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A graph consists of a finite or a countable set V of vertices together with a set of edges E such that each edge has two endpoints in the vertex set V. A simple plane graph is a graph whose vertex set is a point set in the Euclidean plane while the edges are Jordan curves such that two different edges have at most one end point in common. A planar graph is an abstract graph isomorphic to (or represented by) a plane graph. Thomassen showed that every simple planar graph has a straight line representation, that is, a representation in which all edges are straight line segments.

Let \( \mathcal{A} \) be the set of graphs and \( \mathcal{G} \) a suitable \( \sigma \)-algebra. A random graph \( X \) is a measurable mapping from a probability space \( (\Omega, \mathcal{F}, P) \) into \( (\mathcal{A}, \mathcal{G}) \). In the case that \( V \) is a subset of \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) then translations on \( \mathbb{R}^d \) extend to translations on \( \mathcal{A} \), the space of graphs. Such a random graph is said to be stationary if its distribution is invariant under all translations.
In this thesis we show that Thomassen's result does not extend to stationary simple random planar graphs. In particular we construct a stationary simple random plane graph which possesses no stationary straight line representation.
On Straight Line Representations of Random Planar Graphs

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Typed for In-Kyeong Choi
To
My Parents
and
My Parents-In-Law
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“And we know that all things work together for good to those who love God, to those who are the called according to His purpose.”

(Romans 8:28)

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A graph is said to be planar if it can be embedded in the plane such that its edges intersect only at their end points. Such an embedding of a planar graph is called a plane graph. Fáry [5] proved that every finite planar graph has a straight line representation and Thomassen [18] showed that every infinite planar graph has a straight line representation and every connected, locally finite, plane graph with no vertex accumulation point is isomorphic to a subgraph of a straight line triangulation.

We define a stationary random graph and consider the following question: Does every stationary random planar graph have a stationary straight line representation? We show that the answer is negative.

Chapter two is devoted to the definitions and preliminaries of graph theory.

In chapter three, we review the straight line representations of finite and infinite planar graphs [5, 18]. In section 3.1, it is proved
that every simple planar graph is isomorphic to a subgraph of a triangulated simple graph and every triangulated simple graph is isomorphic to a straight line triangulation. In section 3.2, we deal with infinite planar graphs.

We define random graphs and stationary random graphs in section 4.1 and in section 4.2 and 4.3, we construct a stationary random plane graph $\Gamma$ which, as we will show, does not yield a stationary straight line representation.

In chapter five we think about the straight line representation of the given stationary plane graph $\Gamma$. In section 5.1, we show that the structure of $\Gamma$ is preserved to $\Delta$ if we assume that $\Delta$ is stationary. In section 5.2, it is proved that $\Delta$ is not a stationary graph.
CHAPTER 2
PRELIMINARIES OF GRAPH THEORY

The terminologies of graph theory are from [2], [8], [12] and [18] with minor modifications.

A graph $G$ consists of a finite or countable set, $V(G)$ of vertices together with a prescribed set $E(G)$ of edges. Each edge $e$ is associated with an unordered pair (or singleton) of vertices $\{x, y\} \subseteq V(G)$. In this case $e$ is said to join $x$ and $y$ and we say that $x$ and $y$ are adjacent vertices; vertex $x$ and edge $e$ are incident with each other, as are $y$ and $e$. An edge connecting a vertex with itself is called a self-loop. For many studies it is desirable to exclude self-loops and multiple edges. We call graphs resulting from such restrictions simple. In general the term simple is used in graph theory to mean without repetition.

If the set of vertices and the set of edges of a graph $G$ are both finite, the graph is called finite, otherwise infinite. An infinite graph may have infinitely many edges but possibly finitely many vertices (for example, two vertices can be connected by infinitely many edges).

Let $v_0e_0v_1e_1v_2...e_{n-1}v_n$ be a finite sequence whose terms are alternatively vertices and edges such that, for $0 \leq i \leq n-1$, $e_i$ joins $v_i$ and $v_{i+1}$. If all edges $e_0, e_1, ..., e_{n-1}$ are distinct and all vertices $v_0, v_1, ..., v_n$ are also distinct, then $v_0e_0v_1e_1v_2...e_{n-1}v_n$ is called a path (of length $n$). In a simple graph, a path $v_0e_0v_1e_1v_2...e_{n-1}v_n$ is determined by the sequence $v_0v_1...v_n$ of its vertices; hence a path in a simple graph can be specified simply by its vertex sequence. A path whose
origin and terminus are the same is called a **cycle**. A cycle with \( n \) edges is called an **n-cycle**. An infinite set of \( v_i e_i v_{i+1} \) for \( i = 0, 1, 2, \ldots \) and the graph formed by them is called a **1-way** infinite path if \( v_i \neq v_j \) and \( e_i \neq e_j \) for \( i \neq j \); under the same condition the \( v_i e_i v_{i+1} \) for \( i = 0, \pm 1, \pm 2, \ldots \) form a **2-way** infinite path. Here is a simple but fundamental theorem about infinite graphs due to König.

**Lemma** (The Infinity Lemma [10]) Let \( V_0, V_1, V_2, \ldots \) be a countably infinite sequence of finite, nonempty, pairwise disjoint sets of points. Let the points contained in these sets form the vertices of a graph. If \( G \) has the property that every point of \( V_{n+1} \) \( (n = 0, 1, 2, \ldots) \) is joined with a point of \( V_n \) by an edge of \( G \), then \( G \) has a 1-way infinite path \( v_0 v_1 \ldots \), where \( v_n \) \( (n = 0, 1, 2, \ldots) \) is in \( V_n \).

For any vertices \( x \) and \( y \) in a graph, there is a path joining \( x \) and \( y \), then the graph is said to be **connected**. A cycle \( C \) of a connected graph \( G \) is a **separating** cycle if and only if \( G - V(C) \) is not connected.

If every vertex of a graph is incident with a finite number of edges, then the graph is said to be **locally finite** (i.e., finite degree). Locally finite, infinite graphs play an important role in that some properties of finite graphs, which lose their validity (or even their meaning) for arbitrary infinite graphs, can be extended to graphs of finite degree. A locally finite, infinite graph, of course, always has infinitely many vertices.

If the vertex set of the graph \( H \) is contained in the vertex set of the graph \( G \) and the edge set of \( H \) is also contained in the edge set of \( G \) then \( H \) is called a **subgraph** of \( G \). A **spanning** subgraph of \( G \) is a
subgraph $H$ with $V(G) = V(H)$. For any $A \subseteq V(G)$, the induced subgraph of $G$ by $A$ is the graph having vertex set $A$ and whose edge set consists of those edges of $G$ incident with two elements of $A$. For any set $B$ of edges of $G$, the edge-induced subgraph of $G$ by $B$ is the graph whose vertex set consists of those vertices of $G$ incident with at least one edge of $B$ and having edge set $B$.

An edge $e \in E(G)$ is said to be subdivided when it is deleted and replaced by a path of length 2 connecting its ends, the internal vertex of this path being a new vertex. A subdivision $\tilde{G}$ of $G$ is a graph that can be obtained from $G$ by a sequence of edge subdivisions.

Two graphs $G$ and $G'$ are isomorphic if there exists a one-to-one incidence-preserving map taking $V(G)$ onto $V(G')$ and $E(G)$ onto $E(G')$. A plane graph is a graph whose vertex set is a point set in the Euclidean plane while the edges are Jordan curves such that two different edges have at most end points in common. A planar graph is an abstract graph isomorphic to a plane graph. Kuratowski [11] characterized finite, planar graphs by a theorem saying that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$. See Figure 1. Dirac and Schuster [3] generalized it to countable graphs. Wagner [20] characterized all planar graphs and Halin [7] characterized those graphs which are isomorphic to locally finite plane graphs with no vertex accumulation points (to be defined in §3.2). Thomassen [19] summarized the current results about planarity of infinite graphs.

If the graph $G$ is isomorphic to the plane graph $\Gamma$, then $\Gamma$ is a representation of $G$. If the edges of $\Gamma$ are polygonal arcs, then $\Gamma$ is a
polygonal arc representation. If the edges of $\Gamma$ are straight line segments, then $\Gamma$ is a straight line representation (it is defined only when the graph $G$ is simple).

Two finite plane graph $\Gamma$ and $\Gamma'$ are equivalent if there exists a one-to-one incidence-preserving map taking $V(\Gamma)$ onto $V(\Gamma')$, $E(\Gamma)$ onto $E(\Gamma')$, and the set of regions of $\Gamma$ onto the set of regions of $\Gamma'$. If, furthermore, the unbounded region of $\Gamma$ corresponds to the unbounded region of $\Gamma'$, then $\Gamma$ and $\Gamma'$ are strongly equivalent.
A simple planar graph has no self-loops or multiple edges. In this chapter we only deal with simple planar graphs since most of the results in this chapter does not hold for non-simple planar graphs.

3.1 Straight Line Representations Of Finite Planar Graphs

A finite, plane graph $\Gamma$ partitions the plane into a finite number of regions one of which is unbounded. If every region in $\Gamma$ is bounded by a cycle of exactly three edges (3-cycle), then $\Gamma$ is called a triangulation. Such graphs are always simple. In order to see straight line representations of a finite planar graph we need the following lemma by Färy [5].

**LEMMA 3.1.1** Every finite plane graph $\Gamma$ is a spanning subgraph of a triangulation $\Gamma'$.

**PROOF** This result is trivially true when $\Gamma$ has three vertices. So assume that $\Gamma$ has more than three vertices. We construct a triangulation $\Gamma'$ of which $\Gamma$ is a subgraph by adding edges to $\Gamma$ according to the following scheme; select two vertices on the boundary of a region that are not connected by an edge. Give an edge between these two vertices which lies interior to the region.
We can continue in this way until no such pair of vertices can be found. The process will always stop since \( \Gamma \) is finite. The resulting graph is \( \Gamma' \). Then \( \Gamma' \) is connected for if it has separated components there would be vertices on the boundary of a region (i.e., the unbounded region) which should be connected by an edge according to the above scheme. This is a contradiction.

Graph \( \Gamma' \) is triangulated. If there was a finite region of \( \Gamma \) with more than three edges, then the above process could not have terminated which contradicts the definition of \( \Gamma' \). Thus every simple finite plane graph \( \Gamma \) is a subgraph of a triangulation \( \Gamma' \) based on the same set of vertices as \( \Gamma \).

