AN ABSTRACT OF THE THESIS OF

In-Kyeong Choi for the degree of Doctor of Philosophy in Mathematics presented on June 11, 1991.

Title: On Straight Line Representations of Random Planar Graphs

Redacted for Privacy

Abstract approved: ______________________________

Robert M. Burton

A graph consists of a finite or a countable set V of vertices together with a set of edges E such that each edge has two endpoints in the vertex set V. A simple plane graph is a graph whose vertex set is a point set in the Euclidean plane while the edges are Jordan curves such that two different edges have at most one end point in common. A planar graph is an abstract graph isomorphic to (or represented by) a plane graph. Thomassen showed that every simple planar graph has a straight line representation, that is, a representation in which all edges are straight line segments.

Let Λ be the set of graphs and $\mathcal{G}$ a suitable σ-algebra. A random graph X is a measurable mapping from a probability space $(\Omega, \mathcal{F}, P)$ into $(\Lambda, \mathcal{G})$. In the case that V is a subset of d-dimensional Euclidean space $\mathbb{R}^d$ then translations on $\mathbb{R}^d$ extend to translations on Λ, the space of graphs. Such a random graph is said to be stationary if its distribution is invariant under all translations.

In this thesis we show that Thomassen's result does not extend to stationary simple random planar graphs. In particular we construct a stationary simple random plane graph which possesses no stationary straight line representation.

On Straight Line Representations of Random Planar Graphs

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed June 11, 1991

Commencement June 1992

APPROVED:

Redacted for Privacy

Professor of Mathematics in charge of major

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Date thesis is presented June 11, 1991

Typed for In-Kyeong Choi

To

My Parents

and

My Parents-In-Law

ACKNOWLEDGEMENT

"And we know that all things work together for good to those who love God, to those who are the called according to His purpose."

(Romans 8:28)

I would like to express my deepest gratitude to professor Robert M. Burton for his encouragement, guidance and patience throughout my graduate studies and dissertation. I also thank professors M. Ossiander, J. Lee, L. Chen and Dr. R. Meester for serving on my committee and their helpful comments.

I would also like to thank my husband Ill-Keun Rhee and daughter Misoo, for their unending love, encouragement and patience. My sincere appreciation and love are extended to my parents Byung-Yong and Wi-Kyo Choi and my parents-in-law Joon-Soo and Jung-Soon Rhee for their many sacrifices, invaluable encouragement, love and support.

I also thank the Department of Mathematics to provide financial aid for me as a teaching assistant.

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CHAPTER 1
INTRODUCTION

A graph is said to be planar if it can be embedded in the plane such that its edges intersect only at their end points. Such a embedding of a planar graph is called a plane graph. Färy [5] proved that every finite planar graph has a straight line representation and Thomassen [18] showed that every infinite planar graph has a straight line representation and every connected, locally finite, plane graph with no vertex accumulation point is isomorphic to a subgraph of a straight line triangulation.

We define a stationary random graph and consider the following question : Does every stationary random planar graph have a stationary straight line representation? We show that the answer is negative.

Chapter two is devoted to the definitions and preliminaries of graph theory.

In chapter three, we review the straight line representations of finite and infinite planar graphs [5, 18]. In section 3.1, it is proved that every simple planar graph is isomorphic to a subgraph of triangulated simple graph and every triangulated simple graph is isomorphic to a straight line triangulation. In section 3.2, we deal with infinite planar graphs.

We define random graphs and stationary random graphs in section 4.1 and in section 4.2 and 4.3, we construct a stationary random plane graph Γ which, as we will show, does not yield a stationary straight line representation.

In chapter five we think about the straight line representation of the given stationary plane graph Γ. In section 5.1, we show that the structure of Γ is preserved to Δ if we assume that Δ is stationary. In section 5.2, it is proved that Δ is not a stationary graph.

CHAPTER 2

PRELIMINARIES OF GRAPH THEORY

The terminologies of graph theory are from [2], [8], [12] and [18] with minor modifications.

A *graph* G consists of a finite or countable set, $V(G)$ of *vertices* together with a prescribed set $E(G)$ of edges. Each edge e is associated with an unordered pair (or singleton) of vertices $\{x, y\} \subseteq V(G)$. In this case e is said to join x and y and we say that x and y are *adjacent* vertices; vertex x and edge e are *incident* with each other, as are y and e. An edge connecting a vertex with itself is called a *self-loop*. For many studies it is desirable to exclude self-loops and multiple edges. We call graphs resulting from such restrictions *simple*. In general the term simple is used in graph theory to mean without repetition.

If the set of vertices and the set of edges of a graph G are both finite, the graph is called *finite*, otherwise *infinite*. An infinite graph may have infinitely many edges but possibly finitely many vertices (for example, two vertices can be connected by infinitely many edges).

Let $v_0e_0v_1e_1v_2 \dots e_{n-1}v_n$ be a finite sequence whose terms are alternatively vertices and edges such that, for $0 \leq i \leq n-1$, e_i joins v_i and v_{i+1}. If all edges $e_0, e_1, \dots, e_{n-1}$ are distinct and all vertices $v_0, v_1, \dots, v_n$ are also distinct, then $v_0e_0v_1e_1v_2 \dots e_{n-1}v_n$ is called a *path* (*of length* n). In a simple graph, a path $v_0e_0v_1e_1v_2 \dots e_{n-1}v_n$ is determined by the sequence $v_0v_1 \dots v_n$ of its vertices; hence a path in a simple graph can be specified simply by its vertex sequence. A path whose origin and terminus are the same is called a *cycle*. A cycle with n edges is called an *n-cycle.* An infinite set of $v_i e_i v_{i+1}$ for $i = 0, 1, 2, \ldots$ and the graph formed by them is called a *1-way* infinite path if $v_i \neq v_j$ and $e_i \neq e_j$ for $i \neq j$; under the same condition the $v_i e_i v_{i+1}$ for $i = 0, \pm 1, \pm 2, \ldots$ form a *2-way* infinite path. Here is a simple but fundamental theorem about infinite graphs due to König.

LEMMA (The Infinity Lemma [10]) Let $V_0, V_1, V_2, \ldots$ be a countably infinite sequence of finite, nonempty, pairwise disjoint sets of points. Let the points contained in these sets form the vertices of a graph. If G has the property that every point of V_{n+1} ($n = 0, 1, 2, \ldots$) is joined with a point of V_n by an edge of G, then G has a 1-way infinite path $v_0 v_1 \ldots$, where v_n ($n = 0, 1, 2, \ldots$) is in V_n.

For any vertices x and y in a graph, there is a path joining x and y, then the graph is said to be *connected*. A cycle C of a connected graph G is a *separating* cycle if and only if $G - V(C)$ is not connected.

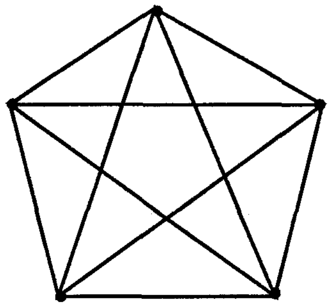
If every vertex of a graph is incident with a finite number of edges, then the graph is said to be *locally finite* (i.e., finite degree). Locally finite, infinite graphs play an important role in that some properties of finite graphs, which lose their validity (or even their meaning) for arbitrary infinite graphs, can be extended to graphs of finite degree. A locally finite, infinite graph, of course, always has infinitely many vertices.

If the vertex set of the graph H is contained in the vertex set of the graph G and the edge set of H is also contained in the edge set of G then H is called a *subgraph* of G. A *spanning* subgraph of G is a subgraph H with $V(G) = V(H)$. For any $A \subseteq V(G)$, the *induced subgraph of G by A* is the graph having vertex set A and whose edge set consists of those edges of G incident with two elements of A. For any set B of edges of G, the *edge-induced subgraph of G by B* is the graph whose vertex set consists of those vertices of G incident with at least one edge of B and having edge set B.

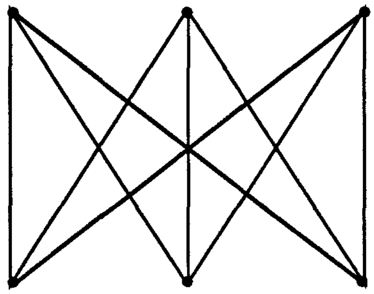
An edge $e \in E(G)$ is said to be *subdivided* when it is deleted and replaced by a path of length 2 connecting its ends, the internal vertex of this path being a new vertex. A *subdivision* $\tilde{G}$ of G is a graph that can be obtained from G by a sequence of edge subdivisions.

Two graphs G and G' are *isomorphic* if there exists a one-to-one incidence-preserving map taking $V(G)$ onto $V(G')$ and $E(G)$ onto $E(G')$. A *plane* graph is a graph whose vertex set is a point set in the Euclidean plane while the edges are Jordan curves such that two different edges have at most end points in common. A *planar* graph is an abstract graph isomorphic to a plane graph. Kuratowski [11] characterized finite, planar graphs by a theorem saying that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. See Figure 1. Dirac and Schuster [3] generalized it to countable graphs. Wagner [20] characterized all planar graphs and Halin [7] characterized those graphs which are isomorphic to locally finite plane graphs with no vertex accumulation points (to be defined in § 3.2). Thomassen [19] summarized the current results about planarity of infinite graphs.

If the graph G is isomorphic to the plane graph Γ, then Γ is a *representation* of G. If the edges of Γ are polygonal arcs, then Γ is a



K_5



$K_{3,3}$

Figure 1

polygonal arc representation. If the edges of Γ are straight line segments, then Γ is a *straight line representation* (it is defined only when the graph G is simple).

Two finite plane graph Γ and Γ' are *equivalent* if there exists a one-to-one incidence-preserving map taking $V(\Gamma)$ onto$V(\Gamma')$, $E(\Gamma)$ onto $E(\Gamma')$, and the set of regions of Γ onto the set of regions of Γ'. If, furthermore, the unbounded region of Γ corresponds to the unbounded region of Γ', then Γ and Γ' are *strongly equivalent.*

CHAPTER 3
STRAIGHT LINE REPRESENTIONS OF PLANAR GRAPHS

A simple planar graph has no self-loops or multiple edges. In this chapter we only deal with simple planar graphs since most of the results in this chapter does not hold for non-simple planar graphs.

3.1 Straight Line Representions Of Finite Planar Graphs

A finite, plane graph Γ partitions the plane into a finite number of regions one of which is unbounded. If every region in Γ is bounded by a cycle of exactly three edges (3-cycle), then Γ is called a triangulation. Such graphs are always simple. In order to see straight line representations of a finite planar graph we need the following lemma by Färy [5].

LEMMA 3.1.1 Every finite plane graph Γ is a spanning subgraph of a triangulation Γ'.

PROOF This result is trivially true when Γ has three vertices. So assume that Γ has more than three vertices. We construct a triangulation Γ' of which Γ is a subgraph by adding edges to Γ according to the following scheme ; select two vertices on the boundary of a region that are not connected by an edge. Give an edge between these two vertices which lies interior to the region.

We can continue in this way until no such pair of vertices can be found. The process will always stop since Γ is finite. The resulting graph is Γ'. Then Γ' is connected for if it has separated components there would be vertices on the boundary of a region (i.e., the unbounded region) which should be connected by an edge according to the above scheme. This is a contradiction.

