AN ABSTRACT OF THE THESIS OF

WILLIAM ADAMS MIDDLETON for the DOCTOR OF PHILOSOPHY
(Name) (Degree)

in APPLIED MATHEMATICS presented on 7 April 1967
(Major) (Date)

Title: : ELECTROMAGNETIC RADIATION AND HEAT CONDUCTION FIELDS NEAR COAXIAL CONICAL STRUCTURES

Abstract approved: Redacted for Privacy
F. Oberhettinger

The effect of a time harmonic electric source ring placed axially symmetric between the walls of a double conical structure of finite slant height is investigated. The bases of the conical structures are spherical caps of radii equal to the slant height.

The Green's function for ideally conducting walls is obtained using the normalized eigenfunction expansion theorem. The magnetic and electric components of the induced electromagnetic field are obtained in the form of a single infinite series.

The solution is investigated for several special cases. The special cases include the finite single cone, semi-infinite single cone, semi-infinite double cone, and biconical antenna.

The heat conduction problem for the identical geometry is solved also. The solution is obtained by employing the theory of Laplace transformations on the Green's function for the electric

source ring. Two cases, finite and semi-infinite slant heights, present two forms of solutions. Finite slant height yields a double infinite series and semi-infinite slant height a single infinite series.

Electromagnetic Radiation and Heat Conduction Fields Near Coaxial Conical Structures

by

William Adams Middleton

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

June 1969

APPROVED:

Redacted for Privacy

Professor of Mathematics

Redacted for Privacy

Chairman of Mathematics Department

Redacted for Privacy

Dean of Graduate School

Date thesis is presented 5 April 1967

Typed by Clover Redfern for William Adams Middleton

ACKNOWLEDGMENTS

The author wishes to express his appreciation for the financial support provided by the U. S. Naval Undersea Warfare Center in Pasadena, California. He would never have been able to return to school without their generous Fellowship program.

He would also like to thank Dr. F. Oberhettinger for the suggestion of the problem and accepting the author as a doctoral candidate. His lectures on boundary value problems and special functions were, and will continue to be, of immeasurable value. It was a real pleasure to work under such a man.

Last, but certainly not least, the author wishes to express his appreciation to his family for their patience and support throughout his advanced studies. He thanks especially his wife, Barbara, for performing her accepted role of both father and mother so well to our three children, Michael, Dennis, and Karen. She worked for and earned her Ph. T.

TABLE OF CONTENTS

ELECTROMAGNETIC RADIATION AND HEAT CONDUCTION FIELDS NEAR COAXIAL CONICAL STRUCTURES

I. INTRODUCTION

The investigation of the field induced by a "vibrating" point or line source in the presence of conical structures has been carried on for many years. Most of the investigations were concerned with semi-infinite cones and various approximations or very special cases were taken to discuss a finite cone. The purpose of this thesis is to investigate the finite conical problem with a double cone structure involved. The single finite cone and semi-infinite cones are obtained as special cases.

One of the first investigations of the field induced by a vibrating point source in the presence of a conical structure was performed by H.S. Carslaw (1914). He divided the field into two parts, a primary and a secondary, where one part satisfied the free-space conditions and the second part satisfied the boundary conditions. The primary field obtained involved evaluating contour integrals using residue theory. The integrands contained conical functions. The secondary field was assumed to be of the same form as the primary only containing an unknown function to be determined.

H. Buchholz (1940) examined the semi-infinite ideally conducting cone using the same procedures as Carslaw used. The field was excited in this investigation by an electric source ring placed axially symmetric to the cone's axis.

The problem did not have much appeal until the advent of the space age. At that time the problem of the finite cone became of importance.

When a re-entry vehicle re-enters the atmosphere at hypersonic speeds a double conical structure of finite length is formed. The interior cone is the re-entry vehicle itself. The outer cone is a plasma sheath formed from a surface induced electromagnetic wave.

An electric dipole antennae radiates most of its power along this sheath. Thus transmission of electromagnetic waves through this conical sheath of plasma is highly attenuated. This attenuation phenomenon is exhibited by a marked reduction in the strength of the radio signals from the re-entry vehicle to the earth's surface.

There have been many investigations of the field in the vicinity of semi-infinite cones. For some of the recent investigations one should look at L. B. Felsen (1957, 1959a, 1959b) or S. Adachi (1960). However, it is clear that the electric dipole results would be greatly modified if a semi-infinite conical structure were replaced by a finite conical structure.

J. B. Keller (1960) investigated the backscattering of an electromagnetic wave incident upon a perfectly-conducting finite cone. The theory was based upon geometrical theory of diffraction. He thus assumes the results are " . . . probably valid for wavelengths as large as cone dimensions or smaller."

From 1964 to 1968 several contributions were made that are all directly related to this problem.

D. C. Pridemore-Brown (1964) investigated the radiation pattern of a small-loop antenna situated on the axis of a thin conical sheet of plasma. He used the same basic concept used by H. S. Carslaw (1914) and H. Buchholz (1940). He took as his secondary field that induced by a surface current on the plasma sheath. The eddy current distribution had to be obtained as the solution of an integral equation. By assuming the cone was highly reflecting, perturbation techniques were developed and applied to the special case of a cone with apex angle 10° and the wavelengths of 2π times the distance from the antenna to the cone apex.

This was not too fruitful so he developed a different perturbation method (D. C. Pridemore-Brown, 1966) and applied it to the integral equation for the eddy current distribution. He developed the equations further using the Kantorovich-Lebedev transform theory to obtain an infinite system of equations involving unknown transforms instead of an integral equation. This process was first employed by A. Leitner and C. P. Wells (1956) for a circular disk. Under the assumption the slant height of the cone is small compared to the wavelength, D. C. Pridemore-Brown obtained a solution to this infinite system (1968).

A. Baños _et al._ investigated this problem along the same lines as discussed above (1964). Their solution had essentially the same restrictions.

The purpose of this investigation is to obtain the character of the field of electromagnetic waves in the conical sheath of the re-entering space vehicle in a "closed form." The conical sheath will be represented as the space between two finite coaxial cones with coincident apexes and bases. The base of the outer cone, and thus the inner one as well, is taken to be a spherical cap of radius equal to the slant height of the cones. The base and outer cone have a smooth joining surface. This geometry approximates the actual re-entry flow geometry experienced by the space vehicle at hypersonic speeds. If we assume the conical shells are perfectly conducting, the field may be investigated by placing a ring source of time harmonic character between the walls and located axially symmetric to the conical axis.

As may be seen from the above, prior investigations have been concerned with determining induced eddy currents, secondary fields, etc. The approach taken here is completely different.

Herein a very simple method is utilized. The Green's function is determined by using the well known Green's function representation as a series involving the eigenvalues and eigenfunctions of the problem (A. Sommerfeld, 1964). This approach leads successfully to a solution for the finite conical structure with no restrictions made upon the wavelengths, body slant height, or apex angles.

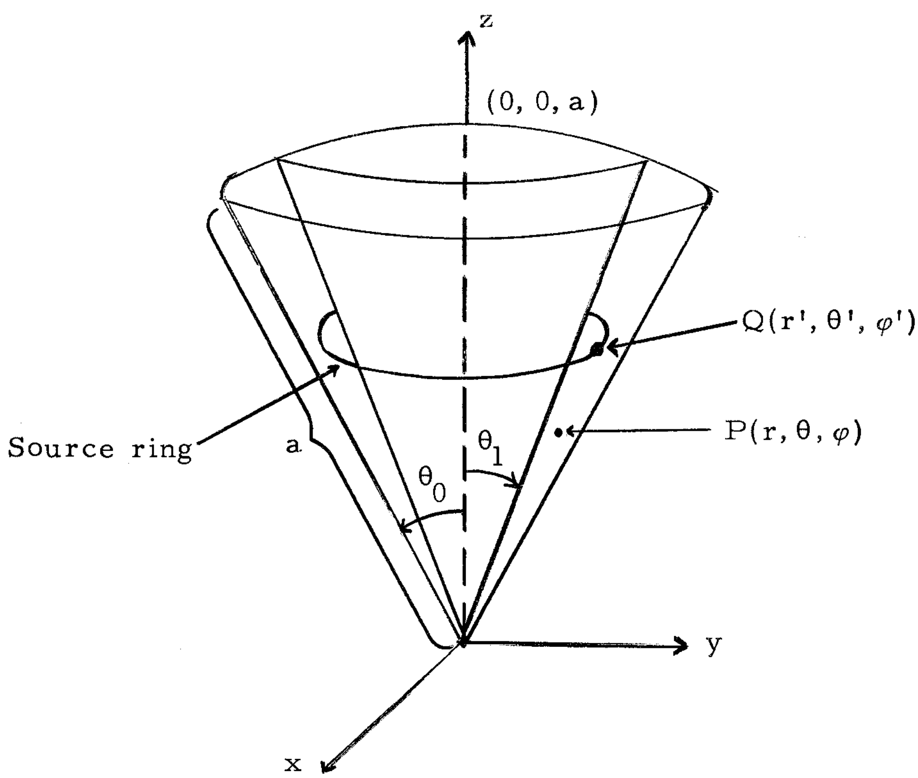
Chapter II is concerned with the derivation of the eigenvalues, eigenfunctions, normalization procedure, and the determination of the vector potential field. The electric and magnetic components of the field are obtained.

