AN ABSTRACT OF THE DISSERTATION OF

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The purpose of this investigation was to explore how differences in an elementary mathematics teacher's subject matter knowledge structure impact classroom teaching and student learning. The study included two phases. Phase 1 focused on the selection of a single case. An open-ended questionnaire and interview were used to identify the subject matter knowledge structure for addition, subtraction, multiplication, and division of three elementary teachers. One teacher was selected who demonstrated clearly different levels of knowledge for multiplication and division. An additional interview provided information on the teacher's specific climate for teaching mathematics and details about the unit on multiplication and division to be observed.

Phase 2 included daily classroom observations for approximately one hour each day of a seven-week unit on multiplication and division. Informal interviews were conducted with the teacher throughout the unit to better understand the lessons and allow the teacher an opportunity to clarify statements and actions. A final teacher interview occurred after the last classroom observation. At the conclusion of the observations, the students were assessed to determine their knowledge of multiplication and division based on the teacher's unit objectives. And six students, representing the range of class performance, were interviewed to provide additional insights into the students' learning.

The teacher's subject matter knowledge of multiplication was strong but her knowledge of division was faulty and incomplete on several topics including the

different meanings of division, the conceptual underpinnings of division procedures, the relationships between symbolic division and real life problems, and the idea of divisibility. Although the translation of the teacher's subject matter knowledge was complex, it seemed to be directly related to classroom teaching and students' learning. The teacher's narrow understandings were associated with an incomplete developing of the full range of division situations. Although the students had significantly more success on the post assessment problems involving multiplication than on those involving division (understandable since the teacher spent more time teaching multiplication than division), a more worrisome concern was that the students in this study exhibited serious misconceptions associated with the meanings of division, division computation, and notions of divisibility.

ELEMENTARY MATHEMATICS TEACHER SUBJECT MATTER KNOWLEDGE AND ITS RELATIONSHIP TO TEACHING AND LEARNING

By

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

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William F. Buckreis, Author

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ELEMENTARY MATHEMATICS TEACHER SUBJECT MATTER KNOWLEDGE AND ITS RELATIONSHIP TO TEACHING AND LEARNING

CHAPTER I

THE PROBLEM

Introduction

To teach the arithmetic-driven curriculum of the past, elementary teachers needed little more than knowledge of basic facts, computational skills with standard algorithms, and textbooks to provide practice. Reformers and researchers concerned about the quality of instruction in United States classrooms (Mathematics Association of America [MAA], 1991; National Council of Teachers of Mathematics [NCTM], 1989, 1991, 1995; National Research Council [NRC], 1989, 1990, 1996; Third International Mathematics and Science Study [TIMSS], 1996) indicate that, for meaningful mathematics to occur, that level of knowledge and manner of instruction is no longer adequate.

Computational algorithms, manipulations of symbols, and memorization of rules no longer dominate school mathematics. Rather, central to school mathematics is the development of mathematical power for all students (NCTM, 1989, 1991). Mathematical power includes having conceptual understanding as well as the ability to apply concepts to new situations: formulate and solve problems, explore, conjecture, reason logically, and communicate mathematically (NCTM, 1989; NRC, 1996). To accomplish this goal, teachers no longer simply "deliver" content, and students are no longer viewed as "empty vessels" or "blank slates" (NCTM, 1991). The constructivist perspective on teaching and learning that has provided the basis for current reform efforts in mathematics education theorizes that learning is an active social process in which students construct their own mathematical knowledge rather than receiving it in finished form from the teacher or a textbook (Mathematics Science Education Board, 1989, 1990; NCTM, 1989, 1991, 1995).

Changing teachers' attitudes about mathematics and habits of teaching, however, requires more than the publication and discussion of reform documents (NCTM, 1989, 1991, 1995; TIMSS, 1996). Although most teachers report familiarity with current recommendations, it appears that only a few teachers apply the key points in their classrooms (TIMSS, 1996). Several powerful forces may be obstacles to making significant changes and need to be addressed before any change can occur (NCTM, 1991).

First, the current vision of what it means to teach mathematics contrasts sharply with how most teachers learned about mathematics and teaching. Reformers indicate that the experiences teachers have while learning mathematics have a powerful impact on the education they provide their students. Through their own learning, teachers develop conceptions of the nature of mathematics, what it means to teach mathematics, and how particular topics are taught (MAA, 1991; NCTM, 1989, 1991, 1995; NRC, 1989, 1990, 1996).

Second, there is some evidence that the subject matter knowledge of elementary mathematics teachers is not strong. After reviewing studies about elementary preservice teachers' subject matter knowledge of mathematics, Brown, Cooney, and Jones (1990) concluded that "research of this type leaves the distinct impression that preservice teachers do not possess a level of mathematical understanding necessary to teach elementary school mathematics as recommended in various proclamations from professional organizations such as NCTM" (p. 643).

Although there exists much rhetoric that reflects strong belief about the importance of mathematical knowledge to teachers, early attempts to relate quantitatively-oriented measures of what teachers know (e.g. number of courses taken in college, GPAs, scores on standardized tests) with measures of effective teaching have not produced relationships of strong, practical significance (School Mathematics

Study Group, 1972; Eisenberg, 1977). In these studies, however, no attempts were made to measure what teachers actually knew about mathematics or to ascertain what mathematics was covered in the various courses completed by the teachers. Consequently, these earlier research paradigms have yielded to more in-depth qualitative measures to investigate questions concerning teachers' knowledge and its potential impact on teaching.

Recent attempts to explore teachers' subject matter knowledge, as reported by Brown, Cooney, and Jones (1990), have used a wide variety of approaches, notably interviews, card sorts, concept maps, questionnaires, classroom observations, and various types of classroom documents. These investigations have concentrated on providing in-depth descriptions of teachers' knowledge and its relationship to teaching. In most of these studies, preservice teachers were described or experts were compared to novice teachers. Often a small number of teachers were investigated with inferences drawn on data collected from several classroom observations.

From data collected using the aforementioned approaches, much has been learned about the impact of preservice teachers' subject matter knowledge on teaching. For instance, several studies have concluded that teachers with weak subject matter knowledge had difficulty making transitions to pedagogical thinking, were unable to connect topics during classroom instruction, and focused on procedural rather than conceptual understanding (Lehrer & Franke, 1992; Leinhardt & Smith, 1985).

Finally, the world of elementary schools has not offered a positive environment for teachers to develop new ways in which to teach. Although many educators believe that teachers learn best through experience, a growing body of research suggests that the typical experiences of teachers in schools are noneducative at best and miseducative at worst (Lanier, J. E. & Little, J. W. , 1986). Compared to Japanese teachers, US teachers have fewer opportunities for professional development and less time to discuss teaching-related issues with colleagues. On average, US teachers devote only about one hour per week to professional development and reading (TIMSS, 1996).

Although school mathematics reformers (MAA, 1991; NCTM, 1989, 1991, 1995; NRC, 1989, 1990, 1996) suggest that professional development is needed at all levels, especially elementary, to help teachers become more "competent," few such solutions to the current state of mathematics education seem to be informed by research (Thompson, 1992). Understanding teachers' conceptions of subject matter knowledge and its relationship to teaching and learning may be fundamental to designing successful professional development programs.

In recent years, elementary mathematics teachers' subject matter knowledge has been the focus of much research attention. These studies have explored teachers' knowledge on topics in multiplication and division (Ball, 1990a, 1990b; Simon, 1993; Tirosh & Graeber, 1988, 1989), fractions (Post, Harel, Behr, & Lesh, 1991; Leinhardt and Smith,1985; Lehrer and Franke, 1992), problem solving (Funkhouser, 1993), and functions and graphing (Stein, Baxter, and Leinhardt,1990). Although these studies have provided important insights into teachers' knowledge and its relationship to classroom practices, some of the methods employed in these studies contained inherent limitations. First, relatively few studies avoided the pitfalls of limiting the assessment of teachers' knowledge to only a few topics. Since understanding is essential to knowing mathematics, and the degree of understanding is determined by the number and strength of the connections made (Hiebert & Carpenter, 1992), an overall understanding of teachers' knowledge may not be achieved by attention to such a narrow focus of content (Flennema & Franke, 1992).

Second, in most studies, teachers were asked to solve problems suggested by the researchers (Ball, 1990a, 1990b; Khoury & Zazkis, 1994; Post, Harel, Behr, & Lesh, 1991; Simon, 1993) or to perform card sorts (Stein, Baxter, Leinhardt, 1990; Scholz, 1996) to demonstrate their knowledge. Although the data yielded were qualitative in nature, restricting subjects to predetermined sets of topics may have compromised the benefits and purpose of using a qualitative research design. A more open-ended methodology sensitive to teachers' personal understandings of content may produce considerably different results (Gess-Newsome & Lederman, 1993; Lederman & Chang, 1997).

Finally, although some researchers (Leinhardt & Smith, 1985) have studied the impact of teachers' knowledge on student learning (ex post facto), most have not. The relationship between teachers' subject matter knowledge and classroom practices has been the focus of much research attention, but whether teachers' subject matter knowledge truly impacts students' learning has not been given similar attention.

Statement of Problem

The purpose of this investigation was to explore how differences in an elementary mathematics teacher's subject matter knowledge structure impact classroom teaching and student learning. In particular, this investigation attempted to answer the following questions:

1. What is the appearance of an elementary mathematics teacher's subject matter knowledge structure of addition, subtraction, multiplication, and division?

2. How do differences in this knowledge structure relate to classroom teaching and student learning?

Addition, subtraction, multiplication, and division were chosen as the content areas for this investigation for several reasons. First, since these operations are fundamental to knowing mathematics (NCTM 1989), they are major content areas in elementary school mathematics. Second, research suggests that teachers' knowledge in some of these areas may not be strong. In particular, elementary teachers' difficulties with multiplication and division have been well documented (Ball, 1990a, 1990b; Simon, 1993; Tirosh & Graeber, 1988, 1989). Third, reform documents (NCTM, 1989, 1991) suggest that the teaching of these operations should include connections (to different representations, to concepts both among and within areas of mathematics, and to experiences both in and out of school) and processes (problem solving, communication, reasoning, patterning, etc.) in order for understanding to occur. These connections and processes involve similar components for each operation. Thus, teachers who lack these connections and processes across operations may impact the learning of students differently than those teachers who have such connections and processes.

A set of numbers (whole numbers, fractions, decimals, integers, or rational numbers) on which these operations are to be performed was not specified. This decision was purposeful to avoid possible sources of bias. Not restricting subjects to a predetermined sets of numbers allows for a more open-ended methodology sensitive to the teacher's personal understanding of addition, subtraction, multiplication, and division.

For the purpose of this investigation, "subject matter knowledge," refers to the comprehension of the subject appropriate to a content specialist in elementary school mathematics. "Knowledge structure" means the knowledge a teacher possesses and the manner in which this knowledge is organized (Lederman & Chang, 1997; Lederman, Gess-Newsome, & Latz, 1994). The research definition of knowledge structure is intentionally broad, and it is recognized that it may be more accurate in describing the teacher's knowledge as "conception" of subject matter as opposed to formal knowledge structure. Whether the label "knowledge structure" or "conception" is preferred, such referents should not distract from the primary focus of this investigation: an elementary mathematics teacher's subject matter knowledge and how it relates to teaching and learning.

Differences in knowledge structure refer to variations in the format of the structure or in the breadth and depth of the structure for particular topics within the structure. The breadth of the structure refers to the topics identified and the depth refers to the detail in which topics are developed. Support for the impact of the teacher's conceptual framework of knowledge on teaching and learning is found in cognitive psychological literature. Cognitive research and theory suggests that a teacher's knowledge is organized and stored in structures in the human mind. Schema theory

(Anderson, 1984) provides one such model for the representation and organization of this knowledge. Schemata are abstract knowledge structures that organize and summarize information about many particular cases and the relationships among them. A fundamental assumption of cognitive psychology is that these existing mental structures allow the learning of new information that provides daily guidance with the common-sense "theories" and behavioral scripts needed to interpret the world (Putnam, Lampert, & Peterson, 1990). Anderson (1984) suggests that teachers with "expert" knowledge have well-developed schemata or structures on which to build knowledge, making the transfer and acquisition of knowledge more efficient, and the potential for the translation of their knowledge into classroom practices more likely.

Significance of the Study

This investigation provides insights into how differences in an elementary mathematics teacher's subject matter knowledge structure relate to classroom teaching and student learning. This understanding is important for several reasons.

First, more research is needed to better understand teachers' subject matter knowledge and its relationships to teaching and learning (Lederman, Gess-Newsome, & Latz, 1994). Although subject matter knowledge and its impact on classroom practices is presently the source of much research attention, how differences in subject matter knowledge relate to teaching and learning has yet to be systematically analyzed. A better understanding of this relationship provides a basis for further research to answer questions related to the relative effectiveness of different knowledge structures and whether these structures truly impact classroom teaching and student learning.

Second, further research is needed to guide the design and development of preservice and inservice programs. Reformers (MAA, 1991; NCTM, 1989, 1991, 1995; NRC, 1989, 1990, 1996) suggest the need for professional development programs at all levels, especially elementary, to help teachers become more "competent." Although most educators agree with this recommendation, it is essential that the types of programs and the designs of these programs be informed by research. If these structures do not impact teaching and learning, the identification of teachers' subject matter structures may be interesting, but an unproductive topic for professional development. If, however, impacts are identified, implications for professional development programs exist. For instance, the formation of a teacher's subject matter structure may significantly affect the ability of the teacher to present subject matter and to assist students in constructing their own knowledge of mathematics. If this case is true, the development of a teacher's subject matter structure is an important component in the design of preservice and inservice programs. Such findings should also stimulate research to identify the best means of facilitating the development of subject matter structures in teachers, thereby fostering the transition from novice to expert teachers. In short, given a clear connection, understanding elementary teachers' subject matter knowledge and its potential impact on teaching and learning is an important step in designing successful professional development programs.

CHAPTER II

REVIEW OF THE LITERATURE

Introduction

The purpose of this investigation was to explore how differences in an elementary mathematics teacher's subject matter knowledge structure impact classroom teaching and student learning. In particular, this investigation attempted to answer the following questions:

1. What is the appearance of an elementary mathematics teacher's subject matter knowledge structure of addition, subtraction, multiplication, and division?

2. How do differences in this knowledge structure relate to classroom teaching and student learning?

The characteristics that identify effective teachers have been investigated for more than five decades. During this time, research on teaching has undergone several periods of reform with various definitions of effective teaching and models of mediating variables. Medley (1979) noted that research first focused on identifying effective teachers through their characteristics, then through the methods they used, next through teacher behavior and classroom climate, and finally through their command of a repertoire of competencies. Research has progressed considerably since Medley made his observation. The research has become more sophisticated in the sense that every aspect of teaching has been looked at in great detail and more attention is being paid to the context of the classroom, especially the specific curriculum content and subject matter studied. (Koehler & Grouws, 1992).

Shulman (1986) and Buchmann (1982) were instrumental in recognizing important variables in the study of teaching and learning. Shulman (1986a) explained:

> Since the events we are coming to understanding occur in classrooms and schools, they invariably occur in the service of teaching something. That something is usually capable of characterization as the content of a subject, . . . a particular set of skills, strategies

> processes or understandings relative to the subject matter, or a set of socializations outcomes. The content ought not be viewed as only a "context variable" comparable to class size or classroom climate, the content and the purpose for which it is taught are the very heart of the teaching-learning process (p. 8).

Additionally, Shulman (1986b) formulated a theoretical framework of teacher understanding for the transmission of knowledge by making a distinction between three kinds of content knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. The literature base that pertains to the present study concerns a teacher's subject matter knowledge and the transmission of this knowledge to teaching and learning. According to Shulman (1986b, p. 9), subject matter knowledge is the "amount and organization of the knowledge per se in the mind of the teacher."

In order to better understand the framework for this study, the review of the literature focused on two areas of investigation: the research on teachers' subject matter knowledge of mathematics, and the research on teachers' subject matter knowledge of mathematics and its relationship to classroom teaching.

Subject Matter Knowledge of Mathematics

In recent years, teachers' subject matter knowledge of mathematics has been the focus of much research attention. This portion of the review provides nine studies that explored and described teachers' subject matter knowledge in specific content topics. These topics include: division (Ball, 1990a, 1990b; Simon, 1993), multiplication and division (Graeber &Tirosh, 1988; Tirosh & Graeber, 1989; Zazkis and Campbell, 1995); fractions (Khoury & Zazkis, 1994; Post, Harel, Behr, & Lesh, 1991), and problem solving (Funkhouser, 1993).

As part of her dissertation, Ball (1990a) examined prospective teachers' knowledge of division and their thinking about what provides a "true" or "reasonable" justification in mathematics. The sample for the study consisted of 10 elementary and nine secondary preservice teachers about to enroll for their first education course. The secondary education students were either mathematics majors or minors. The sample was systematically selected to vary with respect to several key criteria: gender, academic history in college mathematics, and self-reported attitudes toward mathematics. Of the 19 prospective teachers, six were males and 13 were females with one black, one Asian, and the others all Caucasian.

Prospective teachers were interviewed to examine their knowledge of division. Three different mathematical contexts were employed: division with fractions, division by zero, and division with algebraic equations. In the interviews, teachers were asked to explain or generate representations on each of the topics. Additional problems were used to examine prospective teachers' ideas about what it means to justify or to prove a concept in mathematics.

An analysis of the interview questions and responses led to the creation of a set of response categories for each prospective teacher. The categories were then modified in the course of data analysis to accommodate the prospective teachers' responses. Responses were coded on two dimensions, their correctness and the nature of the justification provided.

First, the prospective teachers were asked to calculate and provide a "real world" problem for 1 3/4 ÷ 1/2. While all but two of the teachers could calculate the answer correctly, both elementary and secondary prospective teachers had significant difficulty with its "real world" meaning. Only four of the elementary and seven of the secondary prospective teachers were able to generate a "real world" situation to represent the operation. Some of the prospective elementary school teachers even believed that no "real world" situation existed. Three of the elementary and two of the secondary prospective teachers produced representations that did not correspond to 1 3/4 ÷ 1/2. The most frequent error was to represent the problem as 1/2 x 1 3/4 instead of 1 3/4 ÷ 1/2.

Next, the prospective teachers were asked how they would respond to a student who asked them what seven (7) divided by zero (0) is; they were also asked why they would respond that way. Of the 19 prospective teachers, only five were able to explain the meaning of division by zero. Twelve of the prospective teachers responded by stating rules, five of which were incorrect. Seven of the prospective teachers explained that "you can't divide by zero" but, when probed, they could not provide any mathematical justification for why it was so.

Finally, the prospective teachers were asked how they would respond to a student who asked for help with solving the following equation: $x/0.2 = 5$. The prospective teachers were further asked to justify their response. Four elementary majors were unable to solve the equation and only one prospective teacher, an elementary major, attempted to explain its meaning. Fourteen of the participants focused on the procedure of solving for the variable but when questioned, could not provide a conceptual justification. Most of the participants responded that they were not able to solve the equation because they could not remember the procedure. They further explained that it had been a long time since they had taken algebra.

The author concluded that the difficulties experienced by the prospective teachers indicated a narrow understanding of division. Their knowledge appeared to be based on remembering rules and algorithms rather than being able to make meaningful connections. Although, in most cases, the prospective teachers were able to give a "correct" answer using an algorithmic approach, they lacked the conceptual understandings necessary to convey meaning to their students. Ball suggested that most prospective teachers are not taught the conceptual means needed to support their procedural knowledge.

> The present study highlights that relying on what prospective teachers have learned in their precollege mathematics classes is unlikely to provide adequate subject matter preparation for teaching mathematics for understanding. (p. 142)

The prospective teachers want to be able to give students a "correct" and meaningful answer but they lacked the confidence, subject matter preparation, and understanding necessary to do so. Ball further concluded:

> Attending seriously to the subject matter preparation of elementary and secondary mathematics teachers implies the need to know much more than we currently do about how teachers can be helped to transform and increase their understanding of mathematics, working with what they bring and helping them move toward the kind of mathematical understanding needed in order to teach mathematics well. (pp. 142-143)

In another component of Ball's (1988) dissertation, the author (Ball, 1990b) investigated the mathematical understanding and reasoning held by 252 prospective teachers at the point when they entered teacher education. The sample included 217 elementary education majors and 35 secondary mathematics education majors.

To study prospective teachers' mathematical understanding and reasoning, Ball used the concept of division with fractions since it is a central concept in mathematics throughout the K-12 curriculum and it is most often taught algorithmically. The study design was longitudinal. Two instruments, a questionnaire item and an interview task, were used to investigate how prospective teachers understood division with fractions. Both the questionnaire and the interview task were designed to explore preservice teachers' ideas, feelings, and understandings about mathematics, and the teaching and learning of mathematics.

First, all of the preservice teachers in the sample were administered a questionnaire at repeated intervals. The questionnaire item asked participants to select from among a set of four story problems representing the given division statement.

> Which of the following is a good story problem to illustrate what 4 1/4 ÷ 1/2 means? Choose all that apply.
>
> a) A recipe calls for 4 1/4 cups of milk. How much milk is needed for half a batch?
>
> b) It takes 4 1/4 hours to drive 200 miles. How far will we have gone in half an hour?

c) Jim needs 4 1/4 pounds of lentils. How many half-pound bags should he buy?

d) None of these. Instead:_______________________________.

e) I'm not sure. (p. 453)

Second, a subsample of 35 prospective teachers were selected to participate in an interview task, 25 elementary and 10 secondary. The subjects were asked to explain and demonstrate how they were taught to divide fractions using 1 3/4 ÷ 1/2. They were then told to try to provide a picture, model, story, or real world representation of the same problem.

The author explained that both elementary and secondary students had significant difficulty "unpacking" the meaning of division with fractions. On the questionnaire item, only about 30% of the elementary candidates (n = 217) and 40% of the secondary (n = 35) selected the appropriate response. However, 30% of those that selected the appropriate answer also marked one or more of the inappropriate representations. In addition, about 10% of the elementary and about 6% of the secondary candidates selected the "I don't know" option.

The interviews were used to help the researcher understand the reasons for the participants' responses to the questionnaire items. Almost all of the teacher candidates were able to calculate 1 3/4 ÷ 1/2. However, only 40% of the secondary candidates and none of the elementary candidates were able to generate an appropriate representation. Twelve out of 35 prospective teachers generated representations that did not correspond to the problem and 19 out of 35 candidates were unable to generate any representation at all.

The author concluded that the data suggested that the mathematical understandings that prospective teachers have are inadequate for teaching mathematics for understanding. The teacher candidates thought of division only in partitive terms, forming a certain number of equal parts. This meaning of division is not as easy to use with fractions as the grouping model: forming groups of 1/2 out of 1 3/4. Also, few of the teacher candidates were able to write story problems that modeled a situation for the division operation.

The author indicated that implications from this research suggested the need for changes in preservice training to help transform and increase prospective teachers' understanding of mathematics. Since teachers need to understand mathematics themselves if they are to help students understand mathematics, preservice education must address the subject matter preparation of teachers. Further research on teachers' content knowledge and how they transform their knowledge to the classroom was also recommended.

Simon (1993) investigated prospective elementary teachers' knowledge of division. The study focused on two aspects of prospective elementary teachers' mathematical knowledge: the connectedness of their knowledge and their understanding of units.

The subjects consisted of 41 prospective elementary teachers randomly selected from a list of volunteers solicited from a required mathematics methods course. Thirty-three students were selected for the written phase of data collection and eight for the interview phase. Prior to the study, all students had completed the mathematics content portion of their program but had not yet participated in student teaching.

The instruments for the investigation consisted of two types, written responses to problems and an interview. In the first phase, 33 prospective elementary teachers were administered a set of five problems designed to assess two aspects of their knowledge of division, correctness within and between procedural and conceptual knowledge, and knowledge of units. The students were asked to show all work and to write full explanations in response to the problems.

In the second phase, eight prospective elementary teachers were interviewed as they worked on three problems (#3, 4, and 5) from the original problem set. The interviews were used to obtain a more in-depth understanding of the students' thought processes and understandings with respect to connectedness and units.

The written responses were then analyzed following the phenomenographic method. The responses were arranged in groups and the groups were modified until they were judged to characterize the range of responses. Each problem is stated below followed by a brief discussion.

> 1. *Story Problem:* Write three different story problems that would be solved by dividing 51 by 4 and for which the answer would be, respectively:
>
> a) 12 3/4 b) 13 c) 12
>
> You should have three realistic problems. (Simon, 1993, p. 239)

The results showed that prospective elementary teachers had the most success on part (a), 76% correct, and the least success on part (b), 17% correct.

> 2. *Division by a Fraction:* Write a story problem for which 3/4 divided by 1/4 would represent the operation used to solve the problem. (1993, p. 240)

Seventy percent of prospective elementary teachers were unable to create an appropriate problem. Twelve of these students created problems that would be represented by a different number expression. The most common error consisted of writing a story problem for which 3/4 x 1/4 would represent the operation.

> 3. *Calculator Remainder:* How could you find the remainder of 598,473,947 divided by 98,762 by using a calculator? (p. 240)

Only 24% of the students were able to provide at least one valid method of finding the remainder. None of the students were able to generate two strategies.

> 4. *Cookies:* Serge has 35 cups of flower. He makes cookies that require 3/8 of a cup each. If he makes as many such cookies as he has flour for, how much flour will be left over? (p. 241)

Only 15% of the students were able to provide a correct solution. Thirty percent of the students claimed that there was 1/3 of a cup of flour left over and 30% had other solutions that were incorrect.

> 5. *Long Division:* In long division carried out as in the example below, the sequence divide, multiply, subtract, bring down is repeated. Explain what information the multiply step and the subtraction step provide and how they contribute to arriving at the answer. (p.241)

$$\begin{array}{r} 59 \\ 12\overline{)\ 715} \\ \underline{-600} \\ 115 \\ \underline{-108} \\ 7 \end{array}$$

None of the students were able to explain what information these steps provided. Their responses showed only an algorithmic knowledge. They lacked the understanding of the long division algorithm.

The author reported that the interview data provided a clearer picture of the mathematical knowledge of the students. It confirmed the results found in the written responses and also provided insights into many of the misunderstandings of the students. In some cases, a student's lack of understanding was further revealed by the probing in the interview, and in other cases, the interview process allowed the student to develop an appropriate response. Dawn, for example was unable to offer more than a procedural explanation of the long division algorithm. Asking Dawn to create a word problem solved by 715 divided by 12 was not enough to help Dawn make sense of the problem. Like Dawn, Jane was initially unable to provide meaning for the steps in the division algorithm. However, a real-world context and probing by the interviewer allowed Jane to develop an interpretation of the numbers generated by the long division algorithm.

The author concluded that the prospective teachers showed serious shortcomings in their understanding of division. They seemed to have procedural knowledge of the symbols and algorithms associated with division, but lacked conceptual understanding. Many of the important connections seemed to be missing leaving prospective teachers with a 'sparse web' of mathematical knowledge.

Simon (1993) suggested the need for conceptually-based preservice mathematics courses. These courses should provide students with not only a concrete, contextualized knowledge of division, but also the following: connections between and among concrete situations, symbolic representations, computational procedures, and abstract ideas; an awareness of and connection between the two different types of division (Sharing or partitioning by dividing a collection of objects into a given number of equal parts and grouping or splitting a collection of objects into groups of unknown size.); and an understanding of referential aspects of division. Additional research is also needed to investigate the development of knowledge, beliefs, and attitudes of prospective teachers, and the impact their knowledge, beliefs, and attitudes have on classroom practices.

Multiplication and division were explored in two studies by Tirosh and Graeber (1988, 1989). The first study (1988) investigated preservice elementary school teachers' knowledge and beliefs about multiplication and division with decimals. The following seven questions were addressed in the investigation:

> 1. Do the primitive models of multiplication and division influence preservice teachers' performance in solving word problems?
>
> 2. Are there other apparent differences between the skills or knowledge of preservice teachers who were less successful in solving word problems and those who were more successful?
>
> 3. Do preservice teachers use the primitive partitive and primitive measurement models of division with equal facility?
>
> 4. Are preservice teachers' beliefs about multiplication and division implicit or explicit?
>
> 5. What seems to support preservice teachers' beliefs, reliance on the primitive models, and the related misconceptions?
>
> 6. What appear to be promising strategies for helping preservice teachers overcome their misconceptions?
>
> 7. What are the implications for teacher education? (pp. 263-264)

The sample for the study consisted of 129 preservice elementary teachers. Two instruments were used to collect information about the students: a questionnaire and an interview.

The analysis of the data showed that 99% of the students were able to write correct expressions for a multiplication word problem solved with a whole number operator greater than one (for example, 15 x 2.25), but only 72% were able to write a correct expression for a similar multiplication problem solved with a decimal operator (for example, 1.25 x 15). The results were even worse, only 59%, when the decimal operator was less than one (0.75 x 15). The author reported that the students' performances appeared to be influenced by a primitive (repeated addition) model of multiplication.

The data on division word problems indicated similar difficulties. Although 98% and 89% of the students were able to write correct expressions for partitive word problems solved by 75 ÷ 5 and 96 ÷ 8, respectively; only 51% and 34% were able to write correct expression for partitive word problems solved by 5 ÷ 15 and 5 ÷ 12. The most common errors for the latter operations were ones in which students providing problems that would be solved by the division expressions 15 ÷ 5 and 12 ÷ 5, respectively. The results indicated that the students were more successful writing expressions for word problems that contained division by a whole number greater than the dividend than for those that contained division with a greater divisor than dividend.

Following the paper-and-pencil questionnaire, 33 of the students, including 10 of the highest and 10 of the lowest scores, were interviewed about the expressions they wrote for word problems and the beliefs they held about multiplication and division. Graeber and Tirosh (1988) reported that, after reviewing the word problem expressions, each of the preservice teachers was given additional word problems similar to those on the original paper-and-pencil test.

The author reported that in the interview it was found that students who had scored well on the written questionnaires tended to have more confidence and ability to express their thinking, use a variety of methods, and check their results with the original problem. Students who scored lower, on the other hand, were less confident and unable to determine the reasonableness of their answers. When asked to check their answers, students who scored lower merely checked their computations; they did not check the reasonableness of their solutions. The results of the interviews confirmed the idea that the primitive model for multiplication (repeated addition) and the primitive partitive model for division (dividing a collection of objects into a given number of equal parts) had sustained influences on preservice teachers' performances on these word problems.

As a result of the study, the following implications for preservice teachers were suggested by the authors. Teacher educators must bring preservice teachers to an awareness of their misconceptions and the effect that their misconceptions have on their performance. Teaching techniques need to be used that assist preservice teachers in building a conceptual knowledge of multiplication and division. Problem solving strategies need to be encouraged, and opportunities need to be provided for preservice teachers to explore explicitly the different models of multiplication and division.

In the second study by Tirosh and Graeber (1989), two common misbeliefs about multiplication and division were investigated as well as the sources for the misbeliefs. The study was designed to assess the extent to which the beliefs, "multiplication always makes bigger" and "division always makes smaller," are explicitly held by preservice elementary teachers.

The sample consisted of 135 female students and one male student enrolled in a mathematics method course. The majority of the students were in their third year of university study and had completed at least two mathematics content courses.

The students responded to a paper-and-pencil instrument that included the following six statements:

A. In a multiplication problem, the product is greater than either factor.

B. The product of .45 x 90 is less than 90.

C. In a division problem, the quotient must be less than the dividend.

D. In a division problem, the divisor must be a whole number.

E. The quotient for the problem 60/.65 is greater than 60.

F. The quotient for the problem 70 ÷ 1/2 is less than 70. (p. 81)

The students were asked to label each statement as "True" or "False" and to justify their responses.

Students were reminded of the relationship between quotient, divisor, and dividend prior to answering the questions. Data were also collected on the students' computational skills and on their performances in writing expressions to solve word problems. Two of the exercises, 0.38 x 5.14 and 3.75 ÷ 0.75, provided counter examples to the beliefs under discussion.

About half of the students were interviewed to obtain additional information about their conceptions of multiplication and division. In the interview, the students wrote expressions to solve multiplication or division problems similar to those they had missed on the written word problem instrument. They also explained the logic they used to solve the problems.

The results from both the paper-and-pencil questionnaire and interview instruments suggested that 87% of the students responded correctly to both of the multiplication statements related to the misconception "multiplication always makes bigger" and only 3% of them responded incorrectly to both statements. Although, only 13% of the students explicitly held the misbelief that "multiplication always makes bigger," the data from the interviews suggested that many of them still agree with the statement.

On the four statements related to the misbelief "division always makes smaller," 28% responded correctly to all four of the statements and 3% responded incorrectly to all four. The majority of the students responded incorrectly to statement C, the statement that most closely paralleled the misbelief. This misbelief was also evidenced in the interviews where 45% of the students wrote multiplication expressions for the division word problems with decimal divisors less than one. The question was: "Girls club cookies are packed 0.65 pounds to a box. How many boxes can be filled with 5 pounds of cookies (p. 84)?" Fourteen of the students wrote either 0.65 x 5 or 5 x .65.

Tirosh and Graeber (1989) concluded that the responses students made to the statements of belief about the operations indicated that their conceptual understanding of multiplication was frequently expressed in terms of the repeated addition model and their understanding of division in terms of the primitive partitive model. Furthermore, the discrepancies found among the students' performances on different belief statements, and between their performance on computational exercises and the related belief statements, may be explained by their reliance on procedural knowledge that dominated or, at least was not linked to, correct conceptual knowledge.

