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Title ON THE GENERALIZATION AND APPLICATION OF THE EULER-MACLAURIN FORMULA

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The Euler-MacLaurin sum formula has appeared in the titles of two quite recent papers whose authors were primarily interested in certain applications. In this paper a somewhat different approach to the myriad of formulas for summation, integration, differentiation, etc., is based on the simple identity which defines the set of Bernoulli numbers. Variations of this identity are obtained by the most elementary manipulations, then application of the Laplace transformation leads to the well-known formulas, trapezoidal rule, Simpson's rule, etc., complete with an infinite series of higher derivatives. This type of formula is particularly valuable in carrying out a Frank type of inversion of a Laplace transform. In particular, the Frank method has been extended to the alternating series case. The representation of error of the approximation formula by means of an integral involving a periodic polynomial has been extended to Simpson's rule, with indication of a general method for extending the theory for more general approximation formulas.

ON THE GENERALIZATION AND APPLICATION OF THE EULER MACLAURIN FORMULA

by

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INTRODUCTION

Frank [6, p. 89-91] and Gould and Squire [7, p. 44-52] have made some recent extensions and applications of the well-known Euler-MacLaurin sum formula. Authors of many textbooks and treatises have employed this method to arrive at estimates of such finite sums as

$$\sum_{n=1}^{N} 1/n = \log N + \gamma - 1/2N - 1/12N^2 + 1/120N^4 - \dots,$$

or of the factorial

$$n! = n^n e^{-n} \sqrt{2\pi n}\, \exp[\alpha/12(n-1)], \quad 0 < \alpha < 1.$$

Frank showed that the geometric series expansion of certain types of Laplace transforms may be summed in such a way that the inversion yields a rapidly converging series regardless of whether or not the time variable is taken to be large or small. This method requires the infinite series form of the Euler-MacLaurin formula, rather than a finite series. All integration and differentiation formulas obtained below will have this feature, something not usually found in the standard texts. Gould and Squire developed a second form of the Euler-MacLaurin formula with the important property that the algebraic sign of the estimated remainder term would be opposite that of the corresponding term for the first form. Their formula has been called the second Euler-MacLaurin formula by Hildebrand [8, p. 154].

The purpose of this paper is to (a) develop a number of these types of formulas by elementary operations, (b) extend the Gould and Squire and Frank results to the alternating series case, and (c) study the nature of the error associated with the several approximation formulas.

BERNOULLI IDENTITIES

The basic identity is No. (1) of Erdelyi, et al. [4, p. 51]

$$(1) \qquad \frac{1+e^{-z}}{1-e^{-z}} = 2\sum_{k=o}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k-1}, \qquad |z| < 2\pi,$$

where the B_{2k} are the Bernoulli numbers. A partial list is

$B_o = 1$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$,

$B_{14} = \frac{7}{6}$, $B_{16} = -\frac{3617}{510}$, $B_{18} = \frac{43867}{798}$, $B_{20} = -\frac{174611}{330}$, $B_{22} = \frac{854513}{138}$,

$B_{24} = -\frac{236364091}{2730}$, $B_{26} = \frac{8553103}{6}$, $B_{28} = -\frac{23749461029}{870}$, $B_{30} = \frac{8615841276005}{14322}$,

... . A list complete through B_{60} is given by Fort [5, p. 49] ;probably the most extensive list was published by Adams [1,p. 259-272], all the way through B_{124}. It is interesting to note that B_{120} has 2358255930 for a denominator while B_{122} has 6 for a denominator and a number with 107 digits for numerator.

It is preferable to set $C_{2k} = \frac{B_{2k}}{(2k)!}$, $k = 0, 1, 2, \ldots$, since these are the coefficients of interest in the several developments below:

$C_0 = 1$, $C_2 = \frac{1}{12}$, $C_4 = -\frac{1}{720}$, $C_6 = \frac{1}{30240}$, $C_8 = -\frac{1}{1209600}$, $C_{10} = \frac{1}{47900160}$

$C_{12} = -5.350748 \cdot 10^{-10}, \ldots$. The basic identity may be written as

$$(2) \qquad \frac{1+e^{-z}}{1-e^{-z}} = 2\sum_{k=o}^{\infty} C_{2k} z^{2k-1}, \qquad |z| < 2\pi .$$

The restriction on the value of $|z|$ need not be repeated in the development of additional identities. It is required since the B_{2k} grow very rapidly; an asymptotic form is

$$C_{2k} = \frac{B_{2k}}{(2k)!} \approx \frac{2(-1)^{k-1}}{(2\pi)^{2k}},$$

given by Knopp [9, p. 527].

Additional forms of the basic identity depend on the identity $(1-z)(1+z+\ldots+z^{n-1}) = 1-z^n$. The first such extended form is

$$(4) \qquad \frac{(1+e^{-z})^2}{1-e^{-2z}} - a\,\frac{1+e^{-2z}}{1-e^{-2z}} = 2\sum_{k=o}^{\infty} C_{2k}[1-2^{2k-1}a]\; z^{2k-1}.$$

There are three ideas governing the choice of values of this undetermined parameter, and of others introduced below, the development of (1) an open end integration formula, (2) a closed end integration formula, and (3) a differentiation formula. In each case the complete infinite series expansion is included. Equation (2), as it stands, will lead to the simplest closed end integration formula, the trapezoidal rule. Equation (4) cannot lead to an open end formula but it does lead at once to the well-known Simpson rule and variations of same if it be required that

$$(5) \qquad 1 - 2^{2k-1}a = 0, \quad k = 1, 2, 3, \ldots .$$

It is convenient to consider the first three values of k; for $a = \frac{1}{2}$ Eq. (4) becomes

$$(6) \qquad \frac{1}{3}\,\frac{1 + 4e^{-z} + e^{-2z}}{1-e^{-2z}} = \frac{1}{z} - \frac{4}{3}\sum_{k=2}^{\infty} C_{2k}(2^{2k-2}-1)z^{2k-1}\,.$$

For $a = \frac{1}{8}$,

$$(7) \qquad \frac{1}{15}\,\frac{7 + 16e^{-z} + 7e^{-2z}}{1-e^{-2z}} = \frac{1}{z} + \frac{4}{5}C_2 z - \frac{16}{15}\sum_{k=3}^{\infty} C_{2k}(2^{2k-4}-1)z^{2k-1}\,.$$

For $a = \frac{1}{32}$,

$$(8) \qquad \frac{1}{63}\,\frac{31 + 64e^{-z} + 31e^{-2z}}{1-e^{-2z}} = \frac{1}{z} + \frac{20}{21}C_2 z + \frac{16}{21}C_4 z^3 - \frac{64}{63}\sum_{k=4}^{\infty} C_{2k}(^{2k-6}-1)z^{2k-1}\,.$$