Chapter III examines several cases of the solution. Since no prior solution has been obtained, some of the cases must resort to semi-infinite conical forms to compare with previous results. Included are the single finite cone, single semi-infinite cone, semi-infinite double conical wave guide, and a bi-conical antenna.

The solution of the problem for a time harmonic electric source ring also provides the basis for the examination of the heat conduction inside the sheath. Thus it was felt that this result should be presented as Chapter IV and a few degenerate cases mentioned. This solution is obtained by the use of the inverse Laplace Transform.

II. ELECTRIC SOURCE RING FIELD

The geometry of the boundary value problem is determined by approximating the actual plasma sheath about a re-entry vehicle. The Apollo space vehicle is taken in this investigation. Thus the below figure depicts the geometry which must be used.



The base of the two cones is formed by a spherical cap of radius equal to the slant height of both cones. The cap is fitted smoothly to the outer cone.

The field between the cones is excited by a time-harmonic electric source ring of frequency ω. The ring is placed symmetrically about the axis of the cones.

Let $P = P(r, \theta, \varphi)$ be any point of observation between the two cones and $Q = Q(r', \theta', \varphi')$ be a source point on the source ring. The source element at $Q(r', \theta', \varphi')$ produces a vector potential which is tangent to the source ring. This tangent vector potential $\vec{A}$ has the three components A_r, A_θ, A_φ. But when we integrate about the ring with respect to φ, and the ring is placed symmetrically to the double conical structure's axis, the A_φ component becomes independent of φ and $A_r = A_\theta = 0$.

Maxwell's equations can be reduced for a time-harmonic electrical source to a form that says the vector potential field u must satisfy Helmholtz's equation

$$\Delta u + k^2 u = 0$$

where

$$k = \omega / c$$

$$c = \text{speed of light}$$

Since the vector potential has only the A_φ component, we may write

$$\Delta A_\varphi + k^2 A_\varphi = 0$$

The first form will be used for the analysis and the last form when the field components are desired.

The electric source ring induces a spherical field that has both electric and magnetic components. From Maxwell's equations

$$\vec{E} = \nabla(\nabla \cdot \vec{A}) + k^2\vec{A}, \qquad \text{electric components}$$

$$\vec{H} = \nabla \times \vec{A}, \qquad \text{magnetic components}$$

Introducing spherical coordinates and noting $\vec{A} = (0, 0, A_\varphi)$ we obtain

$$E_r = E_\theta = 0; \quad E_\varphi = k^2 A_\varphi$$

(1)

$$H_r = \frac{1}{r}\left[\frac{\partial A_\varphi}{\partial \theta} + A_\varphi \cot\theta\right]$$

$$H_\theta = -\frac{1}{r} A_\varphi - \frac{\partial A_\varphi}{\partial r}$$

$$H_\varphi = 0$$

The geometry, physics, and above equations all suggest the boundary conditions are expressed as:

(2)

$$\theta = \theta_0, \theta_1 : \quad E_r = E_\varphi = 0$$

$$r = a \quad : \quad E_\varphi = E_\theta = 0$$

These suggest that $A_\varphi = 0$ for $\theta = \theta_0, \theta_1$ and $r = a$. Hence we wish to obtain the first Green's function for this field and geometry due to a point source located at Q.

The first Green's function will be obtained by using the well known expression of it as a series expansion involving the eigenvalues and normalized eigenfunctions (see, for instance, A. Sommerfeld 1964, p. 183):

$$G(P,Q) = \sum_{(j)} \frac{u_j(P)u_j^*(Q)}{k^2-k_j^2} \qquad (3)$$

A remark on the notation should be made. In general this summation is usually over three indices. The $u_j(P)$ are the normalized eigenfunctions corresponding to the eigenvalue k_j evaluated at the point of observation P. The $u_j^*(Q)$ is the complex conjugate of $u_j(P)$, only evaluated at the source element Q.

Since we are in a spherical coordinate system, the Helmholtz equation can be examined using separation of variables. Letting $u(r,\theta,\varphi) = f_1(r)f_2(\theta)f_3(\varphi)$ and substituting into Helmholtz's equation yields the three ordinary differential equations

$$\frac{1}{2}\frac{d^2}{dr^2}(rf_1) + (k^2 - \frac{\nu(\nu+1)}{r^2})f_1 = 0$$

$$\frac{d^2 f_2}{d\theta^2} + \cot\theta \frac{df_2}{d\theta} + (\nu(\nu+1) - \frac{\mu^2}{\sin^2\theta})f_2 = 0$$

$$\frac{d^2 f_3}{d\varphi^2} + \mu^2 f_3 = 0$$

where ν and μ are separation parameters to be determined later. We have taken the Q coordinates as $(r', \varphi', 0)$.

The differential equation for $f_1(r)$ is recognizable as a form of Bessel's differential equation. Its solution is written as

$$f_1(r) = \frac{1}{r^{\frac{1}{2}}} C_{\nu+\frac{1}{2}}(kr)$$

where $C_{\nu+\frac{1}{2}}(kr)$ is any cylindrical function of order $\nu+\frac{1}{2}$. Since we request u be finite at $r = 0$, we select the solution to be

$$f_1(r) = r^{-\frac{1}{2}} J_{\nu+\frac{1}{2}}(kr)$$

where $J_{\nu+\frac{1}{2}}(kr)$ is the ordinary Bessel function of order $\nu+\frac{1}{2}$ and argument kr.

The differential equation for $f_2(\theta)$ may be recognized as Legendre's differential equation. It has the solution of any spherical harmonic function, or any linear combination of them. Solutions for θ real which are single-valued and regular are limited to the associated Legendre functions of the first and second kinds of degree ν and order μ, $P_\nu^\mu(\cos\theta)$ and $Q_\nu^\mu(\cos\theta)$ respectively. Thus a linear combination of these two functions would be a solution for $f_2(\theta)$.

However, a useful substitution is made. The $Q_\nu^\mu(\cos\theta)$ can be written in the form (Magnus and Oberhettinger, 1949, p. 63)

$$Q_{\nu}^{\mu}(\cos\theta) = \csc[\pi(\nu+\mu)]P_{\nu}^{\mu}(-\cos\theta) - \cot[\pi(\nu+\mu)]P_{\nu}^{\mu}(\cos\theta)$$

Thus

$$f_2(\theta) = a_1 P_{\nu}^{\mu}(\cos\theta) + a_2 P_{\nu}^{\mu}(-\cos\theta)$$

The equation for $f_3(\varphi)$ is solved immediately to be

$$f_3(\varphi) = e^{\pm i\mu\varphi}$$

We desire the vector potential field to be single-valued in φ. Furthermore, it must be even in φ. Thus we take $\mu = m = 0, 1, 2, \ldots$ and

$$f_3(\varphi) = \cos(m\varphi)$$

Combining the respective equations for $f_1(r)$, $f_2(\theta)$, $f_3(\varphi)$, we obtain

(4) $$u = \frac{\cos(m\varphi)}{r^{\frac{1}{2}}} J_{\nu}(kr)[a_1 P_{\nu-\frac{1}{2}}^{m}(\cos\theta) + a_2 P_{\nu-\frac{1}{2}}^{m}(-\cos\theta)]$$

The substitution process has involved the changing of the $(\nu+\frac{1}{2})$ to ν on the Bessel function. This, of course, modifies the degree of the associated Legendre functions.

We are now ready to determine the eigenvalues and eigenfunctions. This is accomplished by requiring (4) to satisfy the boundary conditions posed by (2).

Consider first the boundary condition of $u = 0$ at $r = 0$. Since only $J_\nu(kr)$ depends upon r, $u = 0$ for all values of ν such that $J_\nu(ka) = 0$. If we denote the nth positive root of $J_\nu(ka) = 0$ by $\tau_{n,\nu}$, we have the eigenvalues

$$k_{n,\nu} = \frac{\tau_{n,\nu}}{a}$$

We need only take the positive roots as $J_{-\nu}(\zeta)$ has the same roots for real ζ.

The ν-values are obtained from the boundary conditions on θ. We obtain the simultaneous equations of

$$a_1 P^m_{\nu-\frac{1}{2}}(\cos\theta_0) + a_2 P^m_{\nu-\frac{1}{2}}(-\cos\theta_0) = 0$$

$$a_1 P^m_{\nu-\frac{1}{2}}(\cos\theta_1) + a_2 P^m_{\nu-\frac{1}{2}}(-\cos\theta_1) = 0$$

These equations have a non-trivial solution for a_1 and a_2 only if the below determinental equation is satisfied.

$$(5) \qquad M^m_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = \begin{vmatrix} P^m_{\nu-\frac{1}{2}}(\cos\theta_0) & P^m_{\nu-\frac{1}{2}}(-\cos\theta_0) \\ P^m_{\nu-\frac{1}{2}}(\cos\theta_1) & P^m_{\nu-\frac{1}{2}}(-\cos\theta_1) \end{vmatrix} = 0$$

Since m, θ_0, and θ_1 are fixed quantities, this is a transcendental equation in ν.

All the roots are real as they represent eigenvalues of Helmholtz's equation. Furthermore since $P^m_{-\nu-\frac{1}{2}}(\zeta) = P^m_{\nu-\frac{1}{2}}(\zeta)$ we need only examine the positive roots.

As the first positive root is $\nu > 1/2$, $r^{-\frac{1}{2}} J_\nu(dr)$ remains finite for (kr) tending to zero. This substantiates the use of $r^{-\frac{1}{2}} J_\nu(kr)$ as the solution to the differential equation in $f_1(r)$.