Implications of this study indicated that a substantial percent of the students were influenced by misconceptions about multiplication and division. The authors suggested that teacher training programs are needed that provide insights into the status, sources, and support for preservice teachers misbeliefs. Also, instructional strategies need to be developed that can be used in changing students misbeliefs about multiplication and division.

Zazkis and Campbell (1995) investigated preservice elementary school teachers' knowledge of number theory concepts. The objectives of the study were the following:

1. to explore preservice teachers understanding of elementary concepts in number theory with emphasis given to concepts involving divisibility and the multiplicative structure of non-negative integers;

> 2. to analyze and describe cognitive strategies of solving unfamiliar problems involving and combining those concepts;
>
> 3. to adapt and extend a constructivist oriented theoretical framework for the analysis and interpretation of those strategies and the cognitive structures supporting them. (Zazkis & Campbell, 1995, p. 2)

The sample for the study consisted of 21 preservice elementary school teachers who volunteered from those enrolled in a course called "Foundations of Mathematics for Teachers." Data for the study were collected through individual clinical interviews with the preservice teachers. The questions were designed to clarify participants' understandings of procedures and concepts related to divisibility and to investigate their ability of make connections and inferences from them. The questions were as follows:

Questions Set 1

Consider the number M = 33 x 52 x 7.

Is M divisible by 7? Explain.

Is M divisible by 5, 2, 9, 63, 11, 15? Explain.

Questions Set 2

(a) Is 391 divisible by 23?

(b) Is 391 divisible by 46?

(c) What is the next number divisible by 23?

(d) How many positive numbers smaller than 391 are divisible by 23?

Questions Set 3

Consider the numbers 12358 and 12368.

Is there a number between these two numbers that is divisible by 7? by 12?

Questions Set 4

(a) The number 15 has exactly 4 divisors. Can you list them all? Can you think of several other numbers that have exactly 4 divisors?

> (b) The number 45 has exactly 6 divisors. Can you list them all? Can you think of several other numbers that have exactly 4 divisors? (Zazkis & Campbell, 1995, p. 4-5)

The participants were presented with either part (a) or part (b) of this question set depending on their acumen with previous questions.

The authors reported that the finding of this study support the claim that preservice teachers' content knowledge is weak and their conceptual understanding is insufficient in some areas to teach arithmetic at even the elementary school level. A significant percentage of the participants experienced difficulty grasping aspects of mathematical definitions associated with number theory concepts. A frequent claim was that 3 is a multiple of 18, since "you multiply 3 by 6 to get 18 (Zazkis & Campbell, 1995, p. 21)". The participants also had difficulty with the understanding of divisibility in terms of both multiplication and division. The two definitions were a source of conceptual conflicts and confusion to most participants. Furthermore, the majority of participants were not able to discuss divisibility as a property of numbers without performing division. They claimed "you'd have to try to see if it works" or "you cannot be sure that the results is a whole number if you don't know what the result is (Zazkis & Campbell, 1995, p. 6)."

Another difficulty participants had was doing "reversed tasks." The participants found it easier to check whether an object had a certain property than to construct an object that had such a property. Most participants resorted to a "guess and check" strategy to answer the questions in set four.

The authors concluded that the improvement of mathematics education must start with the improvement of mathematical knowledge of teachers. Additionally, conceptual understanding of divisibility and factorization is essential in the development of conceptual understanding of the multiplicative structure of number and the generalizing of such concepts to the study of algebra.

Khoury and Zazkis (1994) investigated preservice teachers' knowledge of fractions by examining their reasoning strategies and arguments given as preservice teachers solved two problems regarding fractions in different symbolic representations. The sample consisted of 100 preservice elementary school teachers and 24 preservice secondary school teachers in their junior or senior year of study. The students had previous experience with whole number representations in different bases but were unfamiliar with the idea of non-integer rational number representations in bases other than ten.

The assessment of students' knowledge was conducted in two parts: written, and clinical interviews. The following two items were administered:

Item 1: Is $(0.2)_{\text{three}}$ equal to $(0.2)_{\text{five}}$?

Item 2: Is the number "one-half" in base three equal to the number "one-half" in base five? (Khoury & Zazkis, 1994, p. 192)

For each of these items, students were asked to explain their decision and, in case of inequality, to choose the larger number. For the first part of the assessment, 124 preservice teachers were asked to respond in writing to each item, to show their computational work, if any, and to provide written explanations for their reasoning. The students' computational work and justifications were analyzed and their explanations and strategies were identified. For the second part of the assessment, a subset of 38 students, that were reported to represent roughly equally various strategies identified in the first part, were asked to discuss the assessment items in individual interviews. The students' protocols were analyzed in order to validate the reasoning strategies identified in the first part of the assessment.

The analysis focused on the identification of the most common explanation arguments or reasoning strategies used. The frequencies of correct performances on the first item were 63 out of 100 elementary education majors and 24 out of 24 secondary mathematics education majors performed correctly. On this item most of the students claimed first that the two numbers were not equal and then used a computational strategy to validate their response. They converted each of the number representations to a decimal fraction or a common fraction and then compared both numbers. On the second item, 26 out of 100 elementary education majors and only 4 out of 24 mathematics education majors performed correctly. While all the mathematics education majors performed correctly on the first item, their percentage of correct responses on the second item was low and even lower than the percentage of correct responses of the elementary education majors. Computational arguments were less frequently used on item two, possibly due to the fact that representing "one-half" in odd bases is not a trivial mathematical task.

The results of the interviews confirmed some understandings and also demonstrated further misunderstandings from the written portion of the assessment. A common correct response in the interviews to justify item one was:

> From the beginning I knew that they couldn't be equal because they are different bases . . . and they are the same number . . . I mean digits . . . [she drew a place value chart] . . . "two times one-third" would be two-thirds, and now I have to compare it to "two times one-fifth" which would be equal to two-fifths. So, 0.2 in base three is greater than 0.2 in base five. (p. 195)

Inadequate responses also surfaced on item one even though the students had applied a correct algorithmic conversion. One example was:

> I: What about if 0.2 is in base 3?
>
> S: . . . [drawing a column chart, and trying to figure out] . . . Well, like in base 10. . . . but I just have a hard time thinking of it in base 3, because, first of all, I don't really know what the columns are called; and so I can't really say that this is "two-tenths" anymore . . . maybe I have to say it's "two-thirds". . . .
>
> I: Okay, and what makes you think of it as "two-thirds". . .
>
> S: Wait, this column may be called the 'three' column, and then, this one is called the "thirtieth,"... and the next one is called the "three hundredth." Well, if I use this same philosophy that I've just used for base 3, then I would have to say "one-fifth". . . "one-fiftieth". . . "one-five hundredth" . . . (p. 195)

A common correct response to justify item two was: "One-half in base three is equal to one-half in base five because one-half is a half of a whole regardless of the different bases used (p. 198)." An example of an inadequate response to item two was:

> S: One-half in base three is less than one-half in base five, because the bases are different. Because in base three you have less numbers, so your one-half is going to be a different answer than in base five that has more numbers . . .
> I: Do you mean by 'answer' the quantity one-half, or the way you write it in base three?
>
> S: . . . they're the same . . . now, I'm confused. (p. 202)

The results were reported to indicate that preservice teachers' knowledge of place value and rational numbers is not conceptual. The fact that the "simplest possible" fraction (one-half) was misunderstood by so many of the students indicated that a high percentage of preservice school teachers have a disconnected knowledge of place value, decimals, and fractions.

Post, Harel, Behr, and Lesh (1991) investigated intermediate elementary school teachers' conceptual knowledge of rational numbers. The purpose of the study was to generate profiles on teachers' conceptions of rational number problems and to determine the adequacy of their explanations. They intended to use the profiles to create teacher training materials.

The sample consisted of 218 intermediate level teachers who were currently teaching mathematics in grades 4, 5 or 6 and who were required to participate in the project. Sixty-seven teachers were selected from a large urban district in Minnesota and 51 teachers from a small rural district in Illinois.

Teachers' profiles were developed using three assessment instruments. Instrument one included two long test versions (A and B) which consisted of 78 short answer items and two shorter versions (AA and BB) contained 58 items each. Teachers were given 75 minutes to complete instrument one. The multiple versions of the tests had some items in common and were used in order to gather a wide variety of information. All versions contained items dealing with fundamental concepts about fractions and decimals and 17 items were one-step multiplications and division problems.

Instrument two of the assessment instrument required teachers to provide explanations to six rational number problems. Teachers were requested to provide as much information as possible relative to their thought processes and solution procedures, and also, to indicate how the information would explained to children. The problems were adapted from a previous study with children. The problems were not typical of those that would appear in the intermediate mathematics curriculum, but rather, would be found at the junior high level. Instrument three of the teacher profile consisted of a two hour structured interview. This part was related to instruments one and two. Fifteen teachers were selected from each third of the distribution of scores on instrument one.

Although some teachers did well, overall mean scores on instrument one of less than 70% were reported. Ten to 25% of the teachers missed items that were at the most rudimentary level. In some cases, almost half the teachers missed fundamental items such as $1/3 \div 3$. Regardless of which item category was selected, a significant percentage of the teachers missed one-half to two-thirds of the items. In general, 20 to 30 percent of the teachers scored less than 50% on the overall instrument.

On instrument two, less than half (47.7%) of the teachers were able to solve the following problem:

> Marissa bought 0.46 of a pound of wheat flour for which she paid \$0.83. How many pounds of flour could she buy for one dollar? (p. 193)

In addition, only 10.5% of those able to solve the problem correctly provided what was considered to be an "acceptable" explanation.

The authors concluded that a multilevel problem associated with teachers' subject matter knowledge exists. First, many teachers do not know enough mathematics. Second, only the minority of those teachers who are able to solve these types of problems correctly are able to explain their solutions in a pedagogically acceptable manner. Implications of this study included the need for effective preservice and inservice programs, alternative delivery systems, curriculum reform, and the possibility of using computer-assisted instruction.

In another study, Funkhouser (1992) investigated inservice teachers' conceptualization of problem solving by compiling and analyzing their responses to the following question:

> You have just had a school-wide meeting with your principal (superintendent, curriculum coordinator). You have been informed that you are to teach problem solving in your mathematics classes as if your job depended on it (it will). What is meant by problem solving? (p. 81, 82)

The question was intentionally open-ended to allow for a variety of answers and to encourage participants to personalize their responses.

The sample for the study included 180 teachers who responded to the question out of approximately 230 who attended the inservices. The question was presented to classroom teachers who attended one of seven inservices in mathematics conducted over a period of one year. The settings for these inservices include: school sites, regional and national meetings of professional teaching associations, and university campuses. Participants at each of the inservices were given five to ten minutes to submit anonymous, written responses to the question posed. Following each inservice, some of the participating teachers volunteered for a debriefing session.

Responses were divided into two categories: vague or precise. A vague response was defined as a definition of problem solving that was circular in nature or tended to use vocabulary in an unclear or ambiguous manner. A precise response was defined as a definition that cited examples, suggested a theoretical basis, and demonstrated a clear understanding of vocabulary related to problem solving. Within the vague category, two subcategories were defined: conceptually vague and terminologically vague. Definitions in the conceptually vague subcategory used the vocabulary of problem solving seemingly without understanding or inappropriately. Definitions in the terminologically vague subcategory used vocabulary associated with problem solving but with no demonstrated understanding of the vocabulary.

Of the 180 teacher participants, 122 teachers responded with a vague definition of problem solving. Of these teachers, 72 were subcategorized to be conceptually vague. This group comprised more than one-third of the total number of teachers. Types of responses in this subcategory included: problem solving is finding a solution to a problem, problem solving is getting an answer to a problem, and problem solving is coming up with a solution to a problem. These teachers were considered to have the least basic understanding about the nature of problem solving.

Fifty of the responses (approximately one-fourth) were placed in the terminologically vague subcategory. Types of responses in this subcategory included: problem solving is knowing a variety of strategies to solve situations or problems, and problem solving is using logic.

The 58 teachers who responded with a precise problem solving definition were further divided into subcategories. Twenty-one of the responses were labeled strategy-based because they used Polya's strategies method, 16 of the responses were labeled skill-based because they used the word "skill" in the definition and then illustrated the skill or gave an example, and the remaining 21 responses were labeled other-based since they did not fall in either of the other subcategories.

Funkhouser (1992) concluded that a large percentage of teachers currently teaching mathematics at some grade level, lack an understanding of basic problem solving concepts and vocabulary. The following recommendations were suggested for teacher training and inservice mathematics instruction: clearer, more explicit definitions of problem solving; more concrete examples of how to apply problem solving models; and more precise use of, and more frequent practice with, problem solving and problem solving vocabulary. It was also suggested that more research is needed on learning and teaching problem solving and its impact on classroom teaching.

Subject Matter Knowledge in Classroom Teaching

Although much research exist concerning teachers' subject matter knowledge of mathematics, only a few researchers have investigated the relationship between this knowledge and classroom teaching. This portion of the review represents three studies that take a closer look at this relationship. The first two studies by Leinhardt and Smith (1985) and Lehrer and Franke (1992) reported the knowledge and organization of knowledge of expert and novice teachers within the domain of fractions and it relationship to classroom teaching. The third study, by Thompson and Thompson (1994), investigated how one teacher's subject matter knowledge of mathematics was reflected in the language he used in teaching the concept of rate.

Leinhardt and Smith (1985) investigated the lesson structure knowledge and subject matter knowledge of expert elementary mathematics teachers and its relationship to classroom behavior. Through the use of semantic and planning nets, the teachers' in-depth knowledge of one topic, fractions, was explored as it occurred in natural teaching settings.

The sample for the study consisted of eight fourth-grade mathematics teachers, four experts and four novices. The expert teachers were selected from a subsample of 12 teachers who had previously participated in another study by Leinhardt (1983) on expert teachers. The expert teachers were selected because their students' scores in mathematics had shown unusual and consistent growth over a five-year period. The novices were student teachers who were in their final year of a teacher training program and who were highly recommended by their supervisors.

Extensive data were collected on the teachers during the first two years of the study. The data were collected through observations, interviews, and a card sort task. The observations occurred over a three month period during each of the first two years of the study. A total of 10 hours of videotaped lessons were collected on each teacher. The teachers were also interviewed on several topics including: taped lessons, planning of the lessons, and knowledge of fractions. In addition, both the teachers and student teachers were given a card sort task consisting of 40 mathematics problems randomly selected from the computational sections of fourth-grade mathematics textbooks.

Two types of analyses were conducted. First, the fraction interviews and card sorts were assessed to determine any consistent patterns of knowledge and understanding as well as any patterns of confusion and misunderstanding. Next, three of the teachers, two considered to have high knowledge and one with middle level knowledge were examined more closely through videotapes of their lessons. The teachers' lessons were taught using the same text, on the same topics, and in approximately the same sequence.

Results of the card sort indicated considerable differences between the high knowledge experts and the novice teachers. High knowledge experts sorted the mathematics topic card into 10 categories and ordered the topics by difficulty to teach or perform. They also grouped the operations of addition and subtraction together. The novices generally made categories for every one or two problems and indicated few internal connections. They also indicated little differentiation in problem difficulty. The authors suggested that, in general, the expert teachers exhibited a more refined hierarchical structure to their knowledge.

Similar distinctions appeared in the interviews with respect to the four items on fractions. The first question required the participants to define a fraction. Seven of the teachers used the relationship of part to whole in their definition. The other teacher defined a fraction as being a point between zero and one or zero and any other whole number, inclusive. This teacher was also the only one to consistently use the number line as a frame for lessons and the only one who saw fractions as having a measurement property.

The second question asked participants to define equivalent fractions. All teachers were able to give a correct definition, but when questioned about the equivalence of 3/7 and 243/567, one expert and two novice teachers stated that the fractions were not equivalent. The less knowledgeable teachers tended to get the number 81 as a factor and then say either that the fractions were not equivalent or that they did not know. In contrast, the two teachers who had greater mathematics skills immediately said the equivalence and reported it.

The third item involved the concept of unit. When asked to draw pictures that represented the fractions 3/4, 5/5, and 5/4, all but one of the teachers did so successfully. They were also asked to indicate the units for each of the fractions, 3/4, 5/5, and 5/4.

The fourth item concerned two concepts: ratio of a set and fraction of a set. Teachers were asked if there were any differences between ratio of a set and fractions of a set. All of the teachers either said that a fraction and a ratio were identical or similar, or said they did not know. This result was unexpected by the researchers.

According to Leinhardt and Smith (1985), the overall analysis of the card sort tasks and the interviews indicated differences among the four experts' subject matter knowledge of mathematics. In spite of high levels of student success for all of these teachers, two of the experts had high math knowledge, one had moderate knowledge, and one had "barely sufficient math knowledge for classroom instruction" (Leinhardt & Smith, 1985, p. 254). The four novices were judged to have moderate to low math knowledge.

An analysis of the videotaped lessons of the three experts having similar knowledge revealed substantial differences in the details of their presentations to students. First, considerable differences existed in the level of conceptual information presented as well as the degree to which algorithmic information was presented. Second, the teachers had decidedly different emphases in the presentations. While one teacher approached reducing fractions by using the identity element, two of the other teachers approached the topic through a contrast with finding equivalent fractions. Finally, there were substantial differences in their uses of different representations of fractions such as number line models, regional models , and numerical models.

The authors reported that findings in this study revealed substantial differences in subject matter knowledge of expert teachers and novice teachers on the topic of fractions. In general, the expert teachers had a more refined and deeper understanding even though one of the expert teachers' knowledge of fractions was reported to be barely sufficient for classroom instruction. The analysis of mathematical content and relationships of three of the expert teachers lessons showed different degrees of knowledge, varied approaches, and different emphasized aspects of the topic in their teaching of fractions. While all three of these teachers were judged to have high levels of student success, they differed greatly in their ability to teach. The expert teacher that was assessed as having insufficient knowledge of mathematics was not included in this portion of the study.

The authors further stated, "as teachers increase their subject matter knowledge and become more fluid in connecting their knowledge to lesson presentation their students' mathematical competence should also improve (p. 270)." The suggested implications of this study were that teachers and textbooks need to provide more complete descriptions of the concepts and relationships in the domain of fractions.

Personal construct psychology was used by Lehrer and Franke (1992) to provide a theoretical and methodological framework to examine the interaction of two teachers' subject matter knowledge of fractions and classroom teaching. The sample for this study consisted of two teachers who were selected because they had been observed, prior to the beginning the study, to be clearly different in their teaching practices. Ms. Hunter, a second grade teacher, had 17 years teaching experience and had received the Presidential Award for teaching of elementary mathematics. Ms. Hunter consistently focused on problem solving in her classroom. She generally displayed a highly improvisational form of teaching often posing problems to her students and listening to their solutions with an eye to understanding. In contrast, Ms. Gardner, a fifth-grade teacher with five years teaching experience, generally followed the order of the textbook in posing problems and providing examples. The authors defined Ms. Hunter and Ms. Gardner, respectively, as more-skilled and less-skilled in the practice of teaching of mathematics.

The research questions were as follows:

> 1. Does personal construct psychology provide a means to elicit the various components of teacher knowledge found in other research?
>
> 2. Are there conditional relationships among the components of a teacher's knowledge?
>
> 3. Is there any relationship between the portrait of teacher knowledge obtained within the personal construct framework and teaching actions in the classroom? (p. 225)

Data were collected through interviews and observations. In the interviews, teachers were presented with several activities. First, the teachers were presented three fraction problems and asked to identify which two problems were more similar to each other yet different than the third in terms of content, how students think about the problem, and pedagogical actions. Next, teachers were presented with 10 triads that included 12 fraction problems. The focus of the triads was to delineate the teachers' notions of fractions related to identification, representation, order, equivalence, and operations. The teachers were then probed for content knowledge through questions such as the following. How are the problems the same or different? Which two problems are alike? Why? Are these two problems different in terms of how students would solve them?

After the presentation of the 10 triads, the teachers were shown a list of their constructs. The constructs were then discussed to be sure the teachers understood the construct as written. Teachers were asked to rate each construct on a 10-point scale. A resulting grid of ratings was formed and analyzed to determine relationships among problem types and teachers' constructs. After the elicitation of constructs, each teacher was observed by one of the researchers on a day when the teacher was teaching a typical lesson on fractions.

Grids were created from the constructs of each teacher. The constructs were classified according to: content knowledge of fractions, general pedagogical knowledge (e.g., concrete materials needed), pedagogical content knowledge associated with the teaching of fractions (e.g., use a related fraction), and cognitional knowledge (e.g., student must have prerequisite knowledge of whole numbers).

The results of the study suggested that wide variability existed between the two teachers' responses to the fraction problem triads. Ms. Hunter provided a total of 33 constructs while Ms. Gardner provided only 18. Ms. Hunter's constructs generally focused on pedagogy as related to the teaching of fractions. About 305 of her constructs were classified as cognitional knowledge. None of her elicited constructs related to the teaching of algorithms or procedures.

Ms. Gardner provided constructs about the concepts underlying fractions and algorithms, and procedures for solving fraction problems. The nine general pedagogical constructs that Ms. Gardner reported included constructs that were applied to all of the problems discussed as well as the pedagogical content constructs related to a specific problem. None of her elicited constructs were classified as cognitional.

In the observation, Ms. Gardner introduced subtraction of fractions with like denominators. Ms. Gardner's actions were reported to be consistent with the constructs elicited. Her focus was on fractions as part of a whole as she indicated with her constructs. She drew pictures and had the students draw pictures, focusing on the fact that the denominator of the fraction determined the number of parts and the same amount of parts as given to make a whole. Whether working with the whole class, small groups, or individuals, her response to children who were having difficulty was to attempt to provide her explanation. She did not necessarily build on the earlier understandings of the children.

Ms. Hunter's lesson was on dividing a number of objects into different fractional parts. The number of objects they worked with built on the number of days they had been in school. Overall, her lesson was student-centered. Ms. Hunter used manipulatives, started with familiar fractions, used knowledge of related fractions, provided pictures and symbols, encouraged verbal interactions and focused on the understanding of larger versus smaller fractions. These activities were all consistent with the constructs elicited as part of Ms. Hunter's knowledge structure.

The authors concluded that personal construct psychology offers a coherent and consistent framework for investigating the interactive roles played by the multiple constructions teachers place on classroom events. This approach to case studies allowed the researcher to elicit teachers' constructs and analyze them in relation to classroom practices. The conditional relationships among the components of teachers' knowledge were echoed in their classroom observations. Ms. Gardner's knowledge was considered to be less "tuned" than Ms. Hunter's. Ms. Gardner confined herself to the presentation of material in a textbook while, on the other hand, Ms. Hunter's lesson was free flowing and presented in a context that the students understood.

Thompson and Thompson (1994) investigated how one teacher's knowledge of mathematics was reflected in the language he used in teaching the concept of rate. The sample for the study was one teacher, called "Bill," who taught mathematics in grades six through eight. At the time of the study, Bill was in his second year of teaching at the middle school level. He had taught high school physics, chemistry, and physical science for six years before teaching at the middle school. Prior to teaching, Bill had worked in business and when he retired in 1986, became interested in teaching. Furthermore, Bill was reported to be adept at problem solving and reasoning proportionally.

The purpose of the study was to examine Bill's way of knowing mathematics and how it was reflected in the language he used in teaching the concept of rate to one sixth grader, called "Ann", during a two day teaching experience. The focus of the teaching experience was to investigate the student's construction of the concept of speed and rate and the relationship between the student's concept of speed and rate. It was reported that the one-on-one teaching experience revealed Bill's difficulty with speaking conceptually about rate. Moreover, Bill was not able to deal with Ann's difficulty with this concept. His explanations were algorithmic in nature and failed to get at the conceptual understandings that Ann lacked. Bill's problems during the teaching experience were further illustrated with his difficulties with the language necessary to facilitate Ann's conceptual grasp of the situation.

The author concluded that although Bill's own conceptualization of rate appeared strong and elaborate, it was encapsulated in the language of numbers and operations, and these numbers and operations undermined his efforts to help Ann understand rates conceptually. Furthermore, Thompson and Thompson (1994) indicated that these results, as in other studies, failed to show a clear link between what the teachers knows and students learn. The authors suggested that more research is needed to expand three issues: what it means for the teacher to have a conceptual understanding of an idea, how those images might be expressed in discourse, and what benefits might accrue to students by addressing the conceptual sources of students difficulties.

Discussion and Conclusions

The review of the literature for this study focused on two major areas: teachers' subject matter knowledge of mathematics, and teachers' subject matter knowledge of mathematics and its relationship to classroom teaching. From the studies reviewed, several patterns emerged.

From the review of research of teachers' knowledge in specific content areas such as division (Ball, 1990a, 1990b; Simon, 1993), multiplication and division (Graeber &Tirosh, 1988; Tirosh & Graeber, 1989; Zazkis and Campbell, 1995); fractions (Khoury & Zazkis, 1994; Post, Harel, Behr, & Lesh, 1991), and problem solving (Funkhouser, 1993), there is strong evidence that the subject matter knowledge of teachers, especially elementary teachers, is not strong. These studies indicated that many elementary teachers have incomplete understandings of mathematics. Their knowledge appears to be founded on remembering rules and algorithms rather than on understanding concepts and being able to make meaningful connections. Prospective teachers, in most problem situations, were able to use algorithms to get "correct" answers, but lacked the conceptual understandings necessary to communicate meaning to their solution.

In particular, teachers showed serious shortcomings in their understanding of mathematical operations. Teachers were unable to demonstrate conceptual understanding or make connections between models associated with these concepts such as connections between real world, concrete, pictorial, and symbolic representations. They were also unable to make connections between conceptual and procedural knowledge (Ball, 1990a, 1990b; Graeber &Tirosh, 1988; Post, Harel, Behr, & Lesh, 1991; Simon, 1993; Tirosh & Graeber, 1989).

Furthermore, these studies suggested that it is unlikely that precollege or university mathematics classes provide adequate subject matter preparation. More professional development programs are needed to help teachers increase their understanding of mathematics and ability to do mathematics. Teaching techniques need to be used that assist teachers in building conceptual knowledge, confronting their misunderstanding, and developing their abilities to problem solve, reason, and communicate ideas effectively (Ball, 1990a, 1990b; Graeber &Tirosh, 1988; Post, Harel, Behr, & Lesh, 1991; Simon, 1993; Tirosh & Graeber, 1989).

From studies of teachers' subject matter knowledge of mathematics and teaching, several patterns were apparent of preservice teachers' learning-to-teach experiences. In these studies, preservice teachers' with limitations in subject matter knowledge had difficulty making transitions to pedagogical thinking, were unable to connect topics during classroom instruction, and focused on procedural rather than conceptual understanding (Lehrer & Franke, 1992; Leinhardt & Smith, 1985). These patterns in novice teachers' suggest that many teachers entering teaching lack the in-depth subject matter knowledge needed to teach mathematics in the manner recommended by current curriculum reform documents (MAA, 1991; NCTM, 1989, 1991, 1995; NRC, 1989, 1990).

Studies of practicing teachers suggest that a sound knowledge of mathematics is neither a necessary condition for students' learning nor a sufficient condition for effective teaching. However, some teachers having sound knowledge of mathematics were able to respond appropriately to students' questions, design appropriate learning activities involving a variety of mathematical representations, and orchestrate mathematical discourse in the classroom (Lehrer & Franke, 1992; Leinhardt & Smith, 1985). Thus, it appears that a sound knowledge of mathematics does not ensure effective teaching and student learning. However, if teaching is a purposeful act, then teachers, clearly, cannot teach what they do not know (Ball, 1988).

Recommendations

The results of these studies highlight three important areas of concern. In recent years, elementary mathematics teachers' subject matter knowledge has been the focus of much research attention. These studies have explored and described teachers' knowledge on topics in division (Ball, 1990a, 1990b; Simon, 1993), multiplication and division (Graeber &Tirosh, 1988; Tirosh & Graeber, 1989; Zazkis and Campbell, 1995); fractions (Khoury & Zazkis, 1994; Post, Harel, Behr, & Lesh, 1991), and problem solving (Funkhouser, 1993). Although these studies have provided important insights into teachers' knowledge of mathematics and its impact on classroom teaching, many of the methods employed in these studies contain inherent limitations. First, relatively few studies have avoided the pitfalls of limiting the assessment of teachers' knowledge to only a few topics. Since understanding is essential to knowing mathematics and the degree of understanding is determined by the number and strength of the connections made (Hiebert & Carpenter, 1992), an overall understanding of teachers' knowledge may not be achieved by attention to such a narrow focus of content (Flennema & Franke, 1992).

Second, in most studies, teachers were asked to solve problems suggested by the researchers (Ball, 1990a, 1990b; Khoury & Zazkis, 1994; Post, Harel, Behr, & Lesh, 1991; Simon, 1993) or to perform card sorts (Stein, Baxter, Leinhardt, 1990; Scholz, 1996) to demonstrate their knowledge. Although the data yielded were qualitative in nature, restricting subjects to predetermined sets of topics may have compromised the benefits and purpose of using a qualitative research design. A more open-ended methodology sensitive to teachers' personal understandings of content may produce considerably different results.

Finally, although some researchers (Leinhardt & Smith, 1985) have studied the impact of teachers' knowledge on student learning (ex post facto), most have not. The relationship between teachers' subject matter knowledge and classroom practices has been the focus of much research attention, but whether teachers' subject matter knowledge truly impacts students' learning has not been given similar attention.

CHAPTER III

DESIGN AND METHOD

Introduction

The purpose of this investigation was to explore how differences in an elementary mathematics teacher's subject matter knowledge relate to classroom teaching and student learning. In particular, the investigation attempted to answer the following questions:

1. What is the appearance of an elementary mathematics teacher's subject matter knowledge structure of addition, subtraction, multiplication, and division?

2. How do differences in this knowledge structure relate to classroom teaching and student learning?

A single-case design was selected to provide an in-depth description and analysis of how teacher subject matter knowledge relates to teaching and learning. The design of the study proceeded in two phases. The purpose of Phase 1 was to select the subjects for the study. These subjects included a teacher and the teacher's class of students. Once the subjects were selected, Phase 2 was used to gather data on the specific teacher's teaching and the student's learning of at least two of the following content areas: addition, subtraction, multiplication, or division.

The selection of subjects and a more detailed description of the two phases of data collection are discussed early in this chapter. This discussion is followed by a description of the research methodology, data sources, and information about the researcher. In addition, a detailed discussion of the data analysis procedures is provided to conclude this chapter.

Subjects

A key feature of this study was the identification of a teacher with clearly different knowledge in at least two content areas. The selection of the specific teacher and students for the study involved an interview that focused on the administration of a questionnaire to gather data on three teachers' subject matter knowledge of addition, subtraction, multiplication, and division. Once the interview data were collected and analyzed, purposeful sampling was used to select one teacher for the study. Six criteria were used to select the teacher. First, the teacher needed to be teaching third, fourth, or fifth grade so that addition, subtraction, multiplication, and division were taught during the school year. Recent studies (TIMSS, 1996) suggested that many elementary students begin to experience difficulties with mathematics during these grade levels. Second, the teacher needed to have 4 to 12 years teaching experience. Generally, a teacher in this range of experience is no longer struggling with classroom management issues but is still developing as a teacher. Third, the teacher needed to have previously taught at the current grade level for at least one year to ensure that the teacher selected was familiar with the important topics at that grade level. Fourth, the teacher needed to have different types (different formats or differences in breadth and depth) of subject matter structures for at least two of the content areas assessed (addition, subtraction, multiplication, or division). Fifth, the teacher needed to be willing to participate in extensive classroom observations. Sixth, permission needed to be granted for the students in the teacher's class to be assessed following the teaching of each content area being observed. The last three criteria were necessary for the researcher to judge how differences in the teacher's knowledge structure related to classroom teaching and student learning. Using these criteria, one teacher was selected. The teacher's students were also subjects in the study.

Method

A single case study design was used that utilized qualitative and quantitative techniques of data collection and analysis. The use of a single case design was purposefully intended for the development of an in-depth description of classroom teaching and student learning when a teacher had differences in her knowledge structure. Multiple sources of data were collected with each type and phase of data being analyzed separately through a constant comparative format in order to derive any patterns or themes of information.

Phase 1: Selecting the Single Case

The initial data collection phase of this study focused on selecting subjects for the study. In order to identify the one teacher for this study, three teachers were considered initially. A questionnaire supplemented by an interview was used to develop a description of each teacher's knowledge structure with regard to addition, subtraction, multiplication, and division. Purposeful sampling was then used to select a teacher that met the purpose of the study.

In the fall of 1998, initial inquiries were directed to third, fourth, and fifth grade teachers to attract teachers willing to participate in the study. These contacts, made by phone, were used to assess the teachers' overall willingness to participate in the investigation. To avoid sensitizing the teachers to the focus of the study, the teachers were told that the selected teacher was to be observed and the teacher's students assessed to determine the ways used to teach elementary mathematics. Since many acceptable variations in the teaching of mathematics exist, it was expected that such an explanation would help reduce the teacher's concerns about critical evaluations and minimize the impact of the observations on classroom instruction.

Once the names of specific teachers expressing an interest were obtained, a letter (Appendix A) was mailed describing the general intent of the study, the types of data to be collected, and the time commitment involved. The teachers interested in participating in the study were asked to complete an information form requesting: name, age, gender, ethnicity, and academic and professional background (Appendix B). A pool of teachers was then formed that met the following three criteria. First, only third, fourth, and fifth grade teachers were selected. Second, each teacher needed to have 4 to 12 years teaching experience. Third, each teacher needed to have previously taught at the current grade level for at least one year.