The formula for differentiation results if k is taken to be zero in Eq. (5), which yields $a = 2$ and Eq. (4) becomes

$$(9) \qquad \frac{1}{6}\,\frac{1 - 2e^{-z} + e^{-2z}}{1 - e^{-2z}} = C_2 z + \frac{1}{3}\sum_{k=2}^{\infty} C_{2k}(2^{2k}-1)z^{2k-1}\,.$$

The next step is to write down the identity

$$(10) \qquad \frac{1 + 2e^{-z} + 2e^{-2z} + e^{-3z}}{1-e^{-3z}} - a\frac{1+e^{-3z}}{1-e^{-3z}} = 2\sum_{k=o}^{\infty} C_{2k}[1-3^{2k-1}a]\,z^{2k-1}\,.$$

and require that

$$(11) \qquad 1 - 3^{2k-1}a = 0, \quad k = 1, 2, 3, \ldots \quad .$$

For $a = \frac{1}{3}$,

$$(12) \qquad \frac{3}{8}\,\frac{1 + 3e^{-z} + 3e^{-2z} + e^{-3z}}{1-e^{-3z}} = \frac{1}{z} - \frac{9}{8}\sum_{k=2}^{\infty} C_{2k}(3^{2k-2}-1)z^{2k-1}\,.$$

For $a = \frac{1}{27}$,

(13) $$\frac{3}{80}\ \frac{13+27e^{-z}+27e^{-2z}+13e^{-3z}}{1-e^{-3z}} = \frac{1}{z}+\frac{9}{10}C_2 z-\frac{81}{80}\sum_{k=3}^{\infty}C_{2k}(3^{2k-4}-1)z^{2k-1}.$$

For $a = \frac{1}{243}$,

(14) $$\frac{3}{728}\ \frac{121+243e^{-z}+243e^{-2z}+121e^{-3z}}{1-e^{-3z}} = \frac{1}{z}+\frac{90}{91}C_2 z+\frac{81}{91}C_4 z^3-\frac{729}{728}\sum_{k=4}^{\infty}C_{2k}(3^{2k-6}-1)z^{2k-1}.$$

The formula for differentiation results if k is taken to be zero in Eq. (11), which yields $a = 3$ and Eq. (10) becomes

(15) $$\frac{1}{4}\ \frac{1-e^{-z}-e^{-2z}+e^{-3z}}{1-e^{-3z}} = C_2 z+\frac{1}{8}\sum_{k=2}^{\infty}C_{2k}(3^{2k}-1)z^{2k-1}.$$

The identity,

(16) $$\frac{1+2e^{-z}+2e^{-2z}+2e^{-3z}+e^{-4z}}{1-e^{-4z}} - a\frac{(1+e^{-2z})^2}{1-e^{-4z}} + b\frac{1+e^{-4z}}{1-e^{-4z}}$$

$$= 2\sum_{k=o}^{\infty}C_{2k}[1-2^{2k-1}a+2^{4k-2}b]z^{2k-1},$$

leads to open end integration formulas if it is required that

(17) $$1-a+b=0$$
$$1-2^{2k-1}a+2^{4k-2}b=0,\ k=1,2,3\ldots\ .$$

The solution is at once

(18) $$a=\frac{2^{4k-2}-1}{2^{4k-2}-2^{2k-1}},\ b=\frac{2^{2k-1}-1}{2^{4k-2}-2^{2k-1}},\ k=1,2,3,\ldots\ .$$

For $a = \frac{3}{2}$, $b = \frac{1}{2}$, the identity of Eq. (16) becomes

$$(19)\quad \frac{4}{3}\,\frac{2e^{-z}-e^{-2z}+2e^{-3z}}{1-e^{-4z}} = \frac{1}{z} + \frac{8}{3}\sum_{k=2}^{\infty} C_{2k}[\,1-3.\,2^{2k-2}+2^{4k-3}]\,z^{2k-1}.$$

For $a = \frac{9}{8}$, $b = \frac{1}{8}$,

$$(20)\quad \frac{4}{15}\,\frac{8e^{-z}-e^{-2z}+8e^{-3z}}{1-e^{-4z}} = \frac{1}{z} - \frac{8}{5}\,C_2 z + \frac{32}{15}\sum_{k=3}^{\infty} C_{2k}[\,1-9.\,2^{2k-4}+2^{4k-5}]\,z^{2k-1}.$$

For $a = \frac{33}{32}$, $b = \frac{1}{32}$,

$$(21)\quad \frac{4}{63}\,\frac{32e^{-z}-e^{-2z}+32e^{-3z}}{1-e^{-4z}} = \frac{1}{z} - \frac{40}{21}\,C_2 z - \frac{32}{3}\,C_4 z^3$$

$$+\frac{128}{63}\sum_{k=4}^{\infty} C_{2k}[\,1-33.\,2^{2k-6}+2^{4k-7}]\,z^{2k-1}.$$

The closed end formulas result if it be required that

$$(22)\quad \begin{aligned} 1 - 2^{2k-1}a + 2^{4k-2}b &= 0 \\ 1 - 2^{2k+1}a + 2^{4k+2}b &= 0, \quad k = 1, 2, 3\ \ldots \end{aligned}$$

The solution is at once

$$(23)\quad a = \frac{5}{2^{2k+1}}, \quad b = \frac{1}{2^{4k}}, \quad k = 1, 2, 3, \ldots\ .$$

For $a = \frac{5}{8}$, $b = \frac{1}{16}$, the identity of Eq. (16) becomes

$$(24)\quad \frac{2}{45}\,\frac{7+32e^{-z}+12e^{-2z}+32e^{-3z}+7e^{-4z}}{1-e^{-4z}} = \frac{1}{z} + \frac{64}{45}\sum_{k=3}^{\infty} C_{2k}[\,1-5.\,2^{2k-4}+2^{4k-6}]\,z^{2k-1}.$$

For $a = \frac{5}{32}$, $b = \frac{1}{256}$,

$$(25)\quad \frac{2}{945}\,\frac{217+512e^{-z}+432e^{-2z}+512e^{-3z}+217e^{-4z}}{1-e^{-4z}} = \frac{1}{z} + \frac{16}{21}C_2 z + \frac{1024}{945}\sum_{k=4}^{\infty} C_{2k}\left[1-5.2^{2k-6}+2^{4k-10}\right]z^{2k-1}.$$