Let us denote the ℓth positive root of $M^m_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = 0$ by $a_{m,\ell}$.

Then the eigenvalues for the boundary value problem are found to be

$$(6) \qquad k_{n,m,\ell} = \frac{\tau_{n,\nu}}{a} \quad \text{with} \quad \nu = a_{m,\ell}$$

The values of n, m and ℓ are defined by

$$m = 0, 1, 2, 3, \ldots \quad \text{and} \quad n, \ell = 1, 2, 3, \ldots$$

With these eigenvalues, the non-normalized eigenfunctions are written as

$$(7) \qquad u_{n,m,\ell} = r^{-\frac{1}{2}} \cos(m\varphi) J_\nu\left(\frac{r}{a}\tau_{n,\nu}\right) L^m_\nu(\theta)$$

where

$$(8) \qquad L^m_\nu(\theta) = a_1 P^m_{\nu-\frac{1}{2}}(\cos\theta) + a_2 P^m_{\nu-\frac{1}{2}}(-\cos\theta)$$

and ν is determined by (5) and (6).

In order to normalize the eigenfunctions we define a numerical quantity N by the equation

$$N^2 = \int_{\varphi=0}^{2\pi} \int_{r=0}^{a} \int_{\theta=\theta_1}^{\theta=\theta_0} u_{n,m,\ell}^2 r^2 \sin\theta d\theta dr d\varphi$$

Then we define the normalized eigenfunctions by

$$v_{n,m,\ell} = \frac{u_{n,m,\ell}}{N}$$

We consider now the determination of the normalization factor N. The triple integral representation of N^2 for (7) can be written as the product of three integrals, namely:

$$N^2 = \int_0^{2\pi} \cos^2(m\varphi) d\varphi \int_{\theta_1}^{\theta_0} [L_\nu^m(\theta)]^2 \sin\theta d\theta \int_0^a [J_\nu(\frac{r}{a}\tau_{n,\nu})]^2 r dr$$

But

$$\int_0^{2\pi} \cos^2(m\varphi) d\varphi = \frac{2\pi}{\varepsilon_m}$$

where ε_m is Neumann's number defined by

$$\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m = 1, 2, 3, \ldots \end{cases}$$

The last integral-factor is evaluated using from Erdélyi (1953, Vol. 2, p. 90)

$$\int \zeta J_\nu^2(\alpha\zeta)d\zeta = \frac{\zeta^2}{2}\left[J_\nu^2(\alpha\zeta) - J_{\nu+1}(\alpha\zeta)J_{\nu-1}(\alpha\zeta)\right]$$

to obtain

$$\int_0^a rJ_\nu^2(rk_{n,\nu})dr = \frac{a^2}{2}\left[J_\nu^2(\tau_{n,\nu}) - J_{\nu+1}(\tau_{n,\nu})J_{\nu-1}(\tau_{n,\nu})\right]$$

$$= \frac{-a^2}{2} J_{\nu+1}(\tau_{n,\nu})J_{\nu-1}(\tau_{n,\nu})$$

From Erdélyi (1953, Vol. 2, p. 12) this can be reduced by the equation

$$\zeta J_{\nu-1}(\zeta) + \zeta J_{\nu+1}(\zeta) = 2\nu J_\nu(\zeta)$$

to yield the final result of

$$\int_0^a rJ_\nu^2(4k_{n,\nu})dr = \frac{a^2}{2} J_{\nu+1}^2(\tau_{n,\nu})$$

The middle integral-factor may be evaluated also. The function $L_\nu^m(\theta)$ is a solution to Legendre's differential equation. Thus using the formula derived in Appendix A,

$$\int_{\theta_1}^{\theta_0} [L_\nu^m(\theta)]^2 \sin\theta d\theta$$

$$= \frac{-1}{2\nu}\Big[\left(\frac{1}{2}-\nu\right)\{L_\nu^m(\theta_0)L_\nu^{m-1}(\theta_0)\sin\theta_0 - L_\nu^m(\theta_1)L_\nu^{m-1}(\theta_1)\sin\theta_1\}$$

(9)

$$+\left(\frac{1}{2}+\nu\right)\{L_{\nu-1}^m(\theta_0)\frac{d}{d\nu}(L_\nu^m(\theta_0)) - L_\nu^m(\theta_0)\frac{d}{d\nu}(L_{\nu-1}^m(\theta_0))$$

$$- L_{\nu-1}^m(\theta_1)\frac{d}{d\nu}(L_\nu^m(\theta_1)) + L_\nu^m(\theta_1)\frac{d}{d\nu}(L_{\nu-1}^m(\theta_1))\}\Big]$$

$$= D(m,\nu,\theta_0,\theta_1)$$

where $\frac{d}{d\nu}(L_\nu^m(\theta_0))$ is interpreted as $\left[\frac{d}{d\nu}(L_\nu^m(\theta_0))\right]_{\nu=\alpha_{m,\ell}}$.

For simplicity, the notation of $D(m,\nu,\theta_0,\theta_1)$ will be used for most of the below discussion.

Substituting these integral factors,

$$N^2 = \frac{\pi a^2}{\varepsilon_m} J_{\nu+1}^2(\tau_{n,\nu}) D(m,\nu,\theta_0,\theta_1)$$

with $\nu = \alpha_{m,\ell}$. Therefore the normalized eigenfunctions are obtained as

$$v_{n,m,\ell} = \frac{\varepsilon_m^{\frac{1}{2}} \cos(m\varphi) J_\nu\left(\frac{r}{a}\tau_{n,\nu}\right) L_\nu^m(\theta)}{[r\pi D(m,\nu,\theta_0,\theta_1)]^{\frac{1}{2}} a J_{\nu+1}(\tau_{n,\nu})}$$

with $\nu = \alpha_{m,\ell} = \ell$th positive root of the equation

$$M^m_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = P^m_{\nu-\frac{1}{2}}(\cos\theta_0)P^m_{\nu-\frac{1}{2}}(-\cos\theta_1) - P^m_{\nu-\frac{1}{2}}(\cos\theta_1)P^m_{\nu-\frac{1}{2}}(-\cos\theta_0) = 0$$

Substituting into (3) and now making the coordinate of Q to be any triple $(r', \theta', \varphi',)$ we get the first Green's function for a finite double cone due to a point source at Q.

$$G(P,Q) = \frac{1}{(rr')^{\frac{1}{2}}\pi a^2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\varepsilon_m \cos[m(\varphi-\varphi')] J_\nu(rk_{n,\nu}) L^m_\nu(\theta) L^m_\nu(\theta') J_\nu(r'h_{n,\nu})}{J^2_{\nu+1}(\tau_{n,\nu}) D(m,\nu,\theta_0,\theta_1)[k^2 - (\frac{\tau_{n,\nu}}{a})^2]}$$

In order to describe an axisymmetric source ring we integrate the Green's function over all source points on the ring. Integrating and noting that

$$\int_0^{2\pi} \cos[m(\varphi-\varphi')] \cos(\varphi-\varphi') d\varphi' = \pi\delta_{1m}$$

we obtain

$$u = \frac{2}{(rr')^{\frac{1}{2}} a^2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_\nu(r'k_{n,\nu}) J_\nu(rk_{n,\nu}) L^1_\nu(\theta) L^1_\nu(\theta')}{J^2_{\nu+1}(\tau_{n,\nu}) D(1,\nu,\theta_0,\theta_1)[k^2 - (\frac{\tau_{n,\nu}}{a})^2]}$$

The series in n is a Fourier-Bessel series and in this case can be summed. From Erdélyi (Vol. 2, p. 104)

$$\sum_{n=1}^{\infty} \frac{J_\nu(r'k_{n,\nu})J_\nu(rk_{n,\nu})}{J^2_{\nu+1}(\tau_{n,\nu})[k^2-\tau^2_{n,\nu}]}$$

$$= \frac{\pi}{4J_\nu(ka)} \begin{cases} J_\nu(kr)[J_\nu(ka)Y_\nu(kr') - Y_\nu(ka)J_\nu(kr')]; & r < r' \\ J_\nu(kr')[J_\nu(ka)Y_\nu(kr) - Y_\nu(ka)J_\nu(kr)]; & r > r' \end{cases}$$

where $Y_\nu(\zeta)$ is Neumann's function defined by

$$Y_\nu(\zeta) = \csc(\nu\pi)[J_\nu(\zeta)\cos(\nu\pi) - J_{-\nu}(\zeta)]$$

It is more convenient to introduce the second Hankel function $H^{(2)}_\nu(\zeta)$ than to work with the Neumann function. Using the definition of

$$H^{(2)}_\nu(\zeta) = J_\nu(\zeta) - iY_\nu(\zeta)$$

we obtain for arbitrary parameters ζ and η

$$(10) \qquad J_\nu(\zeta)Y_\nu(\eta) - J_\nu(\eta)Y_\nu(\zeta) = i[J_\nu(\zeta)H^{(2)}_\nu(\eta) - J_\nu(\eta)H^{(2)}_\nu(\zeta)]$$

Substituting we obtain for the vector potential field

$$(11) \qquad A_\varphi = \frac{i\pi}{(rr')^{\frac{1}{2}}2} \sum_{\ell=1}^{\infty} \frac{L^1_\nu(\theta)L^1_\nu(\theta')}{D(1,\nu,\theta_0,\theta_1)} \, {}_{1,2}F_\nu(kr', kr)$$

where

$$L_\nu^1(\theta) = a_1 P^1_{\nu-\frac{1}{2}}(\cos\theta) + a_2 P^1_{\nu-\frac{1}{2}}(-\cos\theta)$$

(12)

$${}_1F_\nu(kr', kr) = \frac{J_\nu(kr)}{J_\nu(ka)}\,[J_\nu(ka)H_\nu^{(2)}(kr') - J_\nu(kr')H_\nu^{(2)}(ka)]$$

and ${}_2F_\nu(kr', kr)$ is obtained from ${}_1F_\nu(kr', kr)$ by interchanging r and r'. The constant $D(1, \nu, \theta_0, \theta_1)$ may be evaluated from (9).