Interviews were arranged with three teachers to administer a questionnaire that was used to assess the teachers' subject matter knowledge structures of addition, subtraction, multiplication, and division. Following the administration and analysis of the questionnaire and interview data, a teacher was identified who met the first five selection criteria. The teacher was a fourth grade teacher having 4 to 12 years teaching experience, having previously taught at the current grade level for at least one year, having differences (in formats or breadth and depth) in her subject matter structure for at least two of the content areas assessed, and willing to participate in extensive classroom observations. A presentation of the subject matter knowledge structures of the two teachers not selected is discussed briefly in Appendices I and J.

The teacher identified was then contacted to reconfirm interest. Letters were mailed to the district administration, school principal, teacher, and parents of the students in the teacher's class requesting permission for the teacher's students to participate in the study. The teacher's schedule for teaching the unit was also requested. Following permission by all parties involved, one teacher and the teacher's class were then selected as subjects for the study.

Phase 2: Classroom Observations

The primary purpose of the second phase of the study was to generate an in-depth description of classroom teaching and student learning for the content areas of multiplication and division, areas in which the teacher was identified as having differences in her knowledge structure. This phase included semi-structured interviews, classroom observations, the collection of classroom documents, a researcher's journal, informal interviews, and post assessments of students with selected interviews of students.

Once a teacher was selected for the study, a semi-structured interview was conducted that focused on collecting data on the teacher's specific climate for teaching mathematics. At the conclusion of the interview, the researcher requested a copy of the textbook to be used, the curriculum to be followed, and a daily teaching schedule. Classroom observations as well as the collection of classroom documents were arranged at this time. In addition, an interview was arranged prior to the teaching of the unit being observed. The teacher was also asked to recommend six students, two students who consistently performed in the upper third of the class, two who performed in the middle third, and two who performed in the bottom third, to be interviewed following the administration of the post assessment. The details of this interview are discussed in the data sources portion of this chapter.

A second semi-structured interview was conducted with the teacher immediately prior to the teaching of the unit. The interview focused on obtaining information about the unit being observed especially the teacher's objectives. Following the interview, extensive classroom observations were conducted. The class was observed for every lesson taught of a seven week unit on multiplication and division. All materials used in the normal teaching of the class were collected. Videotaped, semi-structured interviews were also conducted to provide the researcher with a better understanding of the lessons and give the teacher an opportunity to clarify statements and actions. The interviews were mostly guided by the researcher's questions and reflections made in the course of the observations recorded in the researcher's journal.

At the conclusion of the observations, the students in the teacher's class were assessed in order to assess their knowledge of the content taught with respect to the teacher's objectives. The post assessment was constructed by the researcher using the teacher's objectives for the specific content areas (multiplication and division) observed. Following the administration of the post assessment, the six students recommended by the teacher were interviewed to provide the researcher additional insights and understanding into the students' learning.

Sources of Data

To investigate the teacher's subject matter knowledge and its impact on classroom teaching and student learning, seven sources of data were utilized: a questionnaire, semi-structured interviews, classroom observations, classroom documents, informal interviews, a researcher's journal, and student assessments. The rationale for using multiple sources of data was that the flaws of one source of data often result in strengths of another. By combining different sources of data, through triangulation, the researcher attempted to achieve the best of each source, while overcoming individual deficiencies (Gall, Borg, & Gall, 1996).

Questionnaire/Interview

During Phase 1 of the study, an audiotaped, semi-structured interview was conducted with each of the three teachers selected to be part of the sampling process. The interview was conducted in each teacher's classroom and the data collected was used to select the teacher subject for the study. At the start of the interview, in accordance with Human Subjects Committee regulations, each teacher was reminded that the data collected would remain confidential and would not be used in any way for evaluation. The researcher and major professor were the only persons having access to all data collected.

The first part of the interview focused on making the teacher feel comfortable. The teacher was asked to use the sign-up form (Appendix B) to talk, in more detail, about her general academic and professional background. A questionnaire (Appendix C) similar to ones used in several studies in science education (Gess-Newsome & Lederman, 1995; Lederman & Chang, 1997) was then administered to gather data on the teacher's subject matter structure of addition, subtraction, multiplication, and division. The teachers were asked to respond in writing (using words, pictures, or diagrams) to the following questions (Appendix C).

1. What are the important topics, concepts, ideas, procedures, or themes that make up the content areas of addition, subtraction, multiplication, and division at the elementary school level? If you were to use these topics to diagram each content area (addition, subtraction, multiplication, and division), what would your diagrams look like?

2. Have you ever thought about these content areas in this way before? Please explain.

The teachers were told that their descriptions may be "represented" by diagrams, concept maps, pictures, or in any ways with which they felt comfortable. They were also told that the questions were intentionally vague with many different ways to respond and no right or wrong answers.

The methodology described was used so that the ideas included in the schematic of each content area were open-ended, removing possible sources of bias perhaps imposed by the researcher. In addition, a set of numbers (whole numbers, fractions, decimals, integers, or rational numbers) on which these operations were performed was not specified. Not imposing a set of numbers on the teacher provided a more open-ended methodology sensitive to the teachers' personal understandings of the content area. Validity for the questionnaire was established by asking the teachers if they understood the questionnaire. If any misunderstandings existed, the researcher clarified the questionnaire and provided the teachers with additional time to answer the questions.

Sufficient time was provided for the teachers to complete the task. When the teachers were comfortable with their answers, the following questions were used to guide a discussion of the teachers' answers to the questionnaire.

Did you understand what you were asked to do in the questionnaire?

How did you select the topics you have included for each content area?

Describe what you have written (drawn) on your paper.

What specifically do you mean by the terms you have used?

Are all of the topics listed of equal importance?

What are the most important topics that should be emphasized at your grade level?

Semi-structured Interview

A semi-structured interview was conducted with the teacher selected at the start of Phase 2. The first part of the interview focused on the teacher's specific climate for teaching mathematics. The following questions were used:

How do you feel about teaching mathematics compared to other subjects?

How often do you teach mathematics compared to other subject?

How do you feel about teaching multiplication and division compared to other mathematics topics?

Do you enjoy teaching one of these topics (multiplication or division) more than the other? Do you find one of these topic harder to teach?

How do you feel your students learn mathematics best?

What are your general goals for teaching mathematics?

What are your goals or objectives for teaching multiplication and division?

How many days will the unit on multiplication and division take and how long will each lesson take?

Will you follow a curriculum (school, district, state, etc.)?

What textbooks and/or supplementary materials will you use?

Following this line of questioning, the researcher requested a copy of the textbook that was used, a copy of the curriculum, and a daily teaching schedule. The teacher was also asked to recommend six students, two students who consistently perform in the upper third of the class, two at the middle third, and at two bottom third, to be interviewed following the completion of the assessment.

Classroom Observations

Classroom observations were used to gather data on the teaching of a seven week unit on multiplication and division. The data were used to generate an in-depth description of the teaching and learning of the unit and how different knowledge structures relate to teaching and learning. The unit integrated the teaching of both multiplication and division and was taught three days per week. The class was observed every day the unit was taught during the seven week period.

During the observations, the researcher focused on all elements of instruction such as the setting, the teacher, the learners, and the activities and interactions happening in the classroom. The focus included such instructional elements as: presentations, discussions, problem solving activities, hands-on activities, assessments, questions, and interactions with students. Of particular interest were the teacher-student and student-student interactions.

All transactions between the teacher and students were videotaped and field notes taken. A special microphone was attached to the teacher in order to record the teacher-student interactions. The field notes recorded information concerning the teacher's movements and apparent enthusiasm, student interest, student behavior, teacher and student actions and interactions, and general classroom tone. All board and overhead work was also recorded as part of the field notes as well as any materials used during the class. A researcher's journal was also kept . This journal included the researcher's thoughts, questions, reactions, and interpretations that were used to guide the weekly informal interviews with the teacher.

Following each lesson, a summary was written including the following information: topic, organizational model, instructional emphasis, general outline of activities, teacher statements/questions about the content, and student statements/questions about the content. In addition, thoughts, questions, reactions, and interpretations made in the course of the lesson were recorded in the researcher's journal for the teacher to clarify. The videotaped observations were transcribed and reviewed weekly.

Classroom Documents

All documents used in the normal course of teaching during the observation phase of the study such as worksheets, textbook activities, hands-on activities, homework assignments, assessments, and lesson plans were collected. The primary purpose for the collection of these data was to provide additional insights into the classes taught by the teacher during the observation phase of the study.

Researcher's Journal

Because the researcher was the principal data collection instrument for the classroom observations, and as such, could be a major threat to the reliability of the data analysis, it was important to establish possible sources of biases or misinterpretations. Thus, a daily journal was kept containing the researcher's reflections on the classroom observations. The journal included: thoughts, questions, reactions, interpretations, and insights made in the course of the observations. These deliberations were then used to guide the weekly interviews of the teacher by providing the teacher an opportunity to clarify observed actions. This process, of allowing the teacher an opportunity to clarify actions, discouraged the researcher from relying on personal interpretations of the behaviors of the teacher and students. By acknowledging personal preconceptions, values, and beliefs, the researcher had an opportunity to challenge the developing notions about how the teacher's subject matter knowledge related to teaching and learning.

Informal Interviews

Weekly audiotaped interviews were conducted in the teacher's classroom to provide the researcher with a better understanding of the lessons and to give the teacher an opportunity to clarify statements and actions. The interviews were transcribed weekly and kept with the other data collected during the week. The interviews, arranged for times convenient for the teacher, were mostly guided by the thoughts, questions, reactions, interpretations, and insights made in the course of the observations that were recorded in the researcher's journal. The following types of questions were also asked:

How did you think the lessons went?

Did you meet the objectives you identified before the lessons? What makes you think so?

Are there any parts of the lessons that you would like to talk about?

What would you change about the lessons?

What will you be doing next week in the lessons?

What will you be expecting your students to do?

How will the students be organized?

Post Assessment

A post assessment was administered to the students in the teacher's class following the teaching of the unit on multiplication and division. The post assessment and scoring rubric (Appendices F and G) were designed by the researcher using the teacher's objectives. The purpose of the post assessment was to gather data on the students' understanding of the content and their ability to apply their knowledge to a variety of problem solving situations. The rationale for giving only a post assessment was to decrease the imposition involved with the assessment as well as the researcher's interest only in the students' knowledge of the content in relation to the teacher's objectives.

Content validity for the post assessment was established by having the questions reviewed by two university mathematics educators and three elementary school mathematics teachers, prior to being administered, to determine if what was assessed was consistent with the objectives stated by the teacher, Appendix F. If more than 20% of the reviewers had agreed that the assessment was invalid, it would have been revised until at least 80% agreement was reached. One-hundred percent agreement was reached on the first attempt.

Assessment reliability was established by calculating a split-half correlation coefficient for the assessment. The assessment was split into two subtests by placing the odd-numbered items in one subtest and the even-numbered items in another subtest. A coefficient of internal consistency was then calculated ($r = 0.910$).

Following the administration and analysis of the post assessment, audiotaped interviews were conducted with six students about their answers to the questions on the assessment. The interviews were conducted within three weeks following the administration of the post assessment to all students. The six students were recommended by the teacher during Phase 2 of the study as consistently performing in the upper third, the middle third, and the bottom third of the class. The purpose of these interviews was to provide the researcher with an extension of the post assessment. The findings from the interviews were also used to judge the generalizability of themes and patterns of students' learning for the unit taught.

In accordance with Human Subject regulations, prior to the start of each student interview, the teacher and students were reminded that the information collected would remain anonymous and would not be used for evaluation purposes.

Each student was asked permission to audiotape the interview session and was shown the recorder that was used to record the interaction. The first few minutes of each interview focused on making the student comfortable and developing a rapport between the researcher and the student. Once rapport was established, the student was asked to read the questions and answers, describe the thinking processes used to answer the questions, and to explain why the answers made sense.

Researcher

In this qualitative study, the researcher was the primary person to collect and analyze data. Since the researcher could be a major threat to the creditability of the study, establishing possible sources of biases or misinterpretations was important.

The researcher holds a Bachelor of Science degree and a Masters of Arts in the teaching of mathematics. He has been a mathematics supervisor, curriculum specialist, computer coordinator, and has taught mathematics in metropolitan and rural communities for over 30 years. His experiences include teaching mathematics at elementary school, middle school, high school, and university levels. He is currently enrolled in a doctoral program at a medium sized university in the northwestern part of the United States.

The researcher has also been involved in supervising preservice teachers, teaching mathematics to prospective elementary teachers, teaching elementary mathematics methods courses, and conducting in-services for K-12 teachers. He has presented at district, county, state, and national conferences on the teaching and assessment of mathematics.

As the researcher observed preservice and experienced teachers, he became interested in their subject matter knowledge of mathematics, and the impact of their knowledge on teaching and learning. After extensive review of the literature, the questions guiding this study evolved.

The researcher recognizes the importance of establishing his own subject matter knowledge structure for the content areas of addition, subtraction, multiplication, and division. For this reason, the researcher completed the questionnaire. Results of the researcher's subject matter knowledge structure are shown in Figure 1.

From the researcher's perspective, knowing mathematics means understanding concepts and procedures, and being able to use them in purposeful ways. Problem solving, communication, and reasoning should be central to all school mathematics. As such, they should be the primary goals of mathematics instruction. These processes are not distinct topics; they should permeate all activities and provide the context in which concepts and skills are taught and learned.

Attention to problem solving, communication, and reasoning, however, does not imply a lack of concern for arithmetic; rather, it necessitates a broader view. Understanding the fundamental operations of addition, subtraction, multiplication, and division is fundamental to knowing mathematics and, in most situations, being able to solve problems. Several components are essential to understanding an operation. These components involve recognizing conditions in real-world situations that indicate that an operation would be useful; knowing different representations of an operation including real-world, verbal, concrete, pictorial, and symbolic representations; having an awareness of models and properties of an operation; seeing relations within and among operations; and being able to connect the operations to all areas of mathematics.

Although technology has drastically changed the methods for computing, knowledge of basic fact and efficient accurate methods for computing are essential to success in most areas of mathematics. Students should be able to recall single-digit addition facts and the counterparts for subtraction, multiplication, and division. Although it is no longer necessary to devote major portions of instructional time to performing computations using paper-and-pencil algorithms, students should be able to use mental strategies, jottings on paper, and in some cases paper-and-pencil algorithms to produce quick and accurate results. Therefore, computational skill should include proficiency with simple calculations, skill in using appropriate technology, mental math skills, ability to estimate, savvy to determine if computed results are reasonable, and accurate methods for computing using paper-and-pencil algorithms.

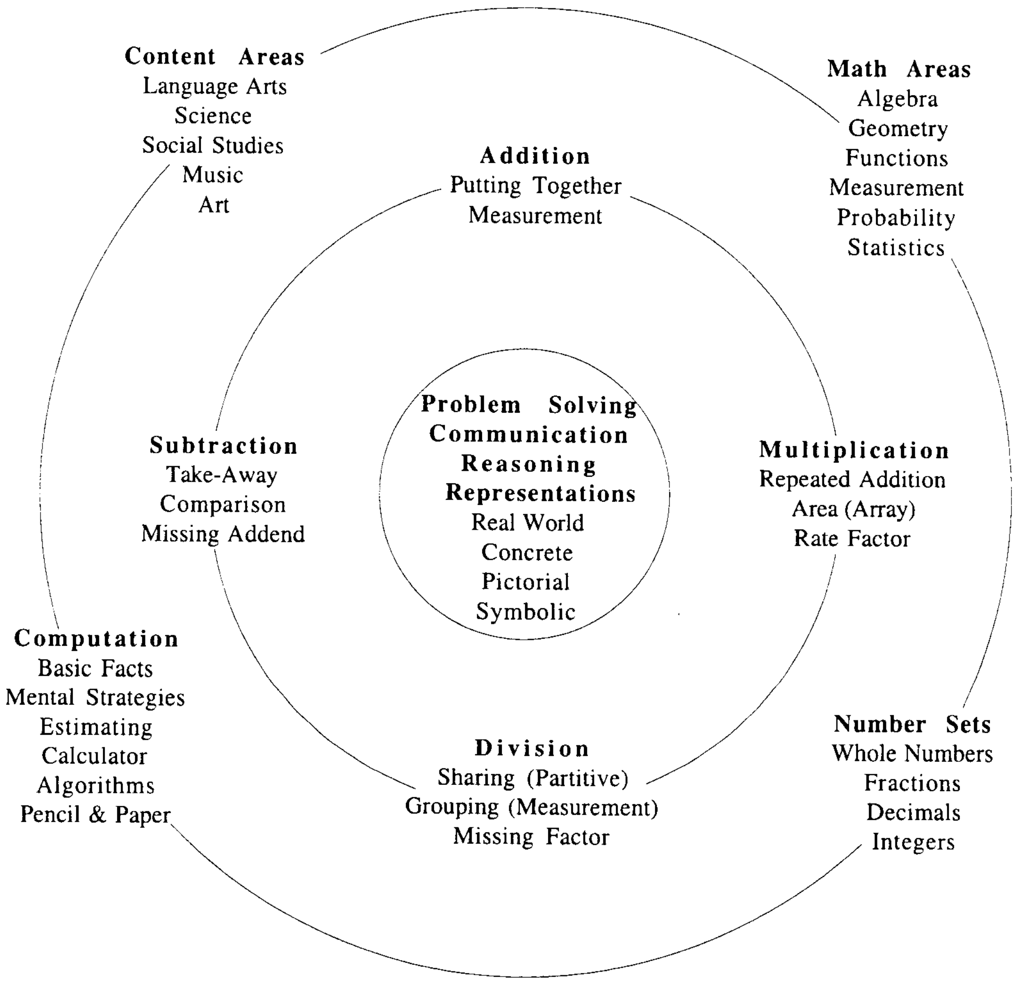


Figure 1. Researcher's Subject Matter Knowledge Structure

Analysis of Data

In an attempt to produce an in-depth description of an elementary mathematics teacher's subject matter knowledge structure of addition, subtraction, multiplication, and division, and how differences in the teacher's knowledge structure related to teaching and learning, six types of data were analyzed: questionnaire and interview of the teacher, semi-structured interviews of the teacher, classroom observations, classroom documents, student assessments, and student interviews. The analysis of these data involved three stages. Briefly, the first stage involved the analysis of Phase 1 data. It was conducted initially to select a case that suited the purpose of the study. These data were also used in answering the research questions. The second stage of analysis included all classroom observation data. Ongoing analysis occurred weekly during observations and a more intensive analysis occurred once the classroom observations were completed. This analysis included developing categories, assigning coding categories to units of data, and searching for patterns and themes in the data. The patterns and themes uncovered were then used in answering the second research question. The final stage of analysis involved the evaluation of the assessment and student assessment interviews. These findings were also used in answering the second research question.

Analysis of Questionnaire Data

The purpose of this stage of analysis was threefold. First, this analysis was conducted so that purposeful sampling could be used to select a teacher that met the intention of the study. Second, this analysis was used in answering the first question of the study that addressed the subject matter knowledge structure of the teacher selected. Finally, the patterns and themes uncovered in this analysis, along with the analysis of classroom observation data and assessment data were used in answering the second question of the study that addressed how differences in the teacher's subject matter knowledge structure related to teaching and learning.

During Phase 1, the questionnaire/interview data on the three teachers' were analyzed separately with each teacher's knowledge structure being analyzed in several ways. First, each teacher's knowledge structure was analyzed in terms of breadth and depth of content. This analysis involved identifying the topics, concepts, ideas, procedures and themes that made up the content areas and the teacher's understanding of these topics. Similarities and differences between the content were then noted. Second, the knowledge structure was analyzed in terms of format. Several studies suggested that the formats illustrated would be either discrete, simple hierarchy, or web-like. In order to decide which format was being used, connections between key elements and processes such as problem solving, communication, mathematical reasoning, and representations were identified. Similarities and differences within the structure were recorded. Finally, a teacher was identified having the most differences in her subject matter knowledge structure for at least two of the content areas assessed (addition, subtraction, multiplication, and division).

Analysis of Classroom Observation Data

Although a more intensive analysis occurred once all of the data were collected, ongoing analysis occurred during the classroom observation phase of the study to uncover patterns, develop working hypotheses, and guide further data collection. In general, this process involved reviewing data from field notes, the researcher's journal, and weekly informal interviews. Ideas and patterns were jotted down, key words and phrases used by the teacher or students circled, and particularly important sections highlighted.

Following Phase 2 of the study, a more intensive analysis of the classroom observation data occurred that included developing categories, assigning coding categories to units of data, and searching for patterns and themes in the data. First, classroom transcripts and documents were placed in chronological order of use and each piece of data numbered sequentially with similar kinds of materials being kept together in order to facilitate locating data. After the data were numerically ordered, the data were read several times in order to develop an initial "picture" of classroom teaching. For each content area, this initial picture included a list of the topics being taught, and the connections between and processes "laid over" the topics. More specifically, the topics being taught included any concepts, ideas, or procedures associated with the content area. *Connections* were defined to be instances in which the teacher specifically or inferentially related one content topic (concept or procedure) to another. Such connections, however, required more than just the use of previously used vocabulary. An example of a connection was using addition and measurement ideas to develop the concept of multiplication. *Processes* were considered to be broad ideas such as problem solving, communication, mathematical reasoning, or patterning that were being taught. Evidence of processes were the inclusion of problem solving and writing activities during the teaching of one of the content areas.

The next step in the analysis process involved searching through the data and writing down words and phases for regularities and patterns in the teaching of the content area. The words and phrases generated from this search were then used as coding categories to sort the data. The following families of categories were of particular interest and provided a starting point for the search. These coding families, developed from the *Professional Standards for Teaching Mathematics* (NCTM, 1991), included: instructional materials and resources, ways to represent mathematical concepts and procedures, instructional strategies and organizational models, and ways to promote discourse.

Instructional materials and resources referred to codes under which problem booklets, concrete materials, textbooks, computer software, calculators, and so on were sorted. Ways to represent mathematical concepts and procedures referred to types of codes involving real world, concrete, pictorial, or symbolic representations. Instructional strategy codes referred to problem solving approaches, various forms of the "tell, show, and do" model (NCTM, 1991) of teaching, manipulative activities, or drill and practice activities. Organizational models were types of codes that included whole-class discussions, small-group collaboration, independent work, peer instruction, or project work.

Ways to promote discourse referred to codes that focused on the role of the teacher in orchestrating students' understanding of mathematics. This family of codes included the posing of questions and tasks by the teacher that elicited, engaged, and challenged students, and that encouraged students' questions or deliberations. It included the teacher's role in deciding what to pursue in-depth, when and how to attach mathematical notation and language to students' ideas, when to provide information, when to clarify an issue, when to provide a model, when to lead, and when to let a student struggle with a difficulty (NCTM, 1991).

Once coding categories were developed, a list was made of each coding category and each code assigned a color. Units of data were assigned by reviewing the data and marking each unit (paragraph, sentence, etc.) with the appropriate coding category color. A number was also placed next to each code that corresponded to the type of data analyzed. The notes were then cut up and the units of data placed in folders that were labeled with one code. Because some of the data were coded for more than one category, several copies of the notes were needed.

After the data were placed in coding category folders, the folders were read in an attempt to better understand the content of each. Further division (subcategories) of the data as well as patterns and themes were sought.

Analysis of Post Assessment Data

Following Phase 2 of the study, assessment data and student interviews were analyzed in order to judge the students' knowledge of each content area based on the teacher's unit objectives. In order to uncover patterns and themes of students' learning, the assessment data on the two content areas (multiplication and division) were analyzed separately and then compared in order to judge similarities and differences between students' understanding of the areas observed. Similar types of questions across content areas, such as those involving conceptual knowledge, procedural knowledge, or connections between conceptual and procedural knowledge, were analyzed. The analysis included item statistics, measures of central tendency, and t-tests. A t-test was used to compare the students' results for the multiplication, division, and integrated portions of the assessment. The number of correct, partially correct, and incorrect responses on the assessment questions were recorded and tables were used to provide the success rates for each item.

Student interviews were also analyzed and attempts were made to uncover additional patterns and themes of students' understanding. The students' thinking processes used to answer the assessment questions and explanations of why their answers made sense were compared and contrasted for each content area as well as across content areas. Points of reaffirmation or contradiction were noted. Potential reasons behind any differences or similarities noted were sought.

Research Questions

Triangulation among phases and types of data was then used to answer the research questions (Gall, Borg, & Gall, 1996). The first question addressed the appearance of the elementary teacher's subject matter knowledge structure for addition, subtraction, multiplication, and division. In order to answer this question, the data from the questionnaire and the first interviews were analyzed in several ways. First, the teacher's subject matter knowledge structure was described in terms of its format (discrete, hierarchy, web-like, or some other form) and its breadth and depth of content. In particular, the teacher's understanding of topics, connections, and themes were noted. Second, a global analysis of the teacher's described knowledge structure was conducted by comparing the topics selected and the key elements understood by the teacher. Patterns between content areas including similarities and differences in the content as well as the breadth and depth of the topics were described.

The second question addressed how differences in the teacher's subject matter knowledge structure related to classroom teaching and student learning. This relationship was addressed by comparing the knowledge structure described by the teacher to patterns and themes generated through classroom observation data and the assessment of students. First, in-depth descriptions of the teaching of each content area was provided using the patterns and themes generated for each topic. In particular, the teaching of each content area was discussed in terms of the instructional materials and resources used, ways used to represent mathematical concepts and procedures, different instructional strategies and organizational models used, and ways used to promote discourse. These descriptions were documented with data taken from field notes and other materials. Whenever possible, quotes from the teacher or students were presented. Second, for each content area, comparisons were made between the teacher's knowledge structure and classroom teaching and student learning. Post assessment results were included with the analysis of scores, student quotes, and other data provided in order to document inferences made. Factors associated with any type of relationship were noted. Third, comparisons were made across content areas with similarities and differences between classroom teaching, and student learning noted as well as factors associated with the teacher's subject matter knowledge structure addressed. Quotes and short sections from the data were used to document these comparisons.

CHAPTER IV

ANALYSIS OF DATA

Introduction

The purpose of this investigation was to explore how differences in an elementary mathematics teacher's subject matter knowledge structure relate to classroom teaching and student learning. One experienced elementary teacher and the teacher's fourth grade mathematics class participated in this study. The teacher, Meg, taught in an elementary school of about 350 students with a rich and diverse ethnic population. Pseudonyms were used to assure the anonymity of the teacher and the students.

An initial interview was conducted for the purpose of assessing Meg's subject matter knowledge structure of addition, subtraction, multiplication, and division. Following the initial interview, an interview was conducted prior to the observation phase of the study to obtain additional information. The interview focused on collecting data on Meg's specific climate for teaching mathematics and obtaining information about the unit on multiplication and division to be observed. Of particular interest, during this interview, were Meg's unit objectives.

Following the interviews, Meg's class was observed three days per week for approximately one hour each day during the teaching of a seven week unit on multiplication and division. A total of 20 classroom observations were conducted. At the conclusion of the observations, all of the students in the class were given a post assessment to evaluate their knowledge of multiplication and division with respect to Meg's unit objectives. Within three weeks after the administration of the post assessment, six students recommended by Meg, with a diversity of success in the mathematics class, were interviewed to provide additional insights and understandings of the students' learning.

All material used in the normal course of teaching were collected. Informal shortened interviews were also conducted with Meg throughout the unit to provide the researcher with a better understanding of the lessons and allow Meg an opportunity to clarify statements and actions. A final interview occurred after the last classroom observation.

The data described were used to generate an in-depth description and analysis of how Meg's subject matter knowledge related to teaching and learning. Several sections are used to describe Meg's background and the teaching and learning of multiplication and division. The first section provides an academic and professional profile of Meg. This information, generated primarily from the second and third interviews, paints a general picture of Meg. The second section presents the subject matter knowledge held by Meg that was developed during the first interview. The next three sections were generated from the analyses of the classroom observation data. Meg's classroom profile, created by the researcher, portrays a generalized class period and an overview of the unit. The following section describes the content and development in the unit for the teaching of multiplication. The next section describes the content and development within the unit for the teaching of division. Following the description of the teaching of multiplication and division, evidence of students' learning is presented. This section is generated from the analyses of the post assessment and student interview data. Finally, a general summary and analyses section is provided. In particular, comparisons are made between Meg's subject matter knowledge, her classroom practices, and student learning.

Meg: Teaching is Eclectic

Academic and Professional Profile

Meg was recommended for inclusion in the study by a local colleague. Meg responded enthusiastically when first contacted about the purpose and intent of the study and she reaffirmed her willingness to participate by returning the letter of confirmation two weeks later. Meg's initial interview was conducted in the fall about one month into the school year.

Meg is a friendly, sensitive, and energetic woman who described herself as "an eclectic person and teacher." Meg seemed as comfortable with expressing her conceptions of teaching as she was in admitting that she considered herself to be "a good mathematics teacher, but not very good in mathematics."

In middle school, high school, and college, Meg found mathematics to be difficult. As Meg explained:

> Math was hard. It wasn't easy in high school or college. I went to middle school in the good old 70's when everything was individualized. The teacher would tell us to work on an assignment and then go and correct it. So my friends and I would just go and write the answers down from the answer keys cause they were setting there . . . and we'd turn them [our assignments] in and get credit. Occasionally, we had to take a test . . . but it was three quarters of the way into the school year before I realized that I didn't know anything. At that point in time, I wanted things to be easy and then I realized that I had to figure out how to do this [the mathematics] and I didn't know how. I felt like a drowning person in mathematics from then on.

When Meg was a sophomore in high school, her attitude about learning mathematics changed. Meg stated:

> I realized that what I was doing was stupid. I knew I wanted to go to college, and unless I learned more mathematics, I wouldn't be able to do well.

But, according to Meg, she had missed too much. She struggled through Algebra I and Geometry in high school; she "never truly understood" either subject.

Meg felt that she passed the classes, Algebra I and Geometry, only because "the teachers thought I was working hard. I didn't get Algebra, I didn't get Geometry." Although Meg wanted to take Algebra over again in high school, she ended up dropping the class. Her Algebra I teacher, the second time around, told her she should take Algebra II rather than repeat Algebra I. Meg felt that she did not know enough Algebra I to go on to the next level so she dropped Algebra I.

Meg graduated from a large midwestern university with a Bachelor of Arts in Elementary Education. In college, Meg was a self proclaimed "party girl." She wanted to major in physical therapy but soon learned that she did not have the necessary mathematics background or work ethic. "I wanted to drink my way through college so I majored in Elementary Education cause I thought it would be an easy degree." According to Meg, Elementary Education was easy. "It was a joke." The only mathematics courses she took in college were Mathematics for Elementary Teachers and Mathematics Review. Meg wanted to take more mathematics but her college advisor told her that she should not because she was "not good" in mathematics and "you don't need it. After I screwed myself up [in mathematics] I never found anybody who would say, Hey, wait a minute, you can do this . . . you need to start here and build on it."

> It wasn't until I started working with kids that I realized how important teaching was . . . and how little content I knew to do it. So I'm constantly striving to figure out how to teach whatever subject it is better. That's why I'm willing to participate in this [study] . . . cause I want to know more.

Meg's opinion of her educational course work was not positive. She described most of the coursework as "a waste of time." She felt that the most significant experience of her college career was student teaching. Meg student taught at two grade levels, fifth grade and kindergarten. Meg described her fifth grade supervising teacher as "wonderful."

> She [her fifth grade supervising teacher] was amazing. She was strict with the kids but the kids loved her . . . cause she loved them.

Although Meg considered her fifth grade supervising teacher a "traditional" mathematics teacher, she felt she learned a lot about teaching. "She [her supervising teacher] was firm but sensitive and caring and that's how I wanted to be. The kids really worked for her."

When asked specifically about what she had learned from her supervising teacher in kindergarten, Meg replied: "She was the first person to introduce me to manipulatives. We used Math Their Way." Meg admitted that she "didn't always understand what was going on" but the experience started her thinking about how important manipulatives are in developing students' understanding of mathematics.

After teaching for several years, Meg earned a Masters of Arts in Multicultural Education from a small southwestern university. Her Masters program, however, did not include any mathematics courses. Although Meg acknowledged that she would "be afraid" of such coursework, she felt strongly that elementary teachers should be required to have more mathematics. In Meg's opinion, she has improved her knowledge of and teaching of mathematics by taking inservice classes, reflecting on her own teaching, reading lots of books, and collaborating with her colleagues. "I'm not afraid to admit I don't know something and ask somebody. But I find most [elementary] teachers don't seem to understand things [mathematics] any better than me."

Meg has been teaching elementary school for eight years and has taught both single grade and multi-age classes. During Meg's career her experiences included teaching the entire range of kindergarten through sixth grade. For this study, Meg was teaching multiplication and division to fourth grade students.

Although Meg felt insecure about her knowledge of mathematics, she seemed to enjoy teaching mathematics. When asked how she felt about teaching mathematics compared to other subjects, Meg replied:

> The more I teach mathematics, the more I like it. When I first started teaching I was very very nervous about teaching math. Now I like it. I really do.

Meg described reading, writing, and math as "truly the three R's."

> . . . everything else [music, art, history, geography, etc.] should be integrated with reading, writing, and math. I also teach math separately, totally by itself. Depending on the schedule, I think it's important for students to have at least an hour of math a day. With our grouping for math, we teach math to our groups, at least one hour, three days a week. Sometimes we run over some but I also try to do other things with the students when I can. I think math is really important and some of our students are really low. They're not on grade level so they need all the math they can get.

Although Meg considers multiplication and division important content areas in elementary school mathematics, they are not her favorites to teach.

