AN ABSTRACT OF THE THESIS OF

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Cellular sets in the Hilbert cube are the intersection of nested sequences of normal
cubes. One way of getting cellular maps on the Hilbert cube is by decomposing the Hilbert
cube into cellular sets and using a quotient map. By using a cellular decomposition of the
Hilbert cube, an example of a cellular map is given to show that the image of the Hilbert
cube under a cellular map can have complex non manifold part, not be a Hilbert cube
manifold, and still be a Hilbert cube manifold factor. The non degenerate decomposition
elements are shown to satisfy the cellularity criteria.

To measure how far the image is from being a Hilbert cube manifold, the idea of
covering codimension in finite dimensions is generalized by using a homological codimen-
sion approach. In finite dimensional settings, the two codimensions are equivalent. The
complexity of the non manifold part of the image space is measured in terms of intrinsic
codimension, which uses the homological codimension of the image of the union of nonde-
generate decomposition elements. The intrinsic codimension of the map in this example
is found to be exactly two.

Using a characterization of Hilbert cube manifolds, it is shown that the decomposi-
tion space is not the Hilbert cube, but is a factor of the Hilbert cube.
An Intrinsic Codimension Two Cellular Decomposition of the Hilbert Cube

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____________________________
Kailash C Ghimire, Author
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To the Loving Memory of My Parents
1. INTRODUCTION AND BACKGROUND

The main goal of this work is to construct a decomposition of the Hilbert cube with some interesting properties. By using this construction, we get a map which has nice preimages, but has a very complicated image. We also measure the complexity of the image by using homology on the image set.

In this chapter, we give some definitions, notation and background of the study.

**Definition 1.0.0.1.** A closed set \( C \) in \( \mathbb{R}^n \) or in an \( n \)-dimensional manifold is said to be cellular if there is a nested sequence \( \{C_1, C_2, \ldots\} \) of \( n \) cells with \( C_{i+1} \) a subset of the interior of \( C_i \) and \( C = \bigcap_{i \geq 1} C_i \). A map \( f : M \to X \) is said to be cellular if \( f^{-1}(x) \) is cellular for every \( x \) in \( X \).

Any \( n \) cell in an \( n \)-dimensional manifold is a trivial example of a cellular set. But the topologist sine curve on \( \mathbb{R}^2 \) given as

\[
C = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{0\} \times [-1, 1]
\]

is also a cellular set in \( \mathbb{R}^2 \). Figures 1.1 and 1.2 illustrate the cellularity of \( C \).

This set \( C \) is not path connected and not even locally path connected. But there is a continuous map \( f \) from \( \mathbb{R}^2 \) onto itself such that \( f^{-1}\{p\} = C \) for some \( p \) in \( \mathbb{R}^2 \). This map is in fact a limit of a sequence of homeomorphisms from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \).

**Definition 1.0.0.2.** A closed subset \( A \) of a manifold \( M \) is said to be cell-like if for every
Abstract. We prove that Bing-Whitehead Cantor sets are equivalently embedded in $\mathbb{R}^3$ if and only if their defining sequences differ by some finite number of Whitehead constructions.

1. Bing and Whitehead links

Let $T$ be a solid torus. Throughout this paper, we assume that the tori we are working with are unknotted in $\mathbb{R}^3$. A Bing link in $T$ is a union of 2 linked tori $F_1 \cup F_2$ embedded in $T$ as shown in Figure 1. A Whitehead link in $T$ is a torus $W$ embedded in $T$ as shown in the Figure. For background details and terminology, see Wright's paper [Wr89].

Let $M = T_1 \cup T_2 \cup \cdots \cup T_n$ be a finite union of pairwise disjoint unknotted and unlinked tori in $\mathbb{R}^3$. The standard Bing (resp. Whitehead) construction in $M$ is to embed a Bing link (resp. Whitehead link) in every component of $M$. Let $X = BW(n_1, n_2, \ldots)$ be a Bing-Whitehead compactum defined by standard Bing and Whitehead constructions as follows: Beginning with an unknotted torus $T \subset \mathbb{R}^3$, put $n_1$ Bing constructions, then $n_2$, and so on.

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open set $U$ of $M$ containing $A$, $A$ can be contracted to a point in $U$. A map $f : M \to X$ is said to be a cell-like (CE) map if $f^{-1}(x)$ is a cell-like set in $M$ for every $x$ in $X$.

Every cellular set is a cell-like set but the Whitehead continuum (Dav86) is cell-like but not cellular in $\mathbb{R}^3$. Figure 1.3 illustrates the first stage of the construction of the Whitehead continuum.