Graph Γ' is triangulated. If there was a finite region of Γ with more than three edges, then the above process could not have terminated which contradicts the definition of Γ'. Thus every simple finite plane graph Γ is a subgraph of a triangulation Γ' based on the same set of vertices as Γ. ♠♠

Now we go the

THEOREM 3.1.2 [5] Every finite planar graph G has a straight line representation Γ.

PROOF Let Γ denote a representation of G. It is clear that every subgraph of a straight line graph is also straight line graph. By Lemma 3.1.1, the theorem will be established if every triangulation Γ' is isomorphic to a straight line triangulation Δ. To do so requires two facts.

<u>FACT (1)</u> Let T be a triangulation with at least four vertices. Let x be a vertex of T and $xv_1, xv_2, \ldots, xv_k$ be all the edges with x as one end vertex. The edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1$ are in T and form a cycle Θ_x which separates x from every other vertex of T.

<u>Proof of fact (1)</u> First we assume that x is not on the boundary of the unbounded region. Then there is a cycle Φ_x such that there is no

cycle inside Φ_x which contains x inside it. We denote the vertices of Φ_x by $u_1, u_2, \dots, u_m$. Since T triangulates the interior of Φ_x, the edges u_iu_{i+1} of Φ_x are adjacent to a region in the interior of Φ_x and the region contains a third vertex y on its boundary. Suppose $x \neq y$. Then the cycle $u_1u_2 \dots u_iyu_{i+1} \dots u_mu_1$ has x in its interior and lies in the interior of Φ_x. This is a contradiction to the definition of Φ_x. Hence $x = y$ and this is true for all the edges of Φ_x.

Suppose that x is on the boundary of the unbounded region. Then there is a cycle Φ_x such that there is no cycle outside Φ_x which has x in its exterior. Using similiar argument used above we can show that Φ_x can be used for the cycle Θ_x and so the fact is fully established.

Let Δ be a triangulation having more than three vertices. To prepare the proof by induction we construct from Δ a not necessarily simple graph Δ^* having one less vertex. A vertex x of Δ which does not lie on the boundary of the unbounded region is removed from Δ. All edges incident on x are also removed. The graph thus obtained contains a cycle Θ_x. Graph Γ' is identical with Δ outside Θ_x and is empty inside Θ_x. We connect vertex v_1 of Θ_x with each of the vertices $v_3, v_4, \dots, v_{k-1}$ of by edges in the interior of Θ_x that do not intersect each other. The resulting graph is denoted by Δ^*.

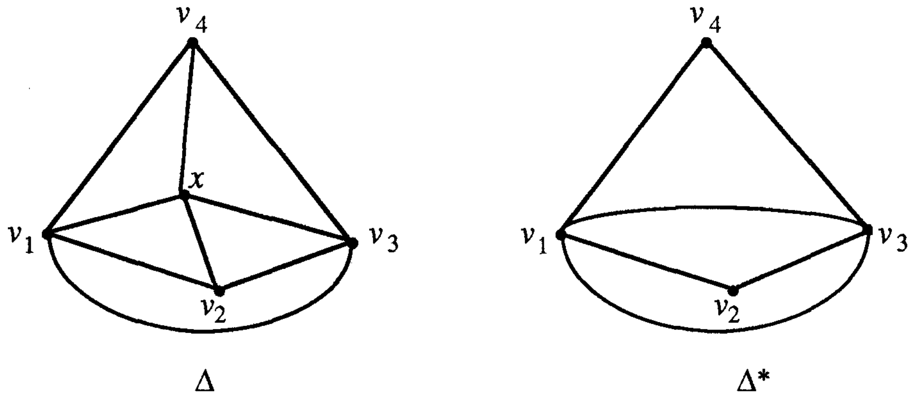
Now we need the second fact.

<u>*FACT (2)*</u> If Δ^* is not simple, then Δ has a separating 3-cycle.

<u>*Proof of fact (2)*</u> Since Δ is a triangulation, Δ^* can not be simple only when a new edge connects two vertices which are also

connected by an edge in the exterior of Θ_x. Since all new edges are incident on v_1, there is a vertex v_i $(2 < i \leq k-1)$ which is joined by an edge in the exterior of Θ_x to v_1. Then the 3-cycle $v_1 x v_i v_1$ in Δ separates v_2 from v_k. For example see Figure 2.

Now we are able to prove the theorem by induction on n, the number of vertices of Δ. If $n = 3$, certainly Δ is a straight line triangulation. So assume that $n > 3$ and Δ is a straight line triangulation with fewer than or equal to n vertices. We consider Δ



The separating 3-cycle $v_1 x v_3 v_1$

Figure 2

with $n+1$ vertices. Let x be a vertex of Δ not on the boundary of the unbounded region. We distinguish two cases according to as Δ^* is simple or not.

Case (i) Δ^* is simple.

In this case Δ^* is a straight line triangulation by the induction hypotheses. We put it into a straight line representation. The edges $v_1 v_3, \ldots, v_1 v_{k-1}$ are removed leaving the interior of Θ_x empty. We may now place x into this region and connect it to each of the vertices $v_1, v_2, \ldots, v_k$ with straight edges. Then Δ is a straight line triangulation.

Case (ii) Δ^* is not simple.

By fact (2), there is a separating 3-cycle in Δ. Denote this cycle byΘ. By induction the subgraph Δ_1 of Δ induced by the vertices of Θ and the vertices in the interior of Θ has a straight line triangulation. The subgraph Δ_2 of Δ induced by the vertices of Θ and the vertices in the exterior of Θ also has a straight line triangulation. Thus $\Delta = \Delta_1 \cup \Delta_2$ is straight line triangulation.

Thus we showed that every planar graph G is isomorphic to a spanning subgraph Γ of a straight line triangulation Δ. ♠♠

3.2 Straight Line Representations of Infinite Planar Graphs

In this section, we begin with the following definitions related to infinite graphs [18].

DEFINITION 3.2.1 A *vertex accumulation point* (respectively *edge accumulation point*) of an infinite plane graph Γ is a point p of the plane such that, for each $\varepsilon > 0$, $B(p,\varepsilon)$ contains (respectively intersects) infinitely many vertices (respectively edges) of Γ where $B(p,\varepsilon)$ denotes the set of points of Euclidean distance less than ε. A *VAP* (abbr. vertex accumulation point) *-free* plane graph is a plane graph with no VAP, and an *EAP* (abbr. edge accumulation point) *-free* plane graph is a plane graph with no EAP.

The *point set* of a graph Γ is the set of vertices and all points on the edges of Γ.

DEFINITION 3.2.2 An infinite, connected, plane graph Γ is a *triangulation* if and only if Γ is VAP-free and, for every vertex x of Γ, Γ contains a cycle Π_x such that x is the only vertex of Γ in the interior of Π_x and x is joined by an edge to every vertex of Π_x. If, in addition, every point not in the point set of Γ is contained in the interior of some cycle of Γ, then Γ is a *triangulation of the plane.*

By definition of infinite triangulation we have the following lemma.

LEMMA 3.2.1 [18] An infinite, plane, VAP-free graph Γ is a triangulation if and only if Γ contains a sequence of mutually disjoint cycles Θ_1, Θ_2, Θ_3, ... such that for each $k \geq 2$, the subgraph of Γ induced by Θ_k and the vertices in the interior of Θ_k is finite and triangulates the interior of Θ_k and contains Θ_{k-1}, and furthermore every vertex of Γ is in the interior of some Θ_m.

PROOF ($\Leftarrow$) It is trivially true by the definition of triangulation. ($\Rightarrow$) Let Γ be any triangulation. We define the sequence of cycles Θ_1, Θ_2, Θ_3, ... recursively as follows: Let Θ_1 be any cycle of Γ. Suppose that we have already defined $\Theta_1, \dots, \Theta_k$. Let Γ_k be the graph $\cup\Pi_x$, where the union is taken over all vertices x of Θ_k. Since Γ_k is 2-connected, the boundary of the unbounded region of Γ_k is a cycle Θ_{k+1}. There are only finitely many vertices of Γ in the interior of Θ_{k+1} since Γ is VAP-free. By the definition of a triangulation, Γ triangulates the interior of Θ_{k+1}. If y is a vertex of Γ in the exterior of Θ_{k+1} then the graphic distance from y to Θ_{k+1} is less than the distance from y to Θ_k. Hence y is in the interior of some Θ_m, and so the proof is complete. ♠♠

We state the following theorem without proof. See Thomassen [18] and Wagner [20] for the proof.

THEOREM 3.2.2 Every infinite planar graph has a straight line representation.

In order to show that a locally finite, connected, VAP-free plane graph is isomorphic to a subgraph of a straight line triangulation, we start with the following lemma.

LEMMA 3.2.3 [18] Let Γ be a countably infinite, locally finite, VAP-free, plane graph. Then there exists a VAP-free, plane graph Ψ isomorphic to Γ such that each edge of Ψ is a polygonal arc and such that no point of an edge of Ψ is an EAP of Ψ.

PROOF Let G denote the abstract graph which is represented by Γ, and let $E(G) = \{e_1, e_2, \ldots\}$. For each edge e_i of G we add two new edges e_i', e_i'' joining the same vertices as e_i, and we denote the resulting multigraph by G'. To each vertex p of Γ we associate a positive real number $\varepsilon(p)$ such that, for any two distinct vertices p, q of Γ, we have that $\bar{B}(p,\varepsilon(p)) \cap \bar{B}(q,\varepsilon(q)) = \emptyset$ where $\bar{B}(p, \varepsilon)$ is the set of points of Euclidean distance less than or equal to ε. We define a sequence $\Psi_0, \Psi_1, \Psi_2, \ldots$ of plane multigraphs such that $\Psi_0 \subseteq \Psi_1 \subseteq \Psi_2 \subseteq \ldots$, and such that, for each k,

(i) Ψ_k is a polygonal arc representation of $G[\{e_1, e_1', e_1'', \ldots, e_k, e_k', e_k''\}]$ with the same vertex set as Γ;

(ii) for each $j \leq k$, the union of the arcs representing e_j', e_j'' is a closed Jordan curve whose interior contains the arc representing e_j and nothing else of the graph;

(iii) for each vertex p of Ψ_k, the intersection of $\bar{B}(p, \varepsilon(p))$ and the point set of Ψ_k either consists of p only, or it consists of straight line segments radiating from p;

(iv) the subgraphs of Ψ_k, respectively Γ, representing $G[\{e_1, e_2, ..., e_k\}]$ are strongly equivalent.

The graph Ψ_0 is defined as the graph with vertex set $V(\Gamma)$ and with empty edge set. Having defined Ψ_{k-1} ($k \geq 1$), we define Ψ_k as follows: The subgraph Ψ'_{k-1} of Ψ_{k-1} representing $G[\{e_1, e_2, ..., e_{k-1}\}]$ is equivalent to the corresponding subgraph of Γ. Hence we can add to Ψ'_{k-1} a polygonal arc representing e_k such that the resulting graph Ψ_k' is strongly equivalent to the subgraph of Γ representing $G[\{e_1, e_2, ..., e_k\}]$. If necessary, we can modify the new arc such that it does not intersect any of the arcs representing edges e_j', e_j'' ($j \leq k$-1), and we can further modify it so that condition (iii) is satisfied. Then we can add polygonal arcs representing e_k', e_k'' such that (ii) is satisfied. The resulting plane multigraph Ψ_k satisfies conditions (i)-(iv). Now the multigraph $\Psi' = \cup \Psi_k$ is a representation of G', and it contains a representation of G with the desired properties. ♠♠

Lemma 3.2.4 [18] Let Γ be an infinite, locally finite, connected, VAP-free plane graph such that each edge is a polygonal arc and no point of an edge is an EAP of Γ. Then Γ is a spanning subgraph of a polygonal arc triangulation.