> I think it's really important that they [the students] know how to multiply and divide because they use it for so many different things. But I think the one [content area] I enjoy teaching the most is geometry. I'm learning so much about geometry. It's probably because I wasn't any good [in Geometry].

When asked about which content area, multiplication or division, she enjoyed teaching the most, Meg replied:

> I probably enjoy multiplication more just because I feel more confident with it [multiplication]. And since I'm not as confident with division, I think it's harder to teach.

Meg also feels that one reason division is more difficult to teach is because there are fewer materials and activities available.

> It seems like when I get materials there are a lot more materials for multiplication [than for division]. You get lots and lots of pages for multiplication and then you throw a little division at them [the students], and ok, they're suppose to understand it [division] now. So it seems like there's more support for multiplication than division.

Meg's conceptions of teaching and learning seem to reflect her own personality and experiences. When asked about how she felt students learn mathematics best, Meg quickly replied:

> Variety. I mean I'm very eclectic [in my teaching]. Students need to have their computational practice but they really need to have that conceptual working with things, figure things out, do projects, work together, learning about each other. They need to have some cooperative as well as solo stuff, cause when they're working with each other, it gets their brains going. To understand math really well

> they [the students] need to be able to explain it to another kid . . . it's like cementing it in their brains. Maybe they can do it, but to actually explain it to another kid or write about it takes them to another level of understanding. I had a fifth grade girl several years ago and when she was doing multiplication and division of fractions she could do it but, she said, "you know, I just don't understand it." And I said, "You know, just keep practicing, stay with me and you'll get it." And the next year when she was in sixth grade she said, "Oh my gosh, I totally get this." Yet, she could do it [get an answer] in fifth grade and she could do it in sixth grade but when she was in sixth grade this light bulb came on where she said "I get it. I understand it. I can explain it now." So I think kids need to revisit things, they need to work with each other, they need to practice their computation and learn about algorithms, practice with manipulatives, and then take it over to that abstract.

For Meg, teaching mathematics involves using a variety of materials and resources. According to Meg, students are not exposed to all of the mathematics they need to learn by following the textbook:

> Lots of times when you just use the textbook you start out at the beginning of the textbook and you work your way through it [the textbook] and you run out of time before the end of the year. So they [students] don't get enough exposure to the stuff that comes at the end of the year [textbook], which typically, is the same stuff [every year]. So kids all of the sudden, by the time they're in intermediate [grades], haven't had geometry, haven't had algebra, haven't had probability, because they all come at the end of the book. So I want them [students] to have exposure to a lot of different things, a lot of different types of math. So I use lots of things [materials].

Meg feels that multiplication and division facts and pencil and paper computation are important but students who have difficulty memorizing the facts or doing paper and pencil computation should not be held back from doing problem solving.

> [In teaching mathematics] . . . my goals are I want them [the students] to feel like they can be problem solvers. They should be able to take their computational stuff and apply it to other areas . . . like problem solving. Some kids, like our LD [Learning Disabled] kids may have a terrible time with computation, but they still need to have exposure to other types of math, not just things like multiplication and division. There's tools for that [doing computation]. I still want them to be able to try other things and use a calculator for computation.

When asked about specific goals and objectives for teaching multiplication and division, Meg included many of the goals she had for mathematics students in general. But, in addition to the importance of process skills and exposure to "lots of different things," Meg included specific content topics for multiplication and division. Meg wanted her students to understand the meaning of multiplication and division, know multiplication as repeated addition or an array, division as equal sharing or repeated subtraction, and relationships between multiplication and division.

> I would like the students to know their basic multiplication and division facts and get faster at doing them [multiplication and division facts]. I would like the students to be able to build and draw arrays, and write the related multiplication and division equations. I would like the students to be able to solve story problems and be able to create their own story problems. I would like the students to be familiar with the vocabulary of multiplication and division. They should understand words like rows, columns, arrays, product, factor, divisor, quotient, dividend, remainder, and multiples. I would like the students to be able to multiply two-digit numbers by one-digit numbers and divide two-digit numbers by one-digit numbers.

Meg had strong opinions on what was essential to teach and how this content should be taught. In deciding what to teach in fourth grade mathematics, Meg first looked at the third, fourth, and fifth grade textbooks and then at the District Assessment Plan and the Standards [NCTM, 1989]. After examining all of these materials and resources, Meg indicated that she throws "it all in a big old paper bag" and shakes it up and comes out with her own version.

In describing the materials she would use to teach multiplication and division, Meg said that she used the adopted textbook mostly as supplementary material. "It [the books] has lots of practice for kids . . . that's mostly what I use it for." Along with some other materials, she would mostly be using a variety of Creative Publications' problem solving resources including: *Work Mats Math: Understanding Multiplication and Division*, *Connections*, and *Constructing Ideas about Multiplication and Division.*

Self-Described Subject Matter Knowledge

As part of the initial interview, Meg was asked to answer a questionnaire (Appendix C) concerning her conceptions of the important topics, concepts, ideas, procedures, and themes that make up the content areas of addition, subtraction, multiplication, and division at the elementary school level. Meg's initial reaction indicated that she was hesitant about completing the subject matter questionnaire. Although she seemed to understand the question and task, she felt hesitant about the organization of her responses. When asked if she understood what she was to do, Meg replied: "I think I do, but there are so many things. How should I organize it?" Meg seemed to relax a little when told that the question was intentionally vague since there were many different ways to respond and no right or wrong answers. She was also told that her descriptions may be "represented" by diagrams, concept maps, pictures, or in any ways with which she feels comfortable.

The subject matter knowledge diagram that Meg created in response to the questionnaire is shown in Figure 2. Meg's representation included three parts. The top part represented Meg's notion of the connections between the four operation of addition, subtraction, multiplication, and division; the middle part represented her conception of the close linkage between addition and subtraction; and the final part indicated her idea of the relationship between multiplication and division.

When Meg had finished completing the questionnaire she was asked again if she understood what she was to do. Meg was additionally concerned about the quality of her responses along with her initial concern about how she should organize them. She answered: "At first I was afraid. I wasn't sure what I should write . . . then I started thinking about what I think kids need to know about addition, subtraction, multiplication, and division and . . . what I think's important.

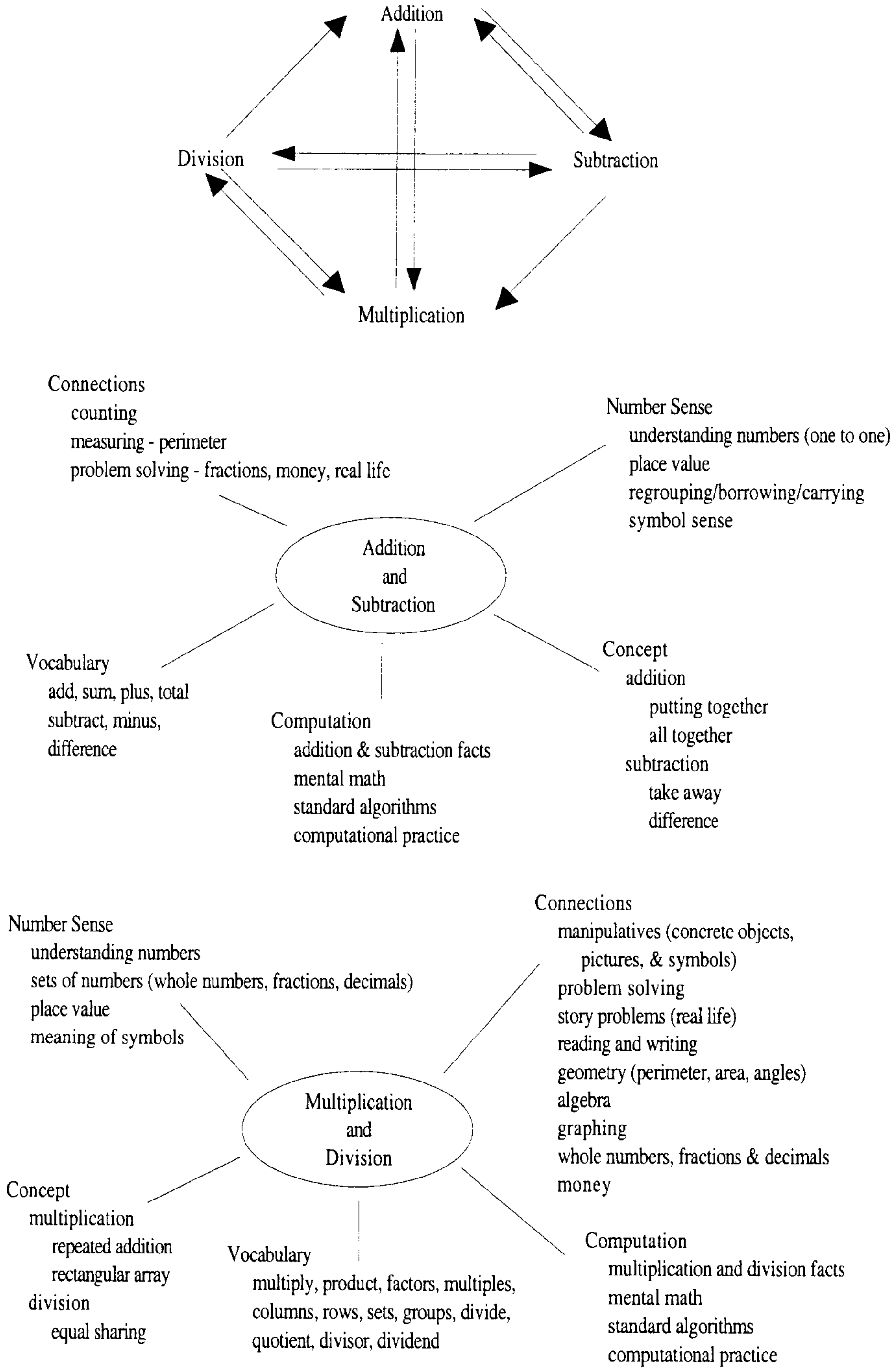


Figure 2. Meg's Subject Matter Knowledge Structure.

Meg began by describing the diagrams she drew in the following manner:

> I think all of these areas [addition, subtraction, multiplication, and division] need a mixture of an understanding of concepts and computational work. And students need to be able to make connections between all four areas. I think of addition and subtraction together and multiplication and division together. Yet, I also think of addition and multiplication together.

Meg explained that the top part, Figure 2, represented her understanding of the relationships among addition, subtraction, multiplication, and division. The arrows were used to show the various connections that could be made. Meg was unable to explain what the arrow between addition and division or the arrow between multiplication and subtraction meant except that they were related.

Meg continued as she described her diagram for addition and subtraction, Figure 2:

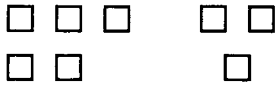
> Basically, the diagram [the diagram of addition and subtraction] is a web of addition and subtraction. I find addition and subtraction very connected together and all of them [addition, subtraction, multiplication, and division] together so I had a hard time separating them out.

Meg included five major components for addition and subtraction: number sense, computation, concepts, vocabulary, and connections. She initially described the components rather quickly:

> Students need to have an understanding of number sense, understanding number . . . one to one, place value, regrouping, borrowing, carrying, and understand symbols. They [students] need to know their facts, be able to do mental math . . . to be working with algorithms. They need to have practice with computational parts of it. They need to understand the concepts by experimenting with manipulatives, putting blocks together . . . taking them away. They need to understand and hear the different vocabulary, and then all of these areas need to be connected into all other parts of the curriculum. I mean not only graphing and measuring, and problem solving, and story problems, but even in their reading and writing, social studies, science . . . giving students a reason why they're doing what they're doing.

For Meg, it seemed that addition and subtraction were connected because, as she put it, "they complement each other."

Say five plus three equals eight [Meg drew a picture and wrote several equations on a sheet of paper, Figure 3]. When you put them together you get eight. Then if you take away three from eight you get five or take away five from eight and you get three. So if you know the answer to an addition problem then you can get the answers to a related subtraction problems.



$5 + 3 = 8$
$8 - 3 = 5$
$8 - 5 = 3$

Figure 3. Relationship Between Addition and Subtraction.

When asked about the concepts of addition and subtraction and the role manipulatives play in understanding the concepts Meg answered:

> Addition is putting things together and subtraction is taking things away or comparing things. I'd use manipulatives like base ten blocks and put them together to add or take away to subtract or compare lengths.

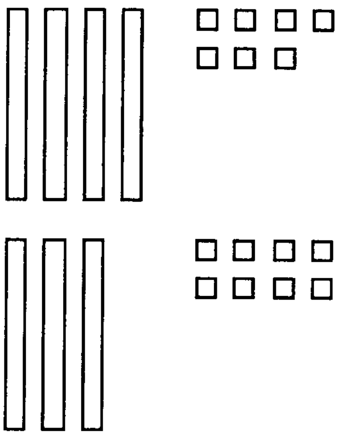
Without the prompting, Meg described the importance of using manipulatives to help students learn addition and subtraction fact:

> I wanted them [students] to see 3 + 5 equals the same as 5 + 3 but when you're building them they're different. So I wanted them to take the blocks and manipulate them, and not just in their heads but have concrete things to move around. They need concrete representations of facts so it cements it into their brains.

Meg indicated that students should know the standard algorithms for each of the operations. She explained that to show the meaning of 47 + 38 she would use base ten blocks. She then drew a picture on a sheet of paper and proceeded to illustrate and explain the meaning of each step in the algorithm, Figure 4.

> I would take four tens and seven ones and you're going to add on to it three tens and eight ones. You first put together the ones. Ok, so you look at the ones column and you say seven ones plus eight ones is 15 ones. Once you have the 15 ones then they need to be regrouped. You

trade ten ones for a 10 and put it in the 10 column. So you have five ones left. Then you add the one ten, four tens, and three tens and you get eight tens. So you have eight tens and five ones . . . eighty-five.

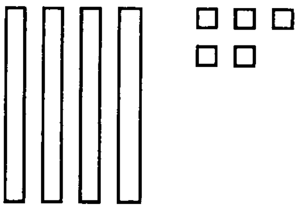


$$\begin{array}{r} 47 \\ \underline{+\,38} \\ 85 \end{array}$$

Figure 4. Addition Algorithm.

Meg used a similar explanation to show the meaning of 45 - 27.

With subtraction, you start with 45 and you want to take 27 away from that. So you take four tens and five ones [Meg drew a picture on a sheet of paper, Figure 5, and then proceeded to illustrate the meaning of each step in the algorithm] and then what you're going to do is take away two tens and seven ones. But you can't take away seven ones from five ones so you have to trade in a 10 for ten ones. Ok, so now you have 15 ones altogether and three tens. So now you can take seven ones away so you have eight ones and you can take two tens away and you have one 10 left. And you get 18.



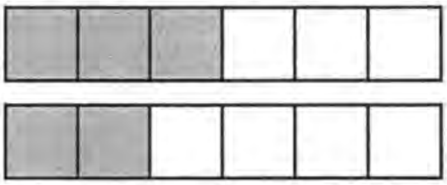
$$\begin{array}{r} 45 \\ \underline{-27} \\ 18 \end{array}$$

Figure 5. Subtraction Algorithm.

Since Meg had mentioned in her diagram that the operations of addition and subtraction needed to be connected to fractions, Meg was asked if algorithms for doing addition and subtraction of fractions were important procedures, and if they were, to give an example for each operation and the meaning of each step in the algorithm. Meg had to think for a few minutes and do some doodling on a sheet of paper before responding.

> Yeah, I think they're important. [Meg wrote on a sheet of paper one-half plus one-third, drew a picture on the sheet of paper, Figure 6, and then proceeded to talk about the meaning of each step in the algorithm]. So if each rectangle is a whole, you want to divide the first one into halves and the second one into thirds. So then you put them together. Let's see, before you get the answer . . . you need the same [size] pieces which are sixths. So you divide the half and you have three sixths and the third, and you have two sixths and you get five sixths. A half plus a third is three-sixths plus two-sixths and you get five sixths.

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$



$$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

Figure 6. Addition and Subtraction of Fractions.

For subtraction, Meg used the problem one-half minus one-third and continued using the picture she had previously drawn.

> So you take away two-thirds from three-thirds and you get one-third. [Meg pointed to the picture, Figure 6, and wrote the equation under picture.]

Since Meg had also mentioned the importance of connecting real life story problems to addition and subtraction, she was asked to tell a real life story problem that could be used. Meg provided the following story for addition of whole numbers:

> Let's say five of the students are going to Ms. Moore's room to help peer tutor and three are going to the library to get books. How many students would be out of the room? So here you would add five and three and you would get eight.

For subtraction of whole numbers, Meg used the same story as above but changed the question to, "How many more students would be going to Ms. Moore's room to help peer tutor?"

Meg used another classroom situation to provide an example of a story for which 1/2 + 1/3 could represent the operation used to solve the problem.

> One half of the class is going to Ms. Moore's room to help peer tutor and a third is going to the library. How much of the class would be out of the room? So you would have one-half plus one-third and you would get five-sixths.

To provide a story for which 1/2 - 1/3 could represent the operation used to solve the problem, Meg used the same story as above but changed the question to: "How much more of the class would be going to Ms. Moore's room?"

In summary, Meg felt that all five of the components (number sense, computation, concepts, vocabulary, and connections) for addition and subtraction were of equal importance. She related the importance of these components to her own experiences in learning mathematics:

> [In elementary school] I learned how to do addition and subtraction without the understanding the concepts and I think it hurt me later. I never really understood base ten and I had no number sense. So it's not that you can't do it [addition or subtraction computation] without understanding it, it just helps you later on when you're adding other things [areas of mathematics] to it [addition or subtraction]. So they're all important . . . plus not only do you need to understand it [addition or subtraction] but the quicker you can do it the more it will help you later on. But you know, you're going to have LD kids who may never be able to get through addition and subtraction. They may never get seven plus eight no matter how much they practice. They need to understand the concept and you need to give them opportunities to do the skill drill, but if they don't get it, you need to give them a calculator. Cause they can still do the problem solving, they can still

> connect it to other things. You know, it's not that it can't be done without parts, it's just harder. I think they're all very connected. So, yes, I think they're all important.

Meg then described the diagram she had drawn connecting multiplication and division. She explained:

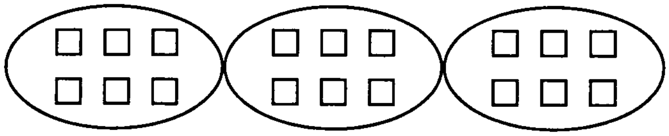
> It's very similar to the one for addition and subtraction where students need to have number sense. They need to understand numbers like whole numbers, fractions and decimals. They need to be able to do algorithms and and develop computational skills. They need to understand the meanings of multiplication and division. They need to have lots of opportunities to problem solve, and I mean, the reason why they do it [multiplication and division], is to connect it to real life problem solving, graphing, story problems, algebra, geometry, and fractions. And, they need to understand the concepts by experimenting with manipulatives, and understand and know the different vocabulary [for multiplication and division].

As Meg explained, her diagram for multiplication and division included the same five major components as her addition and subtraction diagram: number sense, computation, concepts, vocabulary, and connections. Meg explained that she put multiplication and division together because they are opposites:

> I put multiplication and division together cause they're opposites. Division is the opposite of multiplication and multiplication is the opposite of division. If you have a multiplication equation like 3 x 5 = 15 then you know that 15 ÷ 3 = 5 and if you know 15 ÷ 3 = 5 then 3 x 5 = 15.

When asked about the concepts of multiplication and division and the role manipulatives play in the understanding the concepts Meg drew a picture on a sheet of paper, Figure 7, and wrote the equations below the picture. She then replied:

> If you have three times six, you have three groups with six in each one. Multiplication is repeated addition.

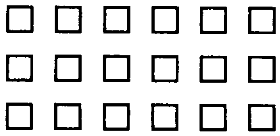


3 x 6 = 18
6 + 6 + 6 = 18

Figure 7. Multiplication as Repeated Addition.

Without the prompting of a request, Meg provided another way of understanding the meaning of the equation 3 x 6 = 18.

> Another way that you can do it is . . . if you have three times six . . . is that you have three rows with six in each row [Meg drew a picture on a sheet of paper, Figure 8, and wrote the equation that follow]. This way you have a rectangular array.



3 x 6 = 18

Figure 8. Multiplication as an Array.

Meg spent about a minute doodling on her paper before providing an interpretation of the meaning for division. After jotting down the picture in Figure 9A, Meg continued:

> With division, you're taking a big group of something, or set, and you're going to break it into even groups. For example, if you have 46 divided by 2, you're dividing the whole group of 46 into two [Meg pointed to the picture on her paper]. Divide the total number of 46 into two equal groups [Meg drew another picture and divided the picture into two groups of 23, Figure 9B, and wrote the computation notation]. If it doesn't go into it equally that's your remainder.

When asked if there were any other important concepts of division, Meg commented: "I don't think so."

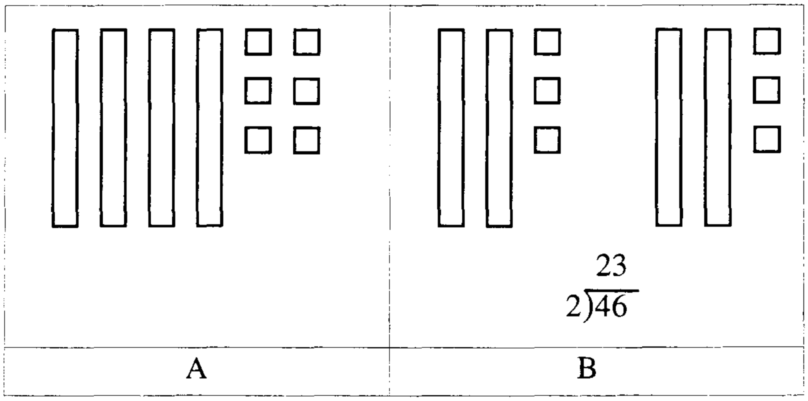


Figure 9. Base Ten Block Model of Division.

In Meg's opinion, multiplication and division facts are important and manipulatives can help students learn these facts:

> They [students] need to see 3 x 6 equals the same as 6 x 3 but when you build them they're different. They [students] need to take the blocks and build the equation. It cements the facts into their brains.

Meg felt that the array model for multiplication was important for students to understand the meaning of multiplication for large numbers and to understand the meanings of the steps in the standard multiplication algorithm. Meg explained:

> When they're [students] multiplying small numbers, they can think of them as groups. I have three groups with six in each group so they can add them all up. With larger numbers, they need to understand multiplication as an array. So if you take 14 times 23, they need to understand that it's 14 rows with 23 in each row. Once again, I think of it with base ten blocks. So, 14 time 23 . . . so you have 14 rows with 23 in each.

Meg drew a picture, Figure 10A, of a 14 x 23 array on a sheet of paper and then continued to explain:

> So you have a hundred block and you're going to add four more rows of ten. And then you have to have twenty-three columns. So it's more than ten so it's going to take two hundred blocks. And then three tens more so you fill in with ones. So 14 times 23 is two hundreds and 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, and 12 more makes 82. So 14 times 23 is 322.

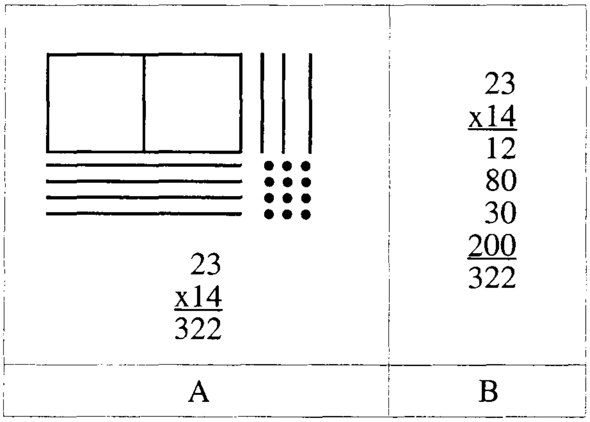


Figure 10. Multiplication Algorithm.

Meg was hesitant about showing how to do the computation for 14 times 23, and tell the meaning of each step in the algorithm. After pausing for a few minutes to study her drawing, Meg replied (using the array model):

> [Meg used the array model, Figure 10A, previously drawn to illustrate what she would do and then wrote the problem 14 x 23, Figure 10B.] So you're now multiplying four times three. It's four times three ones for a total of 12 ones. Which are right there [Meg pointed to the four rows with three in each row in the picture]. And so you could say four times three is 12. And then the next thing you do is you look at four times two or which is four times 20 which is 80 [Meg pointed to the four rows with 20 in each row in the picture]. And that would be put down here. And then you switch over here and you say 10 times three which is 30, which is your three tens right here. [Meg pointed to the 10 rows with three in each row in the picture]. And then you say 10 times 20 [Meg pointed to the ten rows with 20 in each row in the picture], which would be your 10 rows with 20 in each row or two hundreds. And you add all those up together and that 322.

To show 322 ÷ 14, Meg used the array model, Figure 10A, she had used for multiplication. She explained division in the following manner:

> In dividing, you need to look at your total. So for example, I have 322 and I'm going to divide it by 14. Take 322 and divide it into 14 rows. So I use my blocks again . . . my ones, tens, and hundreds blocks. And I take 322 and I want to divide it into 14 rows [Meg pointed to the picture, Figure 10A] so when I divide it into 14 rows, I end up with 23 in each row.

With the prompting of a request, Meg showed the algorithm for long division but when asked to explain the meaning of each step, Meg doodled with the algorithm and picture for several minutes before confessing that she was unable to explain the meaning. She replied:

> So if you have 322 divided by 14. How do I connect it to the algorithm? That's a very good question. I should know this . . . and at this point I'd need to think about it for a while.

Since Meg had mentioned in her diagram that the operations of multiplication and division needed to be connected to fractions, Meg was asked if algorithms for doing multiplication and division of fractions were important procedures, and if they were, to give an example for each operation and the meaning of the algorithm. Meg immediately replied: "I can do this." She took a minute to collect her thoughts, think of a problem, and then continued: "Yes, I think they're important."

Meg wrote on a sheet of paper one-third times one-fourth and responded:

> If I multiply one-third times one-fourth, I would say one-third of one-fourth. I would connect it to fraction [factory] pieces. So I would get a fourth block and you need to break it into thirds. So, if I break a fourth block into thirds . . . what small piece would be one-third of one-fourth, which would be one-twelfth.

After Meg wrote the equation she was prompted to draw a picture to show what she had just said.

> Ok, this is one [pointing to a fraction piece, Figure 11], then with the fraction factory pieces . . . so one-fourth . . . so here would be one-fourth of this and I have to break that one-fourth into thirds. And one-third of that would be your answer. The one [piece] that equals that would be one-twelfth.

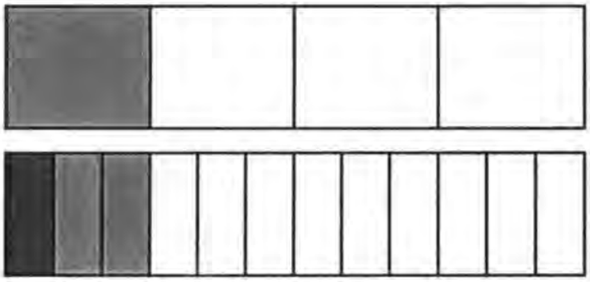


Figure 11. Multiplication with Fraction Pieces.

When asked about division of fractions and what division of fractions means, Meg began with an attempt to explain the meaning of division but then resorted to explaining the "invert and multiply" algorithm:

> If I was to take 3/4 divided by 1/2. I'm going to take 3/4 and break it into one-half. So I have my 3/4 block, so I'm going to take this and I'm going to break it into one-half . . . so I'm going to find the equivalent shape for it. [Meg wrote the equation: $3/4 \div 1/2 = 3/4 \times 2/1 = 6/4 = 3/2$.] So for my algorithm, I just find the reciprocal and I say three-fourths times two over one . . . I did that wrong . . . so three-fourths divided by . . . which means how many halves . . . I'm not sure now. I'll have to think about this for a minute.

After thinking for a few minutes, Meg was not able to to provide a meaningful response.

Since Meg had specified the importance of connecting real life story problems to multiplication and division, she was asked to tell a real life story problem for each operation. Meg supplied the following story for which $1/3 \times 1/4$ could represent the operation used to solve the problem.

> Today, one-fourth of my class is in band and strings . . . and one-third of that group is absent. What part of the class will go to band and strings today?

Meg was unable to supply a story for which $3/4 \div 1/2$ could represent the operation used to solve the problem. Her reply was as follows:

> Three-fourths of the band and strings is . . . let me think . . . [long pause] I need to go back to cooking. I need three-fourths of a cup and

> I'm going to cut the recipe in half. So normally I need three-fourths of a cup . . . [long pause while Meg compared what she had said with the equation she had written, $3/4 \div 1/2 = 3/4 \times 2/1 = 6/4 = 3/2$ that wouldn't match though . . . Cooking . . . [long pauses] I should know this but I'm going to need more time to think about it.

In summary, Meg felt that all five of the components (number sense, computation, concepts, vocabulary, and connections) for multiplication and division were of equal importance. She again related her reasons to her own experiences in learning mathematics:

> Once again, you can do it [multiplication and division computation] without [knowing all of the topics in the diagram] . . . like maybe a student will know how to get 14 times 23 or 322 divided by 14. They [students] may be able to do the computational part but they need to be able to connect it to the other stuff . . . number sense, computation, concepts, vocabulary, and story problems to really understand it. So to do it well, yes, they need all of the parts [components in the diagram, Figure 1].

Although Meg felt that all of the components in her diagrams for addition and subtraction and for multiplication and division were important, she felt that understanding the concepts and being able to apply concepts and procedures to real life problems was of major importance:

> They [students] need to understand the concepts and they need to [be able to] solve real life problems. That's the most important. They need to know the other things but you can let them use a calculator to do the computation. But problem solving and knowing what they all mean [addition, subtraction, multiplication, and division], that's what's really important.

Meg said that she had never thought about these operations in this way before. She responded in the following way:

> It's good for me to do it. It helps to see the connections between all areas and the importance of number sense, computation, understanding concepts, and vocabulary for all of them [addition, subtraction, multiplication, and division]. It makes me think about the Standards [NCTM, 1989], and materials I'm using, and types of activities for all areas. I think this way of looking at content areas is helpful in formulating a clear picture.

Classroom Profile

Meg's class was observed for all 20 lessons of a seven week unit on multiplication and division. The class was taught three days a week, Tuesday, Thursday, and Friday, and was scheduled for one hour a day (from 1:15-2:15). On average, the class lasted about 65 minutes, rarely ending on time. The class included 23 students grouped from four multi-age three-four grade classes. The four multi-age classes were grouped for mathematics according to grade level and efforts were made to balance the classes with respect to gender, ethnicity, behavior, and special education placements. The students spent the remainder of the day in their original three-four multi-age classroom.

Generally, the first 20 minutes of the class followed an established routine. Students began changing classrooms a few minutes before the start of the mathematics period. Several of the students remained in the classroom while other students filtered into the room quietly, taking assigned seats. For the first four weeks of the unit, classroom desks were arranged in groups of six and during the remaining three weeks the desks were arranged in pairs. According to Meg, the desks were arranged in this manner for the entire school day and the grouping for mathematics activities was usually different from these arrangements. The sizes of groups for mathematics varied from two to six students per group.

The students used the first few minutes of the class to organize their materials, and then began working on the warm-up problems that were written on the front chalkboard. A daily warm-up was an established classroom routine. On several occasions, Meg spent a brief moment at the beginning of the class reminding students that they needed to be working on the warm-up, but mostly students began working without prompting. The students were to complete the warm-up on their own during the first 10 minutes of class time. The problems were to be transcribed and solved in their notebooks. Meg mainly used this time to organize her own materials and to observe students working at their desks.

When it appeared that most students had completed the warm-up, Meg had specific students put their answers next to the problems on the chalkboard. Throughout the unit, everyone seemed to have several opportunities to put their answers on the board. All of the students were expected to check their own work with the answers students provided. Meg began a discussion by asking who agreed and who disagreed with the answers to each problem written on the chalkboard. Frequently, several students would disagree with an answer and Meg would ask the student who had provided the answer to explain what he/she had done and why. Sometimes lively and insightful discussions occurred but often students were able to quickly find their errors since their mistakes were most often due to carelessness. After the warm-up problems were discussed, the students would write the number correct next to the problems and put away their notebooks.

Warm-up activities were followed by a variety of activities with no customary routine. These activities included the following components: directed lesson segments, explorations, discussions, practice exercises, and assessments. For Meg, directed lesson segments were interactive and focused on introducing procedures, materials, and language that the students needed for tasks that followed or reviewing and presenting concepts and skills. Explorations consisted of collaborative and independent projects and problem solving tasks. These tasks provided students opportunities to explore and use mathematical ideas, representations, and language. Whole-class and small-group discussions were used to encourage students' questions and deliberations and to provide students opportunities to share their thinking and justify results. Practice exercises were used to give students opportunities to reinforce and practice what they learned, and assessments were used to determine what student knew and were able to do.

It is important to note that the separation of the different types of activities (directed lesson segments, explorations, discussions, practice exercises, and assessments) of the unit does not imply a separation in Meg's instruction. These activities were usually interwoven in lessons. For example, explorations included directed lessons and discussion. Also, according to Meg, a major part of assessment consisted of watching and listening to the students during directed lesson segments, explorations, and discussions.

