitivity matrix in the form [T] = [B] [A], the sensitivity submatrix [B] for identification of damping parameters is given by

$$[B] = \begin{bmatrix} \left[\dfrac{\partial x_*}{\partial \alpha}\right] & \left[\dfrac{\partial x_*}{\partial \beta}\right] \\ \left[\dfrac{\partial \dot{x}_*}{\partial \alpha}\right] & \left[\dfrac{\partial \dot{x}_*}{\partial \beta}\right] \\ \left[\dfrac{\partial \ddot{x}_*}{\partial \alpha}\right] & \left[\dfrac{\partial \ddot{x}_*}{\partial \beta}\right] \end{bmatrix} \tag{3.78}$$

and submatrix [A] is the identity matrix, [I].

For identification of the damping coefficients, the number of parameters to identify, np, is equal to one or two (depending on whether one or both of the coefficients are to be identified). Therefore, the dimensions of the above matrices are

[T] : nd x np

[B] : nd x np

[A] : np x np (Identity Matrix)

The terms of the submatrix [B] shown in Eq. (3.78) are arranged in the following order :

$$\left[\frac{\partial x_*}{\partial \alpha}\right] = \begin{bmatrix} \dfrac{\partial x_{*1}}{\partial \alpha} \\ \dfrac{\partial x_{*2}}{\partial \alpha} \\ \vdots \\ \dfrac{\partial x_{*p}}{\partial \alpha} \end{bmatrix} \quad : \text{ Size } p \times 1 \tag{3.79a}$$

$$\left[\frac{\partial x_*}{\partial \beta}\right] = \begin{bmatrix} \dfrac{\partial x_{*1}}{\partial \beta} \\ \dfrac{\partial x_{*2}}{\partial \beta} \\ \vdots \\ \dfrac{\partial x_{*p}}{\partial \beta} \end{bmatrix} \quad : \text{ Size } p \times 1 \qquad (3.79b)$$

$$\left[\frac{\partial \dot{x}_*}{\partial \alpha}\right] = \text{Same order} \quad : \text{ Size } q \times 1 \qquad (3.79c)$$

$$\left[\frac{\partial \dot{x}_*}{\partial \beta}\right] = \text{Same order} \quad : \text{ Size } q \times 1 \qquad (3.79d)$$

$$\left[\frac{\partial \ddot{x}_*}{\partial \alpha}\right] = \text{Same order} \quad : \text{ Size } s \times 1 \qquad (3.79e)$$

$$\left[\frac{\partial \ddot{x}_*}{\partial \beta}\right] = \text{Same order} \quad : \text{ Size } s \times 1 \qquad (3.79f)$$

The kinematic derivatives in the above expressions are derived later in this chapter.

Forms of the Sensitivity Submatrix, [B]

The sensitivity matrix was decomposed into the product of two submatrices according to [T] = [B] [A], and forms of [B] were established for both free and forced vibration. Equations are presented in this section for the terms contained in the various forms of the [B] submatrix.

Free Vibration. The frequency and mode shape derivatives (sensitivities) indicated in Eqs. (3.69) and (3.70) can be obtained using perturbation techniques (61,62). They have been presented by Collins, et al. (4) and are as follows :

$$\frac{\partial \lambda_i}{\partial M_{uv}} = \frac{-\lambda_i \phi_{ui} \phi_{vi}}{\{\phi\}_i^T [M] \{\phi\}_i} \tag{3.80a}$$

$$\frac{\partial \lambda_i}{\partial K_{uv}} = \frac{\phi_{ui} \phi_{vi}}{\{\phi\}_i^T [M] \{\phi\}_i} \tag{3.80b}$$

$$\frac{\partial \phi_{ij}}{\partial M_{uv}} = \sum_{w=1}^{N} \left[\left(\frac{\lambda_i \phi_{iw} \phi_{uw} \phi_{vj}}{\lambda_j - \lambda_w} \right) \left(\delta_{wj} - 1 \right) - \frac{\phi_{ij} \phi_{uj} \phi_{vj}}{2} \right] \tag{3.81a}$$

$$\frac{\partial \phi_{ij}}{\partial K_{uv}} = \sum_{w=1}^{N} \left(\frac{\phi_{iw} \phi_{uw} \phi_{vj}}{\lambda_j - \lambda_w} \right) \left(1 - \delta_{wj} \right) \tag{3.81b}$$

In the above equations, N is the number of modes that must be included in the summation for convergence to be attained ($N \leq$ number of d.o.f.). Also, the summation convention on repeated indices is used, and δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 1 & : \ i = j \\ 0 & : \ i \neq j \end{cases} \tag{3.82}$$

It has been found by others (1,4), and is confirmed subsequently in this study, that only a few terms are typically needed in the summations to ensure convergence.

Forced Vibration. In this chapter, expressions for the kinematic sensitivities are derived from the relationships used to solve the equations of motion. Because these sensitivities are dependent on the method used to solve for the kinematics, this method will first be presented, followed by the derivation of the sensitivities.

To solve for the unknown kinematics (displacements, velocities, and accelerations), the equations of motion, restated below, are satisfied at discrete time points spaced Δt apart :

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{P(t)\} \tag{1.1}$$

For clarity of the following presentation, matrix notation will not be used in this section, except when presenting the final form of the kinematic sensitivity equations.

Although standard numerical methods for the solution of differential equations can be used to solve for x, $\dot{x}$, and $\ddot{x}$, other techniques are available which take advantage of the special nature of the terms in the above equation. One of the most widely used techniques is the Wilson Theta Method, which is an extension of the linear acceleration method. The Wilson Theta Method is a self-starting, implicit integration scheme and is unconditionally stable under conditions which are easily satisfied (60).

Assume that the displacements, velocities, and accelerations are known at the discrete time steps from time 0 to time t. To obtain the solution for the kinematics at time t+Δt by the Wilson Theta Method, equilibrium is established at some later time t+θΔt. For unconditional stability of the solution, $\theta \geq 1.37$; for convenience, $\theta = 1.4$ is usually used.

The acceleration is assumed to vary linearly from time t to time t+θΔt, as shown in Fig. 3. Thus, the acceleration at some intermediate time τ, measured from time t, is

$$\ddot{x}_{t+\tau} = \ddot{x}_t + \frac{\tau}{\theta\Delta t}\left(\ddot{x}_{t+\theta\Delta t} - \ddot{x}_t\right) \tag{3.83}$$

Integrating twice and evaluating the constants of integration from the boundary conditions

$$@\,\tau = 0 \;:\; \dot{x}_{t+\tau} = \dot{x}_t \tag{3.84a}$$
$$@\,\tau = 0 \;:\; x_{t+\tau} = x_t \tag{3.84b}$$

yields the velocity and displacement at any value of τ as :

$$\dot{x}_{t+\tau} = \dot{x}_t + \ddot{x}_t\tau + \frac{\tau^2}{2\theta\Delta t}\left(\ddot{x}_{t+\theta\Delta t} - \ddot{x}_t\right) \tag{3.85a}$$
$$x_{t+\tau} = x_t + \dot{x}_t\tau + \frac{\tau^2}{2}\ddot{x}_t + \frac{\tau^3}{6\theta\Delta t}\left(\ddot{x}_{t+\theta\Delta t} - \ddot{x}_t\right) \tag{3.85b}$$

For clarity of presentation, let the subscript '**' indicate time 't+θΔt', '*' indicate time 't+Δt', and the absence of a subscript indicate time 't'. This notation allows the accelerations to be written more compactly as

$$\ddot{x} = \ddot{x}_t \tag{3.86a}$$
$$\ddot{x}_* = \ddot{x}_{t+\Delta t} \tag{3.86b}$$
$$\ddot{x}_{**} = \ddot{x}_{t+\theta\Delta t} \tag{3.86c}$$

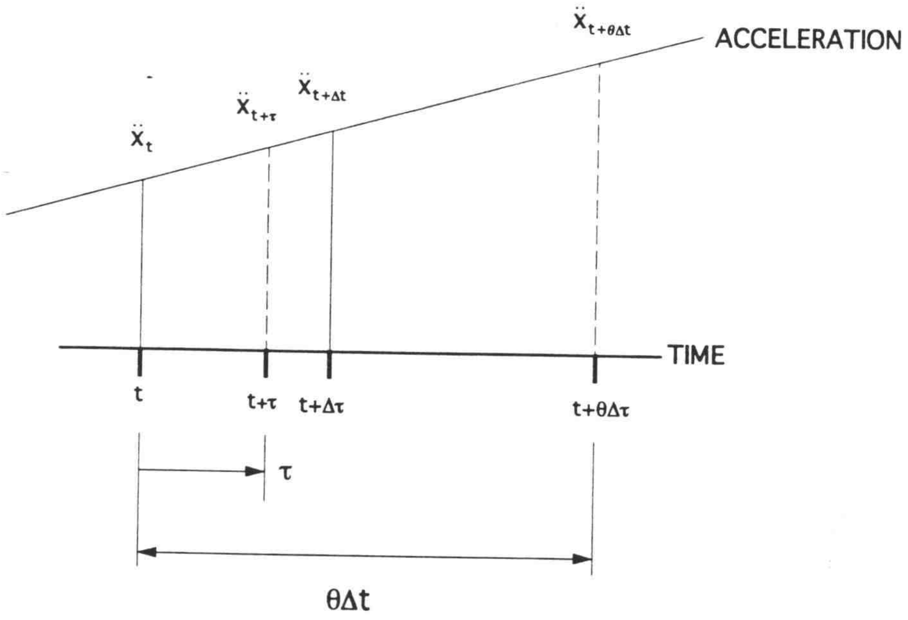


Figure 3. Definition Sketch for Wilson Theta Method

The other kinematic terms can be written in a similar fashion.

Evaluating the velocity and displacement equations at $\tau = \theta\Delta t$ (time $t+\theta\Delta t$), and using the notation defined above, one obtains

$$\dot{x}_{**} = \dot{x}_{t+\theta\Delta t} = \dot{x} + \frac{\theta\Delta t}{2}\left(\ddot{x}_{**} + \ddot{x}\right) \tag{3.87a}$$

$$x_{**} = x_{t+\theta\Delta t} = x + \theta\Delta t\,\dot{x} + \frac{(\theta\Delta t)^2}{6}\left(\ddot{x}_{**} + 2\ddot{x}\right) \tag{3.87b}$$

With expressions for the kinematics at the projected time step, equilibrium is established by substituting these kinematics into the equations of motion, Eq. (1.1) :

$$M\ddot{x}_{**} + C\dot{x}_{**} + Kx_{**} = \bar{R}_{**}(t) \tag{3.88}$$

The loading term is obtained by extrapolating the specified load to time $t+\theta\Delta t$; that is,

$$\bar{R}_{**}(t) = R + \theta(R_{*} - R) \tag{3.89}$$

where "R" denotes the applied loading P(t) in Eq. (1.1).

In order to find a solution to Eq. (3.88), the acceleration and velocity need to be expressed in terms of the displacement, x_{**}. Solve (3.87b) for $\ddot{x}_{**}$ in terms of x_{**} :

$$\ddot{x}_{**} = \frac{6}{(\theta\Delta t)^2}(x_{**} - x) - \frac{6\dot{x}}{\theta\Delta t} - 2\ddot{x} \tag{3.90}$$

Substitution of this equation into (3.87a) allows $\dot{x}_{**}$ to be expressed in terms of x_{**} as

$$\dot{x}_{**} = \frac{3}{\theta\Delta t}(x_{**} - x) - 2\dot{x} - \frac{\theta\Delta t}{2}\ddot{x} \tag{3.91}$$

Define the constants

$$a_0 = \frac{6}{(\theta\Delta t)^2} \qquad a_2 = \frac{6}{\theta\Delta t}$$
$$a_1 = \frac{3}{\theta\Delta t} \qquad a_3 = \frac{\theta\Delta t}{2} \tag{3.92}$$

and substitute Eqs. (3.90), (3.91), and (3.92) into Eq. (3.88). The equations of motion become

$$M(a_0 x_{**} - a_0 - a_2\dot{x} - 2\ddot{x}) + C(a_1 x_{**} - a_1 x - 2\dot{x} - a_3\ddot{x}) + + Kx_{**} = \overline{R}_{**} \tag{3.93}$$

Collect coefficients of x_{**} and bring all other terms to the right hand side of the equation to rewrite this equation as

$$(a_0 M + a_1 C + K)x_{**} = \overline{R}_{**} + M(a_0 x a_2\dot{x} + 2\ddot{x}) + + C(a_1 x + 2\dot{x} + a_3\ddot{x}) \tag{3.94}$$

Define the "effective stiffness" as

$$\hat{K} = a_0 M + a_1 C + K \tag{3.95}$$

and the "effective loads" as

$$\hat{R}_{**} = \overline{R}_{**} + M(a_0 x a_2\dot{x} + 2\ddot{x}) + C(a_1 x + 2\dot{x} + a_3\ddot{x}) \tag{3.96}$$

The equations of motion, Eq. (3.90), can then be written as

$$\hat{K} x_{**} = \hat{R}_{**} \tag{3.97}$$

A solution for x_{**}, the displacement at the projected time $t+\theta\Delta t$, can be obtained from this equation.

In order to recover the kinematics at time step $t+\Delta t$, Eq. (3.90) can be substituted into Eqs. (3.83) and (3.84), with $\tau = \Delta t$, to yield

$$\ddot{x}_* = a_4(x_{**}-x)+a_5\dot{x}+a_6\ddot{x} \tag{3.98a}$$

$$\dot{x}_* = \dot{x}+a_7(\ddot{x}_*+\ddot{x}) \tag{3.98b}$$

$$x_* = x+a_9\dot{x}+a_8(\ddot{x}_*+2\ddot{x}) \tag{3.98c}$$

where

$$a_4 = \frac{6}{\theta^3\Delta t^2} \qquad a_7 = \frac{\Delta t}{2}$$
$$a_5 = -\frac{6}{\theta^2\Delta t} \qquad a_8 = \frac{\Delta t^2}{6} \tag{3.99}$$
$$a_6 = 1-\frac{3}{\theta} \qquad a_9 = \Delta t$$

The kinematic sensitivities can now be derived from the above relationships. The sensitivities are sought for the current time step, denoted by the subscript '*'. It is assumed that all of the kinematics at the current time step have been calculated, and that the kinematics and sensitivities from the previous time steps are available.

<u>Sensitivity $\partial\ddot{x}_*/\partial M_{ij}$:</u> To calculate the acceleration/mass sensitivity, the equations of motion must be expressed in terms of $\ddot{x}_*$ as the only unknown. To do this, Eq. (3.98a) is solved for x_{**} and the result is substituted into Eq. (3.93) to obtain

$$M\left(\frac{a_0}{a_4}\ddot{x}_* - \frac{a_0 a_5}{a_4}\dot{x} - \frac{a_0 a_6}{a_4} - a_2\dot{x} - 2\ddot{x}\right) + \\ + C\left(\frac{a_1}{a_4}\ddot{x}_* - \frac{a_1 a_5}{a_4}\dot{x} - \frac{a_1 a_6}{a_4}\ddot{x} - 2\dot{x} - a_3\ddot{x}\right) + \\ + K\left(\frac{\ddot{x}_*}{a_4} - \frac{a_5}{a_4}\dot{x} - \frac{a_6}{a_4}\ddot{x} + x\right) = \overline{R}_{**} \qquad (3.100)$$

Rearrange terms, and define the constants

$$\begin{array}{lll} b_1 = \dfrac{a_0}{a_4} & b_4 = \dfrac{a_1}{a_4} & b_7 = \dfrac{1}{a_4} \\ b_2 = \dfrac{a_0 a_6}{a_4} & b_5 = \dfrac{a_1 a_5}{a_4} & b_8 = \dfrac{a_5}{a_4} \\ b_3 = \dfrac{a_0 a_6}{a_4} & b_6 = \dfrac{a_1 a_6}{a_4} & b_9 = \dfrac{a_6}{a_4} \end{array} \qquad (3.101)$$

Thus,

$$(b_1 M + b_4 C + b_7 K)\,\ddot{x}_* = \overline{R}_{**} + M\,(b_2\dot{x} + b_3\ddot{x} + a_2\dot{x} + 2\ddot{x}) + \\ + C\,(b_5\dot{x} + b_6\ddot{x} + 2\dot{x} + a_3\ddot{x}) + K\,(b_8\dot{x} + b_9\ddot{x} - x) \qquad (3.102)$$

Combine constants by defining

$$\begin{array}{ll} g_1 = b_5 + 2 & g_3 = b_3 + 2 \\ g_2 = 4a_2 + b_2 & g_4 = b_6 + a_3 \end{array} \qquad (3.103)$$

This allows the equation of motion to be written as

$$(b_1 M + b_4 C + b_7 K)\,\ddot{x}_* = \overline{R}_{**} + M\,(g_2\dot{x} + g_3\ddot{x}) + \\ + C\,(g_1\dot{x} + g_4\ddot{x}) + K\,(b_8\dot{x} + b_9\ddot{x} - x) \qquad (3.104)$$

The acceleration/mass sensitivity can now be determined by taking the partial derivative of both sides of this equation with respect to M_{ij} :

$$\frac{\partial}{\partial M_{ij}}(b_1M + b_4C + b_7K)\,\ddot{x}_* + (b_1M + b_4C + b_7K)\,\frac{\partial \ddot{x}_*}{\partial M_{ij}} =$$
$$= \frac{\partial M}{\partial M_{ij}}(g_2\dot{x} + g_3\ddot{x}) + M\,\frac{\partial}{\partial M_{ij}}(g_2\dot{x} + g_3\ddot{x}) +$$
$$+ \frac{\partial C}{\partial M_{ij}}(g_1\dot{x} + g_4\ddot{x}) + C\,\frac{\partial}{\partial M_{ij}}(g_1\dot{x} + g_4\ddot{x}) + \qquad (3.105)$$
$$+ \frac{\partial K}{\partial M_{ij}}(b_8\dot{x} + b_9\ddot{x} - x) + K\,\frac{\partial}{\partial M_{ij}}(b_8\dot{x} + b_9\ddot{x} - x)$$

Performing the indicated differentiations, and solving for the unknown sensitivity, one obtains

$$\frac{\partial \ddot{x}_*}{\partial M_{ij}} = (b_1M + b_4C + b_7K)^{-1}\left[\left(-b_1\frac{\partial M}{\partial M_{ij}} - b_4\frac{\partial C}{\partial M_{ij}} - b_7\frac{\partial K}{\partial M_{ij}}\right)\ddot{x}_* + \right.$$
$$+ \frac{\partial M}{\partial M_{ij}}(g_2\dot{x} + g_3\ddot{x}) + M\left(g_2\frac{\partial \dot{x}}{\partial M_{ij}} + g_3\frac{\partial \ddot{x}}{\partial M_{ij}}\right) +$$
$$+ \frac{\partial C}{\partial M_{ij}}(g_1\dot{x} + g_4\ddot{x}) + C\left(g_1\frac{\partial \dot{x}}{\partial M_{ij}} + g_4\frac{\partial \ddot{x}}{\partial M_{ij}}\right) + \qquad (3.106)$$
$$\left. + \frac{\partial K}{\partial M_{ij}}(b_8\dot{x} + b_9\ddot{x} - x) + K\left(b_8\frac{\partial \dot{x}}{\partial M_{ij}} + b_9\frac{\partial \ddot{x}}{\partial M_{ij}} - \frac{\partial x}{\partial M_{ij}}\right)\right]$$

In this equation,

$$\frac{\partial K}{\partial M_{ij}} = 0 \qquad (3.107a)$$

since the stiffness matrix is assumed independent of the mass. The term $\partial M/\partial M_{ij}$ is a matrix with all terms equal to zero, except the term in the ij-th position (row i, column j), which is unity. This matrix is denoted as I_{ij}; that is,

$$I_{ij} = \frac{\partial M}{\partial M_{ij}} \qquad (3.107b)$$

With this notation, Eq. (106) can be simplified to

$$\begin{aligned}\frac{\partial \ddot{x}_*}{\partial M_{ij}} = (b_1M + b_4C + b_7K)^{-1} \Bigg[& \left(-b_1 I_{ij} - b_4 \frac{\partial C}{\partial M_{ij}}\right) \ddot{x}_* + \\ & + I_{ij}\,(g_2\dot{x} + g_3\ddot{x}) + M\left(g_2 \frac{\partial \dot{x}}{\partial M_{ij}} + g_3 \frac{\partial \ddot{x}}{\partial M_{ij}}\right) + \\ & + \frac{\partial C}{\partial M_{ij}}\,(g_1\dot{x} + g_4\ddot{x}) + C\left(g_1 \frac{\partial \dot{x}}{\partial M_{ij}} + g_4 \frac{\partial \ddot{x}}{\partial M_{ij}}\right) + \\ & + K\left(b_8 \frac{\partial \dot{x}}{\partial M_{ij}} + b_9 \frac{\partial \ddot{x}}{\partial M_{ij}} - \frac{\partial x}{\partial M_{ij}}\right) \Bigg]\end{aligned} \tag{3.108}$$

Since matrix multiplication is distributive and associative, this equation can be rearranged as

$$\begin{aligned}\frac{\partial \ddot{x}_*}{\partial M_{ij}} = (b_1M + b_4C + b_7K)^{-1} \Bigg[& -\left(b_1 I_{ij} + b_4 \frac{\partial C}{\partial M_{ij}}\right) \ddot{x}_* + \\ & + \left(g_2 I_{ij} + g_1 \frac{\partial C}{\partial M_{ij}}\right)\dot{x} + \left(g_3 I_{ij} + g_4 \frac{\partial C}{\partial M_{ij}}\right)\ddot{x} + \\ & - K \frac{\partial x}{\partial M_{ij}} + (g_2M + g_1C + b_8K)\frac{\partial \dot{x}}{\partial M_{ij}} + \\ & + (g_3M + g_4C + b_9K)\frac{\partial \ddot{x}}{\partial M_{ij}} \Bigg]\end{aligned} \tag{3.109}$$

For Rayleigh damping, recall that the damping matrix is expressed as a linear combination of the mass and stiffness matrices as

$$[C] = \alpha\,[M] + \beta\,[K] \tag{1.2}$$

When C is expressed in this fashion,

$$\frac{\partial C}{\partial M_{ij}} = \frac{\partial}{\partial M_{ij}}(\alpha M + \beta K) = \alpha\, I_{ij} \tag{3.110}$$

Substitute Eqs. (1.2) and (3.110) into Eq. (3.109), and define the constants

$$\begin{aligned} g_5 &= b_1 + \alpha b_4 & g_8 &= g_3 + \alpha g_4 \\ g_6 &= b_7 + \beta b_4 & g_9 &= b_9 + \beta g_4 \\ g_7 &= g_2 + \alpha g_1 & g_0 &= b_8 + \beta g_1 \end{aligned} \tag{3.111}$$

This yields the desired form of the acceleration/mass sensitivity :

$$\begin{aligned} \left\{\frac{\partial \ddot{x}_*}{\partial M_{ij}}\right\} = (g_5[M] + g_6[K])^{-1} \Bigg(& -g_5[I_{ij}]\{\ddot{x}_*\} + g_7[I_{ij}]\{\dot{x}\} + \\ & + g_8[I_{ij}]\{\ddot{x}\} - [K]\left\{\frac{\partial x}{\partial M_{ij}}\right\} + (g_7[M] + g_0[K])\left\{\frac{\partial \dot{x}}{\partial M_{ij}}\right\} \\ & + (g_8[M] + g_9[K])\left\{\frac{\partial \ddot{x}}{\partial M_{ij}}\right\}\Bigg) \end{aligned} \tag{3.112}$$

An examination of the above equation reveals that the acceleration/mass sensitivity vector is of size n x 1. The remaining vectors presented in this section are also this size. If experimental data are not available for all n degrees-of-freedom, then a subset of the n x 1 vector must be extracted. This is discussed in Chapter IV.