Let $\{h_1, h_2, \ldots\}$ be a convergent sequence of homeomorphisms from a manifold $M$ onto itself. Then the limit $h$ of this sequence is a cell-like map (Lac77). Hence, after homeomorphisms, cell-like maps are considered to be the next simplest. One way of constructing a cell-like or cellular map is by using decomposition theory. There are previous examples of decompositions of finite and infinite dimensional manifolds which give cellular maps with interesting properties. The cellularity of a subset of a manifold depends on how the subset is embedded in the manifold. For example, there exist embeddings of the interval $[0,1]$ in $\mathbb{R}^n, n \geq 3$ with non simply connected complement (WR88) and thus noncellular.

Let $B^2$ be the unit disc in $\mathbb{R}^2$, $I$ be the interval $[-1,1]$ and $S^1$ be the unit circle in $\mathbb{R}^2$. We write

$$I^n = \prod_{i=1}^{n} I_i$$

and

$$Q_n = \prod_{i \geq n} I_i,$$

where

$$I_i = [-1,1]$$

The Hilbert cube is a countable product of copies of $I$ and is denoted by $Q$, i.e

$$Q = \prod_{i \geq 1} I_i.$$
and

\[ Q = I^n \times Q_{n+1} \]

The distance between two points \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) in \( Q \) is defined as

\[ \rho(x, y) = \sum_{i \geq 1} \left| \frac{x_i - y_i}{2^i} \right| \]

This generates the product topology in \( Q \). Any manifold modeled on \( Q \) is called a \( Q \)-manifold. We take the \( sup \) metric \( \rho \) on \( I^n \). That is, if \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) are any two points in \( I^n \), then

\[ \rho(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\} \]

The product of \( n \) copies of \( I \) is called an \( n \)-cube and is denoted as \( I^n \). We write \( \pi_n : Q \to I_n \) for the projection of \( Q \) onto its \( n^{th} \) component and \( \tau_n : Q \to I^n \) for the map defined by

\[ \tau(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n) \]

from \( Q \) onto the \( n \)-cube \( I^n \).

The pseudo-interior of the Hilbert cube is denoted as

\[ s = \prod_{i \geq 1} I^0_i, \ I^0_i = (-1, 1) \]

and the pseudo-boundary of the Hilbert cube is \( Q - s \).

Let

\[ f, g : X \to Y, \]

be maps, where \( X \) is a compact space and \( Y \) is a metric space with metric \( \rho \). Then we define \( \rho(f, g) \) as

\[ \rho(f, g) = \sup \{ \rho(f(x), g(x)) \mid x \in X \}. \]

Throughout the thesis, both the word map refers to continuous function. A decomposition \( G \) of a space \( M \) is a partition of \( M \). That is, \( G \) is a subset of the power set of \( M \), and
its elements are pairwise disjoint nonempty subsets of $M$. For any decomposition $G$ of a space $M$, there is a decomposition space $M/G$ associated with the decomposition with the property that each point of the decomposition space is an element of the decomposition $G$. The topology of the decomposition space is induced by the quotient map $\pi : M \to M/G$ that sends $x$ in $M$ to the unique element of $G$ containing $x$. For any decomposition $G$ of a space $M$ we use $H_G$ to denote the set of non-degenerate elements of $G$. We use $N_G$ to denote the union of the elements of $H_G$. If $\pi$ is a quotient map for the decomposition $G$, then we write $N_G = N_\pi$.

We define a 2-disc with holes as a compact planar 2-manifold with boundary and denote such a disc as $D^2$. The boundary of a manifold $M$ is denoted by $\partial M$.

We assume familiarity with basic topological and algebraic topological concepts as presented in (Mun00) and (Mun84). Any terms not explicitly defined in this thesis can be found in (Mun00) and (Mun84). For additional information on other used concepts such as absolute neighborhood retracts (ANRs), characterization of Hilbert cube manifolds see (DW81b), (Zvo80), (Tor80), (LJ80), (Wal83), (Dav81), (CD81), (McM64) and (Koz81).

The motivation of this work is to see how complicated in terms of codimension cellular maps on the Hilbert cube $Q$ can be. We generalize the example from (DG82) to the Hilbert cube by decomposing the Hilbert cube into cellular sets and single points. For this, we use non-standard or wild Cantor sets in the Hilbert cube. The existence of wild Cantor sets in the Hilbert cube was first obtained by Wong in 1968 (Won68). In Chapter 2, we use a simple related technique to construct such wild Cantor sets in the Hilbert cube.