The summation over ℓ is imbedded in the determination of the ℓth positive root $a_{1,\ell}$ of the equation

$$P^1_{\nu-\frac{1}{2}}(\cos\theta_0)P^1_{\nu-\frac{1}{2}}(-\cos\theta_1) - P^1_{\nu-\frac{1}{2}}(\cos\theta_1)P^1_{\nu-\frac{1}{2}}(-\cos\theta_0) = 0 \quad (13)$$

Using the above results, the electric and magnetic components of the electromagnetic field may be obtained from (1). Substituting we obtain for the electric components

$$E_\varphi = k^2 A_\varphi$$

$$E_r = E_\theta = 0$$

and for the magnetic components

$$H_r = \frac{\pi i}{2(r^3 r')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta')\,{}_{1,2}F_\nu(kr', kr)}{D(1, \nu, \theta_0, \theta_1)} \left\{(\nu - \frac{1}{2}) \tan(\frac{\theta}{2}) L_\nu^1(\theta)\right.$$

$$\left. + [\nu(\frac{\cos\theta - 2}{\sin\theta}) + \cot(\frac{\theta}{2})]\, a_2 P^1_{\nu-\frac{1}{2}}(-\cos\theta)\right\}$$

$$H_\theta = \frac{-\pi i}{4(r^3 r')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \{ {}_{1,2}F(kr', kr) + 2\, {}_{1,2}H_\nu(kr', kr) \}$$

with

$${}_{1,2}H_\nu(kr', kr) = r \frac{\partial}{\partial r} [{}_{1,2}F_\nu(kr', kr)]$$

and finally

$$H_\varphi = 0$$

The field components are all real. This can be seen from (10) and (12).

In what follows, the vector potential A_φ will be used. If the magnetic and electric field components are desired, the immediately above equations can be used directly.

III. SPECIAL CASES

The vector potential field representation obtained in the previous chapter is new. Thus a direct comparison with previous solutions can not be made directly. However, by taking some special situations, a few cases have been solved in the past that allow a comparison to be made.

The majority of cases that exist for comparison are single semi-infinite conical structures. Thus direct comparisons will be made with those few cases available.

The cases of finite single cone and a semi-infinite double conical structure are presented not for comparison purposes but for completeness. Up to this time these solutions were not known.

Before considering each of the special cases, we must introduce a different form of the solution. This is due to the behavior of the Bessel and Neumann functions for large arguments.

In order to investigate the cases of the semi-infinite conical structures, the radius "a" must be taken to infinity. We must transform our equations to a monotonically behaving form.

This is accomplished by transforming the solution of Helmholtz's equation to the solution of the modified Helmholtz equation

$$\Delta u - \gamma^2 u = 0$$

This is obviously accomplished by the transformation

(14) $$\gamma = ik \quad \text{or} \quad k = -i\gamma$$

Examination of (11) and (12) show that substitution of (14) affects the ${}_{1,2}F_{\nu}(kr', kr)$ term only. From Magnus and Oberhettinger (1949, p. 17 and 19)

(15) $$J_{\nu}(-i\gamma\zeta) = e^{i\pi\nu} J_{\nu}(i\gamma\zeta) = e^{i\nu 3\pi/2} I_{\nu}(\gamma\zeta)$$

$$H_{\nu}^{(2)}(-i\gamma\zeta) = -e^{i\nu\pi} H_{\nu}^{(1)}(i\gamma\zeta) = \frac{2i}{\pi} e^{i\nu\pi/2} K_{\nu}(\gamma\zeta)$$

Substitution into ${}_{1,2}F_{\nu}(k'r, kr)$ yields

(16) $${}_{1}F_{\nu}(-i\gamma r', -i\gamma r) = \frac{2ie^{i2\pi\nu}}{\pi} \frac{I_{\nu}(\gamma r)}{I_{\nu}(\gamma a)} \{I_{\nu}(\gamma a)K_{\nu}(\gamma r') - I_{\nu}(\gamma r')K_{\nu}(\gamma a)\}$$

$$= {}_{1}R_{\nu}(\gamma r', \gamma r), \qquad r < r'$$

We obtain ${}_{2}R_{\nu}(\gamma r', \gamma r)$ from ${}_{1}R_{\nu}(\gamma r', \gamma r)$ by interchanging r and r'.

Thus we have the vector potential field with an imaginary wave number γ

(17) $$\overline{A}_{\varphi} = \frac{i\pi}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_{\nu}^{1}(\theta)L_{\nu}^{1}(\theta')}{D(1,\nu,\theta_0,\theta_1)} \; {}_{1,2}R_{\nu}(\gamma r', \gamma r)$$

Now from G. N. Watson (1958, p. 202, 203) for $|\zeta| \gg 1$,

$$K_\nu(\zeta) \approx \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2z)^m}, \quad \text{for} \quad |\arg z| < 3\pi/2$$

and

$$I_\nu(\zeta) \approx \frac{e^{\zeta}}{(2\zeta\pi)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(-)^m(\nu, m)}{(2\zeta)^m} + \frac{e^{-\zeta+(\nu+\frac{1}{2})\pi i}}{(2\zeta\pi)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2\zeta)^m}$$

for

$$|\arg \zeta| < \pi/2 \quad \text{and} \quad (\nu, m) = \frac{\Gamma(\nu+m+\frac{1}{2})}{\Gamma(\nu-m+\frac{1}{2})}.$$

Therefore, for a given γ,

(18)
$$\lim_{a \to \infty} I_\nu(\gamma a) = \infty$$

$$\lim_{a \to \infty} K_\nu(\gamma a) = 0$$

Now letting "a" tend toward infinity in (17) and utilizing (18) we obtain the result of

$$\left[{}_1R_\nu(\gamma r', \gamma r)\right]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(\gamma r)K_\nu(\gamma r'), \qquad r < r'$$

and

$$\left[{}_2R_\nu(\gamma r', \gamma r)\right]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(\gamma r')K_\nu(\gamma r), \qquad r > r'$$

Utilizing (15) again we obtain

$$(19) \qquad [{}_1F_\nu(kr', kr)]_\infty = J_\nu(kr)H_\nu^{(2)}(kr'), \qquad r < r'$$

$$[{}_2F_\nu(kr', kr)]_\infty = J_\nu(kr')H_\nu^{(2)}(kr), \qquad r > r'$$

We are now ready to investigate the special cases.

Case 1

The first case we will investigate does not have a known existing solution with which we can compare. However by taking a special case of this case a comparison can be made. The two situations are treated separately as they are both interesting and useful results in their own right.

We let $\theta_1 = 0$. The finite double-cone structure then behaves as a finite single-cone structure. From (11) we have

$$A_\varphi = \frac{\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,0)} \; {}_{1,2}F_\nu(kr', kr)$$

As can be seen, the only term apparently modified is the term $D(1,\nu,\theta_0,0)$. However $\theta_1 = 0$ implies only one boundary is present and thus $a_2 = 0$. Hence, from (8),

$$[L_\nu^1(\theta)]_{\theta_1=0} = P^1_{\nu-\frac{1}{2}}(\cos\theta)$$

Now $a_2 = 0$ and $\theta_1 = 0$ provide from (A.3) in Appendix A the result of

(20) $D(1,\nu,\theta_0, 0)$

$$= \frac{-1}{2\nu}\Big[(\tfrac{1}{2}-\nu)P^1_{\nu-\frac{1}{2}}(\cos\theta_0)P_{\nu-\frac{1}{2}}(\cos\theta_0)\sin\theta_0$$

$$+ (\tfrac{1}{2}+\nu)\{P^1_{\nu-\frac{3}{2}}(\cos\theta_0)\frac{d}{d\nu}(P^1_{\nu-\frac{1}{2}}(\cos\theta_0)) - P^1_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}(P^1_{\nu-\frac{3}{2}}(\cos\theta_0))\}\Big]$$

But from (13), as $a_2 = 0$, the values of ν are now determined from

(21) $$P^1_{\nu-\frac{1}{2}}(\cos\theta_0) = 0$$

Substituting (21) into (20) we obtain

$$D(1,\nu,\theta_0, 0) = \frac{-1}{2\nu}(\tfrac{1}{2}+\nu)P^1_{\nu-\frac{3}{2}}(\cos\theta_0)\frac{d}{d\nu}(P^1_{\nu-\frac{1}{2}}(\cos\theta_0))$$

where $\frac{d}{d\nu}(P^1_{\nu-\frac{1}{2}}(\cos\theta_0))$ is understood to mean

$$\Big[\frac{d}{d\nu}(P^1_{\nu-\frac{1}{2}}(\cos\theta_0))\Big]_{\nu=a_{1,\ell}}$$