<u>Sensitivity $\partial \ddot{x}_*/\partial K_{ij}$:</u> If the form of Rayleigh damping in Eq. (1.2) is substituted into the equation of motion given by Eq. (3.104), along with the constants in Eq. (3.111), the result can be written as

$$\begin{aligned} (g_5 M + g_6 K)\, \ddot{x}_* = \overline{R}_{**} - Kx + (g_7 M + g_0 K)\, \dot{x} + \\ + (g_8 M + g_9 K)\, \ddot{x} \end{aligned} \tag{3.113}$$

Take the partial derivative of both sides of this equation with respect to K_{ij}, and note that

$$\frac{\partial M}{\partial K_{ij}} = 0 \tag{3.114a}$$

$$\frac{\partial K}{\partial K_{ij}} = I_{ij} \tag{3.114b}$$

$$\frac{\partial C}{\partial K_{ij}} = \beta \tag{3.114c}$$

Equation (3.113) then yields the desired form of the acceleration/mass sensitivity :

$$\left\{\frac{\partial \ddot{x}_*}{\partial K_{ij}}\right\} = (g_5[M]+g_6[K])^{-1}\Bigg(g_6[I_{ij}]\{\ddot{x}_*\}-[I_{ij}]\{x\}+ \\ + g_0[I_{ij}]\{\dot{x}\}+g_9[I_{ij}]\{\ddot{x}\}-[K]\left\{\frac{\partial x}{\partial K_{ij}}\right\}+ \\ + (g_7[M]+g_0[K])\left\{\frac{\partial \dot{x}}{\partial K_{ij}}\right\}+(g_8[M]+g_9[K])\left\{\frac{\partial \ddot{x}}{\partial K_{ij}}\right\}\Bigg) \tag{3.115}$$

<u>Sensitivity $\partial\dot{x}_*/\partial M_{ij}$:</u> In order to determine the velocity/mass sensitivity, the equation of motion given by Eq. (3.102) must be rewritten with $\dot{x}_*$ as the only unknown. This is done by solving Eq. (3.98b) for $\ddot{x}_*$ in terms of $\dot{x}_*$ as

$$\ddot{x}_* = \frac{1}{a_7}(\dot{x}_*-\dot{x}) - \ddot{x} \tag{3.116}$$

Substitution of this result into Eq. (3.102) gives

$$
\begin{aligned}
(b_1M + b_4C + b_7K)\,\dot{x}_* = a_7 \Big\{ \overline{R}_{**} &+ \\
+ M\left[\left(b_2 + a_2 + \frac{b_1}{a_7}\right)\dot{x} + (b_3 + 2 + b_1)\,\ddot{x}\right] &+ \\
+ C\left[\left(b_5 + 2 + \frac{b_4}{a_7}\right)\dot{x} + (b_6 + a_3 + b_4)\,\ddot{x}\right] &+ \\
+ K\left[\left(b_8 + \frac{b_7}{a_7}\right)\dot{x} + (b_9 + b_7)\,\ddot{x} - x\right]\Big\}&
\end{aligned}
\tag{3.117}
$$

Define the following constants :

$$
\begin{aligned}
f_1 &= a_7\left(b_2 + a_2 + \frac{b_1}{a_7}\right) & f_4 &= a_7\left(b_6 + a_3 + b_4\right) \\
f_2 &= a_7\left(b_3 + 2 + b_1\right) & f_5 &= a_7\left(b_8 + \frac{b_7}{a_7}\right) \\
f_3 &= a_7\left(b_5 + 2 + \frac{b_4}{a_7}\right) & f_6 &= a_7\left(b_9 + b_7\right)
\end{aligned}
\tag{3.118}
$$

Substitution of these constants into the previous equation yields

$$
\begin{aligned}
(b_1M + b_4C + b_7K)\,\dot{x}_* = a_7\overline{R}_{**} + M\,(f_1\dot{x} + f_2\ddot{x}) &+ \\
+ C\,(f_3\dot{x} + f_4\ddot{x}) + K\,(f_5\dot{x} + f_6\ddot{x} - x)&
\end{aligned}
\tag{3.119}
$$

Take the derivative of both sides of this equation with respect to M_{ij} to obtain

$$
\begin{aligned}
\frac{\partial}{\partial M_{ij}}(b_1M + b_4C + b_7K)\,\dot{x}_* + (b_1M + b_4C + b_7K)\,\frac{\partial \dot{x}_*}{\partial M_{ij}} &= \\
+ \frac{\partial M}{\partial M_{ij}}(f_1\dot{x} + f_2\ddot{x}) + M\left(f_1\frac{\partial \dot{x}}{\partial M_{ij}} + f_2\frac{\partial \ddot{x}}{\partial M_{ij}}\right) &+ \\
+ \frac{\partial C}{\partial M_{ij}}(f_3\dot{x} + f_4\ddot{x}) + C\left(f_3\frac{\partial \dot{x}}{\partial M_{ij}} + f_4\frac{\partial \ddot{x}}{\partial M_{ij}}\right) &+ \\
+ \frac{\partial K}{\partial M_{ij}}(f_5\dot{x} + f_6\ddot{x} - x) + K\left(f_5\frac{\partial \dot{x}}{\partial M_{ij}} + f_6\frac{\partial \ddot{x}}{\partial M_{ij}} - \frac{\partial x}{\partial M_{ij}}\right)&
\end{aligned}
\tag{3.120}
$$

Upon substitution of Eq. (3.107), the above equation can be solved for the sensitivity :

$$\frac{\partial \dot{x}_*}{\partial M_{ij}} = (b_1 M + b_4 C + b_7 K)^{-1} \Bigg[-\left(b_1 I_{ij} + b_4 \frac{\partial C}{\partial M_{ij}} \right) \dot{x}_* + + I_{ij}\,(f_1 \dot{x} + f_2 \ddot{x}) + M\left(f_1 \frac{\partial \dot{x}}{\partial M_{ij}} + f_2 \frac{\partial \ddot{x}}{\partial M_{ij}} \right) + + \frac{\partial C}{\partial M_{ij}} (f_3 \dot{x} + f_4 \ddot{x}) + C\left(f_3 \frac{\partial \dot{x}}{\partial M_{ij}} + f_4 \frac{\partial \ddot{x}}{\partial M_{ij}} \right) + + K\left(f_5 \frac{\partial \dot{x}}{\partial M_{ij}} + f_6 \frac{\partial \ddot{x}}{\partial M_{ij}} - \frac{\partial x}{\partial M_{ij}} \right) \Bigg] \tag{3.121}$$

Recall that

$$[C] = \alpha\,[M] + \beta\,[K] \tag{1.2}$$

$$\frac{\partial C}{\partial M_{ij}} = \alpha\, I_{ij} \tag{3.110}$$

$$\begin{aligned} g_5 &= b_1 + \alpha b_4 \\ g_6 &= b_7 + \beta b_4 \end{aligned} \tag{3.111}$$

and define new constants

$$\begin{aligned} f_7 &= f_1 + \alpha f_3 \qquad & f_9 &= f_2 + \alpha f_4 \\ f_8 &= f_5 + \beta f_3 \qquad & f_0 &= f_6 + \beta f_4 \end{aligned} \tag{3.122}$$

Substitution of the above into Eq. (3.121), and rearrangement of terms, results in the desired form of the velocity/mass sensitivity :

$$\left\{ \frac{\partial \dot{x}_*}{\partial M_{ij}} \right\} = (g_5 [M] + g_6 [K])^{-1} \Bigg(-g_5 [I_{ij}] \{\dot{x}_*\} + f_7 [I_{ij}] \{\dot{x}\} + + f_9 [I_{ij}] \{\ddot{x}\} - a_7 [K] \left\{ \frac{\partial x}{\partial M_{ij}} \right\} + + (f_7 [M] + f_8 [K]) \left\{ \frac{\partial \dot{x}}{\partial M_{ij}} \right\} + (f_9 [M] + f_0 [K]) \left\{ \frac{\partial \ddot{x}}{\partial M_{ij}} \right\} \Bigg) \tag{3.123}$$

Sensitivity $\partial \dot{x}_*/\partial K_{ij}$: Substitution of Rayleigh damping into Eq. (3.119), along with the previously defined constants, allows the equation of motion to be expressed as

$$(g_5 M + g_6 K)\, \dot{x}_* = a_7 \overline{R}_{**} - a_7 K x + (f_7 M + f_8 K)\, \dot{x} + \\ + (f_9 M + f_0 K)\, \ddot{x} \tag{3.124}$$

Taking the partial derivative, and using the simplifications given in Eqs. (3.114), one finds the desired form of the velocity/stiffness sensitivity as

$$\left\{\frac{\partial \dot{x}_*}{\partial K_{ij}}\right\} = (g_5[M] + g_6[K])^{-1}\Bigg(-g_6[I_{ij}]\{\dot{x}_*\} - a_7[I_{ij}]\{x\} + \\ + f_8[I_{ij}]\{\dot{x}\} + f_0[I_{ij}]\{\ddot{x}\} - a_7[K]\left\{\frac{\partial x}{\partial K_{ij}}\right\} + \\ + (f_7[M] + f_8[K])\left\{\frac{\partial \dot{x}}{\partial K_{ij}}\right\} + (f_9[M] + f_0[K])\left\{\frac{\partial \ddot{x}}{\partial K_{ij}}\right\}\Bigg) \tag{3.125}$$

Sensitivity $\partial x_*/\partial M_{ij}$: Before the displacement/mass sensitivity can be derived, the equation of motion must be expressed on terms of x_* as the only unknown. This is done by solving Eq. (3.98c) for $\ddot{x}_*$ as a function of x_*, and substituting the result into Eq. (3.102). Thus

$$\ddot{x}_* = \frac{1}{a_8}(x_* - x - a_9 \dot{x}) - 2\ddot{x} \tag{3.126}$$

and the equation of motion becomes

$$\begin{aligned}(b_1M + b_4C + b_7K)\ x_* = a_8 \Bigg\{ & \bar{R}_{**} + \\ + M\left[\left(\frac{b_1}{a_8}\right)x + \left(b_2 + a_2 + \frac{b_1a_9}{a_8}\right)\dot{x} + (b_3 + 2 + 2b_1)\ \ddot{x}\right] + \\ + C\left[\left(\frac{b_4}{a_8}\right)x + \left(b_5 + 2 + \frac{b_4a_9}{a_8}\right)\dot{x} + (b_6 + a_3 + 2b_4)\ \ddot{x}\right] + \\ + K\left[\left(\frac{b_7}{a_8} - 1\right)x + \left(b_8 + \frac{b_7a_9}{a_8}\right)\dot{x} + (b_9 + 2b_7)\ \ddot{x}\right]\Bigg\}\end{aligned} \qquad (3.127)$$

Define

$$\begin{aligned} h_0 &= a_8\left(\frac{b_7}{a_8} - 1\right) & h_4 &= a_8\left(b_3 + 2 + 2b_1\right) \\ h_1 &= a_8\left(b_2 + a_2 + \frac{b_1\,a_9}{a_8}\right) & h_5 &= a_8\left(b_6 + a_3 + 2b_4\right) \\ h_2 &= a_8\left(b_5 + 2 + \frac{b_4a_9}{a_8}\right) & h_6 &= a_8\left(b_9 + 2b_7\right) \\ h_3 &= a_8\left(b_8 + \frac{b_7\,a_9}{a_8}\right) \end{aligned} \qquad (3.128)$$

and substitute these constants into the previous equation to obtain

$$\begin{aligned}(b_1M + b_4C + b_7K)\ x_* = a_8\bar{R}_{**} + M\,(b_1x + h_1\dot{x} + h_4\ddot{x}) + \\ + C\,(b_4x + h_2\dot{x} + h_5\ddot{x}) + K\,(h_0x + h_3\dot{x} + h_6\ddot{x} - x)\end{aligned} \qquad (3.129)$$

Take the derivative of both sides of the equation :

$$\begin{aligned}\frac{\partial}{\partial M_{ij}}\,(b_1M + b_4C + b_7K)\ x_* + (b_1M + b_4C + b_7K)\,\frac{\partial x_*}{\partial M_{ij}} = \\ = \frac{\partial M}{\partial M_{ij}}\,(b_1x + h_1\dot{x} + h_4\ddot{x}) + M\left(b_1\frac{\partial x}{\partial M_{ij}} + h_1\frac{\partial \dot{x}}{\partial M_{ij}} + h_4\frac{\partial \ddot{x}}{\partial M_{ij}}\right) + \\ + \frac{\partial C}{\partial M_{ij}}\,(b_4x + h_2\dot{x} + h_5\ddot{x}) + C\left(b_4\frac{\partial x}{\partial M_{ij}} + h_2\frac{\partial \dot{x}}{\partial M_{ij}} + h_5\frac{\partial \ddot{x}}{\partial M_{ij}}\right) + \\ + \frac{\partial K}{\partial M_{ij}}\,(h_0x + h_3\dot{x} + h_6\ddot{x}) + K\left(h_0\frac{\partial x}{\partial M_{ij}} + h_3\frac{\partial \dot{x}}{\partial M_{ij}} + h_6\frac{\partial \ddot{x}}{\partial M_{ij}}\right)\end{aligned}$$

(3.130)

With the simplification given by Eqs. (3.107), the above equation can be solved for the sensitivity :

$$\frac{\partial x_*}{\partial M_{ij}} = (g_5 M + g_6 K)^{-1} \left[-\left(b_1 I_{ij} + b_4 \frac{\partial C}{\partial M_{ij}} \right) x_* + \right.$$
$$+ I_{ij}(b_1 x + h_1 \dot{x} + h_4 \ddot{x}) + M\left(b_1 \frac{\partial x}{\partial M_{ij}} + h_1 \frac{\partial \dot{x}}{\partial M_{ij}} + h_4 \frac{\partial \ddot{x}}{\partial M_{ij}} \right) +$$
$$+ \frac{\partial C}{\partial M_{ij}}(b_4 x + h_2 \dot{x} + h_5 \ddot{x}) + C\left(b_4 \frac{\partial x}{\partial M_{ij}} + h_2 \frac{\partial \dot{x}}{\partial M_{ij}} + h_5 \frac{\partial \ddot{x}}{\partial M_{ij}} \right) + \tag{3.131}$$
$$\left. + K\left(h_8 \frac{\partial x}{\partial M_{ij}} + h_3 \frac{\partial \dot{x}}{\partial M_{ij}} + h_6 \frac{\partial \ddot{x}}{\partial M_{ij}} \right) \right]$$

Recall Eqs. (1.2), (3.110), and (3.111), and define

$$\begin{array}{ll} h_7 = h_0 + \beta b_4 & e_0 = h_4 + \alpha h_5 \\ h_8 = h_1 + \alpha h_2 & e_1 = h_6 + \beta h_5 \\ h_9 = h_3 + \beta h_2 & \end{array} \tag{3.132}$$

Substitution of these constants into Eq. (3.131), and subsequent rearrangement of the equation, results in the desired form of the displacement/mass sensitivity :

$$\left\{ \frac{\partial x_*}{\partial M_{ij}} \right\} = (g_5 [M] + g_6 [K])^{-1} \left(-g_5 [I_{ij}] \{x_*\} + \right.$$
$$+ g_5 [I_{ij}] \{x\} + h_8 [I_{ij}] \{\dot{x}\} + e_0 [I_{ij}] \{\ddot{x}\} +$$
$$+ (g_5 [M] + h_7 [K]) \left\{ \frac{\partial x}{\partial M_{ij}} \right\} + \tag{3.133}$$
$$+ (h_8 [M] + h_9 [K]) \left\{ \frac{\partial \dot{x}}{\partial M_{ij}} \right\} +$$
$$\left. + (e_0 [M] + e_1 [K]) \left\{ \frac{\partial \ddot{x}}{\partial M_{ij}} \right\} \right)$$

Sensitivity $\partial x_* / \partial K_{ij}$: If the necessary constants are substituted into the equation of motion given by Eq. (3.129), along with the assumption of Rayleigh damping, the following is obtained :

$$(g_5 M + g_6 K)\, x_* = a_8 \overline{R}_{**} + (g_5 M + h_7 K)\, x + + (h_8 M + h_9 K)\, \dot{x} + (e_0 M + e_1 K)\, \ddot{x} \qquad (3.134)$$

Upon differentiation, and simplification according to Eqs. (3.114), the above equation yields the desired form of the displacement/stiffness sensitivity :

$$\begin{aligned} \left\{ \frac{\partial x_*}{\partial K_{ij}} \right\} = (g_5 [M] + g_6 [K])^{-1} \Bigg(& -g_6 [I_{ij}] \{x_*\} + \\ & + h_7 [I_{ij}] \{x\} + h_9 [I_{ij}] \{\dot{x}\} + e_1 [I_{ij}] \{\ddot{x}\} + \\ & + (g_5 [M] + h_7 [K]) \left\{ \frac{\partial x}{\partial K_{ij}} \right\} + \\ & + (h_8 [M] + h_9 [K]) \left\{ \frac{\partial \dot{x}}{\partial K_{ij}} \right\} + \\ & + (e_0 [M] + e_1 [K]) \left\{ \frac{\partial \ddot{x}}{\partial K_{ij}} \right\} \Bigg) \end{aligned} \qquad (3.135)$$

Sensitivity $\partial \ddot{x}_* / \partial \alpha$: Recall the equation of motion expressed in terms of the acceleration $\ddot{x}_*$:

$$(g_5 M + g_6 K)\, \ddot{x}_* = \overline{R}_{**} - Kx + (g_7 M + g_0 K)\, \dot{x} + + (g_8 M + g_9 K)\, \ddot{x} \qquad (3.113)$$

where the constants in this equation are given in Eq. (3.111). Take the derivative of both sides of this expression with respect to the first Rayleigh damping coefficient, α :

$$\frac{\partial}{\partial \alpha}(g_5M + g_6K)\,\ddot{x}_* + (g_5M + g_6K)\,\frac{\partial \ddot{x}_*}{\partial \alpha} = -K\,\frac{\partial x}{\partial \alpha} - \frac{\partial K}{\partial \alpha}\,x + $$
$$+ (g_7M + g_0K)\,\frac{\partial \dot{x}}{\partial \alpha} + \frac{\partial}{\partial \alpha}(g_7M + g_0K)\,\dot{x} + \tag{3.136}$$
$$+ (g_8M + g_9K)\,\frac{\partial \ddot{x}}{\partial \alpha} + \frac{\partial}{\partial \alpha}(g_8M + g_9K)\,\ddot{x}$$

Note that the constants g_5, g_7, and g_8 are functions of α; their derivatives are given by

$$\frac{\partial g_5}{\partial \alpha} = b_4 \tag{3.137a}$$

$$\frac{\partial g_7}{\partial \alpha} = g_1 \tag{3.137b}$$

$$\frac{\partial g_8}{\partial \alpha} = g_4 \tag{3.137c}$$

Since the mass and stiffness matrices are not functions of α and β, it follows that

$$\frac{\partial M}{\partial \alpha} = 0 \tag{3.138a}$$

$$\frac{\partial K}{\partial \alpha} = 0 \tag{3.138b}$$

Substituting Eqs. (3.137) and (3.138) into Eq. (3.136), one obtains the acceleration/alpha sensitivity as

$$\left\{\frac{\partial \ddot{x}_*}{\partial \alpha}\right\} = (g_5[M] + g_6[K])^{-1}\Bigg(-b_4[M]\{\ddot{x}_*\} + g_1[M]\{\dot{x}\} +$$
$$+ g_4[M]\{\ddot{x}\} - [K]\left\{\frac{\partial x}{\partial \alpha}\right\} + (g_7[M] + g_0[K])\left\{\frac{\partial \dot{x}}{\partial \alpha}\right\} +$$
$$+ (g_8[M] + g_9[K])\left\{\frac{\partial \ddot{x}}{\partial \alpha}\right\}\Bigg) \tag{3.139}$$

<u>Sensitivity $\partial\ddot{x}_*/\partial\beta$:</u> When the derivative with respect to β is taken of Eq. (3.113), the result is

$$\frac{\partial}{\partial\beta}(g_5M + g_6K)\,\ddot{x}_* + (g_5M + g_6K)\,\frac{\partial\ddot{x}_*}{\partial\beta} = -K\,\frac{\partial x}{\partial\beta} - \frac{\partial K}{\partial\beta}\,x + \\ + (g_7M + g_0K)\,\frac{\partial\dot{x}}{\partial\beta} + \frac{\partial}{\partial\beta}(g_7M + g_0K)\,\dot{x} + \\ + (g_8M + g_9K)\,\frac{\partial\ddot{x}}{\partial\beta} + \frac{\partial}{\partial\beta}(g_8M + g_9K)\,\ddot{x} \qquad (3.140)$$

Note that g_6, g_9, and g_0 are functions of β; their derivatives are found to be

$$\frac{\partial g_6}{\partial\beta} = b_4 \qquad (3.141a)$$

$$\frac{\partial g_9}{\partial\beta} = g_4 \qquad (3.141b)$$

$$\frac{\partial g_0}{\partial\beta} = g_1 \qquad (3.141c)$$

The derivatives of the mass and stiffness matrices are

$$\frac{\partial M}{\partial\beta} = 0 \qquad (3.142a)$$

$$\frac{\partial K}{\partial\beta} = 0 \qquad (3.142b)$$

since neither are a function of β. Substitution of the above into Eq. (3.140) will yield the acceleration/beta sensitivity :

$$\left\{\frac{\partial \ddot{x}_*}{\partial \beta}\right\} = (g_5[M]+g_6[K])^{-1}\Bigg(-b_4[K]\{\ddot{x}_*\}+g_1[K]\{\dot{x}\}+ + g_4[K]\{\ddot{x}\}-[K]\left\{\frac{\partial x}{\partial \beta}\right\}+(g_7[M]+g_0[K])\left\{\frac{\partial \dot{x}}{\partial \beta}\right\}+ + (g_8[M]+g_9[K])\left\{\frac{\partial \ddot{x}}{\partial \beta}\right\}\Bigg) \tag{3.143}$$

<u>Sensitivity $\partial\dot{x}_*/\partial\alpha$:</u> Equation (3.124) along with the constants given by Eqs. (3.111) and (3.122) express the equation of motion in terms of $\dot{x}_*$ as the only unknown. The derivative of this equation can be taken with respect to α, recognizing that the terms g_5, f_7, and f_9 are functions of α. The result, after simplification, is the desired velocity/alpha sensitivity :

$$\left\{\frac{\partial \dot{x}_*}{\partial \alpha}\right\} = (g_5[M]+g_6[K])^{-1}\Bigg(-b_4[M]\{\dot{x}_*\}+f_3[M]\{\dot{x}\}+ + f_4[M]\{\ddot{x}\}-a_7[K]\left\{\frac{\partial x}{\partial \alpha}\right\}+(f_7[M]+f_8[K])\left\{\frac{\partial \dot{x}}{\partial \alpha}\right\}+ + (f_9[M]+f_0[K])\left\{\frac{\partial \ddot{x}}{\partial \alpha}\right\}\Bigg) \tag{3.144}$$

<u>Sensitivity $\partial\dot{x}_*/\partial\beta$:</u> Beginning with the equation of motion given by Eq. (3.124), one may derive the velocity/beta sensitivity in a similar fashion :

$$\left\{\frac{\partial \dot{x}_*}{\partial \beta}\right\} = (g_5[M]+g_6[K])^{-1}\Bigg(-b_4[K]\{\dot{x}_*\}+f_3[K]\{\dot{x}\}+ + f_4[K]\{\ddot{x}\}-a_7[K]\left\{\frac{\partial x}{\partial \beta}\right\}+(f_7[M]+f_8[K])\left\{\frac{\partial \dot{x}}{\partial \beta}\right\}+ + (f_9[M]+f_0[K])\left\{\frac{\partial \ddot{x}}{\partial \beta}\right\}\Bigg) \tag{3.145}$$

Sensitivity $\partial x_*/\partial\alpha$: The equation of motion given by Eq. (3.134), in conjunction with the constants defined in Eqs. (3.111) and (3.132), contains the displacement x_* as the only unknown. Following the same procedure as above, one finds the displacement/alpha sensitivity to be

$$\left\{\frac{\partial x_*}{\partial\alpha}\right\} = (g_5[M]+g_6[K])^{-1}\left(-b_4[M]\{x_*\}+b_4[M]\{x\}+\right.$$
$$+\ h_2[M]\{\dot{x}\}+h_5[M]\{\ddot{x}\}+(g_5[M]+h_7[K])\left\{\frac{\partial x}{\partial\alpha}\right\}+ \qquad (3.146)$$
$$\left.+(h_8[M]+h_9[K])\left\{\frac{\partial\dot{x}}{\partial\alpha}\right\}+(e_0[M]+e_1[K])\left\{\frac{\partial\ddot{x}}{\partial\alpha}\right\}\right)$$

Sensitivity $\partial x_*/\partial\beta$: In a similar manner, Eq. (3.134) can be differentiated with respect to β, and the result simplified to yield the displacement/beta sensitivity :

$$\left\{\frac{\partial x_*}{\partial\beta}\right\} = (g_5[M]+g_6[K])^{-1}\left(-b_4[K]\{x_*\}+b_4[K]\{x\}+\right.$$
$$+\ h_2[K]\{\dot{x}\}+h_5[K]\{\ddot{x}\}+(g_5[M]+h_7[K])\left\{\frac{\partial x}{\partial\beta}\right\}+$$
$$\left.+(h_8[M]+h_9[K])\left\{\frac{\partial\dot{x}}{\partial\beta}\right\}+(e_0[M]+e_1[K])\left\{\frac{\partial\ddot{x}}{\partial\beta}\right\}\right) \qquad (3.147)$$

Forms of the Sensitivity Submatrix, [A]

The sensitivity matrix [T], as given by Eq. (3.57), expresses the change in structural response to the change in the parameters as

$$[T] = \left[\frac{\partial Q}{\partial r}\right] = \left[\left[\frac{\partial Q}{\partial M}\right]\left[\frac{\partial Q}{\partial K}\right]\right]\left[\begin{array}{c}\left[\frac{\partial M}{\partial r}\right]\\ \left[\frac{\partial K}{\partial r}\right]\end{array}\right] \qquad (3.57)$$

or

$$[T] = [B][A] \tag{3.59}$$

where

$$[A] = \begin{bmatrix} \left[\dfrac{\partial M}{\partial r}\right] \\ \left[\dfrac{\partial K}{\partial r}\right] \end{bmatrix} \tag{3.58b}$$

In this last matrix, the n^2 x np terms of $[\partial M/\partial r]$ are arranged as follows :

$$\left[\frac{\partial M}{\partial r}\right] = \begin{bmatrix} \dfrac{\partial M_{11}}{\partial r_1} & \dfrac{\partial M_{11}}{\partial r_2} & \cdots & \dfrac{\partial M_{11}}{\partial r_{np}} \\ \dfrac{\partial M_{12}}{\partial r_1} & \dfrac{\partial M_{12}}{\partial r_2} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial M_{nn}}{\partial r_1} & \dfrac{\partial M_{nn}}{\partial r_2} & \cdots & \dfrac{\partial M_{nn}}{\partial r_{np}} \end{bmatrix} \tag{3.148}$$

The n^2 x np terms of $[\partial K/\partial r]$ are arranged in the same order.

The procedure for assembling [A] is now presented. This procedure differs for the identification of element parameters, lumped parameters, and damping parameters.

Element Parameters. In the finite element method, the structure mass and stiffness matrices, [M] and [K], are assembled from mass and stiffness matrices, [m] and [k], calculated at the element level. Similarly, the mass/parameter and stiffness/parameter sensitivities in submatrix [A] can be assembled from sensitivities calculated at the element level.

Consider the uniform, planar, elastic beam element shown in Fig. 4. The six degrees-of-freedom for the element are shown : they are the axial, transverse, and rotational displacements at the nodes. The structural parameters which dictate the mass and stiffness characteristics of the element are shown in the figure and are defined as follows :

I = Moment of inertia

A = Cross-sectional area

$\overline{m}$ = Distributed mass per unit length

E = Modulus of Elasticity

L = Length

The element stiffness matrix for the element (63) is symmetric and is given by

$$[k] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \qquad (3.149)$$

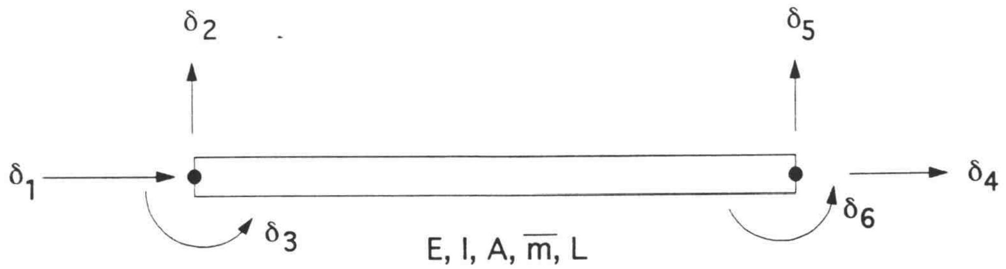


Figure 4. Beam Element

Two approaches are available for forming the element mass matrix, [m] : the consistent mass method, and the lumped mass method (59). The consistent mass method produces a fully-populated mass matrix, while the lumped mass method yields a diagonal matrix. For a large class of structural analysis problems, the lumped mass approach is of sufficient accuracy and is preferred because of the reduced solution effort which results (2). The lumped mass approach is used to form the element mass matrices in this study. The use of consistent mass matrices would not require any modifications to the identification procedure, other than the inclusion of additional off-diagonal terms.

In the lumped mass approach used in this study for forming [m], the mass of the element is lumped at the translational degrees-of-freedom (axial and transverse). The inertial effects corresponding to the rotational d.o.f. are assumed to be negligible. Thus, the element mass matrix is given by

$$[m] = \begin{bmatrix} \frac{\overline{m}L}{2} & & & & & \\ & \frac{\overline{m}L}{2} & & & & \\ & & 0 & & & \\ & & & \frac{\overline{m}L}{2} & & \\ & & & & \frac{\overline{m}L}{2} & \\ & & & & & 0 \end{bmatrix} \qquad (3.150)$$

where all off-diagonal terms are zero (59).

Recall that the sensitivity submatrix [A] is of the form

$$[A] = \begin{bmatrix} \left[\dfrac{\partial M}{\partial r} \right] \\ \left[\dfrac{\partial K}{\partial r} \right] \end{bmatrix} \tag{3.58b}$$

Consider the matrix [$\partial K/\partial r$] in this equation. It represents the stiffness/parameter sensitivities for the entire structure, and is assembled from the individual element sensitivities, [$\partial k/\partial r$], following the same procedure used to assemble the structure stiffness matrix from the element stiffness matrices.

The stiffness/parameter sensitivities for an element are obtained by taking the partial derivative of [k] with respect to the parameter being identified. For example, suppose that the moment of inertia of element "i" is to be identified and that it is the j-th parameter being identified; thus, $r_j = I_i$. Using Eq. (3.149), the element sensitivity is

$$\left[\frac{\partial k_i}{\partial r_j}\right] = \left[\frac{\partial k_i}{\partial I_i}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{12E}{L^3} & \dfrac{6E}{L^2} & 0 & \dfrac{-12E}{L^3} & \dfrac{6E}{L^2} \\ 0 & \dfrac{6E}{L^2} & \dfrac{4E}{L} & 0 & \dfrac{-6E}{L^2} & \dfrac{2E}{L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{-12E}{L^3} & \dfrac{-6E}{L^2} & 0 & \dfrac{12E}{L^3} & \dfrac{-6E}{L^2} \\ 0 & \dfrac{6E}{L^2} & \dfrac{2E}{L} & 0 & \dfrac{-6E}{L^2} & \dfrac{4E}{L} \end{bmatrix} \tag{3.151}$$

The matrix $[\partial M/\partial r]$ appearing in $[A]$ is assembled from the element sensitivities $[\partial m/\partial r]$ in a similar manner. Suppose $r_j = \overline{m}$, the element mass density. Then the mass/parameter sensitivities for the element are obtained by taking the partial derivative of $[m]$ with respect to the parameter $\overline{m}$:

$$\left[\frac{\partial m_i}{\partial r_j}\right] = \left[\frac{\partial m_i}{\partial \overline{m}}\right] = \begin{bmatrix} \frac{L}{2} & & & & & \\ & \frac{L}{2} & & & & \\ & & 0 & & & \\ & & & \frac{L}{2} & & \\ & & & & \frac{L}{2} & \\ & & & & & 0 \end{bmatrix} \tag{3.152}$$

The sensitivity submatrix $[A]$ formed in this fashion is used for both free vibration and forced vibration identification.