For $a = \frac{5}{128}$, $b = \frac{1}{4096}$,

$$(26)\quad \frac{2}{16065}\,\frac{3937+8192e^{-z}+7872e^{-2z}+8192e^{-3z}+3937e^{-4z}}{1-e^{-4z}} = \frac{1}{z} + \frac{112}{119}C_2 z + \frac{256}{357}C_4 z^3 + \frac{16384}{16065}\sum_{k=5}^{\infty} C_{2k}\left[1-5.2^{2k-8}+2^{4k-14}\right]z^{2k-1}.$$

The differentiation formula results if it be required that

$$(27)\quad \begin{aligned} &1 - \frac{a}{2} + \frac{b}{4} = 0 \\ &1 - 2^{2k-1}a + 2^{4k-2}b = 0, \quad k = 2, 3, \ldots \end{aligned}$$

The solution is at once

$$(28)\quad a = \frac{2(2^{4k}-1)}{2^{4k}-2^{2k}}, \quad b = \frac{4(2^{2k}-1)}{2^{4k}-2^{2k}}, \quad k = 2, 3, \ldots .$$

For $a = \frac{17}{8}$, $b = \frac{1}{4}$, Eq. (16) takes the form

$$(29)\quad \frac{1}{36}\,\frac{7-16e^{-z}+18e^{-2z}-16e^{-3z}+7e^{-4z}}{1-e^{-4z}} = C_2 z - \frac{4}{9}\sum_{k=3}^{\infty} C_{2k}\left[1-17.2^{2k-4}+2^{4k-4}\right]z^{2k-1}.$$

For $a = \frac{65}{32}$, $b = \frac{1}{16}$,

$$\frac{1}{180}\,\frac{31-64e^{-z}+66e^{-2z}-64e^{-3z}+31e^{-4z}}{1-e^{-4z}} = C_2 z + \frac{32}{9} C_4 z^3$$

(30)

$$-\frac{16}{45}\sum_{k=4}^{\infty} C_{2k}[1-65.2^{2k-6}+2^{4k-6}]\, z^{2k-1}.$$

It is sufficient to consider even order identities after this point of the analysis, so the next identity with undetermined multipliers goes as

$$\frac{1+2e^{-z}+2e^{-2z}+2e^{-3z}+2e^{-4z}+2e^{-5z}+e^{-6z}}{1-e^{-6z}} - a\frac{1+2e^{-2z}+2e^{-4z}+e^{-6z}}{1-e^{-6z}}$$

(31)

$$+b\frac{(1+e^{-3z})^2}{1-e^{-6z}} - c\,\frac{1+e^{-6z}}{1-e^{-6z}} = 2\sum_{k=o}^{\infty} C_{2k}[1-2^{2k-1}a+3^{2k-1}b-6^{2k-1}c]z^{2k-1}.$$

The open end requirement is

$$1 - a + b - c = 0$$

$$1 - 2^{2k-1}a + 3^{2k-1}b - 6^{2k-1}c = 0$$

$$1 - 2^{2k+1}a + 3^{2k+1}b - 6^{2k+1}c = 0,\ k = 1, 2, \ldots\ .$$

It is not too practicable to solve for arbitrary values of k. For $k = 1$ the solution is $a = \frac{25}{11}$, $b = \frac{15}{11}$, $c = \frac{1}{11}$, and Eq. (31) takes the form

(33) $$\frac{3}{10}\,\frac{11e^{-z}-14e^{-2z}+26e^{-3z}-14e^{-4z}+11e^{-5z}}{1-e^{-6z}}$$

$$=\frac{1}{z}+\frac{33}{10}\sum_{k=3}^{\infty}C_{2k}[1-\frac{25}{11}2^{2k-1}+\frac{15}{11}3^{2k-1}-\frac{1}{11}6^{2k-1}]z^{2k-1}.$$

From the closed end formulas the requirement is that

(34) $$\begin{aligned} 1-2^{2k-1}a+3^{2k-1}b-6^{2k-1}c&=0\\ 1-2^{2k+1}a+3^{2k+1}b-6^{2k+1}c&=0\\ 1-2^{2k+3}a+3^{2k+3}b-6^{2k+3}c&=0,\ k=1,2,3,\ldots\ . \end{aligned}$$

The solution is at once

(35) $$a=\frac{7}{2^{2k+1}}\ ,\ b=\frac{7}{3^{2k+1}}\ ,\ c=\frac{1}{6^{2k+1}}\ ,\ k=1,2,3,\ldots\ .$$

For $a=\frac{7}{8}$, $b=\frac{7}{27}$, $c=\frac{1}{216}$, Eq. (31) takes the form

(36) $$\frac{1}{140}\,\frac{41+216e^{-z}+27e^{-2z}+272e^{-3z}+27e^{-4z}+216e^{-5z}+41e^{-6z}}{1-e^{-6z}}$$

$$=\frac{1}{z}+\frac{54}{35}\sum_{k=4}^{\infty}C_{2k}[1-7.2^{2k-4}+7\cdot 3^{2k-4}-6^{2k-4}]z^{2k-1}.$$

For $a=\frac{7}{32}$, $b=\frac{7}{243}$, $c=\frac{1}{7776}$,

(37) $$\frac{1}{7000}\,\frac{3149+7776e^{-z}+6075e^{-2z}+8000e^{-3z}+6075e^{-4z}+7776e^{-5z}+3149e^{-6z}}{1-e^{-6z}}$$

$$=\frac{1}{z}+\frac{18}{25}C_2z+\frac{972}{875}\sum_{k=5}^{\infty}C_{2k}[1-7\cdot 2^{2k-6}+7\cdot 3^{2k-6}-6^{2k-6}]z^{2k-1}.$$

For $a = \frac{7}{128}$, $b = \frac{7}{2187}$, $c = \frac{1}{279936}$,

$$\frac{1}{272580}\ \frac{132761+279936e^{-z}+264627e^{-2z}+280832e^{-3z}+264627e^{-4z}}{1-e^{-6z}}$$

(38)

$$\frac{+279936e^{-5z}+132761e^{-6z}}{1-e^{-6z}}$$

$$=\frac{1}{z}+\frac{823}{3245}C_2z+\frac{216}{649}C_4z^3+\frac{23328}{22715}\sum_{k=6}^{\infty}C_{2k}[1-7\cdot 2^{2k-8}+7\cdot 3^{2k-8}-6^{2k-8}]z^{2k-1} .$$

The formula for differentiation would result if it be required that

$$1-\frac{a}{2}+\frac{b}{3}-\frac{c}{6} = 0$$

(39)

$$1-2^{2k+1}a+3^{2k+1}b-6^{2k+1}c = 0$$

$$1-2^{2k+3}a+3^{2k+3}b-6^{2k+3}c = 0, \quad k = 1, 2, 3, \ldots .$$

For $k = 1$ the solution is $a = \frac{41}{16}$, $b = \frac{23}{27}$, $c = \frac{7}{432}$, and Eq. (31) takes the form

$$\frac{1}{720}\ \frac{157-432e^{-z}+675e^{-2z}-800e^{-3z}+675e^{-4z}-432e^{-5z}+157e^{-6z}}{1-e^{-6z}}$$

