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We investigate the optimal harvesting strategies for McKendrick type population models. Models of this type endow the population with a continuous age structure. They consist of a partial differential equation with a boundary condition which involves an integral of the solution.

We study two problems, the first concerns the yield on an infinite time interval of a linear model. It is shown that an optimal policy exists, and that it consists of harvesting at no more than three specific ages. The optimal harvesting ages are dependent on the birth and death rates, and the economic parameters of the problem.

The goal of the second problem is to find the harvesting policy that maximizes the sustainable yield of a nonlinear model. It is shown that a maximal sustainable yield exists, and is accomplished by

harvesting at only two ages.

Optimal Harvesting of Continuous Age Structured Populations

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you don't know what you've got 'til it's gone.
They paved paradise,
and they put up a parking lot. "

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OPTIMAL HARVESTING OF CONTINUOUS AGE STRUCTURED POPULATIONS

I. INTRODUCTION

This thesis investigates optimal harvesting policies for age structured populations. We deal with two types of problems. The sustainable yield problem involves finding the highest yield which a population can support indefinitely. The time dependent problem involves maximizing the yield delivered over a fixed time interval.

Chapter two contains brief derivations of the age dependent population models popular in the literature. These models have served as the basis of a number of optimal harvesting studies. We describe the results of these studies in chapter two.

A time dependent optimal harvesting problem is presented in chapter three. The model, since it is both age and time dependent, is based upon a function of two independent variables. It is, therefore, outside the realm of classical control theory. We impose restrictions on the model which allow us, in section 3.2, to transform the balance law of the model, which is a partial differential equation, into an ordinary differential equation. The model is further simplified in section (3.3), so that the techniques of control theory are more readily applicable to the optimization problem.

Section (3.4) includes a discussion of the admissable class of harvest rates. This class includes generalized functions. A general theory of optimization is developed in section (3.5). The classical theory of optimal control is adapted and extended to provide information about the existence of and the form of the optimal control in the harvesting problem.

Finally, in section (3.6), the results of section (3.5) are applied to the optimal harvesting problem of section (3.1). The conclusion is that an optimal yield exists, and it may be obtained by harvesting at no more than three distinct ages. That is, the optimal age distribution is discontinuous, and the optimal harvesting policy is a combination of dirac delta functions.

Chapter four investigates the maximum sustainable yield for a nonlinear model. The model is introduced in section (4.1), and is reformulated into a control problem so that the previous results of section (3.5) may be applied.

In section (4.2), the results of section (3.5) are applied to show that for a fixed population size, there exists an optimal harvesting policy.

Finally, in section (4.3), with the use of previously known results, it is shown that the optimal equilibrium harvesting policy dictates that harvesting occur at no more than two ages. This section also includes a discussion on how to numerically determine these two ages.

II. A BRIEF REVIEW OF OPTIMAL HARVESTING OF AGE STRUCTURED POPULATION MODELS

The many models of populations with age structure can be divided into two categories: those in which age is a continuous variable and those in which the population is subdivided into distinct age classes. We begin our review by deriving models with continuous age structure. The relationship between the two types of models will be discussed at the end of the section.

The first model with a continuous age structure was described by F. R. Sharpe and A. J. Lotka in 1911 [19]. The Sharpe-Lotka model is based on data provided by a life probability function, π, and a fertility function, b, defined as follows.

$\pi(a)$ is the probability an individual lives to age a,

$b(a)$ is the rate at which an individual of age a bears offspring.

Also, $u_0(a)$, the initial distribution of individuals, is known.

The birthrate, $B(t)$, is the number of individuals born at time t, per unit time. (For simplicity, in the derivation of the model we refer to $B(t)$ as the number of individuals born at time t.) To derive the model, we divide the population into two groups, those born before time $t = 0$, and those born after, and consider the contribution to the birthrate, at time t, made by each group. At time t, individuals of age a, $a < t$, are the survivors of those $B(t-a)$ newborns born a units earlier in time. The number of those $B(t-a)$ newborns who survive to age a is $B(t-a)\pi(a)$. Since these individuals are bearing offspring at a rate $b(a)$, the contribution to the birthrate by this group is the integral of the product $b(a)\pi(a)B(t-a)$

$$\int_0^t b(a)B(t-a)\pi(a)\,da\,.$$

The individuals of age a, $a > t$, at time t, are the survivors of the initial $u_0(a-t)$ individuals present t units earlier in time. The number of those $u_0(a-t)$ individuals who survive to age a depends upon the conditional probability of surviving from age $(a-t)$ to age a, and is $u_0(a-t)\frac{\pi(a)}{\pi(a-t)}$. The contribution to the birthrate from this group is the integral

$$\int_t^L b(a)\,u_0(a-t)\frac{\pi(a)}{\pi(a-t)}\,da\,,$$

where L is the maximum lifespan of an individual in the population. This integral is a known function; we will refer to it as $B_0(t)$. We define $B_0(t) = 0, \quad t > L$.

Thus we arrive at the Sharpe-Lotka Equation, or renewal equation:

$$B(t) = \int_0^t b(a)B(t-a)\pi(a)da + B_0(t) \; . \qquad (2.1)$$

The Sharpe-Lotka equation is a volterra integral equation. In fact, it is a convolution, and Laplace transforms have been used to prove the existence of a solution under the hypotheses that $b\pi$ and u_0 are nonnegative, and that the Laplace transforms of $b\pi$ and the integrand of B_0 exist for $s > 0$ (Feller) [6]. Furthermore, solutions of the Sharpe-Lotka equation satisfy

$$\lim_{t\to\infty} \frac{B(t)}{e^{ct}} = \text{constant, where} \int_0^L b(a)\pi(a)e^{-ca}da = 1.$$

Thus the asymptotic behavior of the birthrate and hence the population is exponential.

McKendrick [14] derived a second model with a continuous age structure in 1926. The McKendrick model differs from the Sharpe-Lotka model in that the population is described in terms of an age density. The McKendrick model incorporates a fertility rate, b, identical to that which appears in Sharpe-Lotka, and a death rate, d, defined as follows.

$d(a)$ is the per capita death rate of individuals of age a.

The <u>age distribution</u>, u, which describes the population is defined as

$u(a,t)$ is the number of individuals of age $a \geq 0$, at time $t \geq 0$, per unit age.

The McKendrick model consists of a balance law to account for deaths in the population and a renewal equation to account for the births in the population. (For simplicity, in the derivation of the model we refer to $u(a,t)$ as the number of individuals of age a at time t). We begin with the balance law.

The number of individuals of age a at time t who die in the time interval $[t,t+h]$ is $u(a,t) - u(a+h,t+h)$. Alternately, the rate at which individuals in this class die is the product of the per capita death rate with the number of individuals. By integrating this rate over the time interval $[t, t+h]$

we have an alternate expression for the number of deaths. We equate these two expressions to arrive at

$$u(a,t) - u(a+h,t+h) = \int_0^h u(a+\lambda,t+\lambda)d(a+\lambda)d\lambda. \quad (2.2)$$

Dividing each side of equation (2.2) by h, and letting h tend to zero in the limit results in the balance law,

$$\frac{\partial u(a,t)}{\partial a} + \frac{\partial u(a,t)}{\partial t} + d(a)u(a,t) = 0. \quad (2.3)$$

The birthrate at time t is represented by $u(0,t)$. Similar to the derivation of the Sharpe-Lotka equation, the birth law is obtained by integrating the product of the fertility function and the number of individuals available to produce young:

$$u(0,t) = \int_0^L b(a)u(a,t)da, \quad (2.4)$$

where L is the maximum lifespan of an individual in the population.

The McKendrick Model is actually equivalent to the Sharpe-Lotka model. To see this we solve the partial differential equation (2.3) by integrating along characteristics.

Let $u(a,t)$ be a solution. Choose (a_0,t_0) arbitrarily and define $U(h) = u(a_0+h, t_0+h)$. Then

$$\frac{dU}{dh} = \frac{d}{dh}(u(a_0+h, t_0+h)) = u_a + u_t = -d(a_0+h)U(h),$$

so that $\frac{U'}{U} = -d(a_0+h)$, or $U(h) = U(0)\exp\left[-\int_0^h d(a_0+\lambda)d\lambda\right]$.

We wish to interpret $\exp\left[-\int_0^h d(a_0+\lambda)d\lambda\right]$.

Now, for small h, $hd(a)$ approximates the proportion of individuals of age a, at time t, that die in the time interval $(t, t+h)$. Also, if $\pi(a)$ is the probabilty that an individual lives to age a, then the probability an individual dies before age $a+h$ given it has survived to age a is $1 - \frac{\pi(a+h)}{\pi(a)}$. Thus, $\frac{\pi(a) - \pi(a+h)}{\pi(a)}$ is also the proportion of individuals of age a, at time t, who die in the time interval $(t, t+h)$. In the limit as h tends to zero, the two quantites must be equal. It follows that

$$d(a) = \lim_{h\to 0} \frac{\pi(a) - \pi(a+h)}{h\pi(a)} = -\frac{\pi'(a)}{\pi(a)} .$$

Since $\pi(0) = 1$, we obtain $\pi(a) = \exp\left[-\int_0^a d(\alpha)d\alpha\right]$.

Using this relationship between π and d in the equation for U above we get

$$U(h) = U(0)\exp(\ln\pi(a_0+h) - \ln\pi(a_0)) = U(0)\frac{\pi(a_0+h)}{\pi(a_0)},$$

or
$$u(a_0+h, t_0+h) = u(a_0, t_0)\frac{\pi(a_0+h)}{\pi(a_0)}.$$

Thus the solution u can be written in one of the two following forms. To write u at (a,t), where $a > t$, let $a_0 = a - t$, $t_0 = 0$, $h = t$, so that

$$u(a,t) = \frac{u(a-t,0)\pi(a)}{\pi(a-t)} \quad \text{for} \quad a > t. \qquad (2.5)$$

Also, if we let $a_0 = 0$, $t_0 = 0$, $h = a$, then

$$u(a,t) = u(0,t-a)\pi(a) \quad \text{for} \quad t > a. \qquad (2.6)$$

If we substitute equation (2.5) and equation (2.6) into the birth law (2.8) and rename $u(0,t)$ as $B(t)$, we get exactly the Sharpe-Lotka equation (2.1). Thus a solution to the McKendrick model produces a solution to the Sharpe-Lotka model. Also, a solution to the Sharpe-Lotka model, $B(t)$, generates a solution to the McKendrick model through equations (2.5) and (2.6). The models, then, are equivalent. Thus the McKendrick model shares the asymptotic exponential growth in solutions that we have seen in the Sharpe-Lotka equation. This exponential growth, which is a consequence of linearity, is not appropriate to all biological situations. The first nonlinear model was introduced by Gurtin and MacCamy [8] in 1974. It is an extension of McKendrick's model.

The model due to Gurtin and MacCamy differs from the linear model, (2.3) and (2.4), only in that the deathrate, d, and the birthrate, b, depend on the total population size, P, as well as age. That is,

$$d = d(a,P), \quad b = b(a,P), \quad \text{and} \quad P(t) = \int_0^L u(a,t)\,da .$$

This dependence models a situation in which population size influences the resources available to individual members of the population, and consequently influences the ability of the individual to survive and reproduce. For a given initial distribution there exists a unique solution to the model. The asymptotic behavior is no longer necessarily exponential. In fact, there exists an equilibrium solution corresponding to any P such that net rate of reproduction,

$$\int_0^L b(a,P)\pi(a,P)\,da ,$$

is one. Other nonlinear models of this type have been introduced. We will not attempt to describe them all here.