<u>Lumped Parameters.</u> Consider, now, the estimation of lumped structural parameters; i.e., concentrated masses and stiffnesses incorporated directly at the structure level in $[M]$ and $[K]$. Suppose that a concentrated spring stiffness K^L_{ii} is the j-th parameter to be identified; thus, $r_j = K^L_{ii}$. The K_{ii} term in the structure stiffness matrix consists of all of the contributions from the element stiffness matrices plus the concentrated stiffness K^L_{ii}. This term can be expressed as

$$K_{ii} = \Sigma\ k_{ii} + k^{L}_{ii} \tag{3.153}$$

where $\Sigma\ k_{ii}$ indicates assembly over all of the contributing elements. Thus, the stiffness/parameter sensitivity can be calculated at the structure level, and is given by

$$\left[\frac{\partial K}{\partial r_j}\right] = \left[\frac{\partial K}{\partial K^{L}_{ii}}\right] \tag{3.154}$$

It can be seen from Eq. (3.153) that

$$\frac{\partial K_{ii}}{\partial r_j} = \frac{\partial K_{ii}}{\partial K^{L}_{ii}} = 0 + 1 \tag{3.155}$$

The partial derivatives of all other terms in [K] are zero, since they do not contain K^{L}_{ii}.

A similar argument can be made for a concentrated mass M^{L}_{ii}; the outcome will be analogous.

For lumped parameter identification, submatrix [A] is therefore a matrix containing zeroes everywhere except at the locations corresponding to the concentrated masses and stiffnesses which are to be identified; these locations contain the value unity. The submatrix [A] formed in this manner is used for both free vibration and forced vibration identification.

Damping Parameters. In a previous section, it was shown that the equations which allow for the solution of the kinematics are explicit functions of the damping parameters,

α and β. This allows the sensitivity matrix to be expressed directly in terms of the kinematic/damping sensitivities as

$$[T] = \begin{bmatrix} \left[\dfrac{\partial x_*}{\partial \alpha}\right] & \left[\dfrac{\partial x_*}{\partial \beta}\right] \\ \left[\dfrac{\partial \dot{x}_*}{\partial \alpha}\right] & \left[\dfrac{\partial \dot{x}_*}{\partial \beta}\right] \\ \left[\dfrac{\partial \ddot{x}_*}{\partial \alpha}\right] & \left[\dfrac{\partial \ddot{x}_*}{\partial \beta}\right] \end{bmatrix} = [B][A] \qquad (3.156)$$

Thus, for the identification of the damping parameters, the submatrix [A] is simply the identity matrix, [I].

CHAPTER IV

ALGORITHM AND PROGRAM DEVELOPMENT

Introduction

A computer program was developed to implement the theory and procedures presented in this study. General features of the program are described in this chapter. Special programming techniques and algorithms used to execute the system identification theory are also described. Special attention is given to the calculation of the sensitivity submatrices [A] and [B], and to the assembly of the sensitivity matrix, [T]. Finally, important aspects of the weighting matrix are presented for the three estimation schemes available in the program.

Program Description

The program is divided into two distinct components. The first component uses the finite element method (FEM) for the analysis of plane frame structures. The second component consists of the code needed to set up and execute the parameter estimation using the system identification (SID) techniques presented in this study. Organizing the program in this manner allows the estimation algorithm to be kept entirely separate from the finite element code.

The program is written in FORTRAN and contains approximately 4000 statements. The FEM component consists of a main

The program is written in FORTRAN and contains approximately 4000 statements. The FEM component consists of a main routine and 28 subroutines; the SID component comprises 34 subroutines. This modular construction allows each subroutine to perform a specific, narrowly-defined function. An overlay structure is used to minimize memory requirements, as are a total of 37 files, both sequential and random access. These files are used to store the mass and stiffness matrices, load history (for forced vibration), experimental data, and sensitivity matrices.

The finite element portion of the program performs a linear analysis of two-dimensional, elastic frames composed of prismatic members. The program is capable of static, free vibration, and forced vibration analysis. The input consists of the data typically required for a FEM frame program.

The SID portion of the program is capable of identifying element and lumped parameters for free vibration response, and element, lumped, and damping parameters for forced vibration response. Any number of parameters may be identified simultaneously, and in any combination of the basic types (element, lumped, damping). For example, consider parameter estimation for forced vibration response. The user may request estimation of any or all of the following parameters, for as many elements as desired : moment of inertia (I), cross-sectional area (A), and mass density ($\bar{m}$). One or more concentrated masses (M^L) and/or concentrated stiffnesses (K^L) may also be identified at the same time, along with one or both of the Rayleigh damping coefficients, α and β.

Input for the identification component of the program consists of : 1) the scheme desired (LS, WLS, Bayes), 2) a list of the parameters to be identified, 3) measured data, 4) data and parameter variances, and 5) convergence criteria. Program printout consists of the improved estimates of the parameters and the response of the model based on these parameters.

Parameter estimation proceeds within an iteration loop which is terminated when convergence is attained. To define the convergence criteria, the user specifies the required tolerance on : 1) the percent change in the parameters, and 2) the percent difference between measured and calculated response. To minimize calculations, only quantities that are effected by changes in the parameters are recalculated within the iteration loop.

Free Vibration

A flowchart depicting the general progression of the system identification process for structures was presented in Fig. 2. The detailed flowchart shown in Fig. 5 is for free vibration identification as it is implemented in the computer program. In this figure, "M and K sensitivities" denotes $[\partial M/\partial r]$ and $[\partial K/\partial r]$; "λ and ϕ sensitivities" denotes

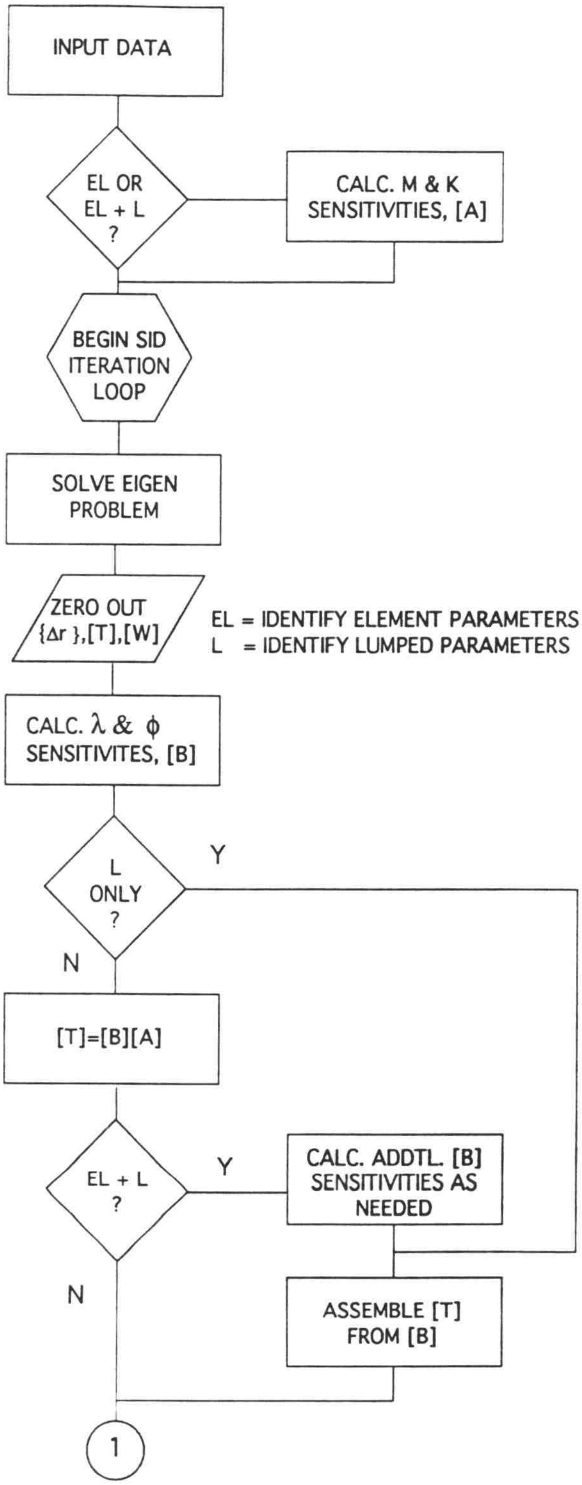


Figure 5. Flowchart for Free Vibration Identification

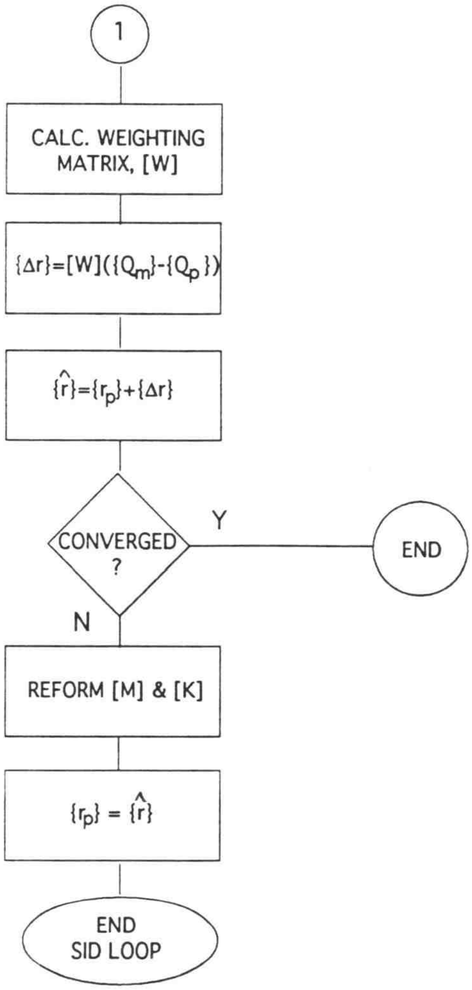


Figure 5 (Cont.). Flowchart for Free Vibration Identification

$[\partial\lambda/\partial M]$, $[\partial\lambda/\partial K]$, $[\partial\phi/\partial M]$, and $[\partial\phi/\partial K]$. Reference should be made to the flowchart for the following discussion.

Calculation of [A] and [B]. The two classes of parameters which may be estimated for a free vibration problem are element parameters and lumped parameters. The program begins by reading in all of the data for the finite element model, followed by the data needed for the system identification calculations. After all data are read, the sensitivity submatrix [A] is formed if any element parameters are to be identified. As previously discussed, [A] is assembled from sensitivities calculated at the element level using the same FEM routines used to assemble the structure mass and stiffness matrices from the element matrices.

Consider the matrix of stiffness/parameter sensitivities given by Eq. (3.151). It is apparent that the terms in this matrix would also be obtained if the stiffness matrix of element "i" is evaluated for a unit value of the moment of inertia; i.e., for $I_i = 1$. Thus, the matrix $[\partial k/\partial r]$ can be calculated by calling the finite element routine which calculates [k]. When calling this routine, a value of unity is passed for the parameter being identified.

The element level sensitivities are calculated in this manner for all of the moments of inertia and areas to be identified. These element level sensitivities are then assembled to form $[\partial K/\partial r]$, the stiffness/parameter sensitivities for the entire structure. The assembly is accomplished by calls to the finite element routine which assembles [K] from the element [k] matrices.

In the same manner, the mass/parameter sensitivities $[\partial M/\partial r]$ are assembled from sensitivities calculated at the element level. These sensitivities are calculated by calling the finite element routine for forming [m]; the routine is called with a unit value for $\overline{m}$, the mass density of the element.

Each call to the [m] or [k] routine calculates one column of submatrix [A]. After a column is calculated, the values are written to file. The logic for the entire process is contained in the SID component of the program.

To minimize computer resources when assembling [A], advantage is taken of the special form of [M] and [K]. Since [M] is a diagonal matrix, only the n sensitivities corresponding to the main diagonal are calculated and stored. Because [K] is a symmetric matrix, only the $(n^2+n)/2$ sensitivities corresponding to the upper triangle are calculated and stored. Thus, the size of [A] is reduced from n^2 x np to $[n + (n^2+n)/2]$ x np.

For the estimation of lumped parameters, [A] differs in content from the matrix used for element estimation. In the previous chapter, [A] was shown to consist mostly of zeroes, with unity appearing in np locations (np = number of parameters). It would be inefficient to form and store such a sparsely populated matrix. Therefore, a more efficient procedure is used to form [T] directly from [B] for lumped parameter identification. This procedure is described in the next section.

The SID iteration loop is now begun. The structure response (eigenvalues, λ and eigenvectors, ϕ) is calculated according to Eq. (3.64). This response is based on the prior (most recent) estimate of the parameters. For the first pass through the loop, the prior values of the parameters are the initial values used to construct the finite element model.

The sensitivities needed to form [B] can now be calculated. If only lumped parameters are to be identified, the λ and ϕ sensitivities of [B] are calculated, as needed, depending on the type of measured data available and the type of lumped parameters. For example, if experimental frequencies are available and concentrated masses are being estimated, only $[\partial\lambda/\partial M]$ is calculated. The sensitivity matrix [T] can be assembled directly from [B]; the procedure to do this is described in the next section.

If only element parameters are to be identified, the λ and ϕ sensitivities are calculated, as needed. For example, if mode shapes have been measured and element areas and mass densities are being estimated, $[\partial\phi/\partial M]$ and $[\partial\phi/\partial K]$ must be calculated.

If a combination of element and lumped parameters are to be identified, the λ and/or ϕ sensitivities required for the element parameters are first calculated. If any of the sensitivities needed for the lumped parameters were not calculated when processing the element parameters, they are now calculated.

In all cases, the λ and ϕ sensitivities which constitute the terms of matrix [B] are calculated according to Eqs.(3.69) and (3.70). As each column of [B] is calculated, it is written to file. Recognition is made of the diagonal form of [M] and the symmetry of [K]. Thus, sensitivities are only calculated and stored for the main diagonal of [M] and the upper triangle of [K]. As a result, the size of [B] is reduced from $p+(q*s) \times 2n^2$ to $p+(q*s) \times [n + (n^2+n)/2]$.

<u>Assembly of [T].</u> The sensitivity matrix [T] can now be formed. If lumped masses and stiffnesses are the only parameters being estimated, [T] can be assembled directly from [B] as follows.

For simplicity, only consider the frequency/stiffness sensitivity $[\partial\lambda/\partial K]$. Suppose the second parameter to be identified, r_2, is the concentrated stiffness $K^L{}_{ii}$. The total stiffness K_{ii} is made up of this term and the contributions from the elements. For this case, the sensitivity matrix is calculated from [T] = [B][A] and is given by

$$[T] = \begin{bmatrix} \frac{\partial\lambda_1}{\partial K_{11}} & \frac{\partial\lambda_1}{\partial K_{12}} & \cdots & \frac{\partial\lambda_1}{\partial K_{nn}} \\ \frac{\partial\lambda_2}{\partial K_{11}} & \frac{\partial\lambda_2}{\partial K_{12}} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial\lambda_p}{\partial K_{11}} & \frac{\partial\lambda_p}{\partial K_{12}} & \cdots & \frac{\partial\lambda_p}{\partial K_{nn}} \end{bmatrix} \begin{bmatrix} \frac{\partial K_{11}}{\partial r_1} & \frac{\partial K_{11}}{\partial r_2} & \cdots & \frac{\partial K_{11}}{\partial r_{np}} \\ \frac{\partial K_{12}}{\partial r_1} & \frac{\partial K_{12}}{\partial r_2} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial K_{nn}}{\partial r_1} & \frac{\partial K_{nn}}{\partial r_2} & \cdots & \frac{\partial K_{nn}}{\partial r_{np}} \end{bmatrix} \tag{4.1}$$

The form of the submatrix [A] used for lumped parameter identification was discussed in the previous chapter. This form allows the above equation to be written as

$$[T] = \begin{bmatrix} \frac{\partial\lambda_1}{\partial K_{11}} & \frac{\partial\lambda_1}{\partial K_{12}} & \cdots & \frac{\partial\lambda_1}{\partial K_{nn}} \\ \frac{\partial\lambda_2}{\partial K_{11}} & \frac{\partial\lambda_2}{\partial K_{12}} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial\lambda_p}{\partial K_{11}} & \frac{\partial\lambda_p}{\partial K_{12}} & \cdots & \frac{\partial\lambda_p}{\partial K_{nn}} \end{bmatrix} \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.2}$$

After multiplication, matrix [T] becomes

$$[T] = \begin{bmatrix} 0 & \frac{\partial\lambda_1}{\partial K_{11}} & \cdots & 0 \\ 0 & \frac{\partial\lambda_2}{\partial K_{11}} & \cdots & \cdots \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{\partial\lambda_p}{\partial K_{11}} & \cdots & 0 \end{bmatrix} \tag{4.3}$$

Thus, the first column of [B] becomes the second column of [T]. This result can be generalized as follows. Suppose $r_j = K^L{}_{ii}$. Determine the row of [A] which would contain the term $\partial K_{ii}/\partial r_j$, say row v. Then column j of [T] is equal to column p of [B]. The remaining columns of [T], corresponding to the remaining lumped stiffnesses, can be assembled in a similar fashion. This is the procedure used by the program to form [T] directly from [B].

The other sensitivities ($\partial\lambda/\partial M$, $\partial\phi/\partial M$, and $\partial\phi/\partial K$) are processed in the same fashion when assembling [T].

If any element parameters are to be estimated, the sensitivity matrix can be constructed according to [T] = [B][A]. However, the program does not perform this calculation using the standard algorithm for matrix multiplication. The complete [B] and [A] matrices are large and would occupy a significant amount of computer memory if they were retrieved from file in their entirety. Also, to perform the matrix multiplication in the usual manner, the values in [B] need to be retrieved by rows in order to multiply the columns of [A]. However, [B] and [A] are written to file by columns, and must be retrieved in the same order. Therefore, a special procedure is needed in the program to assemble [T] from [B] and [A].

The routine which does this reads a column of [B] from file, and then reads the first column of [A] for processing. It locates the terms in [A] which are to be multiplied by the terms in the column of [B]. The multiplications are performed, and the results are saved in [T]. The remaining columns of [A] are read from file one at a time and processed. The procedure is then repeated for the remaining columns of [B].

If both element and lumped parameters are to be identified, [T] is first assembled for all of the element parameters; it is then augmented for the lumped parameters.

Calculation of Improved Parameters. Once [T] is completely assembled, the weighting matrix [W] can be calculated according to the identification scheme chosen; i.e., LS, WLS, or Bayes. Equation (3.54a), (3.54b), or (3.54c) is used for this calculation. The changes in the parameters, $\{\Delta r\}$, are then calculated as

$$\{\Delta r\} = (\{\hat{r}\} - \{r_p\}) = [W](\{Q_m\} - \{Q(r_p)\}) \qquad (4.4)$$

For this calculation, the measured response $\{Q_m\}$ is read from file; $\{Q(r_p)\}$ (denoted as Q_p in the flowchart) is the response calculated by the program for the prior (most recent) values of the parameters. The improved estimate of the parameters, $\{\hat{r}\}$, is determined from

$$\{\hat{r}\} = \{r_p\} + \{\Delta r\} \qquad (4.5)$$

Convergence is now checked; if the convergence criteria have been met, the SID iteration loop is terminated, and the vector $\{\hat{r}\}$ represents the best estimate of the structural parameters. If the process has not converged, the vector $\{\hat{r}\}$ becomes the vector of prior estimates, $\{r_p\}$. If any element parameters have been modified, the corresponding element matrices [m] and [k] are recalculated to reflect the changes. The structure matrices [M] and/or [K] must now be updated, since they are effected by changes to both the element parameters and the lumped parameters. This SID iteration loop is repeated until convergence is attained, or until the maximum number of iterations specified by the user has been reached.

Forced Vibration

Figure 6 presents a detailed flowchart for forced vibration identification, as it is implemented in the computer program. In this figure, "M and K sensitivities" denotes $[\partial M/\partial r]$ and $[\partial K/\partial r]$; "Kin/M and Kin/K sensitivities" denotes $\{\partial x_*/\partial M\}$, $\{\partial x_*/\partial K\}$, etc.; and "Kin/$\alpha$ and "Kin/β sensitivities" denotes $\{\partial x_*/\partial \alpha\}$, $\{\partial x_*/\partial \beta\}$, etc. Reference should be made to the flowchart for the following discussion.

Calculation of [A] and [B]. Three classes of parameters may be estimated for a forced vibration problem : element parameters, lumped parameters, and damping parameters. After the program reads in all data, the sensitivity submatrix [A] is formed if any element parameters are to be identified. This is done using the same procedure described for free vibration identification.

The SID iteration loop is begun; within this loop is another loop on time. At each time step, the equations of