(40)

$$= C_2z-\frac{27}{45}\sum_{k=0}^{\infty}C_{2k}[1-41\cdot 2^{2k-5}+23\cdot 3^{2k-4}-\frac{7}{2}\cdot 6^{2k-4}]z^{2k-1} .$$

To obtain the Gould-Squire or second Euler-MacLaurin formula begin with the identity

$$\frac{1+2e^{-\frac{z}{2}}+2e^{-z}+2e^{-\frac{3z}{2}}+e^{-2z}}{1-e^{-2z}} - a\,\frac{(1+e^{-z})^2}{1-e^{-2z}} + b\,\frac{1+e^{-2z}}{1-e^{-2z}}$$

(41)

$$= 2\sum_{k=o}^{\infty} C_{2k}[1-2^{2k-1}a+2^{4k-2}b]\,(\tfrac{z}{2})^{2k-1}\ .$$

For no over-lapping the open end condition is required; also, the middle term on the left must be zero so the requirement on the co-efficients is that

(42)

$$1 - a + b = 0$$

$$2 - 2a \quad = 0\ .$$

The solution is at once $a = 1$, $b = 0$, which leads to

(43) $$\frac{e^{-\frac{z}{2}}+e^{-\frac{3z}{2}}}{1-e^{-2z}} = \frac{1}{z} - \sum_{k=1}^{\infty} C_{2k}[\,2^{2k-1}-1]\,(\tfrac{z}{2})^{2k-1}\ .$$

The negative sign before the series is quite significant. It is well-known that the Bernoulli polynomials are defined by

(44) $$\frac{z\,e^{zt}}{e^{z}-1} = \sum_{k=o}^{\infty} \frac{z^k}{k!}\,B_k(t)\ .$$

and

(45) $$B_{2k}(\tfrac{1}{2}) = -(1-\frac{1}{2^{2k-1}})B_{2k},\quad k = 1, 2, 3, \ldots\ .$$

Thus Eq. (43) may be written in the form

$$(46)\qquad \frac{e^{-\frac{z}{2}}+e^{-\frac{3z}{2}}}{1-e^{-2z}} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{B_{2k}(\frac{1}{2})}{(2k)!} z^{2k-1} .$$

A second form of this type results if it be required that

$$(47)\qquad \begin{aligned} 1 - a + b &= 0 \\ 2 - 2a \quad &= -4 . \end{aligned}$$

Solution is at once $a = 3$, $b = 2$, and Eq. (41) reduces to

$$(48)\qquad \frac{2}{3}\,\frac{e^{-\frac{z}{2}}-2e^{-z}+e^{-\frac{3z}{2}}}{1-e^{-2z}} = C_2 z + \frac{2}{3}\sum_{k=2}^{\infty} C_{2k}[2^{4k-1}-3\cdot 2^{2k-1}+1](\tfrac{z}{2})^{2k-1} .$$

EULER-MACLAURIN FORMULAS

It is quite simple to pass from the identities of the previous section to the various forms of the Euler-MacLaurin formulas. The fundamental idea is that if F(x) and f(s) are suitably restricted so that they may be regarded as a Laplace transform pair then

$$\mathcal{L}^{-1}\{e^{-sh}f(s)\} = \begin{cases} 0, & -\infty < x < h \\ F(x-h), & x > h \, . \end{cases} \tag{49}$$

An identity based on that of Eq. (2) is at once

$$(1+e^{-sh})f(s) = \frac{1+e^{-sh}}{1-e^{-sh}}(1-e^{-sh})f(s) = 2\sum_{k=o}^{\infty} C_{2k}(sh)^{2k-1}(1-e^{-sh})f(s) \, , \tag{50}$$

and inversion yields

$$F(x)+F(x-h) = \frac{2}{h}\int_{x-h}^{x} F(y)\,dy + 2\sum_{k=1}^{\infty} C_{2k}h^{2k-1}[F^{(2k-1)}(x) - F^{(2k-1)}(x-h)] \, . \tag{51}$$

It is convenient to set

$$x = a + (n+1)h \tag{52}$$

and sum over n from 0 to N-1, N = 1, 2, . . . , obtaining

$$\sum_{n=o}^{N} F(a+nh) = \frac{1}{h}\int_{a}^{a+Nh} F(y)\,dy + \frac{F(a+Nh)+F(a)}{2}$$

$$+\sum_{k=1}^{\infty} C_{2k}h^{2k-1}[F^{(2k-1)}(a+Nh) - F^{(2k-1)}(a)] \, . \tag{53}$$

This is the classical form of the Euler-MacLaurin sum formula and was discussed at length by Gould and Squire. The second form depends on the identity of Eq. (43); begin with

$$(e^{-\frac{sh}{2}}+e^{-\frac{3sh}{2}})f(s)=\frac{e^{-\frac{sh}{2}}+e^{-\frac{3sh}{2}}}{1-e^{-2sh}}(1-e^{-2sh})f(s)$$

(54)

$$=\frac{1}{sh}(1-e^{-2sh})f(s)-\sum_{k=1}^{\infty}C_{2k}(1-\frac{1}{2^{2k-1}})(sh)^{2k-1}(1-e^{-2sh})f(s)$$

and inversion leads to

$$(55)\quad F(x-\frac{h}{2})+F(x-\frac{3h}{2})=\frac{1}{h}\int_{x-2h}^{x}F(y)\,dy-\sum_{k=1}^{\infty}C_{2k}(1-\frac{1}{2^{2k-1}})h^{2k-1}[F^{(2k-1)}(x)-F^{(2k-1)}(x-2h)].$$

It is convenient to set

$$(56)\qquad x = a + 2(n+1)h.$$

then sum over n from 0 to N-1, N = 1, 2, ... A slight change of notation on the left side of the resulting identity leads to

$$(57)\quad \sum_{n=o}^{2N-1}F[a+(n+\frac{1}{2})h]=\frac{1}{h}\int_{a}^{a+2Nh}F(y)dy-\sum_{k=1}^{\infty}C_{2k}(1-\frac{1}{2^{2k-1}})h^{2k-1}[F^{(2k-1)}(a+2Nh)-F^{(2k-1)}(a)].$$

This is the form which appears in Hildebrand. If a is replaced by $a - \frac{h}{2}$ Eq. (57) may be written as

$$\sum_{n=o}^{2N-1} F(a+nh) = \frac{1}{h}\int_{a-\frac{h}{2}}^{a+(2N-\frac{1}{2})h} F(y)\,dy$$

(58)

$$-\sum_{k=1}^{\infty} C_{2k}\left(1-\frac{1}{2^{2k-1}}\right)h^{2k-1}\left[F^{(2k-1)}\left[a+(2N-\tfrac{1}{2})h\right]-F^{(2k-1)}\left(a-\tfrac{h}{2}\right)\right],$$

which is equivalent to Eq. (1.8) of Gould and Squire. Both these formulas imply an even number of terms in the series on the left.