Harvesting may be incorporated into this type of model by augmenting the death term du. Many types of harvesting are possible. If we assume we can freely remove individuals from the population throughout time according their ages, then we must add an arbitrary perturbation $h(a,t)$ to the death rate $d(a,P(t))u(a,t)$. However, in a wild population an arbitrarily chosen harvest rate might be impossible to accomplish. As a wild population dwindles it becomes increasingly hard to harvest individuals. In this case, we assume that the harvest rate is proportional to the number of individuals available to the harvester. This constant of proportionality is called the harvesting <u>effort</u>. Effort may be measured, for example, as the number of fishing boats per day. In this case, we add a term $e(a,t)u(a,t)$ to the death rate $d(a,P(t))u(a,t)$.

An optimal harvesting policy removes individuals from the population in such a way as to optimize an objective function called the yield. To define the yield, we assign a value to individuals according to age, denoted by $w(a)$. For example, if the harvest rate is $h(a,t)$ the yield on a time interval $[0,T)$ is

$$\int_0^T\int_0^L w(a)h(a,t)\,da\,dt.$$

The optimal harvesting problem is to identify the harvest function that removes individuals in such a way as to maximize the yield. At this level of generality, the problem is rather difficult. Thus far only special cases have been solved.

An important class of optimal harvesting problems are those concerned with the maximum sustainable yield. In optimizing the sustainable yield, one maximizes the yield rate a population can sustain in equilibrium. In this case, we seek solutions in which both the harvest rate and the population density are independent of time, and the partial differential equation of the model reduces to an ordinary differential equation. Mathematically, this simplifies the optimization procedure immensely.

Rorres and Fair [17], considered the maximum sustainable yield problem for a linear, McKendrick model. Since the linear model admits infinitely large solutions, any maximization problem must be subject to a constraint on population size. With this constraint, the authors show that the optimal harvesting policy consists of harvesting at most two ages, with the older age being harvested completely. Policies of this type are called bimodal. As we shall see at the end of this section, bimodal harvests have also been found to be optimal in maximum sustainable yield problems based on discrete models. (Rorres and Fair [16], Reed [15], Beddington and

Taylor [2]). We will solve a nonlinear maximum sustainable yield problem based on the McKendrick model in chapter 4. Impulsive optimal controls occur often for linear state equations, but are not typical in nonlinear systems. Nevertheless, we find that the optimal harvest is bimodal in the nonlinear model. The intuitive explanation for this phenomenon is that the nonlinearity in the model does not involve the unknown u, but only its integral, P.

The time dependent optimal harvesting problem was addressed by Gurtin and Murphy [9], [10]. The authors used the model due to Gurtin and MacCamy [8], and modeled harvesting in terms of an age independent effort, that is, $h = E(t)u$. They also made restrictions on the forms of the death rate and the birthrate. The authors showed, on a finite time horizon, that there exists an "ideal" time dependent age distribution. The optimal policy is to harvest so as to keep population size as near to this ideal path as the given initial conditions allow.

Martin Brokate [3], has proved existence of an optimal policy under generally less restrictive assumptions on b, d, and h; however, the harvest rate is constrained. This work unfortunately includes no convenient methods for constructing optimal policies.

Much of the optimal harvesting work that has been done is based on models with discrete age classes. We will now briefly discuss discrete population models in order that we may be able to compare the results of these models with our own model. Discrete models divide the population into age classes and represent the population by a vector. P. H. Leslie was the first to model populations in this way with matrices.

We begin by choosing Δt, Δa such that $\Delta t = \Delta a$. Let $n_t = [n_{t,0}, n_{t,1}, \ldots, n_{t,j}]^T$ be a column vector and $n_{t,j}$ be the number of individuals in the j^{th} age class at time t. Let f_i: $i = 0,1,\ldots,j$ be the expected number of offspring per time unit to an individual in class i, while s_i: $i = 0,1,\ldots,j-1$ represents the probability of survival from class i to class $i+1$, then the expected number of individuals in the various age classes at time $t+1$ is:

$$n_{t+1,0} = \sum_{i=0}^{j} f_i n_{t,i}$$

$$n_{t+1,i} = s_{i-1} n_{t,i-1} \qquad i = 1,\ldots,j.$$

Or, in terms of matrices, $\vec{n}_{t+1} = L\vec{n}_t$

where

$$\begin{bmatrix} f_0 & f_1 & . & . & . & f_n \\ s_0 & 0 & 0 & . & . & 0 \\ 0 & s_1 & 0 & . & . & 0 \\ . & 0 & s_2 & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & 0 & s_{n-1} & 0 \end{bmatrix} = L$$

and $\vec{n}_{t+j} = L^j \vec{n}_t$.

This model is linear and predicts geometric growth in population size. Nonlinear discrete models have been introduced, however, we do not attempt to review them here.

Rorres and Fair [16], find the optimal harvesting policy which will maximize the sustainable yield. Rorres and Fair use a Leslie model and maximize the yield of the harvest under equilibrium conditions, that is, conditons under which the population is to return to a fixed initial age configuration after each harvest. The ideal initial age population configuration was to be determined as well as the ideal harvesting policy. In order to keep the sustainable yield bounded the authors imposed a linear economic constraint on the process of raising individuals. The resulting optimal harvesting policy consisted of harvesting a certain fraction of a primary age group and all of a secondary age group. The optimal policy is thus bimodal, and is consistent with the results of the continuous problem.

A model due to Reed [15], included a density dependent self limiting mechanism. Reed formed a modified Leslie matrix model that assumed the survival of individuals in the first age class was dependent upon the total number of individuals in the first age class, and not on the number of individuals in the older age groups. Also survival, growth, and fertility at other age classes were considered density independent. In effect, Reed assumed that the young occupy a different environmental niche from the older animals. The survival rate of newborns to the second age class was assumed to be a decreasing function of the number of newborns. Also, Reed assumed harvesting occured over a relatively short harvest season, so that natural mortality in the population is negligible during the harvest season. Reed showed that the optimal harvesting policy to maximize the sustainable yield was to harvest at most two age classes.

Finally, Reed's model was extended by Getz [7]. Getz divided each year into two distinct classes: a harvesting season followed by a spawning season. Each age class had a time independent death rate and a time dependent harvest function. During a harvest season then, the harvest was modeled by a system of nonlinear ordinary differential equations and spawning seasons were modeled with a system of difference equations. Getz also assumed that during any given harvest season that the harvesting effort applied to each age class is the same. Getz showed that should an optimal sustainable yield exist, the optimal harvesting policy is to harvest at most two ages.

III. OPTIMAL HARVESTING OF AN AGE STRUCTURED POPULATION

3.1 The Model

In this chapter, we seek an optimal harvesting policy for a population which obeys a linear McKendrick Model [14]. The harvest rate depends on both time and age. The profit to be maximized is a discounted integral of the profit rate, and the profit from a single individual is allowed to depend upon the age of the individual at harvesting. Our basic system is then,

$$\frac{\partial u}{\partial a}(a,t) + \frac{\partial u}{\partial t}(a,t) + m(a)u(a,t) + h(a,t) = 0$$

$$B(t) = u(0,t) = \int_0^\infty b(a)u(a,t)da \qquad (3.1)$$

$$P(t) = \int_0^\infty u(a,t)da$$

supplemented by the initial condition

$$u(a,0) = u_0(a). \qquad (3.2)$$

As usual,

$u(a,t)$ is the <u>age distribution</u> (the number of individuals, per unit age, of age $a \geq 0$ at time $t \geq 0$);

$m(a)$ is the death function (the per capita rate at which individuals of age a die);

$b(a)$ is the birth function (the rate at which an individual of age a produces offspring);

$h(a,t)$ is the harvest function (the number of individuals of age a harvested at time t, per unit age and time).

We assume that b, m, and u_0 are continuous and nonnegative on $[0,\infty)$. We further assume that b and u_0 are bounded on $[0,\infty)$.
We call $B(t)$ the birthrate, and $P(t)$ is the total population size.

$y(a)$ is the yield function (the economic value of a harvested individual of age a),

and $d>0$ is the discount rate (the discounting of future yield is a common practice in economics).

So the economic yield due to a harvesting policy h is

$$\int_0^{\infty}\int_0^{\infty} y(a)e^{-dt}h(a,t)\,dtda \tag{3.3}$$

We assume that $y>0$ and continuous for $0\le a<\infty$. The optimal harvesting problem then consists in choosing $h \geq 0$ to maximize yield (3.3), subject to equations (3.1) and (3.2).

To make the problem tractable, we impose the following additional constraint. First, we choose an age, L, a priori, and insist that an admissible harvesting policy eliminate every individual before they reach this age. Mathematically, this assumption simplifies the analysis guaranteeing both the existence and the form of the optimal harvesting policy. From a biological point of view, there is a maximum age, beyond which so few individuals live that one would never plan to harvest past such a late age. Indeed, even if very old individuals were extremely valuable, one would not want to risk the loss of all profit which would occur if the small population of old individuals was subject to a catastrophe.

3.2 The Balance Law

In this section we transform the partial differential equation stated in the first equation of (3.1) into an ordinary differential equation. To perform these manipulations we assume a priori that the solution u of equation (3.1) is continuously differentiable, and that $e^{-dt}u(a,t)$ possesses a (possibly infinite) limit as $t \to \infty$. We multiply the equation by e^{-dt} and integrate from $t = 0$ to $t = \infty$ to obtain

$$\frac{d}{da}\int_0^\infty e^{-dt}u(a,t)dt + \int_0^\infty e^{-dt}\,\frac{\partial u}{\partial t}(a,t)dt + \int_0^\infty e^{-dt}m(a)u(a,t)dt +$$

$$\int_0^\infty e^{-dt}h(a,t)dt = 0. \tag{3.4}$$

In order to write equation (3.4) as an ordinary differential equation, we define

$$U(a) = \int_0^\infty e^{-dt}u(a,t)\,dt \tag{3.5}$$

and

$$H(a) = \int_0^\infty e^{-dt}h(a,t)\,dt\ . \tag{3.6}$$

We further constrain $u(a,t)$ as follows

$$\int_0^L U(a)\,da \le P. \tag{3.7}$$

We are using a linear model and in order to keep the yield from becoming unbounded it is necessary to require this constraint.

The second term of (3.4) can be integrated by parts to get

$$\lim_{t \to \infty} e^{-dt} u(a,t) - u_0(a) + d\int_0^\infty u(a,t)e^{-dt}dt,$$

and from the constraints on u, the above limit equals zero. Using equation (3.5) and equation (3.6), equation (3.4) can now be written

$$\frac{dU}{da} = u_0(a) - (d + m(a))U(a) - H(a). \qquad (3.8)$$

3.3 The Birth Law

In the last section we eliminated the partial differential equation from our model and replaced it with an ordinary differential equation for U. Our goal here is to further simplify our model, so that the techniques of classical control theory are more readily applicable. We concentrate our efforts on the boundary condition for U, using tricks developed by Rorres and Fair, [17], 1980, who circumvented an awkward initial condition in a similar circumstance with the introduction of an auxiliary variable.

It is clear from equation (3.5) and the second equation in (3.1) that

$$U(0) = \int_0^L b(a)U(a)da \ . \qquad (3.8)$$

The condition (3.8) and constraint (3.7) are awkward to deal with in an optimal control setting. We transform them with the aid of the auxiliary variables V and W defined below.

Let

$$V(a) = \int_L^a b(\alpha)U(\alpha)d\alpha \ ,$$

so that, in view of equation (3.8),

$$V'(a) = b(a)U(a) \quad \text{and} \quad V(0) = -U(0).$$

Then let

$$W(a) = \int_0^a U(\alpha)\,d\alpha$$

so that we may replace equation (3.9) with

$$W'(a) = U(a) \quad \text{and} \quad W(L) \leq P/d.$$

Thus the optimal harvesting problem now consists in choosing a control H on [0,L] to maximize

$$\int_0^L y(a)H(a)\,da$$

subject to

$$\begin{aligned} U'(a) &= u_0(a) - (d + m(a))U(a) - H(a) \\ V'(a) &= b(a)U(a) \\ W'(a) &= U(a) \end{aligned} \tag{3.10}$$

and boundary conditions

$$\begin{aligned} U(0) &= -V(0) \qquad & U(L) &= 0 \\ W(0) &= 0 \qquad & V(L) &= 0 \end{aligned}$$

$$0 \leq W(L) \leq P/d$$

with the auxiliary conditions

$$U(a) \geq 0$$

$$H(a) \geq 0.$$

The class of controls over which we perform this maximization will be discussed in the next section.