Rather than being tied to the content and format of the textbook, Meg had strong opinions on what content should be taught and how this content should be taught. This philosophical stance that she expressed in the interview was obvious in the manner in which she sequenced her unit and the content included. Meg did not follow the sequencing of the proposed text, *Exploring Mathematics* by Scott, Foresman and Company. The sequencing and content of the unit seemed to be more determined by the other materials and resources Meg used including: manipulatives, problem and project booklets, and children's stories. Figure 12, presents Meg's instructional emphasis and sequencing; the unit objectives are in Appendix F.

As displayed in Figure 12, the first two days and last day of the unit focused primarily on assessing students' knowledge of and ability to do multiplication and division. Days three through eight included concepts and skills associated with multiplication. The remaining 11 days of the unit encompassed concepts and skills on both multiplication and division as well as connections between the two operations.

Meg used various instructional materials and resources for posing mathematical tasks. The text was used only as a resource. A variety of manipulatives were used for exploring and developing concepts, ideas, and processes including: base ten blocks, rectangular grid paper, rainbow cubes, and various drawings and pictures. For example, these manipulatives were used to represent multiplication as repeated addition and as an array, division as equal sharing and as an array, as well as model an assortment of multiplication and division problem situations. Problem and project booklets included: *Connections, Constructing Ideas About Multiplication and Division*, and *Workmat Mathematics: Understanding Multiplication and Division* by Creative Publications. Activities from these booklets provided projects and

Week	Tuesday	Thursday	Friday
1	Warm-Up (I, WC/T); Overview of Unit (WC/T); Mad Minute (I)	Warm-Up (I, WC/T); Pre-assessment (WC/T & I)	Warm-Up (I, WC/T); Project: Multiplication Matrix (WC/T, SG/C)
2	Warm-Up (I, WC/T); Project: Multiplication Matrix (WC/T, SG/C)	Warm-Up (I, WC/T); Project: Multiplication Matrix (SG/C)	Warm-Up (I, WC/T); Project: Multiplication Matrix (SG/C, WC/T); Story: *Sea Squares* (WC/T)
3	Warm-Up (I, WC/T); Meaning of Multiplication repeated addition and arrays, concrete representations, and real life story problems (WC/T, I); Story: *One Hundred Hungry Ants* (WC/T)	Warm-Up (I, WC/T); Solving, writing, and concrete modeling real life multiplication story problems (WC/T, I)	Grid paper representation of multiplication facts (WC/T, I) Meaning of division and connection between multiplication and division discussion (SG/T)
4	Mad Minute (I); Meaning of multiplication and division (WC/T)	Warm-Up (I & WC/T); Project: Bug Books (SG/C); Workmat Math (WC/T, I)	Project: Bug Books (SG/C); Building concrete representations of multiplication and division (SG/T)
5	Mid-unit assessment (I); Project: Bug Books (SG/C) Complete make-up assignments (I)	Mad Minute (I); Building concrete representations of division (SG/T); Project: Bug Books (SG/C)	Problem solving: Sharing Marbles (WC/T, SG/C)
6	Warm-Up (I & WC/T); Building representations of multiples (WC/T, I)	Warm-Up (I, WC/T); Building array representations for multiplication and division (WC/T) Wanted Posters (WC/T, I)	Warm-Up (I, WC/T); Building grid paper representations of factors (WC/T, I)
7	Warm-Up (I & WC/T); Multiple, factors, & prime numbers (WC/T); Workmat Math: multiplication and division computation (WC/T, I)	Post-assessment (WC/T, I)	

Coding: Whole-class activity teacher-directed (WC/T), Small-group collaborative activity (SG/C), Small Group teacher directed activity (SG/T), Independent activity (I), Whole-class teacher-directed activity and independent activity (WC/T, I)

Figure 12. Meg's Instructional Emphasis and Sequencing.

problem solving activities. Children's stories were used to model square numbers and provide an array model for multiplication. Real life representations of multiplication and division were developed through solving and writing stories.

Projects. The unit included three projects: creating a Multiplication Matrix, making Bug Books from the resource book *Connections*, and creating a Wanted Poster. The Multiplication Matrix activity involved creating multiplication charts with grid paper rectangles constructed on large sheets of butcher paper. The Bug Books project involved students inventing "crazy bug" collections and writing multiplication and division equations to explain their pictures. The books were to have at least 20 pages and each page was to have any number of bugs, but they were all to be the same kind (same number of legs). Each page was to have a multiplication equation telling about the total number of legs in the bug collection on that page and a counter part division equation expressing the same relationship. The Wanted Poster project consisted of making a Wanted Poster for a favorite number. The students were to be neat, accurate, and creative in writing sentences and drawing pictures that described a number greater than two.

Problem Solving. The unit included several problem solving activities, Rainbow Multiples and Rainbow Factors, from the activity book *Constructing Ideas About Multiplication and Division*, and other problem solving activities involving sharing marbles and solving, telling, and writing real life story problems. In the Rainbow Multiples lesson, students worked in pairs using Rainbow Cubes on hundreds charts to show multiples of two, three, four, five, and six. First multiples were to be found by putting cubes on Rainbow Multiples hundreds chart and then the patterns that emerged were extended to find all of the multiples to 100.

In the Rainbow Factor activity students explored factors of numbers from one to 100 using Rainbow Cubes by building all the possible rectangles for a given number. After cutting out the squares, they completed a recording sheet by pasting down their grid paper rectangles to show their findings. Students then wrote multiplication equations on the rectangles and listed the factors of that number. To conclude the activity, students' findings were displayed and discussed.

The Sharing Marbles problem involved students working together to figure out how to share a bag of 17 marbles that they found. The students were to tell how many marbles each person will get and why their solution makes sense.

Story problems (word problems) were used in various ways. In several activities, the students were presented with problems, orally and in writing, and guided by Meg through solutions to the problems on the overhead using blocks and picture. Multiplication and division equations that could be used to solve the problems were written. In other story problem activities, students were to write their own problems. The story problems were to follow two rules: (1) Each problem must have at least three sentences; (2) Each problem must end in a question.

Literature. Two children's stories were read to the class. The children's story *Sea Squares* by Joy Ann Holm is the story of some creatures that live on the ocean floor. The story presented a real life representation of square numbers. The story *One Hundred Hungry Ants* by Eleanor J. Pinczes is the story of 100 hungry ants hurrying to a nearby picnic. This story presented an array model for multiplication.

Directed Activities. Frequently, directed activities were used with whole-class and small-group discussions to review or present concepts and skills. These lessons included concept development and manipulative worksheet activities. Concept development lessons were usually interactive, involving Meg and students. These interactions were used to review and explore the meanings of multiplication and division concepts and skills and relationships between them.

Manipulative worksheet activities included activities from: *Workmat Mathematics*, *Constructing Ideas About Multiplication and Division*, and the *Exploring Mathematics* textbook. In most of these activities, students explored multiplication and division facts and computation problems by building concrete representations of the problems using cubes, base ten blocks, or grid paper or by drawing pictorial representations. Worksheet or textbook exercises were mostly used to guided the specifics of these activities.

Practice. Practice activities were used daily to review and reinforce skills and concepts for the current topics as well as other mathematics topics. These activities included warm-ups, classwork, and homework. Warm-ups constituted a major portion of seat work. These problems consisted of review problems from other topics as well as problems applicable to current topics. Both skill and concept development problems were included.

Mad Minute worksheets consisting of 30 multiplication facts on one side of the sheet and 30 divisions facts on the other side were used as practice as well as individual assessment activities. The students were to do as many multiplication problems as they could in one minute. After one minute they were to stop and count how many they had completed, write down the number, and then circle it. After 120 seconds they were to stop again, and then again after 180 seconds. When the students finished the multiplication portion of the assessment they were to do the division side in a similar manner. Students' scores over the first 60 seconds were used to evaluate students' improvement.

Seat work activities were mostly extensions of explorations, directed lesson segments, or discussions. These seat work activities generally involved exploration as well as reinforcement and practice. Homework assignment were either the completion of seat work activities or activities from textbook resource practice sheets. The practice assignments consisted mostly of puzzle-like drill work and short problem solving tasks.

Mastery of the basic multiplication and division facts was an on going homework assignment. The students were expected to have their parents or brother and sister drill them on the facts daily. The basic facts for multiplication referred to those combinations where both factors were less than 10 and the basic division facts included the counterparts to the basic multiplication facts.

Assessments. A two part pre assessment, given at the beginning of the unit, included a Mad Minute worksheet and a comprehensive assessment of multiplication and division constructed by Meg. The comprehensive portion of the test included some facts, computation, and vocabulary, but the problems mostly focused on concepts. Students were given about 30 minutes to complete this assessment.

A mid-unit assessment was given that included another Mad Minute worksheet and a second assessment constructed by Meg. The mid-unit assessment constructed by Meg included only problems that focused on concepts. A third Mad Minute assessment was also given at the conclusion of the unit. For the unit final, Meg used the assessment designed by the researcher.

Multiplication Development

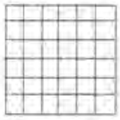
In Meg's unit, multiplication ideas and situations were introduced from several perspectives: concrete, pictorial, symbolic, and through real-life context. Manipulatives were used to introduce students to various grouping and arrays models, repeated addition was used as a numerical model, and stories and problem situations were used to provide a real-life context. The standard mathematical representation of multiplication was introduced in the context of the activities, helping students connect the abstract representation to their own experiences.

As previously stated, warm-up activities were an established classroom routine and about 19% of class time centered around these activities. During the first few weeks of the unit, the warm-up problems consisted mostly of review from other topics with one or two problems involving multiplication. An example of such a warm-up is shown in Figure 13. Later in the unit, the warm-ups consisted mainly of problems applicable to multiplication and division with an occasional review problem included. Figure 14 is an example of one such warm-up.

1) 48,924 +28,492

2) 30,030 -12,045

3) Complete the pattern: 24, 32,36, __, __48

4) Find the perimeter. 

5) Write the multiplication sentence for 7 + 7 + 7 + 7 + 7+ 7 + 7.

Figure 13. Day 6 Warm-up.

1) Write six multiples of 5.

2) Write a multiplication and a division equation.

3) Write 6 + 6 + 6 + 6 as a multiplication sentence.

4) Find 36 ÷ 4.

5) 9,406 -529

Figure 14. Day 13 Warm-up

The analysis of classroom transcripts and field notes indicated that 33% of the warm up problems involved multiplication concepts and skills, 14% involved division, 5% involved concepts associated with both multiplication and division, and 48% of the problems consisted of topics other than multiplication and division. These multiplication problems involved mostly conceptual development. Geometric models of multiplication included finding areas of rectangles and representing multiplication as an array. Numerical problems involved writing multiplication as repeated addition and repeated addition as multiplication. Real-life context problems included solving and writing story problems.

Meg's initial project, creating a Multiplication Matrix, introduced students to a geometric model for multiplication. According to Meg, the reason for creating the Multiplication Matrix was to explore multiplication and, in particular, give students a concrete representation of the multiplication facts. The multiplication tables that emerged were then used to give students something to think about and discuss for the rest of the unit. This array representation of multiplication was also used to explore relationships between and among multiplication and division.

Four lessons (about 12% of class time as determined through the analysis of classroom transcripts and field notes) in the unit centered around the creation of these multiplication charts. At the beginning of this activity, Meg asked students to come up and sit on the floor in front of her. Meg explained what was to follow by first reminding the class about the lively discussion they had just completed in the warm-up activity. The problem consisted of finding the area of a 6 x 4 rectangular grid:

> Earlier today you were saying, "Wait a second. That's 6 x 4 not 4 x 6." Well, we're going to take a look at multiplication today and figure out, what exactly is 6 x 4 or 4 x 6.

Meg then pointed to the Multiplication Matrix she had taped to the chalkboard in the front of the classroom. Only the first few rectangular arrays for the chart had been completed and Meg explained to the students how they would be working in small groups to create multiplication charts with grid paper rectangles later in the class."When you look at this chart, what do you think it shows?" Meg asked Brad. "Multiplication facts," Brad responded. "Why?" Meg asked. Meg continued to discuss the chart pointing to a grid paper rectangle with two rows of three, Meg asked: "How many rows are in the rectangle? How many squares are in each row? How many squares are there altogether? Why?" Although most of Meg's questions during the lesson could be answered with a single word or phrase, Meg usually followed-up the students' responses with the question, "Why?" Generally, the students were expected to raise their hands to be called on, but sometimes students answered in chorus and occasionally student-to-student discourse occurred without the normal protocol.

Meg then discussed the directions for creating the multiplication chart. When she finished discussing the directions, Meg used the next few minutes to remind students about the different roles each group member would play in creating the Multiplication Matrix. At this point in the lesson, Meg seemed pressed for time and her directions were hurried with little opportunity for student interaction. Several students, during the discussion, seemed to lose interest as the questioning and explanations dragged on but the students were not openly disrespectful. The students either listened and responded appropriately or were engaged in off task behaviors such as fidgeting and day dreaming. When Meg finished discussing procedures and materials for creating the charts, the students had about 10 minutes to work on the project before the end of the class.

The next day Meg again discussed the directions and group member's roles in creating the matrix. Meg felt that she had hurried through the directions so she spent most of the second day on the project reiterating what she had said the previous day. The students then had about 20 minutes to work in their groups. As they worked, Meg moved around the room watching and listening to the groups. She stopped to help a few groups organize their rectangles and asked several questions. She reminded one of the groups to write the equations on the rectangles and another group to be sure to lay out the rows and columns before writing the numbers on the sides. Most groups seemed to be actively engaged in what they were doing and worked well together.

The third day and most of the fourth day of the project were used to complete the multiplication charts. Meg felt that the project had taken more time than expected and that time always seemed to be a critical factor in grouping students from several classes. Meg felt that they would have been able to complete the project in two days had the students not been grouped from other classes.

When the groups were done making their charts, near the end of the fourth day of the project, Meg asked students to come up and sit on the floor in front of her. Pointing to the Multiplication Matrix taped to the front chalkboard, the following lesson segment transpired:

> Meg: I want you to take a look at the chart and seeing if you notice any patterns. You know what, if you have a pattern that you would like to share, an observation, please raise your hand, so we can all hear you. Ann, what do you notice?
>
> Ann: The ones that are going across get bigger and that are going down get bigger.
>
> Meg: Okay, as you go across it gets wider and as you go down it gets longer. Great, what else? What other observations do you see, Jack?
>
> Jack: It gets bigger as you go sideways.
> Meg: What do you notice about those diagonals, Jack?
>
> Jack: They are perfect squares and they get bigger.
>
> Meg: They are perfect squares and they get bigger. Good observation, wonderful. Mickey, what did you notice?
>
> Mickey: When you go diagonal, they're all doubles, 7 times 7, 8 times 8, 9 times 9.
>
> Meg: Great, great, good. What other observations do you see? Jackie, what do you notice?

A lively discussion continued in this manner for about 15 minutes. Throughout the discussion, Meg attempted to help students make connections between and among the ideas generated. Meg concluded the discussion by reading the story *Sea Squares*. Although the story provided a real-life representation of square numbers and reinforced the “main diagonal pattern” in the Multiplication Matrix, Meg did not attempt to summarize the many other insightful ideas that were generated by the students.

A writing assignment followed a short discussion of the story. Meg put several words on the chalkboard for the students to use in their writing: product, array, row, column, and equation. The students were to use pictures, diagrams and words to answer:

> What patterns do you see in the chart? What did you learn about multiplication? And could this matrix be used for division?

Eight worksheet homework assignments were made during the four days students were creating their Multiplication Matrix. Seven of the worksheets involved skill practice with place value, addition computation, counting area, and multiplication facts. The other worksheet involved a concept development activity similar to the Multiplication Matrix project. In this assignment, the students made a multiplication facts table by drawing a dot-array for each multiplication fact in the table and writing the product. To Meg, this table was important in reinforcing what the students had learned in creating the Multiplication Matrix.

The day following the completion of the Multiplication Matrix project, a lengthy teacher-directed lesson occurred that included reviewing the meaning of multiplication, reading a story, building arrays on the overhead, solving story problems by building arrays, and writing story problems. Meg began the lesson by asking the students to leave their notebooks open because she wanted them to take some notes. Meg explained that when she had checked what they had written on the Multiplication Matrix assignment, she was very disappointed. When asked about this comment later, Meg said that most of the students in the class had only written one sentence about what they had learned. She told the class that one of the things she had learned over her years of teaching was that if only a couple of students did not understand the lesson then all they needed was a little more practice. But if a whole bunch of students did not understand it, then it usually meant that they needed to go back and take another look at what they had done. Meg then pointed to the multiplication matrix charts hanging in the back of the room:

> When we looked at the arrays last week, we noticed patterns. We have vertical columns . . . follow with your eyes . . . we have

> columns going up and down . . . and rows go sideways, and then we put together a multiplication matrix. Well, what does it mean to multiply?

Meg proceeded to discuss the definition of multiplication with the class. Several times during the discussion, Meg wrote, erased, and rewrote the students' notions of multiplication on the chalkboard. As the students shared their ides, Meg sometimes drew pictures or symbolic expressions to help connect what students were saying as the definition unfolded. When everyone was satisfied with the definition, Meg read what they had developed: "Multiplication is taking groups of equal numbers and adding them? It's a faster way of adding."

In third grade, the students had learned multiplication as repeated addition. Prior to this lesson, several of the warm-up problems had focused on this numerical representation. The definition of multiplication as taking groups of equal numbers and adding them conveyed multiplication as repeated addition. This definition had little to do with the array representation (Multiplication Matrix) that Meg had alluded to when she began the discussion. Despite this conflict, everyone, including Meg, seemed satisfied with what they had developed. Meg then had the students write the two sentences in their notebooks.

Meg next used the story, *One Hundred Hungry Ants* to provide a real-life context for multiplication. The story was also used to motivate an array activity using base ten blocks. In the activity, Meg guided the students as they built several arrays related to the story. When they had finished building the two arrays, Meg told the following story problem to the class: "There were four groups working on matrixes [matrices]. Each group had three students. How many students were there all together." Meg then wrote the story on the chalkboard. Meg had Wanda model the story problem on the overhead using base ten blocks. Even though the story problem used the word "groups," Wanda was to use an array to model the problem. Wanda built a 4 x 3 array on the overhead and then recorded her answer on the chalkboard. The classwork assignment that followed was similar to the activity the students had just done. The students were to write eight real-life story problems with each problem having at least three sentences and ending in a question. They were to draw pictures using arrays and write multiplication sentences to model their stories. The students finished the classwork assignment for homework.

Meg continued the story problem activity the following day. She began the lesson by having several students share their stories and solutions from the homework. After discussing several stories, Meg asked what the students thought about the homework. Jamie said, "It was kind of easy because once you did the first one, you just had to change the words and groups." Several other students agreed. Meg then spent the next 20 minutes telling and solving story problems with the class. The format of the stories was consistent with the assignment the night before; the problems had at least three sentences and each problem ended in a question.

Meg then read another story:

> There are 3 chickens. The 3 chickens had 4 eggs each. How many eggs are there in all? Build it. There's 3 chickens, each chicken had 4 eggs each. How many eggs are there in all?

The students were not clear as to what they were to do so Meg modeled the problem on the overhead by drawing an array of dots. Two students then solved the next two problems on the overhead. Meg read the stories:

> There are 10 ants. They each eat nine crumbs. How many crumbs do they eat altogether?
>
> My mom wants to bring home homemade cookies for my soccer team. The team has nine people. She wants each child to have two cookies. How many cookies does she need to bake?

When Meg finished discussing the third problem, she asked the students to pretend they were going to a concert. "Who's your favorite band? Meg asked. "Back Street Boys." Bill replied. "Back Street Boys?" said Meg. "If you got to go to their concert, where would you like sit?" As the discussion continued, Meg had six students stand in a three by two array to pretend they were seated at the concert. The class acted-out and discussed several other seating arrangements and recorded multiplication equations for each arrangement.

After discussing the concert seating, Meg wrote the problem, 4 x 7 = 28, on the chalkboard horizontally and vertically. She then built and drew array representations for the equation using base ten blocks and dots on the overhead while the groups modeled what Meg was doing at their desks. After doing another example, Meg explained that they would be drawing arrays for homework similar to what they had just done. The assignment was guided by a *Workmat Mathematics* worksheet. It involved drawing arrays of dots to match words and equations.

Seat work the following day involved cutting out array representations for multiplication facts. The activity was adapted from a textbook exercise that consisted of finding multiplication facts. Meg extended the textbook activity to include cutting grid paper arrays to represent the facts. Half of the class did the seat work while the other half joined Meg in a small group to explore the idea of division using arrays.

A whole-class teacher-directed discussion was used the following day to further develop what had been discussed in the small-groups. It was decided in this discussion that since multiplication was taking groups of equal number and adding them, division must be subtracting groups of equal number. And since multiplication was a faster way to add, division must be a faster way to subtract. Everyone in the class seemed to agree with these relationships.

Meg concluded the lesson by discussing the homework. The assignment involved building concrete representations of multiplication facts using base ten pictorial representations. A worksheet from *Workmat Mathematics, Constructing Ideas About Multiplication and Division* was used to guide the specifics of this activity.

The warm-up activity the following day involved writing multiplication and division story problems that could be represented by a three by five array. Meg reminded the students that each story was to have at least three sentences and should end in a question. After several minutes, Bill read his story problem to the class:

> There are three apples. Every apple has five seeds. How many seeds are there altogether?

Everyone agreed that Bill's story was correct as well as several other multiplication stories that were read.

After discussing the warm-up, Meg had students share their solutions to the multiplication homework worksheet from the previous day. Several students drew and built arrays to model multiplication facts on the overhead and recorded their solutions with multiplication equations.

A whole-class teacher-directed lesson segment was then used to introduce procedures, materials, and language that the students needed for making Bug Books. For Meg, the purpose of this activity was to help students construct creative pictorial representations of multiplication and division.

Meg introduced the project by showing a Bug Book she had created. She then discussed how they were to make their own books and how the books would be graded. This small-group activity lasted parts of five days and was done in conjunction with other small-group activities in the unit. The books were collected at the conclusion of the project but they were not discussed in class.

While students were making Bug Books, Meg used several opportunities to work with small-groups of students that were having difficulty with the meanings of multiplication and division. The students would join Meg in the back of the classroom and build concrete representations using base ten blocks on the floor in front of her. When Meg felt the students were able to build arrays for a variety of multiplication and division facts, she sent them back to their seats to continue making the books.

The Rainbow Multiples activity occurred several days later. In this activity, Rainbow Cubes were put on hundreds charts to show multiples of several numbers.

Meg first asked the students what they thought a multiple was. She then used addition to explain multiples of two.

Meg: What's 2 + 2?

Students: 4.

Meg: 2 + 2 + 2 ?

Students: 6.

Meg: 2 + 2 + 2 + 2?

Students: 8.

Meg: 2 + 2 + 2 + 2 + 2?

Students: 10.

Meg: These are multiples of 2.

Meg then explained that they were going to look at multiples of two in another way by looking for patterns on a hundreds chart. Meg started by putting a cube on the number two on the Rainbow Multiples hundreds chart on the overhead. She then pointed to each consecutive square on the chart counting, "One, two, one, two, one, two . . ." each time putting a cube on a square whenever she said, "Two." She continued counting by two's and putting cubes on the numbers 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24, 26, 28, 30. Meg then told the class that the numbers covered were multiples of two and asked them if anyone could see a pattern: "Who can tell me the next number on the chart that's a multiple of two?" Almost everyone in the class responded and Meg called on one of the students who gave several more numbers in the pattern.

Meg had the students do a similar activity in groups of threes. The groups were given Rainbow Multiples hundreds charts and a number from three to six (by the roll of a die). Each group was to find all of the multiples of their given number between one and 100 by first finding several multiples of the number and then extended the pattern. Rainbow cubes were placed on the first few multiples of the number and then each multiple was colored to reveal a pattern. While the students worked in their groups, Meg walked around the room checking and questioning each group.

When the groups were finished, they taped their completed Rainbow Multiples hundreds chart on the chalkboard. Meg then discussed the patterns that the students observed. Meg began by asking if anyone noticed any patterns. Jackie, noticed that the multiples of 10 go straight down under the 10. Mickey noticed that the multiples of nine go across until you get to 81, then the pattern continues over at 90. Mickey was also able to predict the next two numbers beyond the chart, number 108 and 117. Jamie noticed that the threes go in diagonals 3, 12, and 21, then 6, 15, 24, 33, 42, and 51, and then 9, 18, 27, and so on. After several other patterns were generated, Meg ask:

> Meg: Who can tell me what a multiple is? Jill, what do you think a multiple is?
>
> Jill: It's counting by the number it's a multiple of.
>
> Meg: You're on to it. I think I know what you are trying to say. So you are saying you take a number, and what are you doing to that number?
>
> Jill: Adding the number on to it.
>
> Meg: Ok, so you are taking that product, or the answer of the number and another number?
>
> Jill: Yeah.

Although it appeared that Jill was thinking of multiples in terms of addition and not as a product, Jill agreed with Meg. This type of maneuvering in discussions sometimes occurred. Meg appeared to be twisting what Jill had said in order to generate a predetermined definition that she wanted the students to discover. Typically, Meg rarely summarized discussions so many of the ideas generated by the students, whether correct or not, were not made explicit.

When the class finished discussing multiples, Meg quickly discussed the homework assignment. The assignment involved practice with finding multiples and missing factors in multiplication equations.

Wanted
for pairing off
2
Last seen as a couple.
Can be found by dividing
6 by 3 and 10 by 5.
Also known as 1 + 1.
Has only two factors, 1 and itself.
Has been called the loneliest
number since the number 1.

Figure 15. Meg's Wanted Poster.

The Wanted Poster project was done on the seventh day of the unit, a day used for catching up on missing assignments. This independent project involved each student making a Wanted Poster for a favorite number. According to Meg, the project provided students an opportunity to use mathematical ideas, representations, and language. Meg modeled what the students were to do by showing a Wanted Poster for the number two as in Figure 15.

The students were to choose any number other than two, be neat, accurate, and creative. Somewhere on their posters they were to have at least one idea involving multiplication and one involving division. Meg then moved around the room helping students on the project and explaining to them the assignments they had turned in and those that were missing or needed to be redone. The students used the remainder of the class completing missing assignments and working on their posters. A few students had all of their assignments completed and were able to begin at once working on the project but several students worked the entire period on missing assignments and needed to complete the poster for homework.

On the following day, the students explored factors of numbers from one to 100 using Rainbow Cubes by building all the possible rectangles for a given number. Meg began the lesson by telling each student to get 12 Rainbow Cubes and arrange their cubes to make different rectangles. On Meg's request, Jack built a 3 x 4 rectangle on the overhead. Meg then asked the students to write the multiplication sentences for the rectangles they found. With some assistance by Meg, Jack wrote 3 x 4 = 12 and 4 x 3 = 12 on the chalkboard. Next to Jack's rectangle, Brad built a 2 x 6 rectangle and recorded the multiplication sentences that were represented by the rectangle. The discussion continued until they had built all possible different rectangles. Meg then had the students tell the numbers that could be multiplied to get 12. As the students provided the number (not necessarily in order), Meg wrote the numbers in increasing order on the chalkboard: "1, 2, 3, 4, 6, 12." Meg then explained that these numbers were the factors of 12. When the students were unable to generate a definition of a factor, Meg ask the class to turn to the glossary and find the definition. Bill read the definition: "A number that divides evenly into a given number is a factor of that number."

Following a short discussion on the meaning of the word "factor," each group was given a bag of Rainbow Cubes, a sheet of grid paper, and a number. The students were to find all factors of the number by first building different rectangles and then cutting the rectangles out of grid paper. Meg demonstrated cutting out rectangles for the number 12.

As the groups actively worked, Meg moved from group to group listening and talking to the students. After a few minutes, Meg had several students that seemed to be having some difficulty with the activity come to the back of the room so she could help them. For the students having difficulty, Meg suggested that they begin with a rectangle with one row, then try a rectangle with two rows, and so on.

Most groups were unable to complete the Rainbow Factor activity in class so the activity was to be completed for homework.

The next day Meg asked the students to get out their factor sheet from the previous night's homework. The class then discussed the factors of the numbers 13, 17, 24, 39, and 14. For each number, Meg wrote the students' responses on the board and then checked and discussed the factors with the class. The discussion led to those numbers that had exactly two factors, one and the number itself. After a discussion about these numbers, Meg ended the discussion by telling students that the numbers with exactly two factors were called prime numbers.

Meg then explained the seat work and homework by guiding the students through a manipulative worksheet activities from *Workmat Mathematics*. The activity involved exploring one-digit by two-digit multiplication computation problems by building base ten representations of the problem. A worksheet and textbook exercise were used to guide the specifics of this activity.

Meg began by writing the problem 3 x 14 in computational form (vertically) on the overhead and building a base ten block array to represent the product. Meg then discussed each step in the algorithm and its relationship to the concrete representation she had built. After discussing another problem from the worksheet, Meg guided the students through the problem, 3 x 24, represented in the textbook. The classwork and homework that followed consisted of the completion of the Workmat Mathematics worksheet and the textbook exercises.

According to Meg, time was a major factor in not being able to do more computation. Although Meg felt that the multiplication portion of the unit had gone, as Meg put it, "okay," she felt that she had not had sufficient time to complete all of the activities and cover enough material to assure mastery by the students. She explained that though the class would be moving on to other mathematics topics, they would be doing multiplication computation in the warm-up activities for the remainder of the year.

Division Development

Division ideas and situations were introduced from several perspectives: concrete, pictorial, symbolic, and real-life context. However, the extent in which these connections were made was not as extensive for multiplication. Numerical connections and real-life connection were used sparingly. Both multiplication and division were taught on days nine through 19 of the unit, and in many of these lessons, the two operations were interwoven with each other and no one day was devoted entirely to division.

Interestingly, only 14% of the warm-up problems involved division. These problems included three types: basic facts, writing a division equation for a given array, and writing division story problems. None of these types of problems were included in warm-up activities until the eleventh day of the unit.

On the ninth day of the unit, Meg introduced the meaning of division from a geometric perspective. Half of the class worked on an assignment at their seats while the other half joined Meg in a small group in the back of the classroom. Meg sat next to the dry erase board and the students sat on the floor in front of the her. The notion of division as equal grouping came up in the beginning of the discussion but Meg did not allow the students to represent a 3 by 5 array as $15 \div 5 = 3$. In the discussions with both groups, Meg implied the divisor, 5, represented the number of rows (or the number of groups) and the quotient, 3, represented how many are in each row (or how many were in each group). The division representation of a 3 by 5 array was $15 \div 3 = 5$.

Meg began the lesson by drawing a 3 x 5 array made of dots on the dry erase board and asking Jill to tell her what the array represented.

Jill: 5 x 3.

Meg: 5 x 3? On your homework yesterday, how did we write it?
Brad how did we write it yesterday?

Brad: 5 x 3.

Meg: No.

Brad: I mean, 3 x 5.

Meg: Think about what we did for homework yesterday. So we are saying 3 rows with 5 in each row, and how many altogether?

Brad: Fifteen.

Meg: Fifteen altogether. When I look at that array, is it always multiplication?

Mickey: No.

Meg: What else could this be, Mickey?

Mickey: Division.

Meg: How do you figure?

Mickey: Cause it can be reversed.

Meg: Show me what you mean, explain it to me.

Mickey: 15 divided by 5.

Meg: Now, you are saying 15 divided by 5. What's the 15?

Mickey: The answer.

Meg: The answer? Well let's look at this. What is the 15?

Mickey: It's how many we have altogether.

Meg: Good! So there's 15 altogether. So, you say 15 ÷ 5. What does the 5 represent?

Mickey: Rows.

Meg: Really? Do you want to rethink that? There's 15 altogether. Raise your hand if you agree with Mickey that there's 15 altogether. He says there's 5 rows. Donny do you see 5 rows?

Donny: No.

Meg: Mickey how many rows do you see?

Mickey: Three.

Meg: Yeah, you do. So you are saying, it's 15 divided by 3. So, if I wrote 3 rows, 5 in each row, 15 altogether, how would I write it for division?

Mickey: 15 ÷ 3.

Jamie: No, it's not.

Meg: Jamie, if I write this, [Meg pointed to the array and multiplication equation written on the dry erase board] this is 3 rows, 5 in each row, 15 altogether, 3 x 5 equals 15. How would I write this as division? How many do I have altogether?

Jamie: Fifteen.

Meg: How may are in each row?

Jamie: Five.

Meg: How many rows are there?

Jamie: Three.

Meg: So, what do you think I should write next? 3 rows? Or 5 in each row?

Jamie: Three rows.

Meg: Why would you say 3 rows? You're right. But why do you say 3 rows would come next? Kim.

Kim: Because that is what you divided it by.

Meg: What does it mean to divide it by then? You say that's what you are dividing it by?

Kim: You're putting it into how many groups you're dividing by?

Meg: Oh, so you're saying that those 3 rows mean 3 groups. Right? Raise your hand if you agree with that. Donny, what do you think?

Donny: I don't know.

Meg: You don't know. So you need a little bit more time? So we have 15 altogether, 3 rows, and how many are in each row? Donny.

Donny: Five.

> Meg: There's 5 in each row. And we would write it just like this: 15 ÷ 3 = 5. So this [Meg pointed to the 3] is telling you how many groups. Right?
>
> Donny: Yeah.

After about 20 minutes, the two groups switched places. The students working at their desks joined Meg in the back of the classroom. The discussion with the second group of students was similar. Meg began by drawing a 4 by 5 array on the dry erase board and eventually Jeff suggested representing the array as 20 ÷ 5.