The summation of terms with alternating signs depends on the implementation of the identities of Eqs. (9) and (48). Begin with Eq. (9) in the form

(59)

$$(1-2e^{-sh}+e^{-2sh})f(s) = \frac{1-2e^{-sh}+e^{-2sh}}{1-e^{-2sh}}(1-e^{-2sh})f(s)$$

$$= 2\sum_{k=1}^{\infty} C_{2k}(2^{2k}-1)(sh)^{2k-1}(1-e^{-2sh})f(s);$$

inversion yields

(60)

$$F(x)-2F(x-h)+F(x-2h) = 2\sum_{k=1}^{\infty} C_{2k}(2^{2k}-1)h^{2k-1}\left[F^{(2k-1)}(x)-F^{(2k-1)}(x-2h)\right].$$

Let

(61)

$$x = a + 2(n+1)h$$

then summation with respect to n from 0 to $N-1, N=1,2,\ldots,$ leads to

(62)

$$\sum_{n=o}^{2N}(-1)^n F(a+nh) = \frac{F(a+2Nh)+F(a)}{2}$$

$$+\sum_{k=1}^{\infty} C_{2k}(2^{2k}-1)h^{2k-1}\left[F^{(2k-1)}(a+2Nh)-F^{(2k-1)}(a)\right].$$

This result is proposed as an exercise by Knopp [9, p. 553] and is written out by Hildebrand [8, p. 157] for $N \to \infty$. It should be checked out for the simple polynomial case $F(x) = x^p$, $p = 0, 1, 2, \ldots$. For $p = 0$ Eq. (62) is obviously true, and the case $p = 1$ yields at once

$$\sum_{n=o}^{2N} (-1)^n (a+nh) = a+Nh \; ; \tag{63}$$

the case $p = 2$ yields

$$\sum_{n=o}^{2N} (-1)^n (a+nh)^2 = \frac{(a+2Nh)^2 + a^2}{2} + \frac{h(a+2Nh-a)}{2} = a^2 + Nh(2a+h) + 2N^2 h^2 . \tag{64}$$

The special situation $a = h = 1$ is of interest; final forms are

$$\sum_{n=o}^{2N} (-1)^n (1+n)^2 = (1+N)(1+2N) \; ,$$

$$\sum_{n=o}^{2N} (-1)^n (1+n)^3 = (1+N)^2 (1+4N) \; , \tag{65}$$

$$\sum_{n=o}^{2N} (-1)^n (1+n)^4 = (1+N)(1+2N)(1+6N+4N^2) \; .$$

These results may be checked out by induction or by any of several other methods. In his most recently published work Polya [11, p. 60-98] has discussed

$$\sum_{n=o}^{N} (1+n)^p \; ,$$

p a positive integer, at considerable length but does not include the corresponding formulas for alternating sign. For p taking negative integer values the series representation is no longer finite. That is, for $p = -q$, $q = 1, 2, \ldots$,

(66) $$F^{(2k-1)}(x) = -\frac{(q)_{2k-1}}{x^{q+2k-1}} ,$$

where

(67) $$(q)_o = 1, \quad (q)_n = q(q+1)(q+2)\ldots\ldots(q+n-1).$$

Thus, for $h = 1$ and a understood to be a positive integer, Eq. (62) yields

(68) $$\sum_{n=o}^{2N} \frac{(-1)^n}{(a+n)^q} = \sum_{n=a}^{2N+a} \frac{(-1)^{n+a}}{n^q} = \frac{1}{2}\left(\frac{1}{(a+2N)^q} + \frac{1}{a^q}\right)$$
$$-\sum_{k=1}^{\infty} C_{2k}(2^{2k}-1)(q)_{2k-1}\left[\frac{1}{(a+2N)^{q+2k-1}} - \frac{1}{a^{q+2k-1}}\right].$$

A short table may be quickly constructed to show the nature of this approximation. Let

(69) $$\sum_{n=a}^{\infty} \frac{(-1)^{n+a}}{n} \approx A_1 = \frac{1}{2a} + \frac{1}{4a^2}$$

and

$$\sum_{n=a}^{\infty} \frac{(-1)^{n+a}}{n} \approx A_2 = \frac{1}{2a} + \frac{1}{4a^2} - \frac{1}{8a^4} ,$$

where A_1 includes only the first term of the series of Eq. (68) while A_2 includes the first two terms. Since $\ln 2 = 0.6931471$

five place accuracy is maintained.

a	exact	A_1	A_2
1	.69315	.75000	.62500
2	.30685	.31250	.30469
3	.19315	.19444	.19290
4	.14019	.14062	.14013
5	.10981	.11000	.10980
10	.052488	.052500	.052487

Following the lead of Gould and Squire it seems that another formula might be available for this type of asymptotic expansion. Begin with the identity which depends on Eq. (48),

$$(e^{-\frac{sh}{2}}-2e^{-sh}+e^{-\frac{3sh}{2}})f(s) = \frac{e^{-\frac{sh}{2}}-2e^{-sh}+e^{-\frac{3sh}{2}}}{1-e^{-2sh}}(1-e^{-2sh})f(s)$$

(71)

$$= \sum_{k=1}^{\infty} C_{2k}[2^{4k-1}-3\cdot 2^{2k-1}+1](\frac{sh}{2})^{2k-1}(1-e^{-2sh})f(s);$$

inversion yields

$$F(x-\frac{h}{2}) - 2F(x-h)+F(x-\frac{3h}{2})$$

(72)

$$= \sum_{k=1}^{\infty} C_{2k}[2^{4k-1}-3\cdot 2^{2k-1}+1](\frac{h}{2})^{2k-1}[F^{(2k-1)}(x)-F^{(2k-1)}(x-2h)],$$

and it is convenient to set

(73) $$x = a + (n+\frac{3}{2})h .$$

Summation with respect to n from 0 to N-1 yields

$$\sum_{n=o}^{2N}(-1)^n F(a+\frac{nh}{2}) = \frac{F(a+Nh)+F(a)}{2}$$

(74)
$$+\frac{1}{2}\sum_{k=1}^{\infty} C_{2k}[2^{4k-1}-3\cdot 2^{2k-1}+1](\frac{h}{2})^{2k-1}\Big[F^{(2k-1)}[a+(N+\frac{1}{2})h]$$

$$+F^{(2k-1)}[a+(N-\frac{1}{2})h]-F^{(2k-1)}(a+\frac{h}{2})-F^{(2k-1)}(a-\frac{h}{2})\Big].$$