3.4 Admissable Harvesting Policies

The nature of the optimal policy is determined by the class of admissible controls. Realistically, the rate at which a population may be harvested is bounded, according to the resources of the harvester. However, the true nature of the optimum policy is often more transparent when the harvest rate is unconstrained. For example, Rorres and Fair [17], solve the time independent maximum sustainable yield problem with continuous age dependence for both constrained and unconstrained harvest rates. When the harvest rate is unbounded, the optimal harvesting policy consists of impulses at two ages. When the harvest rate is bounded, and the bound is sufficiently large, the optimal policy consists of harvesting at the maximum rate during two age intervals which contain the two optimal harvesting ages determined in the unbounded harvest rate case.

As we shall see, the time dependent problem investigated here is optimized by impulsive harvesting. Consequently, we do not bound admissible controls. Our class of optimal controls will be a subset of the set of measures obtained by taking the generalized derivatives of functions of bounded variation. This approach is modelled on a similar development by Lee and Markus, [12]. We begin our development with the following definitions.

Let $I = [l_1, l_2]$ be a closed interval of R, the reals, and $g: I \rightarrow R$, then the total variation of g, var g, is defined to be

$$\operatorname*{var}_{I} g = \sup \sum_{0}^{k} | g(a_{j+1}) - g(a_j) |$$

where the supremum is computed over all finite sequences $l_1 = a_0 < a_1 < . . . < a_k < a_{k+1} = l_2$ in I. The function g is of bounded variation in I if and only if var $g < \infty$. A function of bounded variation may have a countable number of (first kind) discontinuities. Without loss of generality, we will define BV to be the set of functions with finite variation on $I = [l_1, l_2]$, which are continuous from the right on $(l_1, l_2]$. If g is of bounded variation on a closed interval I, we define the generalized derivative, Dg, of g as

$$Dg(a_j, a_{j+1}] = g(a_{j+1}) - g(a_j) \qquad \text{if} \quad a_j \neq l_1,$$

$$\text{and} \quad Dg(l_1, a_{j+1}] = g(a_{j+1}) - g(l_1+)$$

on each subinterval $a_j < a \leq a_{j+1}$. If g is continuous then the signed measure Dg assigns a zero weight to each point of I, but if g is discontinuous at a', then

$$Dg[a'] = g(a') - g(a'-) \qquad \text{if} \quad a' \in (l_1, l_2],$$

$$\text{and} \quad Dg[l_1] = g(l_1+) - g(l_1) \qquad \text{if} \quad a' = l_1.$$

Then Dg is a signed measure on I, this class of measures includes impulses, and $\int_I Dg$ is an ordinary Riemann - Stieltjes integral.

Let g be of bounded variation on a closed interval I in R. Then we can consider the restriction of the signed measure Dg to any subset of I.

In particular, the norm of Dg on a compact interval [b,c] in I = $[l_1, l_2]$ is defined to be

$$||Dg|| = \int_b^c |Dg| = \operatorname*{var}_{[b,c]} g + |g(b) - g(b-)| \quad \text{if} \quad b \neq l_1,$$

and

$$||Dg|| = \int_{l_1}^c |Dg| = \operatorname*{var}_{[l_1,c]} g .$$

We now generalize the model developed in the previous sections by allowing H to be a generalized derivative and interpret the ordinary differential system given in equation (3.10) as a generalized differential system. We want the profit function to be a norm, so we require y(a)H to be the generalized derivative of a monotone increasing function of bounded variation.

Our class of admissible functions then is

$\mu^+ = \{\, g \mid$ g is right continuous on (0,L], monotone increasing and of bounded variation on [0,L], and $g(0) = 0 \,\}$.

We assign $g(0) = 0$ so that the possible jump in g at 0 is just $g(0+)$, that is, $Dg[0] = g(0+)$.

Finally, we now write our optimal harvesting problem in its final form: Choose a control $g \in \mu^+$ to maximize

$$||Dg||$$

subject to the differential equation

$$\begin{bmatrix} DU \\ DV \\ DW \end{bmatrix} = \begin{bmatrix} -(d+m(a)) & 0 & 0 \\ b(a) & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} + \begin{bmatrix} -1/y(a) \\ 0 \\ 0 \end{bmatrix} Dg + \begin{bmatrix} u_0(a) \\ 0 \\ 0 \end{bmatrix}, \qquad (3.11)$$

boundary conditions $U(0) = -V(0)$ $\quad U(L) = 0$

$W(0) = 0$ $\quad V(L) = 0$

$0 \leq W(L) \leq P/d$,

and the auxiliary condition $\quad U(a) \geq 0$.

3.5 The General Theory

In this section we develop a general theory for the linear impulse control process. The development is an extension of the theory in Lee and Markus, [12]. Later we will apply results of this section to equation (3.11).

Consider the linear impulse process in R^n

$$Dx = A(a)x + c(a)Du + v(a) \qquad (3.12)$$

For all $a \in [0,L]$, the coefficient A(a) is a known nxn matrix, and the coefficients c(a) and v(a) are known nx1 vectors. Given a nonempty, convex, compact, initial set, $X_0 \subset R^n$, and a nonempty, compact, target set, $X_1 \subset R^n$, the goal is to maximize profit by choosing an initial point, $x^0 \in X_0$, and a scalar controller u of bounded variation with generalized derivative Du to steer the response x from x^0 to any point of the target set X_1. We assume that the coefficients of equation (3.12), A, c, and v, are continuous on $0 \leq a \leq L$.

Additionally, we require the controllers, u, to be right continuous on (0,L], monotone increasing and of bounded variation on [0,L]. Each such controller defines a positive measure Du on $0 \leq a \leq L$.

Also, let $\Phi(a)$ be the fundamental matrix solution of the homogeneous differential system, that is, $D\Phi = A\Phi$, $\Phi(0) = I$. Then according to the variation of parameters formula, the solution of equation (3.12) is

$$x(a) = \Phi(a)x^0 + \Phi(a)\int_0^a \Phi^{-1}(s)c(s)Du(s) + \Phi(a)\int_0^a \Phi^{-1}(s)v(s)ds \quad (3.13)$$

where $x(0) = x^0$.

We seek a controller $u \in \mu^+$ (or corresponding positive measure Du) and a initial point $x^0 \in X_0$ such that that u steers x^0 to any point of the target set X_1 with maximal profit

$$||Du|| = \int_0^L |Du| = \operatorname*{var}_{[0,L]} u.$$

We begin our analysis by defining the set of points that can be attained by steering the initial set X_0, by the process of equation (3.12), with controllers $u \in \mu^+$. We define $K(\alpha,\beta)$, the set of attainability, as follows.

$$K(\alpha,\beta) = \{x_1 \in R^n \mid \text{there exists } u \in \mu^+,\ x^0 \in X_0 \text{ such that } \alpha \le ||Du|| \le \beta \text{ and } x(L) = x_1\}.$$

It is from the nature of K that we derive the properties necessary for optimality. We begin the analysis of K with theorem 1.

Theorem 1. For all $0 \leq \alpha \leq \beta$, and any convex initial set X_0, $K(\alpha,\beta)$ is convex.

Proof.

Let $x_1, x_2 \in K(\alpha,\beta)$. Then there exists $x^0{}_1$, $x^0{}_2 \in X^0$ and u_1, $u_2 \in \mu^+$ such that $\lambda x_1 + (1-\lambda)x_2 =$

$$\lambda\left[\Phi(L)x^0{}_1 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du_1 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da\right] +$$

$$(1-\lambda)\left[\Phi(L)x^0{}_2 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du_2 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da\right]$$

$$= \Phi(L)(\lambda x^0{}_1 + (1-\lambda)x^0{}_2) +$$

$$\Phi(L)\int_0^L \Phi^{-1}(a)c(a)(\lambda Du_1 + (1-\lambda)Du_2) + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da.$$

Now, $\lambda x^0{}_1 + (1-\lambda)x^0{}_2 \in X_0$, since X_0 is convex. Also, if $u = \lambda u_1 + (1-\lambda)u_2$, then $\alpha \leq \|Du\| \leq \beta$, so that $x = \lambda x_1 + (1-\lambda)x_2 \in K(\alpha,\beta)$. Thus $K(\alpha,\beta)$ is convex.

This theorem will be used later in theorem 3 to aid in the development of an optimal control. We state two key theorems that will be used extensively in the analysis of the optimal control problem. The first theorem is referred to as " Helly's convergence theorem ", and is found in Kolmogorov and Fomin [11], chapter 10, section 36.

<u>Helly's convergence theorem</u>. Let g_n be a sequence of functions of bounded variation on [a,b], converging pointwise to g on [a,b]. Suppose the sequence of total variations $\{ \operatorname{var}_{[a,b]} (g_n) \}$ is bounded; that is, there exists a

k > 0 such that

$$\operatorname{var}_{[a,b]} (g_n) \leq k \qquad (n = 1, 2, \ldots).$$

Then g is also of bounded variation on [a,b], and if f is any continuous function on [a,b]

$$\lim_{n\to\infty} \int_a^b f \, dg_n = \int_a^b f \, dg .$$

The second theorem is referred to as " Helly's selection theorem ", and is found in Taylor [20], chapter 9, section 6.

<u>Helly's selection theorem</u>. Let $\{g_n\}$ be a sequence of functions of bounded variation on [a,b]. Suppose there is a constant k such that

$$|g_n(t)| \leq k \text{ for } t \in [a,b] \text{ and } n = 1,2,\ldots$$

and

$$\underset{[a,b]}{\mathrm{var}} (g_n) \leq k \quad \text{for each } n.$$

Then there is a subsequence of $\{g_n\}$ which converges pointwise on $[a,b]$ to a limit function g which is of bounded variation on $[a,b]$.

We now prove several lemmas about $K(\alpha,\beta)$ that will be useful in proving the existence of an optimal control.

Lemma 1. Given any compact initial set X_0, $K(\alpha,\beta)$ is compact for all $0 \leq \alpha \leq \beta$.

Proof.

We will show that $K(\alpha,\beta)$ is compact by proving that every sequence of points $\{x_r\}$ in $K(\alpha,\beta)$ has a subsequence which converges to some limit point $\overline{x} \in K(\alpha,\beta)$. If $x_r \in K(\alpha,\beta)$, then there exists a $u_r \in \mu^+$ and $x^0{}_r \in X_0$ such that $\alpha \leq \|Du_r\| \leq \beta$ and

$$x_r = \Phi(L)\, x^0{}_r + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du_r(a) + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da.$$

Since u_r is monotone increasing, $\underset{[0,L]}{\mathrm{var}}\; u_r = u_r(L) - u_r(0)$ and $u_r(L) \geq u_r(a)$ for $a \in [0,L]$. Now $\beta \geq \|Du_r\| = \underset{[0,L]}{\mathrm{var}}\; u_r = u_r(L) - u_r(0) = u_r(L) \geq u_r(a)$ for all $a \in [0,L]$. Therefore, $\beta \geq u_r(a)$ for all $a \in$

$[0,L]$, and we may apply Helly's selection theorem to infer the existence of a subsequence u_{r_i} that converges to a limit $\overline{u} \in \mu^+$ and $\alpha \leq ||D\overline{u}|| \leq \beta$. Also, by Helly's convergence theorem, since $\Phi^{-1}(a)c(a)$ is continuous,

$$\lim_{r_i \to \infty} \int_0^L \Phi^{-1}(a)c(a)Du_{r_i} = \int_0^L \Phi^{-1}(a)c(a)D\overline{u} \ .$$

X_0 is compact, so there exists a subsequence of $x^0{}_{r_i}$, $x^0{}_{r_j}$, such that $x^0{}_{r_j} \to \overline{x}^0 \in X_0$.