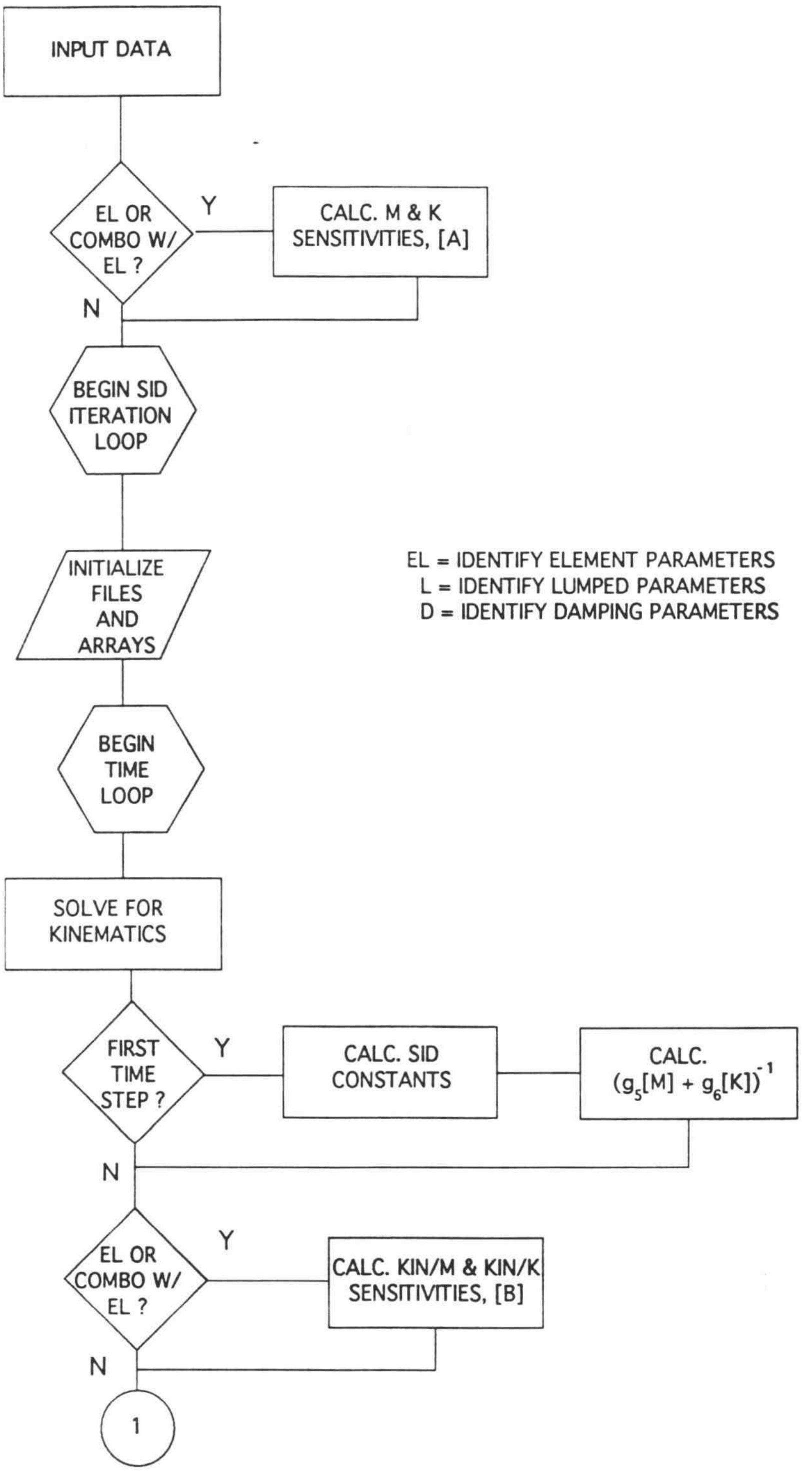


Figure 6. Flowchart for Forced Vibration Identification

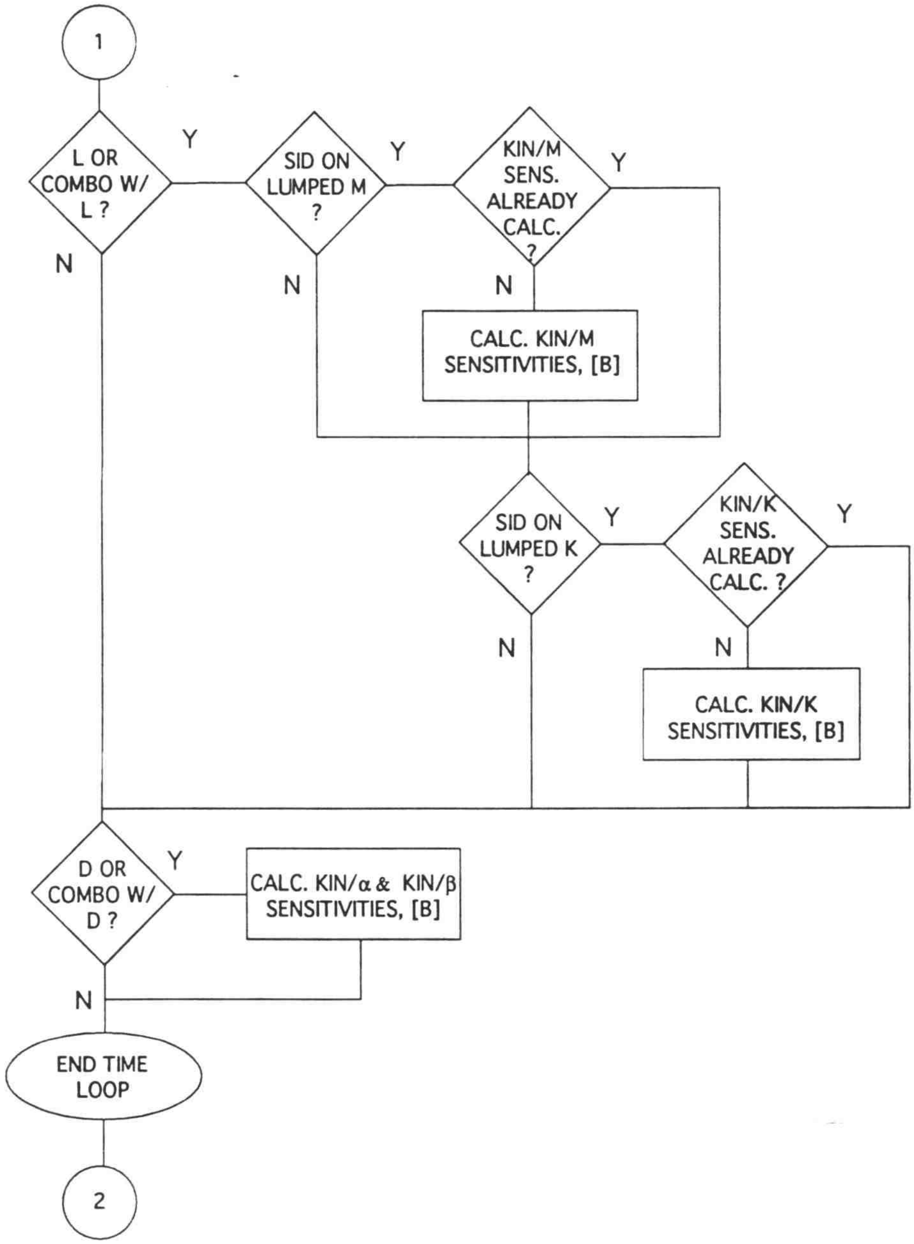


Figure 6 (Cont.). Flowchart for Forced Vibration Identification

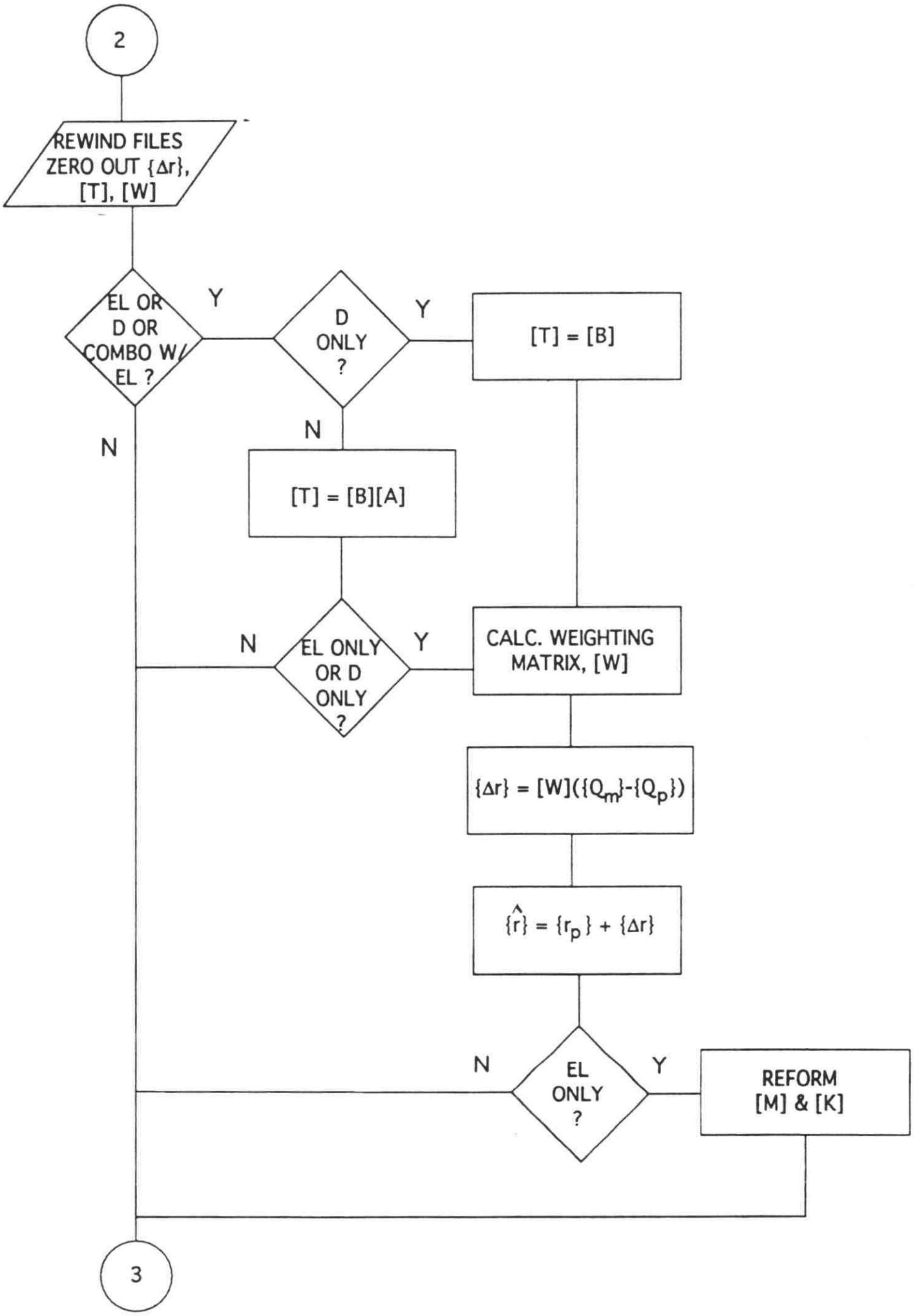


Figure 6 (Cont.). Flowchart for Forced Vibration Identification

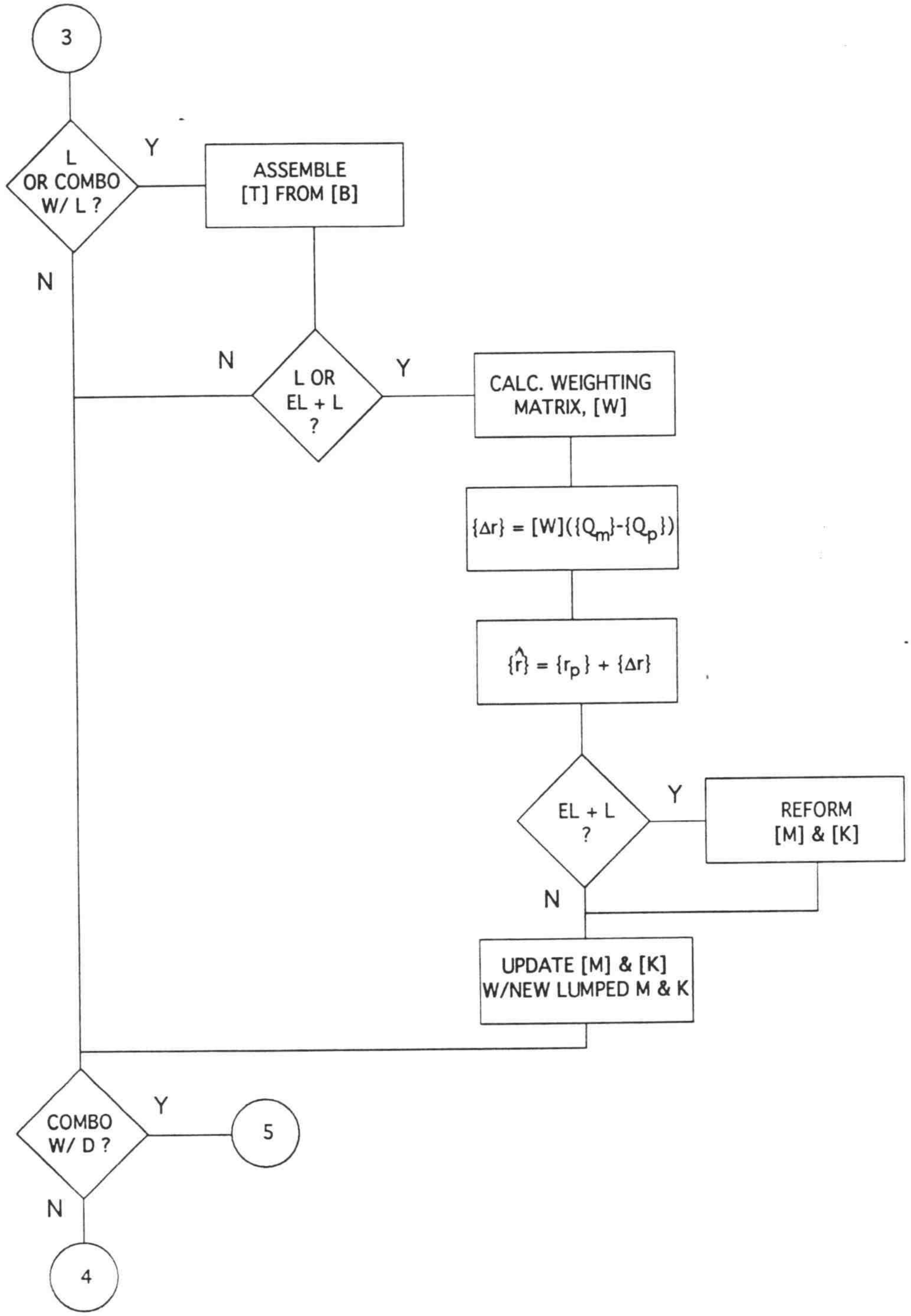


Figure 6 (Cont.). Flowchart for Forced Vibration Identification

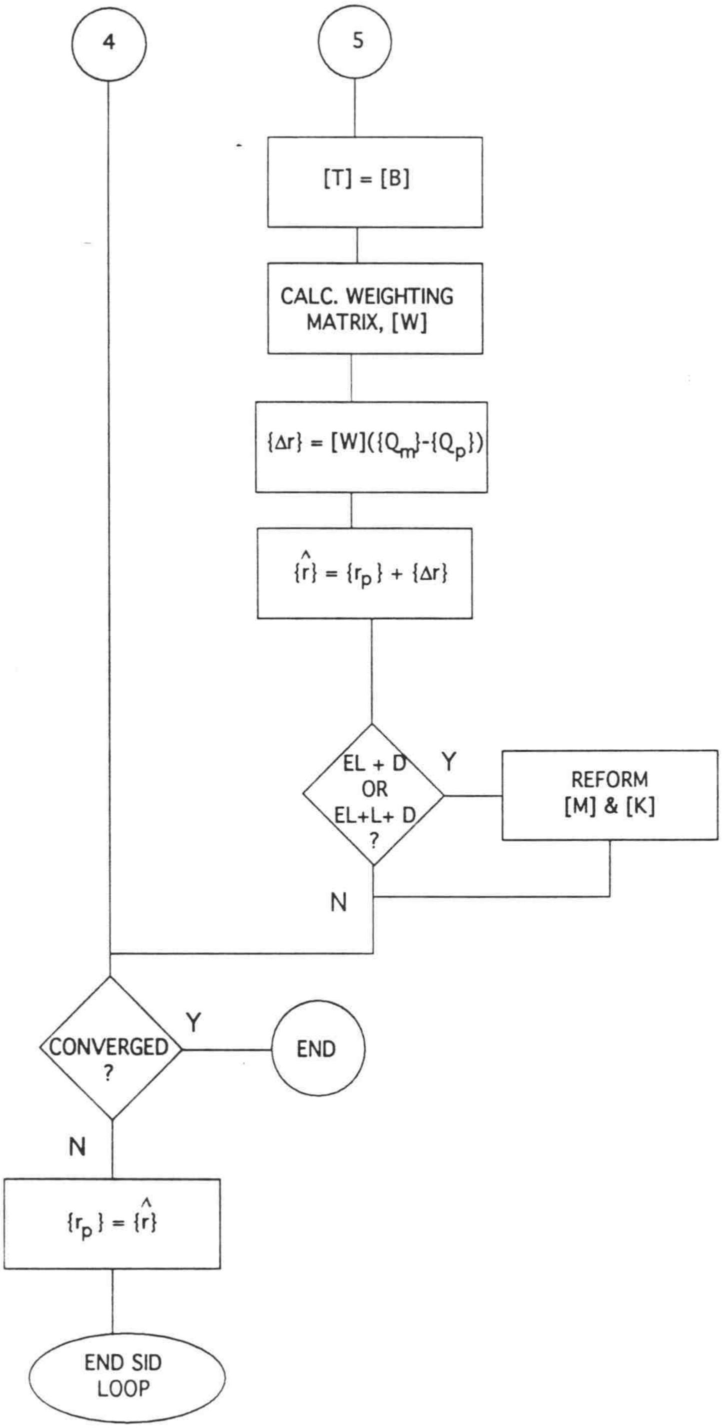


Figure 6 (Cont.). Flowchart for Forced Vibration Identification

motion, Eq. (1.1), are solved for the kinematics using Eqs. (3.87) and (3.98). Since the constants appearing in the kinematic sensitivity expressions do not vary with time, they are calculated during the first time step and saved. Also, the matrix $(g_5M + g_6K)^{-1}$ is calculated and saved; this matrix appears in all of the equations for the kinematic sensitivities.

The sensitivity submatrix [B] must now be formed. The system identification can be performed for any number and type of parameters, and in any combination. Consider, first, identification problems where all parameters are from the same class (e.g., all element parameters).

For element parameters, Eqs. (3.112), (3.123), and (3.133) are used to calculate the kinematic/mass sensitivities if any mass densities are to be identified. If any moments of inertia or element areas are to be identified, the kinematic/stiffness sensitivities are calculated by Eqs. (3.115), (3.125), and (3.135). These sensitivities constitute the columns of submatrix [B]; they are calculated in the order shown in Eq. (3.76) and are saved to file.

For lumped parameters, the kinematic/mass and kinematic/stiffness sensitivities are calculated from the same equations used for the element parameters. If any of these sensitivities were determined when the element parameters were processed, the calculations are skipped.

If any damping parameters are being estimated, the kinematic/damping sensitivities given by Eq. (3.139) and Eqs. (3.143) through (3.147) are calculated, as needed.

In all cases, the sensitivities which constitute the columns of submatrix [B] are calculated and stored to file in the order shown in Eqs. (3.76) and (3.79). The kinematic sensitivities are calculated and saved for each time step. After all time steps have been processed, the time loop is terminated; the sensitivity matrix [T] can now be assembled.

Assembly of [T]. Recall that all of the kinematic sensitivities which were calculated and stored on file are of size n x 1, where n = number of d.o.f.. Thus, they contain values for the sensitivity of the kinematics at all n degrees-of-freedom. For example, $\{\partial\ddot{x}_*/\partial M_{ij}\}$ is made up of values corresponding to $\ddot{x}_{*1},\ddot{x}_{*2},\ldots,\ddot{x}_{*n}$. If measured accelerations are only available for "p" of the d.o.f., only those sensitivities corresponding to the p d.o.f. are to be included when forming [T]. Therefore, in assembling [T], the program extracts the subset of sensitivities corresponding to the d.o.f. at which measured data are available.

For the estimation of element parameters or lumped parameters, [T] is assembled in the same manner as for free vibration identification. For the estimation of the damping parameters α and β, submatrix [A] was shown to be an identity matrix. Therefore, the sensitivity matrix [T] is simply submatrix [B].

If estimation is to be carried out for a combination of parameters, the sensitivity matrix [T] is assembled in stages, with each stage corresponding to one of the three classes of parameters. The processing take place in the following order : 1) element parameters, 2) lumped parameters, and 3) damping parameters. For example, suppose parameters from all three classes are being identified. All of the element parameters are treated first, and the portion of [T] corresponding to these parameters is assembled. The lumped parameters are then processed, and [T] is augmented with the appropriate sensitivities. Finally, the portion of [T] corresponding to the damping parameters is formed and is joined to the portions already assembled. The sensitivity matrix [T] is now complete.

Calculation of Improved Parameters. The remainder of the identification process is identical to the procedure used for free vibration estimation. Note that the structure mass and stiffness matrices, [M] and [K], do not have to be reassembled if the Rayleigh Damping coefficients are the only parameters being identified, since these matrices are not functions of α and β.

Comments on the Three Estimation Schemes

Three identification schemes are available in the program : Least Squares, Weighted Least Squares, and Bayes. Other than minor differences in the input to the program (data and parameter variances), the only difference among the three schemes appears in the calculation of the weighting matrix, [W].

From Eq. (3.54), it can be seen that each scheme requires a different matrix to be inverted for the calculation of [W]. Let this matrix be denoted by [N]. For the three estimation schemes, the matrix that must be inverted is :

$$\text{LS} : [N]^{-1} = ([T]^T[T])^{-1} \tag{4.6a}$$

$$\text{WLS} : [N]^{-1} = ([T]^T[V_{\epsilon\epsilon}][T])^{-1} \tag{4.6b}$$

$$\text{BAYES} : [N]^{-1} = ([T][V_{rr}][T]+[V_{\epsilon\epsilon}])^{-1} \tag{4.6c}$$

For the first two schemes, the size of [N] is np x np, where np = number of parameters. For Bayesian estimation, [N] is an nd x nd matrix, where nd = number of measured data.

For free vibration problems, np and nd are usually of the same order of magnitude. However, for forced vibration, nd>>np since measurements are typically available at several locations and for many time steps. Therefore, [N] will be very large; its inverse may be difficult to obtain and is subject to numerical inaccuracies.

The weighting matrix [W] is used to calculate the changes in the parameters $\{\Delta r\}$ which, in turn, are used to obtain the improved estimate of the parameters $\{\hat{r}\}$. In order to calculate [W], the inverse of [N] must exist; therefore, [N] must be a nonsingular matrix (64).

From the equations above, it is seen that the calculation of [N] involves the sensitivity matrix [T]. The size of [T] is nd x np and its rank is np (18). Therefore, singularity of [N] is dictated by the magnitude of nd relative to the magnitude of np. Three distinct possibilities exist and will now be described.

<u>Case 1 : nd > np .</u> If there are more data than parameters being identified, then nd > np. The matrix [N] is nonsingular for both Least Squares and Weighted Least Squares estimation; [N] will be nonsingular for Bayesian estimation only if $[V_{\epsilon\epsilon}] \neq [0]$.

For Bayesian identification, the matrix product $[T][V_{rr}][T]^T$ is singular. In Eq. (4.1c), $[V_{\epsilon\epsilon}]$ is used to remove the singularity. Therefore, this matrix, which is the variance matrix for the experimental error, must be nonsingular. When a nonsingular $[V_{\epsilon\epsilon}]$ matrix is specified, [N] is nonsingular and may be inverted. Note, however, that different values of $[V_{rr}]$, the variance matrix expressing the uncertainties in the parameters, will cause different values of [W] to be calculated. Thus, the values of the expected parameters, $\{\hat{r}\}$, are greatly affected by the choice of $[V_{\epsilon\epsilon}]$ and $[V_{rr}]$.

Finally, [W] as calculated by Least Squares and Weighted Least Squares will be identical if all of the data measurements possess the same uncertainty; i.e., if $[V_{\epsilon\epsilon}] = c[I]$, where c is any positive number.

<u>Case 2 : nd = np .</u> If there as many data as parameters, then nd = np. For this case, all three estimation schemes produce a nonsingular [N] matrix for any values of $[V_{\epsilon\epsilon}]$ and $[V_{rr}]$. In addition, the Least Squares and Weighted Least Squares schemes result in identical weighting matrices. Bayesian estimation will produce this same matrix if $[V_{\epsilon\epsilon}] = [0]$, regardless of the value of $[V_{rr}]$. Finally, for $[V_{rr}] = [V_{\epsilon\epsilon}] = c[I]$, the weighting matrix calculated according to the Bayes scheme is independent of the value of c.

<u>Case 3 : nd < np .</u> If the number of parameters to be estimated exceeds the amount of available data, then nd < np. Least Squares and Weighted Least Squares identification is not possible for this situation, since [N] will be singular. However, Bayesian estimation is possible, subject to the limitations and observations presented in Case 1.

CHAPTER V

EXAMPLE PROBLEMS

Introduction

Example problems are presented in this chapter to demonstrate the validity and capability of the parameter identification procedures presented in the previous chapters. The computer program described in Chapter IV was used to execute all of the problems.

As can be seen from Eq. (3.52) and Fig. 2, the identification procedure relies heavily on the difference between measured response and predicted response. To test convergence of the identification process, fictitious experimental data were generated analytically for the examples in this chapter. This was done for two reasons : 1) suitable experimental data were not available for the types of problems considered, and 2) known baseline responses were desired so that convergence of the identification process could be more readily detected.

In generating the pseudo-experimental data for each example structure, a finite element model was formulated with all structural parameters fixed at their true, or target, values. The program was then used (in its non-identification mode) to calculate the response based on these values. This response was then treated as experimental, or measured, data in the parameter estimation process.

In the second example problem, zero-mean Gaussian noise was added to the response to simulate actual experimental measurements.

After experimental data were generated for a structure, various (prior) analytical models were postulated to be different from the model used to generate the data. These prior models had one or more structural parameters with values that were different from those used to generate the measured response. The program was then run to identify improved estimates of the parameters.

Since the measured data were generated under controlled conditions (i.e., for known target values of the parameters), the quality of the obtained solution can be easily judged. This is done by comparing the final parameter estimates with the target values, and by comparing the final calculated response with the measured response.

Six example problems are presented in this chapter. These problems demonstrate the major capabilities and features of the identification techniques for both free and forced vibration.

The first example is a one-story frame acted upon by a time-varying load. Parameter identification is performed, in turn, for the column moment of inertia, the floor mass, and the damping coefficient. Studies are made to determine the effects of the type, quantity, and spacing of the measured kinematics. The effect of noisy data on convergence is also investigated.

The second example problem presents studies of a two-story frame for free and forced vibrations. Simultaneous identification of two or more parameters is demonstrated, and the effect of the type of measured data are shown. The column inertias, floor masses, and damping coefficients are the parameters estimated. A comparison is made of the Least Squares, Weighted Least Squares, and Bayesian estimation schemes.

A cantilever beam is used as the third example to further illustrate the ability to identify two or more parameters at the same time. Free vibration is considered, and identification is performed at the element level for the moments of inertia and mass densities.

A cantilever beam with two lumped masses is studied in the fourth example. Parameter estimation is performed for the element cross-sectional areas using free vibration data. The beam is then subjected to a suddenly-applied load at the tip, and time-history identification is executed. Simultaneous identification is made of the element mass densities and moments of inertia, followed by simultaneous identification of the element inertias and one of the Rayleigh damping coefficients.

The fifth example presents a tower with a concentrated mass at the top. The tower is supported at mid-height by a pair of guys, modeled as a linear spring (concentrated stiffness). The foundation is partially restrained against rotation : this effect is modeled by a rotational spring (concentrated stiffness). The identification of these stiffnesses is executed individually and simultaneously, for both free vibration and forced response.

An ocean-based guyed tower is presented as the sixth example problem. The tower supports a deck mass (platform) and is acted upon by a train of regular waves. The displacement of the platform was calculated for the hydrodynamic load given by a form of Morison's equation which includes a nonlinear, relative velocity term. This displacement was taken as the measured response of the structure. The model was then executed for the same conditions, using the same hydrodynamic and structural parameters, but using a form of Morison's equation which does not contain the relative velocity term. In this model, identification was performed on one of the Rayleigh damping coefficients in an attempt to correct for the missing relative velocity dependence. A second model was formed for which estimates of the hydrodynamic parameters were in error, and which did not include relative velocity effects. Multiple structural parameters were identified in an attempt to account for all modeling errors.

Example 1 : One-Story Frame

For the first example, studies are made to determine the effects of the type, quantity, and spacing of the measured data on forced vibration identification. The effect of noisy data on convergence is also studied. A summary of the studies performed for the structure in Fig. 7 is given in Table 1.

The single story frame in Fig. 7 consists of two steel columns supporting a rigid floor. The columns have a moment of inertia of 61.9 in^4, are 20 feet in height, and form a 30-foot bay. The total floor load is 1000 lb/ft, giving a mass at the floor level of 932 slugs. The mass of the columns is small in comparison to the floor, and may be neglected in the analysis.

A sinusoidal, time-varying force is applied at the floor level. This loading function has a maximum amplitude of 4300 lb and a period of 1.95 sec. The response of the frame is damped, with damping in the first mode equal to 10 percent of critical damping ($\alpha = 1.288$).

A time step size of 0.10 sec was chosen for the analysis, and pseudo-experimental response data were generated for the structure based on the above values. These data consisted of the displacement, velocity, and acceleration of the first floor. In Fig. 8, one curve, labeled "pseudo-experimental data", depicts the acceleration time-history for the structure.

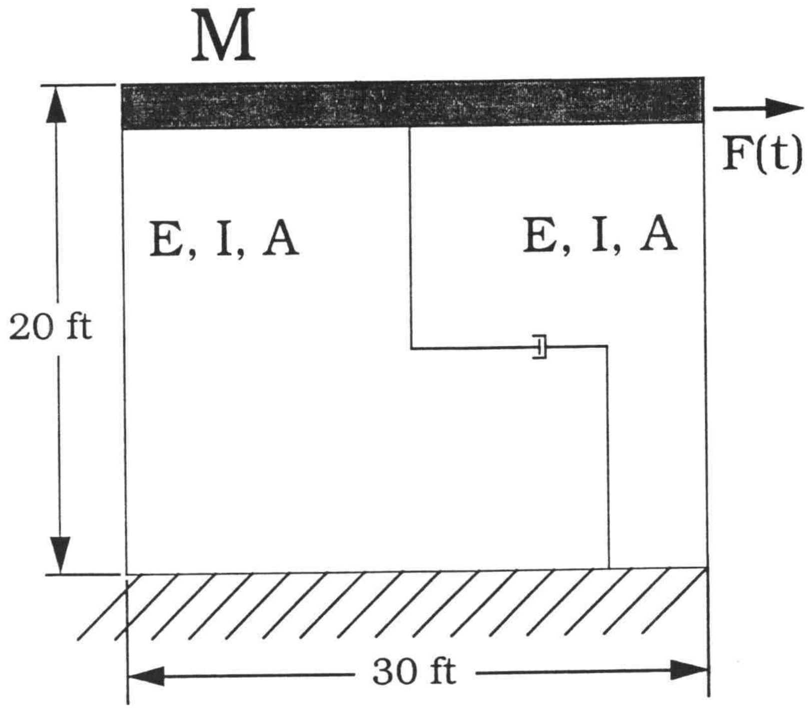


Figure 7. One-Story Frame

Study	Param.	Scheme	Data	Effect Studied	No. of Cases
1	I	LS	$\ddot{x}$	Number of data	4
2	M	LS	$\ddot{x}$	Number of data	4
3	α	LS	$\ddot{x}$	Number of data	4
4	α	LS	$\ddot{x}$	Spacing of data	7
5	α	LS	Combs. of x, $\dot{x}$, $\ddot{x}$	Type of data	6
6	M	LS	$\ddot{x}$	Noisy data	7

Table 1. Studies for Example 1

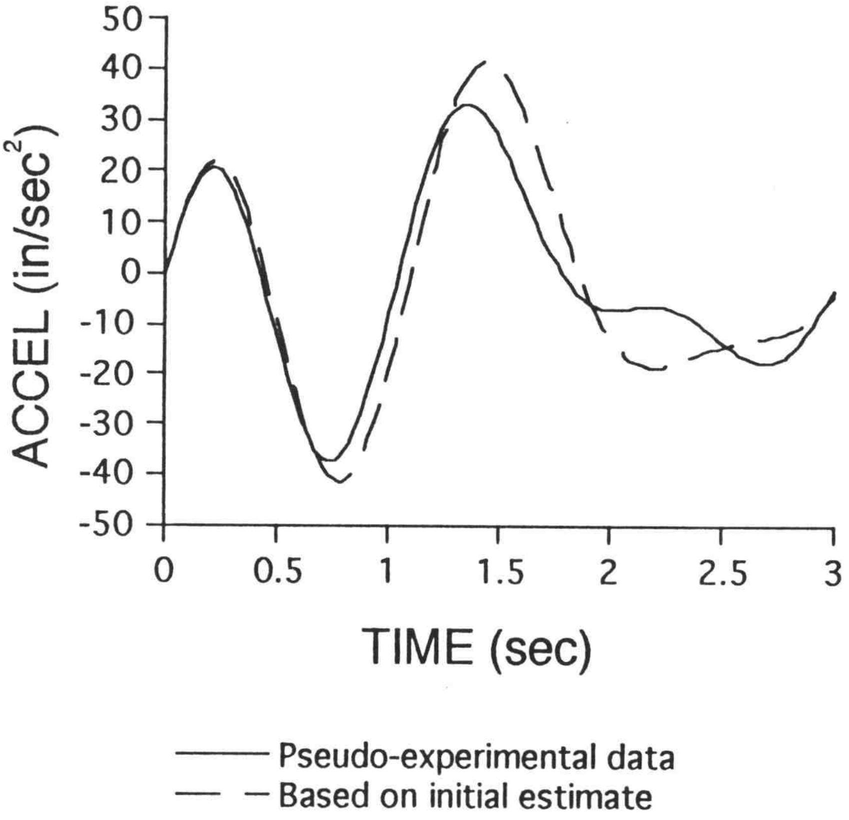


Figure 8. Calculated Acceleration Based on Initial Estimate vs Experimental Acceleration - Inertia

The Least Squares estimation scheme was chosen for this problem. Unlike the other two schemes, this method doesn't rely on user-supplied variances for the parameters or for the data. Therefore, the behavior of the solution will be a direct measure of the capabilities of the identification procedure; the solution will not be affected by subjective estimates of the variances.