PROOF Let G denote the abstract graph represented by Γ and let G' be the multigraph obtained from G by adding, for every edge e of G, two new edges e', e'' with the same ends as e. Since no point of point set of Γ contains EAP of Γ, we can extend Γ to a polygonal arc representation Γ' of G'. If e is an edge of G', we denote by $\bar{e}$ the arc of Γ' representing e. Now assume that for each edge e of G, the interior of $\bar{e}' \cup \bar{e}''$ (denoted by $\Omega(e)$) contains $\bar{e}$ (except for the ends) and nothing else of Γ', and to each vertex p of Γ' we associate a positive real number $\varepsilon(p)$ such that $B(p,\varepsilon(p)) \cap B(q,\varepsilon(q)) = \emptyset$ for any two distinct vertices p, q and such that the intersection of point set of Γ' and $B(p,\varepsilon(p))$ consists of straight line segments radiating from p for each vertex p of Γ'. Denote by Ω the union of the sets $B(p,\varepsilon(p))$, where $p \in V(\Gamma)$ and the sets $\Omega(e)$, where $e \in E(G)$. The graph Γ has the property that if x and y are two non-adjacent vertices with a common neighbour z and the edges joining z to x and y are consecutive in the clockwise ordering of the edges incident with z, then we can add to Γ a polygonal arc representing the edge $e = xy$ such that $\bar{e} \subseteq \Omega$. Any plane graph obtained from Γ by adding finitely many edges in Ω satisfies the assumption of the above lemma.

Now we want to show that, for any vertex z of Γ, we can add finitely many edges in Ω so as to obtain a plane graph Θ with the property that z is contained in the interior of some cycle of Θ. Since Γ is connected, z has degree at least one, and if z has degree 1, we can add an edge in Ω joining z to a vertex of graphic distance two from z. So we can assume that z has degree at least two. We can also assume that the ends (distinct from z) of any two edges which are incident with z and consecutive in the clockwise ordering of the edges incident with z are adjacent. So if z has degree at least 3 in Γ, then Γ contains a cycle containing the vertices adjacent to z and no other vertex. If z is in the interior of this cycle, then we have finished, so assume the opposite. Then Γ contains two adjacent vertices, x and y say, each of which is adjacent to z such that each vertex adjacent to z (other than y and x) is in the interior of the 3-cycle $zxyz$. This is also true if z has degree 2. Since Γ is infinite, locally finite, and connected, it has a 1-way infinite path starting at any vertex by the Infinity Lemma. So there is in Γ a 1-way infinite path starting at z. This path contains either x or y (or both) since there are only finitely many vertices in the interior of the 3-cycle $xyzx$. Therefore Γ has a 1-way infinite path starting at x or y (say y) and containing none of x and z. Let y_1 be the vertex adjacent to y such that the edge yy_1 succeeds the edge yz in the clockwise ordering of the edges adjacent to y. Clearly we have $y_1 \neq z$ and $y_1 \neq x$. If x and y_1 are non-adjacent in Γ, we can add an edge (in Ω) joining y_1 and x such that z is in the interior of the cycle xy_1yx. So assume that y_1 and x are adjacent and that z is in the exterior of the cycle xyy_1x. Let y_2 be the vertex adjacent to y_1 such that the edge y_1y_2 succeeds y_1y in the clockwise ordering of the edges incident with y_1. So we can conclude that y_2 is distinct from z, y and x, and also we may assume that x is adjacent to y_2 and that z is in the exterior of the cycle xyy_1y_2x. We continue like this defining the vertices y_3, y_4, ... Since x has finite degree, there is a (smallest) k such that x is not adjacent to y_k. Then we can add in Ω an edge joining y_k and x such that z is in the interior of the cycle $\Theta: xyy_1y_2 \ldots y_kx$.

Let $\{p_1, p_2, \ldots\}$ be the vertex set of Γ. By adding finitely many edges (in Ω) to Γ we obtain a graph containing a cycle Θ such that p_1 is in the interior of Θ. The subgraph of the resulting graph consisting of Θ and the vertices and edges in the interior of Θ is a finite, plane graph, so by adding finitely many edges in the interior of Θ, we obtain a graph Γ_1 containing a cycle Ψ_1 whose vertices are precisely the neighbours of p_1 and whose interior contains p_1 and no other vertex of Γ_1. The plane graph Γ_1 has the same vertex set as Γ and no point of point set of Γ_1 is an EAP of Γ_1. Hence we can repeat the argument with Γ_1 instead of Γ and p_2 instead of p_1. We can obtain a graph Γ_2 and a cycle Ψ_2 of Γ_2 such that p_2 and no other vertex of Γ_2 is in the interior of Ψ_2. Since Γ_2 has the same vertex set as Γ_1 and Γ, none of the edges added to Γ_1 in order to obtain Γ_2 intersects the interior of Ψ_1. We continue like this defining $\Gamma_3, \Gamma_4, \ldots$, and finally the union $\bigcup_{i=1}^{\infty}\Gamma_i$ is a polygonal arc triangulation containing Γ as a subgraph. ♠♠

Now we need the following result about finite triangulations without the proof, in order to represent an infinite triangulation by a straight line triangulation. To begin with, we need also some definitions. If xy is an edge of a graph Γ, then we say that the *contraction* Γ' of Γ by the edge xy is obtained from $\Gamma \setminus \{x, y\}$ by adding a new vertex z ($z \notin V(\Gamma)$) and joining z to those vertices which are adjacent to x or y (or both) in Γ. We also say that Γ is obtained from Γ' by *splitting* z into x and y.

PROPOSITION 3.2.5 [18] Let Γ be a finite polygonal arc triangulation such that the boundary of the unbounded region is the cycle $\Theta_0 : x_0y_0z_0x_0$, and suppose that Γ has a region whose boundary is a cycle $\Theta_1 : x_1y_1z_1x_1$ disjoint from Θ_0. Let T_0, T_1 be disjoint triangles of the plane with vertices p_i, q_i, r_i for $i = 0, 1$, such that T_1 is contained in the interior of T_0 and such that each straight line segment connecting a vertex of T_0 with a vertex of T_1 has only its ends in common with $T_0 \cup T_1$. Then there is a straight line triangulation Δ which is strongly equivalent to Γ such that x_1, y_1 and z_1 are represented by p_1, q_1 and r_1 respectively and Θ_0 is represented by T_0, and such that each vertex of Δ other than p_1, q_1 and r_1 has Euclidean distance less than ε from one of p_0, q_0 and r_0 for $\varepsilon > 0$.

THEOREM 3.2.6 [18] Let Γ be an infinite, locally finite, connected, VAP-free plane graph. Then there exists an infinite straight line triangulation Δ of the plane such that Γ is isomorphic to a subgraph of Δ.

PROOF From Lemma 3.2.3 and Lemma 3.2.4, we may assume that Γ is an infinite polygonal arc triangulation, and so we need to show that there is a straight line triangulation Δ isomorphic to Γ. By Lemma 3.2.1, there is a sequence of mutually disjoint cycles Θ_1, Θ_2, Θ_3, ... in Γ such that for each k, the interior of Θ_k and Θ_k form a finite, plane graph containing Θ_{k-1} and triangulating the interior of

Θ_k. For each $k \geq 1$, let Γ_k denote the subgraph of Γ induced by the vertices of Θ_{3k}, Θ_{3k-3} and the vertices in the interior of Θ_{3k} and the exterior of Θ_{3k-3}.

Let $v_1, v_2, \ldots, v_m$ be vertices of Θ_{3k-1}. If the edge v_1v_2 is not contained in a separating 3-cycle of Γ_k whose interior contains the interior of Θ_{3k-2}, then we delete the interior of every separating 3-cycle (if any) of Γ_k containing the edge v_1v_2, and then we contract Γ_k by the edge v_1v_2. We repeat this procedure with the edge v_1v_3 instead of v_1v_2. Then in a finite number of steps we have a separating 3-cycle Θ'_{3k-1} whose interior contains the interior of Θ_{3k-2}. The resulting graph is Γ'_k. All edges of Γ_k which are not incident with a vertex of Θ_{3k-1} are also present in Γ'_k. Also Θ_{3k} and Θ_{3k-3} remain unchanged, so the union $\bigcup_{k=1}^{\infty} \Gamma'_k$ is an infinite triangulation Γ'.

Let T_k denote the triangle with vertices $(-k, -k)$, $(0, k)$, $(k, -k)$. Let Γ_k^* denote the induced subgraph of Γ' by the vertices of Θ'_{3k-1} and Θ'_{3k-4} and all vertices in the interior of Θ'_{3k-1} and the exterior of Θ'_{3k-4}. We represent Γ_1^* such that Θ'_2 is represented by T_1 and all other vertices of Γ_1^* are in the interior of T_1. Using Proposition 3.2.5 we can extend this to a straight line representation of $\Gamma_1^* \cup \Gamma_2^*$ such that Θ'_5 is represented by T_2. Similarily, we represent Γ_3^*, Γ_4^*, ... The union $\bigcup_{k=1}^{\infty} \Gamma_k^*$ is denoted by Δ'. Then Δ' is a straight line triangulation of the plane, and Δ' is isomorphic to Γ.

Let Φ_{3k} denote the cycle of Δ' corresponding to Θ_{3k} in Γ' and let Δ'_k denote the induced subgraph of Δ' by the vertices of Φ_{3k}, Φ_{3k-3} and the vertices in the interior of Φ_{3k} and the exterior of Φ_{3k-3}. Then Δ'_k

is isomorphic to Γ'_k. We split vertices of Δ'_k and add finite triangulations to Δ'_k in order to obtain a straight line representation Δ_k of Γ_k in a finite number of steps. Only the cycles Φ_{3k-1}, Φ_{3k-2} are affected. Let Δ denote the union $\bigcup_{k=1}^{\infty} \Delta_k$. Then Δ is straight line triangulation of the plane, and Δ is isomorphic to Γ. So the proof is complete. ♠♠

CHAPTER 4

STATIONARY RANDOM PLANAR GRAPHS

4.1. Random Graphs

Let $\mathbb{R}^d$ be d-dimensional Euclidean space. Let $\mathcal{B}^d$ be the collection of all Borel subsets of $\mathbb{R}^d$. Let $\tilde{\mathcal{B}}^d$ denote the subset of $\mathcal{B}^d$ consisting of all bounded sets (i.e., sets with compact closures). We say that a measure ν on $(\mathbb{R}^d, \mathcal{B}^d)$ is *Radon* if $\nu(B) < \infty$ for all sets $B \in \tilde{\mathcal{B}}^d$. Let M denote the set of all such measures on $(\mathbb{R}^d, \mathcal{B}^d)$ and further let N denote the subset of M consisting of all nonnegative integer-valued measures, that is $N = \{\nu \in M \mid \nu(B) \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ and $\nu(B) < \infty$ for all $B \in \tilde{\mathcal{B}}^d\}$. So N is the set of all counting measures on $(\mathbb{R}^d, \mathcal{B}^d)$. Then N may be uniquely identified with the set of all finite or infinite configurations of points in $\mathbb{R}^d$ with no limit points but where repetitions are allowed.