Now let $\overline{x}(a)$ be the response to $D\overline{u}$, $\overline{x}^0$, on $[0,L]$:

$$\overline{x}(a) = \Phi(a)\overline{x}^0 + \Phi(a)\int_0^a \Phi^{-1}(\alpha)c(\alpha)D\overline{u} + \Phi(a)\int_0^a \Phi^{-1}(\alpha)v(\alpha)d\alpha$$

Thus, $\lim_{r_j \to \infty} x_{r_j}(L) = \overline{x}(L) \in K(\alpha,\beta)$, consequently $K(\alpha,\beta)$ is compact.

<u>Lemma</u> <u>2</u>. Given any nonempty, compact initial set, X_0, and any nonempty, compact set, X_1, if $\Phi(L)\Phi^{-1}(a)c(a)$ has at least one component which is never zero, and there exists a γ such that $X_1 \subseteq K(0,\gamma)$, then there exists an (α,β) such that $X_1 \subseteq K(0,\beta)$, but $X_1 \cap K(\alpha,\beta) = \emptyset$.

Proof.

We denote by $|x|$ the (Euclidean) norm of $x \in R^n$, and the minimum distance between sets A and B as

$$\min d(A,B) = \min \{ |a - b| ; a \in A, b \in B \}.$$

Let X_0 and X_1 satisfy the hypothesis of the lemma, and note $K(0,0) = X_0$. Denote the components of $\Phi(L)\Phi^{-1}(a)c(a)$ as ϕ_i, for $i = 1,2,\ldots,n$.

For convenience, let $M_1 = \max_{x^0, x^{0\prime} \in K(0,0)} |\Phi(L)(x^0 - x^{0\prime})|$,

$$M_2 = \Big[\sum_{i=1}^{n} (\max_{0 \le a \le L} |\phi_i(a)|)^2 \Big]^{1/2}, \text{ and}$$

denote the minimum of the absolute value of the always nonzero component of $\Phi(L)\Phi^{-1}(a)c(a)$ as m.

Now let $\epsilon > 0$, $\alpha \ge \max\big[(M_1 + M_2 + \epsilon)/m, \gamma\big]$, $\beta \ge \alpha$ be arbitrary, and choose any $x \in K(\alpha,\beta)$ and any $x' \in X_1$. Since x and x' are both attainable, we may infer the existence of x^0, $x^{0\prime}$, u, and u', and by virtue of equation (3.13), $|x - x'| =$

$$|\Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du - \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du' - \Phi(L)(x^0 - x^{0\prime})|,$$

where $x^0, x^{0\prime} \in X^0$ and u steers x^0 to x, while u' steers $x^{0\prime}$ to x'. Note that $\| Du \| \ge \alpha$, while $\| Du' \| \le \gamma$. So, $|x - x'| \ge$

$$|\Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du| - |\Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du'| - |\Phi(L)(x^0 - x^{0\prime})| \ge$$

$$m\int_0^L Du - \Big[\sum_{i=1}^{n}\big(\max_{0\le a\le L}|\phi_i(a)|\big)^2\Big]^{1/2}\int_0^L Du' - M_1$$

$$\ge m\alpha - M_2\gamma - M_1 \ge \epsilon .$$

Therefore, min $d(K(\alpha,\beta),X_1) \ge \epsilon$, so that $X_1 \cap K(\alpha,\beta) = \emptyset$. Also, $\beta \ge \gamma$, and consequently, $X_1 \subseteq K(0,\beta)$.

Lemma 3. If $\Phi(L)\Phi^{-1}(a)c(a)$ is not identically the zero vector on $[0,L]$, then $K(\alpha,\beta)$ is nonincreasing and continuous in α.

Proof.

That $K(\alpha,\beta)$ is nonincreasing in α follows directly from the definition. $K(\cdot,\beta)$ is continuous at α_1 if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\alpha_2 - \alpha_1| < \delta$ and $x_1 \in \partial K(\alpha_1,\beta)$ implies there exists $x_2 \in \partial K(\alpha_2,\beta)$ such that $|x_1 - x_2| < \epsilon$.

Note that $x_1 \in \partial K(\alpha_1,\beta)$ implies $x_1 \in K(\alpha_1,\alpha_1)$ or $x_1 \in K(\beta,\beta)$. If $x_1 \in K(\beta,\beta)$ then $x_1 \in \partial K(\alpha_2,\beta)$. So we need only consider $x_1 \in K(\alpha_1,\alpha_1) \cap \partial K(\alpha_1,\beta)$.

If $x_1 \in K(\alpha_1,\alpha_1) \cap \partial K(\alpha_1,\beta)$, then there exists $u \in \mu^+$ such that $||Du_1|| = \alpha_1$ and

$$x_1 = \Phi(L)x^0 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Du_1 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da .$$

Let $\epsilon > 0$ be arbitrary. Choose

$$\delta = \epsilon / (\max_{0 \leq a \leq L} |\Phi(L)\Phi^{-1}(a)c(a)|)$$

and let α_2 be any number such that $|\alpha_1 - \alpha_2| < \delta$. Define

$$x_2 = \Phi(L)x^0 + \Phi(L)\frac{\alpha_2}{\alpha_1}\int_0^L \Phi^{-1}(a)c(a)Du_1 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da .$$

We first claim $x_2 \in \partial K(\alpha_2,\beta)$. Clearly, $x_2 \in K(\alpha_2,\alpha_2) \subseteq K(\alpha_2,\beta)$. Suppose $x_2 \notin \partial K(\alpha_2,\beta)$, then for some $\gamma > 0$ there exists a ball of radius γ centered at x_2 contained in $K(\alpha_2,\beta)$. Since $x_1 \in \partial K(\alpha_1,\beta)$, there exists $\tilde{x}_1$ such that $|x_1 - \tilde{x}_1| < \frac{\alpha_1}{\alpha_2}\gamma$ and $\tilde{x}_1 \notin K(\alpha_1,\beta)$. Now define $\tilde{x}_2$ by

$$\tilde{x}_2 = (\tilde{x}_1 - \Phi(L)x^0 - \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da)\frac{\alpha_2}{\alpha_1} + \Phi(L)x^0 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da ,$$

then $|\tilde{x}_2 - x_2| < \gamma$, so that $\tilde{x}_2 \in K(\alpha_2,\beta)$ and there exists $\tilde{u} \in \mu^+$ such that $||D\tilde{u}|| = \alpha_1$ and

$$\tilde{x}_2 = \Phi(L)x^0 + \Phi(L)\frac{\alpha_2}{\alpha_1}\int_0^L \Phi^{-1}(a)c(a)D\tilde{u} + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da .$$

Thus

$$\tilde{x}_1 = \Phi(L)x^0 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)D\tilde{u} + \Phi(L)\int_0^L \Phi^{-1}(a)\ v(a)da$$

which implies $\tilde{x}_1 \in K(\alpha_1,\beta)$, since $||D\tilde{u}|| = \alpha_1$. This is a contradiction, therefore $x_2 \in \partial K(\alpha_2,\beta)$.

Furthermore,

$$|x_1 - x_2| = |\int_0^L \Phi(L)\Phi^{-1}(a)c(a)\left(1 - \frac{\alpha_2}{\alpha_1}\right)Du_1| =$$

$$|1-\frac{\alpha_2}{\alpha_1}||\int_0^L \Phi(L)\Phi^{-1}(a)c(a)Du_1| \leq |\frac{\alpha_1-\alpha_2}{\alpha_1}| \max_{0\leq a\leq L} |\Phi(L)\Phi^{-1}(a)c(a)|\alpha_1$$

$$< \delta \max_{0\leq a\leq L} |\Phi(L)\Phi^{-1}(a)c(a)| < \epsilon.$$

Thus, $K(\alpha,\beta)$ is continuous in α.

<u>Theorem</u> <u>2</u>. Given any nonempty, compact initial set X_0, and any nonempty, compact target set X_1, if $\Phi(L)\Phi^{-1}(a)c(a)$ has at least one component which is never zero, and there exists γ such that the target set, $X_1 \subseteq K(0,\gamma)$, then there exists an initial point $x^0 \in X_0$, and an optimal controller $u^* \in \mu^+$, such that u steers x^0 to a point of the target set X_1, with maximal profit $||Du^*|| = \alpha^*$.

Proof.

Since $K(\alpha,\beta)$ is compact, continuous in α, grows to $K(0,\beta)$ as α decreases to 0, and $X_1 \subseteq K(0,\beta)$, $X_1 \cap K(\alpha,\beta) = \emptyset$, there exists a maximum value $\alpha = \alpha^*$ for which $X_1 \cap K(\alpha,\beta)$ is not empty. In fact, if $x_1 \in$

$X_1 \cap K(\alpha^*,\beta)$, then x_1 lies on the boundary of $K(\alpha^*,\alpha^*)$.

Q.E.D.

We now show that every point of $\partial K(\alpha^*,\alpha^*)$ is attainable by a admissible control with maximal profit α^* that is a convex combination of n α^*-weighted scalar delta functions, where by a scalar delta function $\delta(a - a')$ we mean the signed measure corresponding to the step function

$$s(a) = \begin{cases} 0 & \text{for } a < a' \\ 1 & \text{for } a \geq a' \end{cases}$$

where $0 \leq a' \leq L$.

That is, $\delta(a - a')$ has a weight of 1 at $a = a'$, and zero weight otherwise. We refer to $\alpha\delta(a - a')$ as an α-weighted scalar delta function.

<u>Theorem</u> <u>3</u>. Consider the controllable impulse process in R^n given by equation (3.12), if $\Phi(L)\Phi^{-1}(a)c(a)$ has at least one component which is never zero, the initial set X_0, is convex, nonempty, and compact in R^n and the target set X_1, is nonempty and compact in R^n, then there exists an initial point $x^0 \in X_0$, and a controller $u^*(a) \in \mu^+$ whose corresponding measure Du^* is a convex combination of n α^*-weighted scalar delta functions steering x^0 to a point of X_1 with maximal profit α^*.

Proof.

The existence of a maximal profit follows from the previous corollary. In order to see that such a controller exists as a convex combination of n+1 scalar delta functions, let $x^0 \in X_0$, $g \in \mu^+$ such that $\|Dg\| = \alpha^*$ and let x_g be the endpoint resulting from g. Let $\epsilon > 0$ and choose a finite time partition $0 = a_0 < a_1 < \dots < a_r = L$ such that $\epsilon >$.

$$\left|\int_0^L \Phi(L)\Phi^{-1}(a)c(a)Dg(a) - \sum_{\sigma=0}^{\nu-1} \Phi(L)\Phi^{-1}(a_\sigma)c(a_\sigma)(g(a_{\sigma+1}) - g(a_\sigma))\right|.$$

Let $$Dg_\Delta(a) = \sum_{\sigma=0}^{\nu-1} \left(\frac{g(a_{\sigma+1}) - g(a_\sigma)}{\alpha^*} \right) (\alpha^*\delta(a - a')),$$

and observe $$\sum_{\sigma=0}^{\nu-1} \left(\frac{g(a_{\sigma+1}) - g(a_\sigma)}{\alpha^*} \right) = \frac{g(a_\nu) - g(a_0)}{\alpha^*} = 1.$$

Thus, $Dg_\Delta(a)$ is a convex combination of α^*-weighted scalar delta functions. Also the response endpoint, x_Δ, with initial point x^0, approximates x_g, that is, $|x_g - x_\Delta| < \epsilon$.

Let D be the set of all points in $K(\alpha^*,\alpha^*)$ that are attainable by α^*-weighted scalar delta function controllers. Given any $x_g \in K(\alpha^*,\alpha^*)$ and $\epsilon > 0$, we can find a point x_Δ, that is a convex combination of points attained by α^*-weighted scalar delta functions that is within ϵ of x_g. Thus, the convex hull $\mathcal{H}(D)$ is dense in $K(\alpha^*,\alpha^*)$, and every point interior to $K(\alpha^*,\alpha^*)$ lies in $\mathcal{H}(D)$.