In Chapter 3, we use the construction of the wild Cantor sets and a generalized construction of (DG82) to get a defining sequence in the Hilbert cube. The defining sequence ultimately gives us a decomposition in the Hilbert cube such that all the decomposition elements are cellular in the Hilbert cube.
In Chapter 4, it is shown that the decomposition space $Q/G$ associated with the defining sequence in the Hilbert cube $Q$ is not a Hilbert cube manifold but $Q/G \times I \cong Q$. The decomposition space $Q/G$ does not satisfy the disjoint disc property but interestingly, the space $Q/G \times I$ satisfies the Disjoint $n$-Disc Property for every $n \geq 1$. In 1981, Daverman and Walsh characterized Hilbert cube manifolds by using the Čech homology argument (DW81b). We use some results from their paper in order to prove $Q/G \times I \cong Q$.

To measure the intrinsic codimension of the decomposition space, a theory of finite codimension in the Hilbert cube is developed in Chapter 5. It is proved that the two approaches, homological and covering dimensional, of codimension in the finite dimensional setting are equivalent. Some results analogous to finite dimensional results are also established in Chapter 5. These results are used to prove that the intrinsic codimension of the decomposition space we construct is exactly two.

1.1. Finite Dimensions and Infinite Dimensions

There are examples of cell-like, totally non-cellular maps [(CD81) and (DW81a)] on $\mathbb{R}^n, n \geq 3$ with non manifold images and with the non-manifold part of the image space being $n$-dimensional. These examples are constructed by using cell-like, totally non cellular decompositions $G$ of $\mathbb{R}^n, n \geq 3$, such that the decomposition space $\mathbb{R}^n/G$ is not a manifold but $\mathbb{R}^n/G \times I \cong \mathbb{R}^{n+1}$. Cellular maps are not as complicated as cell-like maps. Bing’s dog-bone space (Bin57) in 1957 gives an example of a cellular map with the property that the non-manifold part of the image set has zero dimension. Later, Daverman and Garity use cellular decompositions $G$ of $\mathbb{R}^n$ to get cellular maps on $\mathbb{R}^n$ with non-manifold images [ (DG83) and (DG82)]. The construction in (DG83) has the property that the set $\pi(N_\pi)$ is intrinsically $n - 1$ dimensional (to be defined later in section 5.2) while the construction in (DG82) has the property that the set $\pi(N_\pi)$ has intrinsic
dimension $n - 2$. Both of the examples also satisfy the property that $\mathbb{R}^n/G \times \mathbb{R} \cong \mathbb{R}^{n+1}$.

In infinite dimensional manifolds, particularly in Hilbert cube manifolds, there are examples of cellular maps using cellular decompositions of Hilbert cube manifolds such that the non manifold part of the decomposition space has infinite codimension (Cha76). In fact, this nonmanifold part is zero dimensional. In 1980, T. Lay generalized the examples from (CD81) and (DW81a) to Hilbert cube manifolds to get a cell-like totally non cellular decomposition of Hilbert cube manifolds with the decomposition having zero intrinsic codimension (Lay80).

In $\mathbb{R}^n$, $n \leq 2$, if $X$ is a space with the property that $X \times I \cong \mathbb{R}^{n+1}$, then $X$ is homeomorphic to $\mathbb{R}^n$ (Wil63). The previous examples of show some interesting factorizations of finite dimensional as well as infinite dimensional manifolds. The other interesting fact in the case of Hilbert cube is that the Hilbert cube can be factored in to a non manifold and a one dimensional manifold. The main result of this thesis is the construction of a cellular decomposition of the Hilbert cube $Q$ such that the decomposition space $Q/G$ is not a Hilbert cube manifold, $Q/G \times \cong Q$ and the decomposition has intrinsic codimension two.
2. CONSTRUCTING WILD CANTOR SETS IN THE HILBERT CUBE

2.1. Introduction

The main goal of this section is to construct certain non standard or wild Cantor sets in the Hilbert cube. Wong (Won68) in 1968 constructed wild Cantor sets in the Hilbert cube by generalizing the construction of wild Cantor sets in $E^n$ described by Blankenship (Bla51) in 1951. In this chapter, we will use a related simpler method to construct such a wild Cantor set in the Hilbert cube. The technique Wong used to detect the non-standardness of his wild Cantor set was to look at the complement of the Cantor set. Blankenship proved that this traditional method of detecting wildness is not applicable in the pseudo-interior of the Hilbert cube by proving every compact subset of pseudo-interior of the Hilbert cube has simply connected complement.