Let $A(B,k) = \{\nu \in M \mid \nu(B) < k$ for $0 \leq k < \infty\}$ for $B \in \tilde{\mathcal{B}}^d$. Let $\mathcal{M}$ be the σ-algebra on M generated by the sets $A(B,k)$ for all subsets $B \in \tilde{\mathcal{B}}^d$. Likewise $\mathcal{N}$ is the σ-algebra generated by such sets of measures in N. Then we have $\mathcal{N} \subset \mathcal{M}$ and so $\mathcal{N}$ is the restriction of $\mathcal{M}$ to N (see Kallenberg [9]). Now from $(N, \mathcal{N})$ we can count the points in regions of $\mathbb{R}^d$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

DEFINITION 4.1.1 A measurable mapping X from $(\Omega, \mathcal{F}, P)$ into $(M, \mathcal{M})$ or $(N, \mathcal{N})$ is called a *random measure* or a *point process*, respectively.

Since $(N, \mathcal{N})$ is a restriction of $(M, \mathcal{M})$ a point process may alternatively be considered as an N-valued random measure, and conversely any a.s. N-valued random measure coincides a.s. with a point process. Note that the set of point processes on $\mathbb{R}^d$ is closed under addition and under multiplication by $\mathbb{Z}^+$-valued random variables (Kallenberg [9]) and is also closed under weak limits.

In particular, any probability measure on $(N, \mathcal{N})$ yields a point process. For $B \in \mathcal{B}^d$, we think of $X(B)$ as counting the number of points in the set B. If a function $f: \mathbb{R}^d \to \mathbb{R}^+$ is continuous with compact support then we set $X(f) = \int f\, dX$.

The *distribution* of a point process X is by definition the induced probability measure $P \circ X^{-1}$ on $(N, \mathcal{N})$ given by

$$P \circ X^{-1}(A) = P(X^{-1}(A)) = P\{\omega \in \Omega \mid X(\omega) \in A\} = P_X(A), \quad A \in \mathcal{N}.$$

DEFINITION 4.1.2 A transformation $T: \Omega \to \Omega$ is *measure-preserving* on $(\Omega, \mathcal{F}, P)$ if T is measurable and

$$P(T^{-1}(A)) = P(A) \text{ for } A \in \mathcal{F}.$$

The measure P is said to be an *invariant measure* for T.

DEFINITION 4.1.3 A set $A \in \mathcal{F}$ is *invariant* if $P(T^{-1}(A) \triangle A) = 0$. The transformation T (or more properly, the system $(\Omega, \mathcal{F}, P, T)$) is called *ergodic* if every invariant set has measure 0 or 1.

For any $u \in \mathbb{R}^d$, let $T_u : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $T_u(x) = x + u$ for all $x \in \mathbb{R}^d$. Translations in $\mathbb{R}^d$ act naturally on point processes. So we abuse notation and extend a translation in $\mathbb{R}^d$ to a mapping $T_u : N \to N$, given by $T_u(\nu)$, for $\nu \in N$, which denotes the translation of occurrences in ν by u. That is, for $A \in \mathcal{B}^d$, $\nu(T_u(A)) = (T_{-u}(\nu))(A)$.

DEFINITION 4.1.3 We say that X is a *stationary point process* if for every $u \in \mathbb{R}^d$, P_X is an invariant measure for T_u.

For details on (stationary) point processes see Franzosa [6], Kallenberg [9], Lewis [13] and Rolski [15].

Let N° denote the subset of N without multiple occurrences. That is, $N^\circ = \{ \nu \in N \mid \nu(\{x\}) = 0 \text{ or } 1 \text{ for any } x \in \mathbb{R}^d\}$. For $\nu \in N^\circ$, let A_ν be the finite or countable subset of $\mathbb{R}^d$ such that $A_\nu = \{x \in \mathbb{R}^d \mid \nu(\{x\}) = 1\}$. Let E be a subset of $E_\nu{}^* = \{e \subseteq A_\nu \mid \mathrm{card}(e) = 2\}$. Each $e \in E$ has the form $e = \{x, y\}$ for some $x \neq y$ in A_ν and is thought of as an edge with end points x and y. Then we may think of $G = (A_\nu, E)$ as a simple graph. Let $\Lambda = \{G = (A_\nu, E) \mid \nu \in N^\circ, E \subseteq E_\nu{}^*\}$ be the set of simple graphs whose vertex set is an accumulation point free subset of $\mathbb{R}^d$.

Let $\Lambda(B_1, B_2; k_1, k_2, l) = \{G \in \Lambda \mid \nu(B_1) = k_1, \nu(B_2) = k_2$ and there are l edges between B_1 and $B_2\}$ for $0 \le k_1, k_1, l < \infty$ and $B_1, B_2 \in \tilde{\mathcal{B}}^d$.

Let $\mathcal{G}$ be the σ-algebra on Λ generated by the sets $A(B_1, B_2; k_1, k_2, l)$ for all $B_1, B_2 \in \tilde{\mathcal{B}}^d$. Note that this definition is consistent with the measurable space of point processes which is a projection of the space of graphs on $\mathbb{R}^d$.

DEFINITION 4.1.4 A measurable mapping X from a probability space $(\Omega, \mathcal{F}, P)$ into the set of graphs $(\Lambda, \mathcal{G})$ is called a *random* graph.

The *distribution* of X is the induced measure P_X on $\mathcal{G}$ by $P_X = P \circ X^{-1}$. As before $\mathbb{R}^d$ acts by translations on the set of graphs on $\mathbb{R}^d$. Define a map $T_u : \Lambda \to \Lambda$ by $T_u(G) = (T_u(A_v), T_u(E))$ for $v \in N^\circ$, and $G = (A_v, E) \in \Lambda$, where $T_u(A_v) = \{T_u(x) \in \mathbb{R}^d \mid v(\{x\}) = 1\}$ and $T_u(E) = \{T_u(x), T_u(y)\}$, for some distinct points x and y in A_v and $\{x, y\} \in E$.

DEFINITION 4.1.5 A random graph X is said to be *stationary* if for every $u \in \mathbb{R}^d$, P_X is an invariant measure for T_u.

That is, the finite dimensional distributions are invariant under all T_u, $u \in \mathbb{R}^d$.

4.2 Construction of an $\mathbb{R}^d$-action T

In this section we use the cutting and stacking method to construct a stationary (and ergodic) $\mathbb{R}^d$-action T and we will use it when we construct a stationary plane graph in $\mathbb{R}^2$ in next section.

DEFINITION 4.2.1 If H is a group, then an *action* of H on $(\Omega, \mathcal{F}, P)$ is a measurable map ϕ from $H \times \Omega$ to Ω with the properties that

(i) $\phi(e, \omega) = \omega$ for every $\omega \in \Omega$ (e is the identity element of H),

and

(ii) $\phi(h, \phi(h', \omega)) = \phi(hh', \omega)$ for every $h, h' \in H$ and for every $\omega \in \Omega$.

Every action of the integers $\mathbb{Z}$ is determined by a bijection $T : \Omega \to \Omega$ by the formula $\mathbb{Z} \times \Omega \to \Omega : (n, \omega) \to T^n(\omega)$.

We will give a construction of an $\mathbb{R}^d$-action T, due to Rudolph [17].

Set $\Omega = [0, 1)^{d+1}$ with Lebesgue probability measure P. For $\omega \in \Omega$, we write $\omega = (u, x)$, $u \in \mathbb{R}^d$. If (u, x) and $(u+u', x)$ are both in Ω, we set

$$T_{u'}(u, x) = (u+u', x).$$

Then T_u is measure-preserving on $\{\omega \in \Omega \mid (u+u', x) \in \Omega\}$.

Now we subdivide Ω into 2^d sets $A_1^k = [0, 1)^d \times [\frac{k}{2^d}, \frac{k+1}{2^d})$, for $k = 0, 1, 2, \dots, 2^d-1$. Let f_1 be the map of $\Omega \leftrightarrow \Omega_1 = [0, 2)^d \times [0, \frac{1}{2^d})$, linear on each A_1^k which accomplishes stacking to form a $\frac{1}{2^d} \times 2 \times \dots \times 2$ rectangle. For example these stacking for $d = 2$ is as follows (see Figure 3):

$$f_1(A_1^0) = [0,1) \times [1,2) \times [0, \tfrac{1}{4}),$$

$$f_1(A_1^1) = [1,2) \times [1,2) \times [0, \tfrac{1}{4}),$$

$$f_1(A_1^2) = [0,1) \times [0,1) \times [0, \tfrac{1}{4}),$$

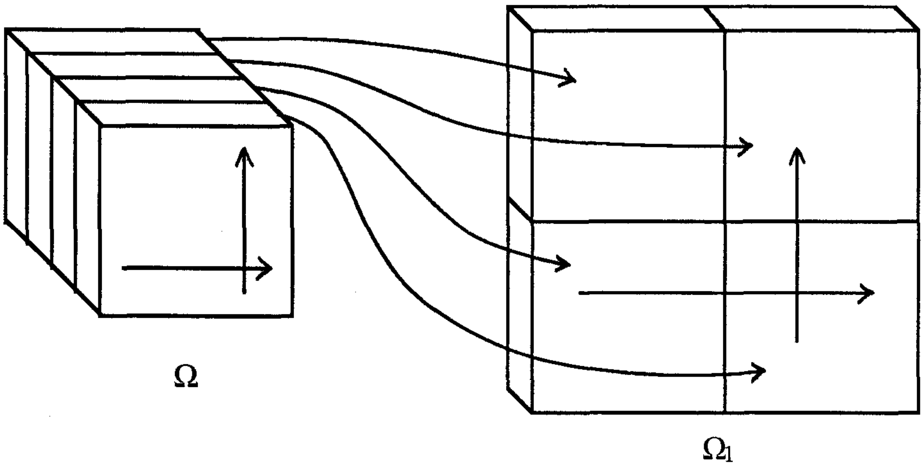
and

$$f_1(A_1^3) = [1,2) \times [0,1) \times [0, \tfrac{1}{4}).$$

If (u, x) and $(u+u', x)$ are both in Ω_1, we set

$$T_{u'}(f_1^{-1}(u, x)) = f_1^{-1}(u+u', x).$$

Let $A \subseteq \Omega_1$ and $A + u' = \{(u+u', x) \mid (u, x) \in A\}$. If $A + u' \subseteq \Omega_1$ then $P(f_1^{-1}(A)) = P(f_1^{-1}(A + u')) = P(T_{u'}(f_1^{-1}(A)))$. This extends the definition of T_u to more of Ω.



Cutting and stacking for d=2 and n=1

Figure 3

Inductively suppose that the map $f_n : \Omega \leftrightarrow \Omega_n = [0,\ 2^n)^d \times [0, \frac{1}{2^{dn}})$ has been defined, linear on each $A_n^k = [0,\ 1)^d \times [\frac{k}{2^{dn}}, \frac{k+1}{2^{dn}})$, and if (u, x) and $(u+u', x)$ are both in Ω_n, we set

$$T_{u'}(f_n^{-1}(u, x)) = f_n^{-1}(u+u', x).$$

Then T_u is measure-preserving on $\{\omega \in \Omega \mid f_n(\omega) \in \Omega_n,\ f_n(T_{u'}(\omega)) \in \Omega_n\}$. Now we subdivide Ω_n into 2^d sets of the form

$$\tilde{A}_{n+1}^k = [0,\ 2^n)^d \times [\frac{k}{2^{d(n+1)}}, \frac{k+1}{2^{d(n+1)}}), \quad k = 0, 1, 2, \ldots, 2^d\text{-}1.$$

Now note that the sets

$$A^k_{n+1} = [0,\ 1)^d \times [\,\frac{k}{2^{d(n+1)}},\ \frac{k+1}{2^{d(n+1)}}\,), \qquad k = 0,\ 1,\ 2,\ \dots,\ 2^{d(n+1)}-1,$$

are precisely set of the form $A^{k'}_n \cap \mathrm{f}^{-1}_n(\tilde{A}^{k''}_{n+1})$.