To continue the proof we make use of the following theorem from Eggelston [5], Chapter 2.

Caratheodory's Theorem. If y is a point of the convex hull $\mathcal{H}(\mathfrak{S})$ there is a set of s points $x_1, x_2, \ldots, x_s$ all belonging to $\mathfrak{S}$ with $s \leq n + 1$ such that y is a point of the simplex whose vertices are $x_1, x_2, \ldots, x_s$.

We also note that an n-simplex in R^n is the convex hull of $n + 1$ independent points in R^n. Thus, every point in the interior of $K(\alpha^*,\alpha^*)$ lies in a simplex with at most $n + 1$ vertices located in D. Each simplex lies in R^n, so each point in the interior of $K(\alpha^*,\alpha^*)$ is attainable by a convex combination of at most $n+1$ α^*- weighted scalar delta functions. Thus each point interior to $K(\alpha^*,\alpha^*)$ is attainable by a convex combination of at most $n+1$ α^*-weighted scalar delta functions. That is, each x_g in the interior of $K(\alpha^*,\alpha^*)$ is attainable by a controller u^* with corresponding measure Du^* of the form

$$Du^*(a) = \sum_{i-1}^{n+1} \alpha_i \delta(a - a_i^m),$$

with $\sum_{i=1}^{n+1}\alpha_i = \alpha^*$ and $a_i \in [0,L]$, for $i=1,2,\ldots,n+1$.

From here we extend the result to include points on $\partial K(\alpha^*,\alpha^*)$, that is, we now show that every point $x \in \partial K(\alpha^*,\alpha^*)$ is attainable by a convex combination of n+1 α^*-weighted scalar delta functions.

Let $x \in \partial K(\alpha^*,\alpha^*)$ and let x_m be a sequence lying in the interior of $K(\alpha^*,\alpha^*)$ with corresponding controllers $u_m{}^*$ and measures $Du_m{}^*$ approaching $x \in \partial K(\alpha^*,\alpha^*)$. Each x_m lying in the interior of $K(\alpha^*,\alpha^*)$ is attainable by a controller $u_m{}^*$ such that

$$Du_m{}^* = \sum_{i=1}^{n+1}\alpha_i{}^m\, \delta(a - a_i{}^m),$$

with $\sum_{i=1}^{n+1}\alpha_i{}^m = \alpha^*$ and $a_i{}^m \in [0,L]$, for $i = 1,2,\ldots,n+1$.

Thus,

$$x_m = \Phi(L)x^0{}_m + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)\Big(\sum_{i=1}^{n+1}\alpha_i{}^m\, \delta(a - a_i{}^m)\Big) +$$

$$\Phi(L)\int_0^L \Phi^{-1}(a)v(a)da$$

where $x^0{}_m \in X_0$.

Then

$$x_m = \Phi(L)x^0_m + \Phi(L)\sum_{i=1}^{n+1}\alpha_i^m\Phi^{-1}(a_i^m) + \Phi(L)\int_0^L\Phi^{-1}(a)v(a)da.$$

Now, $x^0_m \in X_0$, $\alpha_i^m \in [0,\alpha]$, and $a_i^m \in [0,L]$ for $i = 1,\dots, n+1$, so $(x^0_m,\alpha_1^m,\dots,\alpha_{n+1}^m,a_1^m\dots,a_{n+1}^m)$ is an infinite sequence lying in a compact set of R^{3n+2}, and there must exist a convergent subsequence

$$(x^0_{m_k},\alpha_1^{m_k},\dots,\alpha_{n+1}^{m_k},a_1^{m_k},\dots,a_{n+1}^{m_k}),$$

converging to

$$(x^0,\alpha_1,\dots,\alpha_{n+1},a_1,\dots,a_{n+1}),$$

where $x_0 \in X_0$, $\alpha_i \in [0,\alpha]$ and $a_i \in [0,L]$, for $i = 1,2,\dots,n+1$.

Also, $\alpha_i^{m_k} \to \alpha_i$ and $\sum_{i=1}^{n+1}\alpha_i^{m_k} = \alpha^*$, therefore

$$\alpha^* = \sum_{i=1}^{n+1}\alpha_i^{m_k} \to \sum_{i=1}^{n+1}\alpha_i\ , \quad \text{or} \quad \sum_{i=1}^{n+1}\alpha_i = \alpha^*.$$

Now let

$$x_{m_k} = \Phi(L)x^0_{m_k} + \Phi(L)\sum_{i=1}^{n+1}\alpha_i^{m_k}\ \Phi^{-1}(a_i^{m_k})c(a_i^{m_k}) +$$

$$\Phi(L)\int_0^L\Phi^{-1}(a)v(a)da,$$

and recalling Φ^{-1} and c are continuous, it follows

$$x_{m_k} \rightarrow \Phi(L)x^0 + \Phi(L)\sum_{i=1}^{n+1}\alpha_i\Phi^{-1}(a_i)c(a_i) + \Phi(L)\int_0^L\Phi^{-1}(a)v(a)da,$$

where $\sum_{1=1}^{n=1}\alpha_i = \alpha^*$.

However, if $x_m \rightarrow x$ and

$$x_{m_k} \rightarrow \Phi(L)x^0 + \Phi(L)\sum_{i=1}^{n+1}\alpha_i\Phi^{-1}(a_i)c(a_i) + \Phi(L)\int_0^L\Phi^{-1}(a)v(a)da \ ,$$

then

$$x = \Phi(L)x_0 + \Phi(L)\sum_{i=1}^{n+1}\alpha_i\Phi^{-1}(a_i)c(a_i) + \Phi(L)\int_0^L\Phi^{-1}(a)v(a)da \ ,$$

where $\sum_{i=1}^{n+1}\alpha_i = \alpha^*$.

Therefore, if we choose $Du^*(a) = \sum_{i=1}^{n+1}\alpha_i\delta(a - a_i)$

and $x^0 \in X_0$, we attain

$$x = \Phi(L)x^0 + \Phi(L)\sum_{i=1}^{n+1}\alpha_i\Phi^{-1}(a_i)c(a_i) + \Phi(L)\int_0^L\Phi^{-1}(a)v(a)da \ .$$

Now, $Du^*(a) = \sum_{i=1}^{n+1}\frac{\alpha_i}{\alpha^*}(\alpha^*\delta(a - a_i))$ and $\sum_{i=1}^{n+1}\alpha_i = \alpha^*$,

thus, each $x \in \partial K(\alpha^*,\alpha^*)$ is attainable by a convex combination of n+1 α^*-weighted scalar delta functions, and $||Du|| = \alpha^*$, the maximal profit.

Theorem 1 implies that $K(\alpha^*,\alpha^*)$ is convex, and since x lies on the boundary of $K(\alpha^*,\alpha^*)$, it cannot lie in the interior of the n-simplex whose vertices correspond to the α^*-weighted scalar delta function controllers. Thus x can be obtained by a convex combination of only n α^*-weighted scalar delta function controllers.

Q.E.D.

In the next section we will apply the result of this theorem to our optimal harvesting problem.

3.6 An Optimal Harvesting Policy

In this section we apply the results of the previous section to the optimal harvesting problem defined by equation (3.11). Equation (3.11) is a special case of equation (3.12). The continuity of y, c, m, and b guarantees that the matrix

$$A(a) = \begin{bmatrix} -(d+m(a)) & 0 & 0 \\ b(a) & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and the vectors

$$c(a) = \begin{bmatrix} -1/y(a) \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v(a) = \begin{bmatrix} u_0(a) \\ 0 \\ 0 \end{bmatrix},$$

which are the coefficients of equation (3.11), are continuous on $[0,L]$.

The fundamental matrix for equation (3.11), $\Phi(a)$, is

$$\begin{bmatrix} E(a) & 0 & 0 \\ \int_0^a b(\alpha)E(\alpha)d\alpha & 1 & 0 \\ \int_0^a E(\alpha)d\alpha & 0 & 1 \end{bmatrix}$$

where $E(a) = \exp\{-\int_0^a (d + m(\alpha))\,d\alpha\}$. Since m and b are continuous on [0,L], $E(a) > 0$ for every $a \in [0,L]$, and Φ is nonsingular on [0,L].

From equation (3.11), our initial set is

$$\{(U(0), -U(0), 0) \mid U(0) \geq 0\}.$$

This initial set doesn't satisfy the hypothesis of the general theory since it is unbounded, and consequently, not compact. We redefine the initial set, X_0, to include only those initial values from which the solution can reach a desired endpoint. That is,

$$X_0 = \{(U(0), -U(0), 0) \mid U(0) \geq 0, \text{ and there exists } g \in \mu^+ \text{ which steers } (U(0), -U(0), 0) \text{ to a point of } (0, 0, [0, P/d])\}.$$

We now show that this initial set is nonempty, compact, and convex. We start by showing X_0 is nonempty. Let g be the control generated by the measure $Dg = u_0(\alpha)y(\alpha)d\alpha$; g is absolutely continous. The solution of equation (3.11), with initial point (0,0,0) is

$$U(a) = E(a)\left(U(0) + \int_0^a \frac{u_0(\alpha)}{E(\alpha)}d\alpha - \int_0^a \frac{Dg}{E(\alpha)y(\alpha)}\right), \quad (3.14)$$

$$V(a) = V(0) + \int_0^a b(\alpha)U(\alpha)\,d\alpha\,,$$

$$W(a) = W(0) + \int_0^a U(\alpha)\,d\alpha\,.$$

Clearly $(U(L), V(L), W(L)) = (0,0,0) \in X_1$. Therefore, $(0,0,0) \in X_0$, and X_0 is nonempty.

We now show that X_0 is compact. Recall that (3.11) implies that

$$U(0) = \int_0^L b(a)U(a)da\,.$$

Now b is continuous, so there exists an upper bound, M, for b on [0,L]. Combining this bound with the last component of equation (3.11) yields

$$U(0) \leq M\int_0^L u(a)da = MW(L) \leq MP/d.$$

Consequently, X_0 is bounded.

Now let $\{x^0_n\} \subseteq X_0$ be a sequence converging to x^0. Then there exists $\{g_n\} \subseteq \mu^+$, $\{x^1_n\} \subseteq X_1$ such that, for every n,

$$x^1_n = \Phi(L)x^0_n + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Dg_n + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)da.$$

Solving the top equation of equation (3.11) we get

$$U_n(L) = E(L)\left(U_n(0) + \int_0^L \frac{u_0(a)}{E(a)}da - \int_0^L \frac{Dg_n}{y(a)E(a)}\right) = 0.$$

Since $E(L) > 0$ and $0 \le U_n(0) \le MP/d$,

$$\int_0^L \frac{Dg_n}{y(a)E(a)} \le MP/d + \int_0^L \frac{u_0(a)}{E(a)}da.$$

Furthermore,

$$\int_0^L \frac{Dg_n}{\max\limits_{0\le a\le L}\{y(a)\}} \le \int_0^L \frac{Dg_n}{y(a)E(a)},$$

so that

$$\int_0^L Dg_n \le k, \quad \text{where} \quad k = \max_{0\le a\le L}\{y(a)\}\left(MP/d + \int_0^L \frac{u_0(a)}{E(a)}da\right).$$

So $k \ge \operatorname*{var}\limits_{[0,L]} g_n$. Also, since g_n is monotone increasing, $\operatorname*{var}\limits_{[0,L]} g_n = g_n(L) - g_n(0)$, and $g_n(0) = 0$, so $k \ge g_n(L) \ge g_n(a)$, for all $a \in [0,L]$. Therefore, by Helly's selection theorem, there exists a $g \in \mu^+$ such that g_{n_k} converges to g, and by Helly's convergence theorem

$$x^1{}_{n_k} = \Phi(L)x^0{}_{n_k} + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Dg_{n_k} + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da$$

converges to

$$x^1 = \Phi(L)x^0 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Dg + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da\,.$$

X_1 is compact, so $x^1 \in X_1$, which implies $x^0 \in X_0$. Therefore, the limit points of X_0 are contained in X_0, so X_0 is closed as well as bounded and consequently, X_0 is compact.