A Cantor set $C'$ in $\mathbb{R}^n$ or in the Hilbert cube $Q$ is said to be tame if there is a homeomorphism that takes $C'$ to the standard Cantor set:

$$C \subset I \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-1} \cong \mathbb{R}^n$$

or

$$C \subset I \times \{0\} \subset I \times Q_2 \cong Q$$

Otherwise, $C'$ is said to be non standard or wild. If the complement of the Cantor set is not simply connected then it is wild since the standard Cantor set has simply connected complement in $\mathbb{R}^n, n \geq 3$. However, R. Osborne and D. DeGryse in 1974 (DeG74) ([Kir58] in $\mathbb{R}^3$) constructed a wild Cantor set in $\mathbb{R}^n$ with simply connected complement, and there have been other such constructions since then. We will find a defining sequence in the Hilbert cube which eventually gives us Cantor sets with non-simply connected complement.
2.2. Geometrical Centrality

Let $N$ be an $n$-manifold with or without boundary and let $f : D^2 \to B^2 \times N$ with $f(\partial D^2) \subset \partial B^2 \times N$. The map $f$ is said to be interior inessential (I-inessential) if there is a map $g : D^2 \to \partial B^2 \times N$ such that $f = g$ on $\partial D^2$. Otherwise, $f$ is said to be I-essential. This can be used as a helpful tool to test the geometric centrality of a subset of a manifold.

A subset $A$ of $B^2 \times N$ is said to be geometrically central in $B^2 \times N$ if $f(D^2) \cap A \neq \phi$ for every I-essential map $f$.

A collection $\mathcal{C}$ of subsets of $B^2 \times N$ is said to be geometrically central in $B^2 \times N$ if the union of elements of $\mathcal{C}$ is geometrically central in $B^2 \times N$. Some related literature can be found in (DE87) and (DG82).

2.2.1 Standard Examples of Geometrically Central Subsets

Let $N$ be $n$-manifold, and identify $A$ as $\{0\} \times N$. Suppose

$$f : D^2 \to B^2 \times N$$

is an I-essential map with $f(D^2) \cap A = \phi$. Since $\partial B^2 \times N$ is a retract of $(B^2 \times N) - A$, then there is a map

$$g : D^2 \to \partial B^2 \times N$$

satisfying

$$g \mid \partial D^2 = f \mid \partial D^2.$$ 

It contradicts the fact that $f$ was I-essential. This gives us that $A$ is geometrically central in $B^2 \times N$.

Also let $f : B^2 \to B^2 \times N$ be given by $f(x) = (x, a)$ for some fixed $a$ in $N$. Then $f$ is an example of an I-essential map. Note that $f(\partial B^2) \subset \partial B^2 \times N$. If $f$ is I-inessential, then there is a map $g : B^2 \to \partial B^2 \times N$, with $f = g$ on $\partial B^2$. Then $\pi g : B^2 \to \partial B^2$ is a retraction of $B^2$ onto $\partial B^2$, which is impossible.
2.2.2 Preview of Our Plan for Constructing a Wild Cantor Set in $Q$

We start with a construction, say $A_3$, a set in $I^3$ which is geometrically central in $I^3$. With the help of this set, we will find a decreasing nested sequence $\{C_i\}$ of subsets of $Q$ such that each $C_i$ has a non-simply connected complement in $Q$. Finally, we will show that the intersection $C$ of the sets $C_i$ also has a non-simply connected complement. For this, we will consider a non-trivial loop $f(\partial B^2)$ in $I^3 \setminus A_3$. Then $f(\partial B^2) \times \{pt\}$ will be shown to be non-trivial in $Q \setminus C$.

2.2.3 Results on Geometric Centrality

In order to construct the Cantor set in the Hilbert cube, we need to use the following results related to geometric centrality.

**Lemma 2.2.3.1.** Given $B^2 \times I$ and given $\epsilon > 0$, there is a family

$$\{C_1, C_2, ..., C_k\},$$

with

$$C_1 \cong C_k \cong B^2 \times I$$

and for $i = 2, 3, ..., k - 1$,

$$C_i \cong B^2 \times S^1$$

such that the family is geometrically central in $B^2 \times I$ and for each $C_i$, the diameter of $C_i < \epsilon$. Similarly, given $B^2 \times S^1$ and given $\epsilon > 0$, there is a geometrically central family

$$\{C_1, C_2, ..., C_k\}$$

with $C_1 \cong B^2 \times S^1$ and the diameter of $C_i < \epsilon$.

Figure 2.1 and figure 2.2 illustrate Lemma 2.2.3.1 for $B^2 \times S^1$ and $B^2 \times I$.