Now we go the

**THEOREM 3.1.2** [5] Every finite planar graph \( G \) has a straight line representation \( \Gamma \).

**PROOF** Let \( \Gamma \) denote a representation of \( G \). It is clear that every subgraph of a straight line graph is also straight line graph. By Lemma 3.1.1, the theorem will be established if every triangulation \( \Gamma' \) is isomorphic to a straight line triangulation \( \Delta \). To do so requires two facts.

**FACT (1)** Let \( T \) be a triangulation with at least four vertices. Let \( x \) be a vertex of \( T \) and \( xv_1, xv_2, \ldots, xv_k \) be all the edges with \( x \) as one end vertex. The edges \( v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1 \) are in \( T \) and form a cycle \( \Theta_x \) which separates \( x \) from every other vertex of \( T \).

**Proof of fact (1)** First we assume that \( x \) is not on the boundary of the unbounded region. Then there is a cycle \( \Phi_x \) such that there is no
cycle inside $\Phi_x$ which contains $x$ inside it. We denote the vertices of $\Phi_x$ by $u_1, u_2, \ldots, u_m$. Since $T$ triangulates the interior of $\Phi_x$, the edges $u_iu_{i+1}$ of $\Phi_x$ are adjacent to a region in the interior of $\Phi_x$ and the region contains a third vertex $y$ on its boundary. Suppose $x \neq y$. Then the cycle $u_1u_2 \ldots u_iy_u_{i+1} \ldots u_mu_1$ has $x$ in its interior and lies in the interior of $\Phi_x$. This is a contradiction to the definition of $\Phi_x$. Hence $x = y$ and this is true for all the edges of $\Phi_x$.

Suppose that $x$ is on the boundary of the unbounded region. Then there is a cycle $\Phi_x$ such that there is no cycle outside $\Phi_x$ which has $x$ in its exterior. Using similar argument used above we can show that $\Phi_x$ can be used for the cycle $\Theta_x$ and so the fact is fully established.

Let $\Delta$ be a triangulation having more than three vertices. To prepare the proof by induction we construct from $\Delta$ a not necessarily simple graph $\Delta^*$ having one less vertex. A vertex $x$ of $\Delta$ which does not lie on the boundary of the unbounded region is removed from $\Delta$. All edges incident on $x$ are also removed. The graph thus obtained contains a cycle $\Theta_x$. Graph $\Gamma'$ is identical with $\Delta$ outside $\Theta_x$ and is empty inside $\Theta_x$. We connect vertex $v_1$ of $\Theta_x$ with each of the vertices $v_3, v_4, \ldots, v_{k-1}$ of by edges in the interior of $\Theta_x$ that do not intersect each other. The resulting graph is denoted by $\Delta^*$.

Now we need the second fact.

**FACT (2)** If $\Delta^*$ is not simple, then $\Delta$ has a separating 3-cycle.

**Proof of fact (2)** Since $\Delta$ is a triangulation, $\Delta^*$ can not be simple only when a new edge connects two vertices which are also
connected by an edge in the exterior of $\Theta_x$. Since all new edges are incident on $v_1$, there is a vertex $v_i (2 < i \leq k-1)$ which is joined by an edge in the exterior of $\Theta_x$ to $v_1$. Then the 3-cycle $v_ixv_1$ in $\Delta$ separates $v_2$ from $v_k$. For example see Figure 2.

Now we are able to prove the theorem by induction on $n$, the number of vertices of $\Delta$. If $n = 3$, certainly $\Delta$ is a straight line triangulation. So assume that $n > 3$ and $\Delta$ is a straight line triangulation with fewer than or equal to $n$ vertices. We consider $\Delta$

![Diagram of $\Delta$ and $\Delta^*$](image)

The separating 3-cycle $v_1xv_3v_1$

Figure 2

with $n+1$ vertices. Let $x$ be a vertex of $\Delta$ not on the boundary of the unbounded region. We distinguish two cases according to as $\Delta^*$ is simple or not.

Case (i) $\Delta^*$ is simple.

In this case $\Delta^*$ is a straight line triangulation by the induction hypotheses. We put it into a straight line representation.

The edges $v_1v_3$, $\ldots$, $v_1v_{k-1}$ are removed leaving the interior
of \( \Theta_x \) empty. We may now place \( x \) into this region and connect it to each of the vertices \( u_1, u_2, \ldots, u_k \) with straight edges. Then \( \Delta \) is a straight line triangulation.

Case (ii) \( \Delta^* \) is not simple.

By fact (2), there is a separating 3-cycle in \( \Delta \). Denote this cycle by \( \Theta \). By induction the subgraph \( \Delta_1 \) of \( \Delta \) induced by the vertices of \( \Theta \) and the vertices in the interior of \( \Theta \) has a straight line triangulation. The subgraph \( \Delta_2 \) of \( \Delta \) induced by the vertices of \( \Theta \) and the vertices in the exterior of \( \Theta \) also has a straight line triangulation. Thus \( \Delta = \Delta_1 \cup \Delta_2 \) is a straight line triangulation.

Thus we showed that every planar graph \( G \) is isomorphic to a spanning subgraph \( \Gamma \) of a straight line triangulation \( \Delta \). □□

3.2 Straight Line Representations of Infinite Planar Graphs

In this section, we begin with the following definitions related to infinite graphs [18].

**Definition 3.2.1** A *vertex accumulation point* (respectively *edge accumulation point*) of an infinite plane graph \( \Gamma \) is a point \( p \) of the plane such that, for each \( \varepsilon > 0 \), \( B(p, \varepsilon) \) contains (respectively intersects) infinitely many vertices (respectively edges) of \( \Gamma \) where \( B(p, \varepsilon) \) denotes the set of points of Euclidean distance less than \( \varepsilon \). A VAP (abbr. vertex accumulation point) -free plane graph is a plane graph with no VAP, and an EAP (abbr. edge accumulation point) -free plane graph is a plane graph with no EAP.
The point set of a graph \( \Gamma \) is the set of vertices and all points on the edges of \( \Gamma \).

**Definition 3.2.2** An infinite, connected, plane graph \( \Gamma \) is a triangulation if and only if \( \Gamma \) is VAP-free and, for every vertex \( x \) of \( \Gamma \), \( \Gamma \) contains a cycle \( \Pi_x \) such that \( x \) is the only vertex of \( \Gamma \) in the interior of \( \Pi_x \) and \( x \) is joined by an edge to every vertex of \( \Pi_x \). If, in addition, every point not in the point set of \( \Gamma \) is contained in the interior of some cycle of \( \Gamma \), then \( \Gamma \) is a triangulation of the plane.

By definition of infinite triangulation we have the following lemma.

**Lemma 3.2.1** An infinite, plane, VAP-free graph \( \Gamma \) is a triangulation if and only if \( \Gamma \) contains a sequence of mutually disjoint cycles \( \Theta_1, \Theta_2, \Theta_3, \ldots \) such that for each \( k \geq 2 \), the subgraph of \( \Gamma \) induced by \( \Theta_k \) and the vertices in the interior of \( \Theta_k \) is finite and triangulates the interior of \( \Theta_k \) and contains \( \Theta_{k-1} \), and furthermore every vertex of \( \Gamma \) is in the interior of some \( \Theta_m \).

**Proof** \( (\Rightarrow) \) It is trivially true by the definition of triangulation. \( (\Rightarrow) \) Let \( \Gamma \) be any triangulation. We define the sequence of cycles \( \Theta_1, \Theta_2, \Theta_3, \ldots \) recursively as follows: Let \( \Theta_1 \) be any cycle of \( \Gamma \). Suppose that we have already defined \( \Theta_1, \ldots, \Theta_k \). Let \( \Gamma_k \) be the graph \( \cup \Pi_x \), where the union is taken over all vertices \( x \) of \( \Theta_k \). Since \( \Gamma_k \) is 2-connected, the boundary of the unbounded region of \( \Gamma_k \) is a cycle \( \Theta_{k+1} \). There are only finitely many vertices of \( \Gamma \) in the interior of \( \Theta_{k+1} \) since \( \Gamma \) is VAP-free. By the definition of a triangulation, \( \Gamma \) triangulates the interior of \( \Theta_{k+1} \). If \( y \) is a vertex of \( \Gamma \) in the exterior of \( \Theta_{k+1} \) then the graphic distance from \( y \) to \( \Theta_{k+1} \) is less than the
distance from $y$ to $\Theta_k$. Hence $y$ is in the interior of some $\Theta_m$, and so the proof is complete.

We state the following theorem without proof. See Thomassen [18] and Wagner [20] for the proof.

**Theorem 3.2.2** Every infinite planar graph has a straight line representation.

In order to show that a locally finite, connected, VAP-free plane graph is isomorphic to a subgraph of a straight line triangulation, we start with the following lemma.

**Lemma 3.2.3** [18] Let $\Gamma$ be a countably infinite, locally finite, VAP-free, plane graph. Then there exists a VAP-free, plane graph $\Psi$ isomorphic to $\Gamma$ such that each edge of $\Psi$ is a polygonal arc and such that no point of an edge of $\Psi$ is an EAP of $\Psi$.

**Proof** Let $G$ denote the abstract graph which is represented by $\Gamma$, and let $E(G) = \{e_1, e_2, \ldots\}$. For each edge $e_i$ of $G$ we add two new edges $e'_i, e''_i$ joining the same vertices as $e_i$ and we denote the resulting multigraph by $G'$. To each vertex $p$ of $\Gamma$ we associate a positive real number $\epsilon(p)$ such that, for any two distinct vertices $p, q$ of $\Gamma$, we have that $\bar{B}(p, \epsilon(p)) \cap \bar{B}(q, \epsilon(q)) = \emptyset$ where $\bar{B}(p, \epsilon)$ is the set of points of Euclidean distance less than or equal to $\epsilon$. We define a sequence $\Psi_0, \Psi_1, \Psi_2, \ldots$ of plane multigraphs such that $\Psi_0 \subseteq \Psi_1 \subseteq \Psi_2 \subseteq \ldots$, and such that, for each $k$,

(i) $\Psi_k$ is a polygonal arc representation of $G[\{e_1, e'_1, e''_1, \ldots, e_k, e'_k, e''_k\}]$ with the same vertex set as $\Gamma$.
(ii) for each \( j \leq k \), the union of the arcs representing \( e_j', e_j'' \) is a closed Jordan curve whose interior contains the arc representing \( e_j \) and nothing else of the graph;

(iii) for each vertex \( p \) of \( \Xi_k \), the intersection of \( B(p, \varepsilon(p)) \) and the point set of \( \Xi_k \) either consists of \( p \) only, or it consists of straight line segments radiating from \( p \);

(iv) the subgraphs of \( \Xi_k \), respectively \( \Gamma \), representing \( G[[e_1, e_2, \ldots, e_k]] \) are strongly equivalent.