Finally, we show that X_0 is convex. Let $x^0{}_1$, $x^0{}_2 \in X_0$, then there exists g_1, $g_2 \in \mu^+$ such that

$$x' = \Phi(L)x^0{}_1 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Dg_1 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da$$

$$x'' = \Phi(L)x^0{}_2 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Dg_2 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da\,.$$

So, $x_1 = \lambda x' + (1 - \lambda)x'' =$

$$\lambda(\Phi(L)x^0{}_1 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Dg_1 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da) +$$

$$(1 - \lambda)(\Phi(L)x^0{}_2 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)Dg_2 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da)$$

$$= \Phi(L)(\lambda x^0{}_1 + (1 - \lambda)x^0{}_2) +$$

$$\Phi(L)\int_0^L \Phi^{-1}(a)c(a)(\lambda Dg_1 + (1 - \lambda)Dg_2) + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da .$$

Now, x', $x'' \in \{(0,0,[0,P/d]\}$, and $\{(0,0,[0,P/d])\}$ is convex, so $x_1 \in \{0,0,[0,P/d]\}$.
Let $g = \lambda g_1 + (1 - \lambda)g_2$, then $g \in \mu^+$ and g steers $x^0 = \lambda x^0{}_1 + (1 - \lambda)x^0{}_2$ to x_1. Therefore, $x^0 \in X_0$, and X_0 is convex.

We now define the target set. From equation (3.14), and boundary condition $U(L) = 0$, we observe

$$\int_0^L \frac{Dg}{E(a)y(a)} = U(0) + \int_0^L \frac{u_0(a)}{E(a)}\,da .$$

Since, as previously shown, $U(0) \leq MP/d$,

$$\int_0^L Dg \leq \max_{0\leq a\leq L}\{y(a)\}\left[MP/d + \int_0^L \frac{u_0(a)}{E(a)}da\right] = \gamma, \text{ or } \|Dg\| \leq \gamma .$$

Hence, if control $g \in \mu^+$ and satisfies equation (3.11), as well as the associated boundary conditions and initial conditions, then the norm of Dg is bounded by γ.

Therefore all possible sets of attainability for such controls g, are contained in $K(0,\gamma)$, and observing the boundary conditions of equation (3.11), we define the target set X_1 as

$$X_1 = \{ K(0,\gamma) \cap (0,0,[0,P/d]) \}.$$

The target set X_1, then contains all points that can be attained by any control $g \in \mu^+$ that satisfies equation (3.11) and the associated boundary conditions and initial conditions. $K(0,\gamma)$ is convex and compact by our theorems of the previous section, so the target set is also convex and compact. Also, the target set is nonempty since, as previously shown, the control g with corresponding measure $Dg = u_0(a)y(a)da$ steers $(0,0,0)$ to $(0,0,0) \in (0,0,[0,P/d])$.

Finally, as required by theorem 2 and theorem 3 of the previous section, $\Phi(L)\Phi^{-1}(a)c(a)$ has a component which is always nonzero, since the first component of $\Phi(L)\Phi^{-1}(a)c(a)$ for equation (3.11) is

$$-\frac{E(L)}{E(a)y(a)}, \text{ which is nonzero for all } a \in [0,L].$$

Our optimal harvesting problem now satisfies the hypotheses of theorems 1, 2, and 3 of the previous section. Thus, there exists an optimal control, g^*, with corresponding Dg^* of the form

$$Dg^*(a) = \sum_{i=1}^{3} \alpha_i \delta(a - a_i).$$

The corresponding optimal harvest rate has the form

$$H^*(a) = \sum_{i=1}^{3} \beta_i \delta(a - a_i),$$

where $\beta_i = (\alpha_i / y(a_i))$ for $i = 1, 2, 3$.

Recall that $H(a) = \int_0^\infty e^{-dt} h(a,t)dt$, so the optimal harvest function h^* is of the form

$$h^*(a,t) = \sum_{i=1}^{3} l_i(t) \delta(a - a_i),$$

where $\int_0^\infty e^{-dt} l_i(t) = \beta_i$ for $i = 1, 2, 3$.

Therefore a optimal harvesting policy for the system of equations given in (3.1) and (3.2) is one in which we harvest at most three ages. The actual ages a_i, and the time dependent intensities l_i are still unknown. The a_i and β_i can be determined numerically.

Without loss of generality, we may assume

$$H(a) = \sum_{i=1}^{3} \gamma_i U(a_i -) \delta(a - a_i),$$

where $\gamma_i \in [0,1]$, $a_i \in [0,L]$ and $a_i- = 0$ if $a_i = 0$, for $i = 1,2,3$, so that $U \geq 0$ on $[0,L]$. The corresponding solution to the top equation of (3.11) according to equation (3.14) is

$$U(a) = E(a)(U(0) + G(a) - \sum_{i=1}^{3} \gamma_i \frac{U(a_i-)}{E(a_i)} s_i(a)),$$

$$\text{where} \quad G(a) = \int_0^a \frac{u_0(\alpha)}{E(\alpha)} d\alpha \quad \text{and}$$

$$s_i(a) = \begin{cases} 1 & \text{for } a \geq a_i \text{ if } a_i \in (0,L], \text{ and } a > a_i \text{ if } a_i = 0 \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1,2,3$.

If we define K_1, K_2, B_i, for $i = 1,2,3$, as follows,

$$K_1 = \int_0^L b(a)E(a)da, \quad K_2 = \int_0^L b(a)E(a)G(a)da, \text{ and}$$

$$B_i = \frac{1}{E(a_i)} \int_0^L s_i(a) b(a) E(a) da,$$

then the second equation of (3.11) has solution

$$U(0) = U(0)K_1 + K_2 + \sum_{i=1}^{3} \gamma_i U(a_i-) B_i .$$

If we define K_3, and C_i for $i = 1,2,3$ as

$$K_3 = \int_0^L E(a)G(a)\,da, \quad K_4 = \int_0^L E(a)\,da, \text{ and}$$

$$C_i = \frac{1}{E(a_i)}\int_0^L E(a)s_i(a)\,da,$$

the third equation of (3.11), with its associated boundary conditions implies

$$K_3 + U(0)K_4 - \sum_{i=1}^{3}\gamma_i U(a_i-)C_i \leq P.$$

Finally, if we define A_i for $i = 1,2,3$, as

$A_i = \frac{1}{E(a_i)}$, the boundary condition, $U(L) = 0$, becomes

$$E(L)(U(0) + G(L) - \sum_{i=1}^{3}\gamma_i U(a_i-)A_i) = 0.$$

Thus, we have reduced the optimization problem of equation (3.11) to the following:

Choose $(\gamma_1,\gamma_2,\gamma_3,a_1,a_2,a_3,U(0))$, to maximize the

yield, $$\sum_{i=1}^{3} y(a_i)\gamma_i U(a_i-),$$

subject to

$$U(0) + G(L) - \sum_{i=1}^{3} A_i\gamma_i U(a_i-) = 0$$

$$U(0) = U(0)K_1 + K_2 + \sum_{i=1}^{3} B_i\gamma_i U(a_i-) \qquad (3.15)$$

$$K_3 + U(0)K_4 - \sum_{i=1}^{3} C_i \gamma_i U(a_i-) \quad \leq \quad P$$

with the constraints

$U(0) \geq 0$, $\gamma_i \in [0,1]$, and $a_i \in [0,L]$, where $a_i- = 0$ if $a_i = 0$, for $i = 1,2,3$.

The optimization problem stated in equation (3.15) is a constrained nonlinear programming problem. Given the functions b, m, y, u_0, and the constants d, L, and P, the optimization could be done numerically by a nonlinear optimal search technique. Suitable nonlinear optimal search techniques are discussed by Bazarra and Shetty [1], chapters 9 and 10, and by Luenberger [13], chapters 11 and 12.

Once the optimal values for the parameters U(0), γ_i, and a_i, for $i = 1,2,3$, have been numerically determined, β_i and a_i, for $i = 1,2,3$, are specified, and the optimization is reduced to determining the $f_i \geq 0$, for $i = 1,2,3$, such that

$$\int_0^{\infty} f_i(t) e^{-dt} dt = 1, \qquad i = 1,2,3,$$

subject to equations (3.1), (3.2), and (3.3), with the control

$$h(a,t) = \sum_{i=1}^{3} \beta_i f_i(t) \delta(a - a_i).$$

This is an open problem, and investigating the ages as well as the time dependent intensities of the harvest function will be the study of future work.

IV. MAXIMUM SUSTAINABLE YIELD OF AN AGE STRUCTURED POPULATION

4.1 The Model

In this chapter, we investigate the maximum sustainable yield problem for a population with age structure whose dynamics are density dependent. Our study is based on the nonlinear version of the McKendrick [14] model introduced by Gurtin and MacCamy [8]. A steady state solution to this model satisfies the equations

$$u'(a) + d(a,P)u(a) + h(a) = 0$$

$$u(0) = \int_0^\infty b(a,P)u(a)da \qquad (4.1)$$

$$P = \int_0^\infty u(a)da$$

where

u(a) is the equilibrium age distribution (the number of individuals, per unit age, of age a, $a \geq 0$); it follows that P is the total population size.

d(a,P) is the <u>death function</u> (the rate at which individuals of age a die, when population is P);

b(a,P) is the <u>birth function</u> (the expected number of births, per unit time, to an individual of age a, when population is P);

h(a) is the <u>harvest rate</u> (the rate at which individuals are harvested at age a, per unit age).

We assume that $b(a,P)$ and $d(a,P)$ are nonnegative and continuous on $[0,\infty) \times [0,\infty)$. We further assume that there exists an age past which individuals are no longer fertile, that is, there exists a K such that $b(a,P) = 0$ for every P and for all $a \geq K$.

In order to describe the profit associated with a harvesting policy, we introduce the following function,

y(a) is the <u>yield function</u> (the economic value of a harvested individual of age a).

We incorporate into y any cost associated with the harvest of such an individual and assume that y is positive and continuous on $[0,\infty)$. The rate at which a harvest policy h yields profit is then,

$$\int_0^\infty y(a)h(a)da\,. \tag{4.2}$$

Our goal is to choose h to maximize this yield. The class of functions admissable as harvest policies will be discussed in section (4.2). However, in order to reformulate the problem of equation (4.1) into a more tractable form, we hereby impose the constraint that the harvesting policy eliminate every individual before it reaches a predetermined age $L < \infty$. From a biological point of view, there may be a maximum age beyond which so few individuals live that one would never plan to harvest past such a late age. Mathematically, if y does not grow "too fast" as $a \to \infty$ and if L, chosen greater than K, is chosen large enough, this constraint does not significantly affect the yield.

In the absence of harvesting, the first equation of (4.1) becomes

$$u'(a) = -d(a,P)u(a)$$

and, for a given P, has solution

$$u(a) = u(0)\pi(a,P),$$

where
$$\pi(a,P) = \exp\left[-\int_0^a d(\alpha,P)d\alpha\right].$$

The function $\pi(a,P)$ is the probability that an individual will survive to age a, if the population is constant at P, and is not subject to harvesting. Now let

$$R(a,P) = \int_0^a b(\alpha,P)\pi(\alpha,P)d\alpha .$$

The function R is an age dependent <u>net rate of reproduction</u>. Specifically, $R(a,P)$ is the expected number of offspring born to an individual in an unharvested population of constant size P, before the indivdual reaches age a. $R(a,P)$ is monotone increasing in a, and for fixed P, is constant for $a \geq K$. If $R(K,P) < 1$, the population cannot maintain itself at size P, and of course, a positive yield cannot be sustained.