The following technical lemma from Daverman and Edwards (DE87) is used to prove Lemma 2.2.3.1. For completeness, we include the proof of this technical lemma at the end of this chapter.
Lemma 2.2.3.2. (DE87) Let $D^2$ be a disc with holes and $f : D^2 \to B^2 \times I$ a map and let $P$ be a bicollared subset of $D^2 \times I$ and assume that $K = f^{-1}(P)$ is closed in $D^2$. If $F : K \times I \to P$ is a homotopy with $F_0 = f|_K$ and $U$ is a neighborhood of $F(K \times I)$ in $B^2 \times I$, then there is a neighborhood $V$ of $K$ in $D^2$ and a map $\tilde{f} : D^2 \to B^2 \times I$ such that:

1. $\tilde{f}|_{D^2-V} = f|_{D^2-V}$.
2. $\tilde{f}|_K = F_1$.
3. $\tilde{f}(V - K) \subset (U - P)$.

Proof of Lemma 2.2.3.1: The idea of the proof of this lemma is similar to one in (DE87). We will construct a chain of linked arcs and circles in $B^2 \times I$. Then we will show that this chain is geometrically central in $B^2 \times I$. Then the thickened three dimensional neighborhood of each component of the chain will give us the desired family. Let $\alpha$ be an arc with end points on one of the components of $B^2 \times \partial I$ and $\beta$ be another arc with the end points in another component of $B^2 \times \partial I$. Let us link those two arcs with a finite numbers of linked circles. Let the whole construction be denoted as $L$. Let $L_\alpha$ be the arc on $B^2 \times \partial I$ that joins the end points of $\alpha$ and $L_\beta$ be the arc on $B^2 \times \partial I$ for $\beta$.

To prove $L$ geometrically central in $B^2 \times I$, we use Lemma 2.2.3.2.

Let us denote the circles in the chain by $C_1, C_2, ..., C_n$ and let $C_0$ be $\alpha \cup L_\alpha$ and $C_{n+1}$ be $\beta \cup L_\beta$. Let us insert planar discs in every other circle of $L$ starting from $C_0$. If $n$ is odd, we also insert one additional disc bounded by $C_{n+1}$. Denote the union of $L$ and these discs by $L^*$ and the family of these discs by $\{D_k\}$. Now $L^*$ contains a copy of a core of $B^2 \times I$, hence from one of the standard examples in the previous section 2.2.1, it is geometrically central in $B^2 \times I$.

If possible, let $f : D^2 \to B^2 \times I$ be an I-essential map such that $f(D^2) \cap L = \phi$. We may assume $f(D^2)$ intersects $L^*$ in general position. Hence, for each disc $D_k$, $f^{-1}(D_k)$ is a collection of a finite number of simple closed curves. For a fixed $D_k$, if $f(D^2) \cap D_k \neq \phi$, then
FIGURE 2.1: Geometrically Central Collection on $B^2 \times I$

FIGURE 2.2: Geometrically Central Collection on $B^2 \times S^1$
there is an innermost component $J$ in $D^2$ of $f^{-1}(D_k)$ such that $D^2 - J$ has a component $E$ for which $J \cup E$ is a disc $B'$ with holes and $f^{-1}(D_k) \cap B' = J$. Let $C'_k = D_k \cap (\bigcup_{i=0}^{n+1} C_i)$. Also note that each $D_k$ is bicollared in $B^2 \times I$. $f(J)$ is null-homotopic on $D_k - (\partial D_k \cup C'_k)$. Otherwise, it will contradict the assumption that $f(D^2) \cap L = \phi$. Now let us extend the disc $D_k$ to another disc $D'_k$ such that $D_k \subset \text{int} D'_k$. Then we will be able to find a homotopy $F : J \times I \to D'_k - C'_k$ such that, $F_0 = f$ and $F_1(J \times I) \subset D'_k - D_k$.

Hence, by using the above Lemma 2.2.3.2, there is a map $f_2 : D^2 \to B^2 \times I$ such that $f = f_2$ on $\partial D^2$. Now we replace the map $f$ by $f_2$. After repeated use of the above process in every innermost disc, we will be able to get a map $g : D^2 \to B^2 \times I$ such that $g = f$ on $\partial D^2$ and $g(D^2) \cap L^* = \phi$. This contradicts that $L^*$ is geometrically central in $B^2 \times I$. Similar arguments follow for $B^2 \times S^1$. We may take the number of components of $L$ large enough to get the diameter of each component $C_i$ of $L$ less than $\epsilon$. Taking $n + 1 = k$ proves the theorem.

Lemma 2.2.3.3. If $N$ is a subset of an $n$-manifold $M \cong B^2 \times X$ which is geometrically central in $M$, then $N \times I$ is geometrically central in $M \times I \cong B^2 \times X \times I$, and similarly, if $N$ is geometrically central in $M \cong B^2 \times X$, then $N \times S^1$ is geometrically central in $M \times S^1 \cong B^2 \times X \times S^1$.