> Meg: 20 ÷ 5? But you just told me that 4 meant how many rows. So the 4 tells you how many rows. OK. Let me try and understand you. How can you think you can switch that? Raise your hand if you think I write it like this? This right here [Meg pointed to the expression 20 ÷ 5]. How many of you think this would be 20 ÷ 5?
>
> Jeff: I figure you can't do this.
>
> Meg: You are saying you can't? Why not?
>
> Jeff: Well, you probably could, but it would be kind of hard to divide it into five rows.
>
> Meg: If I said 20 ÷ 5, what would the 5 mean?
>
> Ann: It would mean you divided it into 5 groups.
>
> Meg: What would the 5 mean? If you are saying 20 ÷ 5, what does the 5 represent? Sue?
>
> Sue: Rows.
>
> Meg: If it's rows, how many rows do I have there? Four rows huh? So can she do that?
>
> Kim: No. She has to flip it the other way.
>
> Meg: Right, she would have to flip it the other way.

Meg made no attempt to summarize what had occurred during the discussions until the following class. In a whole-class teacher-directed lesson, Meg reviewed and further developed what had been discussed in the small groups. Meg began the lesson by reviewing what had been discussed. That is, 15 ÷ 3 meant 15 altogether with 3 rows. The quotient was the number in each row. On request, Jeff read the definition of multiplication the students had written in their notebooks: "Multiplication is taking groups of equal number and adding them. It is a faster way of adding." Meg then asked the students to compare multiplication and division: "Since multiplication is adding groups of equal numbers, and division is like multiplication, what would division be?" Ann responded: "It would be subtracting groups of equal numbers." After a short discussion, the class agreed so Meg had the students write the definition in their notebook. Meg went on to say the following:

> You need to be writing it down. I am going to tell you something. When you come up with a definition, and you write it down, it is cemented into your brains. Because not only are you seeing it, you're thinking it, you're reading it, and you're writing it, so you're using a lot of things [senses] to remember it. Whereas if you're only reading it, you're just using one way to get it.

When the students finished writing the definition in their notebooks, the discussion continued:

> Meg: Division is subtracting groups of equal numbers. Now, people, you said that multiplication is a faster way of adding. And if multiplication is a faster way of adding, what would division be?
>
> Donny: It's a faster way of subtracting.

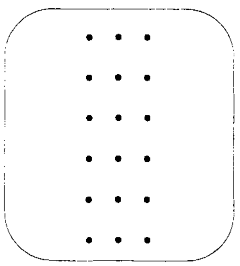
Although several students seemed confused with the idea of division as being a faster way to subtract, most of the class agreed. Meg, again, made no attempt to summarize what had been discussed.

The warm-up activity on the next day involved writing multiplication and division story problems for a three by five array. Brad read his story problem for division:

> There are three people going on a hike. There are 15 water bottles. How many water bottles will each person get?

Several other students read their stories and all of the stories involved equal sharing.

After discussing the warm-up, Meg had several students share their solutions to the multiplication homework worksheet from the previous day. Meg then guided the students through a similar activity on the overhead that involved exploring division fact by building concrete representations of the problems using base ten blocks. She began by taking 18 blocks and making an array with six rows. The students built six by three arrays at their desks. Meg then wrote $18 \div 6 = 3$ on the chalkboard and had the students write the equation on their papers. After discussing a few more examples, a worksheet from *Workmat Mathematics* was used to guide the specifics of the seat work activity that followed. The worksheet followed the same format used by Meg as described in Figure 16.



18 all together.

6 rows.

3 in each row.

$18 \div 6 = 3$.

Figure 16. Seat Work Problems.

The Bug Book project was also part of division instruction. The activity that occurred next in the teaching of division was to help students construct creative pictorial representations of both multiplication and division. While students were making Bug Books, Meg worked with small-groups of students that were having some difficulty with the meanings of multiplication and division. As Meg wrote division facts on the dry erase board, the students built concrete representations using base ten blocks on the floor in front of her. Division as equal sharing was reinforced by the manner in which each problem was solved. When Meg felt the students understood what they were doing, she sent them back to their seats to continue making Bug Books.

The warm-up the following day involved doing a Mad Minute worksheet for multiplication and division. On average, the students completed about the same number correct for division as they did for multiplication, 50% and 51% respectively.

Later in the same lesson, Meg worked with small groups of students solving equal sharing and equal grouping problems using Rainbow Cubes. Meg began by taking 12 cubes and making groups of three. She then wrote the equation $12 \div 3 = 4$ in computational form on the chalkboard to record the solution. After discussing several other equal grouping problems, some with remainders, Meg guided the students through an equal sharing problem by dividing 30 cubes into seven groups. The students recorded the quotient in computational form with the answer and the remainder included. Although several of the homework assignments included division with remainders, the activity just discussed was the only one involving remainders. This activity, from the textbook, was also the only activity that included grouping types of division. Array models were not used in this lesson.

The next lesson involved the problem of Sharing Marbles. Meg began by passing out a bag of colored cubes (marbles) to each group of five or six students. Meg then posed the following story problem:

> You are walking to school with your friends. You find a bag of marbles. In the bag there are 17 marbles. When you get to school you turn in the marbles to Mrs. Wilson because you know that when you find something, and it doesn't belong to you, you should turn it in. So you turn your bag of marbles in, being the good, honest, great people that you are. A week later, Mrs. Wilson gets on the intercom and says, "Wilson, Brad, Jill and Lauren, come to the office please." She tells you that you may keep the marbles if you can share them equally. You are to tell me as a group using pictures, diagrams, and words, how you will share the marbles equally. Then you will answer these questions.

Meg then wrote on the chalkboard the following two statements: "Each person gets ______ marbles. We think this because ______." The groups then began actively working on solving the problem. Meg moved around the room watching and listening as the students worked. She stopped to helped two of the groups and questioned them on how they were solving the problem. When it appeared that the groups were finished solving the problem, Meg had the students return to their seats. The class then discussed the problem together. Meg began by allowing the groups to share their answers and how they solved the problem. One student in each group shared the group's solution but most of the students had a chance to speak. Meg did not explicitly summarize or interpret the results of this activity. An interpretation and summary of the ideas and relationships generated in the discussion were left to the students.

The Rainbow Multiples activity discussed under multiplication also involved division. In the discussion, several students suggested that division was a quicker way than multiplication to determine if 44 was a multiple of three. Jamie suggested, and convinced Bill and Meg, that divisibility by three could be determined by simply looking at the ones digit. Meg asked if there were other ways to know if 44 was a multiple of 3:

> Bill: Divide it.
>
> Meg: So I could take 44 and divide it by three. If it comes out with no remainder, would it be a multiple?
>
> Bill: Yes.
>
> Jamie: Wait.
>
> Meg: Yes, Jamie?
>
> Jamie: Well, an easy way to do it is by taking the last number . . . and like three can't go into four because if you multiply by three it can't equal four, so you just know it couldn't go into 44.
>
> Meg: And would that work for every number?
>
> Bill: No.
>
> Meg: Well, let's see. We can try some numbers. Give me a number.
>
> Jamie: 38.
>
> Meg: Okay. Is 38 a multiple of three?
>
> Jamie: No.

Meg: Why?

Jamie: Because three can't go into eight.

Meg: How about 39.

Jamie: Yes, because three times three is nine. See, it works.

Bill: Three won't go into 39.

Meg: Jamie says it would work for 39. Three goes into three once, three goes into nine, three times. No remainder.

Bill: Oh. I get it now.

Meg was unable to provide an example to the class that showed that the student's conjecture was incorrect. Although Meg did not interpret or summarize the results of this discussion, the implied notion was that a number is divisible by 3 if and only if its ones digit is divisible by 3.

When the class finished discussing multiples, Meg quickly discussed the homework assignment. The assignment consisted of two practice worksheets involving finding multiples and missing factors in multiplication equations.

The Rainbow Factors activity included a brief discussion about division. When the students were unable to generate a definition for a factor of a number, Meg had Bill read the definition: "A number that divides evenly into a given number is a factor of that number." In the activity that followed on finding factors of numbers, Meg suggested that students first try building a rectangle with two rows, then three rows, and so on. The approach illustrated the equal sharing concept of division emphasized in the unit.

The day following the Rainbow Factors activity, the nineteenth day of the unit, was the final day of instruction. During this class, prime numbers were discussed in terms of both multiplication and division. The ideas generated in the discussion were that prime numbers had exactly two factors, and thus, were divisible by only themselves and one. Meg ended this portion of the discussion by telling students that such numbers were called prime numbers.

The rest of the lesson was spent developing the meaning of each step in the standard (textbook) algorithm for multiplication. Although both the multiplication and division algorithms were used several times in solving warm-up problems consisting of finding products and quotients, Meg did not explain the algorithm for division at any time during the unit.

Meg felt that the division portion of the unit had not gone as well as multiplication According to Meg, time was a major influence in not being able to do more computation. She explained that the class would be moving on to other mathematics content areas, but most warm-up activities for the remainder of the year would include some division computation and application.

Student Learning Profile

The students for this study consisted of 23 fourth grade children grouped with Meg for mathematics. The class was comprised of 11 boys and 12 girls with a rich and diverse ethnic population: 11 students were Caucasian, seven were Native American, three were Hispanic, and two were Black. Within this diverse classroom community, nine students were identified ESL (English as a Second Language), three were Resource (Full Inclusion) students, and four were identified as gifted.

The post-assessment designed by the researcher based on Meg's unit objectives, Appendix G, was administered to 21 of the students on the twentieth day of the unit. One student moved during the last week of the unit and a second student was ill. The students were given approximately 40 minutes to complete the assessment. Following the assessment, six students were interviewed. The students interviewed were asked to tell how they solved the problems and why they thought their answers made sense.

Thirty-six problems were designed to provide a view of students' knowledge of multiplication and division related to the Meg's objectives for the unit. Of the 36 problems, 16 problems were related to multiplication, 14 problems involved division, and six problems were a combination of multiplication and division. The assessment was divided into two parts and given separately, since in part one of the assessment, students were asked to solve six story problems involving multiplication and division, and in part two, students were asked to write their own story problems.

The analysis of students' scores, Table 1, on the three types of problems indicated that students had significantly more success on the multiplication problems than on the other two types of problems. A comparison of the students' scores on the multiplication problems and division problems suggested that the students scored significantly higher ($t = 2.690$, p-value = 0.007) on the multiplication problems than on the division problems. A comparison of students' scores on the multiplication problems and the problems involving both multiplication and division yielded similar results. That is, there was a significant difference between the means of students' scores on the multiplication problems and the problems involving both multiplication and division ($t = 2.419$, $p = 0.021$). However, there was no significant difference between the means of students' scores on the division problems, and the problems involving both multiplication and division ($t = -0.256$, $p = 0.798$).

Assessment Content Area	Mean	Standard Deviation
Multiplication	$\mu_1 = 0.729$	0.174
Division	$\mu_2 = 0.560$	0.210
Both mult. & div.	$\mu_3 = 0.577$	0.228
Tests of Significance	**p-value**	**t-value**
$\mu_1 \neq \mu_2$	$p = 0.007$	$t = 2.690$
$\mu_1 \neq \mu_3$	$p = 0.021$	$t = 2.419$
$\mu_2 \neq \mu_3$	$p = 0.798$	$t = -0.256$

Table 1. Summary of Quantitative Student Assessment Data (numbers rounded).

Students' success rates were about 17 percentage points higher on the multiplication problems than division problems and 15 percentage points higher than problems involving both multiplication and division. These results are summarized in Table 1.

Write the product.		
1. 2 x 4	2. 5 x 8	3. 6 x 7
Write the quotient.		
7. 16 ÷ 2	8. 30 ÷ 5	9. 28 ÷ 7

Figure 17. Multiplication and Division Facts.

Problem	Correct	Partial	Incorrect
1	100%	0%	0%
2	90%	0%	10%
3	76%	0%	24%
Mean for Mult.	89%	0%	11%
Problem	Correct	Partial	Incorrect
7	90%	0%	10%
8	86%	0%	14%
9	81%	0%	19%
Mean for Div.	86%	0%	14%

Table 2. Summary of Quantitative Data on Problems 1-3 and 7-9 (% rounded).

Students' success rates on the problems are summarized with a table for each set of problems. Problems 1-3 and 7-9 were designed to assess students' knowledge of multiplication and division facts, Figure 17. A problem was judged to be correct if a correct product or quotient was written. Overall students' success rates on the multiplication and division facts were about the same. On average, 89% of problems 1-3 were correct. Twenty-four percent (five students) had Problem 3 incorrect and 10% (two students) had Problem 2 incorrect. On average 86% of problems 7-9 were correct. Ten percent (2 students) of the students had Problem 7 incorrect, 14% (3 students) had Problem 8 incorrect and 19% (4 students) had Problem 9 incorrect. These results are summarized in Table 2.

Write the product.

4. $\begin{array}{r} 21 \\ \underline{\times 3} \end{array}$ 5. $\begin{array}{r} 13 \\ \underline{\times 4} \end{array}$ 6. $\begin{array}{r} 46 \\ \underline{\times 7} \end{array}$

Write the quotient and remainder.

10. $4\overline{)11}$ 11. $8\overline{)41}$ 12. $6\overline{)82}$

Figure 18. Multiplication and Division Computation.

Problem	Correct	Partial	Incorrect
4	100%	0%	0%
5	100%	0%	0%
6	52%	0%	48%
Mean for Mult.	84%	0%	16%
Problem	Correct	Partial	Incorrect
10	29%	19%	52%
11	52%	10%	38%
12	29%	5%	67%
Mean for Div.	37%	11%	52%

Table 3. Summary of Quantitative Data on Problems 1-3 and 7-9 (% rounded).

Problems 4-6 and 10-12 were designed to assess students' knowledge of multiplication and division computation, Figure 18. Problems 4-6 were judged to be correct if correct products were written. There were no partially correct answers. Problems 10-12 were judged to be correct if the correct quotients and remainders were both written. They were judged to be partially correct if a correct quotient was written and an incorrect remainder.

Students had more success on the multiplication computation problems than on the division computation. On average 84%, of Problems 4-6 were answered correctly as compared to only 37% on Problems 10-12. Eleven percent of Problems 10-12 were partially correct leaving 52% incorrect. Whereas only 16% of multiplication computation Problems 4-6 were incorrect. All students were able to answer Problems 4 and 5 correctly but only slightly more than half of the students were able to provide a correct response for Problem 6. Problem 11, however, was the only division computation problem that over half of the students answered correctly. These results are summarized in Table 3.

Problems 13 and 14 were designed to assess students' ability to write multiplication and division equation that modeled an array, Figure 19.

13. Write a multiplication equation that describes the array .
14. Write a division equation that describes the array below.

● ● ●
● ● ●
● ● ●
● ● ●
● ● ●
● ● ●

Figure 19. Representing an Array as Multiplication and Division.

Problem	Correct	Partial	Incorrect
13	76%	14%	10%
14	71%	5%	24%

Table 4. Summary of Quantitative Data on Problems 13 and 14 (% rounded).

Problem 13 was considered to be correct if 6 x 3 = 18 or 3 x 6 = 18 was written either horizontally or vertically. Problem 14 was deemed to be correct if the students response was 18 ÷ 3 = 6 or 18 ÷ 6 = 3 written horizontally or in computational (long division) form. Both problems were judged to be partially correct if an expression was written correctly but the product or quotient was not given.

Seventy-six percent of students (16 students) were able to write a multiplication equation and 71% (15 students) were able to write a division equation that described the array. Although students' success rates were similar for multiplication and division, the mistakes made on Problem 14 were more serious than those made on Problem 13. Two students were unable to write a division equation that made any sense. Tom wrote the expression 6 ÷ 3 and Will wrote 6 + 3 = 9 and 6 - 3 = 3. Jim reversed the dividend and divisor and wrote 3 ÷ 18 =6. Mistakes made on Problem 13 appeared to be careless errors resulting from counting the rows, columns, or circles incorrectly. These results are summarized in Table 4.

Problems 15-20 were designed to assess students' ability to solve real-life multiplication and division story problems. Students were also expected to write an equation that could be used to solve the problem. Problems 16, 17, and 20 involved multiplication and Problems 15, 18, and 19 involved division, Figure 20. A response, for Problems 15-20, was judged to be correct if it showed an equation that could be used to solve the problem and included a correct answer to the question. A response was deemed to be partially correct if it contained an equation that could be used to solve the problem but the student failed to answer the question correctly or it included the correct answer but did not contain an equation that could be used to solve the problem.

Solve the story problem. Write an equation that could be used to solve the problem. Make sure to answer the question.

15. Marie has 20 apples. She wants to share them equally among 6 of her friends. How many apples will each friend receive?

16. Mark has 4 bags of apples. There are 6 apples in each bag. How many apples does Mark have altogether?

17. Jill has a box of candy. There are 6 rows with 8 pieces of candy in each row. How many pieces of candy does Jill have altogether?

18. Kate is cooking omelettes for a class breakfast. She has 24 eggs in the refrigerator. If it takes four eggs for each omelette, how many omelettes can Kate make?

19. The school vans can hold 8 students. How many vans will it take to carry 25 students for the field trip?

20. Pencils cost 12¢ each. How much will it cost to buy 6 pencils?

Figure 20. Multiplication and Division Story Problems.

Students' success rates were slightly higher on the multiplication story problems than on the division problems, Table 5, the biggest difference occurring between incorrect answers. On average, students had 32% of the division problems incorrect as compared to only 19% incorrect for multiplication. Most of the students' errors on the multiplication stories seemed to be ones of omission. The differences between correct and incorrect responses were greater among the different types of division problem. Problem 15 involved equal sharing division and Problems 18 and 19 involved equal grouping. Forty-eight percent of the students were able to get the equal sharing problem (Problem 15) correct and, on average, only 31% of the students were able to provide correct solutions for the equal grouping problems (Problems 18 and 19). On average, twice as many students had incorrect solutions to the equal grouping problems as compared to those involving equal sharing.

Problem	Correct	Partial	Incorrect
16	48%	38%	14%
17	33%	52%	14%
20	43%	29%	29%
Mean for Mult.	41%	40%	19%

Problem	Correct	Partial	Incorrect
15	33%	19%	48%
18	48%	33%	19%
19	29%	43%	29%
Mean for Div.	37%	32%	32%

Table 5. Summary of Quantitative Data on Problems 15-20 (% rounded).

Problems 21 and 22 were used to assess students' understanding of multiplication as repeated addition, Figure 21.

Find the sum. Then write the related multiplication equation.

21. $8 + 8 + 8 + 8$
22. $3 + 3 + 3 + 3 + 3 + 3$

Figure 21. Multiplication as Repeated Addition.

Problems 21 and 22 were considered to be correct if 4 x 8 = 32 and 6 x 3 = 18 were written. Students seemed to understand multiplication as repeated addition with most partially correct responses involving only an incorrect sum. These results are summarized in Table 6.

Problem	Correct	Partial	Incorrect
21	81%	19%	0%
22	71%	10%	19%

Table 6. Summary of Quantitative Data on Problems 21 and 22 (% rounded).

Problems 23-26 were used to assess students' ability to identify multiples and factors, Figure 21. On average, 74% of the students were able to answer the problems involving multiples correctly as compared to only 53% for those problems involving factors. However, these data may be misleading, since 81% were able to answer Problem 25 correctly as compared to only 24% for Problem 26, Table 7. Problem 26 had the more incorrect answers than any other problem on the assessment. Students' interviews, discussed later in this section, provide further insight into students' thinking on each of these problems.

Answer yes or no.	
23. Is 18 a multiple of 4?	24. Is 20 a multiple of 4?
Answer yes or no.	
25. Is 7 a factor of 21?	26. Is 16 a factor of 4?

Figure 22. Factors and Multiples

Problem	Correct	Partial	Incorrect
23	71%	0%	29%
24	76%	0%	24%
25	81%	0%	19%
26	24%	0%	76%

Table 7. Summary of Quantitative Data on Problems 23-26 (% rounded).

Problems 27 and 28 were used to assess students' knowledge of multiplication and division vocabulary, Figure 23. Students did about the same (one student difference) in identifying the product as they did in identifying the quotient, Table 8. These data are interesting since the term quotient was not used as frequently in the teaching of the unit. Student interviews provide some insight.

Answer yes or no.

27. What is the product in the equation $4 \times 6 = 24$?
28. What is the quotient in the equation $12 \div 3 = 4$?

Figure 23. Multiplication and Division Vocabulary.

Problem	Correct	Partial	Incorrect
27	67%	0%	33%
28	62%	0%	38%

Table 8. Summary of Quantitative Data on Problems 27 and 28 (% rounded).

Problems 29 and 30 were used to assess students' understanding of a relationship between multiplication and division, Figure 24. Problems 29 and 30 were judged to be correct if at least two other "fact family" members were written and deemed to be partially correct if exactly one other "fact family" member was written. Two more students (9%) were able to write at least two other fact members when starting with a multiplication equation than were able to write at least two members when starting with a division equation. However, the same number of students were able to write exactly one other fact family.

29. Write the other members of the fact family for 6 x 8 = 48.

30. Write the other members of the fact family for 27 ÷ 9 = 3.

Figure 24. Fact Families.

Problem	Correct	Partial	Incorrect
29	38%	38%	24%
30	29%	38%	33%

Table 9. Summary of Quantitative Data on Problems 29 and 30 (% rounded).

Problems 31 and 32 were used to assess students' ability to write their own story problems, Figure 25. Problems 31 and 32 were judged to be correct if the stories met both rules with a correct equation written that could be used to solve the problem. The problems were deemed partially correct if the problems met both rules without a correct equation that could be used to solve the problem. Although students' success rates were about the same for Problem 31 (Writing Multiplication

Story Problem) and Problem 32 (Writing a Division Story Problem) Table 10, does not show that all of the division stories written by the students involved equal sharing.

31. Write a story problem that follows two rules: (1) It must end in a question. (2) The question must be one that is possible to answer by using multiplication. Write the multiplication equation that you would use to solve the story problem that you wrote.

32. Write a story problem that follows two rules: (1) It must end in a question. (2) The question must be one that is possible to answer by using division.Write a division equation that you could use to solve the story problem that you wrote.

Figure 25. Writing Multiplication and Division Story Problems.

Problem	Correct	Partial	Incorrect
31	57%	5%	38%
32	62%	14%	24%

Table 10. Summary of Quantitative Data on Problems 31 and 32 (% rounded).

Problems 33 and 34, Figure 26, assessed students' ability to represent multiplication and division as an array. The problems were judged to be correct if a correct grid and equation were written. The problems were deemed partially correct if a correct grid and an incorrect equation or an incorrect grid and a correct equation were written.

Twenty students (95%) were able to use grid paper to show the multiplication problem 6 x 13 and 17 students (81%) were able to show the division problem 63 ÷ 7. Only one student was unable to provide at least a partially correct solution to

Problem 33 and three students were unable to provide a partially correct solution for Problem 34. The most common errors for these two problems consisted of failure to write a complete equation. Table 11 summarizes correct, partially correct, and incorrect response for problems 33 and 34.

33. Use grid paper to show the multiplication problem 6 x 13. Show the problem by cutting and pasting part of the grid paper in the space below. Make sure to show the complete equation somewhere in the space below.

34. Use grid paper to show the division problem 63 ÷ 7. Show the problem by cutting and pasting part of the grid paper in the space below. Make sure to show the complete equation somewhere in the space below.

Figure 26. Representing Multiplication and Division as an Array.

Problem	Correct	Partial	Incorrect
33	43%	52%	5%
34	38%	48%	14%

Table 11. Summary of Quantitative Data on Problems 33 and 34 (% rounded).

Problems 35 and 36, Figure 27, were used to assess students' understanding of multiplication and division and their ability to write at least several ideas explaining what they know. Problems 35 and 36 were judged to be correct if the students expressed at least two correct ideas about multiplication or division using pictures, numbers, or words. The problems were judged partially correct if the students expressed exactly one correct idea. Only four students (19%) were able to provide at least two ideas about multiplication and none on the students were able to give more than one idea about division. The most common responses about multiplication involved its relationship to addition. Most students wrote: "Multiplication is a faster way to add." A few students wrote a repeated addition equation and a multiplication equation next to it expressing the same idea. Only 38% of the students provided at least one idea on division. The most common correct idea for Problem 36 involved dividing a collection of objects into a given number of equal groups. A few responses about division indicated that: "Division is repeated subtraction." Many incorrect responses suggested that: "Division is a faster way to subtract."

35. What do you know about multiplication? Use pictures, numbers, or words to explain your answer in as many ways as you can.

36. What do you know about division? Use pictures, numbers, or words to explain your answer in as many ways as you can.

Figure 27. Meanings of Multiplication and Division.

Problem	Correct	Partial	Incorrect
35	19%	62%	19%
36	0%	38%	62%

Table 12. Summary of Quantitative Data on Problems 35 and 36 (% rounded).

Student Interviews. The interview data provided a more detailed picture of the multiplication and division knowledge of the students. Aspects of the six interviews were chosen for discussion because they were judged to show significant examples of the understandings and difficulties uncovered by the assessment.

The six students interviewed included Bill, Kim, Ann, Jamie, Jackie, and Tom. Bill and Kim consistently performed in the upper third of the class, Ann and Jamie performed in the middle third, and Jackie and Tom performed in the bottom third. Overall, the mean scores of the students interviewed were slightly higher than those for the whole class. These data are summarized in Table 13.

Students	Multiplication	Division	Mult. and Div.	Total Score
Bill	93%	84%	62%	87%
Kim	86%	86%	100%	89%
Ann	90%	68%	88%	81%
Jamie	90%	80%	38%	79%
Jackie	72%	44%	75%	61%
Tom	62%	32%	38%	47%
Total Score	82%	66%	67%	74%

Table 13. Summary of Quantitative Data on Students Interviewed (% rounded).

Jackie and Tom (students who performed in the bottom third) had incorrect responses on both Problems 3 and 6 involving multiplication and division facts. Both students said that for Problem 3, 6 x 7, they did not remember the facts so they tried to get the answer by adding. On Problem 9, 28 ÷ 7, remembering the facts was also the obstacle. Jackie said, "I went 7 x 1 is 7, 7 x 2 is 14, and then 7 x 3 is 27 [this is where Jackie made her mistake] and I couldn't add another 7 into that so there was one remainder." The other students indicated that they knew the answers because the problems involved basic facts. Bill also shared a strategy for doing Problem 3 that Meg had taught him. As Bill put it, "Ms. Magee, she always taught us that it is pretty easy. If you know that 6 x 6 is 36 then all you have to do is add another 6."

On the multiplication and division computation set, all six students answered Problems 4 and 5 correctly and they all used the standard algorithm. For example, on Problem 5, 4 x 13, Kim said: "4 times 3 is 12 so I put down the 2, and I added the 1 up here [pointing next to the one], and 4 times 1 is 4 plus 1 is 5. So the answer is

52." Jackie and Tom had incorrect responses for Problems 6, 10, 11, and 12, and their difficulties again seemed to be in remembering the basic facts.

Ann was able to do Problems 4-6 (multiplication computation) correctly, but missed all three division computation problems. Her erroneous strategy was the same for each of the problems. For example, in Problem 10, Ann figured a quotient of 3 and a remainder of 1 by multiplying 4 times 3 to get 12 and then subtracted 11 from 12 to get a remainder of 1.

Jamie and Kim used thinking strategies similar to the standard algorithm to answer the three division computation problems correctly, although neither student showed her work on her paper. Bill, also correctly answered Problems 10-12 and used multiplication. For Problem 12, Bill multiplied 6 by 10, 11, 12, and 13, until he identified the number 76, then he subtracted 76 from 82 to find the remainder.

All of the students interviewed, except Tom, seemed to have an understanding of Problems 13 and 14 on representing an array as multiplication and division. Although Jackie had Problem 14 incorrect, she saw her mistake while she was explaining her solution. Tom had no idea on Problem 14. He insisted that 6 ÷ 3 = 18 was correct. It should be noted, however, that only Bill wrote 18 ÷ 3 = 6 in response to Problem 14. When asked why he wrote the equation this way, Bill said that you always divide by the smaller number. Even though Bill was considered one of the brightest students in the class, he seemed be be having some difficulty with the array concept of division.

On average, the success rates of the six students interviewed was higher on Problems 15-20, solving multiplication and division story problems, than the success rate on the same problems for the whole class. Jackie and Tom, again, seemed to have the most difficulty, especially with the problems involving division. Although the class as a whole had higher success rates for the equal sharing types of division problems compared to the equal grouping problems, Jackie and Tom had difficulty was both types. On Problem 15, Tom wrote "20 ÷ 6 = 4 r 2." Tom' error, again, seemed to be linked to not remembering his multiplication (and counterparts for division) facts. For Problem 19, Tom seemed to be able to use a real-life understanding of the situation to reason the answer, "3 vans with one student left over", but was unable to make any sense mathematically out of what he had written, "25 ÷ 3 = 24 r 1."

Bill, had all six of the multiplication and division story problems correct but still seemed confused about Problem 18. Bill needed to use the real-life situation to help him get the correct answer. As Bill put it:

> Kate can make 6 omelets because we have 24 eggs, and you can only put 4 eggs in each omelet. So you have to have 24 ÷ 4 = 6, because if you turn it around like I thought, it would be 6 x 4 = 24. And she can't make 24 omelets.

Although Bill's strategy of using the real-life context to check mathematical meaning was a good strategy, Bill did not seem to be able to make sense of abstract aspects of division.

All of the students interviewed seemed to have a strong understanding of multiplication as repeated addition. In their own ways, they all expressed the notion that the first factor meant how many groups and the second factor meant how many were in each group. For Problem 21, Kim had written 8 x 4 = 32 but when asked the meaning of the equation, 8 x 4 = 32 as repeated addition, she changed her answer to 4 x 8 = 32.

Although students' responses rated high on the problems involving multiples and factor, they seemed to have particular difficulty with the wording of Problem 26. Jackie and Tom both said that they guessed on Problem 26 because as Jackie put it "I don't really know what factors are, so I was guessing on that one." Although Jamie had "no" (the correct answer) written on her paper, before she finished reading Problem 26 said, "Oops, it's yes because 4 x 4 = 16." But after thinking for a few seconds, Jamie decided that her original answer was correct.

Ann was the only student interviewed that seemed confident about her answer and she had no trouble explaining why her answer made sense. Ann put it this way:

"No, it's because the one that they are asking is it a factor of, is bigger than the number which they are asking if the factor is being . . ."

On multiplication and division vocabulary problems, four of the six students interviewed, provided the correct answers, 24 and 4 respectively, and said that the product was the answer in a multiplication problem and the quotient was the answer in a division problem. Tom simply rewrote the two equations exactly as they were written. He said he did not remember what the words 'product' or 'quotient' meant. Ann had an incorrect answer to Problem 27 written on her paper, but as she read the question, she realized her mistake.

The success rates for the entire class were low for the questions involving fact families, about one-third of students responses were judged correct, one-third partially correct, and one-third incorrect. Although the summary indicated the students' lack of understanding of the relationship between multiplication and division, the interviews suggested otherwise. When the wording for Problem 29 was changed to "write as many correct multiplication and division equations as you can using only the number given in the equation, 6 x 8 = 48," four of the six students interviewed were able to write at least two members of the fact family. The results were similar for Problem 30, the equation 27 ÷ 9 = 3. Several of the students interviewed also recalled that they had found fact families for addition and subtraction.

The students interviewed felt that writing multiplication and division story problems, Problems 31 and 32, were easy and they were able to write them successfully. However, all of the story problems written involved sharing a collection of objects. None of the stories involved grouping.

All six students interviewed were able to represent Multiplication and Division as an array. Several students did not complete the equation, but were able to do so when reminded. Interestingly, all of the students represented Problem 34, 63 ÷ 7, as a 7 by 9 array. When asked, only Ann felt that it did not make any difference how the rectangle was placed on the paper.

On the questions involving the meanings of multiplication and division, all six students interviewed were able to provide at least one correct idea about multiplication. The most common response concerned its relationship to addition. For Problem 35, Kim wrote, "Multiplication is a faster way of adding." She was able to show and explain the meaning of her statement.

Ann provided two correct statements for Problem 35 and she was able to explain the meaning of each. She wrote, "Multiplication is taking groups of equal number and adding them together. It is a faster addition. It is the oposit [opposite] of division." When asked about what she meant by her third statement, Ann showed the equations 4 x 2 = 8 and 8 ÷ 2 = 4 and explained how the two equations were related.

Only two students interviewed were able to provide at least one idea on division. Ann said that division was repeated subtraction. When asked about the meaning of her statement Ann responded:

> Well, let's say I have 20 altogether, right here, and let's say it is 20 ÷ 4, so you take 4 away, and 4 away, and then 4 away, and I am putting them into groups of four, 4 away, 4 away, and the amount of times I took 4 away in order to make 0 is the answer. Let me see, I have how many groups of 4, it would be 5. So the answer would be 5.

Several students conveyed the notion that division is a faster way to subtract. For example Bill said, "Division is an easier way of subtracting. Instead of doing this [subtraction] (Bill wrote 32-8-8-8-8 = 0 on his paper) you can do this [divide] (Bill wrote 32 ÷ 8 = 4)." When Bill asked how 32 ÷ 8 = 4 made the problem 32 - 8 - 8 - 8 - 8 = 0 easier to do he responded:

> Well, it's something we had in our math class. We wrote it down, and we put this [Bill pointed to the statement on his assessment, Division is an easier way of subtracting.], and Ms. Magee . . . she wouldn't tell us, and this is what I remembered from it.