Typically, the environment limits the population size. If P is too large, resources are strained, and both fertility and average lifetime duration are reduced. We assume there exists a $P = P^c$, such that $R(L,P) < 1$ for every $P > P^c$ and $R(L,P) \geq 1$ for at least one $P \in [0,P^c]$. Thus populations with size greater than P^c are incapable of sustaining a yield.

We reformulate the optimization problem in a manner similar to that used in section (3.3). The boundary condition stated in the second equation of (4.1) can be simplified with the auxiliary variable

$$v(a) = \int_L^a b(\alpha,P)u(\alpha)d\alpha .$$

We may now replace the second equation of (4.1) with

$$v'(a) = b(a,P)u(a) \quad \text{and} \quad v(0) = -u(0).$$

Also, let

$$w(a) = \int_0^a u(\alpha)d\alpha ,$$

so that we may replace the third equation of (4.1) with

$$w'(a) = u(a) \quad \text{and} \quad w(L) = P.$$

Thus, the maximal sustainable yield problem consists in choosing the harvest rate h and population size $P \in [0,P^c]$, to maximize

$$\int_0^L y(a)h(a)da$$

subject to

$$
\begin{aligned}
u'(a) &= -d(a,P)u(a) - h(a) \\
v'(a) &= b(a,P)u(a) \\
w'(a) &= u(a)
\end{aligned}
\tag{4.3}
$$

and boundary conditions

$$
\begin{aligned}
u(0) &= -v(0) \quad & u(L) &= 0 \\
w(0) &= 0 \quad & v(L) &= 0 \\
& & w(L) &= P
\end{aligned}
$$

with the auxiliary conditions

$$
\begin{aligned}
u(a) &\geq 0 \\
h(a) &\geq 0 .
\end{aligned}
$$

4.2 Existence Of An Optimal Harvesting Policy For A Fixed Population Size.

We will begin our search for an optimal policy by first considering a problem in which the total population size P is arbitrarily fixed at some $P_0 \in [0,P^c]$, such that $R(L,P_0) \geq 1$. As in section 3.4, we allow the harvest rate to be unbounded. Specifically, we require yh to be the generalized derivative of a monotone increasing function of bounded variation, so that the profit function is a norm. Also as in section 3.4, we define our class of admissable controls, μ^+ as follows,

$$\mu^+ = \{\ g \mid g \text{ is right continuous on } (0,L], \text{ monotone increasing and of bounded variation on } [0,L], \text{ and } g(0) = 0\ \}.$$

Thus our optimal harvesting problem is to choose a control $g \in \mu^+$, with corresponding generalized derivative, $dg = yh$, to maximize

$$||dg||$$

subject to the differential equation

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} -d(a,P_0) & 0 & 0 \\ b(a,P_0) & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} -1/y(a) \\ 0 \\ 0 \end{bmatrix} dg, \quad (4.4)$$

the boundary conditions $u(0) = -v(0) \quad u(L) = 0$

$$w(0) = 0 \qquad v(L) = 0$$

$$w(L) = P_0$$

and auxiliary condition $u(a) \geq 0$.

Define A, c, and v on $[0,\infty)$ as follows,

$$A(a) = \begin{bmatrix} -d(a,P_0) & 0 & 0 \\ b(a,P_0) & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$c(a) = \begin{bmatrix} -1/y(a) \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v(a) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Under our earlier assumptions that y is continous on $[0,\infty)$, and d and b are continuous on $[0,\infty) \times [0,\infty)$, the matrix A and the vectors c and v are continuous on $[0,\infty)$. The optimal harvesting problem of equation (4.4), then is of the same form as equation (3.12). We now precede to show that the current problem satisfies the hypothesis of section (3.5), and the results derived there apply.

The fundamental matrix for the process in equation (4.4), $\Phi(a)$, is

$$\begin{bmatrix} \pi(a,P_0) & 0 & 0 \\ R(a,P_0) & 1 & 0 \\ \int_0^a \pi(\alpha,P_0)\,d\alpha & 0 & 1 \end{bmatrix}.$$

Since d is continuous, $\pi(a,P_0) > 0$ for all $a \in [0,L]$, and Φ is nonsingular on $[0,L]$.

Our optimal harvesting problem is to maximize the profit by choosing a control $g \in \mu^+$, which steers a point of an initial set, to a target set, by the process of equation (4.4).

Refering to equation (4.4), our target set is the point $x_1 = (0,0,P_0)$, which is nonempty, convex, and compact in R^3. Also from equation (4.4), our initial set is

$$\{ (u(0),-u(0),0) \mid u(0) \geq 0 \}.$$

This initial set doesn't satisfy the hypothesis of the theory of section (3.5) since it is unbounded, and consequently, not compact. So we redefine the initial set, X_0, as

$$X_0 = \{ (u(0), -u(0), 0) \mid u(0) \geq 0, \text{ and there exists } g \in \mu^+ \text{ such that } g \text{ steers } (u(0), -u(0), 0) \text{ to } x_1 \}.$$

We now show that this initial set is nonempty, compact, and convex. We start by showing X_0 is nonempty.

For convenience, define

$$C(a,P) = \int_0^a \pi(\alpha,P)\, d\alpha .$$

Let the control g be defined as

$$g(a) = \frac{(R(L,P_0) - 1)P_0 y(0)}{C(L,P_0)} s(a) + \frac{\pi(L,P_0)P_0 y(L)}{C(L,P_0)} s(a - L),$$

where $s(a) = 1$ if $a \geq 0$, and $s(a) = 0$ if $a < 0$.

The corresponding generalized derivative then is

$$dg = \frac{(R(L,P_0) - 1)P_0 y(0)}{C(L,P_0)} \delta(a) + \frac{\pi(L,P_0)P_0 y(L)}{C(L,P_0)} \delta(a - L).$$

Consider the point

$$x^0 = \left(\frac{P_0 R(L,P_0)}{C(L,P_0)}, - \frac{P_0 R(L,P_0)}{C(L,P_0)}, 0 \right).$$

We claim $x^0 \in X_0$.

We need only show that the solution to equation (4.4), with the initial condition $x^0 = (u(0), v(0), w(0))$ and control g satisfies the indicated boundary conditions.

The first component of the solution of equation (4.4) is,

$$u(a) = \pi(a,P_0)\left(u(0) - \int_0^a \frac{dg}{\pi(\alpha,P_0)\,y(\alpha)} \right).$$

Substitution of the aforementioned control g yields,

$$u(a) = \pi(a,P_0)\left[u(0) - \frac{(R(L,P_0) - 1)P_0}{C(L,P_0)}\right] ,$$

for $0 \le a < L$, and

$$u(L) = \pi(L,P_0)[u(0) - \frac{(R(L,P_0) - 1)P_0}{C(L,P_0)} - \frac{P_0}{C(L,P_0)}].$$

It is clear from the two preceding equations that if

$u(0) = \frac{P_0 R(L,P_0)}{C(L,P_0)}$, then $u \ge 0$ on $[0,L]$ and $u(L) = 0$.

From the second component of equation (4.4) we get,

$$v(L) = v(0) + \int_0^L b(a,P_0)\,u(a)\,da =$$

$$v(0) + \int_0^L b(a,P_0)\frac{\pi(a,P_0)P_0}{C(L,P_0)}da = v(0) + \frac{P_0R(L,P_0)}{C(L,P_0)} = 0 .$$

From the third component of equation (4.4) we get,

$$w(L) = w(0) + \int_0^L u(a)\,da = \int_0^L \frac{\pi(a,P_0)P_0}{C(L,P_0)}\,da = P_0 .$$

The system of equations of (4.4) is satisfied, as well as the associated boundary conditions. Therefore,

$$x^0 = \left(\frac{P_0R(L,P_0)}{C(L,P_0)},\ -\frac{P_0R(L,P_0)}{C(L,P_0)},\ 0 \right) \in X_0, \text{ and } X_0 \text{ is nonempty.}$$

We now show that X_0 is compact. Let u and v be the first two components of a solution to equation (4.4), satisfying the associated boundary conditions, then

$$v(a) = \int_0^a b(\alpha,P_0)\,u(\alpha)\,d\alpha ,$$

and since $v(0) = -u(0)$ and $v(L) = 0$, we derive the relation

$$u(0) = \int_0^L b(a,P_0)\,u(a)\,da .$$

Since b is continuous, there exists an upper bound, M, for $b(\cdot,P_0)$ on $[0,L]$. Therefore,

$$u(0) \le M\int_0^L u(a)\,da = MP_0.$$

Consequently, X_0 is bounded.

Now let $\{x^0{}_n\} = \{(x^1{}_n, x^2{}_n, x^3{}_n)\} \subseteq X_0$, be a sequence converging to x^0. Then there exists $\{g_n\} \subseteq \mu^+$, such that, for all n

$$(0,0,P_0)^T = \Phi(L)x^0{}_n + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)\,dg_n + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da.$$

The first component of the equation above is

$$0 = u_n(L) = \pi(L)\Big(x^1{}_n - \int_0^L \frac{dg_n}{y(a)\pi(a)} \Big).$$

Since $0 < \pi(L) \le 1$, and $0 \le x^1{}_n \le MP_0$,

$$\int_0^L \frac{dg_n}{y(a)\pi(a)} \le MP_0.$$

It follows that

$$\int_0^L dg_n \le k, \quad \text{where} \quad k = \max_{0\le a\le L}\{y(a)\}MP_0.$$

Now since each g_n is monotone increasing and $g_n(0) = 0$, $g_n(a) \le g_n(L)$ for all $a \in [0,L]$,

$$\text{and} \quad g_n(L) \;=\; \operatorname*{var}_{[0,L]} g_n \;=\; \int_0^L dg_n \;\le\; k.$$

Since g_n and $\operatorname*{var}_{[0,L]} g_n$ are uniformly bounded in n, we may apply Helly's selection theorem, to infer the existence of a $g \in \mu^+$, and a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that g_{n_k} converges to g. By Helly's convergence theorem,

$$\Phi(L)\int_0^L \Phi^{-1}(a)c(a)\,dg_{n_k} \quad \text{converges to} \quad \Phi(L)\int_0^L \Phi^{-1}(a)c(a)\,dg,$$

so that

$$x_1 = \Phi(L)x^0 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)\,dg + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da.$$

Therefore, $x^0 \in X_0$, and the limit points of X_0 are contained in X_0, so X_0 is closed as well as bounded, and consequently, compact.

Finally, we show that X_0 is convex. Let $x^0{}_1$, $x^0{}_2$ $\in X_0$, then there exists $g_1, g_2 \in \mu^+$ such that

$$x_1 = \Phi(L)x^0{}_1 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)\,dg_1 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da =$$

$$\Phi(L)x^0{}_2 + \Phi(L)\int_0^L \Phi^{-1}(a)c(a)\,dg_2 + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da .$$

So that,

$$x_1 = \lambda x_1 + (1-\lambda)x_1 = \Phi(L)\ (\lambda x^0{}_1 + (1-\lambda)x^0{}_2) +$$

$$\Phi(L)\int_0^L \Phi^{-1}(a)c(a)(\lambda dg_1 + (1-\lambda)dg_2) + \Phi(L)\int_0^L \Phi^{-1}(a)v(a)\,da .$$

Let $g = \lambda g_1 + (1-\lambda)g_2$, then $g \in \mu^+$ and g steers $x^0 = \lambda x^0{}_1 + (1-\lambda)x^0{}_2$ to x_1. Therefore, $x^0 \in X_0$, and X_0 is convex.

Note that for our chosen control g, the norm of Dg is $||Dg|| = P_0((R(L,P_0) - 1)y(0) + \pi(L,P_0)y(L))/C(L,P_0) = \gamma$, so there exists a γ such that $x_1 \in K(0,\gamma)$.