Next let $\tilde{f}_{n+1}$ be a map $\Omega_n \leftrightarrow \Omega_{n+1} = [0,\ 2^{n+1})^d \times [0,\ \frac{1}{2^{d(n+1)}})$, linear on each $\tilde{A}^k_{n+1}$, stacking them to fill Ω_{n+1}. For example the stacking for $d = 2$ is as follows :

$$\tilde{f}_{n+1}(\tilde{A}^{0}_{n+1}) = [0,\ 2^n) \times [2^n,\ 2^{n+1}) \times [0,\ \frac{1}{4^{n+1}}),$$

$$\tilde{f}_{n+1}(\tilde{A}^{1}_{n+1}) = [2^n,\ 2^{n+1}) \times [2^n,\ 2^{n+1}) \times [0,\ \frac{1}{4^{n+1}}),$$

$$\tilde{f}_{n+1}(\tilde{A}^{2}_{n+1}) = [0,\ 2^n) \times [0,\ 2^n) \times [0,\ \frac{1}{4^{n+1}}),$$

and

$$\tilde{f}_{n+1}(\tilde{A}^{3}_{n+1}) = [2^n,\ 2^{n+1}) \times [0,\ 2^n) \times [0,\ \frac{1}{4^{n+1}}).$$

Let $f_{n+1} = \tilde{f}_{n+1} \circ f_n$. Then f_{n+1} is the map of $\Omega \leftrightarrow \Omega_{n+1}$ which is linear on each A^k_{n+1} and accomplishes the stacking. For any $(u,\ x)$ and $(u+u',\ x)$ are both in Ω_{n+1}, we set

$$T_{u'}(f^{-1}_{n+1}(u,\ x)) = f^{-1}_{n+1}(u+u',\ x).$$

The T_u is also measure-preserving on $\{\omega \in \Omega \mid f_{n+1}(\omega) \in \Omega_{n+1}$ and $f_{n+1}(T_{u'}(\omega)) \in \Omega_{n+1}\}$. This extends the definition of $T_{u'}$. Continue inductively.

Consider the set S of $(u, x) \in \Omega$ for which T_u is defined. For some $M > 0$, let S^M denote the subset of $(u, x) \in \Omega$ for which $T_{u'}$ is defined for every $u' \in \mathbb{R}^d$ with $|u'| < M$. Let

$$S_n^M = \{(u, x) \mid d(f_n(u, x), \partial(\Omega_n)) \geq M\}.$$

Then $S_n^M \subseteq S_{n+1}^M$ and $S^M = \bigcup_{n=1}^{\infty} S_n^M$. Now

$$P(S_n^M) = \frac{1}{2^{dn}}(2^n - 2M)^d = (1 - \frac{2M}{2^n})^d \to 1 \quad \text{as } n \to \infty.$$

So we have $P(S^M) = 1$ for some $M > 0$ and further $P(S) = 1$ since $S = \bigcup_M S^M$. Thus the measure-preserving (and ergodic) action $T = \{T_u\}_{u \in \mathbb{R}^d}$ is defined a.s. on Ω.

4.3 Construction of a Stationary Random Plane Graph

Let $\Omega = [0, 1)^3$. Let Ω_n^i, i = 0, 1, 2, be a subset of Ω such that

$$\Omega_n^0 = \{\frac{1}{5}(\frac{4}{5})^n\} \times \{\frac{1}{5}(\frac{4}{5})^n\} \times [\frac{2\sum_{i=1}^{n} 4^{i-1}}{4^n}, \frac{2\sum_{i=1}^{n} 4^{i-1}+1}{4^n}),$$

$$\Omega_n^1 = \{\frac{1}{5}(\frac{4}{5})^n\} \times \{1 - \frac{1}{5}(\frac{4}{5})^n\} \times [0, \frac{1}{4^n}),$$

and

$$\Omega_n^2 = \{1 - \frac{1}{5}(\frac{4}{5})^n\} \times \{\frac{1}{5}(\frac{4}{5})^n\} \times [\frac{4^n-1}{4^n}, 1)$$

for $n \geq 0$.

Using the measure-preserving $\mathbb{R}^2$-action T constructed in section 4.2, we define a stationary (and ergodic) point process X on $\mathbb{R}^2$ as follows. For a.e. $\omega = (u, x) \in \Omega$, $u \in \mathbb{R}^2$ and for any $u' \in \mathbb{R}^2$

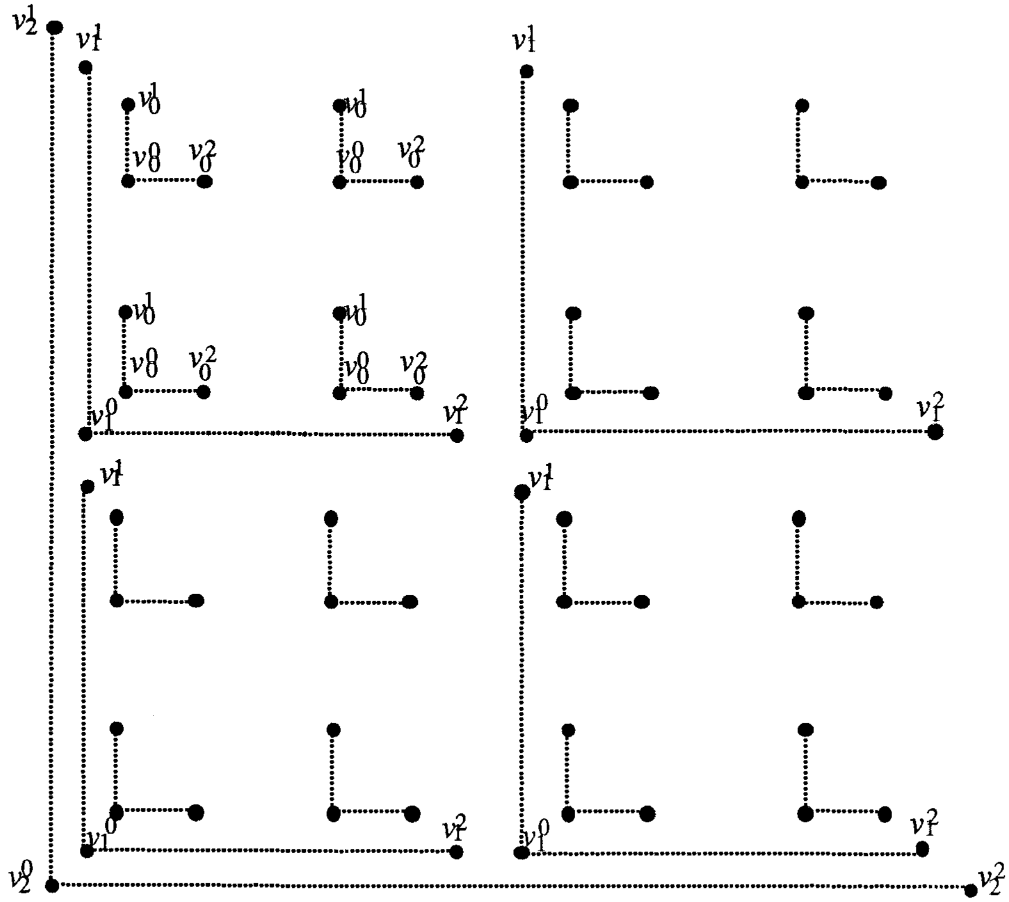
$$X(\{u'\}) = 1 \text{ if and only if } T_{u'}(u, x) \in \Omega_n^i \qquad (*)$$

for some $i = 0, 1, 2$ and $n \geq 0$. A point v in $\mathbb{R}^2$ with $X(\{v\}) = 1$ is a point of the point process X.

For the construction of the graph Γ, a point of X will be a vertex of Γ. We write $L(u') = v_n^i$ if u' satisfies (*). Let P_n^i be the set $\{v \in X \mid L(v) = v_n^i\}$. Then there are infinitely many points in P_n^i, and they form a rectangular grid in the plane such that the distance from each point to its nearest neighbors is 2^n, for all $i = 0, 1, 2$ and $n \geq 0$. For a convenience we abuse notations and write $v = v_n^i$ if $v \in P_n^i$. See Figure 4. The edges of Γ are chosen as follows, with a restriction that every edge does not intersect any other edges except two end points.

(1) we connect any two points v_n^0 and v_n^1 if $d(v_n^0, v_n^1) = 2^n - \frac{2}{5}(\frac{4}{5})^n$, any two points v_n^0 and v_n^2 if $d(v_n^0, v_n^2) = 2^n - \frac{2}{5}(\frac{4}{5})^n$, and any two points v_n^1 and v_n^2 if $d(v_n^1, v_n^2) = \sqrt{2}\,[2^n - \frac{2}{5}(\frac{4}{5})^n]$, for $n \geq 0$, so that the interior of all 3-cycles $v_0^0 v_0^1 v_0^2 v_0^0$ in clockwise order is empty and for $n \geq 1$, the interior of every n^{th} level 3-cycle $v_n^0 v_n^1 v_n^2 v_n^0$ contains exactly four $(n-1)^{\text{th}}$ level 3-cycles and so 4^n $v_0^0 v_0^1 v_0^2 v_0^0$'s.