Consistent with the results of the whole class, the six students interviewed did significantly better on the multiplication portion of the assessment. Their responses provided additional support for this conclusion and their difficulties seemed to be directly linked to classroom instruction. The students had considerable difficulty with division computation, a topic that involved limited instruction. Another difficult topic consisted of solving division story problems and instruction in this area, for the most part, included only division sharing types of problems. Although students had difficulty with the fact family problems, when probed in the interviews, most of the students seemed to understand the concepts but were unfamiliar with the 'fact family' language used in the directions of the problem. When reminded the students were able to relate the multiplication and division "fact family" idea to what they had previously learned for addition and subtraction.

Summary

The subject matter knowledge structure that Meg held at the beginning of the investigation included three parts: the connections between the operations of addition, subtraction, multiplication, and division; the relationship between addition and subtraction; and the relationship between multiplication and division. To Meg, the four operations were interrelated, and she felt strongly that multiplication and division as well as addition and subtraction needed to be taught together. As Meg put it:

> I teach them [multiplication and division] together. The textbook separates multiplication and division and that's one reason why I don't follow the book. It saves a lot of time if you do them together and they're opposites [inverses] so it's easier to teach them at the same time and it helps [students] see how they're related.

Meg's diagram of multiplication and division can be described as a web consisting of five primary components extending out on axes: concepts, computation, vocabulary, number sense, and connection. The lines of connections simply represented a convenient manner of organizing the topics listed. Meg's understanding of these components and the absence of ideas within these components were reflected in her teaching and the resulting student learning.

Consistent with her knowledge structure, Meg introduced the concepts of multiplication and division from several perspectives by investigating various groups and arrays using concrete, pictorial, numerical, and real-life representations. In teaching multiplication, attention was devoted to the concept of grouping or repeated addition, but the primary representation of multiplication was that of an array.

Although division has different meanings, depending on the context, instruction focused mostly on the concept of division as equal sharing, a view of separating things into equal size groups. As previously reported, Meg's self-described subject matter knowledge prior to the investigation did not include an understanding of division as equal grouping, splitting a collection of things into groups of a known size. The absence of division as grouping in Meg's knowledge structure seemed to prevent her from developing the full range of division situations. For example, in one class discussion, Meg conveyed the notion that it was inaccurate to represent a 3 by 5 array as $15 \div 5 = 3$. Meg's interpretation was that the divisor, 5, represented the number of rows (or the number of groups) and the quotient, 3, represented how many were in each row (or how many were in each group). Thus, the notion imparted to students was that the only division equation associated with a 3 by 5 array was $15 \div 3 = 5$.

A student's conception of this invalid notion came up in an interview following the post-assessment when Bill was asked to explain his understanding of division in relation to a 3 by 5 array:

> Researcher: Tell me an equation for this three by five array [The researcher drew a 3 by 5 array of dots on the board.].
>
> Bill: Three times five is 15.
>
> Reacher: What else can you tell me about the array?
>
> Bill: There are three rows. Five in each row. Five columns. Fifteen altogether?

Reacher: Does this array only represent multiplication?

Bill: No. Fifteen divided by three is five. [Bill wrote $15 \div 3 = 5$.]

Reacher: What does the three mean?

Bill: Rows.

Reacher: And the 15?

Bill: How many altogether.

Reacher: Is there any other way to write the array as division?

Bill: If you turn it . . . $15 \div 5 = 3$. Ms. Magee said that you have to turn it.

Reacher: Why?

Bill: Cause 5 has to be the rows.

Reacher: Could you write $15 \div 5 = 3$ if you didn't turn it.

Bill: No, cause Ms. Magee taught us that.

Although Bill did not seem to understand division as equal grouping in this situation, he had no difficulty solving story problems that involved both types of division. On the questions involving solving story problems on the post-assessment, Bill was able to get all three division problems correct. When asked about this dilemma, Bill said: "It's different when it's a story [problem]. Ms. Magee taught us to do story problems this way."

Several times during the unit Meg referred to division as "repeated subtraction" and as a "faster way to subtract." Equal grouping division is also referred to as repeated subtraction, the equal size groups are "subtracted" from the total. When asked at the end of the unit what she meant by the notion of division as "repeated subtraction" and the statement that division was a "faster way to subtract," Meg was unable to provide answers. Meg shared that she "really didn't understand division as repeated subtraction." She said that the subject matter knowledge assessment in the initial interview had made her think and she realized when she was preparing for the unit that one of the books talked about repeated subtraction. She thought that since multiplication was repeated addition, then it was only logical that division was repeated subtraction. Although Meg was unable to provide the meaning of division as repeated subtraction, Ann was able to explain what it meant in the post-assessment interview.

Overall, on questions involving an understanding of the concepts, students had more success on multiplication (representing an array as multiplication, representing multiplication as an array, multiplication as repeated addition, multiplication story problems, writing multiplication story problems, and the meaning of multiplication) than on division (representing an array as division, representing division as an array, division story problems, writing division story problems, and the meaning of division). On average, 52% of the multiplication concept items were answered correctly as compared to only 40% on the items related to division. These results are summarized in Table 14.

Problem	Correct	Partial	Incorrect
Mult. Concepts	52%	31%	16%
Div. Concepts	40%	29%	31%

Table 14. Multiplication and Division Concepts.

A second component in Meg's knowledge structure of multiplication and division consisted of computation: multiplication and division facts, mental math, standard algorithms, and computational practice. To Meg, knowledge of basic multiplication and division facts was fundamental in enabling students to solve problems. Although in the unit the memorization of these facts was an on going homework assignment and little specific class time was given to this endeavor, a major activity of the unit did involve creating a Multiplication Matrix. The main purpose of this project was to give students a concrete representation of the multiplication facts.

Most homework assignments involved practice of basic facts for multiplication and division, but only a few assignments included computation beyond the basics. According to Meg, time was a major factor in not being able to do more computation:

> When I used to teach only my own class, I could teach math all day long. If I didn't finish I'd just keep doing math. Now, I need to be more organized and do a better job preparing. I never seem to have enough time to get everything done.

Near the end of the unit, the standard algorithm for multiplication used in the textbook was presented with the meaning of each step in the algorithm explored. In the subject matter knowledge interview, Meg was able to show the steps and the meaning of each step in a standard multiplication algorithm. Although the standard algorithm for division was used several times in solving warm-up problems that consisted of finding quotients, the meaning of each step was not developed. This result is not surprising since, in the subject matter knowledge interview, Meg was unable to explain the meaning of the steps in an algorithm for division.

Although students success rates of correct responses on problems involving basic facts for multiplication and division were similar, 89% to 86% respectively, their overall success rates for correct responses for computation differed by about 25 percentage point. These results are summarized in Table 15.

Problem	Correct	Partial	Incorrect
Mult. Computation	86%	0%	14%
Div. Computation	61%	6%	33%

Table 15. Multiplication and Division Computation.

No pre or post test teacher designed data were available for computation, but students' scores on the Mad Minute activities did show an increase for every student. On average, students scores increased 25 percentage points for multiplication and 28 percentage points for division. Interestingly, students did slightly better on the final Mad Minute activity for division than they did for multiplication, 63% to 61% respectively. The results for the three Mad Minute assessments are summarized in Table 16.

Assessments	Multiplication	Division
Test 1	36%	35%
Test 2	51%	50%
Test 3	61%	63%

Table 16. Mad Minute Data.

A third component of Meg's subject matter knowledge for multiplication and division included an understanding of and an ability to identify terms such as multiply, product, factor, multiple, column, row, set, group, divide, quotient, divisor, and dividend. Although Meg used the vocabulary for multiplication frequently, she rarely used the terms divisor, dividend, and quotient in class.

Several activities in the unit involving vocabulary also focused on developing students' number sense, a fourth component in Meg's knowledge structure. Various activities consisted of exploring and describing arrays to show how a product was related to its factors and exploring factors and multiples to discover properties of numbers (e.g. finding that some numbers are prime).

In a discussion during the Rainbow Multiples activity, several students suggested that division was a quicker way than multiplication to determine if 44 was a multiple of three. Jamie erroneously conjectured (in less formal terms) that a number is divisible by 3 if and only if its ones digit is divisible by 3. Meg suggested that they try some numbers, so they tried 38 and 39. They concluded that Jamie's method worked. Although Meg did not interpret or summarize the results of this discussion, Meg never provided a counter example nor was the conjecture ever discussed again. When asked later about rules for divisibility, Meg said that she only knows rules for 2, 5, and 10. She said:

> I didn't think Jamie was right, but I was afraid to go there. I should have come back to it later but I forgot. I should write things like that down.

The final component in Meg's knowledge structure for multiplication and division consisted of connections. To Meg, these connections involved manipulatives, problem solving, and other content areas. In the development of the unit, various manipulatives were used in representing concepts, procedures, and processes: base ten blocks, rectangular grid paper, rainbow cubes, and various drawings and pictures. These manipulatives were used to represent multiplication as repeated addition and as an array, division as equal sharing and as an array, as well as an assortment of multiplication and division problem situations. The standard mathematical representations (symbols) of multiplication and division were also introduced in the context of the activities, helping students connect the abstract representation to their own experiences.

Meg's notion of the interrelatedness of the operations was shown in more ways than just by teaching multiplication and division together. Several activities were used to explore the notions, the operations of multiplication and division were related in that one was the inverse [Meg incorrectly used the term opposite] of the other, multiplication can be thought of as repeated addition, and division can be thought of as repeated subtraction. However, several misconceptions, already discussed, surfaced as a result of the idea of division as repeated subtraction.

An assortment of activities involving problem solving, reading, writing, and solving story problems, various writing activities, and children's stories were also used to provide a real-world context. As already discussed, students' success rates were slightly higher on the multiplication story problems than on the division problems, Table 5, with the biggest difference occurring between incorrect answers. However, the differences between correct and incorrect responses were greater among the different types of division problems with 48% of the students able to get the equal sharing problem (Problem 15) correct and only 31% of the students able to provide correct solutions for the equal grouping problems (Problems 18 and 19). Twice as many students had incorrect solutions to the equal grouping problems as compared to those involving equal sharing. Although students' success rates were about the same, as in Table 10, for Writing Multiplication and Division Story Problems, all of the division stories written by the students involved equal sharing.

Students also had difficulty with the writing assignment that was used to summarize the ideas learned in the Multiplication Matrix project. Although a discussion generated many insightful ideas, the students were unable to share these ideas in their writings. Meg later revealed that most students had written only a short sentence about the matrix or about what they had learned. Meg felt that the discussion "went well" but she needed to do a better job helping students summarize what they had learned.

The results of this study indicate that Meg's subject matter knowledge of multiplication was relatively strong but her knowledge of division was faulty and incomplete on several topics including an understanding of division only as sharing, the conceptual underpinnings of long division, the relationship between symbolic division and real life problems (particularly with fractions), and notions of divisibility. Although it was not clear whether Meg's subject matter knowledge structure affected teaching or whether her teaching affected her subject matter structure, the data suggested that it was directly related to classroom teaching and students' learning. Most importantly, her faulty and incomplete understanding of division seemed to be related to a negative outcome of teaching and learning. The analysis indicated that students had significantly more success on topics involving multiplication than on ideas associated with division.

CHAPTER V

DISCUSSION AND CONCLUSIONS

Introduction

This study investigated how differences in an elementary mathematics teacher's subject matter knowledge structure relate to classroom teaching and student learning. Two general questions were posed at the beginning of this study. These were:

1. What is the appearance of an elementary mathematics teacher's subject matter knowledge structure of addition, subtraction, multiplication, and division?

2. How do differences in this knowledge structure relate to classroom teaching and student learning?

The study included two phases. Phase 1 focused on the selection of a single case. An open-ended questionnaire and interview were used to identify the subject matter knowledge structure for addition, subtraction, multiplication, and division of three elementary teachers. One teacher, Meg, was selected who demonstrated clearly different knowledge for multiplication and division. An additional interview provided information on the teacher's specific climate for teaching mathematics and about the unit on multiplication and division to be observed.

Phase 2 consisted of 20 classroom observations. The class was observed three days per week (every day the class was taught) for approximately one hour each day during the teaching of a seven week unit on multiplication and division. Informal interviews were also conducted with the teacher throughout the unit to acquire a better understanding of the lessons and allow the teacher an opportunity to clarify statements and actions. A final interview occurred after the last classroom observation.

At the conclusion of the observations, the students were assessed to determine their knowledge of multiplication and division with respect to the teacher's unit objectives. Within three weeks after the administration of the student assessment, six students recommended by the teacher (two of the students recommended consistently performed in the upper third of the class, two performed in the middle third, and two performed in the bottom third) were interviewed to provide additional insights into the students' learning.

Conclusions concerning the answers to the research questions are addressed in the following section. These conclusions are drawn from data collected throughout the case study. In addition to the conclusions and the attending discussion, comments concerning the limitations of the study, recommendations for further research, and implications of this study for the field of elementary mathematics teacher education are addressed.

Meg's Subject Matter Knowledge Structure

The first research question addressed the appearance of Meg's subject matter knowledge structure for addition, subtraction, multiplication, and division. The interpretation of her generated subject matter knowledge structure was derived from both an analysis of the visual representation and her comments made in the interview related to her diagram.

The subject matter knowledge diagram that Meg created in response to the questionnaire included three parts. The top part expressed Meg's notion of how addition, subtraction, multiplication, and division were related to each other; the middle part represented her conception of the close linkage between addition and subtraction; and the final part indicated her idea of the relationship between multiplication and division.

Meg actually considered all four operations closely related. She felt it was important to understand the meaning of operations, relationships among them, and have computational skill so that the operations could be effectively used in other settings both in and out of mathematics. Meg viewed especially close relationships between addition and subtraction and between multiplication and division. In her opinion the operations should be learned in pairs. Thus, from her perspective, addition and subtraction should be taught together and multiplication and division should be taught together.

Although Meg was not observed during the teaching of addition and subtraction, her subject matter knowledge structure with respect to addition and subtraction was assessed. Meg's diagram of addition and subtraction was a web consisting of five primary components extending out on axes: concepts, computation, vocabulary, number sense, and connections. The lines of connections merely represented a convenient manner of organizing the topics listed. For Meg, addition and subtraction were closely related because one was the inverse of the other. To Meg, the concept of addition was putting things together and subtraction was taking things away or comparing things. She considered manipulatives as the basic tools for providing meaning to operations and connecting the meanings to symbols. She showed a basic understanding of addition and subtraction by solving and recording simple "putting-together" and "take-away" problems using counters. Meg also explained and justified the meanings of the standard algorithms for addition and subtraction. Similarly, Meg's subject matter knowledge included an understanding of addition and subtraction of fractions and, using fraction pieces, she was able to show and explain how to add and subtract fractions. Real-life story problems were another basic tool to provide meaning to addition and subtraction. Meg was able to provided meaningful stories for both addition and subtraction of whole numbers and fractions.

Similar to addition and subtraction, Meg organized multiplication and division into five essential components: concepts, computation, vocabulary, number sense, and connection. Meg considered multiplication and division to be inverses of each other. Her knowledge structure consisted of two ways of modeling multiplication:

repeated addition and an array. To Meg, the most common representation of multiplication was that of repeated addition. For example, the equation 6 + 6 + 6 = 18 and the equation 3 x 6 = 18 represented the same thing, both modeled by three groups of six objects. She was also able to represent the equation 3 x 6 = 18 as a 3 by 6 rectangular array and explain what she meant. To Meg, the 3 meant there were three rows of squares, 6 meant there were six squares in each row, and the product, 18, was how many squares there were in the 3 x 6 rectangle filled in with squares.

Meg's knowledge structure for division had only one meaning, equal sharing (partitive). Equal sharing meant separating a collection of objects into equal size groups. To Meg, the equation 46 ÷ 2 = 23 meant dividing the total 46 into two equal groups of 23. Meg's knowledge structure did not include an understanding of division as grouping (repeated subtraction or measurement), splitting a collection of objects into groups of a known size. Several times during the teaching of the unit Meg referred to division as "repeated subtraction," but she never explained in class exactly what she meant. When asked following the teaching of the unit what repeated subtraction meant, Meg was unable to give a meaningful response. Meg also referred to division simply as "a faster way to subtract." She never explained in class what she meant nor was she able to explain this idea at the conclusion of the unit. Ball (1990a, 1990b) and Simon (1993) found that prospective elementary teacher in their studies exhibited serious shortcomings, similar to Meg's, in their understanding of division.

A second component in Meg's knowledge structure was computation. To Meg, knowing multiplication and division facts was necessary in solving problems. These facts included the single-digit multiplication combinations and the counterparts for division. Manipulatives were a basic tool for understanding and visualizing the facts. Meg felt that although the notion of multiplication as repeated addition was sufficient for visualizing products of small numbers, the array model was more efficient for seeing products of larger numbers and doing computation.

Meg was hesitant about explaining a method for doing two-digit by two-digit multiplication, but she was able to build an array to model the product and justify the meaning of each step in an extended algorithm. Meg's structure also included multiplication of fractions and she was able to show an algorithm for finding the product of factions as well as explain the meaning of the operation using manipulatives.

Meg could calculate the quotient of a three-digit number divided by a two-digit number by building an array and doing the long division algorithm but she was unable to explain the meaning of the steps in the algorithm. Simon (1993) found that prospective elementary teachers had similar difficulty. They seemed to have appropriate knowledge of the symbols associated with division, but appeared to be missing the conceptual underpinnings of the division algorithm.

Meg also had difficulty with division of fractions. She was able to find the quotient by the "invert and multiply" algorithm but she was unable to extend any meaning to this idea. Although Meg tried to relate her understanding of division of whole numbers to division of fractions, her sharing notion of division corresponded less easily to division with fractions than grouping would have. Ball (1990a, 1990b) also found that prospective elementary teachers could calculate a quotient involving fractions but had difficulty with the meaning of division with fractions. The prospective teachers in her study perceived the task to be about fractions not division.

Other components in Meg's structure included a knowledge of vocabulary and number sense. The terms she considered to be essential for multiplication and division were: multiply, product, factor, multiple, column, row, set, group, divide, quotient, divisor, and dividend. Although Meg was able to explain the meanings of such terms and many of the properties and relationships among them, She had difficulty with divisibility as a property of numbers without performing division. Meg knew rules for divisibility by 2, 5, and 10, but she had no knowledge of other rules for divisibility. She seemed to think that if the ones digit of a number was divisible by three then the number was divisible by three. Zazkis and Campbell (1995) found that prospective elementary teachers had difficulty with ideas of divisibility as a property of numbers without performing division but most participants were familiar with the divisibility rules for 2, 3, 5, and 10. Meg's knowledge of divisibility seemed to be more limited than students in this study.

A final component in Meg's knowledge structure was connections. Meg was able to use manipulatives to show meanings of multiplication and division and to link the meanings of these operations to symbols. Another important connection, according to Meg, was to connect real life story problems to multiplication and division. Meg provided a story for which multiplication of fractions could represent the operation used to solve the problem but she was unable to supply a story for which division of fractions could represent the operation used. Ball (1990a, 1990b) and Simon (1993) also found that most prospective elementary teacher were unable to create appropriate story problems for expressions involving division of fractions. Simon (1993) found that the most common errors consisted of writing a story for which multiplication of fractions represented the operation used to solve the problem rather than division of fractions. Ball (1990a, 1990b) suggested that the difficulty stemmed from the fact that most were able to consider division only in partitive terms.

Subject Matter Knowledge and Its Relationship to Teaching and Learning

The second research question addressed how Meg's subject matter knowledge related to teaching and learning. The understandings and philosophical orientations that Meg held toward the teaching of multiplication and division were directly related to her teaching and students' learning. However, it was unclear whether her knowledge structure affected her teaching or visa versa.

Rather than being tied to the content and format of the textbook, Meg designed her own unit based on the content she knew and how she thought the content should be taught. Meg's selection and organization of materials, ways to represent concepts and procedures, instructional strategies, and ways to promote discourse reflected how she organized mathematics for herself. A major focus of the unit involved developing the meanings of multiplication and division and how they were related to each other. Although this focus was consistent with recommendations by current reforms in mathematics education (Mathematics Association of America [MAA], 1991; National Council of Teachers of Mathematics [NCTM], 1989, 1991, 1995; National Research Council [NRC], 1989, 1990, 1996; Third International Mathematics and Science Study [TIMSS], 1996), the limits of Meg's knowledge structure and time inhibited the development of the full range of multiplication and division situations in her instruction.

Teaching and Learning

Consistent with Meg's knowledge structure, she introduced the concepts of multiplication and division from several perspectives: concrete, pictorial, symbolic, and real-life contexts using projects, problem solving activities, and guided manipulative activities. Models were the basic tools Meg used to develop the meaning of these concepts. Attention was devoted to the concept of multiplication as repeated addition, but the primary representation of multiplication was that of an array. Arrays were used to provide meaning to the basic multiplication facts and to connect conceptual and procedural knowledge. Division instruction focused mostly on the idea of sharing, separating a collection of objects into equal size groups. Although Meg introduced division as the inverse of multiplication (Meg called it the opposite) by using arrays, the primary model for division involved sharing countable objects.

Computation was also evident in Meg's teaching. The memorization of facts was an on-going homework assignment and a major activity of the unit involved creating multiplication charts with grid paper rectangles to give students a concrete representation of the facts. Most homework assignments involved practice of basic facts for multiplication and division and Mad Minute activities were frequently used throughout the unit to give students additional practice with the facts and to assess their skills. Although few activities involved computational practice beyond the basic facts, the standard algorithm for multiplication was presented with the meanings of steps explained and justified.

Connections were used to foster understanding and relationships. Various manipulatives were used to represent concepts, procedures, and processes: base ten blocks, rectangular grid paper, rainbow cubes, and various drawings and pictures. These manipulatives were used to explore multiplication as repeated addition and as an array, division as sharing and as an array as well as an assortment of multiplication and division problem situations. Equations and computational representations were also introduced in the context of the activities to provide connections between the operations and symbols.

Meg's notion of the connectiveness between the operations of addition, subtraction, multiplication, and division was evident in more than just the teaching of multiplication and division together. Several activities were used to explore the notions that the operations of multiplication and division were related in that one was the inverse (Meg incorrectly used the term opposite) of the other, multiplication can be thought of as repeated addition, and division can be thought of repeated subtraction. An assortment of activities involving problem solving, reading, writing, and solving story problems, various writing activities, and children's stories were also used to provide a real-world context.

Although students generally had more success on questions involving an understanding of multiplication, students were successful representing arrays as multiplication and division, representing multiplication and division as arrays, and solving and writing multiplication and division story problems. The students interviewed were able to relate physical materials, pictures, and story problems to the meaning of multiplication and the sharing idea of division.

On the post assessment, students success rates of correct responses on problems involving basic facts for multiplication and division were similar and both fairly high. Scores on the Mad Minute activities showed an increase for every students and, interestingly, students did slightly better on the final Mad Minute activity for division than they did for multiplication.

Meg was insecure about her knowledge of mathematics and in particular her knowledge of division. She knew only one type for division, equal sharing (partitive). Her knowledge structure did not include an understanding of division as grouping. Research by Ball (1990a, 1990b) and Simon (1993) concur that preservice teachers have a narrow understanding of division that only included division in partitive terms. Meg's narrow understanding of division seemed to prevent her from developing the full range of division situations. For example, Meg suggested in several discussions that the only division equation associated with an array was one in which the divisor represented the number of rows and the quotient represented how many were in each row. Although students suggested an alternative interpretation in which the divisor represented the number in each row, Meg was unable to develop their ideas. As a result, manipulative models and story problems focused only on the sharing meaning of division. Students were not provided opportunities to recognize, model, and solve both division sharing and grouping types of problems as recommended by NCTM (1989).

Meg's limited understanding of division seemed to be linked to students' success on the post assessment. The students did consistently better on problems involving the meaning of multiplication than on those involving division. The students also did considerably better on solving story problems involving the sharing type of division as opposed to those story problems involving grouping. Interestingly, all of the division story problems written by the students consisted of division sharing ideas.

Another situation in which Meg's insecurity with mathematics prevented her from developing the full range of multiplication and division situations occurred in class discussions. According to NCTM (1991), the teacher has a central role in orchestrating discussions that contribute to students' meaningful understanding of mathematics. Several researchers (Cobb et al., 1991) further suggested that a role of the teacher in classroom discourse was to legitimize aspects of contributions to a discussion in light of their potential fruitfulness for further mathematical constructs, redescribe students explanations in more sophisticated terms that students can still understand, and guide the development of taken-to-be-shared ideas. In summarizing lessons, Meg was able to orchestrate lively discussions but she was unable to guide the development of taken-to-be-shared ideas and provide closure to what had been learned. Other researchers have reached similar conclusions that teachers with weak conceptual understanding of mathematics have difficulty orchestrating mathematical discourse in the classroom (Lehrer & Franke, 1992; Leinhardt & Smith, 1985).

The students' performances on assignments and the post assessment seemed to reflect this lack of closure to activities. Although discussions generated insightful ideas, the students were unable to communicate what they had learned on written assignments. Students also had difficulty writing about multiplication and division on the post assessment. Only four students were able to provide at least two ideas about multiplication and none of the students were able to give more than one idea about division. The six students interviewed were able to provide at least one correct idea about multiplication and only two students interviewed were able to provide at least one correct idea on division.

Homework assignments involved practice of basic facts for multiplication and division, but only a few assignments included computation that required an algorithm. Although Meg had no difficulty explaining the multiplication algorithm to the students and the meaning of each step in the algorithm, she did not present a method for doing long division nor provide opportunities for students to explore their own algorithms for division.

The students' success on the computation portion of the post assessment reflected the lack of attention given to division. The students had more success on the multiplication computation problems than on the division computation. On average, 84% of multiplication computation problems were answered correctly as compared to only 37% for division. The students interviewed were mostly able to do the multiplication computation problems correctly and justify their answers but most of the students were unable to even compute a correct answer on the problems consisting of division computation. The students did not seem to have adequate strategies for computing division computation especially problems with remainders.

Several times during the unit, Meg allowed aspects of discussions to resulted in the development of taken-to-be-shared ideas that were incorrect. For example, Meg frequently referred to division as "repeated subtraction" and as a "faster way to subtract." Meg developed the idea of repeated subtraction in a discussion as an analogy to multiplication as repeated addition. Although the notion of division as repeated subtraction was correct, Meg never explained exactly what she meant. In fact, Meg implicitly legitimized the notion that it meant "a faster way to subtract." In the post assessment, several students indicated that division was repeated subtraction but only one student was able to provide a meaningful explanation of the statement. Several students stated that division was a faster way to subtract but their responses indicated a lack of understanding.

In a discussion involving multiples, several students suggested that division was a quicker way than multiplication to determine if a number was a multiple of three. Additionally, a student conjectured (in less formal terms) that a number was divisible by 3 if and only if its ones digit was divisible by 3. When this incorrect generalization appeared, Meg was unable to provide counter examples to help the students recognize their misconception. By not confronting students about their misconception, she allowed the students to believe that their construct was correct. Although the post assessment was not designed to uncover misconceptions associated with divisibility, classroom discussions revealed that students' misunderstandings existed.

Time

Time seemed to have a huge influence on Meg's teaching of the unit: time to teach, time to prepare and time to reflect. Each of these constraints were mentioned by other researchers as forces that influenced the teaching that occurred in the translation of the teacher's subject matter knowledge structure (Gess-Newsome & Lederman, 1995; Thompson, 1984). Gess-Newsome and Lederman found that time was identified by all of the teachers in their study to have a tremendous influence on their preparation, teaching, and reflection.

For Meg, time to teach simply meant having enough time to complete all of the activities that she wanted to do in order to cover the material and assure mastery by the students. Meg's struggle with time typically seemed to occur in terms of introducing activities, providing students enough time to work on the activities, and discussing them in class in the time allotted. Meg typically extended the original time schedule but, even then, was rarely able to bring topics to closure and assure students' understanding before moving on to the next topic.

Time may have been more critical in terms of Meg's planning and reflection. Since Meg designed her own unit and did not follow the textbook, she selected and organized almost all of her activities from other sources. Such a time commitment may have been critical in terms of Meg's ability to have well-formed expected outcomes for her students. In addition, time may have reduced Meg's opportunities to reflect on the learning and teaching that was taking place in her classroom. The *Professional Standards for Teaching Mathematics* (1991) that teachers of mathematics should take an active role in their professional development by reflecting on learning and teaching individually and with colleagues.

Summary

Meg's subject matter knowledge of addition, subtraction, and multiplication were quite strong, but several circumstances indicated faulty and limited understandings of division. In examining the data, Meg did not seem to understand the idea of division as grouping (measurement, quotitive, repeated subtraction). Meg's difficulties seemed to stem from her awareness of division only as sharing and her reliance on sharing for all division situations. Her knowledge included the conceptual underpinnings of the standard algorithm for multiplication but she seemed to lack the conceptual foundation of long division. Her notion of divisible by three was also faulty.

Meg did not follow the textbook. She designed her own unit for multiplication and division based on her incomplete knowledge of division. The materials she selected, ways to represent concepts and procedures, instructional strategies, and ways to promote discourse were mostly consistent with recommendations by current reforms in mathematics education (MAA, 1991; NCTM, 1989, 1991, 1995; NRC, 1989, 1990, 1996; TIMSS, 1996). However, Meg's incomplete subject matter knowledge of division was actively translated into her planning, classroom teaching, and students' learning for her unit on multiplication and division.

In teaching, Meg's incomplete knowledge of division seemed to prevent her from developing the full range of division situations. Students were not provided opportunities to recognize, model, and solve division grouping types of problems. The manipulative models and story problems focused only on sharing division. Division computation was also not developed and faulty ideas were presented related to divisibility.

Students' learning seemed to be directly linked to the incomplete understandings held by Meg and the instruction she presented as a result of her limited knowledge. Although students' scores were significantly inferior on the division portion of the assessment, a more worrisome concern was the unresolved misconceptions held by the students. The notion of division only in terms of sharing and as "a faster way to subtract," and a faulty rule for divisibility were echoed by students on the post assessment and during the assessment interviews. These misconceptions may go undetected and impede students' ability in problem solving and learning new concepts and procedures. Determining at some later date the root of students' difficulties may be challenging.

Limitations of the Study

Several aspects limit the generalizability of the findings reported: the representativeness of the teacher, the content areas selected to be investigated, the manner in which the teacher's subject matter knowledge was derived, the student assessment, the classroom situation, the length of the study, and the limitations of the researcher. Only one volunteer teacher from a small geographical area was selected for inclusion in this investigation. However, attempts were made to assure that the teacher was representative of experienced intermediate elementary mathematics teachers for the content areas observed. First, only third, fourth, and fifth grade teachers were considered. Second, the teacher needed to have 4 to 12 years teaching experience. Third, the teacher needed to have previously taught at the current grade level for at least one year. Finally, the teacher needed to have differences (in formats or in breadth and depth) in her subject matter knowledge structure for at least two of the content areas assessed. No other specific attempts were made to assure that the teacher was representative of elementary teachers of the four fundamental operations. To further strengthen the generalizability of these findings a much larger teaching community would need to be studied.

In addition, this study purposefully narrowed its focus to the study of the subject matter knowledge structure of the fundamental operations of addition, subtraction, multiplication, and division held by an elementary mathematics teacher. No generalizations can be made concerning the subject matter knowledge structures for other topics or the constraints that may exist for the implications of such structures on teaching and learning. Although the generalizability of these findings may not be an issue since the context of teaching and personal history of the teacher seemed to influence the results obtained to such a great extent, little evidence indicated that the life and experience of this teacher were so unique as to preclude the use of these findings as the stimulus and basis for further investigation.

With the assessment of the subject matter knowledge, two-dimensional diagrams may be inadequate for representing the complex interactions and interwoven substance of a teacher's subject matter knowledge structure. It is also possible that such statements were merely the parroting of answers that were considered to be appropriate, but stated without a true understanding or philosophical commitment. However, interviews were conducted in which the teacher was asked to explain and justify her understandings of the topics mentioned in her diagrams. Although the methods used in this investigation seemed to be superior to those used in other studies, such limitations must be recognized.

The student assessment was created using the teacher's unit objectives. Although content validity and reliability were established, the types of questions asked and the paper-and-pencil manner in which the whole-class data were gathered, may have been inadequate considering the complex and integrated nature of the students' mathematical knowledge and skills. Furthermore, only six students were interviewed following the administration of the assessment to provide a more detailed picture of the multiplication and division knowledge of the students. Attempts were made, however, to assure that the six students were representative of the entire class:

two of the students selected consistently performed in the upper third of the class, two performed in the middle third, and two performed in the bottom third.

The complexity of the classroom situation may have included several factors that skewed the results. First, the class was taught only three days a week and lasted about 65 minutes per day. The class included 23 students grouped from four multi-age third-fourth grade classes and the students spent the remainder of the day in their original third-fourth multi-age classrooms. With this limited exposure to students, the teacher may have been unable to provide the breadth and depth of instruction she had intended.

The length of the study may have also misrepresented the findings. The study was conducted over a seven week period and, at the conclusion of the unit, the teacher was not satisfied with the students' ability to do computation. She intended to continue practice on this topic for the remainder of the year. Although many of the misunderstandings presented in the lessons seemed to have little to do with the short term nature of the study, a year-long study may provide more reliable results.