Finally, as required by theorem 2, $\Phi(L)\Phi^{-1}(a)c(a)$ has at least one component which is always nonzero, since $\Phi(L)\Phi^{-1}(a)c(a)$ for equation (4.4) is

$$\frac{[\pi(L,P_0),(R(L,P_0)-R(a,P_0)),(C(L,P_0)-C(a,P_0))]^T}{(\pi(a,P_0)y(a))} ,$$

and the first component is nonzero on $[0,L]$.

We have shown that our optimal harvesting problem now satisfies the hypothesis of theorem 2 in section

(3.5), which guarantees the existence of an optimal control. Thus, for every $P_0 \in [0,P^c]$ such that $R(L,P_0) \geq 1$, there exists an optimal harvesting policy, $g \in \mu^+$, which maximizes the yield, $\|dg\|$, subject to equation (4.4) and the associated boundary conditions.

4.3 The Optimal Harvesting Policy

As mentioned in chapter 3, Rorres and Fair [17], discovered that when the linear model has a maximum sustainable yield, it is attained by a bimodal optimal harvesting policy. If we prescribe the population size P, our problem is also linear. We have shown that for a fixed population size P, an optimal harvesting policy exists, therefore we may apply the result of Rorres and Fair [17], to guarantee that the optimal harvesting policy is bimodal. To treat the nonlinear case, we need to examine the dependence of the optimal policy on the prescribed value of P_0. For that reason, we now describe Rorres and Fair's result more extensively, beginning with the definition of several auxiliary variables.

Let $\Gamma:\{P \in [0,P^c]$ such that $R(L,P) \geq 1\}$. Now let $P \in \Gamma$, and define $\hat{a}(P)$ to be the age a such that $R(\hat{a}(P),P) = 1$, the function $\hat{a}$ is the <u>replacement age.</u> That is, $\hat{a}(P)$ is the average age at which an individual first replaces itself by giving birth to one viable offspring, if the population has constant size P. Such an $\hat{a}(P)$ exists as defined for every $P \in \Gamma$, since $R(a,P)$ is monotone increasing in a and $R(L,P) \geq 1$ for $P \in \Gamma$.

We now show that Γ is a compact set. Let P_n be a sequence in Γ, such that P_n converges to $\tilde{P}$. Since $b(a,P)$ and $\pi(a,P)$ are both continuous for every P, so is $R(L,P)$, so $R(L,P_n)$ converges to $R(L,\tilde{P})$. However, $R(L,P_n) \geq 1$, so $R(L,\tilde{P})$ is also ≥ 1. Therefore, $\tilde{P} \in \Gamma$, and Γ is a closed set. Also, $\Gamma \subseteq [0,P^c]$, so Γ is bounded and consequently, Γ is compact.

Also, we define

$$X(a,P) = C(a,P)/[1 - R(a,P)],$$

and

$$Y(a,P) = Py(a)\pi(a,P)/[1 - R(a,P)].$$

X and Y are defined on $\{[0,\hat{a}(P)) \cup (\hat{a}(P),L]\} \times [0,\infty)$.

If we fix population size at P_0, the optimal harvesting policy is, according to Rorres and Fair [17], bimodal. That is, $h^*(a) = h_1\delta(a - a_1) + h_2\delta(a - a_2)$, where δ is the dirac delta function. Further, the two optimal harvesting ages, a_1 and a_2, maximize

$$E(a_1,a_2,P_0) = \frac{Y(a_1,P_0) - Y(a_2,P_0)}{X(a_1,P_0) - X(a_2,P_0)},$$

subject to $0 \leq a_1 \leq \hat{a}(P_0) \leq a_2 \leq L$. In fact, this maximum is also the maximum sustainable yield.

The problem of finding the optimal harvesting policy for equation (4.4) for fixed P_0, then reduces to finding the maximum value of $E(a_1,a_2,P_0)$ when a_1 and a_2 vary over the rectangular region $0 \le a_1 \le \hat{a}(P_0) \le a_2 \le L$.

Thus the problem of finding the optimal harvesting policy, for the nonlinear (P dependent) problem, then reduces to maximizing $E(a_1,a_2,P)$ over the region in R^3 defined by

$$\Lambda = \{ P \in \Gamma,\ 0 \le a_1 \le \hat{a}(P),\ \hat{a}(P) \le a_2 \le L \}.$$

Since the maximization is over a compact subset of R^3 we need only show that E is continuous on Λ.

Since b and π are continuous in both of their arguments, so are R and C. Further,

$$X(a,P) = \frac{C(a,P)}{[1 - R(a,P)]} \quad \text{and} \quad Y(a,P) = \frac{y(a)\pi(a,P)}{[1 - R(a,P)]}$$

are continuous for every (a,P) such that $a \neq \hat{a}(P)$. Therefore, the quotient

$$E(a_1,a_2,P) = \frac{Y(a_1,P) - Y(a_2,P)}{X(a_1,P) - X(a_2,P)}$$

is continuous for every (a_1,a_2,P) such that

$$0 \le a_1 < \hat{a}(P) < a_2 \le L, \quad \text{and} \quad P \in \Gamma.$$

(Note that $X(a_2,P) < 0$ and $X(a_1,P) \ge 0$, so that the denominator of E is nonzero.)

We will now show that for every $P \in \Gamma$,

$$E(a_1,a_2,P) \to \frac{y(\hat{a}(P))\pi(\hat{a}(P),P)P}{C(\hat{a}(P),P)} \quad \text{as} \quad (a_1,a_2) \to (\hat{a}(P),\hat{a}(P)).$$

Suppose $a_1 \neq \hat{a}(P)$, $a_2 \neq \hat{a}(P)$, $P \in \Gamma$ and recall

$$E(a_1,a_2,P) = \frac{\dfrac{y(a_1)\pi(a_1,P)P}{[1 - R(a_1,P)]} + \dfrac{y(a_2)\pi(a_2,P)P}{[R(a_2,P) - 1]}}{\dfrac{C(a_1,P)}{[1 - R(a_1,P)]} + \dfrac{C(a_2,P)}{[R(a_2,P) - 1]}} =$$

$$\frac{y(a_1)\pi(a_1,P)P + y(a_2)\pi(a_2,P)P\dfrac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]}}{C(a_1,P) + C(a_2,P)\dfrac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]}} . \quad (4.5)$$

Since $C(a_2,P) \geq C(a_1,P)$,

$$\frac{\left(y(a_1)\pi(a_1,P) + y(a_2)\pi(a_2,P)\dfrac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]}\right) P}{C(a_2,P)\left(1 + \dfrac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]}\right)}$$

$$\leq E(a_1,a_2,P) \leq$$

$$\frac{\left(y(a_1)\pi(a_1,P) + y(a_2)\pi(a_2,P)\dfrac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]}\right) P}{C(a_1,P)\left(1 + \dfrac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]}\right)} ,$$

therefore

$$\frac{\min\left[y(a_1)\pi(a_1,P),y(a_2)\pi(a_2,P)\right]\left(\frac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]} + 1\right)P}{C(a_2,P)\left(1 + \frac{[1 - R(a_2,P)]}{[R(a_2,P) - 1]}\right)}$$

$$\leq E(a_1,a_2,P) \leq$$

$$\frac{\max\left[y(a_1)\pi(a_1,P),y(a_2)\pi(a_2,P)\right]\left(\frac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]} + 1\right)P}{C(a_1,P)\left(1 + \frac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]}\right)}.$$

Cancelling the like terms in the denominator and the numerator of the above inequalities, we get

$$\frac{\min\left[y(a_1)\pi(a_1,P),y(a_2)\pi(a_2,P)\right]P}{C(a_2,P)} \leq E(a_1,a_2,P) \leq$$

$$\frac{\max\left[y(a_1)\pi(a_1,P),y(a_2)\pi(a_2,P)\right]P}{C(a_1,P)}.$$

If we take the limit of the above inequalities as (a_1,a_2) approach $(\hat{a}(P),\hat{a}(P))$ we obtain

$$E(a_1,a_2,P) \rightarrow \frac{y(\hat{a}(P))\pi(\hat{a}(P),P)P}{C(\hat{a}(P),P)}.$$

We will now show for every $P \in \Gamma$,

$$E(a_1,a_2,P) \to \frac{y(\hat{a}(P))\pi(\hat{a}(P),P)P}{C(\hat{a}(P),P)} \quad \text{as} \quad (a_1,a_2) \to (\hat{a}(P),\alpha),$$

where $\alpha \neq \hat{a}(P)$.

From equation (4.5) we see that as $(a_1,a_2) \to (\hat{a}(P),\alpha)$, where $\alpha \neq \hat{a}(P)$,

$$E(a_1,a_2,P) \to \frac{y(\hat{a}(P))\pi(\hat{a}(P),P)P}{C(\hat{a}(P),P)},$$

since

$$\frac{[1 - R(a_1,P)]}{[R(a_2,P) - 1]} \to \frac{0}{[R(\alpha,P) - 1]} = 0,$$

as $(a_1,a_2) \to (\hat{a}(P),\alpha)$, and the conclusion holds.

By a similar argument one can show

$$E(a_1,a_2,P) \to \frac{y(\hat{a}(P))\pi(\hat{a}(P),P)P}{C(\hat{a}(P),P)} \quad \text{as} \quad (a_1,a_2) \to (\alpha,\hat{a}(P)),$$

for every $P \in \Gamma$ and $\alpha \neq \hat{a}(P)$.

Therefore, if we define $E(a_1,\hat{a}(P),P)$ and $E(\hat{a}(P),a_2,P)$ equal to

$$\frac{y(\hat{a}(P))\pi(\hat{a}(P),P)P}{C(\hat{a}(P),P)},$$

then $E(a_1,a_2,P)$ is continuous on all of Λ. Since Λ

is compact, E attains a maximum in Λ. Hence, there exists a maximum sustainable yield for the optimal harvesting problem. Thus, the harvester of a population that roughly meets the assumptions used here need only seek ages a_1 and a_2, and population size P, that maximize the function E.

That is, the optimal harvesting problem is now reduced to: choose $(a_1,a_2,P) \in \Lambda$ to maximize

$$E(a_1,a_2,P). \tag{4.6}$$

The optimization problem stated in equation (4.6) is a constrained nonlinear programming problem. Given the functions b, m, and y, and the constant L, the optimization could be done numerically by a nonlinear optimal search technique. Suitable nonlinear optimal search techniques for this optimization are discussed by Bazarra and Shetty [1], chapters 9 and 10, and by Luenberger [13], chapters 11 and 12.

Once the optimal values for a_1, a_2, and P are numerically determined, they may be used in the system of differential equations of equation (4.4) to determine the values of $u(0)$, and the coefficients of $\delta(a - a_1)$ and $\delta(a - a_2)$, h_1 and h_2, respectively, which appear in the optimal harvesting function.

Solving the first component of equation (4.4) and using the boundary condition $u(L) = 0$, we get

$$u(0) = \frac{h_1}{C(a_1,P)} + \frac{h_2}{C(a_2,P)}. \tag{4.7}$$

Solving the second component of equation (4.4) and using the boundary conditions $u(0) = -v(0)$, and $v(L) = 0$, we have equation (4.8):

$$u(0)(R(L,P)-1) = h_1\frac{(R(L,P)-R(a_1,P))}{C(a_1,P)} + h_2\frac{(R(L,P)-R(a_2,P))}{C(a_2,P)}.$$

Finally, solving the third component of equation (4.4), and using the boundary conditions, $w(0) = 0$ and $w(L) = P$, we have equation (4.9):

$$u(0)R(L,P) = h_1\frac{(R(L,P)-R(a_1,P))}{C(a_1,P)} + h_2\frac{(R(L,P)-R(a_2,P))}{C(a_2,P)} + P.$$

Equations (4.7), (4.8), and (4.9) form a 3 by 3 system of linear equations from which $u(0)$, h_1 and h_2 can easily be determined, thereby specifying the optimal harvesting function,

$$h^*(a) = h_1\delta(a - a_1) + h_2\delta(a - a_2).$$

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