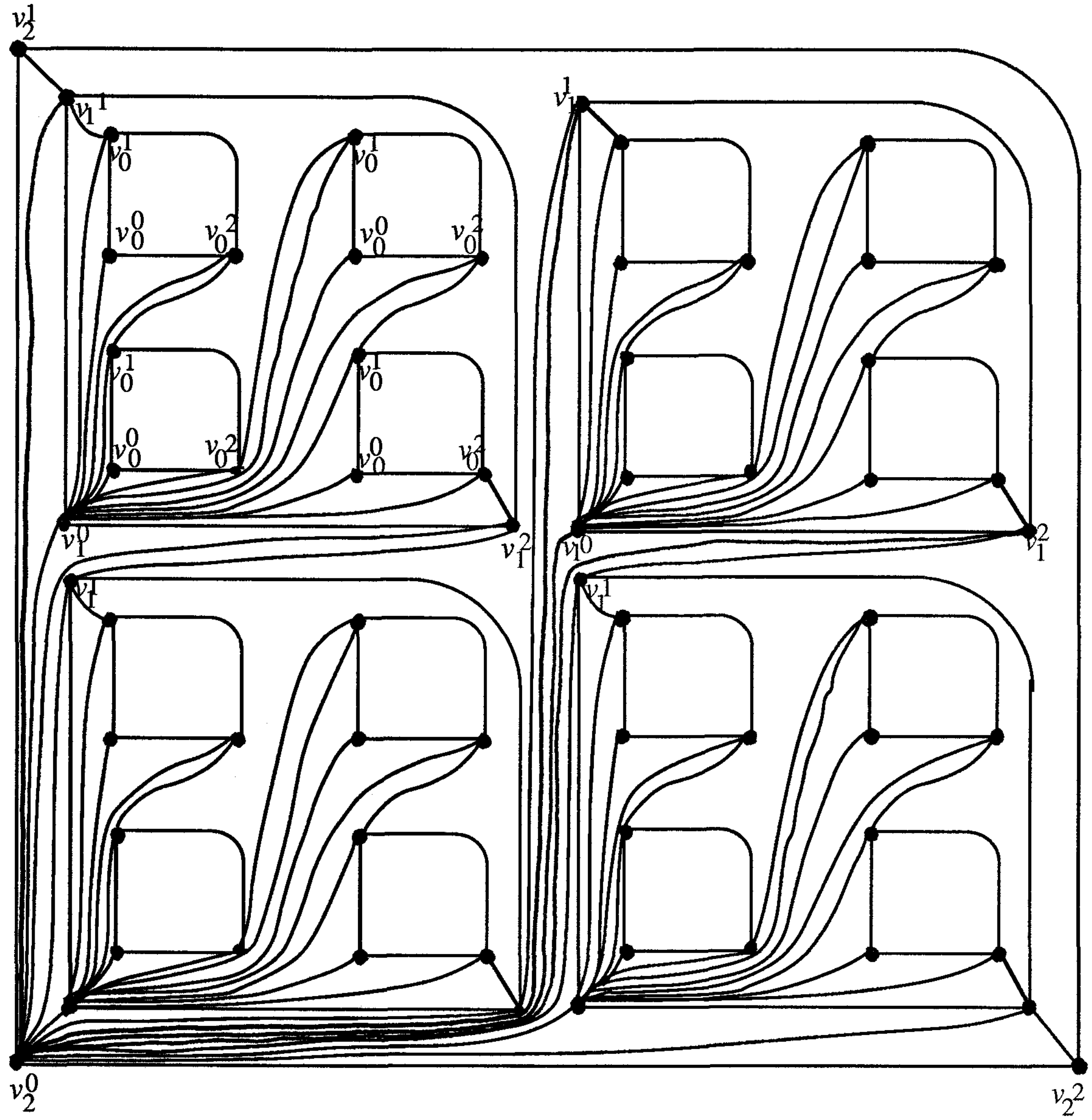
(2) We connect any two points v_n^1 and v_n^2 which are contained in the same interior of $(n+1)^{\text{th}}$ level 3-cycle if (i) the distance between



The dotted-lines are added to the point process to show the self-similar structure

Figure 4

the 1st coordinate of v_n^1 and the 1st coordinate of v_n^2 is $2^n - \frac{2}{5}(\frac{4}{5})^n$, and the distance between the 2nd coordinate of v_n^1 and the 2nd coordinate of v_n^2 is $\frac{2}{5}(\frac{4}{5})^n$, or (ii) the distance between the 1st coordinate of v_n^1 and the 1st coordinate of v_n^2 is $\frac{2}{5}(\frac{4}{5})^n$, and the



The random graph Γ in the interior of 2nd level 3-cycle

Figure 5

distance between the 2nd coordinate of v_n^1 and the 2nd coordinate of v_n^2 is $2^{n+1} - \frac{2}{5}\left(\frac{4}{5}\right)^n$, for $n \geq 0$.

(3) We connect v_n^1 and v_{n+1}^1 if $d(v_n^1, v_{n+1}^1) = \frac{1}{5}\left(1 - \left(\frac{4}{5}\right)^{n+1}\right)\sqrt{2}$,

and connect v_n^2 and v_{n+1}^2 if $d(v_n^2, v_{n+1}^2) = \frac{1}{5}\left(1 - \left(\frac{4}{5}\right)^{n+1}\right)\sqrt{2}$, for $n \geq 0$.

(4) We connect υ^{0}_{n+1} to all vertices of 4 of the $n^{\underline{th}}$ level 3- cycles which are in the interior of $(n+1)^{\underline{th}}$ level 3-cycle $\upsilon^{0}_{n+1}\upsilon^{1}_{n+1}\upsilon^{2}_{n+1}\upsilon^{0}_{n+1}$, for $n \geq 0$.

Now we have the resulting graph Γ. See Figure 5. By definition of stationary random graphs, we know that the graph Γ is a stationary random plane graph.

CHAPTER 5

REPRESENTATION OF STATIONARY RANDOM PLANE GRAPHS

5.1 Straight Line Representations of Γ

Let Λ be the set of all simple graphs, Λ_p the subset of Λ consisting of all planar graphs and Λ_s the subset of Λ consisting of all straight line representations. Theorem 3.2.6 gives a constructive procedure for making a straight line representation from a simple planar graph. This gives a measurable map $\Phi : \Lambda_p \to \Lambda_s$. Let $\Gamma : \Omega \to \Lambda_p$ be a random planar graph. Then the composition $\Phi \circ \Gamma$ gives a random graph, P-a.s. (graph theoretically) isomorphic (defined in chapter 2) to Γ such that $\Phi \circ \Gamma$ is a straight line representation of Γ.

$$\begin{tikzcd} \Omega \arrow[r, "\Gamma"] \arrow[dr, "\Phi \circ \Gamma"'] & \Lambda_p \arrow[d, "\Phi"] \\ & \Lambda_s \end{tikzcd}$$

DEFINITION 5.1.1 If a map $\Gamma : \Omega \to \Lambda_p$ is a random planar graph, then a map $\Delta : \Omega \to \Lambda_s$ is called a stationary straight line representation of Γ if

(i) Δ is a stationary random graph

(ii) for P-a.e. $\omega \in \Omega$, $\Gamma(\omega)$ and $\Delta(\omega)$ are (graph theoretically) isomorphic.

By Theorem 3.2.6, we know that there exists a plane graph with straight edges which is isomorphic to the plane graph $\Gamma(\omega)$ constructed in section **4.3**. In this section we assume that a stationary straight line representation of the stationary random plane graph Γ exists. We describe some properties that this representation must possess. In section 5.2 we will show that no such representation exists. For simplicity, we will only consider ergodic stationary representation. This is not a real loss of generality and it will be convenient in the sequel.

Let Δ be a typical straight line representation of Γ. For $n = 0, 1, 2, \ldots$, let Δ_n denote a subgraph with straight edges (say n-triangle) of Δ representing a $n^{\underline{th}}$ level 3-cycle $C_n : v_n^0 v_n^1 v_n^2 v_n^0$ of Γ and we represent v_n^0 by x_n, v_n^1 by y_n and v_n^2 by z_n. For $n \geq 1$, the vertex x_n of Δ_n is joined by straight edges to all vertices of $\Delta_{n\text{-}1}$'s representing $C_{n\text{-}1}$'s contained in the interior of the C_n in Γ.

For $x, y \in \mathbb{R}^2$, let $\overline{xy}$ denote the straight line segment between x and y.

DEFINITION 5.1.2 If Γ is a plane graph and x is a point not in Γ^o where Γ^o denotes the point set of Γ, then x is called an *admissible point* with respect to Γ if and only if for every vertex y of Γ adjacent to the region containing x we have;

$$\overline{xy} \cap \Gamma^o = \{y\}.$$

In a straight line representation Δ of Γ, the vertex x_n of Δ_n is an admissible point with respect to the Δ_{n-1}'s representing the C_{n-1}'s contained in the interior of C_n in Γ for all $n \geq 1$.

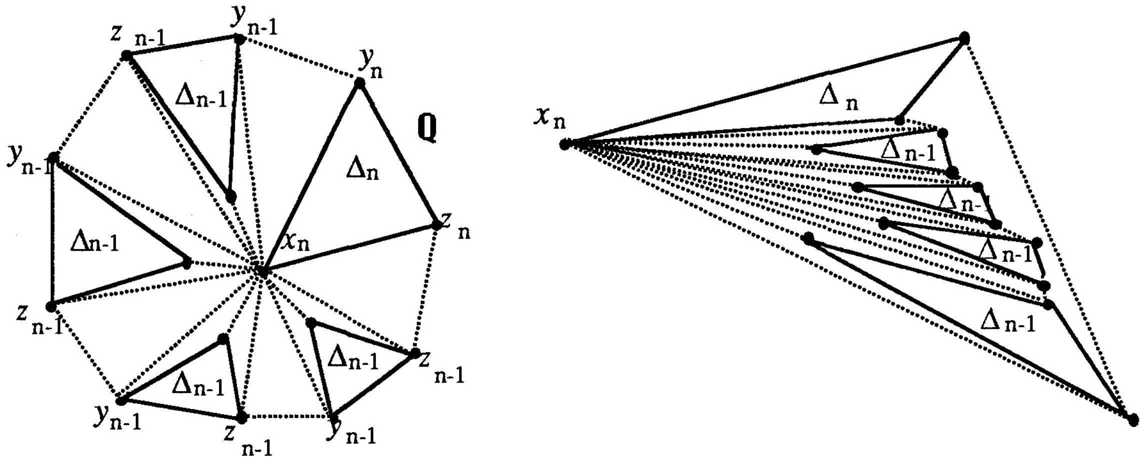
LEMMA 5.1.1 Suppose that Δ is a stationary straight line representation of Γ. Let C_m and C_n be, respectively, $m^{\underline{th}}$ level and $n^{\underline{th}}$ level cycles in Γ, and let Δ_m and Δ_n denote the subgraphs of Δ with straight edges representing C_m and C_n, respectively. Then if C_m is contained in the interior of C_n then Δ_m is contained in the interior of Δ_n.

PROOF We prove the lemma in two steps.

(1) There is no Δ_k such that the interior of Δ_k contains Δ_n for $n \geq k \geq 0$.

Suppose that there is a Δ_k such that the interior of Δ_k contains Δ_n for $n \geq k \geq 0$. Then all vertices adjacent to those of Δ_n should be contained in the interior of the Δ_k and furthermore the vertices adjacent to those which are adjacent to vertices of Δ_n should also be contained in the interior of the Δ_k. If we continue this, then all vertices of Γ except those of the chosen Δ_k are in the interior of the Δ_k. This contradicts the stationarity of Δ.

(2) The interior of n-triangle Δ_n contains exactly the Δ_{n-1}'s of which vertices are joined to x_n by straight edges, where x_n, y_n and z_n are vertices of Δ_n and x_n is an admissible point with respect to the Δ_{n-1}'s, for $n = 1, 2, 3, \ldots$.



case(i) case(ii)

When all Δ_{n-1} 's adjacent to x_n are in the exterior of Δ_n

Figure 6

There are only 4 Δ_{n-1}'s of which vertices are adjacent to $x_n \in V(\Delta_n)$. If one of the Δ_{n-1}'s is in the interior of Δ_n, then the remaining 3 Δ_{n-1}'s must also lie in the interior of the Δ_n.

So, we suppose that all Δ_{n-1}'s adjacent to x_n are in the exterior of Δ_n. Then only two cases are possible. See Figure 6.

Case(i) x_n is in the interior of a finite cycle **Q** with vertices consisting of y_n, z_n, y_{n-1}'s and z_{n-1}'s, and there is a 1-way infinite path $x_nx_{n+1}x_{n+2}\ldots$ by the property of Γ, where x_{n+i} is a vertex of Δ_{n+i} respectively for $i \geq 0$. By planarity, the only region where the infinite path can be is the interior of Δ_n since y_n and z_n should be joined to x_{n+1} by straight edges without intersecting any edges of Δ. This is a contradiction to (1).