The graph \( \Xi_0 \) is defined as the graph with vertex set \( V(\Gamma) \) and with empty edge set. Having defined \( \Xi_{k-1} \) \((k \geq 1)\), we define \( \Xi_k \) as follows: The subgraph \( \Xi'_k \) of \( \Xi_{k-1} \) representing \( G[[e_1, e_2, \ldots, e_{k-1}]] \) is equivalent to the corresponding subgraph of \( \Gamma \). Hence we can add to \( \Xi'_k \) a polygonal arc representing \( e_k \) such that the resulting graph \( \Xi'_k \) is strongly equivalent to the subgraph of \( \Gamma \) representing \( G[[e_1, e_2, \ldots, e_k]] \). If necessary, we can modify the new arc such that it does not intersect any of the arcs representing edges \( e_j', e_j'' \) \((j \leq k-1)\), and we can further modify it so that condition (iii) is satisfied. Then we can add polygonal arcs representing \( e_k', e_k'' \) such that (ii) is satisfied.

The resulting plane multigraph \( \Xi_k \) satisfies conditions (i)-(iv). Now the multigraph \( \Xi' = \cup \Xi_k \) is a representation of \( G' \), and it contains a representation of \( G \) with the desired properties. 

**Lemma 3.2.4** [18] Let \( \Gamma \) be an infinite, locally finite, connected, VAP-free plane graph such that each edge is a polygonal arc and no point of an edge is an EAP of \( \Gamma \). Then \( \Gamma \) is a spanning subgraph of a polygonal arc triangulation.
PROOF  Let G denote the abstract graph represented by \( \Gamma \) and let \( G' \) be the multigraph obtained from G by adding, for every edge \( e \) of G, two new edges \( e', e'' \) with the same ends as \( e \). Since no point of point set of \( \Gamma \) contains EAP of \( \Gamma \), we can extend \( \Gamma \) to a polygonal arc representation \( \Gamma' \) of \( G' \). If \( e \) is an edge of \( G' \), we denote by \( \bar{e} \) the arc of \( \Gamma' \) representing \( e \). Now assume that for each edge \( e \) of \( G \), the interior of \( \bar{e}' \cup \bar{e}'' \) (denoted by \( \Omega(e) \)) contains \( \bar{e} \) (except for the ends) and nothing else of \( \Gamma' \), and to each vertex \( p \) of \( \Gamma' \) we associate a positive real number \( \varepsilon(p) \) such that \( B(p,\varepsilon(p)) \cap B(q,\varepsilon(q)) = \emptyset \) for any two distinct vertices \( p, q \) and such that the intersection of point set of \( \Gamma' \) and \( B(p,\varepsilon(p)) \) consists of straight line segments radiating from \( p \) for each vertex \( p \) of \( \Gamma' \). Denote by \( \Omega \) the union of the sets \( B(p,\varepsilon(p)) \), where \( p \in V(\Gamma) \) and the sets \( \Omega(e) \), where \( e \in E(G) \). The graph \( \Gamma \) has the property that if \( x \) and \( y \) are two non-adjacent vertices with a common neighbour \( z \) and the edges joining \( z \) to \( x \) and \( y \) are consecutive in the clockwise ordering of the edges incident with \( z \), then we can add to \( \Gamma \) a polygonal arc representing the edge \( e = xy \) such that \( \bar{e} \subseteq \Omega \). Any plane graph obtained from \( \Gamma \) by adding finitely many edges in \( \Omega \) satisfies the assumption of the above lemma.

Now we want to show that, for any vertex \( z \) of \( \Gamma \), we can add finitely many edges in \( \Omega \) so as to obtain a plane graph \( \Theta \) with the property that \( z \) is contained in the interior of some cycle of \( \Theta \). Since \( \Gamma \) is connected, \( z \) has degree at least one, and if \( z \) has degree 1, we can add an edge in \( \Omega \) joining \( z \) to a vertex of graphic distance two from \( z \). So we can assume that \( z \) has degree at least two. We can also assume that the ends (distinct from \( z \)) of any two edges which are
incident with \( z \) and consecutive in the clockwise ordering of the edges incident with \( z \) are adjacent. So if \( z \) has degree at least 3 in \( \Gamma \), then \( \Gamma \) contains a cycle containing the vertices adjacent to \( z \) and no other vertex. If \( z \) is in the interior of this cycle, then we have finished, so assume the opposite. Then \( \Gamma \) contains two adjacent vertices, \( x \) and \( y \) say, each of which is adjacent to \( z \) such that each vertex adjacent to \( z \) (other than \( y \) and \( x \)) is in the interior of the 3-cycle \( zxyz \). This is also true if \( z \) has degree 2. Since \( \Gamma \) is infinite, locally finite, and connected, it has a 1-way infinite path starting at any vertex by the Infinity Lemma. So there is in \( \Gamma \) a 1-way infinite path starting at \( z \). This path contains either \( x \) or \( y \) (or both) since there are only finitely many vertices in the interior of the 3-cycle \( xyzx \). Therefore \( \Gamma \) has a 1-way infinite path starting at \( x \) or \( y \) (say \( y \)) and containing none of \( x \) and \( z \). Let \( y_1 \) be the vertex adjacent to \( y \) such that the edge \( yy_1 \) succeeds the edge \( yz \) in the clockwise ordering of the edges adjacent to \( y \). Clearly we have \( y_1 \neq z \) and \( y_1 \neq x \). If \( x \) and \( y_1 \) are non-adjacent in \( \Gamma \), we can add an edge (in \( \Omega \)) joining \( y_1 \) and \( x \) such that \( z \) is in the interior of the cycle \( xyzx \). This assumes that \( y_1 \) and \( x \) are adjacent and that \( z \) is in the exterior of the cycle \( xyzx \). Let \( y_2 \) be the vertex adjacent to \( y_1 \) such that the edge \( y_1y_2 \) succeeds \( y_1y \) in the clockwise ordering of the edges incident with \( y_1 \). So we can conclude that \( y_2 \) is distinct from \( z \), \( y \) and \( x \), and also we may assume that \( x \) is adjacent to \( y_2 \) and that \( z \) is in the exterior of the cycle \( xyxy_2x \). We continue like this defining the vertices \( y_3, y_4, \ldots \). Since \( x \) has finite degree, there is a (smallest) \( k \) such that \( x \) is not adjacent to \( y_k \). Then we can add in \( \Omega \) an edge
joining \( y_k \) and \( x \) such that \( z \) is in the interior of the cycle \( \Theta : xy y_1 y_2 \ldots y_k x \).

Let \( \{p_1, p_2, \ldots \} \) be the vertex set of \( \Gamma \). By adding finitely many edges (in \( \Omega \)) to \( \Gamma \) we obtain a graph containing a cycle \( \Theta \) such that \( p_1 \) is in the interior of \( \Theta \). The subgraph of the resulting graph consisting of \( \Theta \) and the vertices and edges in the interior of \( \Theta \) is a finite, plane graph, so by adding finitely many edges in the interior of \( \Theta \) we obtain a graph \( \Gamma_1 \) containing a cycle \( \Psi_1 \) whose vertices are precisely the neighbours of \( p_1 \) and whose interior contains \( p_1 \) and no other vertex of \( \Gamma_1 \). The plane graph \( \Gamma_1 \) has the same vertex set as \( \Gamma \) and no point of point set of \( \Gamma_1 \) is an EAP of \( \Gamma_1 \). Hence we can repeat the argument with \( \Gamma_1 \) instead of \( \Gamma \) and \( p_2 \) instead of \( p_1 \). We can obtain a graph \( \Gamma_2 \) and a cycle \( \Psi_2 \) of \( \Gamma_2 \) such that \( p_2 \) and no other vertex of \( \Gamma_2 \) is in the interior of \( \Psi_2 \). Since \( \Gamma_2 \) has the same vertex set as \( \Gamma_1 \) and \( \Gamma \), none of the edges added to \( \Gamma_1 \) in order to obtain \( \Gamma_2 \) intersects the interior of \( \Psi_1 \). We continue like this defining \( \Gamma_3, \Gamma_4, \ldots \), and finally the union \( \bigcup_{i=1}^{\infty} \Gamma_i \) is a polygonal arc triangulation containing \( \Gamma \) as a subgraph.

Now we need the following result about finite triangulations without the proof, in order to represent an infinite triangulation by a straight line triangulation. To begin with, we need also some definitions. If \( xy \) is an edge of a graph \( \Gamma \), then we say that the \textit{contraction} \( \Gamma' \) of \( \Gamma \) by the edge \( xy \) is obtained from \( \Gamma \setminus \{x, y\} \) by adding a new vertex \( z \) (\( z \in V(\Gamma) \)) and joining \( z \) to those vertices which
are adjacent to $x$ or $y$ (or both) in $\gamma$. We also say that $\gamma$ is obtained from $\gamma'$ by splitting $z$ into $x$ and $y$.

**Proposition 3.2.5** [18] Let $\gamma$ be a finite polygonal arc triangulation such that the boundary of the unbounded region is the cycle $\Theta_0 : x_0 y_0 z_0 x_0$, and suppose that $\gamma$ has a region whose boundary is a cycle $\Theta_1 : x_1 y_1 z_1 x_1$ disjoint from $\Theta_0$. Let $T_0, T_1$ be disjoint triangles of the plane with vertices $p_i, q_i, r_i$ for $i = 0, 1$, such that $T_1$ is contained in the interior of $T_0$ and such that each straight line segment connecting a vertex of $T_0$ with a vertex of $T_1$ has only its ends in common with $T_0 \cup T_1$. Then there is a straight line triangulation $\Delta$ which is strongly equivalent to $\gamma$ such that $x_1, y_1$ and $z_1$ are represented by $p_1, q_1$ and $r_1$ respectively and $\Theta_0$ is represented by $T_0$, and such that each vertex of $\Delta$ other than $p_1, q_1$ and $r_1$ has Euclidean distance less than $\epsilon$ from one of $p_0, q_0$ and $r_0$ for $\epsilon > 0$.

**Theorem 3.2.6** [18] Let $\gamma$ be an infinite, locally finite, connected, VAP-free plane graph. Then there exists an infinite straight line triangulation $\Delta$ of the plane such that $\gamma$ is isomorphic to a subgraph of $\Delta$.