We shall use this convention in the below analysis. But from Erdélyi (1953, Vol. 1, p. 161) and from (21)

$$-(\frac{1}{2}+\nu)P^{1}_{\nu-\frac{3}{2}}(\cos\theta_0) = \sin\theta_0 P^{2}_{\nu-\frac{1}{2}}(\cos\theta_0) - (\nu-\frac{3}{2})\cos\theta_0 P^{1}_{\nu-\frac{1}{2}}(\cos\theta_0)$$

$$= \sin\theta_0 P^{2}_{\nu-\frac{1}{2}}(\cos\theta_0)$$

where the superscript 2 denotes second order and not a squared quantity. Therefore we obtain for the vector potential field

$$(22)\qquad (A_\varphi)_{\theta_1=0} = \frac{\pi i}{(rr')^{\frac{1}{2}}\sin\theta_0}\sum_{\ell=1}^{\infty} \nu U_\nu(\theta,\theta',\theta_0)\,{}_{1,2}F_\nu(kr',kr)$$

where

(23)

$$U_\nu(\theta,\theta',\theta_0) = \frac{P^{1}_{\nu-\frac{1}{2}}(\cos\theta)P^{1}_{\nu-\frac{1}{2}}(\cos\theta')}{P^{2}_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}P^{1}_{\nu-\frac{1}{2}}(\cos\theta_0)}$$

$${}_{1,2}F_\nu(kr',kr) = \begin{cases} \dfrac{J_\nu(kr)}{J_\nu(ka)}\{J_\nu(ka)H^{(2)}_\nu(kr') - J_\nu(kr')H^{(2)}_\nu(ka)\}; r < r' \quad \text{i.e., } {}_1F_\nu \\[2ex] \dfrac{J_\nu(kr')}{J_\nu(ka)}\{J_\nu(ka)H^{(2)}_\nu(kr) - J_\nu(kr)H^{(2)}_\nu(ka)\}; \; r > r', \quad \text{i.e., } {}_2F_\nu \end{cases}$$

The index ℓ on the summation is determined as the ℓth positive root of

$$P^{1}_{\nu-\frac{1}{2}}(\cos\theta_0) = 0$$

with respect to ν.

Now using a contiguous relation of the associated Legendre functions (Erdélyi, 1953, Vol. 1, p. 161, (11)) the equation for $U_\nu(\theta, \theta', \theta_0)$ may be rewritten as

$$U_\nu(\theta, \theta', \theta_0) = \frac{P^{-1}_{\nu-\frac{1}{2}}(\cos\theta)P^{-1}_{\nu-\frac{1}{2}}(\cos\theta')}{P^{1}_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}(P^{-1}_{\nu-\frac{1}{2}}(\cos\theta_0))}$$

Case 2

This case is a further specialization of the first case. Here $\theta_1 = 0$ and $a = \infty$. We obtain this case immediately upon using (19) in (22). The vector potential field becomes for the semi-infinite single cone

$$(A_\varphi)_\infty = \frac{\pi i}{(rr')^{\frac{1}{2}}\sin\theta_0} \sum_{\ell=1}^{\infty} \nu U_\nu(\theta, \theta', \theta_0) \begin{cases} J_\nu(kr)H_\nu^{(2)}(kr'), & r < r' \\ J_\nu(kr')H_\nu^{(2)}(kr), & r > r' \end{cases}$$

where $U_\nu(\theta, \theta', \theta_0)$ is given by (23).

This is the identical result obtained by H. Buchholtz (1940).

The electric and magnetic components of the vector field are obtained by substitution into (1). The results compare with those found by F. E. Bornis and C. H. Papas (1958, p. 356-358). Bornis and Papas discuss this particular field in detail, especially its physical characteristics.

Case 3

The two cases above may be described as internal problems, i. e., the source ring is located inside its prescribed boundaries.

The case of an external source ring is obtained simply by letting $\theta_1 = 0$ and $a = \infty$, as in Case 2, only now letting θ_0 be greater than $\pi/2$. This case was investigated by L. B. Felsen (1959a) but his solution is in such a complicated form it is not possible to make a direct comparison. He has relied upon the geometrical optics, selected contour paths and use of asymptotics applied to the positioning of the source ring, observation point, and conical structure.

Case 4

We consider now the double cone with $a = \infty$. This corresponds to a semi-infinite double conical wave guide.

The vector potential field is now obtained using (11) and substituting (19) instead of (12). This process yields

$$(24) \quad (A_\varphi)_\infty = \frac{\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \begin{cases} J_\nu(kr) H_\nu^{(2)}(kr'); & r < r' \\ J_\nu(kr') H_\nu^{(2)}(kr); & r > r' \end{cases}$$

If the normalization factor $D(1,\nu,\theta_0,\theta_1)$ were to be ignored, this result would compare favorably with that which would be obtained by R. F. Harrington if his solution was completed (1961, p. 281). Again, the values of ℓ are the ℓth positive roots to the transcendental equation in ν of

(13)

$$M^{1}_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = P^{1}_{\nu-\frac{1}{2}}(\cos\theta_0)P^{1}_{\nu-\frac{1}{2}}(-\cos\theta_1) - P^{1}_{\nu-\frac{1}{2}}(-\cos\theta_0)P^{1}_{\nu-\frac{1}{2}}(\cos\theta_1) = 0$$

Case 5

As a final case, the biconical antenna is examined. This special case is obtained by letting $\theta_0 = \pi - \theta_1$ and $a = \infty$. From the geometry it is apparent we have again an external vector potential field.

The vector potential can be written almost immediately if one accepts a simple substitution of $\theta_0 = \pi - \theta_1$ into $D(1, \nu, \theta_0, \theta_1)$ and the deterministic equation (13) for determining the ν-values. However, care must be taken. Since $\theta_0 = \pi - \theta_1$, we have

$$\cos\theta_0 = -\cos\theta_1 \quad \text{and} \quad \sin\theta_0 = \sin\theta_1$$

Therefore

$$M^{1}_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = [P^{1}_{\nu-\frac{1}{2}}(-\cos\theta_1)]^2 - [P^{1}_{\nu-\frac{1}{2}}(\cos\theta_1)] = 0$$

or, the ν-values are the ℓth positive roots of the transcendental equation

$$(25) \qquad P^1_{\nu-\frac{1}{2}}(-\cos\theta_1) = P^1_{\nu-\frac{1}{2}}(\cos\theta_1)$$

Now substituting (25) into (A. 3) of Appendix A we obtain

$$D(1,\nu,\pi-\theta_1,\theta_1) = \frac{(a_1^2-a_2^2)}{2\nu}\Bigg[2(\frac{1}{2}-\nu)P^1_{\nu-\frac{1}{2}}(\cos\theta_1)P_{\nu-\frac{1}{2}}(\cos\theta_1)\sin\theta_1$$

$$-(\frac{1}{2}+\nu)\left\{P^1_{\nu-\frac{1}{2}}(\cos\theta_1)\right\}^2\frac{d}{d\nu}\left\{\frac{P^1_{\nu-\frac{3}{2}}(\cos\theta_1)-P^1_{\nu-\frac{3}{2}}(-\cos\theta_1)}{P^1_{\nu-\frac{1}{2}}(\cos\theta_1)}\right\}\Bigg]$$

$$= E(\nu,\theta_1)$$

This expression is derived in Appendix B.

Substituting $E(\nu,\theta_1)$ into (24) in place of $D(1,\nu,\theta_0,\theta_1)$, we obtain the vector potential

$$(A_\varphi)_\infty = \frac{\pi i}{2(rr')^{\frac{1}{2}}}\sum_{\ell=1}^{\infty}\frac{L^1_\nu(\theta)L^1_\nu(\theta')}{E(\nu,\theta_1)}\begin{cases}J_\nu(kr')H^{(2)}_\nu(kr); & r' < r\\ J_\nu(kr)H^{(2)}_\nu(kr'); & r' > r\end{cases}$$

where now ν is selected, and thus ℓ, as to (25) instead of (13).

The biconical antenna is discussed in S. Adachi (1960). His results are restricted to very thin and very wide-angle cones.

We have investigated several cases, however many more can be chosen. But these cases chosen seem to provide a spectrum of the more interesting situations and/or situations which provide a comparison of the main results of this investigation.

IV. HEAT CONDUCTION PROBLEM

The previous two chapters have been concerned with the investigation of a vector potential field induced by a time harmonic electric source ring placed in the vicinity of finite and semi-infinite double and single conical structures.

This chapter is concerned with the identical geometries. However in this chapter the source ring, and thus the induced field, have a much different character.

Let the source ring be a heat source of unit strength. Further let the field surrounding the ring and between the boundaries be of a known constant temperature. For simplicity we let it be zero degrees. Now at time $t = 0$ we let the heat source ring emit its energy into its surrounding medium. We desire to investigate the conduction of the thermal energy in the surrounding medium if the boundaries are assumed to be of zero temperature all the time. This corresponds to solving the first boundary value problem for the heat conduction equation

$$\frac{\partial u}{\partial t} = \kappa \Delta u$$

where κ is the thermal conductivity of the medium.

Needless to say, this appears as a second and totally unrelated problem, in fact a problem unto itself. However, this is not the case. The determination of Green's function in cylindrical coordinates for heat conduction was made from the Green's function for the modified Helmholtz equation by H. S. Carslaw and J. C. Jaeger (1940), using the theory of Laplace transformations. F. Oberhettinger has used this same process for general shapes in his class lectures for several years.