Note that for the trivial case N = 1 the two middle terms in the bracket on the right add up to zero. If $\frac{h}{2}$ is replaced by h there results

$$\sum_{n=o}^{2N}(-1)^n F(a+nh) = \frac{F(a+2Nh)+F(a)}{2}$$

(75)
$$+\frac{1}{2}\sum_{k=1}^{2N} C_{2k}[2^{4k-1}-3\cdot 2^{2k-1}+1]h^{2k-1}\Big[F^{(2k-1)}[a+(2N+1)h]$$

$$+F^{(2k-1)}[a+(2N-1)h]-F^{(2k-1)}(a+h)-F^{(2k-1)}(a-h)\Big].$$

This should be checked out for, say, $F(x) = x^4$, a = h = 1. That is

$$\sum_{n=o}^{2N}(-1)^n(1+n)^4 = \frac{(1+2N)^4+1}{2} + \frac{(2N+2N)^3+(2N)^3-2^3}{2} - 7\frac{(2+2N)+2N-2}{4}$$

(76)
$$= (1+N)(1+2N)(1+6N+4N^2),$$

which agrees with Eq. (65). For $F(x) = x^{-q}$, h = 1, a an integer > 0, the formula of Eq. (75) yields

$$(77)\quad \sum_{n=o}^{2N} \frac{(-1)^n}{(a+n)^q} = \sum_{n=a}^{2N+a} \frac{(-1)^{n+a}}{n^q} = \left(\frac{1}{2}\ \frac{1}{(a+2N)^q} + \frac{1}{a^q}\right)$$

$$-\frac{1}{2}\sum_{k=1}^{\infty} C_{2k}[2^{4k-1}-3\cdot 2^{2k-1}+1](q)_{2k-1}\left[\frac{1}{(a+1+2N)^{q+2k-1}} + \frac{1}{(a-1+2N)^{q+2k-1}} - \frac{1}{(a+1)^{q+2k-1}} - \frac{1}{(a-1)^{q+2k-1}}\right]$$

This blows up for $a = 1$, which discourages further study along this line although it is not too bad an approximation for $q = 1$, $a \geq 5$.

As example of a Frank-line inversion of a Laplace transform consider the pair

$$(78)\quad f(s) = \frac{\sinh\sqrt{s}}{s^{3/2}\cosh\sqrt{s}}\ ,\qquad F(t)=1-\frac{8}{\pi^2}\sum_{n=o}^{\infty}\frac{e^{-(n+\frac{1}{2})^2\pi^2 t}}{(2n+1)^2}\ ,$$

where $F(t)$ represents the variables separable form of the solution of a problem proposed by Churchill [3, p. 217]. Clearly $F(t)$ behaves like $2\sqrt{\frac{t}{\pi}}$ as $t \to 0_+$ and is asymptotic to unity from below as $t \to \infty$. The geometric series expansion takes the form

$$(79)\quad f(s) = \frac{1-e^{-2\sqrt{s}}}{s^{3/2}(1+e^{-2\sqrt{s}})} = \frac{1}{s^{3/2}}\left[1-2(e^{-2\sqrt{s}}-e^{-4\sqrt{s}}+e^{-6\sqrt{s}}-e^{-8\sqrt{s}}+\ldots)\right];$$

for $a = h = 2\sqrt{s}$ and $F(x) = e^{-x}$, Eq. (62) yields

$$(80)\quad \sum_{n=o}^{\infty}(-1)^n e^{-2(n+1)\sqrt{s}} = \frac{1}{2}e^{-2\sqrt{s}} + \sum_{k=1}^{\infty} C_{2k}(2^{2k}-1)(2\sqrt{s})^{2k-1}e^{-2\sqrt{s}}\ ,$$

so

$$(81)\quad \begin{aligned} f(s) &= \frac{1}{s^{3/2}}\left[1-e^{-2\sqrt{s}}-2e^{-2\sqrt{s}}\sum_{k=1}^{\infty}C_{2k}(2^{2k}-1)(2\sqrt{s})^{2k-1}\right] \\ &= \frac{1-e^{-2\sqrt{s}}}{s^{3/2}} - \frac{e^{-2\sqrt{s}}}{s} + e^{-2\sqrt{s}}\left[\frac{1}{3}-\frac{2s}{15}+\frac{17s^2}{315}-\frac{682s^3}{31185}+\ldots\ldots\right] \end{aligned}$$

The transform pairs here are not too well tabulated. The iterated error function may be defined by

(82)
$$i^{o}\operatorname{erfc} x = \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-y^{2}}\, dy ,$$
$$i^{n}\operatorname{erfc} x = \int_{x}^{\infty} i^{n-1}\operatorname{erfc} y\, dy, \quad n \geq 1 ,$$

and by inductive methods it is easy to establish that

(83)
$$(4t)^{\frac{n}{2}}\, i^{n}\operatorname{erfc} \frac{x}{2\sqrt{t}} , \quad \frac{e^{-x\sqrt{s}}}{s^{\frac{n}{2}+1}} , \quad n \geq 0,$$

is a transform pair. Thus a second form of the $F(t)$ under discussion, suitable for small values of t, is

(84)
$$F(t) = 2\sqrt{\frac{t}{\pi}} \left[1 + \sqrt{\pi} \sum_{n=1}^{\infty} (-1)^{n}\, \operatorname{ierfc} \frac{n}{\sqrt{t}} \right] .$$

To multiply a suitably restricted transform by s is equivalent to taking the negative derivative of the object function with respect to t, so the continuation of the table of transforms goes as

$$\frac{x}{2\sqrt{\pi t^3}}\, e^{-\frac{x^2}{4t}}, \quad e^{-x\sqrt{s}},$$

$$\frac{x}{4\sqrt{\pi t^5}}\left(\frac{x^2}{2t} - 3\right) e^{-\frac{x^2}{4t}}, \quad s\, e^{-x\sqrt{s}},$$

(85)

$$\frac{x}{16\sqrt{\pi t^7}}\left(\frac{x^4}{2t^2} - \frac{10x^2}{t} + 30\right) e^{-\frac{x^2}{4t}}, \quad s^2 e^{-x\sqrt{s}},$$

$$\frac{x}{64\sqrt{\pi t^9}}\left(\frac{x^6}{2t^3} - \frac{21x^4}{t^2} + \frac{210x^2}{t} - 420\right) e^{-\frac{x^2}{4t}}, \quad s^3 e^{-x\sqrt{s}}.$$