In order to study the behavior of the identification process for the column inertia, an initial estimate to a finite element model was postulated to be different from the one used to generate the pseudo-experimental data. The new model possessed the same mass, area, Young's Modulus, and damping coefficient as the original, but the value of the column inertia was considered to be uncertain. An initial estimate of $I^0 = 52.5$ in^4 was made for the inertia. This value is 15 percent lower than the true (target) value and represents an amount of error typical in an engineering analysis.

The pseudo-experimental (measured) acceleration was used with the Least Squares identification process in order to select a "better" estimate for the inertia. The selection is guided by the difference between the "measured" response and the calculated response based on the prior estimate(s) of the parameter(s). The response curve plotted in Fig. 8, labeled "initial estimate", depicts the acceleration of the frame obtained using the initial estimate I^0.

The first study demonstrates the effect of the number of acceleration data points on the identification of the column inertia. Successive runs of the computer program were made using the accelerations at the first two, three, four, and five time steps. As can be seen in Fig. 9, in all cases the value of the column moment of inertia converged to the exact value in one iteration, and the solution was stable. The number of data points had no effect on the rate of convergence or on the converged value. Figure 10 illustrates the convergence of the calculated response to the measured response. The percent error between the calculated and pseudo-experimental accelerations is plotted for two models : one for which the column inertia is equal to the initial estimate, and one for which it is equal to the identified value. The analysis was repeated using a different initial estimate for the inertia which was 15 percent higher than the target value. The solution converged in a similar fashion to the same value of inertia.

For the next study, the same procedure was followed for the identification of the first floor mass. An initial estimate of the mass was established at M^0 = 1072 slugs (15% high). Figure 11 shows that the identified mass converged to the exact value in two iterations, and the number of experimental data points had only a minor effect on the behavior of the solution. Similar results were also obtained

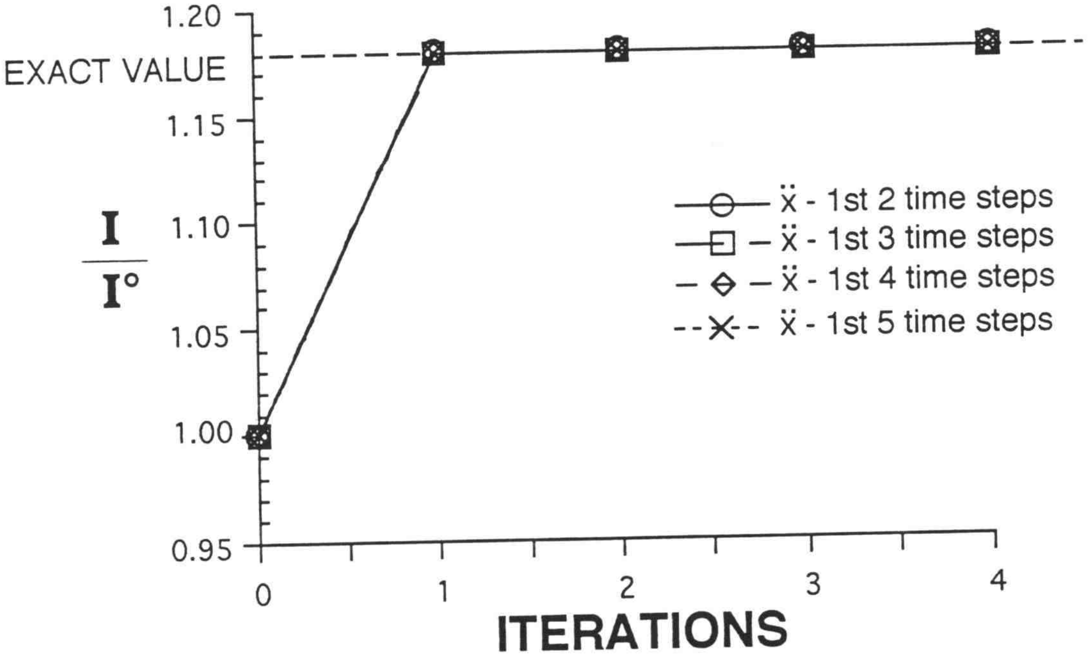


Figure 9. Effect of Number of Data Points on Convergence of Inertia

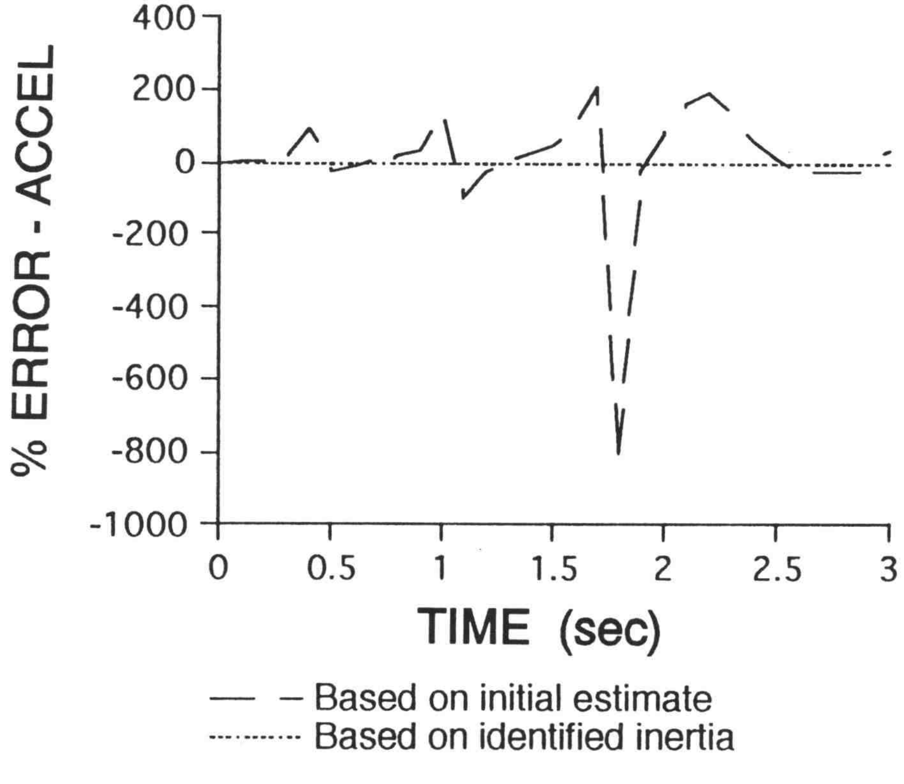


Figure 10. Error in Calculated Accelerations - Inertias

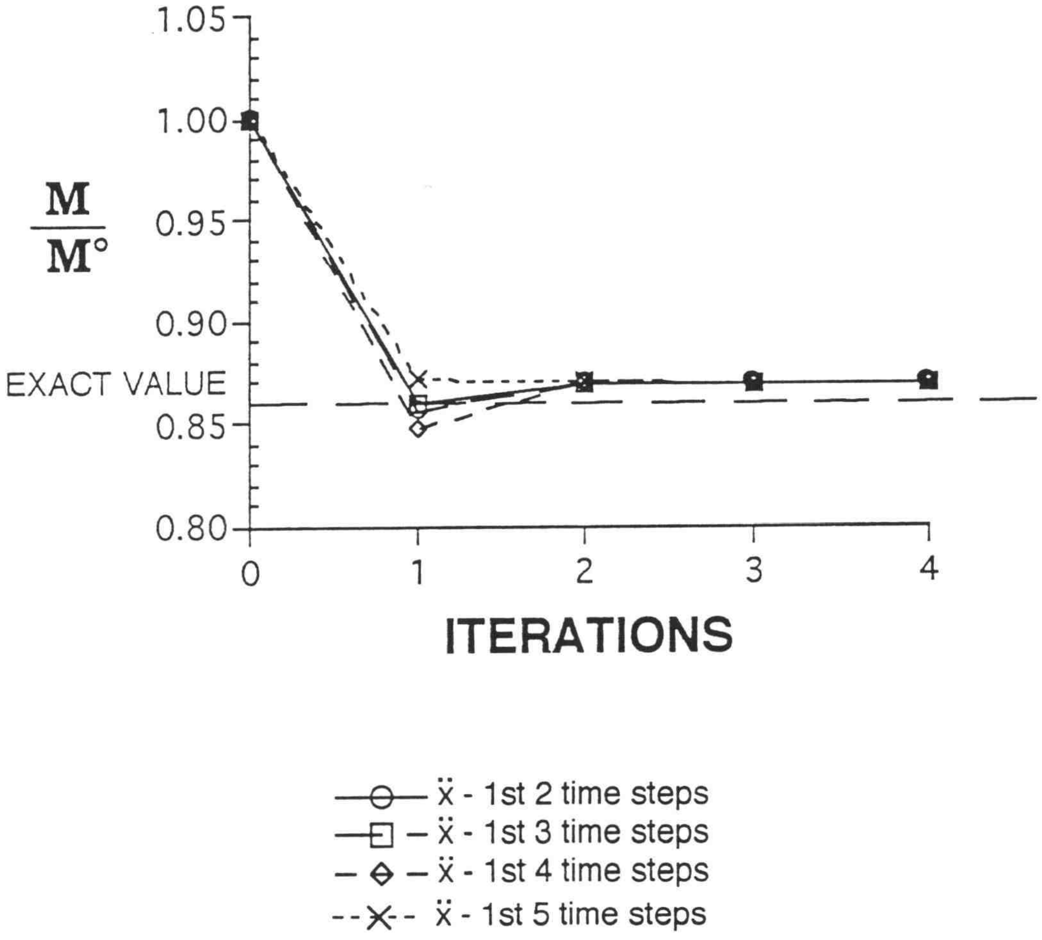


Figure 11. Effect of Number of Data Points on Convergence of Mass

for a different initial estimate that was 15 percent lower than the target value.

This procedure was repeated once again to investigate the identification of the Rayleigh damping coefficient, α. Figure 12 demonstrates the nature of this solution. An initial estimate of $\alpha^0 = 1.095$ was assumed; this value is 15 percent lower than the one used to generate the experimental data. Although convergence is slower for the damping coefficient than for inertia or mass, the convergence is well-behaved and stable. In assessing the quality of this solution, the convergence of the predicted response to the pseudo-measured response must also be considered. For this study, the value of α is in error by about 7 percent after one iteration : however, the maximum error in the response is less than 2 percent if that value is used. After five iterations, this maximum response error is less than 0.2 percent. An examination of the results of this and several other studies indicates that the kinematics are less sensitive to changes in the damping coefficients than to changes in the other structural parameters. This analysis was repeated for a different initial estimate of α that was 15 percent higher than the true value; results were similar.

The next two studies performed were investigations into the effects of the spacing and type of experimental data used. Since the Rayleigh damping coefficient exhibited the slowest rate of convergence, it was selected for these

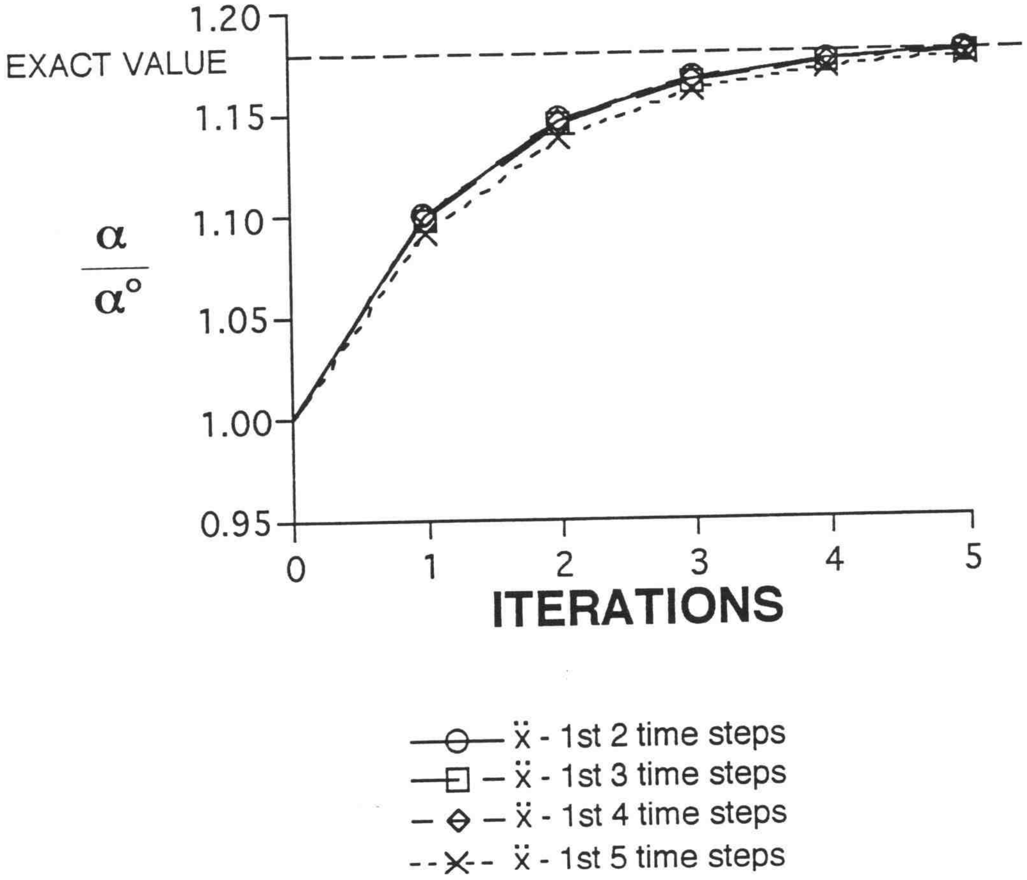


Figure 12. Effect of Number of Data Points on Convergence of Damping Coefficient

studies. The initial estimate of α was 15 percent lower than the true value.

The effect of the spacing of data points was studied by specifying pseudo-experimental accelerations at five time steps selected by various criteria. Referring to the experimental data plotted in Fig. 8, the five points were first chosen at every other time step, and identification of α was performed. Next, the accelerations at the first five time steps were used as the measured response for the identification process. Other distributions of the data included selecting five points from the first peak of response, five from the second peak of response, and dividing the five points between the two peaks. The results of these analyses, along with a description of the location of the points, is given in Fig. 13. It can be seen that the effect of data spacing is negligible. The rate and nature of convergence, as well as the identified value, are nearly identical to the results reported in Fig. 12. It should also be noted that using twice as many data points (at the first ten time steps) does not improve the solution.

It was desired next to determine if the type of kinematic data had a significant effect on the identification process. Once again, the damping coefficient α was chosen because of its slower convergence properties. Experimental data was sampled at every other time step, for a total of five steps. Identification of α was executed

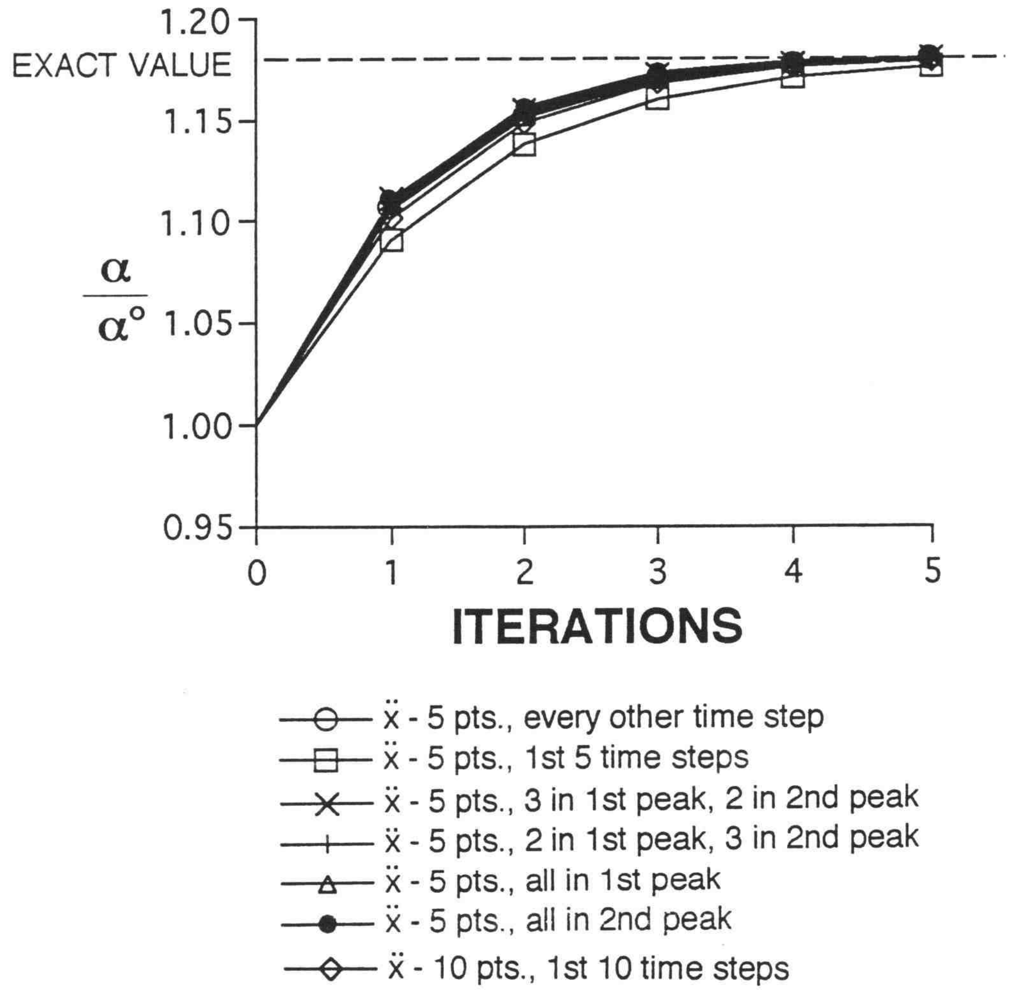


Figure 13. Effect of Spacing of Data Points on Convergence of Damping Coefficient

using just the displacements at these steps, just the velocities, and combinations of displacements, velocities, and accelerations. The results plotted in Fig. 14 show that the identification process is insensitive to the type of kinematic data used.

The final study for this structure involved identification using noisy data. The pseudo-experimental acceleration data in Fig. 8 were polluted with zero-mean Gaussian noise with a variance of 7 percent to simulate typical response data from a structure test. Identification of the mass was performed using the same prior value used for the study depicted in Fig. 11. The results of the analysis for noisy data are shown in Fig. 15. Various numbers of data points were considered. The rate of convergence is unaffected by the presence of noise; however, the solution converges to slightly different values of mass depending on the amount of noise present in the data points used as measured response.

The differences in pseudo-measured and calculated response are seen in Fig. 16. The curves based on the initial estimate and on the identified value of mass are plotted for the case that used data from the first ten time steps. Similar curves were obtained for the other cases detailed in the previous figure. The plots shown in Fig. 17 show the error between the predicted responses and the noisy acceleration data. There is a significant improvement in the

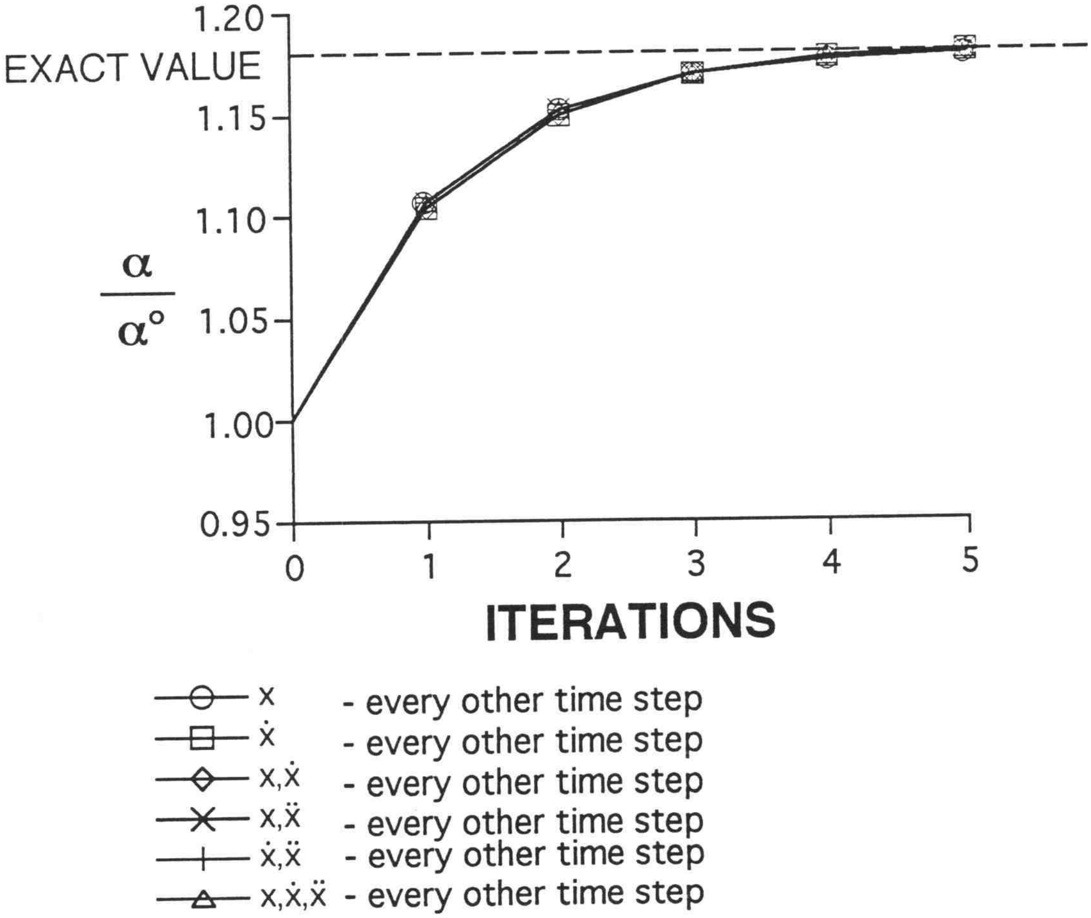


Figure 14. Effect of Type of Data on Convergence of Damping Coefficient

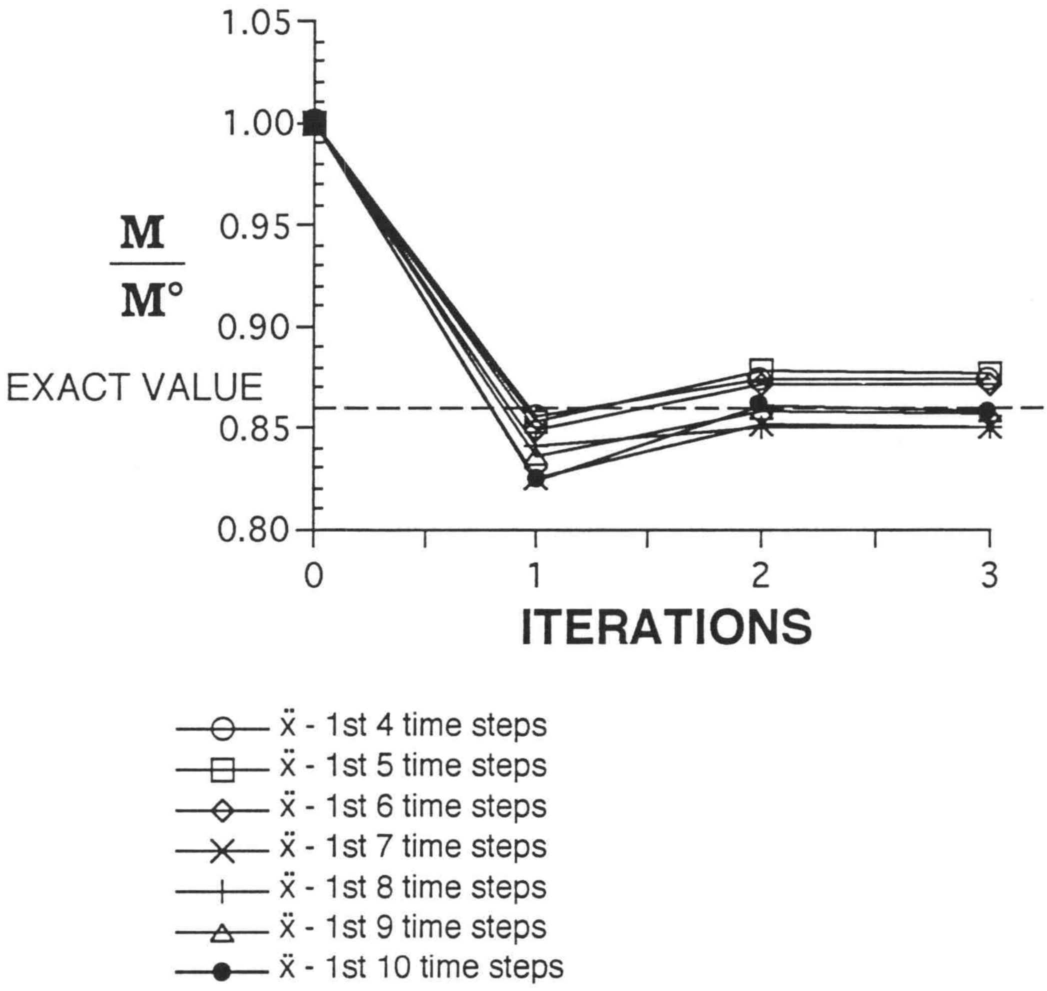


Figure 15. Effect of Number of Data Points on Convergence of Mass - Noisy Data

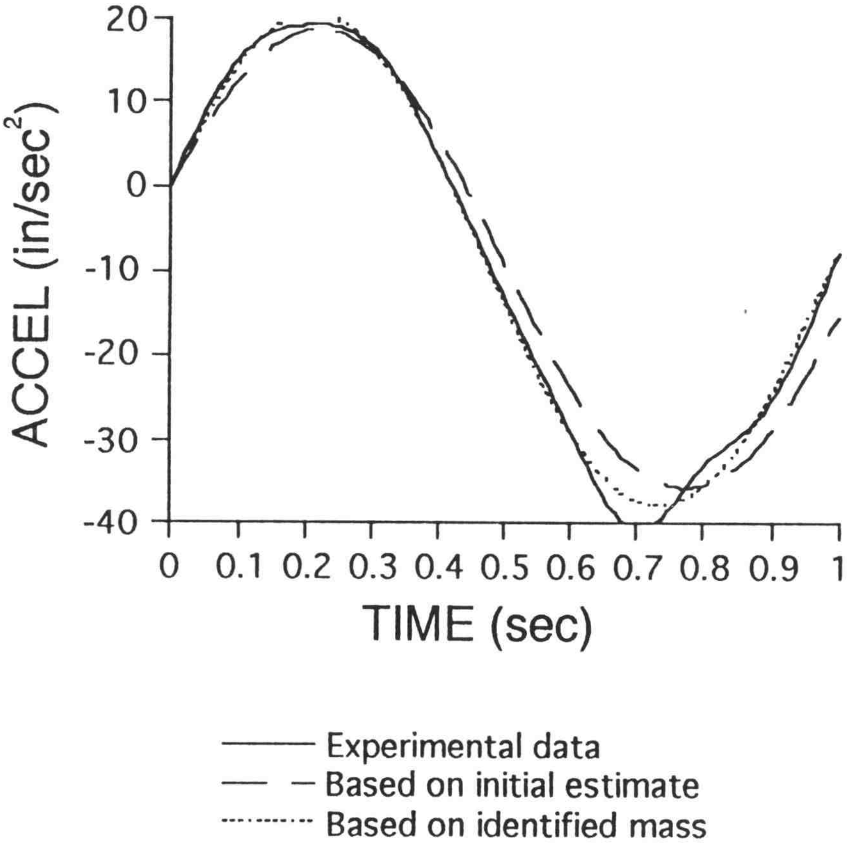


Figure 16. Calculated Accelerations vs Experimental Acceleration - Mass, Noisy Data

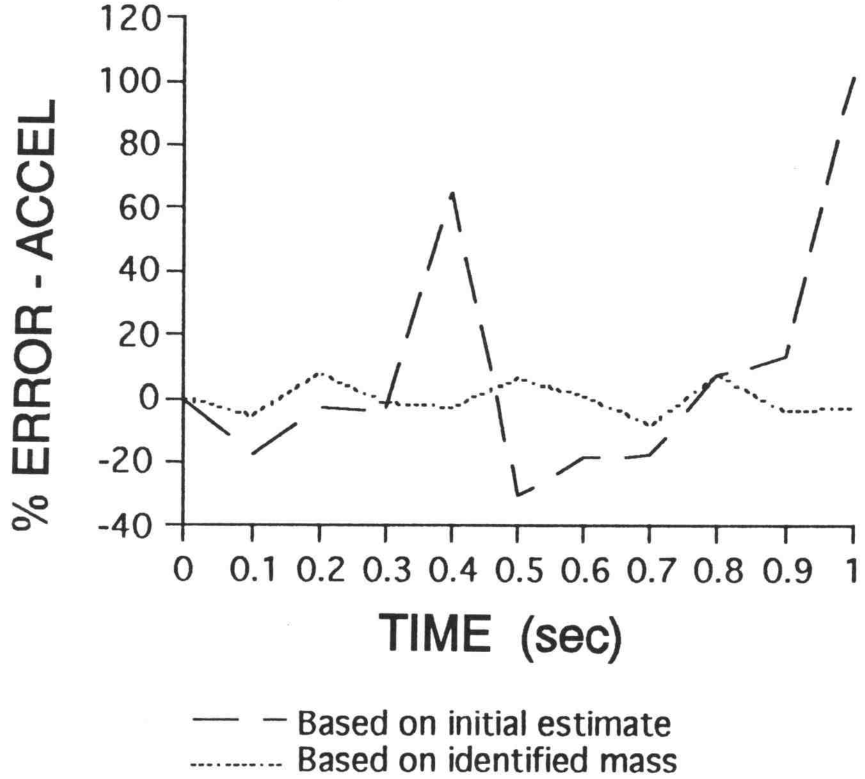


Figure 17. Error in Calculated Accelerations - Mass, Noisy Data

response based on the identified mass, with the maximum error reduced by a factor of ten.