Proof. Let $D^2$ be a disc with holes. If possible, let $f : D^2 \to M \times I$ be an I-essential map such that $f(D^2) \cap N \times I = \phi$. Decompose $f$ into two factors $f_M$ and $f_I$ from $D^2$ to $M$ and $I$ respectively. We claim that $f_M$ is I-essential into $M$. If not, there is a map $g : D^2 \to \partial B^2 \times X$ such that $g = f_M$ on $\partial D^2$. We can then define a map $h : D^2 \to \partial(M) \times I$ by $h = (g, f_I)$, then $f = g$ on $\partial D^2$. That contradicts that $f$ was I-essential. Similar arguments follow to prove that $N \times S^1$ is geometrically central in $M \times S^1$.

To proceed, we need to understand what happens if we iterate the process of placing geometrically central sets in our construction. The following lemma shows how geometrical centrality is preserved.
Lemma 2.2.3.4. Let $A \cong B^2 \times X_1 \times X_2 \times \ldots \times X_n$, where each $X_i$ is $I$ or $S^1$. Let $\mathcal{C} = \{C_i : C_i \cong B^2 \times Y_{i1} \times \ldots \times Y_{in}\}$, where each $Y_{ij}$ is $I$ or $S^1$, be a finite collection of pairwise disjoint subsets of $A$ which is geometrically central in $A$. Also assume that for each $C_i$, there is a finite collection $\mathcal{D}_i = \{D_j : D_j = B^2 \times Z_{j1} \times \ldots \times Z_{jn}\}$, where each $Z_{jk}$ is $I$ or $S^1$, of disjoint subsets of $C_i$, which is geometrically central in $C_i$. Then the collection $\mathcal{D} = \bigcup \mathcal{D}_i$ is geometrically central in $A$.

Proof. Let $D^2$ be a disc with holes and $f : D^2 \to B^2 \times X_1 \times X_2 \times \ldots \times X_n$ be an I-essential map. After a slight adjustment of $f$, we may consider $K = f^{-1}\{C\}$ to be a 2-manifold in $D^2$ with a finite number of components. Hence $f^{-1}(\partial B^2 \times X_1 \times \ldots \times X_n)$ is a finite collection of simple closed curves in $D^2$. This implies that each component of $f^{-1}(C_i)$ is a disc with holes in $D^2$. Then $K$ must have a component $H$ such that $f|_H$ is I-essential into $C_i$ for some $i$.

For, if there is no $C_i$ for which $f|_H$ is I-essential for some component $H$ of $K$, let $H$ be given and let $f(H) \subset C_i$ for some fixed $i$. Also note that $f(\partial H) \subset \partial B^2 \times Y_{i1} \times \ldots \times Y_{in}$ for some $Y_{i1}, Y_{i2}, \ldots, Y_{in}$. Since $f|_H$ is not I-essential, there is a map $g : H \to \partial B^2 \times Y_{i1} \times \ldots \times Y_{in}$ and $g = f|_H$ on $\partial H$ and $C_i \cap C_j = \emptyset$ if $i \neq j$. Hence, we can push $g(H)$ off of $\partial B^2 \times X_{i1} \times \ldots \times X_{in}$ without intersecting any other $C_j$. Now doing the same process with other components of $K$ we can get a new map $h : D^2 \to \partial B^2 \times X_1 \times \ldots \times X_n$ that misses all $C_i$ and $f = h$ on $\partial D^2$. That contradicts the fact that $\mathcal{C}$ was geometrically central in $A$. Hence $f|_H$ must be I-essential on some $C_i$ for some component $H$ and some $i$. But $\mathcal{D}_i$ is geometrically central in $C_i$, hence $f|_H$ must intersect some elements of $\mathcal{D}_i$, that is $f(D^2)$ must intersect $\mathcal{D}$. Therefore, $\mathcal{D}$ is geometrically central in $A$. □

In the setting of this Lemma 2.2.3.4, we can write the following as a corollary.

Corollary 2.2.3.5. Let $f : D^2 \to A$ be an I-essential map, in general position with respect to $\mathcal{C}$. Then there is a disc $H$ with holes in $D^2$ such that $f|_H : H \to \mathcal{D}_i$ is I-essential for
Lemma 2.2.3.6. If \( f = (f_{B^2}, f_1, \ldots, f_{n-1}, f_n) : D^2 \rightarrow B^2 \times X_1 \times \ldots \times X_{n-1} \times X_n \) is \( I \)-essential, then \( \tilde{f} = (f_{B^2}, f_1, \ldots, f_{n-2}, f_n) : D^2 \rightarrow B^2 \times X_1 \times \ldots \times X_{n-2} \times X_n \times X_{n-1} \) is \( I \)-essential.