Case(ii) The only region where the infinite path $x_n x_{n+1} x_{n+2} \ldots$ should be is also the interior of Δ_n since x_{n+1} should be joined to y_n and z_n by straight edges without intersecting any edges of Δ. So it contradicts (1). ♠♠

Now we know that the interior of any n-triangle Δ_n contains 4 of the Δ_{n-1}'s, 4^2 of the Δ_{n-2}'s, ... and 4^n of the Δ_0's for $n = 1, 2, 3, \ldots$. This will be an important ingredient in the proof in the next section.

5.2 Non-existence of a Stationary Straight Line Representation of Γ

In this section we will show that the stationary plane graph Γ given in section 4.3 does not have a stationary straight line representation. We will derive a contradiction, by assuming that there does exist a stationary straight line representation Δ of Γ.

We start with the following definition, whose significance will become clear soon.

DEFINITION 5.2.1 Let T be a triangle with vertices a, b and c, arranged in lexicographical ordering. The *shape* S(T) is a mapping from the set of all triangles into $\mathbb{R}^4$, defined by

$$S(T) = (\vec{b} - \vec{a}, \vec{c} - \vec{a}).$$

Note that S is invariant under translations.

Now let X be a stationary process of disjoint triangles in $\mathbb{R}^2$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Define $F : \Omega \to \mathbb{R}^4$ to be the random vector such that

$$F(\omega) = \begin{cases} S(T^\circ(\omega)) & \text{if the origin is contained in any triangle in } X(\omega) \\ \underline{0} & \text{otherwise} \end{cases}$$

where $T^\circ(\omega)$ is the triangle in $X(\omega)$ which contains the origin.

LEMMA 5.2.1 Let T_u, $u \in \mathbb{R}^2$ be defined by $T_u(v) = v + u$ for $v \in \mathbb{R}^2$. Then given $F(\omega) = (\vec{x}, \vec{y}) = S(T)$ for $x, y \in \mathbb{R}^2$ and some triangle T, the origin is uniformly distributed over T. That is, for any ball B and $u \in \mathbb{R}^2$ such that both B and $T_u(B)$ are in the shape $(\vec{x}, \vec{y})$,

$$P(\ (0,0) \in B \mid F(\omega) = (\vec{x}, \vec{y})\ ) = P(\ (0,0) \in T_u(B) \mid F(\omega) = (\vec{x}, \vec{y})\ )\ F\text{- a.s.}$$

PROOF Let $B(z, \varepsilon)$, for $z \in \mathbb{R}^4$, be the set $\{z' \in \mathbb{R}^4 \mid d(z, z') \le \varepsilon\}$, where d is Euclidean distance in $\mathbb{R}^4$. First we show that

$$(1)\quad P(\ (0,0) \in B \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\ ) = P(\ (0,0) \in T_u(B) \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\ ),$$

for ε so small that B and $T_u(B)$ contained in the shape $(\vec{x}, \vec{y})$, are also contained in all shapes in $B((\vec{x}, \vec{y}), \varepsilon)$.

<u>Proof of (1)</u>

$$P(\ (0,0) \in B \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\ ) = P(\ u \in T_u(B) \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\ )$$

$$= P(\ (0,0) \in T_u(B) \mid F(\omega) \in T_u(B((\vec{x}, \vec{y}), \varepsilon))\ )$$ since X is a stationary process,

$$= \frac{P(\ (0,0) \in T_u(B)\ )}{P(\ F(\omega) \in T_u(B((\vec{x}, \vec{y}), \varepsilon))\ )}$$

$$= \frac{P(\ (0,0) \in T_u(B)\ )}{P(\ F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\ )}$$ since X is a stationary process,

$$= P(\ (0,0) \in T_u(B) \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\ ).$$

So, in order to prove the lemma we need to show that

(2) $$\lim_{\varepsilon \downarrow 0} P(\ (0,0) \in B \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\ ) = P(\ (0,0) \in B \mid F(\omega) = (\vec{x}, \vec{y})\ ) \quad F\text{- a.s.}$$

To show (2) we will prove the following fact.

FACT: Let Y, Z be random vectors on $(\Omega, \mathcal{F}, P)$ and Y be integrable. Then

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[\ Y \mid Z \in B(z, \varepsilon)\ ] = \mathbb{E}[\ Y \mid Z = z\ ] \qquad Z\text{- a.s.}$$

Proof of fact

Let $P_Z(A) = P\{Z \in A\}$. Define a measure ν by

$$\nu\{Z \in A\} = \int_{\{Z \in A\}} Y\, dP = \int_A Y\, dP_Z$$

and let $\nu_Z(A) = \nu\{Z \in A\}$. Then $\dfrac{\nu_Z(A)}{P_Z(A)}$ is the average value of Y on $\{Z \in A\}$. This measure ν_Z is finite because Y is integrable, and it is absolutely continuous with respect to P_Z. By the Radon-Nikodym theorem, there exists a density f for ν with respect to P_Z such that $\nu_Z(A) = \int_A f\, dP_Z$ (see Billingsley [1], section 32-34 for details). The density (called the Radon-Nikodym derivative of ν_Z with respect to P_Z) is often denoted by $\frac{d\nu_Z}{dP_Z}$. Thus, we may write $\frac{d\nu_Z}{dP_Z}(z) = \mathbb{E}[Y \mid Z = z]$, Z- a.s. By Rudin [16] (chapter 8), we get

$$\frac{d\nu_Z}{dP_Z}(z) = \lim_{\varepsilon \downarrow 0} \frac{\nu\{Z \in B(z, \varepsilon)\}}{P\{Z \in B(z, \varepsilon)\}} = \lim_{\varepsilon \downarrow 0} \frac{\int_{\{Z \in B(z, \varepsilon)\}} Y \, dP}{P\{Z \in B(z, \varepsilon)\}}$$

$$= \lim_{\varepsilon \downarrow 0} \mathbb{E}[\, Y \mid Z \in B(z, \varepsilon)\,].$$

Now we have

$$\lim_{\varepsilon \downarrow 0} P(\, (0,0) \in B \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\,)$$
$$= \lim_{\varepsilon \downarrow 0} \mathbb{E}[\, \mathbf{1}_{\{(0,0) \in B\}} \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\,]$$
$$= \mathbb{E}[\, \mathbf{1}_{\{(0,0) \in B\}} \mid F(\omega) = (\vec{x}, \vec{y})\,] \quad F\text{- a.s. by fact}$$
$$= P(\, (0,0) \in B \mid F(\omega) = (\vec{x}, \vec{y})\,) \; F\text{- a.s.}$$

Similarly to (2), we have

$$\lim_{\varepsilon \downarrow 0} P(\, (0,0) \in T_u(B) \mid F(\omega) \in B((\vec{x}, \vec{y}), \varepsilon)\,)$$
$$= P(\, (0,0) \in T_u(B) \mid F(\omega) = (\vec{x}, \vec{y})\,) \; F\text{- a.s.}$$ So the proof of lemma is finished. ♠♠

Let $A_n = \{(x, y) \in \mathbb{R}^2 \mid (x, y)$ is in the interior of Δ_n for some $\Delta_n\}$ for $n = 0, 1, 2, \ldots$. Then $\{A_n\}$ is an increasing sequence. Now we have the following lemma.

Lemma 5.2.2. Let Δ be a stationary (and ergodic) straight line representation of Γ. Then

$$\bigcup_{n=0}^{\infty} A_n = \mathbb{R}^2 \text{ a.s.}$$

PROOF Let $A = \bigcup_{n=0}^{\infty} A_n$, and let E be an event that $A \neq \mathbb{R}^2$. Then E is translation invariant. By ergodicity, $P(E) = 0$ or 1. Assume $P(E) = 1$. Then $\exists\, z \in \mathbb{R}^2$ such that $z \notin A$ a.s. Note that A is an open convex subset of $\mathbb{R}^2$. For if $x, y \in A$ then $\exists\, n$ such that $x, y \in A_n$ and so $\exists\, k, l$ such that x is in the interior of Δ_{n_k} and y is in the interior of Δ_{n_l}. By construction of Δ, $\exists\, N > n$ such that both Δ_{n_k} and Δ_{n_l} are in the interior of Δ_N. Then for $0 < t < 1$, $tx + (1-t)y$ is also in the interior of Δ_N since Δ_N is convex. Since A is open, A is an open convex subset of $\mathbb{R}^2$.

Then there exists a line L in $\mathbb{R}^2$ such that L separates z from A (see Eggleston [4], section 2.7). So there is a half-plane H_Z (determined by L) containing z such that

$$A \cap H_Z = \varnothing. \qquad (*)$$

Choose u with $|u| = 1$ (not parallel to L) such that T_u acts ergodically (this is possible according to Theorem 1 in Pugh and Shub [14]). Choose v with $|v| = 1$ such that $u \bullet v = 0$ where $\bullet$ is the inner product. Let R be the unit box with respect to the base (u, v) and let $R_{m,n} = T_{(m,n)}(R) = R + (m, n)$ for any $m, n \in \mathbb{Z}$, where the coordinates are with respect to the base (u, v). Define a random variable $X_{m,n}$ by

$$X_{m,n} = \begin{cases} 1 & \text{if } A \cap R_{m,n} \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Then the whole plane is tiled with 0-1 valued unit boxes.

For at least one $n \in \mathbb{Z}$, the sequence of unit boxes $R_n = \bigcup_{m \in \mathbb{Z}} R_{m,n}$ must have positive probability to intersect A. We choose such an n. Now we consider the ergodic process Y in $\{0, 1\}^{\mathbb{Z}}$ by

$$Y(m) = 1 \text{ if and only if } X_{m,n} = 1.$$

Then $P(\ Y(m) = 1\ ) > 0$ since $P(\ R_n \cap A \neq \varnothing\ ) > 0$. But by (*) there exists an N such that $Y(m) = 0\ \forall\ m > N$ with positive probability. This is a contradiction. ♠♠

The height of a triangle T is the minimum distance between a vertex and the line through the opposite side. Denote the minimum height of a triangle T by $h(\mathrm{T})$.

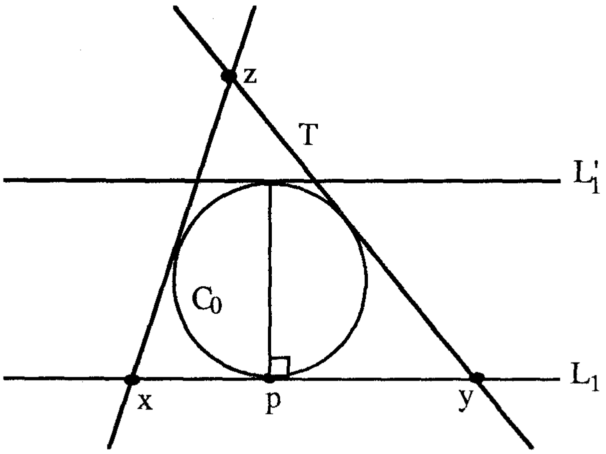
LEMMA 5.2.3 Suppose that Δ is a stationary straight line representation of Γ. Then $\forall\ \varepsilon > 0$, $\forall\ M > 0$, $\exists\ N > 0$ such that $P(h(\Delta_N) > M \mid (0, 0) \in A_N\ ) > 1 - \varepsilon$, where Δ_N is the $n^{\underline{\text{th}}}$ level triangle containing the origin.