That is, the $F(t)$ defined by Eq. (78) and by Eq. (84) is also represented asymptotically by

$$F(t) = 2\sqrt{\frac{t}{\pi}} - 2\sqrt{t}\,\text{ierfc}\,\frac{1}{\sqrt{t}} - \text{erfc}\,\frac{1}{\sqrt{t}} + \frac{e^{-\frac{1}{t}}}{\sqrt{\pi t^3}}\left[\frac{1}{3} - \frac{1}{15t}\left(\frac{2}{t} - 3\right)\right.$$

(86)

$$\left. + \frac{17}{2520t^2}\left(\frac{8}{t^2} - \frac{40}{t} + 30\right) - \frac{341}{498960t^3}\left(\frac{32}{t^3} - \frac{336}{t^2} + \frac{840}{t} - 420\right) + \ldots\right]$$

$$= 2\sqrt{\frac{t}{\pi}}\left(1 - e^{-\frac{1}{t}}\right) + \text{erfc}\,\frac{1}{\sqrt{t}} + \frac{2\,e^{-\frac{1}{t}}}{\sqrt{\pi t^3}}\left[\frac{1}{6} + \frac{1}{10t} + \frac{29}{840t^2} + \frac{13}{1512t^3} + \ldots\right],$$

a form well adapted to computation by means of the table in Carslaw and Jaeger [2, p. 373]. It is at once clear that for small positive values of t $F(t)$ behaves like the parabola $2\sqrt{\frac{t}{\pi}}$, and for large values of t $F(t)$ is asymptotic to unity from below.

INTEGRATION PROCEDURES

Such familiar integration formulas as Simpson's rule, the three-eighths rule, the Milne predictor-corrector formulas, are all readily obtained from the various Bernoulli identities of the first section. For example, Eq. (6) yields

$$\frac{1}{3}(1+4e^{-sh}+e^{-2sh})f(s)=\frac{1}{3}\,\frac{1+4e^{-sh}+e^{-2sh}}{1-e^{-2sh}}(1-e^{-2sh})f(s)$$

(87)

$$=\frac{1}{sh}(1-e^{-2sh})f(s)-\frac{4}{3}\sum_{k=2}^{\infty}C_{2k}(2^{2k-2}-1)(sh)^{2k-1}(1-e^{-2sh})f(s);$$

and inversion yields the Simpson rule in the form

$$\frac{1}{3}[F(x)+4F(x-h)+F(x-2h)] = \frac{1}{h}\int_{x-2h}^{x} F(y)\,dy$$

(88)

$$-\frac{4}{3}\sum_{k=2}^{\infty}C_{2k}(2^{2k-2}-1)h^{2k-1}[F^{(2k-1)}(x)-F^{(2k-1)}(x-2h)] \ .$$

The infinite series represents the successive correction at the two endpoints. A similar development of Eq. (7) leads to

$$\frac{1}{15}[7F(x)+16F(x-h)+7F(x-2h)]=\frac{1}{h}\int_{x-2h}^{x} F(y)\,dy+\frac{h}{15}[F'(x)-F'(x-2h)]$$

(89)

$$-\frac{16}{15}\sum_{k=3}^{\infty}C_{2k}(2^{2k-4}-1)h^{2k-1}[F^{(2k-1)}(x)-F^{(2k-1)}(x-2h)] \ ,$$

a form which is mentioned by Milne [10, p. 78]. As another example note that Eq. (24) yields

(90)
$$\frac{2}{45}[7F(x)+32F(x-h)+12F(x-2h)+32F(x-3h)+7F(x-4h)]=\frac{1}{h}\int_{x-4h}^{x}F(y)\,dy$$
$$+\frac{64}{45}\sum_{k=3}^{\infty}C_{2k}[1-5\cdot 2^{2k-4}+2^{4k-6}]h^{2k-1}[F^{(2k-1)}(x)-F^{(2k-1)}(x-2h)].$$

which is also mentioned by Milne [10, p. 48].

A different approach to the various integration formulas and their respective error terms is to begin with a set of even and odd periodic polynomials, expressed in terms of their Fourier series,

(91)
$$P_{2r}(x,h) = 2(-1)^{r-1}\sum_{n=1}^{\infty}\frac{\cos 2n\pi\frac{x}{h}}{(2n\pi)^{2r}}, \quad r \geq 1,$$
$$P_{2r+1}(x,h)=2(-1)^{r-1}\sum_{n=1}^{\infty}\frac{\sin 2n\pi\frac{x}{h}}{(2n\pi)^{2r+1}}, \quad r \geq 0.$$

The period is $(0,h)$; Knopp [9, p. 522-523] lists the polynomials defined over the fundamental period as

$$P_1(x,h) = \frac{x}{h} - \frac{1}{2} \qquad = \frac{x}{h} + B_1$$

$$P_2(x,h) = \frac{x^2}{2h^2} - \frac{x}{2h} + \frac{1}{12} \qquad = \frac{x^2}{2h^2} + B_1\frac{x}{h} + \frac{B_2}{2}$$

(92)

$$P_3(x,h) = \frac{x^3}{6h^3} - \frac{x^2}{4h^2} + \frac{x}{12} \qquad = \frac{x^3}{6h^3} + B_1\frac{x^2}{2h^2} + \frac{B_2 x}{2h}$$

$$P_4(x,h) = \frac{x^4}{24h^4} - \frac{x^3}{12h^3} + \frac{x^2}{24h^2} - \frac{1}{720} = \frac{x^4}{24h^4} + B_1\frac{x^3}{6h^3} + \frac{B_2 x^2}{4h^2} + \frac{B_4}{24}$$

. . . .

$$P_r(x,h) = \frac{1}{r!}\sum_{k=o}^{r}\binom{r}{k}\left(\frac{r}{h}\right)^{r-k}B_k .$$

A symbolic notation is common,

(93) $$P_r(x,h) = \frac{1}{r!}\left(\frac{x}{h} + B\right)^r ,$$

where, of course, $B^k = B_k$. Note that $B_1 = -\frac{1}{2}$ and $B_k = 0$ for all other odd values of k. Also, for all values of r,

(94) $$hP_r'(x,h) = P_{r-1}(x,h)$$

The usual form of the Euler-MacLaurin formula may be obtained by ordinary integration by parts. For all values of x, $P_1(x,h) = \frac{x}{h} - [\frac{x}{h}] - \frac{1}{2}$, where $[x] = n$, $n \leq x < n+1$, so

$$\int_{nh}^{(n+1)h} P_1(x,h)\ F'(x)\ dx = \int_{nh}^{(n+1)h} (\frac{x}{h} - n - \frac{1}{2})\ F'(x)\ dx$$