Proof. If \( \tilde{f} \) is not \( I \)-essential, then there is a map \( g = (g_{B^2}, g_1, \ldots, g_{n-2}, g_n, g_{n-1}) : D^2 \rightarrow \partial B^2 \times X_1 \times \ldots \times X_{n-2} \times X_n \times X_{n-1} \) such that \( \tilde{f} = g \) on \( \partial D^2 \). Then let us consider a map \( \tilde{g} = (g_{B^2}, g_1, \ldots, g_{n-2}, g_{n-1}, g_n) : D^2 \rightarrow \partial B^2 \times X_1 \times \ldots \times X_{n-1} \times X_n \). Then \( \tilde{g} = f \) on \( \partial D^2 \), which is a contradiction. \( \square \)

We now generalize Lemma 2.2.3.1 to include more factors.

Lemma 2.2.3.7. Given \( \epsilon > 0 \) and \( A = B^2 \times X_1 \times \ldots \times X_n \), where each \( X_i = I \) or \( S^1 \), then there is a finite collection \( C = \{C_1, C_2, \ldots, C_n\} \), where each \( C_i \cong B^2 \times Y_1 \times \ldots \times Y_n \) with each \( Y_i = I \) or \( S^1 \}, \), of disjoint subsets of \( A \) such that \( C \) is geometrically central in \( A \) and the diameter of \( C_i < \epsilon \) for every \( i \).

Proof. We prove this lemma by using induction over \( n \). We have already shown this for \( n = 1 \). Assume the lemma is true for \( n = k \). Consider \( A = B^2 \times X_1 \times \ldots \times X_k \times X_{k+1} \). Consider \( B = B^2 \times X_1 \times \ldots \times X_k \). Then by assumption, there is a finite collection \( D = \{D_i : D_i = B^2 \times X_{i1} \times \ldots \times X_{ik}\} \) of disjoint subsets of \( B \), which is geometrically central in \( B \). By Lemma 2.2.3.3, the collection \( G = \{G_i : G_i = D_i \times X_{k+1}\} \) is geometrically central in \( A \). Now switch the \( k^{th} \) and \((k+1)^{st}\) co-ordinate of \( G_i \) to get \( H_i \). Consider the first \( k \) components of \( H_i \). Then we will have a finite collection \( L_i \) such that \( L_i \times X_k \) is geometrically central in \( H_i \). Now switching back the \( k^{th} \) and \((k+1)^{st}\) coordinates of this collection and using Lemma 2.2.3.5 we will get another collection \( K_i \) which is geometrically central in \( G_i \). Consider \( C \) as the union of these collections \( K_i \). Then by above Lemma 2.2.3.5, \( C \) is geometrically central in \( A \). We may take enough components so that the diameter of each component is less than \( \epsilon \). \( \square \)
Lemma 2.2.3.8. \( A = B^2 \times X_3 \times \ldots \times X_n \) be geometrically central in \( B^2 \times I_2 \times I_4 \times \ldots \times I_n \cong I^n \), where \( B^2 \) is viewed as \( I_1 \times I_3 \) and is geometrically central in \( I^n \). Then \( A \times Q^{n+1} \) has non-simply connected complement in \( Q \). In other words, if \( A \) has non-simply connected complement in \( I^n \), then \( A \times Q_{n+1} \) has a non-simply connected complement in \( Q \).

**Proof.** Let \( f = (f_{\partial B^2}, f_3, \ldots, f_n) : \partial B^2 \to \partial B^2 \times I_3 \times I_4 \times \ldots \times I_n \) be a non-trivial loop. Now if we extend this map to a map \( F : \partial B^2 \to Q \) by \( F(x) = (f_{\partial B^2}(x), f_3(x), \ldots, f_n(x), 0, 0, \ldots) \), then \( F(\partial B^2) \) is a non-trivial loop in \( Q \setminus (A \times Q^{n+1}) \). If possible, let there be a map \( H : \partial B^2 \times I \to Q \setminus (A \times Q^{n+1}) \). such that \( H(x,0) = F \) and \( H(x,1) = c = (c_1, c_2, \ldots) \). Then \( \tau_n \circ H : \partial B^2 \to I^n \setminus A \) is a trivial loop that contradicts that \( A \) was geometrically central in \( I^n \).