**Proof** From Lemma 3.2.3 and Lemma 3.2.4, we may assume that $\gamma$ is an infinite polygonal arc triangulation, and so we need to show that there is a straight line triangulation $\Delta$ isomorphic to $\gamma$. By Lemma 3.2.1, there is a sequence of mutually disjoint cycles $\Theta_1, \Theta_2, \Theta_3, \ldots$ in $\gamma$ such that for each $k$, the interior of $\Theta_k$ and $\Theta_{k-1}$ form a finite, plane graph containing $\Theta_{k-1}$ and triangulating the interior of
\(\Theta_k\). For each \(k \geq 1\), let \(\Gamma_k\) denote the subgraph of \(\Gamma\) induced by the vertices of \(\Theta_{3k}\), \(\Theta_{3k-3}\) and the vertices in the interior of \(\Theta_{3k}\) and the exterior of \(\Theta_{3k-3}\).

Let \(v_1, v_2, \ldots, v_m\) be vertices of \(\Theta_{3k-1}\). If the edge \(v_1v_2\) is not contained in a separating 3-cycle of \(\Gamma_k\) whose interior contains the interior of \(\Theta_{3k-2}\), then we delete the interior of every separating 3-cycle (if any) of \(\Gamma_k\) containing the edge \(v_1v_2\), and then we contract \(\Gamma_k\) by the edge \(v_1v_2\). We repeat this procedure with the edge \(v_1v_3\) instead of \(v_1v_2\). Then in a finite number of steps we have a separating 3-cycle \(\Theta'_{3k-1}\) whose interior contains the interior of \(\Theta_{3k-2}\). The resulting graph is \(\Gamma'_k\). All edges of \(\Gamma_k\) which are not incident with a vertex of \(\Theta_{3k-1}\) are also present in \(\Gamma'_k\). Also \(\Theta_{3k}\) and \(\Theta_{3k-3}\) remain unchanged, so the union \(\bigcup_{k=1}^{\infty} \Gamma'_k\) is an infinite triangulation \(\Gamma'\).

Let \(T_k\) denote the triangle with vertices \((-k, -k), (0, k), (k, -k)\). Let \(\Gamma'_k\) denote the induced subgraph of \(\Gamma'\) by the vertices of \(\Theta'_{3k-1}\) and \(\Theta'_{3k-4}\) and all vertices in the interior of \(\Theta'_{3k-1}\) and the exterior of \(\Theta'_{3k-4}\). We represent \(\Gamma'_1\) such that \(\Theta'_2\) is represented by \(T_1\) and all other vertices of \(\Gamma'_1\) are in the interior of \(T_1\). Using Proposition 3.2.5 we can extend this to a straight line representation of \(\Gamma'_1 \cup \Gamma'_2\) such that \(\Theta'_5\) is represented by \(T_2\). Similarly, we represent \(\Gamma'_3, \Gamma'_4, \ldots\). The union \(\bigcup_{k=1}^{\infty} \Gamma'_k\) is denoted by \(\Lambda'\). Then \(\Lambda'\) is a straight line triangulation of the plane, and \(\Lambda'\) is isomorphic to \(\Gamma'\).

Let \(\Phi_{3k}\) denote the cycle of \(\Lambda'\) corresponding to \(\Theta_{3k}\) in \(\Gamma'\) and let \(\Lambda'_{3k}\) denote the induced subgraph of \(\Lambda'\) by the vertices of \(\Phi_{3k}\), \(\Phi_{3k-3}\) and the vertices in the interior of \(\Phi_{3k}\) and the exterior of \(\Phi_{3k-3}\). Then \(\Lambda'_{3k}\)
is isomorphic to $\Gamma'_k$. We split vertices of $\Delta'_k$ and add finite triangulations to $\Delta'_k$ in order to obtain a straight line representation $\Delta_k$ of $\Gamma_k$ in a finite number of steps. Only the cycles $\Phi_{3k-1}$, $\Phi_{3k-2}$ are affected. Let $\Delta$ denote the union $\bigcup_{k=1}^{\infty} \Delta_k$. Then $\Delta$ is straight line triangulation of the plane, and $\Delta$ is isomorphic to $\Gamma$. So the proof is complete.
4.1. Random Graphs

Let $\mathbb{R}^d$ be $d$-dimensional Euclidean space. Let $\mathcal{B}^d$ be the collection of all Borel subsets of $\mathbb{R}^d$. Let $\mathcal{B}^d$ denote the subset of $\mathcal{B}^d$ consisting of all bounded sets (i.e., sets with compact closures). We say that a measure $\mu$ on $(\mathbb{R}^d, \mathcal{B}^d)$ is Radon if $\mu(B) < \infty$ for all sets $B \in \mathcal{B}^d$. Let $M$ denote the set of all such measures on $(\mathbb{R}^d, \mathcal{B}^d)$ and further let $N$ denote the subset of $M$ consisting of all nonnegative integer-valued measures, that is $N = \{\mu \in M \mid \mu(B) \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \text{ and } \mu(B) < \infty \text{ for all } B \in \mathcal{B}^d\}$. So $N$ is the set of all counting measures on $(\mathbb{R}^d, \mathcal{B}^d)$. Then $N$ may be uniquely identified with the set of all finite or infinite configurations of points in $\mathbb{R}^d$ with no limit points but where repetitions are allowed.

Let $A(B,k) = \{\mu \in M \mid \mu(B) < k \text{ for } 0 \leq k < \infty\}$ for $B \in \mathcal{B}^d$. Let $\mathcal{M}$ be the $\sigma$-algebra on $M$ generated by the sets $A(B,k)$ for all subsets $B \in \mathcal{B}^d$. Likewise $\mathcal{N}$ is the $\sigma$-algebra generated by such sets of measures in $N$. Then we have $\mathcal{N} \subset \mathcal{M}$ and so $\mathcal{N}$ is the restriction of $\mathcal{M}$ to $N$ (see Kallenberg [9]). Now from $(N, \mathcal{N})$ we can count the points in regions of $\mathbb{R}^d$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space.
**Definition 4.1.1** A measurable mapping $X$ from $(\Omega, \mathcal{F}, P)$ into $(M, \mathcal{M})$ or $(N, \mathcal{N})$ is called a random measure or a point process, respectively.

Since $(N, \mathcal{N})$ is a restriction of $(M, \mathcal{M})$ a point process may alternatively be considered as an $N$-valued random measure, and conversely any a.s. $N$-valued random measure coincides a.s. with a point process. Note that the set of point processes on $\mathbb{R}^d$ is closed under addition and under multiplication by $\mathbb{Z}^+$-valued random variables (Kallenberg [9]) and is also closed under weak limits.

In particular, any probability measure on $(N, \mathcal{N})$ yields a point process. For $B \in \mathcal{B}^d$, we think of $X(B)$ as counting the number of points in the set $B$. If a function $f : \mathbb{R}^d \to \mathbb{R}^+$ is continuous with compact support then we set $X(f) = \int f \, dX$.

The distribution of a point process $X$ is by definition the induced probability measure $P \circ X^{-1}$ on $(N, \mathcal{N})$ given by

$$P \circ X^{-1}(A) = P(X^{-1}(A)) = P(\omega \in \Omega \mid X(\omega) \in A) = P_X(A), \quad A \in \mathcal{N}.$$  

**Definition 4.1.2** A transformation $T : \Omega \to \Omega$ is measure-preserving on $(\Omega, \mathcal{F}, P)$ if $T$ is measurable and

$$P(T^{-1}(A)) = P(A) \text{ for } A \in \mathcal{F}.$$  

The measure $P$ is said to be an invariant measure for $T$. 
**Definition 4.1.3** A set $A \in \mathcal{F}$ is **invariant** if $P(T^{-1}(A) \cap A) = 0$.

The transformation $T$ (or more properly, the system $(\Omega, \mathcal{F}, \mathbb{P}, T)$) is called **ergodic** if every invariant set has measure 0 or 1.

For any $u \in \mathbb{R}^d$, let $T_u : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $T_u(x) = x + u$ for all $x \in \mathbb{R}^d$. Translations in $\mathbb{R}^d$ act naturally on point processes. So we abuse notation and extend a translation in $\mathbb{R}^d$ to a mapping $T_u : N \to N$, given by $T_u(v)$, for $v \in N$, which denotes the translation of occurrences in $v$ by $u$. That is, for $A \in \mathcal{B}^d$, $v(T_u(A)) = (T_u(v))(A)$.

**Definition 4.1.3** We say that $X$ is a **stationary point process** if for every $u \in \mathbb{R}^d$, $P_X$ is an invariant measure for $T_u$.

For details on (stationary) point processes see Franzosa [6], Kallenberg [9], Lewis [13] and Rolski [15].

Let $N^\circ$ denote the subset of $N$ without multiple occurrences. That is, $N^\circ = \{ v \in N | v(\{x\}) = 0 \text{ or } 1 \text{ for any } x \in \mathbb{R}^d \}$. For $v \in N^\circ$, let $A_v$ be the finite or countable subset of $\mathbb{R}^d$ such that $A_v = \{ x \in \mathbb{R}^d | v(\{x\}) = 1 \}$. Let $E$ be a subset of $E_v^* = \{ e \subseteq A_v | \text{card}(e) = 2 \}$. Each $e \in E$ has the form $e = \{x, y\}$ for some $x \neq y$ in $A_v$ and is thought of as an edge with end points $x$ and $y$. Then we may think of $G = (A_v, E)$ as a simple graph. Let $\Lambda = \{ G = (A_v, E) | v \in N^\circ, E \subseteq E_v^* \}$ be the set of simple graphs whose vertex set is an accumulation point free subset of $\mathbb{R}^d$.

Let $\Lambda(B_1, B_2; k_1, k_2, l) = \{ G \in \Lambda | v(B_1) = k_1, v(B_2) = k_2 \text{ and there are } l \text{ edges between } B_1 \text{ and } B_2 \}$ for $0 \leq k_1, k_2, l < \infty$ and $B_1, B_2 \in \mathcal{B}^d$. 
Let $\mathcal{G}$ be the $\sigma$-algebra on $\Lambda$ generated by the sets $A(B_1, B_2; k_1, k_2, l)$ for all $B_1, B_2 \in \mathbb{R}^d$. Note that this definition is consistent with the measurable space of point processes which is a projection of the space of graphs on $\mathbb{R}^d$.