Example 2 : Two-Story Frame

Studies are performed for both free and forced vibration for the structure in Fig. 18. The effect of the type of data on the simultaneous identification of two or more parameters is studied. In addition, a comparison is made of the Least Squares, Weighted Least Squares, and Bayesian estimation schemes. Table 2 presents a summary of the studies performed.

A second floor was added to the structure in Example 1 to create the two-story frame shown in Fig. 18. The columns for the first floor are unchanged, but the floor mass was reduced to 745 slugs. The second story columns have a moment of inertia of 22.1 in^4 and support a floor mass of 559 slugs.

The force from the previous example is applied at the first floor level, and a second force with the same period is applied at the upper level. This force has a maximum amplitude of 3200 lb.

Damping was specified as 4 percent of critical damping in the first mode, and 10 percent in the second mode. These damping ratios result in Rayleigh coefficients with values $\alpha = -0.024$ and $\beta = +0.018$.

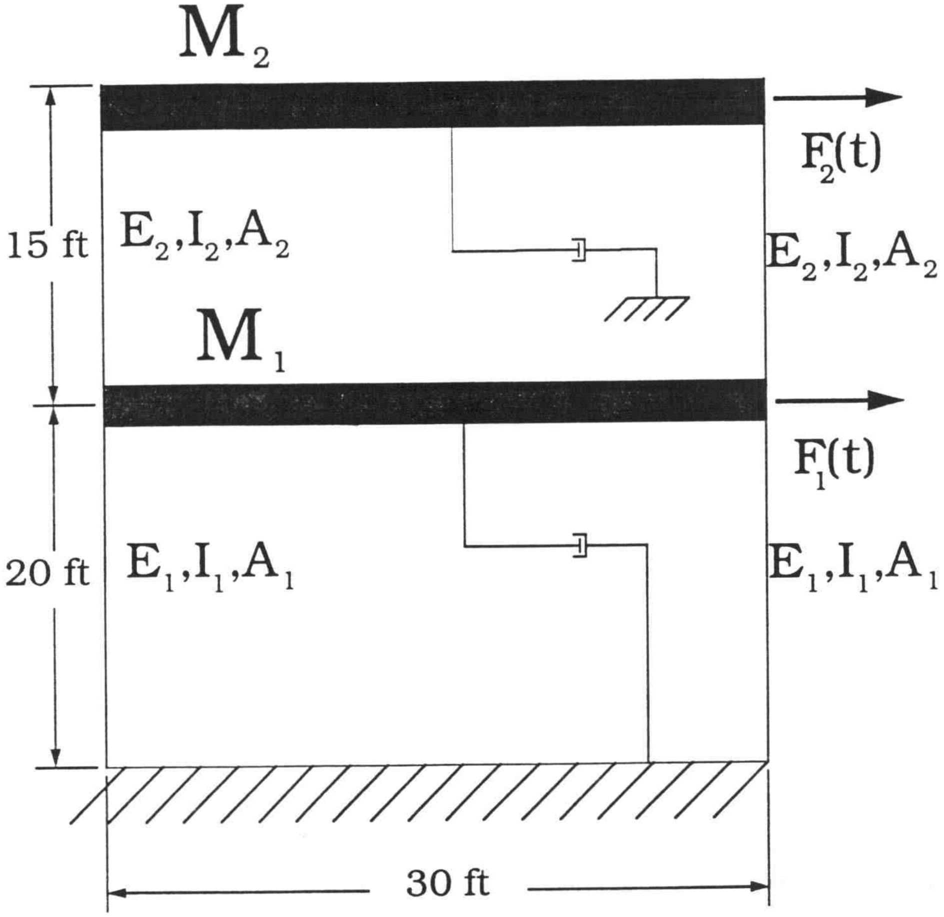


Figure 18. Two-Story Frame

Study	Params.	Scheme	Data	Effect Studied	No. of Cases
1	I_1, I_2	LS	Combs. of $\ddot{x}_1$, $\ddot{x}_2$	Type of data	3
2	M_1, M_2	LS	Combs. of $\ddot{x}_1$, $\ddot{x}_2$	Type of data	3
3	α, β	LS	Combs. of $\ddot{x}_1$, $\ddot{x}_2$	Type of data	3
4	α, β, I_1, I_2	LS	$\ddot{x}_1$, $\ddot{x}_2$	SID on 4 params.	1
5	I_1, I_2	LS	Combs. of λ_1, λ_2, ϕ_1, ϕ_2	Type of data	3
6	M_1, M_2	LS	Combs. of λ_1, λ_2, ϕ_1, ϕ_2	Type of data	3
7	M_1, M_2	LS, WLS, Bayes	λ_1, λ_2	$[V_{\epsilon\epsilon}]$, $[V_{rr}]$	5

Table 2. Studies for Example 2

A time step of 0.10 sec was used in this example, and the calculated accelerations based on these data were taken as the pseudo-experimental response for forced vibration identification. The natural frequencies and mode shapes were also calculated, and used as experimental data for free vibration identification.

For the first three forced vibration studies, the accelerations at the two floor levels were used as the pseudo-measured response, and the effect of the type of data was examined. Identification was performed for two structural parameters simultaneously using : 1) only the acceleration of the first floor, 2) only the acceleration of the second floor, and 3) both accelerations. Accelerations were sampled at the first six time steps, and Least Squares estimation was used.

For the first study, initial estimates of the two column moments of inertia were made : $I^0_1 = 74.3$ in^4 (20% high) and $I^0_2 = 24.3$ in^4 (10% high). Figure 19 shows that both inertias converged to their exact values in four iterations. Note that when only the acceleration of the second floor was used, the initial adjustment to I_2 was large and was in a direction opposite from the correct value. Also, the adjustment to I_1 for this case was substantially more than the adjustment when using the other acceleration data. However, both inertias converged to the correct value, and at a rate that was only slightly slower.

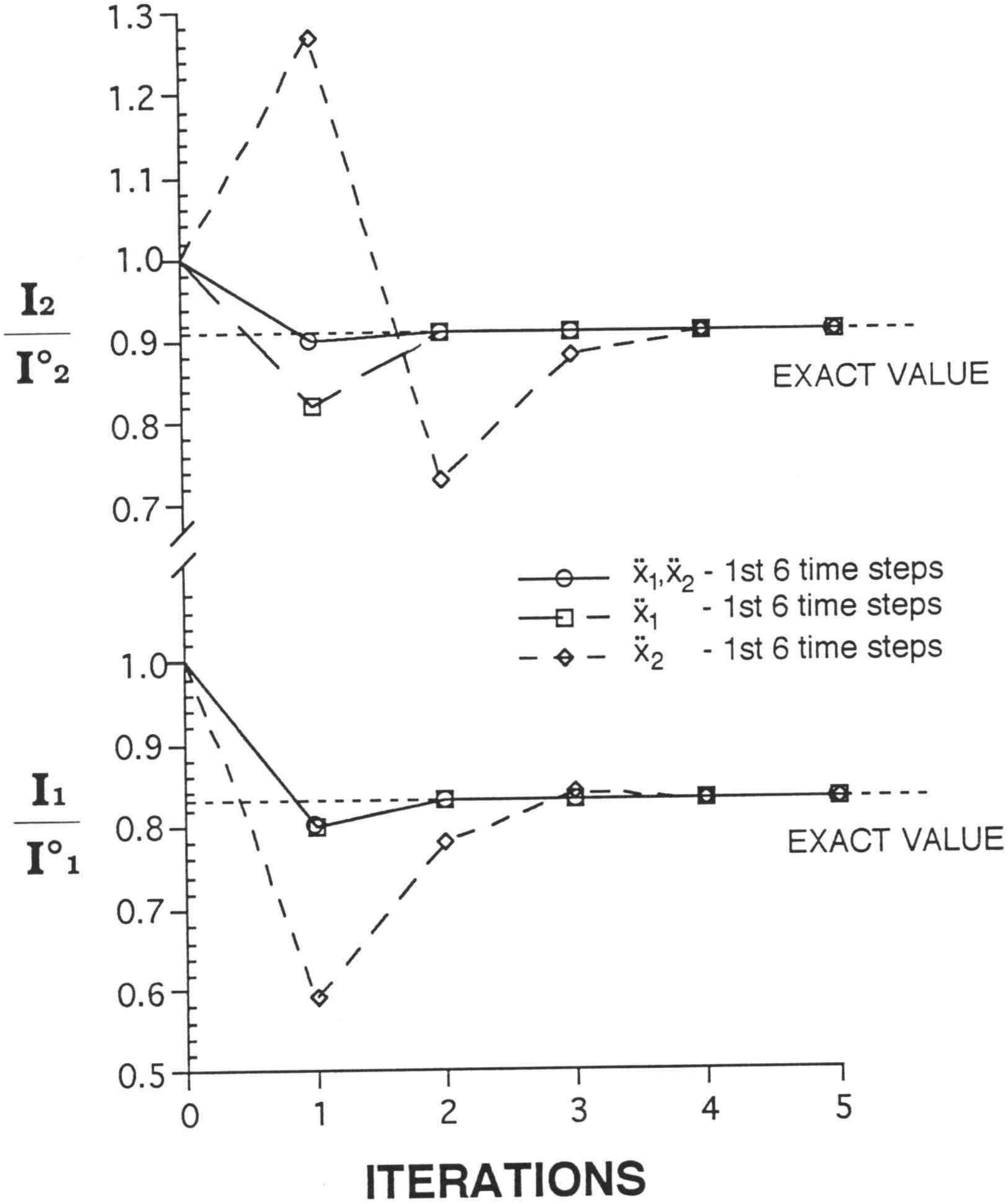


Figure 19. Effect of Type of Data on Convergence of Inertias - Forced Vibration

The two floor masses were identified in the next study. Initial estimates of these parameters were taken as M^0_1 = 596 slugs (20% low) and M^0_2 = 503 slugs (10% low). Convergence to the correct values occurs within four iterations, and is shown in Fig. 20. If only the pseudo-measured acceleration of the first floor is used, the two mass values are over-adjusted in the first two iterations, but converge quickly afterwards.

In the third study of this group, the simultaneous identification of the two Rayleigh damping coefficients was performed. Figure 21 shows the convergence characteristics for this analysis. Initial estimates of the two parameters were taken as α^0 = -0.0300 (25% high) and β^0 = +0.0198 (10% high). As was demonstrated in Example 1, estimation of the damping parameters converges at a slower rate than estimation of the other structural parameters. Note from the figure that the poorest convergence behavior occurs when only the second floor acceleration is used. However, it was found that the calculated response for this case converges to the same accuracy as for the other two cases shown.

Next, the ability of the identification process to simultaneously operate on four parameters was investigated. The two column inertias and two damping coefficients were chosen for this study, and the initial estimates of these parameters were set to the values previously given. The results of the Least Squares estimation are presented in

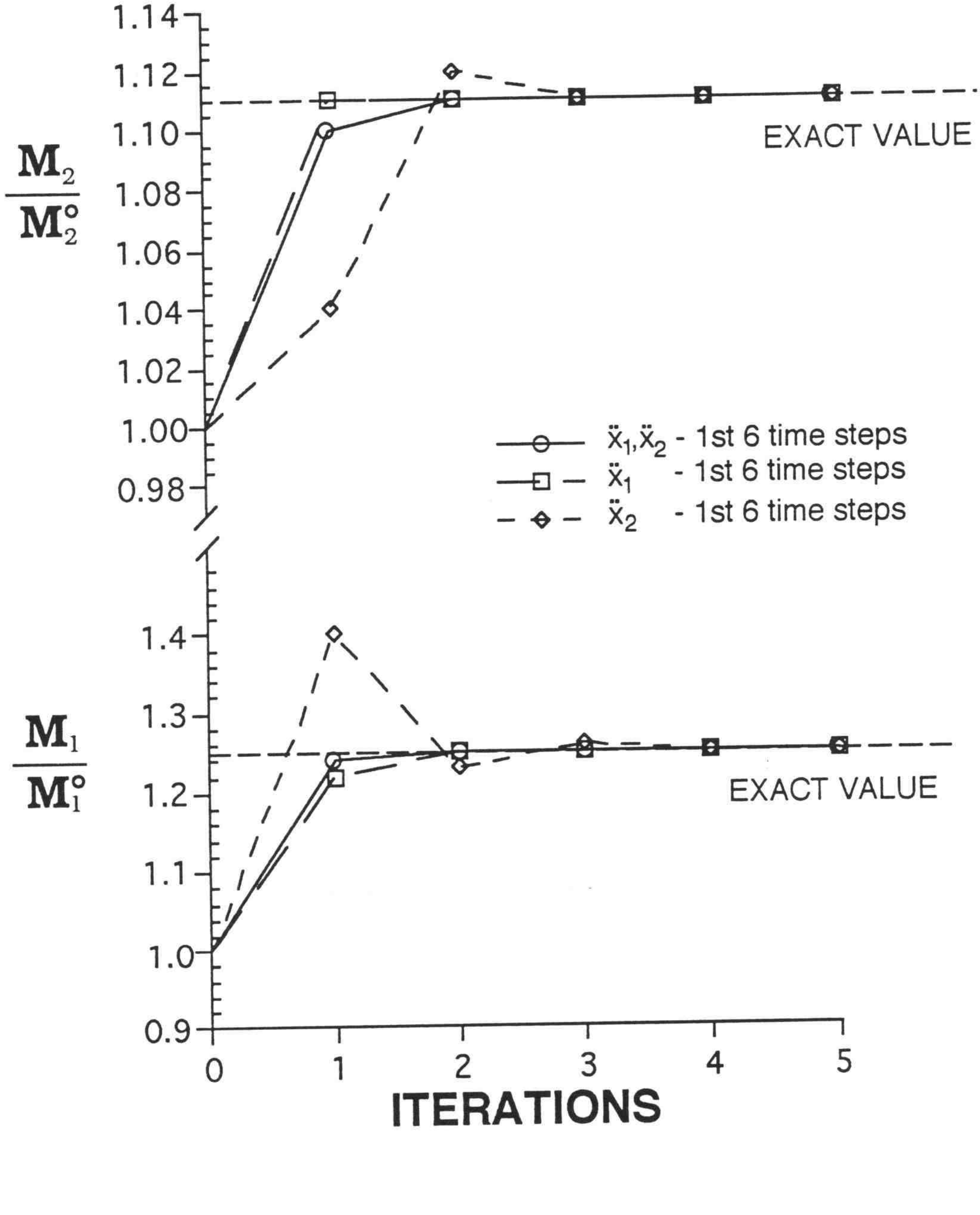


Figure 20. Effect of Type of Data on Convergence of Masses- Forced Vibration

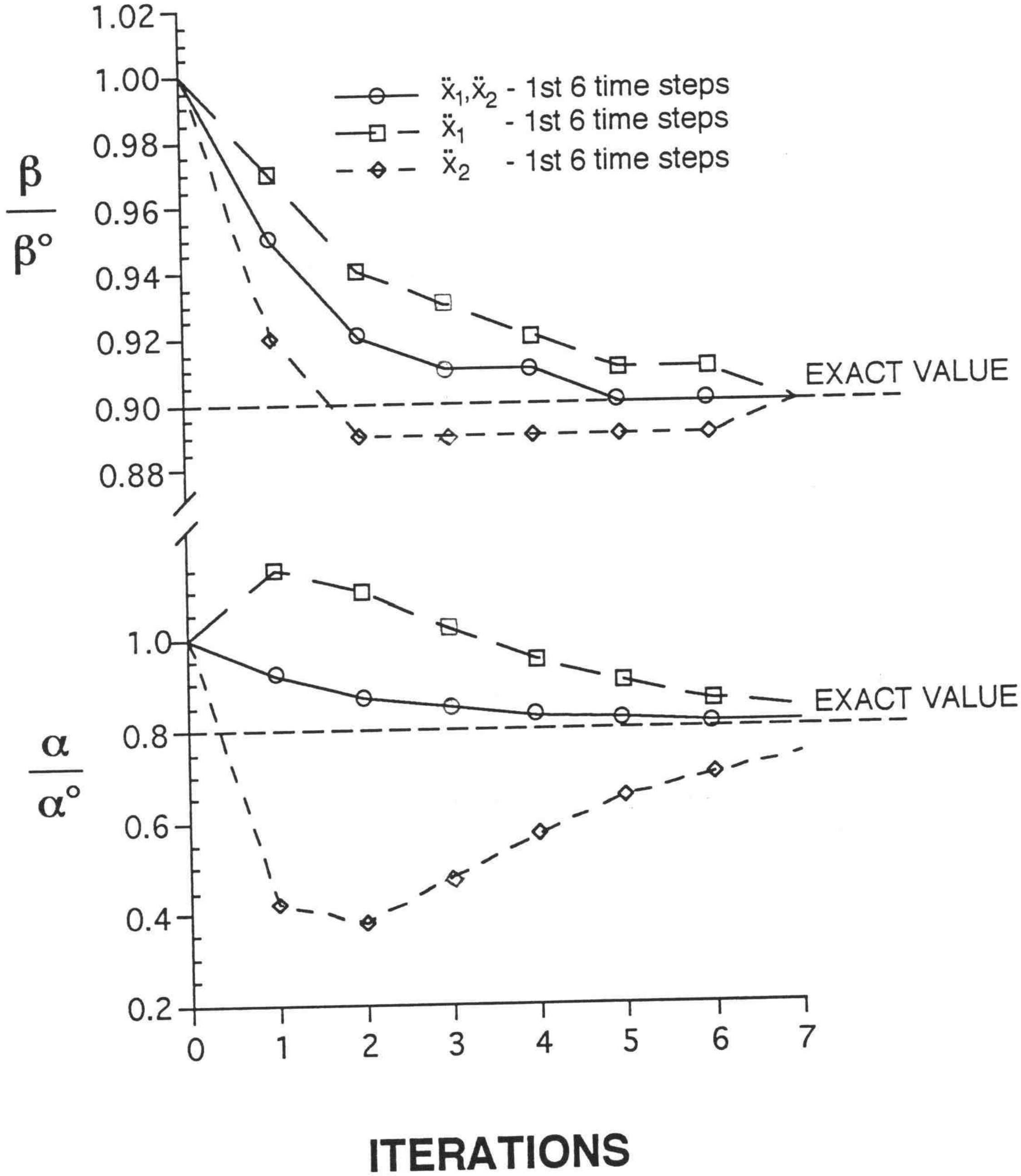


Figure 21. Effect of Type of Data on Convergence of Damping Coefficients - Forced Vibration

Fig. 22; the accelerations at the two floor levels were used to guide the process. As was seen in previous studies, the inertias converge rapidly while the damping coefficients proceed at a slower rate. Note that the shape of the convergence curves are unchanged for the inertias, but have been altered for the damping parameters.

Although the Rayleigh coefficients appear to converge slowly (12 iterations), the calculated response actually converges quite rapidly. This can best be seen by plotting the root-mean-square (RMS) error of the two floor accelerations relative to the pseudo-experimental accelerations. This error can be defined as

$$\text{RMS Error} = \frac{1}{n}\sqrt{\Sigma\left(\ddot{X}_{model} - \ddot{X}_{exp}\right)^2} \qquad (5.1)$$

where n = number of time steps at which experimental data are specified, and the subscripts indicate the response predicted by the model and the experimental response. The plots of the RMS error are presented in Fig. 23, where it can be seen that the error is essentially zero by the third iteration. This sort of behavior was observed for all problems which involved the identification of damping coefficients.

In addition to the forced vibration studies just presented, three studies for free vibration were executed. The first two used Least Squares estimation, while the third

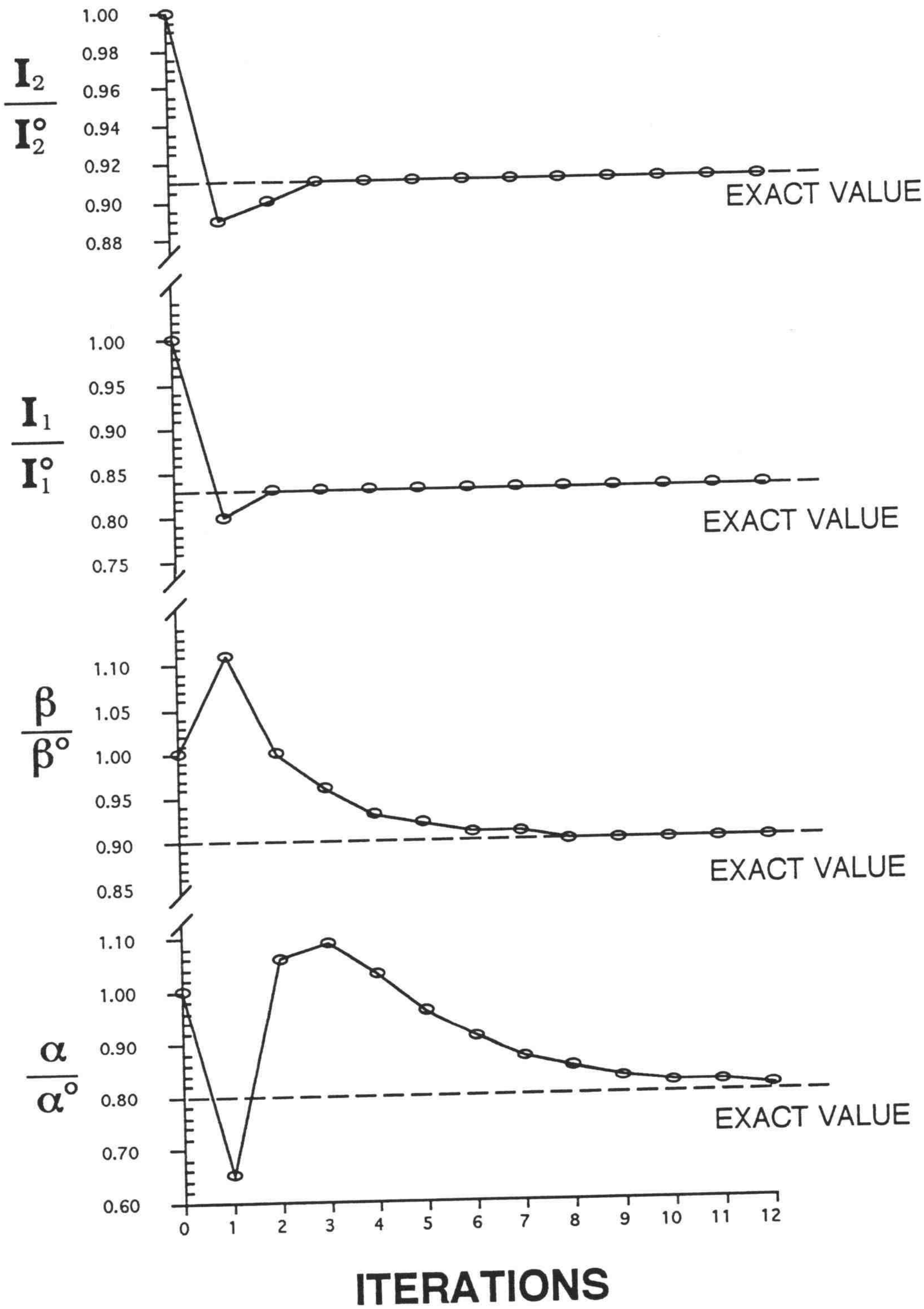


Figure 22. Simultaneous Identification of Inertias and Damping Coefficients

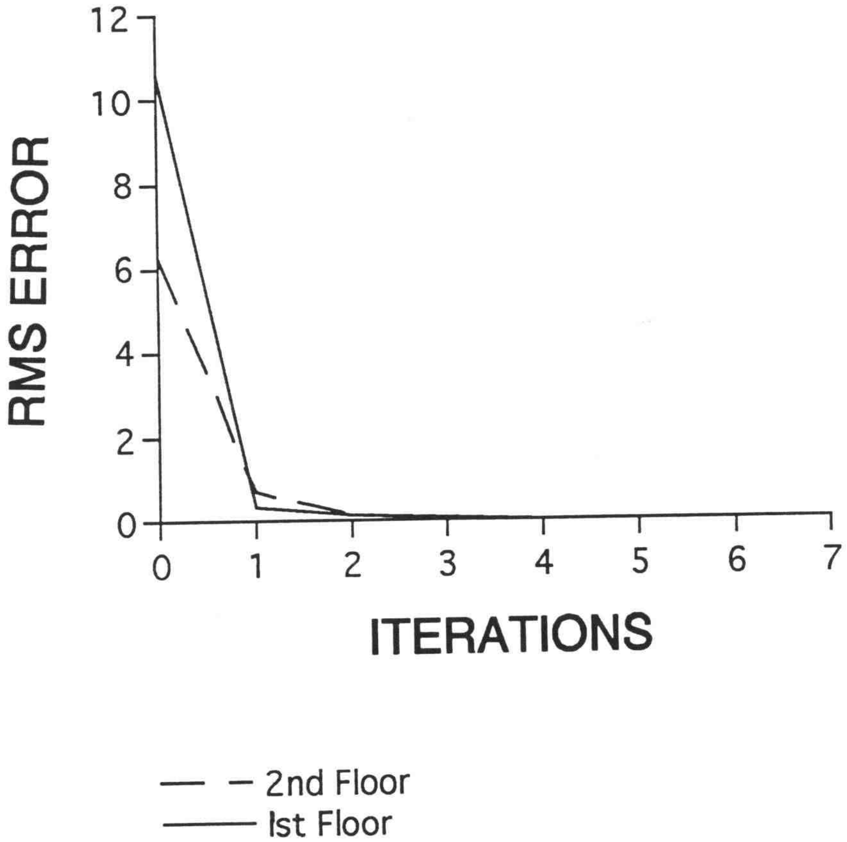


Figure 23. RMS Error - Calculated Accelerations

study compared results from Least Squares, Weighted Least Squares, and Bayesian estimation techniques.