The process is theoretically simple. The Green's function for the heat conduction, $G(P, Q, t)$, is obtained from the modified Helmholtz Green's function, $\overline{G}(P, Q, \gamma)$, by the below equation

$$G(P, Q, t) = -\mathcal{L}^{-1}\{\overline{G}(P, Q, \gamma^{\frac{1}{2}}); \gamma\}_{t \to \kappa t} \tag{26}$$

The "$t \to \kappa t$" refers to substituting "t" by "κt" in the resulting equation after the inverse Laplace transform of the modified Green's function is taken with respect to γ.

We should note that $\overline{G}(P, Q, \gamma^{\frac{1}{2}})$ is the same as $\overline{G}(P, Q, \gamma)$ only with γ replaced by $\gamma^{\frac{1}{2}}$. Further, $\overline{G}$ is obtained for the time harmonic condition on the source element.

We consider now the representation of the vector potential A_φ in terms of γ. From (17) we have upon substituting into (26)

$$G(P, Q, t) = \frac{-\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \mathcal{L}^{-1}\{{}_{1,2}R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}}); \gamma\}_{t \to \kappa t}$$

where ${}_{1,2}R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})$ is defined by

(27)

$${}_1R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}}) = \frac{2ie^{2\pi i\nu}}{\pi} \cdot \frac{I_\nu(\gamma^{\frac{1}{2}} r)}{I_\nu(\gamma^{\frac{1}{2}} a)} \{I_\nu(a\gamma^{\frac{1}{2}})K_\nu(r'\gamma^{\frac{1}{2}}) - I_\nu(r'\gamma^{\frac{1}{2}})K_\nu(a\gamma^{\frac{1}{2}})\}$$

for $r < r'$. We interchange r and r' to obtain for $r > r'$ ${}_2R_\nu(\gamma^{\frac{1}{2}} r', \gamma^{\frac{1}{2}} r)$. As is easily seen this approach in solving the heat conduction problem does not seem to be fruitful.

However if we take two simple cases the method can be used successfully. We divide the class of problems into the classes of "a" finite and "a" infinite.

<u>"a" Infinite</u>

In the case of $a = \infty$ the function ${}_{1,2}R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})$ as expressed by (27) reduces to

$$[{}_1R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(r\gamma^{\frac{1}{2}})K_\nu(r'\gamma^{\frac{1}{2}}); \quad r < r'$$

$$[{}_2R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(r'\gamma^{\frac{1}{2}})K_\nu(r\gamma^{\frac{1}{2}}); \quad r > r'$$

The parameter γ enters the vector potential A_φ only in these terms. Thus we need to obtain $\mathcal{L}^{-1}\{I_\nu(p\gamma^{\frac{1}{2}}, q\gamma^{\frac{1}{2}}), \gamma\}$ for p and q arbitrary real positive quantities.

We consider the transformation of the form

$$p = \alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}} \quad \text{and} \quad q = \alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}$$

Thus we now require $p > q$. Solving for α and β we obtain

$$\alpha = \frac{1}{4}(p+q)^2 > 0 \quad \text{and} \quad \beta = \frac{1}{4}(p-q)^2 > 0,$$

as p and q are both real positive quantities. Since α and β are both positive, we obtain from Erdélyi (1954, p. 284, (56))

$$\mathcal{L}^{-1}\{I_\nu(p\gamma^{\frac{1}{2}})K_\nu(q\gamma^{\frac{1}{2}}); \gamma\}_{t\to\kappa t}$$

$$= \mathcal{L}^{-1}\{I_\nu[(\alpha\gamma)^{\frac{1}{2}} - (\beta\gamma)^{\frac{1}{2}}]K_\nu[(\alpha\gamma)^{\frac{1}{2}} + (\beta\gamma)^{\frac{1}{2}}]; \gamma\}_{t\to\kappa t}$$

$$= \frac{I_\nu(\frac{\alpha-\beta}{2\kappa t})}{2\kappa t} e^{-\frac{(\alpha+\beta)}{2\kappa t}} = \frac{1}{2\kappa t} I_\nu(\frac{pq}{4\kappa t}) e^{-\frac{(p^2+q^2)}{4\kappa t}}$$

We note the resulting inverse Laplace transform is symmetric in p and q. Thus the first Green's function in a semi-infinite double-conical structure may be written as

(28)

$$G(P,Q,t) = \frac{-\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \cdot \frac{2ie^{2\pi i\nu}}{2\kappa t} I_\nu(\frac{rr'}{4\kappa t}) e^{-\frac{(r^2+r'^2)}{4\kappa t}}$$

$$= \frac{\pi e^{-\frac{(r^2+r'^2)}{4\kappa t}}}{2\kappa t(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta)}{D(1,\nu,\theta_0,\theta_1)} e^{2\pi i\nu} I_\nu(\frac{rr'}{4\kappa t})$$

where the index ℓ is connected with the ν-values as the ℓth positive root of the transcendental equation in ν of

$$(13) \qquad P^1_{\nu-\frac{1}{2}}(\cos\theta_0)P^1_{\nu-\frac{1}{2}}(-\cos\theta_1) - P^1_{\nu-\frac{1}{2}}(-\cos\theta_0)P^1_{\nu-\frac{1}{2}}(\cos\theta_1) = 0$$

The quantities $L^1_\nu(\theta)$ and $D(1,\nu,\theta_0,\theta_1)$ are determined, as before, from (12) and (9). We see that the solution has the typical exponential decay behavior.

The cases examined in Chapter III for the semi-infinite conical structures may be re-examined again only for the heat conduction instead of the vector potential due to a time harmonic electric source ring. The only difference would lie in use of (28) instead of (11).

The specific case of a single semi-infinite cone (Case 2 of Chapter III) was investigated by R. Muki and E. Sternberg (1960). They approached solving the problem by using the Mellin transform and integration by parts to reduce the partial differential equation to an ordinary differential equation of the Sturm-Louiville type in terms of the transformation. The solution obtained is in the form of evaluating the inverse transformation. This leads to an improper integral whose integrand contains the ratio of two conical functions whose degree involves the variable of integration. The solution must be obtained by using numerical integration techniques of a result obtained from numerical integrations of the conical functions. This is quite cumbersome.

Application of the same processes used in Chapter III provide us with the solution in closed form.

Letting $\theta_1 = 0$, the below equation is valid:

$$\frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} = \frac{P^{-1}_{\nu-\frac{1}{2}}(\cos\theta)P^{-1}_{\nu-\frac{1}{2}}(\cos\theta')}{P_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}[P^{-1}_{\nu-\frac{1}{2}}(\cos\theta_0)]} \cdot \frac{2\nu}{\sin\theta_0}$$

$$= \frac{2\nu}{\sin\theta_0} U_\nu(\theta,\theta',\theta_0)$$

Substituting into (28) we have the representation of the heat conduction in a semi-infinite single cone expressed by

$$G(P,Q,t) = \frac{\pi e^{-\frac{(r^2+r'^2)}{4\kappa t}}}{\kappa t(rr')^{\frac{1}{2}}\sin\theta_0} \sum_{\ell=1}^{\infty} \nu U_\nu(\theta,\theta',\theta_0)e^{2\pi i\nu} I_\nu(\frac{rr'}{4\kappa t})$$

where under these conditions the correlation between ℓ and ν is given by the ℓ being the ℓth positive root of the transcendental equation in ν of

$$P^1_{\nu-\frac{1}{2}}(\cos\theta_0) = 0$$

The other cases examined in Chapter III for $a = \infty$ can be expressed similarly for the heat conduction problem.

"a" Finite

The problem of obtaining an inverse Laplace transform of (27) was mentioned earlier. But upon investigation of where (27) originates shows that we obtain it by transforming the summed series in n to a function of γ instead of k. Returning to this point we have

$$A_\varphi = \frac{2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \cdot \frac{J_\nu(r'k_{n,\nu}) J_\nu(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu})[k^2 - k_{n,\nu}^2]}$$

where

$$k_{n,\nu} = \tau_{n,\nu}/a$$

and ν and ℓ are related by ℓ being the ℓth positive root of (13).

Substitution of $k = -i\gamma$ and using (28) gives the first Green's function for the heat conduction between two coaxial finite conical structures as

$$G(P,Q,t) = \frac{+2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \frac{J_\nu(r'k_{n,\nu}) J_\nu(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu})}$$

$$\times \mathcal{L}^{-1}\left\{\frac{1}{\gamma + k_{n,\nu}^2} ; \gamma\right\}_{t \to \kappa t}$$

From Erdélyi (1954, P. 229, (1))

$$\mathcal{L}^{-1}\left\{\frac{1}{\gamma+k_{n,\nu}^{2}}\,;\ \gamma\right\}_{t\to\kappa t} = e^{-\kappa t k_{n,\nu}^{2}}$$

Thus

$$G(P,Q,t) = \frac{2}{\pi a^{2}(rr')^{\frac{1}{2}}}\sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty}\frac{L_{\nu}^{1}(\theta)L_{\nu}^{1}(\theta')J_{\nu}(r'k_{n,\nu})J_{\nu}(rk_{n,\nu})e^{-\kappa t k_{n,\nu}^{2}}}{D(1,\nu,\theta_{0},\theta_{1})J_{\nu+1}^{2}(\tau_{n,\nu})}$$

where the index ℓ is the ℓth positive root of the transcendental equation in ν expressed by (13).