**Definition 4.1.4** A measurable mapping $X$ from a probability space $(\Omega, \mathcal{F}, P)$ into the set of graphs $(\Lambda, \mathcal{G})$ is called a random graph. The distribution of $X$ is the induced measure $P_X$ on $\mathcal{G}$ by $P_X = P \circ X^{-1}$.

As before $\mathbb{R}^d$ acts by translations on the set of graphs on $\mathbb{R}^d$. Define a map $T_u : \Lambda \rightarrow \Lambda$ by $T_u(G) = (T_u(A_v), T_u(E))$ for $v \in N^o$, and $G = (A_v, E) \in \Lambda$, where $T_u(A_v) = \{T_u(x) \in \mathbb{R}^d \mid \nu([x]) = 1\}$ and $T_u(E) = \{T_u(x), T_u(y)\}$, for some distinct points $x$ and $y$ in $A_v$ and $\{x, y\} \in E$.

**Definition 4.1.5** A random graph $X$ is said to be stationary if for every $u \in \mathbb{R}^d$, $P_X$ is an invariant measure for $T_u$.

That is, the finite dimensional distributions are invariant under all $T_u, u \in \mathbb{R}^d$.

4.2 Construction of an $\mathbb{R}^d$-action $T$

In this section we use the cutting and stacking method to construct a stationary (and ergodic) $\mathbb{R}^d$-action $T$ and we will use it when we construct a stationary plane graph in $\mathbb{R}^2$ in next section.

**Definition 4.2.1** If $H$ is a group, then an action of $H$ on $(\Omega, \mathcal{F}, P)$ is a measurable map $\phi$ from $H \times \Omega$ to $\Omega$ with the properties that
(i) $\phi(e, \omega) = \omega$ for every $\omega \in \Omega$ ($e$ is the identity element of $H$),

and

(ii) $\phi(h, \phi(h', \omega)) = \phi(hh', \omega)$ for every $h, h' \in H$ and for every $\omega \in \Omega$.

Every action of the integers $\mathbb{Z}$ is determined by a bijection $T: \Omega \to \Omega$ by the formula $\mathbb{Z} \times \Omega \to \Omega: (n, \omega) \to T^n(\omega)$.

We will give a construction of an $\mathbb{R}^d$-action $T$, due to Rudolph [17].

Set $\Omega = [0, 1)^{d+1}$ with Lebesgue probability measure $P$. For $\omega \in \Omega$, we write $\omega = (u, x)$, $u \in \mathbb{R}^d$. If $(u, x)$ and $(u+u', x)$ are both in $\Omega$, we set

$$T_{u'}(u, x) = (u+u', x).$$

Then $T_{u}$ is measure-preserving on $|\omega \in \Omega| (u+u', x) \in \Omega$.

Now we subdivide $\Omega$ into $2^d$ sets $A_k = [0, 1)^d \times [k \frac{1}{2^d}, k+1 \frac{1}{2^d})$, for $k = 0, 1, 2, \ldots, 2^{d-1}$. Let $f_1$ be the map of $\Omega \leftrightarrow \Omega_1 = [0, 2)^d \times [0, \frac{1}{2^d})$, linear on each $A_k$ which accomplishes stacking to form a $\frac{1}{2^d} \times 2 \times \ldots \times 2$ rectangle. For example these stacking for $d = 2$ is as follows (see Figure 3):

$$f_1(A_0) = [0,1) \times [1,2) \times [0, \frac{1}{4}),$$

$$f_1(A_1) = [1,2) \times [1,2) \times [0, \frac{1}{4}),$$

$$f_1(A_2) = [0,1) \times [0,1) \times [0, \frac{1}{4}),$$

and

$$f_1(A_3) = [1,2) \times [0,1) \times [0, \frac{1}{4}).$$

If $(u, x)$ and $(u+u', x)$ are both in $\Omega_1$, we set
Let $A \subseteq \Omega_1$ and $A + u' = \{(u + u', x) | (u, x) \in A\}$. If $A + u' \subseteq \Omega_1$ then

$$P(f_1^{-1}(A)) = P(f_1^{-1}(A + u')) = P(T_u(f_1^{-1}(A))).$$

This extends the definition of $T_u$ to more of $\Omega$.  

![Diagram](image-url)

Cutting and stacking for $d=2$ and $n=1$

**Figure 3**

Inductively suppose that the map $f_n : \Omega \leftrightarrow \Omega_n = [0, 2^n]^d \times [0, \frac{1}{2^{dn}}]$ has been defined, linear on each $A_n = [0, 1)^d \times [\frac{k}{2^{dn}}, \frac{k+1}{2^{dn}})$, and if $(u, x)$ and $(u + u', x)$ are both in $\Omega_n$ we set

$$T_u(f_n^{-1}(u, x)) = f_n^{-1}(u + u', x).$$

Then $T_u$ is measure-preserving on $\{\omega \in \Omega | f_n(\omega) \in \Omega_n, f_n(T_u(\omega)) \in \Omega_n\}$.

Now we subdivide $\Omega_n$ into $2^d$ sets of the form

$$\tilde{A}_{n+1}^k = [0, 2^n]^d \times [\frac{k}{2^{dn+1}}, \frac{k+1}{2^{dn+1}}), \quad k = 0, 1, 2, \ldots, 2^d-1.$$
Now note that the sets

\[ A_{n+1}^k = [0, 1)^d \times \left[ \frac{k}{2^{d(n+1)}}, \frac{k+1}{2^{d(n+1)}} \right), \quad k = 0, 1, 2, \ldots, 2^{d(n+1)} - 1, \]

are precisely set of the form \( A_n \cap \tilde{f}_n^{-1}(\tilde{A}_{n+1}^k) \).

Next let \( \tilde{f}_{n+1} \) be a map \( \Omega_n \leftrightarrow \Omega_{n+1} = [0, 2^{n+1})^d \times [0, \frac{1}{2^{d(n+1)}}) \), linear on each \( \tilde{A}_{n+1}^k \), stacking them to fill \( \Omega_{n+1} \). For example the stacking for \( d = 2 \) is as follows:

\[
\tilde{f}_{n+1}(\tilde{A}_{n+1}^0) = [0, 2^n) \times [2^n, 2^{n+1}) \times [0, \frac{1}{4^{n+1}}),
\]

\[
\tilde{f}_{n+1}(\tilde{A}_{n+1}^1) = [2^n, 2^{n+1}) \times [2^n, 2^{n+1}) \times [0, \frac{1}{4^{n+1}}),
\]

\[
\tilde{f}_{n+1}(\tilde{A}_{n+1}^2) = [0, 2^n) \times [0, 2^n) \times [0, \frac{1}{4^{n+1}}),
\]

and

\[
\tilde{f}_{n+1}(\tilde{A}_{n+1}^3) = [2^n, 2^{n+1}) \times [0, 2^n) \times [0, \frac{1}{4^{n+1}}).\]

Let \( f_{n+1} = \tilde{f}_{n+1} \circ f_n \). Then \( f_{n+1} \) is the map of \( \Omega \leftrightarrow \Omega_{n+1} \) which is linear on each \( A_{n+1}^k \) and accomplishes the stacking. For any \((u, x)\) and \((u+u', x)\) are both in \( \Omega_{n+1} \), we set

\[
T_{u}(\tilde{f}_{n+1}(u, x)) = \tilde{f}_{n+1}(u+u', x).
\]
The \( T_u \) is also measure-preserving on \( \{ \omega \in \Omega \mid f_{n+1}(\omega) \in \Omega_{n+1} \text{ and } f_{n+1}(T_u(\omega)) \in \Omega_{n+1} \} \). This extends the definition of \( T_u \). Continue inductively.

Consider the set \( S \) of \((u, x) \in \Omega\) for which \( T_u \) is defined. For some \( M > 0 \), let \( S^M \) denote the subset of \((u, x) \in \Omega\) for which \( T_{u'} \) is defined for every \( u' \in \mathbb{R}^d \) with \(|u'| < M\). Let

\[
S_n^M = \{(u, x) \mid d(f_n(u, x), \partial(\Omega_n)) \geq M\}.
\]

Then \( S_n^M \subseteq S_{n+1}^M \) and \( S^M = \bigcup_{n=1}^{\infty} S_n^M \). Now

\[
P(S_n^M) = \frac{1}{2^d} (2^n - 2M)^d = (1 - \frac{2M}{2^n})^d \to 1 \quad \text{as } n \to \infty.
\]

So we have \( P(S^M) = 1 \) for some \( M > 0 \) and further \( P(S) = 1 \) since \( S = \bigcup M S^M \). Thus the measure-preserving (and ergodic) action \( T = \{T_{u'}\}_{u' \in \mathbb{R}^d} \) is defined a.s. on \( \Omega \).

4.3 Construction of a Stationary Random Plane Graph

Let \( \Omega = [0, 1)^3 \). Let \( \Omega_n^i, i = 0, 1, 2, \) be a subset of \( \Omega \) such that

\[
\Omega_n^0 = \left\{ \frac{1}{5} (\frac{4}{5})^n \right\} \times \left\{ \frac{1}{5} (\frac{4}{5})^n \right\} \times \left\{ \frac{2 \sum_{i=1}^{\infty} 4^{i-1}}{4^n} \right\} = \left\{ \frac{2 \sum_{i=1}^{n} 4^{i-1}}{4^n} \right\},
\]

\[
\Omega_n^1 = \left\{ \frac{1}{5} (\frac{4}{5})^n \right\} \times \left\{ 1 - \frac{1}{5} (\frac{4}{5})^n \right\} \times \left\{ 0, \frac{1}{4^n} \right\},
\]

and

\[
\Omega_n^2 = \left\{ 1 - \frac{1}{5} (\frac{4}{5})^n \right\} \times \left\{ \frac{1}{5} (\frac{4}{5})^n \right\} \times \left\{ \frac{4^{n-1}}{4^n}, 1 \right\}
\]

for \( n \geq 0 \).
Using the measure-preserving $\mathbb{R}^2$-action $T$ constructed in section 4.2, we define a stationary (and ergodic) point process $X$ on $\mathbb{R}^2$ as follows. For a.e. $\omega = (u, x) \in \Omega$, $u \in \mathbb{R}^2$ and for any $u' \in \mathbb{R}^2$

$$X([u']) = 1 \text{ if and only if } T_u . (u, x) \in \Omega^i_n \quad (*)$$

for some $i = 0, 1, 2$ and $n \geq 0$. A point $v$ in $\mathbb{R}^2$ with $X([v]) = 1$ is a point of the point process $X$.