For the first investigation, the two column inertias were identified. Initial values of the parameters used were the same as for the forced vibration studies. Pseudo-experimental data consisted of the first two natural frequencies and mode shapes. Figure 24 presents the convergence characteristics for three cases based on different amounts of experimental data : 1) frequencies only, 2) frequencies and first mode shape, and 3) frequencies and both mode shapes. Both values of the column moments of inertia converge in one iteration. Inclusion of the mode shapes has negligible effect. This is consistent with the findings of other researchers (1,4), and will be seen again in Example 3.

Similar behavior is exhibited in the identification of the two floor masses, as can be seen in Fig. 25. For this analysis, the same prior mass values were used as for the forced vibration study.

The final investigation for this structure examined the effect of the parameter and data variance matrices, $[V_{rr}]$ and $[V_{\epsilon\epsilon}]$, on the performance of the Bayesian estimation scheme. Also included for comparison are the Least Squares and Weighted Least Squares estimation schemes. The two floor masses were used for this analysis, and the same initial values were assumed as for the previous studies. The outcome of this investigation is presented in Fig. 26, and is based

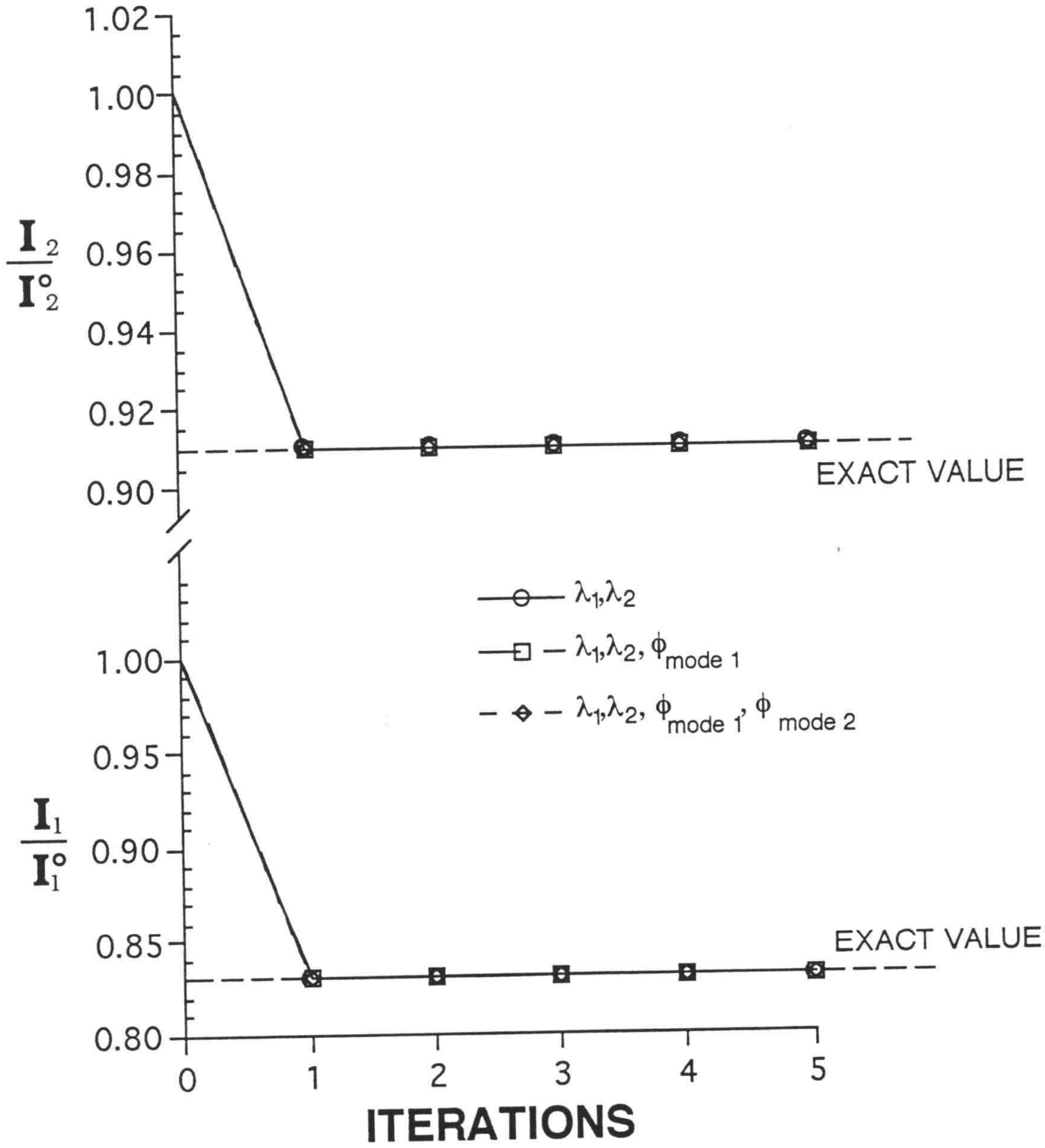


Figure 24. Effect of Type of Data on Convergence of Inertias - Free Vibration

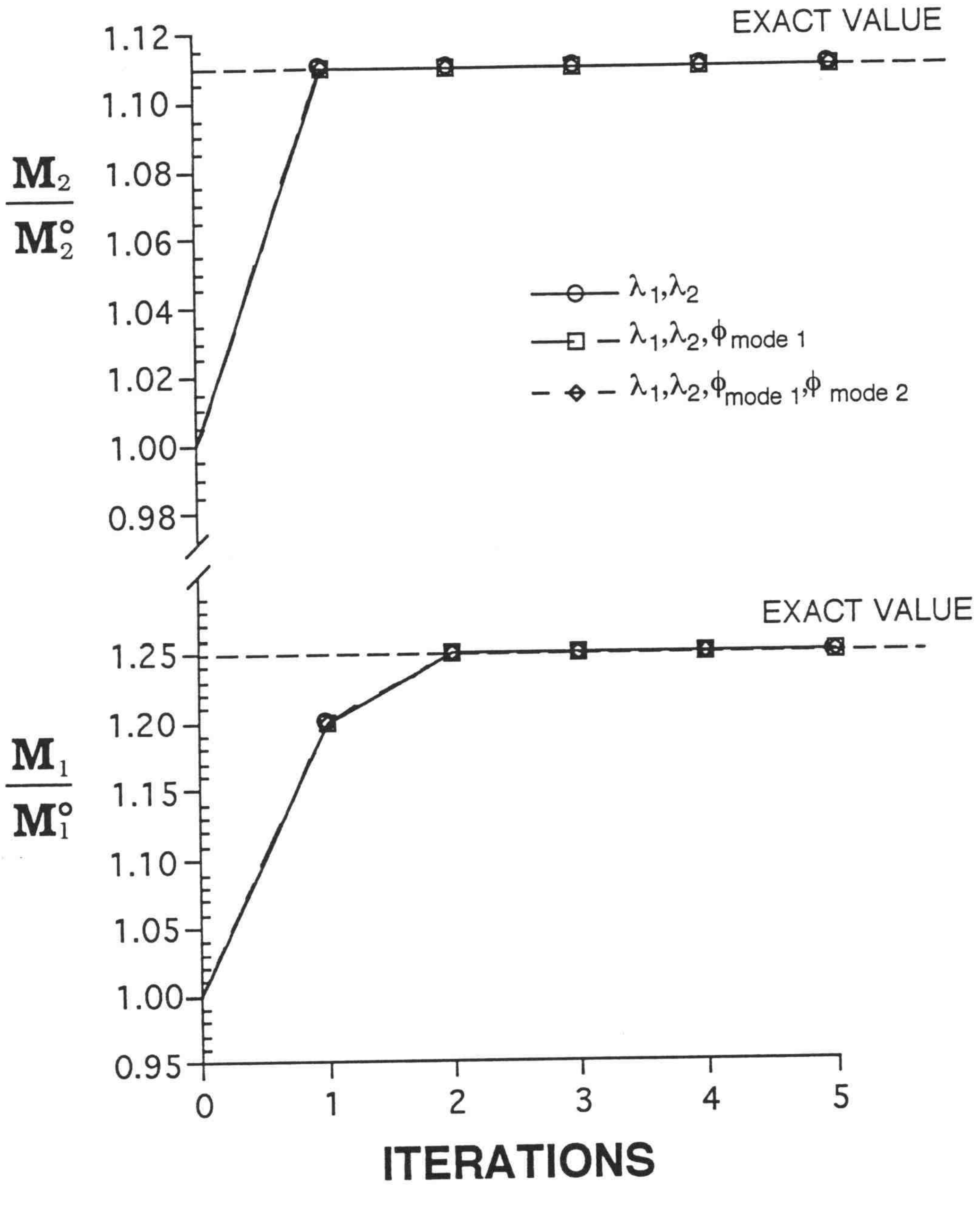


Figure 25. Effect of Type of Data on Convergence of Masses - Free Vibration

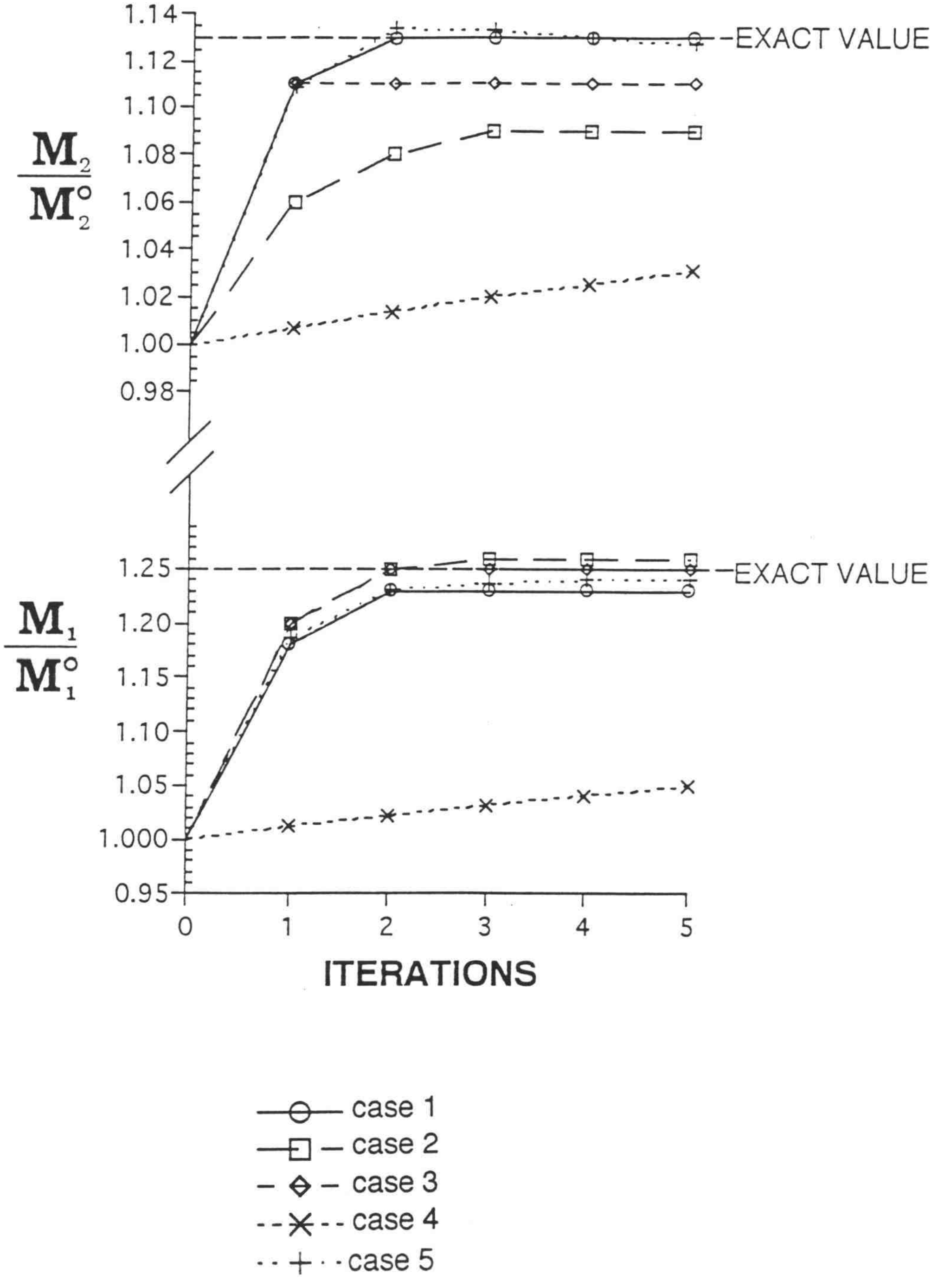


Figure 26. Effect of Assumed Variances on Convergence of Masses - Free Vibration

on the data given in Table 3. All of the parameter estimations use the first two natural frequencies as the measured response.

Case 3 represents the Weighted Least Squares (WLS) solution which, for this problem, produces exactly the same results as the Least Squares (LS) method. It has already been noted in Chapter III that both schemes produce identical results when the number of data equals the number of parameters being operated on, regardless of the makeup of the data variance matrix, $[V_{\epsilon\epsilon}]$. It can be seen from the figure that this solution is also the best one for this problem. No subjective information is being used (i.e., engineer-supplied estimates of the quality of the data) : therefore, the results will not be skewed by any judgements made by the user.

The Bayes solution in Case 4 represents the poorest solution. From the figure, it can be seen that there is very little revision of the parameters. This behavior is a direct consequence of the content of the two variance matrices. The parameters have been assumed to contain little error relative to the data : thus, the identification process will allow little modification to the parameters.

The best Bayes solution is represented by Case 5, for which both parameters and both measured responses are equally weighted. In addition, the parameters are assumed to be more in error than the data. Of the four Bayesian

Case	Scheme	$V_{\epsilon\epsilon}$	V_{rr}
1	Bayes	0.10, 0.05	0.10, 0.10
2	Bayes	0.10, 0.05	0.10, 0.05
3	WLS (=LS)	0.10, 0.05	Not Applicable
4	Bayes	0.20, 0.20	0.02, 0.02
5	Bayes	0.05, 0.05	0.10, 0.10

Table 3. Assumed Variances for Data and Parameters

estimation cases, this case converges to mass values which are closest to the true ones.

For Case 1, the identified value for the first floor mass is farther from the true value than the identified solution for the second floor mass. This is attributable to the relationship between the assumed parameter and data variances for M_1 and M_2. For M_1, the uncertainty in data and parameter is assumed equal. For M_2, the paramater value is more uncertain than the data; thus, this mass will experience more adjustment.

To understand the behavior of the Case 2 solution, comparison should be made to Case 1. The only change from Case 1 is in the assumed variance of M_2 : this value has been cut in half and is now equal to the assumed data variance. Since neither parameter is more uncertain than the other (relative to the data), they both participate equally in the adjustment process. The result is that M_1 now converges to a solution very near the exact value. Note that the solution for M_2 still exhibits about the same amount of error, but it now converges to a value below, rather than above, the true one.

Several other cases were also investigated for this problem. For these cases, the same relationships between the data and parameter variances were used, but the magnitudes of the variances were changed (by as much as a factor of 100). Results which are similar to the behavior detailed above were observed.

Observations from this study illustrate an inherent difficulty in Bayesian estimation : the identification process is very sensitive to the assumptions made regarding the variance matrices $[V_{\epsilon\epsilon}]$ and $[V_{rr}]$. Unless the analyst has great, justifiable confidence in the content of these matrices, Least Squares and Weighted Least Squares estimation are to be preferred.

This study of the effects of the variance matrices on parameter estimation was repeated for forced vibration identification. Similar behavior resulted and the same conclusions can be drawn.

Example 3 : Cantilever Beam

Figure 27 shows a non-prismatic, steel cantilever beam consisting of two-elements. Free vibration identification for this structure has been reported by Collins, et al. (4) and by Leonard and Warren (1).

The element cross-sectional areas were both fixed at 100 in^2. The first two natural frequencies and corresponding mode shapes were calculated for the following target values : I_1 = 888 in^4, I_2 = 988 in^4, and $\bar{m}_1 = \bar{m}_2 = 0.888$ slugs/in. These frequencies and mode shapes correspond to lateral modes of vibration and were used as the experimental response for the Bayesian identification process.

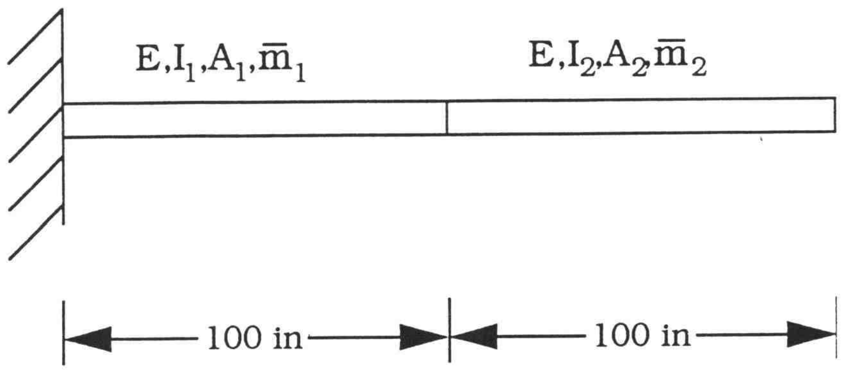


Figure 27. Cantilever Beam

For the first part of this example, the two element inertias were established at initial values of $I^0_1 = 957$ in^4 and $I^0_2 = 1148$ in^4. Compared to the true values of the inertias, these "best guess" values are 7.7 percent and 16.2 percent high, respectively.

Using the first two natural frequencies(λ_1, λ_2), the two inertias converged exactly to the target values in two iterations. The convergence characteristics can be seen in Table 4. The table shows that the calculated frequencies (based on the revised values of the parameters) also converged to the pseudo-experimental values. These results are in agreement with those reported by other researchers (1,4).

The identification process was repeated using the first two natural frequencies and the first two mode shapes to guide to process. The inclusion of the modes shapes resulted in negligible differences to the behavior seen in the table. This is also in agreement with findings previously reported in the literature.

The identification of the inertias was performed using Bayesian estimation with variances $[V_{rr}] = [I]$ for the parameters and $[V_{\epsilon\epsilon}] = [0]$. These variance matrices were used in the previous studies, and indicate complete confidence in the data and large, equal uncertainty in the parameters. However, it should be noted that Bayesian estimation for this problem using these variances is identical to Least

Iter	I_1	% Err	I_2	% Err	λ_1	% Err	λ_2	% Err
0	956.6	+7.73	1148.0	+16.2	2449	+8.31	69810	+11.6
1	887.8	-0.02	985.5	-0.25	2260	-0.04	62500	-0.13
2	888.0	0.00	988.0	0.00	2261	0.00	62580	0.00

Table 4. Convergence of Inertias and Squared Frequencies

Squares and Weighted Least Squares estimation, the number of parameters is equal to the number of data. This was noted in the previous chapter.

For the second part of this example, Bayesian identification was performed for the element mass densities, using the first two natural frequencies to guide the identification process. Initial values of these parameters were taken as $\overline{m}^0{}_1$ = 1.021 slugs/in (15% high) and $\overline{m}^0{}_2$ = 1.066 slugs/in (20% high). The results of this analysis are shown in Table 5. It is seen that the convergence rate and convergence accuracy is similar to the identification of the inertias.

The same variance matrices were used for this part of the example; thus, the results are identical to those obtained from either of the two other identification schemes. The previous studies referenced above did not attempt identification of the element mass densities.

<u>Example 4 : Cantilever Beam with Concentrated Mass</u>

The two-element cantilever beam shown in Fig. 28 is used in conjunction with Least Squares estimation for this example problem. Free vibration identification on the two cross-sectional areas is performed first. Next, the beam is subjected to a suddenly-applied load and forced vibration identification is executed. Simultaneous identification is made of the element mass densitites and moments of inertia,

Iter	$\bar{m}_1$	% Err	$\bar{m}_2$	% Err	λ_1	% Err	λ_2	% Err
0	1.021	+15.0	1.066	+20.0	1891	-16.4	53060	-15.2
1	0.869	-2.14	0.853	-3.94	2351	+3.98	64670	+3.34
2	0.888	0.00	0.887	-0.11	2264	+0.13	62640	+0.10
3	0.888	0.00	0.888	0.00	2261	0.00	62580	0.00

Table 5. Convergence of Mass Densities and Squared Frequencies

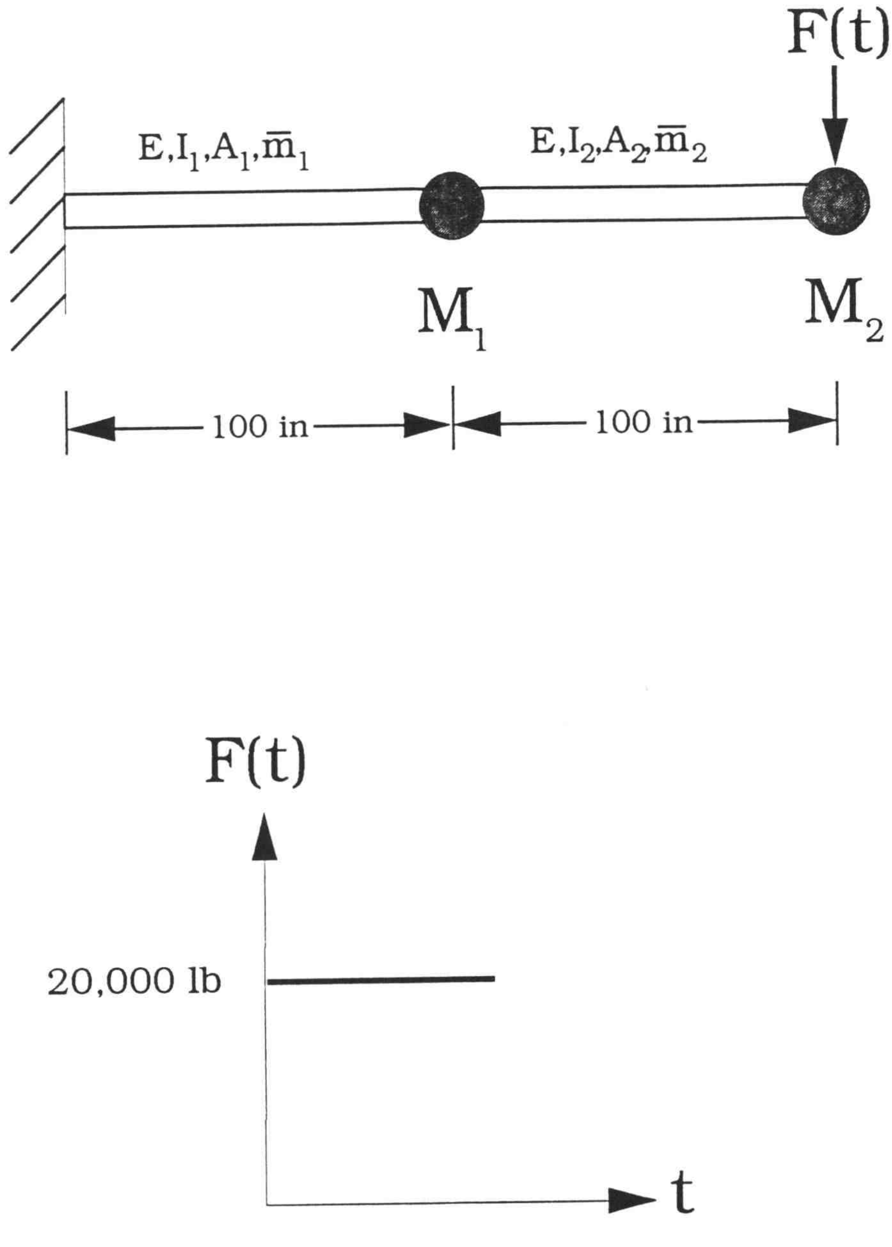


Figure 28. Cantilever Beam with Concentrated Mass

followed by simultaneous identification of the element inertias and the Rayleigh damping coefficient α.

The beam has concentrated masses at midspan and at the tip. Pseudo-experimental results were obtained using the following data :

$$I_1 = 888 \text{ in}^4 \qquad A_1 = 24 \text{ in}^2$$

$$I_2 = 988 \text{ in}^4 \qquad A_2 = 15 \text{ in}^2$$

$$\overline{m}_1 = 0.888 \text{ slugs/in} \qquad \overline{m}_2 = 0.888 \text{ slugs/in}$$

$$M_1 = 124 \text{ slugs} \qquad M_2 = 77.6 \text{ slugs}$$

$$E = 30 \times 10^6 \text{ psi}$$

The first study involves free vibration of the beam. The undamped natural frequencies (load $F(t) = 0$) corresponding to the first two axial modes of vibration were calculated and used as pseudo-experimental data for identification of the element areas. Initial estimates of the areas were taken as $A^0{}_1 = A^0{}_2 = 30$ in^2, which are 25 percent and 100 percent higher than their true values. Both areas converged to the exact values in three iterations, as shown in Table 6. The table also shows that the calculated frequencies converged to the pseudo-experimental values in the same number of iterations. This behavior is consistent with the results presented in the previous example problem.

The second study involves forced vibration of the beam. Forced vibration identification with damping was performed, using a time step of 0.01 sec. For this study, the tip of

Iter	A_1	% Err	A_2	% Err	λ_1 x 10^6	% Err	λ_2 x 10^6	% Err
0	30.00	+25.0	30.00	+100	2.766	+39.1	16.230	+79.8
1	22.83	-4.88	15.43	+2.87	1.933	-2.77	9.083	+0.61
2	23.94	-0.25	15.02	+0.13	1.985	-0.15	9.031	+0.03
3	24.00	0.00	15.00	0.00	1.988	0.00	9.028	0.00

Table 6. Convergence of Areas and Squared Frequencies

the cantilever is subjected to a suddenly applied force of 20,000 lbs, as shown in Fig. 28. The Rayleigh damping coefficient α was set to 2.90; this corresponds to damping in the first mode equal to 5 percent of critical.