The special cases for finite "a" examined in Chapter III can be treated easily using the above equation.

BIBLIOGRAPHY

Adachi, S. 1960. A theoretical analysis of semi-infinite conical antennas. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-8:534-547.

Baños, Jr., A. et al. 1965. The radiation field of an electric dipole antenna in a conical sheath. Journal of Mathematics and Physics 44:189-213.

Borgnis, F.E. and C.H. Papas. 1958. Electromagnetic wave guides and resonators. In: Encyclopedia of Physics. Vol. 16. Berlin, Springer Verlag. p. 284-422.

Buchholz, H. 1940. Die Bewegung elektromagnetischen Wellen in einem keglförmigen Horn. Annalen der Physik, ser. 5, 37: 173-225.

Carslaw, H.S. 1914. The scattering of sound waves by a cone. Mathematische Annalen 75:133-147.

Carslaw, H.S. and J.C. Jeager. 1940. The determination of Green's function for the conduction of heat in cylindrical coordinates by the Laplace transformation. London Mathematical Society, Journal 15:273-281.

Erdélyi, A. et al. 1953. Higher transcendental functions. New York, McGraw-Hill. 3 vols.

____________ 1954. Tables of integral transforms. Vol. 1. New York, McGraw-Hill. 391 p.

Felsen, F.B. 1957. Alternative field representations in regions bounded by spheres, cones, and planes. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-5: 109-121.

____________ 1959a. Radiation from ring sources in the presence of a semi-infinite cone. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-7:168-180.

____________ 1959b. Electromagnetic properties of wedge and cone surfaces with linearly varying surface impedances. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-7:S231-S243.

Harrington, R. F. 1961. Time-harmonic electromagnetic fields. New York, McGraw-Hill. 480 p.

Keller, J. B. 1960. Backscattering from a finite cone. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-8:175-182.

Leitner, A. and C. P. Wells. 1956. Radiation by disks and conical structures. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-4:637-640.

Magnus, W. and F. Oberhettinger. 1949. Formulas and theorems for the special functions of mathematical physics. New York, Chelsea. 172 p.

Muki, R. and E. Sternberg. 1960. Steady-state heat conduction in a circular cone. Zeitschrift für Angewandte Mathematik und Physik 11:306-315.

Pridemore-Brown, D. C. 1964. The radiation field of a magnetic dipole antenna in a conical sheath. Journal of Mathematics and Physics 43:199-217.

____________ 1966. Electric dipole radiation through a finite conical sheath. Institute of Electrical and Electronics Engineers, Transactions on Antennas and Propagation AP-14:428-433.

____________ 1968. A Wiener-Hopf solution of a radiation problem in conical geometry. Journal of Mathematics and Physics 47:79-94.

Sommerfeld, A. 1964. Partial differential equations in physics. New York, Academic. 335 p.

Watson, G. N. 1958. A treatise on the theory of Bessel functions. Cambridge, Cambridge University. 2d ed. 804 p.

APPENDICES

APPENDIX A

The normalization factor for determining the normalized eigenfunctions involves the evaluation of a definite integral with the products of associated Legendre functions of the same order and degree. These relationships are not available in the literature and thus are derived in this appendix.

The Legendre differential equation is written as

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx}\left[\nu(\nu+1)-\frac{\mu^2}{1-x^2}\right]u = 0 \tag{A. 1}$$

and has the solutions of $P^{\mu}_{\nu}(x)$, Q^{μ}_{ν}, or any linear combination of these two linearly independent transcendental functions over the real interval $(-1, 1)$. We take μ and ν as completely arbitrary parameters.

Let $w^{\mu}_{\gamma}(x)$ and $v^{\rho}_{\sigma}(x)$ be any solutions to the differential equation (A. 1), with ν and μ replaced by σ, ρ respectively for $v^{\rho}_{\sigma}(x)$. Then

$$(1-x^2)\frac{d^2w^{\mu}_{\nu}}{dx^2} - 2x\frac{dw^{\mu}_{\nu}}{dx} + [\nu(\nu+1)-\mu^2(1-x^2)^{-1}]w^{\mu}_{\nu} = 0$$

$$(1-x^2)\frac{d^2v^{\rho}_{\sigma}}{dx^2} - 2x\frac{dv^{\rho}_{\sigma}}{dx} + [\sigma(\sigma+1)-\rho^2(1-x^2)^{-1}]v^{\rho}_{\sigma} = 0$$

Multiplying the first equation by v_σ^ρ and the second by w_ν^μ and then subtracting the resulting equations gives the single equation

$$v_\sigma^\rho \frac{d}{dx}[(1-x^2)\frac{dw_\nu^\mu}{dx}] - w_\nu^\mu \frac{d}{dx}[(1-x^2)\frac{dv_\sigma^\rho}{dx}]$$

$$+ [\nu(\nu+1)-\sigma(\sigma+1)+(1-x^2)^{-1}(\rho^2-\mu^2)]w_\nu^\mu v_\sigma^\rho = 0$$

This can be rewritten as

$$-\frac{d}{dx}\ (1-x^2)[v_\sigma^\rho \frac{dw_\nu^\mu}{dx} - w_\nu^\mu \frac{dv_\sigma^\rho}{dx}]$$

$$= [(\nu-\sigma)(\nu+\sigma+1)+(\rho^2-\mu^2)(1-x^2)^{-1}]\ w_\nu^\mu v_\sigma^\rho$$

Integrating both sides we obtain, for $-1 < a < b < 1$,

$$\int_a^b [(\nu-\sigma)(\nu+\sigma+1)+(\rho^2-\mu^2)(1-x^2)^{-1}]w_\nu^\mu(x)v_\sigma^\rho(x)dx$$

$$= (1-x^2)[w_\nu^\mu \frac{dv_\sigma^\rho}{dx} - v_\sigma^\rho \frac{dw_\nu^\mu}{dx}]\Bigg|_a^b$$

$$= [x(\nu-\sigma)w_\nu^\mu v_\sigma^\rho + (\sigma+\rho)w_\nu^\mu v_{\sigma-1}^\rho - (\nu+\mu)v_\sigma^\rho w_{\nu-1}^\mu]_a^b$$

using the recurrence formula from Erdélyi (1953, Vol. 1, p. 161).

Now letting $\rho = \mu$ and $v_\sigma^\rho = w_\sigma^\rho$ we obtain

$$\text{(A. 2)} \qquad \int_a^b w_\nu^\rho w_\sigma^\rho \, dx = \left. \frac{x w_\nu^\rho w_\sigma^\rho}{(\nu+\sigma+1)} + \frac{(\sigma+\rho) w_\nu^\rho w_{\sigma-1}^\rho - (\nu+\rho) w_{\nu-1}^\rho w_\sigma^\rho}{(\nu-\sigma)(\nu+\sigma+1)} \right|_a^b$$

If $\nu = \sigma$, the second term has an indeterminant form. Thus, for $\sigma \neq -1/2$, we have from l'Hospital's rule

$$\lim_{\nu \to \sigma} \frac{(\sigma+\rho) w_\nu^\rho w_{\sigma-1}^\rho - (\nu+\rho) w_{\nu-1}^\rho w_\sigma^\rho}{(\nu-\sigma)(\nu+\sigma+1)}$$

$$= \frac{(\sigma+\rho)}{(2\sigma+1)} \left[w_{\sigma-1}^\rho \frac{dw_\sigma^\rho}{d\sigma} - w_\sigma^\rho \frac{dw_{\sigma-1}^\rho}{d\sigma} \right] - \frac{w_\sigma^\rho w_{\sigma-1}^\rho}{2\sigma+1}$$

Combining this limit with (A. 2) for $\nu = \sigma$ we obtain

$$\int_a^b [w_\sigma^\rho(x)]^2 dx = \frac{1}{2\sigma+1} \left\{ x(w_\sigma^\rho(x))^2 - w_\sigma^\rho(x) w_{\sigma-1}^\rho(x) \right.$$

$$\left. \left. + (\sigma+\rho) \left[w_{\sigma-1}^\rho(x) \frac{dw_\sigma^\rho(x)}{d\sigma} - w_\sigma^\rho(x) \frac{dw_{\sigma-1}^\rho(x)}{d\sigma} \right] \right\} \right|_a^b$$

where $w_\sigma^\rho(x)$ is any solution of Legendre's differential equation on the cut $-1 < x < 1$.