For the construction of the graph $\Gamma$, a point of $X$ will be a vertex of $\Gamma$. We write $L(u') = v_n^i$ if $u'$ satisfies $(*)$. Let $P_n^i$ be the set $\{v \in X \mid L(v) = v_n^i\}$. Then there are infinitely many points in $P_n^i$, and they form a rectangular grid in the plane such that the distance from each point to its nearest neighbors is $2^n$, for all $i = 0, 1, 2$ and $n \geq 0$. For a convenience we abuse notations and write $v = v_n^i$ if $v \in P_n^i$. See Figure 4. The edges of $\Gamma$ are chosen as follows, with a restriction that every edge does not intersect any other edges except two end points.

1. We connect any two points $v_n^0$ and $v_n^1$ if $d(v_n^0, v_n^1) = 2^n - \frac{2}{5} \left(\frac{4}{5}\right)^n$.
2. Any two points $v_n^0$ and $v_n^2$ if $d(v_n^0, v_n^2) = 2^n - \frac{2}{5} \left(\frac{4}{5}\right)^n$, and any two points $v_n^1$ and $v_n^2$ if $d(v_n^1, v_n^2) = \sqrt{2} \left[2^n - \frac{2}{5} \left(\frac{4}{5}\right)^n\right]$, for $n \geq 0$, so that the interior of all 3-cycles $v_0^0 v_0^1 v_0^2$ in clockwise order is empty and for $n \geq 1$, the interior of every $n^{th}$ level 3-cycle $v_n^0 v_n^1 v_n^2 v_n^0$ contains exactly four $(n-1)^{th}$ level 3-cycles and so $4^n v_n^0 v_n^1 v_n^2 v_n^0$'s.

2. We connect any two points $v_n^1$ and $v_n^2$ which are contained in the same interior of $(n+1)^{th}$ level 3-cycle if (i) the distance between
The dotted-lines are added to the point process to show the self-similar structure

Figure 4

the 1st coordinate of \( v_n^1 \) and the 1st coordinate of \( v_n^2 \) is \( 2^n - \frac{2}{5} \left( \frac{4}{5} \right)^n \).

and the distance between the 2nd coordinate of \( v_n^1 \) and the 2nd coordinate of \( v_n^2 \) is \( \frac{2}{5} \left( \frac{4}{5} \right)^n \), or (ii) the distance between the 1st coordinate of \( v_n^1 \) and the 1st coordinate of \( v_n^2 \) is \( \frac{2}{5} \left( \frac{4}{5} \right)^n \), and the
The random graph $\Gamma$ in the interior of 2nd level 3-cycle

Figure 5

distance between the 2nd coordinate of $v_n^1$ and the 2nd coordinate of $v_n^2$ is
\[ 2^{n+1} - \frac{2}{5} \left( \frac{4}{5} \right)^n, \text{ for } n \geq 0. \]

(3) We connect $v_n^1$ and $v_{n+1}^1$ if
\[ d(v_n^1, v_{n+1}^1) = \frac{1}{5} \left( 1 - \left( \frac{4}{5} \right)^{n+1} \right) \sqrt{2}, \]
and connect $v_n^2$ and $v_{n+1}^2$ if
\[ d(v_n^2, v_{n+1}^2) = \frac{1}{5} \left( 1 - \left( \frac{4}{5} \right)^{n+1} \right) \sqrt{2}, \text{ for } n \geq 0. \]
(4) We connect $v_{n+1}^0$ to all vertices of 4 of the $n^{th}$ level 3-cycles which are in the interior of $(n+1)^{th}$ level 3-cycle $v_{n+1}^0 v_{n+1}^1 v_{n+1}^2 v_{n+1}^0$, for $n \geq 0$.

Now we have the resulting graph $\Gamma$. See Figure 5. By definition of stationary random graphs, we know that the graph $\Gamma$ is a stationary random plane graph.
CHAPTER 5
REPRESENTATION OF STATIONARY RANDOM PLANE GRAPHS

5.1 Straight Line Representations of $\Gamma$

Let $\Lambda$ be the set of all simple graphs, $\Lambda_p$ the subset of $\Lambda$ consisting of all planar graphs and $\Lambda_s$ the subset of $\Lambda$ consisting of all straight line representations. Theorem 3.2.6 gives a constructive procedure for making a straight line representation from a simple planar graph. This gives a measurable map $\Phi: \Lambda_p \to \Lambda_s$. Let $\Gamma: \Omega \to \Lambda_p$ be a random planar graph. Then the composition $\Phi \circ \Gamma$ gives a random graph, $P$-a.s. (graph theoretically) isomorphic (defined in chapter 2) to $\Gamma$ such that $\Phi \circ \Gamma$ is a straight line representation of $\Gamma$.

\[
\begin{array}{ccc}
\Omega & \xrightarrow{\Gamma} & \Lambda_p \\
& \searrow^{\Phi \circ \Gamma} & \downarrow \Phi \\
& & \Lambda_s
\end{array}
\]

**DEFINITION 5.1.1** If a map $\Gamma: \Omega \to \Lambda_p$ is a random planar graph, then a map $\Delta: \Omega \to \Lambda_s$ is called a stationary straight line representation of $\Gamma$ if

(i) $\Delta$ is a stationary random graph

(ii) for $P$-a.e. $\omega \in \Omega$, $\Gamma(\omega)$ and $\Delta(\omega)$ are (graph theoretically) isomorphic.
By Theorem 3.2.6, we know that there exists a plane graph with straight edges which is isomorphic to the plane graph $\Gamma(\omega)$ constructed in section 4.3. In this section we assume that a stationary straight line representation of the stationary random plane graph $\Gamma$ exists. We describe some properties that this representation must possess. In section 5.2 we will show that no such representation exists. For simplicity, we will only consider ergodic stationary representation. This is not a real loss of generality and it will be convenient in the sequel.

Let $\Lambda$ be a typical straight line representation of $\Gamma$. For $n = 0, 1, 2, \ldots$, let $\Lambda_n$ denote a subgraph with straight edges (say $n$-triangle) of $\Lambda$ representing a $n$th level 3-cycle $C_n^0 v_n^1 v_n^2 v_n^0$ of $\Gamma$ and we represent $v_n^0$ by $x_n$, $v_n^1$ by $y_n$ and $v_n^2$ by $z_n$. For $n \geq 1$, the vertex $x_n$ of $\Lambda_n$ is joined by straight edges to all vertices of $\Lambda_{n-1}$'s representing $C_{n-1}$'s contained in the interior of the $C_n$ in $\Gamma$.

For $x, y \in \mathbb{R}^2$, let $\overline{xy}$ denote the straight line segment between $x$ and $y$.

**Definition 5.1.2** If $\Gamma$ is a plane graph and $x$ is a point not in $\Gamma^o$ where $\Gamma^o$ denotes the point set of $\Gamma$, then $x$ is called an admissible point with respect to $\Gamma$ if and only if for every vertex $y$ of $\Gamma$ adjacent to the region containing $x$ we have:

$$\overline{xy} \cap \Gamma^o = \{y\}.$$
In a straight line representation $\Delta$ of $\Gamma$, the vertex $x_n$ of $\Delta_n$ is an admissible point with respect to the $\Delta_{n-1}$'s representing the $C_{n-1}$'s contained in the interior of $C_n$ in $\Gamma$ for all $n \geq 1$.

**Lemma 5.1.1** Suppose that $\Delta$ is a stationary straight line representation of $\Gamma$. Let $C_m$ and $C_n$ be, respectively, $m^{th}$ level and $n^{th}$ level cycles in $\Gamma$, and let $\Delta_m$ and $\Delta_n$ denote the subgraphs of $\Delta$ with straight edges representing $C_m$ and $C_n$, respectively. Then if $C_m$ is contained in the interior of $C_n$ then $\Delta_m$ is contained in the interior of $\Delta_n$.

**Proof** We prove the lemma in two steps.

(1) There is no $\Delta_k$ such that the interior of $\Delta_k$ contains $\Delta_n$ for $n \geq k \geq 0$.

Suppose that there is a $\Delta_k$ such that the interior of $\Delta_k$ contains $\Delta_n$ for $n \geq k \geq 0$. Then all vertices adjacent to those of $\Delta_n$ should be contained in the interior of the $\Delta_k$ and furthermore the vertices adjacent to those which are adjacent to vertices of $\Delta_n$ should also be contained in the interior of the $\Delta_k$. If we continue this, then all vertices of $\Gamma$ except those of the chosen $\Delta_k$ are in the interior of the $\Delta_k$. This contradicts the stationarity of $\Delta$.

(2) The interior of $n$-triangle $\Delta_n$ contains exactly the $\Delta_{n-1}$'s of which vertices are joined to $x_n$ by straight edges, where $x_n$, $y_n$, and $z_n$ are vertices of $\Delta_n$ and $x_n$ is an admissible point with respect to the $\Delta_{n-1}$'s, for $n = 1, 2, 3, \ldots$. 
When all $\Delta_{n-1}$'s adjacent to $x_n$ are in the exterior of $\Delta_n$

Case (i) $x_n$ is in the interior of a finite cycle $Q$ with vertices consisting of $y_n$, $z_n$, $y_{n-1}$'s and $z_{n-1}$'s, and there is a 1-way infinite path $x_n x_{n+1} x_{n+2} \ldots$ by the property of $\Gamma$, where $x_{n+i}$ is a vertex of $\Delta_{n+i}$ respectively for $i \geq 0$. By planarity, the only region where the infinite path can be is the interior of $\Delta_n$ since $y_n$ and $z_n$ should be joined to $x_{n+1}$ by straight edges without intersecting any edges of $\Delta$. This is a contradiction to (1).

Case (ii)
Case (ii) The only region where the infinite path $x_nx_{n+1}x_{n+2} \ldots$
should be is also the interior of $\Delta_n$ since $x_{n+1}$ should be
joined to $y_n$ and $z_n$ by straight edges without
intersecting any edges of $\Delta$. So it contradicts (1). ▲▲

Now we know that the interior of any n-triangle $\Delta_n$ contains 4
of the $\Delta_{n-1}$'s, $4^2$ of the $\Delta_{n-2}$'s, ... and $4^n$ of the $\Delta_0$'s for $n = 1, 2, 3, \ldots$.
This will be an important ingredient in the proof in the next section.

5.2 Non-existence of a Stationary Straight Line Representation of $\Gamma$

In this section we will show that the stationary plane graph $\Gamma$
given in section 4.3 does not have a stationary straight line
representation. We will derive a contradiction, by assuming that
there does exist a stationary straight line representation $\Delta$ of $\Gamma$.