Two cases were examined. First, the two element inertias were identified simultaneous with the two element mass densities. The initial values of these parameters were assumed as :

$$I^0_1 = 957 \ \text{in}^4 \quad (7.7\% \text{ high})$$
$$I^0_2 = 1148 \ \text{in}^4 \quad (16.2\% \text{ high})$$
$$\overline{m}^0_1 = 1.021 \text{ slugs/in} \quad (15\% \text{ high})$$
$$\overline{m}^0_2 = 1.066 \text{ slugs/in} \quad (20\% \text{ high})$$

The pseudo-experimental data used for the identification of these four parameters consisted of the lateral accelerations at midspan and at the tip. These data were provided at the first five time steps. Using these response data, the parameters converged to within 0.1 percent of their target values in two iterations, as did the calculated accelerations.

The second case studied involved the simultaneous identification of the two mass densities and the damping coefficient. The initial densities given above were used, along with an initial estimate for the damping given by $\alpha^0 = 3.50$ (20% high). The identification process used the same kinematics as the previous case. The densities and the calculated accelerations converged to within 0.2 percent of the exact values in four iterations; the damping coefficient converged within 1 percent.

Example 5 : Tower with Concentrated Mass and Stiffness

The identification of concentrated stiffness is the subject of this example. The structure shown in Fig. 29 models a guyed tower on an elastic foundation with partial rotational restraint at the base. The nonlinear effect of the guys are replaced by a concentrated stiffness, approximated by a linear spring of stiffness K_2. Rotational flexibility at the base is provided by pinning the structure at this node and attaching a rotational spring of stiffness K_1.

Least Squares identification of the two stiffnesses was performed for both free and forced vibration. Structural data used to generate the pseudo-experimental responses are as follows :

$I = 144 \text{ in}^4$ $\qquad K_1 = 9.0 \times 10^8$ lb-in/rad

$A = 7.61 \text{ in}^2$ $\qquad K_2 = 3.0 \times 10^4$ lb/in

$\overline{m} = 0.500$ slugs/in $\qquad M = 2485$ slugs

$E = 30 \times 10^6$ psi

For free vibration identification, the pseudo-experimental data consisted of the first two natural frequencies corresponding to lateral vibration. Three cases were

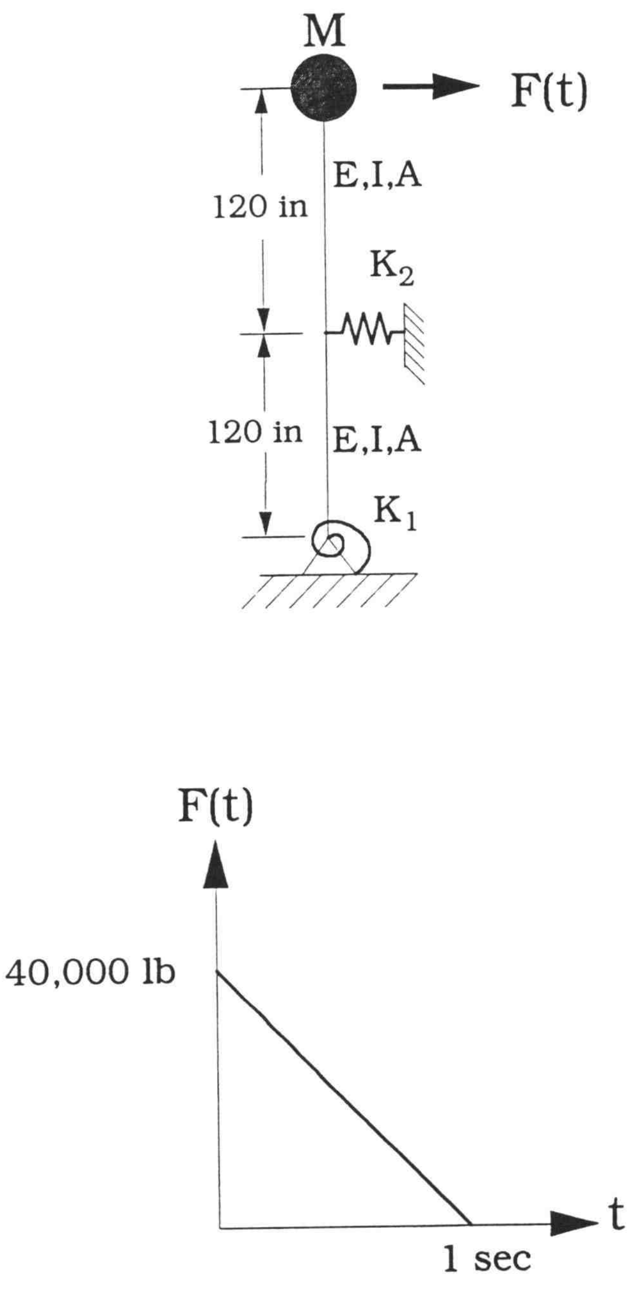


Figure 29. Tower with Concentrated Mass and Stiffness

investigated : 1) identification of K_1 alone, 2) identification of K_2 alone, and 3) simultaneous identification of K_1 and K_2. Initial values were set to $K^0_1 = 18.0 \times 10^8$ lb-in/rad and $K^0_2 = 6.0 \times 10^4$ lb/in, which are 100 percent higher than the target values.

For the first case (identify K_1 only), the value of the rotational stiffness converged to within 0.1 percent of the exact value in three iterations, as did the calculated frequencies. The second case (identify K_2 only) provided convergence of the linear concentrated stiffness and of the two calculated frequencies in one iteration. Convergence was within 0.1 percent. In the final case, simultaneous identification of both stiffness values K_1 and K_2 produced convergence in three iterations. Convergence within 0.1 percent was achieved for K_1, K_2, and both natural frequencies.

The forcing function shown in the figure was used in the forced vibration study with a 0.025 sec time step. The lateral accelerations at midspan and at the tip were used as pseudo-experimental data in the identification process. These accelerations were provided at the first four time steps.

The three cases described above were also used for this part of the problem. When identifying K_1 alone, convergence of the concentrated stiffness and of the two accelerations occurred in three iterations to within one percent of the exact values. For K_2 alone, convergence of the same quantities to the same tolerance occurred in two iterations. Simultaneous identification resulted in convergence of the two concentrated stiffness values and of the calculated accelerations to within 0.3 percent. Convergence occurred in four iterations.

Example 6 : Two-Dimensional Guyed Tower in Waves

A structure subjected to deep water waves is investigated in this example. System identification techniques are used to account for uncertainties and simplifications related to the hydrodynamic loading and response. Least Squares identification is used.

Two studies were performed for this structure. The first study compares the response obtained with the relative velocity form of Morison's equation to the response obtained from the linearized form of this equation. The second study investigates the use of effective mass and damping matrices to account for uncertainties in C_D and C_M and for inaccuracies introduced by the linearization of Morison's equation.

The ocean-based guyed tower (65) is shown in Fig. 30. The tower is 1600 feet tall and supports a deck of mass M_D at its top. The nonlinear guys were approximated by a linear spring with a concentrated stiffness, K_G. The structure data are as follows :

$I = 1.571 \times 10^5\ ft^4$ $\qquad$ $M_D = 466.4$ kslugs

$A = 62.80\ ft^2$ $\qquad$ $K_G = 109$ k/ft

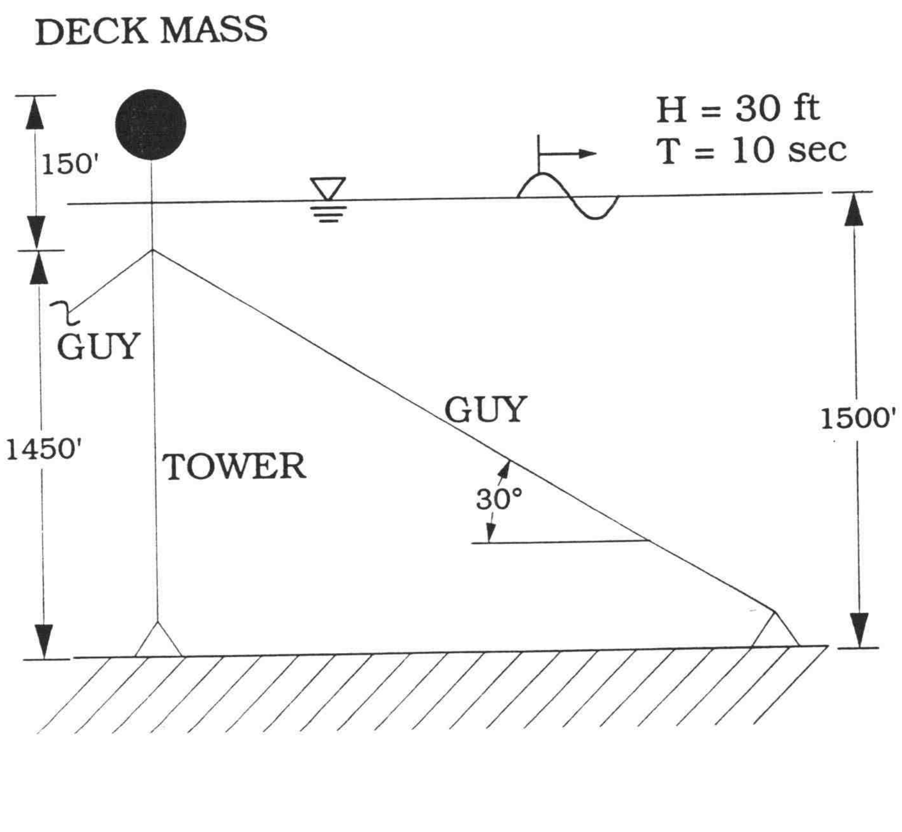


Figure 30. Two-Dimensional Guyed Tower in Waves

$\alpha = 0$ $\beta = 0.136$

$E = 4.176 \times 10^6$ ksf

$\overline{m} = 0.778$ kslugs/ft (including trapped water)

The values of α and β correspond to structural damping equal to 2 percent of critical damping in the first mode.

The tower is located in 1500 feet of water and is subjected to a train of regular waves with a height of 30 feet and a period of 10 seconds. Airy (linear) wave theory was used to calculate the wave kinematics (9,66). The basic hydrodynamic data are :

Drag coeff. : $C_D = 0.7$

Inertia coeff. : $C_M = 1.8$

Fluid density : $\rho = 0.0020$ kslug/ft^3

Drag width (projected width) = 115 ft

Displaced volume = 312.5 ft^3/ft

The drag width and displaced volume are used to calculate the projected area $\overline{A}$ and displaced volume $\overline{V}$ in the equations which follow. The structure was modeled with two beam elements. Five degrees-of-freedom were included : two lateral displacements and three rotations. A time step of $\Delta t = 0.5$ seconds was used.

As discussed in Chapter I, the load on the structure is given by Morison's equation, Eq. (1.3). The loading specified by this equation is dependent on the relative velocity between the fluid and the structure :

$$\{P(t)\} = \rho C_M[\overline{V}]\{\ddot{u}\} - \rho(C_M-1)[\overline{V}]\{\ddot{x}\} + \\ + (0.5\rho C_D[\overline{A}])\{|\dot{u}-\dot{x}|\}\{\dot{u}-\dot{x}\}^T \qquad (1.3)$$

This form of the load, when substituted into the equations of motion given by Eq. (1.1), results in the following equations for the response of the structure :

$$([M] + \rho(C_M-1)[\overline{V}])\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \\ = \rho C_M[\overline{V}]\{\ddot{u}\} + (0.5\rho C_D[\overline{A}])\{|\dot{u}-\dot{x}|\}\{\dot{u}-\dot{x}\}^T \qquad (6.1)$$

If Morison's equation is linearized to eliminate the relative velocity term, the structure loading and response is given by Eq. (1.4) :

$$([M] + \rho(C_M-1)[\overline{V}])\{\ddot{x}\} + ([C] + 0.5\rho C_D\Psi[\overline{A}])\{\dot{x}\} + \\ + [K]\{x\} = \rho C_M[\overline{V}]\{\ddot{u}\} + (0.5\rho C_D[\overline{A}])\{|\dot{u}|\}\{\dot{u}\}^T \qquad (1.4)$$

For both studies, pseudo-measured response was generated for the structure using the data given above; the relative velocity form of the equations was used, Eq. (6.1). In this equation, the force at each time step is a function of the relative velocity $\dot{u}-\dot{x}$. However, this term cannot be calculated since the structure velocity $\dot{x}$ is unknown at the current time step. To overcome this difficulty, a technique was used which "lags" the structure velocities by one time step; i.e., $\dot{x}$ from the previous time step is used with the fluid velocity $\dot{u}$ from the current time step. The kinematic response of the structure was calculated, and the displacement of the deck was taken as the pseudo-measured response. The displacement time-history is plotted in Figs. 31 and 32.

The first study investigated the effects of linearizing the relative velocity term in Morison's equation. The response of the structure was calculated for the given data using Eq. (1.4). The displacement is plotted in Fig. 31, and is labeled "response based on initial estimate".

The response error which is introduced by this simplification can be reduced, to some extent, by increasing the amount of structure damping (12). Therefore, system identification was used to estimate a new value of the Rayleigh damping coefficient ß.

Using the pseudo-measured displacement for the first 10 time steps, an improved value of $\beta = 1.0173$ was identified in 5 iterations. The response based on this value is plotted in Fig. 31 and is labeled "response based on "best" estimate". The increase in damping produces a calculated response which is in close agreement with the response based on the relative velocity form of Morison's equation.

The second case studied also investigates the modification of structural damping to account for relative velocity effects. In addition, initial values of the fluid force coefficients were selected to be different from the values used to generate the pseudo-measured response. This simulates the typical modeling situation in which these values are subject to significant uncertainty. Both the drag

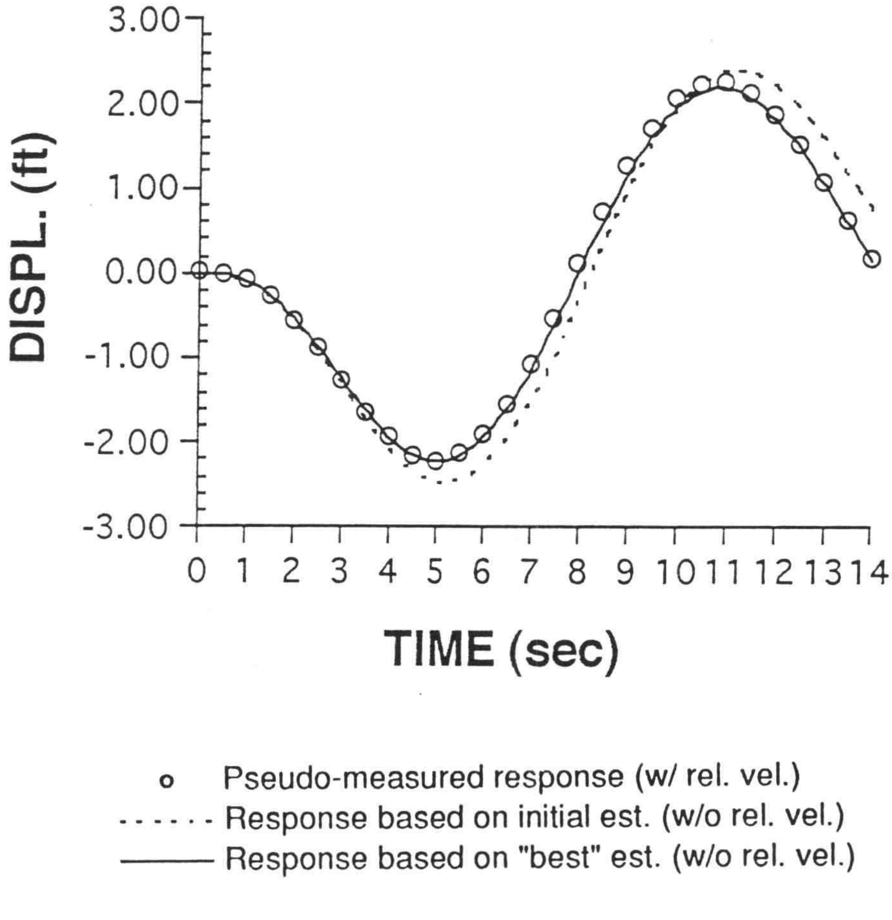


Figure 31. Calculated Displacement vs Experimental Displacement - No Relative Velocity Term

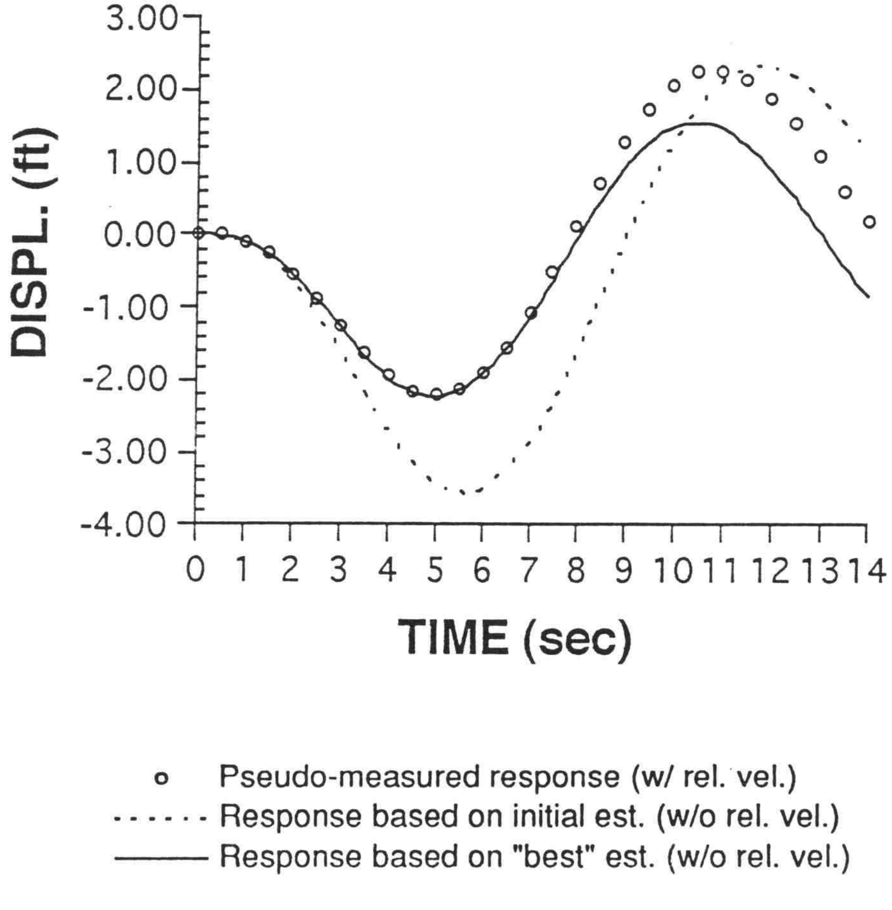


Figure 32. Calculated Displacement vs Experimental Displacement - Mass, Stiffness, and Damping Identified

coefficient C_D and the inertia coefficient C_M are affected by real fluid effects and are not time-invariant (9); appropriate values are therefore difficult to establish.

The response of the structure was calculated using the structure data previously given and initial values of $C_D = 1.2$ (71% high) and $C_M = 2.0$ (11% high). The response was based on Eq. (1.4), the linearized form of Morison's equation. This form does not include the relative velocity term. The curve in Fig. 32 labeled "response based on initial estimate" shows the time-history of this response. The pseudo-measured response generated with the relative velocity form of Morison's equation is also shown in this figure. The difference in these responses is apparent from the figure.

Parameter identification was next used to allow the finite element model to more accurately predict the response of the structure. The effective mass and damping matrices in Eq. (1.4) were modified to account, to some extent, for the inaccuracies introduced by C_D, C_M, and the lack of the relative velocity term. This was accomplished through estimation of the Rayleigh damping coefficients and a concentrated (lumped) mass. The guy stiffness K_G was also identified. The initial value of the lumped mass was set equal to zero.

The same data points used for the first study were also used for this identification problem. Converged values for the four parameters were obtained in 5 iterations. These values are : $\alpha = 0.290$, $\beta = 0.143$, $M^L = 128$, and $K_G = 83.8$. The response based on these improved estimates is shown in Fig. 32 and is labeled "response based on "best" estimate". It can be seen that the finite element model is a better predictor of the response when these identified values of the parameters are used.

CHAPTER VI

CONCLUSION

Summary

System identification techniques for free and forced vibrations of finite element models have been presented and validated in the previous chapters. These techniques were shown to be a useful tool for the systematic improvement of finite element models when data from tests of structures are available.

Classical linear estimation theory was presented, and the weighting matrices were derived for Least Squares, Weighted Least Squares, and Bayesian identification. The estimation equations for structural parameter identification were derived from a first order Taylor Series. These equations were shown to be of the same form as the equations for linear estimation when the parameter vector is random. Forms of the sensitivity matrix were presented for both free and forced vibration. The sensitivity matrix was expressed as the product of two submatrices. One of these relates changes in structural response to changes in the structure mass and stiffness matrices; the other relates changes in these matrices to changes in the basic structural parameters.

Specific forms of the first submatrix were given for free vibration identification. For identification with forced vibration, various forms of this matrix were derived from the basic relationships used in the Wilson Theta method. One of these forms relates the change in the calculated kinematics to the structure mass and stiffness matrices. This matrix is to be used for identification of all parameters except the Rayleigh damping coefficients. Another form of this submatrix was derived for identification of the damping parameters. The derived expressions relate the change in kinematics directly to changes in the coefficients.

For identification of element parameters, it was shown that the second submatrix can be formed from the element stiffness matrix by setting the parameter values to unity. When performing estimation for concentrated mass and stiffness, the submatrix was shown to consist of zeroes and ones, the latter located at positions corresponding to the concentrated parameters. For identification of damping, it was seen that the second submatrix is the identity matrix.

Implementation of the system identification procedure in a computer program was described. In this program, the identification calculations and manipulations are isolated from the finite element routines. To minimize computer memory requirements, an overlay system was used along with offline storage of the experimental data, sensitivity matrices, and load vector. Efficient techniques for forming the various required matrices were detailed.

The present study was shown to be a significant and useful addition to the current techniques of system identification as applied to finite element models. Most of the methods previously developed operate on specialized models subject to many restrictions and limitations; only a few allow system identification techniques to be applied to standard finite element models. Of these, none have addressed identification for forced vibration. Previous studies of free vibration identification have not addressed mass density, concentrated mass, or concentrated stiffness. The current study is also the first to investigate the identification of structural parameters to account for uncertainties in the hydrodynamic loading.

Six example problems were presented. A general discussion of these examples follows.

Discussion

A one-story frame acted upon by a time-varying load was examined in the first example. The column moment of inertia, floor mass, and damping coefficient were the parameters identified. The effects of the quantity, type, and spacing of experimental data were investigated. The behavior of the solution was found to be relatively unaffected. The improved estimates of inertia and mass were seen to converge rapidly; the damping coefficient converged at a rate which was slightly slower. The identification procedure operated successfully with noisy data.

A two-story frame was investigated in the second example. Simultaneous identification of two or more parameters was studied. The two column inertias and two floor masses converged rapidly to the correct values using the floor accelerations as the measured response. Convergence of the two damping coefficients was slightly slower. In all cases, the predicted response was in close agreement with the measured response. The quality of the solution and the rate of convergence were not affected by the type of data (accelerations of first floor only, second floor only, or both floors). Identification of four parameters simultaneously was also presented.

Free vibration studies were also presented in the second example. Convergence of the inertias and masses was seen to be faster than for forced vibration identification. An investigation was made of the effects of parameter and data variances on the behavior of Bayesian estimation. It was concluded that the Bayes scheme is quite sensitive to the content of the variance matrices. If the analyst is unsure of the values to use in these matrices, Least Squares or Weighted Least Squares estimation is to be preferred.

The cantilever beam in the third example was used to demonstrate free vibration identification of element mass density and moment of inertia. Rapid convergence of these parameters to the exact values was achieved using the first two natural frequencies as measured response. Inclusion of the mode shapes did not noticeably improve the solution.

In the fourth example, a cantilever beam with two concentrated (lumped) masses was analyzed. The two cross-sectional areas were successfully identified using free vibration natural frequencies. Forced vibration identification was performed for four element parameters simultaneously : the two inertias and the two mass densities. Convergence was achieved rapidly using the midspan and tip accelerations. The analysis was repeated, this time for the two inertias and one of the Rayleigh damping coefficients : quick convergence was exhibited.

A tower supported by a linear spring and a rotational spring was presented as the fifth example. These concentrated stiffnesses were identified singly and simultaneously, for both free and forced vibration. Even though the initial estimates of these parameters were in error by 100 percent, convergence to the exact values was achieved within a few iterations.

The final example was used to investigate hydrodynamic effects on a guyed tower subjected to ocean waves. Pseudo-experimental response data were generated using the relative velocity form of Morison's equation. The model was then solved for the same data, but using a form of the equation without the relative velocity term. One of the Rayleigh damping coefficients was selected to be modified by the identification process. The improved estimate of this parameter resulted in a very close match between the predicted and experimental responses. This suggests that the relative velocity effects in Morison's equation can be accounted for (in some instances) by an adjustment to the damping of the model. A second study was presented for this example, in which modeling errors were present (in the mass and drag coefficients), and relative velocity effects were neglected in the fluid loading. By allowing the identification procedure to operate on four structural parameters (concentrated mass, concentrated stiffness, and the two Rayleigh damping coefficients), the predicted response was brought into closer agreement with the experimental data. Thus, uncertainties in the hydrodynamic parameters and loading can be partially corrected for by adjustments to the mass, damping, and stiffness characteristics of the structure.

Possible Extensions

Several possible extensions to the present study suggest themselves :

(1) Storage requirements and execution time can be extensive for forced vibration identification. A large portion of this is attributable to the amount of calculations necessary to evaluate the

kinematic sensitivities. The sensitivity of the response must be calculated with respect to each degree-of-freedom : however, the experimental data are typically specified at a small subset of these d.o.f. Significant reductions in storage space and execution times could be realized if a reduced set of kinematic sensitivities could be used.

(2) Although convergence of the Rayleigh damping coefficients is easily achieved, the rate of convergence is slower than for other parameters. Techniques to speed the convergence of damping analyses should be investigated.

(3) For free vibration identification, rapid convergence was obtained when the natural frequencies were specified as the measured data. Inclusion of the mode shapes has negligible effect on the process. Further studies are needed to investigate the mode shape sensitivities and their effect on parameter estimation, with special attention directed toward increasing the participation of these terms.

(4) In some situations encountered in forced vibration identification, the adjustments made to the parameters during the first iteration is larger than is needed. Convergence characteristics could

be improved if a relaxation method were developed to inhibit this overcompensation.

(5) The element mass densities were used in the lumped mass form of the element mass matrix. The program could be easily modified to allow an option to use the consistent mass formulation for structures where this is deemed necessary. However, the use of consistent mass matrices will result in a full structure mass matrix, with nonzero, off-diagonal terms. This will increase the execution time and storage requirements for both free and forced vibration problems.

(6) It was shown that errors and simplifications in the hydrodynamic load terms can be accounted for, to some extent, by performing identification of the structural parameters. Further investigations of this technique are needed.

(7) Although the effect of noisy data was examined, more insight into the identification process could be gained by conducting tests of real structures under closely controlled conditions. The measured responses from these tests could be used with the identification program to further investigate the effect of uncertainties in the experimental data.

(8) System identification was performed for parameters which appear as linear terms in the stiffness matrix. The work presented in this study can be extended to include the estimation of nonlinear terms, such as the product of parameters and parameters which appear with mixed powers. An example of the former is the product EI for a beam subjected to lateral bending. The latter case occurs, for example, when analyzing a thin plate for lateral bending and membrane effects. In this case, bending is a function of the cube of the plate thickness, while the in-plane effects are a function of the thickness.

(9) The present study was limited to finite element models based on linear behavior theory. Further investigations should be made in which nonlinear behavior is considered.

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