Finally, as the main instrument in collecting and analyzing data, the researcher introduced several limitations. A daily journal was kept containing the researcher's reflections on classroom observations and used to guide weekly interviews providing the teacher opportunities to clarify observed actions. This process was an attempt to discourage the researcher from relying on personal interpretations of the behaviors of the teacher and students. Although such design attempts were made to prevent as many threats to validity as possible, the researcher's background, experiences, and biases still limit the conclusions drawn.

Implications and Recommendations for Mathematics Teacher Education

This study extends the information available on teachers' subject matter knowledge of multiplication and division and how teacher knowledge relates to classroom teaching and student learning. In particular, the study provides a direction to the question concerning the relative effectiveness of different knowledge structures and the potential impact of these structures on teaching and learning. In addition, the results suggest implications for both preservice and inservice education as well as several avenues for research.

Examining Meg's subject matter knowledge of mathematics provides a disturbing picture of precollege, college, and inservice mathematics education for teachers. As suggested by the examples in this study, Meg's subject matter knowledge of division tended to be incomplete. However, her limited understanding of division was not surprising since Meg had only a few high school, college, and inservice mathematics courses. For the most part, it seemed that Meg developed her mathematical understandings, perceptions of what it meant to know and be able to do mathematics, and self concept as a doer of mathematics in her own classroom. Considering her limited educational background, it was remarkable that she was able to facilitate the level of instruction and learning presented in this investigation.

Previous research studies have noted that preservice elementary teachers have incomplete subject matter knowledge of mathematics (Ball, 1990a, 1990b; Graeber &Tirosh, 1988; Simon, 1993; Tirosh & Graeber, 1989; Zazkis and Campbell, 1995). These studies suggested that teacher education programs cannot assume that preservice teachers have a comprehensive and well-articulated knowledge of mathematics sufficient to teach elementary mathematics. This present study suggests that inservice teachers may also have faulty and incomplete knowledge of mathematics, especially division. As studies suggest (Lanier, J. E. & Little, J. W. , 1986), the world of elementary schools may not offer a positive environment for teachers to develop their knowledge of mathematics.

Although it was not clear whether Meg's subject matter knowledge structure affected teaching or whether teaching affected her subject matter knowledge structure, the data reported in this study suggested that the scope of Meg's knowledge of multiplication and division was directly related to classroom teaching and students' learning. Therefore, educational programs are needed to identify the mathematical understandings of prospective teachers and develop programs to more adequately prepare them for teaching. Although improving educational programs for preservice teachers may result in changing the number of courses prospective teachers are required to take, efforts must be made to provide prospective teachers the opportunities to understand the concepts underlying the mathematics that they will teach and how these concepts are related. Simon (1993) suggested that a focus on prospective teachers' understandings of concepts and relationships should make the development of dense webs of understandings a higher priority than vertical content coverage. This study supports such a focus.

Meg's pedagogical content knowledge also seemed to be directly related to her subject matter knowledge of multiplication and division. The activities she designed, the ways she represented and formulated concepts and procedure, and the students' suggestions she followed were consistent with her strong understanding of multiplication and her narrow understanding of division. Perhaps, along with developing prospective teachers' subject matter knowledge, teacher preparation programs should focus specific attention on the pedagogical content knowledge that is apparently needed to implement the current popular reforms.

Reformers (MAA, 1991; NCTM, 1989, 1991, 1995; NRC, 1989, 1990, 1996) suggest the need for professional development programs for inservice elementary mathematics teachers to help them become more "competent." This study suggests two aspects of the design of such programs should be considered: diagnosis of teachers' knowledge and intervention to help replace faulty or limited knowledge with appropriate understandings. It seems extremely important that inservice programs bring teachers to an awareness of their knowledge of mathematics and the possible effects that their knowledge has on teaching and learning. Interestingly, although Meg was insecure about her knowledge of mathematics, she was unaware of her faulty and limited understandings of division. Unless teachers become aware and dissatisfied with existing conceptions, change cannot possibly occur.

Since the teacher's knowledge structure was directly related to teaching and learning, classroom observations may be one vehicle to identify such misconceptions. However, observing a teacher several times throughout the year or even several times during a unit may not uncover faulty or incomplete knowledge. It was necessary for the researcher in this study to observe the teacher everyday during the teaching of the unit to identify the reported difficulties.

In light of current recommendations in mathematics education, a few words of caution are in order. It would appear that curriculum reform efforts, that include a wider range of content and a greater emphasis on conceptual understanding (NCTM, 1989, 1991, 1995), presume that elementary teachers have, or at least will have, the mathematical knowledge necessary to provide instruction and develop curriculum as envisioned. A critical question arises as to whether it makes sense to expect, or desire, all elementary teachers to have such a high level of knowledge. It would appear from this study that to improve the quality of mathematics instruction in US schools, it is time to have specialists for teaching mathematics for grades three and beyond.

It is also a widely accepted belief in education today, especially in elementary school, that "good teachers" do not use or follow a textbook, but rather design their own units of instruction. Publishing textbooks is a business, and although publishers enlist as authors mathematics educators and teachers who are quite knowledgeable, there are frequently significant gaps between what the authors think is good and what the publishers think will sell. However, elementary teachers with faulty or limited understanding of mathematics may not be the best judges of what ought to be in the curriculum and, if this study is any indication, elementary teachers may not be aware of their misconceptions. Rather than advising elementary teachers to design their own units of instruction without assistance, teachers should be encouraged to supplement textbook materials but not discouraged from using them.

Furthermore, if teachers are to improve, it is critical that they be given the time and opportunity for reflection and planning. Time to reflect and prepare to teach seemed to have had a tremendous influence on Meg's teaching. This study supports current reform efforts in mathematics education recommending that the professional development of teachers should include reflecting on learning and teaching both individually and with colleagues (NCTM, 1991).

Finally, this study provides a stimulus for further investigation concerning the relative effectiveness of different knowledge structures and the potential impact of these structures on teaching and learning. Given that the data generated involved only one teacher, research is needed on a much larger scale involving teachers with characteristics closely aligned with the nation's teaching force. This study purposefully narrowed its scope to the subject matter of the fundamental operations held by an elementary mathematics teacher. More research is needed concerning the subject matter knowledge held by elementary teachers for other areas of mathematics and the impact of those structures on teaching and learning. This study suggests a link between the incomplete subject matter knowledge held by a teacher and misconceptions acquired by students. Additional studies are needed to verify this link. In particular, research is needed to study the misconceptions held by students and see if those misconceptions relate back to classroom instruction and teachers' subject matter knowledge.

References

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education.* Unpublished doctoral dissertation, College of Education, Michigan State University, East Lansing, MI.

Ball, D. L. (1990a). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21* (2), 132-144.

Ball, D. L. (1990b). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90,* 449-467.

Berliner, D. C. (1988). *The development of expertise in pedagogy.* Paper presented at the annual meeting of the American Association of Colleges for Teacher Education, New Orleans.

Borg, W. (1987). *Applying educational research: A practical guide for teachers, 2nd ed.* New York & London: Longman.

Carpenter T. P., Fennema E., Peterson P. L., Carey, D. A. (1988). Teachings' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19,* 385-401.

Cooney, T. J. (1994). Research and teacher education: In search of common ground. *Journal for Research in Mathematics Education, 25* (6), 608-636.

Eisenberg, T. A. (1977). Begle revisited: Teacher knowledge and student achievement in algebra. *Journal for Research in Mathematics Education, 8,* 216-222.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21,* 521-544.

Even, R. & Lappan, G. (1994). Constructing meaningful understanding of mathematics content. In Aichele, D. & Coxford, A. (Ed.), *Professional development for teachers of mathematics* (1994 Yearbook of the National Council of Teachers of Mathematics). Reston, VA: Author.

Fennema, E., & Loef Franke, M. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-163). New York, NY: Macmillan Publishing Company.

Funkhouser, C. P. (1993). An examination of the problem-solving conceptions of inservice teachers. *School Science and Mathematics, 93* (3), 81-85.

Gage, N. L., & Berliner, D. C. (1998). *Educational psychology* (6th ed.). Boston: Houghton Mifflin.

Gall, M., Borg, W., & Gall, J. (1996) *Educational research: An introduction, 6th ed.* New York: Longman

Gess-Newsome, J., & Lederman, N. G. (1995). Biology teachers' perceptions of subject matter and its relationship to classroom practices. *Science Education, 32*(3), 301-325.

Gess-Newsome, J., & Lederman, N. G. (1993). Preservice biology teachers' knowledge structures as a function of professional teacher education: A year-long assessment. *Science Education, 77*(1), 25-45.

Graeber, A. O., & Tirosh, D. (1988). Multiplication and division involving decimals: Preservice elementary teachers' performance and beliefs. *Journal for Research in Mathematics Education, 7,* 263-280.

Greeno, J. G. (1977). Process of understanding in problem solving. In N. J. Castellan, Jr., D. B. Pisoni, & G. R. Potts (Eds.), *Cognitive theory* (Vol. 2, pp. 43-83). Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Hiebert & Carpenter, (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* (pp. 65- 97). New York, NY: Macmillan.

Khoury, H. A., & Zazkis, R. (1994). On fractions and non-standard representations: Preservice teachers' concepts. *Educational Studies in Mathematics, 27,* 191-204.

Koehler & Grouws, (1992). Mathematics teaching practices and their effects. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* (pp. 115-125). New York, NY: Macmillan.

Lanier, J. E. & Little, J. W., (1986). Research on teacher education. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.. pp. 527-569). New York, NY: Macmillan.

Lappan, G., & Theule-Lubienski, S. (1994). Training teachers or educating professionals? What are the issues and how are they solved? In D. Robitaille, D. Wheeler, & C. Kieran (Eds.), *Selected lectures from the 7th International congress on Mathematical Education* (pp. 249-261). Sainte-Foy, Quebec: Les Presses de L'Universite Laval.

Lederman, N. G. & Chang, H. (1997). An international investigation of preservice teachers' pedagogical and subject matter knowledge structures. *National Science Council Part D: Mathematics, Science, and Technology Education. 7*(2), 110-122.

Lederman, N. G., Gess-Newsome, J., & Latz. M. (1994). The nature and development of preservice teachers' conceptions of subject matter and pedagogy. *Journal of Research in Science Teaching, 31*(2), 129-146.

Lehrer, R., & Franke, M. L. (1992). Applying personal construct psychology to the study of teachers' knowledge of fractions. *Journal for Research in Mathematics Education, 23* (3), 223-241.

Leinhardt, G. (1986). *Math lessons: A contrast of novice and expert competence.* Paper presented at the annual meeting of the Psychology of Mathematics Education, East Lansing, MI.

Leinhardt, G. (1983, April). *Overview of research on teachers' and students' routines, thoughts, and execution of plans.* Paper presented at the meeting of American Educational Research Association, Montreal, Canada.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), 247-271.

Lesh, Post, & Behr, (1987). Representation and translations among representation in mathematics learning and problem solving. In C. Janvier (Ed.) *Problems of representation in the teaching and learning of mathematics* (pp, 33-40). Hillsdale, NJ: Lawrence Erlbaum.

Mathematical Association of America (MAA). *A call for change: Recommendations for the mathematical preparation of teachers of mathematics.* United States: Author.

Mathematics Science Education Board. (1989). *Everybody counts.* Washington, DC: National Academy Press.

Mathematics Science Education Board. (1990). Reshaping school mathematics: A philosophy and framework for curriculum. Washington, DC: National Academy Press.

National Council of Teachers of Mathematics. (1980). *Agenda for action: Recommendations for school mathematics of the 1980s.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

National Research Council. (1989). *Everybody Counts! A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press.

Polya, G. (1957). *How to solve it* (2nd ed.). garden City, NY: Doubleday Anchor Books.

Post, T. R., Harel, G., Behr, M. J., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177-198). Albany, New York: State University of New York Press.

Postman, N. & Weingartner, C. (1969). *Teaching as a subversive activity.* New York, NY: Deacorte Press.

Putnam, R. T., Lampert, M., & Peterson, P. L. (1990). Alternative perspectives on knowing mathematics in elementary schools. In C. Cazden (Ed.), *Review of research in education : Vol. 16.* (pp. 57-150). Washington, D. C.: American Educational Research Association.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20,*, 8-27.

School Mathematics Study Group. (1972) *Correlates of mathematics achievement: Teacher background and opinion variables.* J. W. Wilson & E. A. Begle (Eds.), *NLSMA Reports* (No. 23, Part A). Palo Alto, CA: Author

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24* (3), 233-254.

Shaw, K. (1989). Contrasts of teacher ideal and actual beliefs about mathematics understanding: Three cases. (University of Georgia, Athens, 1989). *Dissertation Abstract International, 50,* 2818A.

Shulman, L. S. (1986a). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1986b). Paradigms and research programs in the study of teaching: A contemporary perspective. In M. C. Wittrock (Ed.), *Handbook of Research on Teaching,* 3rd Ed. (pp. 3-36). New York: Macmillan Publishing Company.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Shulman, L. S., & Grossman, P. L. (1988). *Knowledge growth in teaching: A final report to the Spencer Foundation.* Stanford, CA: Stanford University.

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions to graphing. American *Educational Research Journal, 27*(4), 639-663.

Stigler, James W., and James Hiebert. "Understanding and Improving Classroom Mathematics Instruction: An Overview of the TIMSS Video Study." *Phi Delta Kappan* 79, no. 1 (1997): 14-21.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-147). New York, NY: Macmillan Publication Company.

Thompson, A.G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practices. *Educational Studies in Mathematics*, 15, 105-127.

Tirosh, D., & Graeber, A. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics, 20* (1), 79-96.

Zazkis, R., & Campbell, S. (1994, April). *Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding.* Paper presented at the annual meeting of American Educational Research Association, New Orleans, LA.

APPENDICES

APPENDIX A

Letter of Introduction

Dear Colleague:

Thank you for considering participation in a research project designed to explore ways used to teach elementary school mathematics. Participation will be for the winter quarter of the 1998 - 1999 school year. Your commitment for this investigation will involve several interviews, answering a paper and pencil questionnaire, providing classroom documents used in your teaching, and at least twenty videotaped classroom observations. The questionnaire and an interview will be used to assess your subject matter knowledge of mathematics in the content areas of addition, subtraction, multiplication and division. Following the teaching of each topic observed, a test will be given to your students to determine what they have learned. Several students will also be interviewed to provide additional information on their understanding of the content taught.

During the fall quarter of the 1998 - 1999 school year, teachers will be interviewed by the researcher to collect data on their subject matter knowledge of several content areas of elementary school mathematics. At this time, the teachers will be asked to complete a questionnaire. A teacher will then be selected based on the data collected in these interviews.

The researcher and major professor will be the only persons with access to all data collected. Confidentiality will be maintained through the use of coding. Pseudonyms will be used for the university and all subjects when reporting the results of the research. Video tapes will be kept in a secure place until analysis is completed, at which time they will be erased.

Participation is voluntary; refusal to participate will involve no penalty or loss of benefits to which the subject is otherwise entitled. Furthermore, you may discontinue participation at any time.

Questions about the research, personal rights, or research-related injuries should be directed to: Dr. Margaret Niess at Oregon State University (541-737-1818).

Thank you for your time and participation in this research project.

Sincerely,

Bill Buckreis

I agree to participate in this research project and understand the general intent of the study, the types of data to be collected, and the time commitments involved in the study.

______________________________ ______________

Signature Date

APPENDIX B

Sign-up Form

Name: ______________________________ Age: ________ Gender: ________

Please answer the following questions:

1. What college(s) have you attended?

2. What degrees do you have?

3. What high school mathematics courses have you taken?

4. What undergraduate mathematics courses have you taken?

5. What graduate mathematics courses have you taken?

6. What professional development have you participated in related to your teaching of mathematics?

7. What was your most difficult mathematics course in high school? In college?

8. What was your easiest mathematics course in high school? In college?

9. What grade levels have you taught and for how many years?

10. Have you ever worked on curriculum projects? If so, when?

APPENDIX C

Questionnaire

1. What are the important topics, concepts, ideas, procedures, or themes that make up the content areas of addition, subtraction, multiplication, and division at the elementary school level? If you were to use these topics to diagram each content area (addition, subtraction, multiplication, and division), what would your diagrams look like?

2. Have you ever thought about these content areas in this way before? Please explain.

APPENDIX D

Letter to Parents

Dear Parent:

I am requesting permission for your son/daughter, ________(name)________, to participate in a research project designed to explore the ways used by your teacher to teach elementary school mathematics. The research will be conducted in your son's/daughter's mathematics class. Your son's/daughter's teacher has agreed to participate in the study. Participation will be for the winter quarter of the 1998 - 1999 school year. This investigation will involve at least twenty videotaped observations of the class. Following the teaching of each topic observed, an assessment will be given to all students in the class. Several students will also be interviewed to provide additional information on their understanding of the content taught.

The researcher and major professor will be the only persons with access to all data collected. Confidentiality will be maintained through use of coding. Pseudonyms will be used for the university and all subjects when reporting the results of the research. Video tapes will be kept in a secure place until analysis is completed, at which time they will be erased.

Participation is voluntary; refusal to participate will involve no penalty or loss of benefits to which your child is otherwise entitled. Furthermore, your child may discontinue participation at any time.

Questions about the research, personal rights, or research-related injuries should be directed to: Dr. Margaret Niess at Oregon State University (541-737-1818).

Thank you for your time and participation in this research project.

Sincerely,

Bill Buckreis

I agree to allow my son/daughter, ____________(name)____________, to participate in this research project and understand the general intent of the study, the types of data to be collected, and the time commitments involved in the study.

______________________________ Signature   ______________ Date

APPENDIX E

Letter to Students

Dear Student:

I am requesting permission for you to participate in a research project designed to explore the ways used by your teacher to teach elementary school mathematics. The research will be conducted in your mathematics class. Your teacher has agreed to participate in the study. Participation will be for the winter quarter of the 1998 - 1999 school year. This investigation will involve at least twenty video taped observations of the class. Following the teaching of each topic observed, you will be given a test to determine what you have learned. Several students will also be interviewed to provide additional information on their understanding of the content taught.

The researcher and major professor will be the only persons with access to all data collected. Confidentiality will be maintained through use of coding. Pseudonyms will be used for the university and all subjects when reporting the results of the research. Video tapes will be kept in a secure place until analysis is completed, at which time they will be erased.

Participation is voluntary; refusal to participate will involve no penalty or loss of benefits to which you are otherwise entitled. Furthermore, you may discontinue participation at any time. Questions about the research, personal rights, or research-related injuries should be directed to: Dr. Margaret Niess at Oregon State University (541-737-1818). Thank you for your time and participation in this research project.

Sincerely,

Bill Buckreis

I, ______________(name)______________ , agree to participate in this research project and understand the general intent of the study, the types of data to be collected, and the time commitments involved in the study.

______________________________   ______________

Signature   Date

APPENDIX F

Meg's Unit Objectives: Multiplication and Division

1. I would like the students to understand the meaning of multiplication and division. The students should know multiplication is repeated addition or an array, division is equal sharing or repeated subtraction, and multiplication and division are related operations.

2. I would like the students to know their basic multiplication and division facts and get faster at doing them [multiplication and division facts].

3. I would like the students to be able to build and draw arrays, and write the related multiplication and division equations.

4. I would like the students to be able to solve story problems and be able to create their own story problems.

5. I would like the students to be familiar with the vocabulary of multiplication and division. The students should understand words like rows, columns, arrays, product, factor, divisor, quotient, dividend, remainder, and multiples.

6. I would like the students to be able to multiply a two-digit number by a one-digit number and divide a two-digit number by a one-digit number.

APPENDIX G

Student Post Assessment

Part 1

Name ______________________________ Date ________________

Write the product.

1. 2×4 2. 5×8 3. 6×7

4. $\begin{array}{r} 21 \\ \underline{\times\, 3} \end{array}$ 5. $\begin{array}{r} 13 \\ \underline{\times\, 4} \end{array}$ 6. $\begin{array}{r} 46 \\ \underline{\times\, 7} \end{array}$

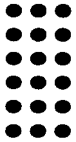
Write the quotient.

7. $16 \div 2$ 8. $30 \div 5$ 9. $28 \div 7$

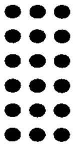
Write the quotient and remainder.

10. $4\overline{)11}$ 11. $8\overline{)41}$ 12. $6\overline{)82}$

13. Write a multiplication equation that describes the array below.



14. Write a division equation that describes the array below.



Solve the story problem. Write an equation that could be used to solve the problem. Make sure to answer the question.

15. Marie has 20 apples. She wants to share them equally among 6 of her friends. How many apples will each friend receive?

16. Mark has 4 bags of apples. There are 6 apples in each bag. How many apples does Mark have altogether?

17. Jill has a box of candy. There are 6 rows with 8 pieces of candy in each row. How many pieces of candy does Jill have altogether?

18. Kate is cooking omelettes for a class breakfast. She has 24 eggs in the refrigerator. If it takes four eggs for each omelette, how many omelettes can Kate make?

19. The school vans can hold 8 students. How many vans will it take to carry 25 students for the field trip?

20. Pencils cost 12¢ each. How much will it cost to buy 6 pencils?

Part 2

Name ______________________________ Date ______________

Find the sum. Then write the related multiplication equation.

21. 8 + 8 + 8 + 8

22. 3 + 3 + 3 + 3 + 3 + 3

Answer yes or no.

23. Is 18 a multiple of 4?

24. Is 20 a multiple of 4?

25. Is 7 a factor of 21?

26. Is 16 a factor of 4?

Answer each question.

27. What is the product in the equation 4 x 6 = 24?

28. What is the quotient in the equation 12 ÷ 3 = 4?

29. Write the other members of the fact family for 6 x 8 = 48.

30 Write the other members of the fact family for 27 ÷ 9 = 3.

Write your own story problems.

31. Write a story problem that follows two rules: (1) It must end in a question. (2) The question must be one that is possible to answer by using multiplication.

Write the multiplication equation that you would use to solve the story problem that you wrote.

32. Write a story problem that follows two rules: (1) It must end in a question. (2) The question must be one that is possible to answer by using division.

Write a division equation that you could use to solve the story problem that you wrote.

33. Use grid paper to show the multiplication problem 6 x 13. Show the problem by cutting and pasting part of the grid paper in the space below. Make sure to show the complete equation somewhere in the space below.

34. Use grid paper to show the division problem 63 ÷ 7. Show the problem by cutting and pasting part of the grid paper in the space below. Make sure to show the complete equation somewhere in the space below.

35. What do you know about multiplication? Use pictures, numbers, or words to explain your answer in as many ways as you can.

36. What do you know about division? Use pictures, numbers, or words to explain your answer in as many ways as you can.

APPENDIX H

Multiplication and Division Assessment Scoring Rubric

For each question on the assessment, the answer was judged to be correct, partially correct, or incorrect.

Questions	Points	Rationale
1 - 3	Correct (1 pts.)	Correct product
	Incorrect (0 pts.)	Incorrect product or fails to respond to the item
4 - 6	Correct (2 pts.)	Correct product
	Incorrect (0 pts.)	Incorrect product or quotient or fails to respond to the item
7 - 9	Correct (1 pts.)	Correct quotient
	Incorrect (0 pts.)	Incorrect quotient or fails to respond to the item
10 - 12	Correct (2 pts.)	Correct quotient and remainder
	Partial (1 pt.)	Correct quotient and incorrect remainder
	Incorrect (0 pts.)	Incorrect quotient or fails to respond to the item
13	Correct (2 pts.)	Answer of 6 x 3 = 18, 3 x 6 = 18 or $\begin{array}{r} 3 \\ \underline{\times 6} \\ 18 \end{array}$ $\begin{array}{r} 6 \\ \underline{\times 3} \\ 18 \end{array}$
	Partial (1 pt.)	Correct expression and incorrect product.
	Incorrect (0 pts.)	Incorrect equation or fails to respond to the item
14	Correct (2 pts.)	$18 \div 6 = 3$ or $18 \div 3 = 6$ or $6\overline{)18}$ with quotient 3 or $3\overline{)18}$ with quotient 6
	Partial (1 pt.)	Correct expression and incorrect quotient
	Incorrect (0 pts.)	Incorrect equation or fails to respond to the items
15 - 20	Correct (2 pts.)	Correct equation and answers the question correctly
	Partial (1 pt.)	Correct equation but fails to answer the question correctly or answers the question correctly but fails to write an equation.
	Incorrect (0 pts.)	Incorrect equation and incorrect answer to the question, or fails to respond to the item

21	Correct (2 pts.)	Answer of 4 x 8 = 32 or $\begin{array}{r} 8 \\ \underline{\text{x } 4} \\ 32 \end{array}$
	Partial (1 pt.)	Correct expression or product but not both.
	Incorrect (0 pts.)	Incorrect product or fails to respond to the item
22	Correct (2 pts.)	Answer of 6 x 3 = 18 or $\begin{array}{r} 3 \\ \underline{\text{x } 6} \\ 18 \end{array}$
	Partial (1 pt.)	Correct expression or product but not both.
	Incorrect (0 pts.)	Incorrect product or fails to respond to the item
23 - 26	Correct (1 pts.)	Correct yes or no answer
	Incorrect (0 pts.)	Incorrect yes or no answer or fails fo respond to the item
27 - 28	Correct (1 pts.)	Correct product or quotient
	Incorrect (0 pts.)	Incorrect product or quotient or fails to respond to the item
29 - 30	Correct (2 pts.)	Correct equations for the other three members of the fact family
	Partial (1 pt.)	Correct equation for at least one other member of the fact family
	Incorrect (0 pts.)	Incorrect equations or fails to respond to the item
31 - 32	Correct (3 pts.)	Story problem meets both rules correctly with a correct equation that could be used to solve the story problem
	Correct (2 pts.)	Story problem meets both rules correctly without a correct equation that could be used to solve the story problem
	Incorrect (0 pts.)	Incorrect story problem or fails to respond to the item
33 - 34	Correct (2 pts.)	Correct grid and correct equation
	Partial (1 pt.)	Correct grid and incorrect equation or fails to include an equation
	Incorrect (0 pts.)	Incorrect grid or fails to respond to the item
35 - 36	Correct (2 pts.)	At least two correct ideas using pictures, numbers, or words
	Partial (1 pt.)	One correct idea using pictures, numbers, or words
	Incorrect (0 pts.)	Incorrect ideas or fails to respond to the item

APPENDIX I

Ann's Subject Matter Knowledge Structure

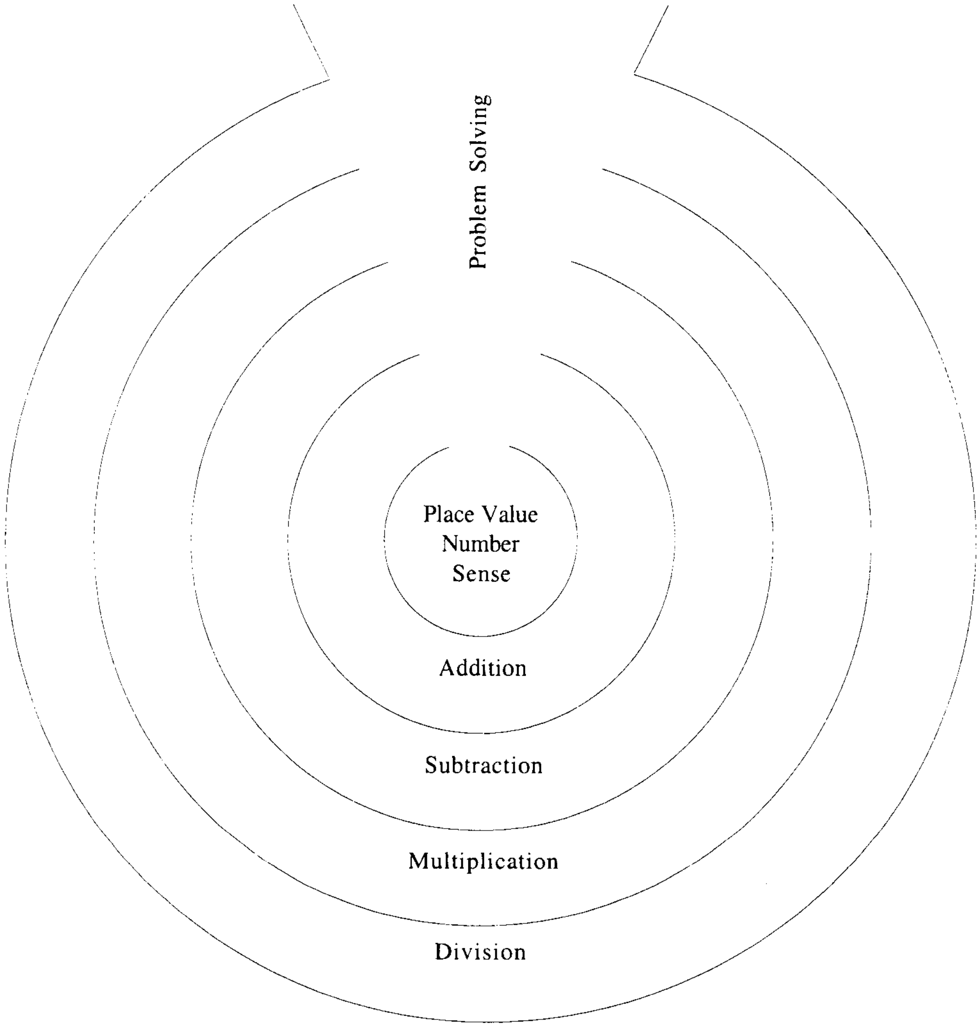


Figure 28. Ann's Subject Matter Knowledge Structure.

Ann's subject matter knowledge structure, Figure 28, consisted of five concentric circles with a funnel like shape extending down through the circles. For

Ann, the funnel represented problem solving. Problem solving was the reason for doing mathematics and, in particular, the reason for learning the four operations. The circles represented a hierarchy of the topics: number sense and place value, addition, subtraction, multiplication, and division extending out from the center, respectively. The inner circle, number sense and place value, was the foundation for the operations.

According to Ann, the content areas were highly interrelated. She saw the content presented in one area as the foundational knowledge for the content area that followed. Ann seemed to have only a few weaknesses in her knowledge structure. Her understandings of addition, subtraction, and multiplication were strong but her knowledge of division was incomplete. Ann did not seem to have a clear understanding of division as grouping (repeated subtraction) and the relationship between symbolic division and real world story problems. For example, Ann was able to explain division of whole numbers in terms of repeated subtraction, but she was unable to justify the meaning of division of fractions. She was also unable to provide a real world story problem for which division of fractions could be used to solve the problem. Rather, her story problem was one that would be solved with multiplication of fractions. Although her knowledge structure made her a possible candidate, Ann was reluctant to participate in extensive classroom observations. Thus, Ann was not considered to be the best subject for the study because she would likely change her normal teaching practices to impress the researcher.

APPENDIX J

Dianne's Subject Matter Knowledge Structure

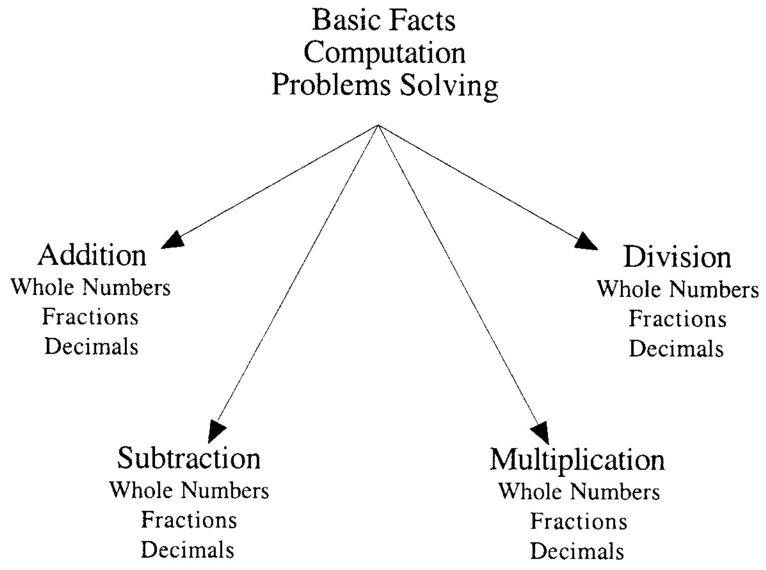


Figure 29. Dianne's Subject Matter Knowledge Structure.

Dianne described her subject matter knowledge diagram, Figure 29, as a hierarchy having four components: addition, subtraction, multiplication, and division with each component made up of whole numbers, fractions, and decimals taught in the respective order. The staring point for her hierarchy was with basic facts, computation, and problem solving. Computation was the major focus of Dianne's understanding of the operation and there was a set sequence for teaching each operation with skill practice an essential part of instruction. For example, addition facts came first, followed by one-digit plus two-digit addition with and without "carrying," then two-digit plus two-digit addition, next multiple-digit problems, and finally problem solving (word problems). Her sequences for teaching subtraction, multiplication, and division were similar. Although Dianne was able to explain the meanings of the algorithms for addition and subtraction of whole numbers, she was unable to provide the conceptual underpinnings of any other computation for whole numbers, fractions, or decimals.

According to Dianne, each of the operations had exactly one meaning: addition was combining objects, subtraction was taking objects away, multiplication was adding sets of objects that were all the same size (repeated addition), and division was putting objects into groups. Since Dianne was unaware of other meanings for the four operations and her knowledge of computation seemed mostly founded on remembering rules (algorithms) that were unattached to the concepts, Dianne was judged to have a narrow understanding of all four operations rather than the diversity of understanding desired for this study. She was also nervous about participating in extensive classroom observations. Thus, Dianne was not considered to be a good candidate for the study.