We start with the following definition, whose significance will
become clear soon.

**Definition 5.2.1** Let $T$ be a triangle with vertices $a$, $b$ and $c$, arranged in lexicographical ordering. The *shape* $S(T)$ is a mapping from the set of all triangles into $\mathbb{R}^4$, defined by

$$S(T) = (\vec{b} - \vec{a}, \vec{c} - \vec{a}).$$

Note that $S$ is invariant under translations.
Now let $X$ be a stationary process of disjoint triangles in $\mathbb{R}^2$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Define $F : \Omega \to \mathbb{R}^4$ to be the random vector such that

$$F(\omega) = \begin{cases} S(T^\circ(\omega)) & \text{if the origin is contained in any triangle in } X(\omega) \\ 0 & \text{otherwise} \end{cases}$$

where $T^\circ(\omega)$ is the triangle in $X(\omega)$ which contains the origin.

**Lemma 5.2.1** Let $T, u \in \mathbb{R}^2$ be defined by $T_u(v) = v + u$ for $v \in \mathbb{R}^2$. Then given $F(\omega) = (x, y) = S(T)$ for $x, y \in \mathbb{R}^2$ and some triangle $T$, the origin is uniformly distributed over $T$. That is, for any ball $B$ and $u \in \mathbb{R}^2$ such that both $B$ and $T_u(B)$ are in the shape $(x, y)$,

$$P\left( (0,0) \in B \mid F(\omega) = (x, y) \right) = P\left( (0,0) \in T_u(B) \mid F(\omega) = (x, y) \right) \text{ a.s.}$$

**Proof** Let $B(z, \epsilon)$, for $z \in \mathbb{R}^4$, be the set $\{z' \in \mathbb{R}^4 \mid d(z, z') \leq \epsilon\}$, where $d$ is Euclidean distance in $\mathbb{R}^4$. First we show that

$$(1) \quad P\left( (0,0) \in B \mid F(\omega) \in B(\vec{x}, \vec{y}, \epsilon) \right) = P\left( (0,0) \in T_u(B) \mid F(\omega) \in B(\vec{x}, \vec{y}, \epsilon) \right),$$

for $\epsilon$ so small that $B$ and $T_u(B)$ contained in the shape $(\vec{x}, \vec{y})$, are also contained in all shapes in $B((\vec{x}, \vec{y}), \epsilon)$.

**Proof of (1)**

$$P\left( (0,0) \in B \mid F(\omega) \in B(\vec{x}, \vec{y}, \epsilon) \right) = P\left( u \in T_u(B) \mid F(\omega) \in B((\vec{x}, \vec{y}), \epsilon) \right)$$
\[ \begin{align*}
&= P\left( (0,0) \in T_u(B) \mid F(\omega) \in T_u(B((x, \ y), \ \varepsilon)) \right) \text{ since } X \text{ is a stationary process,} \\
&= \frac{P\left( (0,0) \in T_u(B) \right)}{P\left( F(\omega) \in T_u(B((x, \ y), \ \varepsilon)) \right)} \\
&= \frac{P\left( (0,0) \in T_u(B) \right)}{P\left( F(\omega) \in B((x, \ y), \ \varepsilon) \right)} \text{ since } X \text{ is a stationary process,} \\
&= P\left( (0,0) \in T_u(B) \mid F(\omega) \in B((x, \ y), \ \varepsilon) \right).
\end{align*} \]

So, in order to prove the lemma we need to show that

\[ (2) \quad \lim_{\varepsilon \downarrow 0} P\left( (0,0) \in B \mid F(\omega) \in B((x, \ y), \ \varepsilon) \right) = P\left( (0,0) \in B \mid F(\omega) = (x, y) \right) \ F\text{-a.s.} \]

To show (2) we will prove the following fact.

**FACT:** Let \( Y, Z \) be random vectors on \((\Omega, \mathcal{F}, P)\) and \( Y \) be integrable. Then

\[ \lim_{\varepsilon \downarrow 0} \mathbb{E}\left[ Y \mid Z \in B(z, \ \varepsilon) \right] = \mathbb{E}\left[ Y \mid Z = z \right] \quad Z\text{-a.s.} \]

**Proof of fact**

Let \( P_Z(A) = P(Z \in A) \). Define a measure \( \nu \) by

\[ \nu(Z \in A) = \int_{\{Z \in A\}} Y \ dP = \int_A Y \ dP_Z \]

and let \( \nu_Z(A) = \nu(Z \in A) \). Then \( \frac{\nu_Z(A)}{P_Z(A)} \) is the average value of \( Y \) on \( \{Z \in A\} \). This measure \( \nu_Z \) is finite because \( Y \) is integrable, and it is absolutely continuous with respect to \( P_Z \). By the Radon-Nikodym theorem, there exists a density \( f \) for \( \nu \) with respect to \( P_Z \) such that

\[ \nu_Z(A) = \int_A f \ dP_Z \] (see Billingsley [1], section 32-34 for details). The
density (called the Radon-Nikodym derivative of \( \nu_z \) with respect to \( P_z \)) is often denoted by \( \frac{d\nu_z}{dP_z} \). Thus, we may write \( \frac{d\nu_z(z)}{dP_z} = \mathbb{E}[Y \mid Z = z] \), Z-a.s. By Rudin [16] (chapter 8), we get

\[
\frac{d\nu_z(z)}{dP_z} = \lim_{\varepsilon \downarrow 0} \frac{\nu\{Z \in B(z, \varepsilon)\}}{P\{Z \in B(z, \varepsilon)\}} = \lim_{\varepsilon \downarrow 0} \frac{\int_{\{Z \in B(z, \varepsilon)\}} Y \, dP}{P\{Z \in B(z, \varepsilon)\}}
\]

\[
= \lim_{\varepsilon \downarrow 0} \mathbb{E}[Y \mid Z \in B(z, \varepsilon)].
\]

Now we have

\[
\lim_{\varepsilon \downarrow 0} P(0,0) \in B \mid F(\omega) \in B((\hat{x}, \hat{y}), \varepsilon)
\]

\[
= \lim_{\varepsilon \downarrow 0} \mathbb{E}[1_{\{0,0\} \in B} \mid F(\omega) \in B((\hat{x}, \hat{y}), \varepsilon)]
\]

\[
= \mathbb{E}[1_{\{0,0\} \in B} \mid F(\omega) = (\hat{x}, \hat{y})] \text{ F-a.s. by fact}
\]

\[
= P(0,0) \in B \mid F(\omega) = (\hat{x}, \hat{y}) \text{ F-a.s.}
\]

Similarly to (2), we have

\[
\lim_{\varepsilon \downarrow 0} P(0,0) \in T_{u}(B) \mid F(\omega) \in B((\hat{x}, \hat{y}), \varepsilon)
\]

\[
= P(0,0) \in T_{u}(B) \mid F(\omega) = (\hat{x}, \hat{y}) \text{ F-a.s. So the proof of lemma is finished.}
\]

Let \( A_n = ((x, y) \in \mathbb{R}^2 \mid (x, y) \text{ is in the interior of } A_n \text{ for some } A_n) \) for \( n = 0, 1, 2, \ldots \). Then \( \{A_n\} \) is an increasing sequence. Now we have the following lemma.

**Lemma 5.2.2.** Let \( \Delta \) be a stationary (and ergodic) straight line representation of \( \Gamma \). Then

\[
\bigcup_{n=0}^{\infty} A_n = \mathbb{R}^2 \text{ a.s.}
\]
PROOF Let $A = \bigcup_{n=0}^{\infty} A_n$ and let $E$ be an event that $A \neq \mathbb{R}^2$. Then $E$ is translation invariant. By ergodicity, $P(E) = 0$ or 1. Assume $P(E) = 1$. Then $\exists z \in \mathbb{R}^2$ such that $z \notin A$ a.s. Note that $A$ is an open convex subset of $\mathbb{R}^2$. For if $x, y \in A$ then $\exists n$ such that $x, y \in A_n$ and so $\exists k, l$ such that $x$ is in the interior of $A_{nk}$ and $y$ is in the interior of $A_{nl}$. By construction of $A$, $\exists N > n$ such that both $A_{nk}$ and $A_{nl}$ are in the interior of $A_N$. Then for $0 < t < 1$, $tx + (1- t)y$ is also in the interior of $A_N$ since $A_N$ is convex. Since $A$ is open, $A$ is an open convex subset of $\mathbb{R}^2$.

Then there exists a line $L$ in $\mathbb{R}^2$ such that $L$ separates $z$ from $A$ (see Eggleston [4], section 2.7). So there is a half-plane $H_z$ (determined by $L$) containing $z$ such that

$$A \cap H_z = \emptyset. \quad (*)$$

Choose $u$ with $|u| = 1$ (not parallel to $L$) such that $T_u$ acts ergodically (this is possible according to Theorem 1 in Pugh and Shub [14]). Choose $v$ with $|v| = 1$ such that $u \cdot v = 0$ where $\cdot$ is the inner product. Let $R$ be the unit box with respect to the base $(u, v)$ and let $R_{m,n} = T_{(m,n)}(R) = R + (m, n)$ for any $m, n \in \mathbb{Z}$, where the coordinates are with respect to the base $(u, v)$. Define a random variable $X_{m,n}$ by

$$X_{m,n} = \begin{cases} 1 & \text{if } A \cap R_{m,n} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then the whole plane is tiled with 0-1 valued unit boxes.
For at least one $n \in \mathbb{Z}$, the sequence of unit boxes $R_n = \bigcup_{m \in \mathbb{Z}} R_{n,m}$ must have positive probability to intersect $A$. We choose such an $n$.

Now we consider the ergodic process $Y$ in $\{0, 1\}^2$ by

$$Y(m) = 1 \text{ if and only if } X_{m,n} = 1.$$

Then $P( Y(m) = 1 ) > 0$ since $P( R_n \cap A \neq \emptyset ) > 0$. But by (*) there exists an $N$ such that $Y(m) = 0 \forall m > N$ with positive probability.

This is a contradiction.

The height of a triangle $T$ is the minimum distance between a vertex and the line through the opposite side. Denote the minimum height of a triangle $T$ by $h(T)$.

**Lemma 5.2.3** Suppose that $A$ is a stationary straight line representation of $\Gamma$. Then $\forall \varepsilon > 0, \forall M > 0, \exists N > 0$ such that $P(h(\Delta_N) > M \mid (0, 0) \in A_N) > 1 - \varepsilon$, where $\Delta_N$ is the $n^{th}$ level triangle containing the origin.