(95)

$$= (\frac{x}{h} - n - \frac{1}{2}) F(x) \Bigg|_{nh}^{(n+1)h} - \frac{1}{h} \int_{nh}^{(n+1)h} F(x) dx.$$

That is,

$$(96) \quad \frac{1}{2}[F[(n+1)h] + F(nh)] = \frac{1}{h} \int_{nh}^{(n+1)h} F(x)\ dx + \int_{nh}^{(n+1)h} P_1(x,h)\ F'(x)\ dx.$$

Summation with respect to n from 0 to $N-1$ leads to

$$(97) \quad \sum_{n=o}^{N} F(nh) = \frac{1}{h} \int_{0}^{Nh} F(x)\ dx + \frac{F(Nh) + F(0)}{2} + \int_{0}^{Nh} P_1(x,h)\ F'(x)\ dx.$$

This may be compared with Eq. (53), except that the a of that formula is taken to be zero. This amounts to a choice of the origin of the coordinate system. Further integration by parts yields

$$\int_{0}^{Nh} P_1(x,h)\ F\ (x)\ dx = hP_2(x,h)F'(x) \Bigg|_{0}^{Nh} - h \int_{0}^{Nh} P_2(x,h)F''(x) dx$$

(98)

$$= [\ hP_2(x,h)F'(x) - h^2 P_3(x,h)F''(x)] \Bigg|_{0}^{Nh} + h^2 \int_{0}^{Nh} P_3(x,h)\ F'''(x) dx.$$

Since the odd degree periodic functions are assigned the value zero at the cross-over points and the even order functions take the value $B_k/k!$, the extension of Eq. (97) yields

$$\sum_{n=o}^{N} F(nh) = \frac{1}{h}\int_0^{Nh} F(x)\,dx + \frac{F(Nh) + F(0)}{2} + \frac{B_2}{2}[F'(Nh) - F'(0)]$$

(99)

$$+ h^2 \int_0^{Nh} P_3(x, h)\, F'''(x)\,dx\,.$$

Such integration by parts may be extended as far as desired, so the final form of the Euler-MacLaurin formula may be written as

$$\sum_{n=o}^{N} F(nh) = \frac{1}{h}\int_0^{Nh} F(x)\,dx + \frac{F(Nh)+F(0)}{2} + \sum_{k=1}^{K} C_{2k} h^{2k-1}[F^{(2k-1)}(Nh) - F^{(2k-1)}(0)]$$

(100)

$$+ h^{2K}\int_0^{Nh} P_{2K+1}(x, h) F^{(2K+1)}(x)\,dx.$$

To arrive at the Simpson integration formula one need only write (for $N = 2$, $K = 1$)

$$F(2h) + 2F(h) + F(0) = \frac{2}{h}\int_0^{2h} F(x)\,dx + 2C_2 h[F'(2h) - F'(0)]$$

(101)

$$+ 2h^2 \int_0^{2h} P_3(x, h)\, F'''(x)\,dx\,,$$

and (for $N = 1$, $K = 1$, h taken to be twice as large)

(102) $$F(2h)+F(0) = \frac{1}{h}\int_0^{2h} F(x)\,dx + 4C_2 h[F'(2h) - F'(0)] + 8h \int_0^{2h} P_3(x, 2h) F'''(x)\,dx;$$

multiplication of the second form by 1/2 and subtraction from the first yields

$$(103)\quad \frac{h}{3}[F(2h)+4F(h)+F(0)] = \int_0^{2h} F(x)\,dx + \frac{4h^3}{3}\int_0^{2h}[P_3(x,h)-2P_3(x,2h)]F'''(x)\,dx.$$

It is sometimes better to reduce each term of the error integral to an integration over the fundamental period. That is,

$$\int_0^{2h} P_3(x,h)F'''(x)\,dx = \int_0^{2h} P_3(x,h)F''(x)\,dx + \int_0^{2h} P_3(x,h)\,F'''(x+h)\,dx$$

(104)

$$= \int_0^{2h} P_3(x,h)[F'''(x) + F''(x+h)]\,dx\ ,$$

and the Simpson rule may be written as

$$\frac{h}{3}[F(2h) + 4F(h) + F(0)] = \int_0^{2h} F(x)\,dx$$

$$+\frac{4h^3}{3}\left[\int_0^{h} P_3(x,h)[F(x)+F(x+h)]dx - 2\int_0^{2h} P_3(x,2h)\,F(x)\,dx\right].$$

It is easy to illustrate with, say, $F(x) = x^4$.

$$\frac{20h^5}{3} = \int_0^{2h} x^4 dx + 32h^3\left[\int_0^{h} P_3(x,h)(2x+h)dx - 2\int_0^{2h} P_3(x,2h)x\,dx\right]$$

$$= \int_0^{2h} x^4 dx + 32h^3\Big[[h(2x+h)P_4(x,h) - h^2P_5(x,h)]_0^h$$

$$-2[2hxP_4(x,2h) - 4h^2P_5(x,2h)]_0^{2h}\Big]$$

$$= \int_0^{2h} x^4 dx + 32h^3\left[-\frac{h^2}{360} + \frac{h^2}{90}\right].$$

That is,

$$\int_0^{2h} x^4 dx = \frac{20h^5}{3}[1 - \frac{1}{25}] \;,$$

and the result of applying the ordinary Simpson rule for this special case is four percent too high.

It is clear that an extended form of Simpson's rule is available which would be comparable to Eq. (298) of Knopp [9, p. 524]. Rewrite Eq. (100) with N replaced by $\frac{N}{2}$, h replaced by 2h, multiply by $\frac{1}{2}$, subtract from the original form, arriving finally at

$$\frac{h}{3}[F(0)+4F(h)+2F(2h)+4F(3h)+\ldots+4F(N-1)h]+F(Nh)] = \int_0^{Nh} F(x)dx$$

(106)
$$-\sum_{k=2}^{K} C_{2k}(2^{2k-2}-1)h^{2k}[F^{(2k-1)}(Nh)-F^{(2k-1)}(0)]$$

$$+\frac{4h^{2k+1}}{3}\int_0^{Nh}[P_{2K+1}(x,h)-2^{2K-1}P_{2K+1}(x,2h)]F^{(2K+1)}(x)\,dx.$$

Knopp [9, p. 531-534] devotes several pages to the discussion of the behavior of the $P_{2K+1}(x,h)$, with illustrations of how best to evaluate the remainder term. If $F(x)$ is defined for all $x \geq 0$ and, together with all its derivatives, tends monotonely to zero as x increases, the error term may be quite easily estimated since the alternating sign associated with the C_{2k} would imply an alternating series.

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