Proof Let C be any circle with diameter M, containing the origin. Since $\bigcup_{n=0}^{\infty} A_n = \mathbb{R}^2$ a.s., we can choose three points a, b, c in $\mathbb{R}^2$ such that a, b, c $\in A_n$ and C is covered by a triangle T_1 with vertices a, b, and c. By the structure of Δ, there exist i, j, k such that a, b and c are in the interior of Δ_{n_i}, Δ_{n_j} and Δ_{n_k}, respectively and furthermore there exists an $N > 0$ such that the interior of Δ_N contains Δ_{n_i}, Δ_{n_j}, Δ_{n_k} and so T_1. Then we claim that

(1) $$h(\Delta_N) \to \infty \text{ a.s.}$$

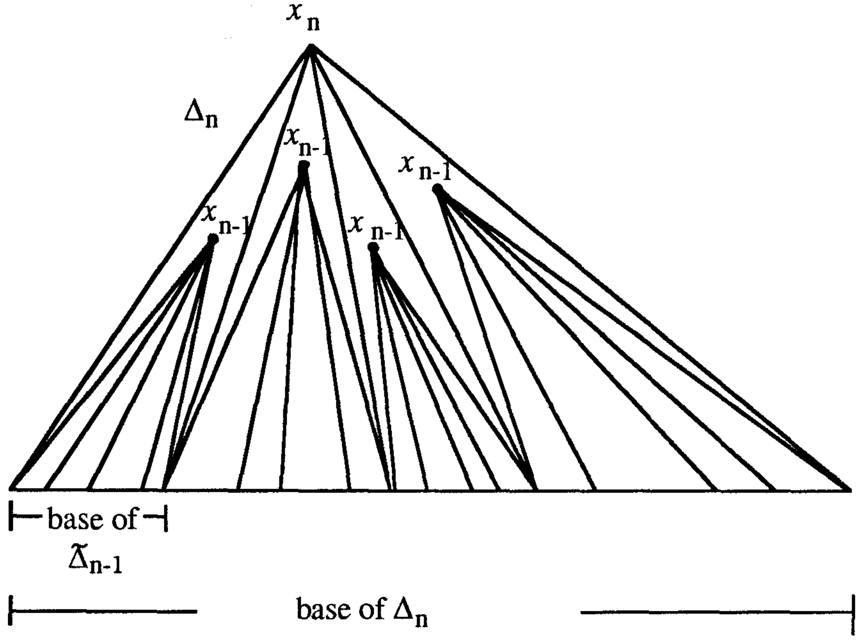
This is enough to prove the lemma.

Proof of (1) we will show that the diameter of inscribed circle of a triangle T is always less than h(T), since the biggest circle covered by T is the inscribed circle of T. Let C_0 be a circle with the tangent line L_1 at p. Let x and y be two points on the line L_1 lying on the opposite side of p. Let the line L_1' be parallel to L_1 and tangent to C_0. The distance between the lines L_1 and L_1' (i.e., $d(L_1, L_1') = \min\{d(l, l') \mid l \in L_1, l' \in L_1'\}$) is equal to the diameter of C_0. Let x, y, z be the vertices of the triangle T inscribing C_0. Then z is in the side of L_1' that does not contain C_0 since if not there is no triangle inscribing C_0 with vertices x, y, z. See Figure 7. So the minimum distance from z to L_1 is greater than or equal to the diameter of C_0. The minimum distance from x to the line passing y and z (or from y to the line passing x and z) is also greater than or equal to the diameter of C_0. So we have h(T) $\geq$ the diameter of C_0. Since C is any circle with diameter M covered by Δ_N, we have $h(\Delta_N) \geq M$. ♠♠



The inscribed circle C_0 of T

Figure 7



The interior of Δ_n

Figure 8

Now we go back to a straight line representation Δ of Γ. We know that the interior of each Δ_n contains four Δ_{n-1}'s and the vertices of these Δ_{n-1}'s are joined to $x_n (\in V(\Delta_n))$ by straight edges. We call the opposite side of x_n the base of Δ_n. We can subdivide the angle at x_n of Δ_n into four subangles and the base of Δ_n into four subbases such that we have 4 subtriangles of Δ_n and each of them contains one Δ_{n-1} in their interior. So the interior of each subtriangle contains a vertex x_{n-1} of a Δ_{n-1}. Denote the triangle with a vertex x_{n-1} and the same base as a subtriangle of Δ_n containing the x_{n-1} by $\tilde{\Delta}_{n-1}$. Then $\tilde{\Delta}_{n-1}$ and Δ_{n-1} has a common vertex x_{n-1} and the interior of $\tilde{\Delta}_{n-1}$ contains Δ_{n-1}. Now we subdivide each triangle $\tilde{\Delta}_{n-1}$ into 4 subtriangles in the same way as above. See Figure 8. Then the interior of each subtriangle of $\tilde{\Delta}_{n-1}$ contains a Δ_{n-2} and thus a vertex x_{n-2} of Δ_{n-2}. So we can get a triangle $\tilde{\Delta}_{n-2}$ with a vertex x_{n-2} and the same base as a subtriangle of $\tilde{\Delta}_{n-1}$ containing the x_{n-2}. We continue this procedure until we have 4^{n-1} of the $\tilde{\Delta}_1$'s in the interior of Δ_n. The base of each $\tilde{\Delta}_1$ is one of 4^{n-1} subbases of the base of Δ_n. Now we also subdivide $\tilde{\Delta}_1$ into 4 subtriangles in the same way as above such that the interior of each subtriangle of $\tilde{\Delta}_1$ contains a Δ_0. We denote such a subtriangle of $\tilde{\Delta}_1$ by $\tilde{\Delta}_0$. The $\tilde{\Delta}_0$ is a triangle with a vertex x_1 and a subbase of $\tilde{\Delta}_1$ as the base of $\tilde{\Delta}_0$, and there are 4^n of the $\tilde{\Delta}_0$ in the interior of Δ_n. This construction and notations will be used in the following theorem.

We define the *diameter of a triangle* T (denoted by diam(T)) to be the maximum distance between vertices of T. Now we are ready to prove the main theorem of this section.

THEOREM 5.2.4 There does not exist a stationary (and ergodic) straight line representation of Γ.

PROOF Suppose that Δ is a stationary straight line representation of Γ. Let E_n be the event that A_n contains the origin for $n \geq 0$. Let Δ_n^0 denote the $n^{\underline{th}}$ level triangle in Δ that contains the origin for $n \geq 0$. Note that $\forall\, \varepsilon > 0$, $\exists\, L > 0$ s.t. $P(\text{ diam } (\Delta_0^0) < L \mid E_0)$ $> 1 - \varepsilon$. By Lemma 5.2.2, $1 = P(\bigcup_{n=0}^{\infty} E_n) = \lim_{n\to\infty} P(E_n)$. So $\forall\, \varepsilon > 0$, $\exists\, N > 0$ s.t. $P(E_N) > 1 - \varepsilon$. Without loss of generality we can assume that

$$P(E_0) > 0.999.$$

Let E_0^L be the event that A_0 contains the origin and diam(Δ_0^0) is less than L. Then $E_0^L \uparrow E_0$ as $L \to \infty$, so $\lim_{L\to\infty} P(E_0^L) = P(E_0)$. So we may assume that $P(E_0^L) > 0.999$.

Let, for $N > 0$,

$$X_N = \begin{cases} \text{the percentage area of all } \Delta_0^0\text{'s in } \Delta_N^0 \text{ with } \mathrm{diam}(\Delta_0^0) < L & \text{if } E_N \text{ occurs} \\ 0 & \text{if } E_N^c \text{ occurs.} \end{cases}$$

Then $0 \le X_N \le 1$. Now we have

$$P(E_0^L \mid E_N) = \mathbb{E}[\mathbf{1}_{E_0^L} \mid E_N]$$

$$= \mathbb{E}[X_N \mid E_N] \quad \text{by Lemma 5.2.1 .}$$

This yields

$$P(E_0^L) = \mathbb{E}[X_N].$$

Now we have, for $0 < \gamma < 1$,

$$0.999 < \mathbb{E}[X_N] = \int_{\{X_N \le \gamma\}} X_N \, dP + \int_{\{X_N > \gamma\}} X_N \, dP$$

$$\le \gamma P(X_N \le \gamma) + P(X_N > \gamma)$$

$$= \gamma + (1 - \gamma) P(X_N > \gamma),$$

so $P(X_N > \gamma) \ge \dfrac{0.999 - \gamma}{1 - \gamma}$. Let $\gamma = 0.9$. Then

(1) $$P(X_N > 0.9) \ge 0.99, \text{ for any } N > 0.$$

Now we choose $M > 10L$. By Lemma 5.2.3, we can choose an N such that

(2) $$P(h(\Delta_N^0) > M \mid E_N) > 0.99.$$

Let Z_N be the ratio of the area of all points in $\tilde{\Delta}_0$'s in $\overset{0}{\Delta}_N$ that are also below the line parallel to the base of $\overset{0}{\Delta}_N$ with distance $(0.1)M$ from this base (say, the $(0.1)M$ line) to the area of $\overset{0}{\Delta}_N$ if E_N occurs, or zero if E_N does not occur. Then

(3) $$Z_N \leq 1 - (0.9)^2 = 0.19.$$

Let the height of $\tilde{\Delta}_0$ be the minimum distance from the vertex x_1 to the line through the base of $\tilde{\Delta}_0$, where $x_1 \in V(\Delta_1)$ is an admissible point with respect to all Δ_0's contained in the interior of the Δ_1. Let $\tilde{\Delta}_0{}^*$ be the region of $\tilde{\Delta}_0$ below the $(0.1)M$ line. Then the area of $\tilde{\Delta}_0{}^*$ is greater than or equal to the area of any other region of $\tilde{\Delta}_0$ between two lines with distance $(0.1)M$, parallel to the base of $\tilde{\Delta}_0$. Since $L <$ $(0.1)M$, a triangle $\overset{0}{\Delta}_0$ with $\mathrm{diam}(\overset{0}{\Delta}_0) < L$ is contained in such a region. Thus the area of a triangle $\overset{0}{\Delta}_0$ with $\mathrm{diam}(\overset{0}{\Delta}_0) < L$ contained in a $\tilde{\Delta}_0$ is always less than the area of $\tilde{\Delta}_0{}^*$. By (2), we have $P(Z_N \geq X_N \mid E_N) >$ 0.99. Thus

$$\begin{aligned} 0.99 &< P(Z_N > X_N) \\ &= P(0.19 \geq Z_N > X_N), \text{ by } (3) \\ &\leq P(X_N < 0.19) \end{aligned}$$

This is a contradiction to (1). ♠♠

CHAPTER 6

CONCLUSION

The random graph Γ constructed in section 4.3 is a connected, locally finite, VAP-free, stationary plane graph. By theorem 3.2.6 there exists a random straight edge graph Δ such that for a.e. $\omega \in \Omega$, $\Delta(\omega)$ is isomorphic to $\Gamma(\omega)$.

In this dissertation, we showed that there does not exist a stationary straight line representation of Γ. For the proof, we assume that there exists a stationary straight line representation Δ of Γ. Then we know from Lemma 5.1.1 that the nested structure of Γ is preserved by Δ. Further we know by stationarity and ergodicity that (i) all Δ_0's fill the plane with positive density and (ii) with high probability, given that the origin is contained in a triangle Δ_0, the diameter of the Δ_0 is of moderate size. In section 5.2 we showed that Δ can not satisfy (i) and (ii) at the same time.

The results of this thesis set the stage for further research. In particular, it is important to find reasonsable conditions which ensure that a stationary straight line representation of a stationary planar (plane) graph will exist.

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