**Proof** Let $C$ be any circle with diameter $M$, containing the origin. Since $\bigcup_{n=0}^{\infty} A_n = \mathbb{R}^2$ a.s., we can choose three points $a, b, c$ in $\mathbb{R}^2$ such that $a, b, c \in A_n$ and $C$ is covered by a triangle $T_1$ with vertices $a$, $b$, and $c$. By the structure of $\Delta$, there exist $i, j, k$ such that $a, b$ and $c$ are in the interior of $\Delta_{n_i}, \Delta_{n_j}$ and $\Delta_{n_k}$, respectively and furthermore there exists an $N > 0$ such that the interior of $\Delta_N$ contains $\Delta_{n_i}, \Delta_{n_j}, \Delta_{n_k}$ and so $T_1$. Then we claim that

$$(1) \quad h(\Delta_N) \to \infty \text{ a.s.}$$

This is enough to prove the lemma.
Proof of (1) we will show that the diameter of inscribed circle of a triangle $T$ is always less than $h(T)$, since the biggest circle covered by $T$ is the inscribed circle of $T$. Let $C_0$ be a circle with the tangent line $L_1$ at $p$. Let $x$ and $y$ be two points on the line $L_1$ lying on the opposite side of $p$. Let the line $L_1'$ be parallel to $L_1$ and tangent to $C_0$. The distance between the lines $L_1$ and $L_1'$ (i.e., $d(L_1, L_1') = \min\{d(l, l') \mid l \in L_1, l' \in L_1'\}$) is equal to the diameter of $C_0$. Let $x, y, z$ be the vertices of the triangle $T$ inscribing $C_0$. Then $z$ is in the side of $L_1'$ that does not contain $C_0$ since if not there is no triangle inscribing $C_0$ with vertices $x, y, z$. See Figure 7. So the minimum distance from $z$ to $L_1$ is greater than or equal to the diameter of $C_0$. The minimum distance from $x$ to the line passing $y$ and $z$ (or from $y$ to the line passing $x$ and $z$) is also greater than or equal to the diameter of $C_0$. So we have $h(T) \geq$ the diameter of $C_0$. Since $C$ is any circle with diameter $M$ covered by $\Delta_N$, we have $h(\Delta_N) \geq M$. 

![The inscribed circle $C_0$ of $T$](image)

Figure 7
Now we go back to a straight line representation $\Delta$ of $\Gamma$. We know that the interior of each $\Delta_n$ contains four $\Delta_{n-1}$'s and the vertices of these $\Delta_{n-1}$'s are joined to $x_n (\in V(\Delta_n))$ by straight edges. We call the opposite side of $x_n$ the base of $\Delta_n$. We can subdivide the angle at $x_n$ of $\Delta_n$ into four subangles and the base of $\Delta_n$ into four subbases such that we have 4 subtriangles of $\Delta_n$ and each of them contains one $\Delta_{n-1}$ in their interior. So the interior of each subtriangle contains a vertex $x_{n-1}$ of a $\Delta_{n-1}$. Denote the triangle with a vertex $x_{n-1}$ and the same base as a subtriangle of $\Delta_n$ containing the $x_{n-1}$ by $\tilde{\Delta}_{n-1}$. Then $\tilde{\Delta}_{n-1}$ and $\Delta_{n-1}$ has a common vertex $x_{n-1}$ and the interior of $\tilde{\Delta}_{n-1}$ contains $\Delta_{n-1}$. Now we subdivide each triangle $\tilde{\Delta}_{n-1}$ into 4 subtriangles in the same way as above. See Figure 8.
subtriangle of $\Delta_{n-1}$ contains a $\Delta_{n-2}$ and thus a vertex $x_{n-2}$ of $\Delta_{n-2}$. So we can get a triangle $\tilde{\Delta}_{n-2}$ with a vertex $x_{n-2}$ and the same base as a subtriangle of $\Delta_{n-1}$ containing the $x_{n-2}$. We continue this procedure until we have $4^{n-1}$ of the $\tilde{\Delta}_i$'s in the interior of $\Delta_n$. The base of each $\tilde{\Delta}_i$ is one of $4^{n-1}$ subbases of the base of $\Delta_n$. Now we also subdivide $\tilde{\Delta}_1$ into 4 subtriangles in the same way as above such that the interior of each subtriangle of $\tilde{\Delta}_1$ contains a $\Delta_0$. We denote such a subtriangle of $\tilde{\Delta}_1$ by $\tilde{\Delta}_0$. The $\tilde{\Delta}_0$ is a triangle with a vertex $x_1$ and a subbase of $\tilde{\Delta}_1$ as the base of $\tilde{\Delta}_0$, and there are $4^n$ of the $\tilde{\Delta}_0$ in the interior of $\Delta_n$. This construction and notations will be used in the following theorem.

We define the diameter of a triangle $T$ (denoted by $\text{diam}(T)$) to be the maximum distance between vertices of $T$. Now we are ready to prove the main theorem of this section.

**Theorem 5.2.4** There does not exist a stationary (and ergodic) straight line representation of $\Gamma$.

**Proof** Suppose that $\Delta$ is a stationary straight line representation of $\Gamma$. Let $E_n$ be the event that $A_n$ contains the origin for $n \geq 0$. Let $A_n^0$ denote the $n^{th}$ level triangle in $\Delta$ that contains the origin for $n \geq 0$. Note that $\forall \epsilon > 0, \exists L > 0$ s.t. $P(\text{diam}(A_n^0) < L \mid E_0) > 1 - \epsilon$. By Lemma 5.2.2, $1 = P(\bigcup_{n=0}^{\infty} E_n) = \lim_{n \to \infty} P(E_n)$. So $\forall \epsilon > 0, \exists N > 0$ s.t. $P(E_N) > 1 - \epsilon$. Without loss of generality we can assume that $P(E_0) > 0.999$. 


Let $E_0^L$ be the event that $A_0$ contains the origin and $\text{diam}(A_0^0)$ is less than $L$. Then $E_0^L \uparrow E_0$ as $L \to \infty$, so $\lim_{L \to \infty} P(E_0^L) = P(E_0)$. So we may assume that $P(E_0^L) > 0.999$.

Let, for $N > 0$,

$$X_N = \begin{cases} \text{the percentage area} \\ \text{of all } A_0^0 \text{'s in } A_N^0 \text{ with } \text{diam}(A_0^0) < L & \text{if } E_N \text{ occurs} \\ 0 & \text{if } E_N^C \text{ occurs.} \end{cases}$$

Then $0 \leq X_N \leq 1$. Now we have

$$P(E_0^L | E_N) = \mathbb{E}[1_{E_0^L} | E_N]$$

$$= \mathbb{E}[X_N | E_N] \quad \text{by Lemma 5.2.1}.$$ This yields

$$P(E_0^L) = \mathbb{E}[X_N].$$

Now we have, for $0 < \gamma < 1$,

$$0.999 < \mathbb{E}[X_N] = \int_{\{X_N \leq \gamma\}} X_N \, dP + \int_{\{X_N > \gamma\}} X_N \, dP$$

$$\leq \gamma P(X_N \leq \gamma) + P(X_N > \gamma)$$

$$= \gamma + (1 - \gamma) P(X_N > \gamma),$$

so $P(X_N > \gamma) \geq \frac{0.999 - \gamma}{1 - \gamma}$. Let $\gamma = 0.9$. Then

$$P(X_N > 0.9) \geq 0.99,$$ for any $N > 0$.

Now we choose $M > 10L$. By Lemma 5.2.3, we can choose an $N$ such that

$$P(h(A_0^N) > M | E_N) > 0.99.$$
Let $Z_N$ be the ratio of the area of all points in $\tilde{A}_0$'s in $\Delta_N^0$ that are also below the line parallel to the base of $\Delta_N^0$ with distance $(0.1)M$ from this base (say, the $(0.1)M$ line) to the area of $\Delta_N^0$ if $E_N$ occurs, or zero if $E_N$ does not occur. Then

\begin{equation}
Z_N \leq 1 - (0.9)^2 = 0.19.
\end{equation}

Let the height of $\tilde{A}_0$ be the minimum distance from the vertex $x_1$ to the line through the base of $\tilde{A}_0$, where $x_1 \in V(\Delta_1)$ is an admissible point with respect to all $\Delta_0$'s contained in the interior of the $\Lambda$. Let $\tilde{A}_0^*$ be the region of $\tilde{A}_0$ below the $(0.1)M$ line. Then the area of $\tilde{A}_0^*$ is greater than or equal to the area of any other region of $\tilde{A}_0$ between two lines with distance $(0.1)M$, parallel to the base of $\tilde{A}_0$. Since $L < (0.1)M$, a triangle $\Delta_0^0$ with diam$(\Delta_0^0) < L$ is contained in such a region. Thus the area of a triangle $\Delta_0^0$ with diam$(\Delta_0^0) < L$ contained in a $\tilde{A}_0$ is always less than the area of $\tilde{A}_0^*$. By (2), we have $P( Z_N \geq X_N \mid E_N ) > 0.99$. Thus

\begin{align*}
0.99 &< P( Z_N > X_N ) \\
&= P( 0.19 \geq Z_N > X_N ), \text{ by (3)} \\
&\leq P( X_N < 0.19 )
\end{align*}

This is a contradiction to (1).
The random graph \( \Gamma \) constructed in section 4.3 is a connected, locally finite, VAP-free, stationary plane graph. By theorem 3.2.6 there exists a random straight edge graph \( \Delta \) such that for a.e. \( \omega \in \Omega \), \( \Delta(\omega) \) is isomorphic to \( \Gamma(\omega) \).

In this dissertation, we showed that there does not exist a stationary straight line representation of \( \Gamma \). For the proof, we assume that there exists a stationary straight line representation \( \Delta \) of \( \Gamma \). Then we know from Lemma 5.1.1 that the nested structure of \( \Gamma \) is preserved by \( \Delta \). Further we know by stationarity and ergodicity that (i) all \( \Delta_0 \)'s fill the plane with positive density and (ii) with high probability, given that the origin is contained in a triangle \( \Delta_0 \), the diameter of the \( \Delta_0 \) is of moderate size. In section 5.2 we showed that \( \Delta \) can not satisfy (i) and (ii) at the same time.

The results of this thesis set the stage for further research. In particular, it is important to find reasonable conditions which ensure that a stationary straight line representation of a stationary planar (plane) graph will exist.
BIBLIOGRAPHY


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