Chapter II required the evaluation of an integral of this form. Since (8) is a solution of Legendre's differential equation, we can substitute into the above result to obtain

$$\int_{\theta_1}^{\theta_0} [L_\nu^1(\theta)]^2 \sin\theta\, d\theta$$

$$= \frac{-1}{2\nu}\left\{[L_\nu^1(\theta)]^2\cos\theta - L_\nu^1(\theta)L_{\nu-1}^1(\theta) + (\tfrac{1}{2}+\nu)\left[L_{\nu-1}^1(\theta)\frac{dL_\nu^1(\theta)}{d\nu} - L_\nu^1(\theta)\frac{dL_{\nu-1}^1(\theta)}{d\nu}\right]\right\}\Bigg|_{\theta_1}^{\theta_0}$$

Now

$$\cos\theta[L_\nu^1(\theta)]^2 - L_\nu^1(\theta)L_{\nu-1}^1(\theta) = (\tfrac{1}{2}-\nu)\sin\theta L_\nu^1(\theta)L_\nu^0(\theta)$$

Thus

$$\int_{\theta_1}^{\theta_0} [L_\nu^1(\theta)]^2 \sin\theta\, d\theta$$

$$= \frac{-1}{2\nu}\left\{(\tfrac{1}{2}-\nu)[L_\nu^1(\theta_0)L_\nu^0(\theta_0)\sin\theta_0 - L_\nu^1(\theta_1)L_\nu^0(\theta_1)\sin\theta_1]\right.$$

$$\left. + (\tfrac{1}{2}+\nu)\left[L_{\nu-1}^1(\theta_0)\frac{dL_\nu^1(\theta_0)}{d\nu} - L_\nu^1(\theta_0)\frac{dL_{\nu-1}^1(\theta_0)}{d\nu} - L_{\nu-1}^1(\theta_1)\frac{dL_\nu^1(\theta_1)}{d\nu} + L_\nu^1(\theta_1)\frac{dL_{\nu-1}^1(\theta_1)}{d\nu}\right]\right\}$$

From (8) we obtain the desired integral solution. By making the substitutions of

$$P_{\nu-\frac{1}{2}}^1(\cos\theta) = P_\mu^1(\theta)$$

we obtain the formula expressed as (9) only in its expanded form

(A. 3)

$$-2\nu D(1,\nu,\theta_0,\theta_1)$$

$$= -2\nu\int_{\theta_1}^{\theta_0}[L_\nu^1(\theta)]^2\sin\theta\,d\theta = -2\nu\int_{\theta_1}^{\theta_0}[a_1P_\mu^1(\theta)+a_2P_\mu^1(-\theta)]^2\sin\theta\,d\theta$$

$$= a_1^2\Big\{(-\mu)[P_\mu^1(\theta_0)P_\mu(\theta_0)\sin\theta_0 - P_\mu^1(\theta_1)P_\mu(\theta_1)\sin\theta_1]$$

$$+ (1+\mu)\left[P_{\mu-1}^1(\theta_0)\frac{dP_\mu^1(\theta_0)}{d\nu} - P_\mu^1(\theta_0)\frac{dP_{\mu-1}^1(\theta_0)}{d\nu} - P_{\mu-1}^1(\theta_1)\frac{dP_\mu^1(\theta_1)}{d\nu} + P_\mu^1(\theta_1)\frac{dP_{\mu-1}^1(\theta_1)}{d\nu}\right]\Big\}$$

$$+ a_1a_2\Big\{(-\mu)[P_\mu^1(\theta_0)P_\mu(-\theta_0)+P_\mu^1(-\theta_0)P_\mu(\theta_0)-P_\mu^1(\theta_1)P_\mu(-\theta_1)-P_\mu^1(-\theta_1)P_\mu(\theta_1)]$$

$$+ (1+\mu)\Big[P_{\mu-1}^1(\theta_0)\frac{dP_\mu^1(-\theta_0)}{d\nu} + P_{\mu-1}^1(-\theta_0)\frac{dP_\mu^1(\theta_0)}{d\nu} - P_{\mu-1}^1(\theta_1)\frac{dP_\mu^1(-\theta_1)}{d\nu} - P_\mu^1(-\theta_1)\frac{dP_{\mu-1}^1(\theta_1)}{d\nu}$$

$$- P_\mu^1(\theta_0)\frac{dP_{\mu-1}^1(-\theta_0)}{d\nu} - P_\mu^1(-\theta_0)\frac{dP_{\mu-1}^1(\theta_0)}{d\nu} + P_\mu^1(\theta_1)\frac{dP_{\mu-1}^1(-\theta_1)}{d\nu} + P_\mu^1(-\theta_1)\frac{dP_{\mu-1}^1(\theta_1)}{d\nu}\Big]\Big\}$$

$$+ a_2^2\Big\{(-\mu)[P_\mu^1(-\theta_0)P_\mu(-\theta_0)\sin\theta_0 - P_\mu^1(-\theta_1)P_\mu(-\theta_1)\sin\theta_1]$$

$$+ (1+\mu)\left[P_{\mu-1}^1(-\theta_0)\frac{dP_\mu^1(-\theta_0)}{d\nu} - P_\mu^1(-\theta_0)\frac{dP_{\mu-1}^1(-\theta_0)}{d\nu} - P_{\mu-1}^1(-\theta_1)\frac{dP_\mu^1(-\theta_1)}{d\nu} + P_\mu^1(-\theta_1)\frac{dP_{\mu-1}^1(-\theta_1)}{d\nu}\right]\Big\}$$

This expanded form is quite useful for examining degenerate cases.

APPENDIX B

The biconical antenna case for the comparison of the main results of this investigation with known results requires the evaluation of $D(1, \nu, \pi-\theta_1, \theta_1)$, which was redefined as $E(\nu, \theta_1)$.

Substituting $\theta_0 = \pi - \theta_1$ provides the equations of

(B. 1)
$$\cos \theta_0 = - \cos \theta_1$$
$$\sin \theta_0 = \sin \theta_1$$

Thus from these equations, (13) is rewritten as

$$[P^1_{\nu-\frac{1}{2}}(-\cos \theta_1)]^2 - [P^1_{\nu-\frac{1}{2}}(\cos \theta_1)]^2 = 0$$

Thus the roots are obtained from

(B. 2)
$$P^1_{\nu-\frac{1}{2}}(-\cos \theta_1) = P^1_{\nu-\frac{1}{2}}(\cos \theta_1)$$

Using the notation introduced in the previous appendix, i. e.,

$$P^1_{\nu-\frac{1}{2}}(\cos \theta) = P^1_{\mu}(\theta),$$

substitution of (B. 1) into (A. 3) provides the equation

$$-2\nu E(\nu,\theta_1) = a_1^2\Big\{(-\mu)[P_\mu^1(-\theta_1)P_\mu(-\theta_1)\sin\theta_1 - P_\mu^1(\theta_1)P_\mu(\theta_1)\sin\theta_1]$$

$$+ (1+\mu)\Big[P^1_{\mu-1}(-\theta_1)\frac{dP^1_\mu(-\theta_1)}{d\nu} - P^1_\mu(-\theta_1)\frac{dP^1_{\mu-1}(-\theta_1)}{d\nu}$$

$$- P^1_{\mu-1}(\theta_1)\frac{dP^1_\mu(\theta_1)}{d\nu} + P^1_\mu(\theta_1)\frac{dP^1_{\mu-1}(\theta_1)}{d\nu}\Big]\Big\}$$

$$+ a_2^2\Big\{(-\mu)[P^1_\mu(\theta_1)P_\mu(\theta_1)\sin\theta_1 - P^1_\mu(-\theta_1)P_\mu(-\theta_1)\sin\theta_1]$$

$$+ (1+\mu)\Big[P^1_{\mu-1}(\theta_1)\frac{d}{d\nu}P^1_\mu(\theta_1) - P^1_\mu(\theta_1)\frac{dP^1_{\mu-1}(\theta_1)}{d\nu}$$

$$- P^1_{\mu-1}(-\theta_1)\frac{d}{d\nu}P^1_\mu(-\theta_1) - P^1_\mu(-\theta_1)\frac{dP^1_{\mu-1}(-\theta_1)}{d\nu}\Big]\Big\}$$

Substituting (B. 2) and combining terms one obtains

$$E(\nu,\theta_1) = \frac{(a_2^2 - a_1^2)}{2\nu}\Big\{(-\mu)P^1_\mu(\theta_1)\sin\theta_1(P_\mu(-\theta_1) - P_\mu(\theta_1))$$

$$+ (1+\mu)(P^1_\mu(\theta_1))^2\frac{d}{d\nu}\left[\frac{P^1_{\mu-1}(\theta_1) - P^1_{\mu-1}(-\theta_1)}{P^1_\mu(\theta_1)}\right]\Big\}$$

Now from (B. 2) and Erdélyi (1953, Vol. 1, p. 148)

$$P^1_\mu(\cos\theta_1) = -\sin\theta_1\frac{dP_\mu(\cos\theta_1)}{d(\cos\theta_1)} = \sin\theta_1\frac{dP_\mu(-\cos\theta_1)}{d(\cos\theta_1)} = P^1_\mu(-\cos\theta_1)$$

Thus

$$\frac{d}{d(\cos\theta_1)}\left[P_\mu(\cos\theta_1) + P_\mu(-\cos\theta_1)\right] = 0$$

This implies either

$$P_\mu(\cos\theta_1) + P_\mu(-\cos\theta_1) = \text{constant}$$

for all μ for any given θ_1 or

$$P_\mu(\cos\theta_1) + P_\mu(-\cos\theta_1) = 0$$

Since the first case is absurd, thus we obtain

$$P_\mu(\cos\theta_1) = -P_\mu(-\cos\theta_1)$$

Substituting we obtain the desired result of

$$E(\nu,\theta_1) = \frac{(a_1^2 - a_2^2)}{2\nu}\left\{2(-\mu)P_\mu^1(\theta_1)P_\mu(\theta_1)\sin\theta_1 - (1+\mu)(P_\mu^1(\theta_1))^2 \times \frac{d}{d\nu}\left[\frac{P_{\mu-1}^1(\theta_1) - P_{\mu-1}^1(-\theta_1)}{P_\mu^1(\theta_1)}\right]\right\}.$$