Let \( A_n \) be a collection of sets homeomorphic to \( B^2 \times X_1 \times X_2 \times \ldots \times X_{n-2} \), where \( X_i = I \) or \( S^1 \), that is geometrically central in \( I^n \). Then \( A_n \) has a non-simply connected complement in \( I^n \). By the Lemma 2.2.3.8, \( A_n \times Q^{n+1} \) has non-simply connected complement in \( Q \). For the construction of Cantor sets in \( Q \), we will write a generalized version of the “Theorem on iteration of I-essentiality” by Daverman and Edwards (DE87)

**Theorem 2.2.3.9.** Let \( \{A_i\}, i \geq 3 \) be a sequence of collections as defined above that satisfies the following:

1. \( A_{i+1} \subset A_i \times I_{i+1} \)

2. \( A_i \) is geometrically central in \( I^i \)

Then, \( \bigcap_{i \geq 3} (A_i \times Q_{i+1}) \) has non-simply connected complement in \( Q \).

**Proof.** By above Lemma 2.2.3.8, \( A_i \times Q_{i+1} \) has a non-simply connected complement in \( Q \). Let \( f : \partial B^2 \to Q \) be a map such that \( f(\partial B^2) \) is a nontrivial loop in \( Q - A_n \times Q_{n+1} \) for some \( n \). We show that this is non-trivial in \( A_{n+1} \times Q_{n+2} \). Note that \( A_{n+1} \subset A_n \times I_{n+1} \subset I^{n+1} \).
with $A_{n+1}$ is geometrically central in $I^{n+1}$, and $A_n \times I^{n+1}$ is also geometrically central in $I^{n+1}$. Therefore, $A_{n+1}$ must be geometrically central in $A_n \times I_{n+1}$. If the loop $f : \partial B^2 \to I^n$ is non-trivial on $I^n - A_i$, then the loop $g : \partial B^2 \to I^{n+1} \cong I^n \times I_{n+1}$ given by $g(x) = (f(x), c)$ is non-trivial on $I^{n+1} - A_n \times I_{n+1}$ and hence non-trivial on $I^{n+1} - A_{n+1}$. That implies that if $g : \partial B^2 \to I^3$ is a non-trivial loop on $I^3 - A_3$, then $G : \partial B^2 \to Q$ given by $G(x) = (g(x), c_4, c_5, \ldots)$ is a non-trivial loop in $Q - A_i$ for every $i = 3, 4, \ldots$. This proves that the intersection has non-trivial complement.

2.2.4 Construction of the Cantor Set.

In our setting, a Cantor set is any set that is zero dimensional, totally disconnected, compact and perfect. The main result of this part is Theorem 2.2.4.2 that gives us a wild Cantor set in the Hilbert cube $Q$. With careful control of the size of the components of the construction in each step, we will get zero dimensionality of the Cantor set. We start controlling the size from the very beginning of the construction i.e. from $I^3$. The Lemma 2.2.3.1 ensures such a construction on $I^3$. The goal of this part is to construct the Cantor set that is not standardly embedded in the Hilbert cube. The above discussion and construction now lead us to the following major results:

**Theorem 2.2.4.1.** There is a decreasing sequence $\{C_n = A_n \times Q^{n+1}\}$ of subsets of $Q$ such that $C_n = B^2 \times X_2 \times \ldots \times X_n$ where each $X_i = I$ or $S^1$ has non-simply connected complement in $Q$ and $\text{dia}(C_n) < \frac{1}{n}$ for every $n$.

**Proof.** It suffices to show the diameter of $C_n < 1/n$. Note that the metric on the Hilbert cube is given as

$$\rho(x, y) = \sum_{i \geq 1} \left| \frac{x_i - y_i}{2^i} \right|$$

We can write
\[
\sum_{i \geq 1} \frac{|x_i - y_i|}{2^i} = \sum_{i=1}^{n} \frac{|x_i - y_i|}{2^i} + \sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i}
\]

Note that,

\[
\sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i} \leq \frac{1}{2^n}
\]

We can take the number of components large enough to make the diameter of each \( C_n < 1/n \). That gives us the total disconnectedness of the Cantor set.

Since the intersection is of closed and compact sets with the non-empty finite intersection property, the intersection itself is compact and the Theorem 2.2.4.1 assures the total disconnectedness of the intersection. The following theorem is the summarization of the construction.

**Theorem 2.2.4.2.** Let \( Q, A_n \) and \( C_n \) be as above. Then \( C = \cap_{i=1}^{\infty} C_i \) is a wild Cantor set in \( Q \) i.e. given \( B^2 \subset I_2 \times I_3 \). Then there is a wild Cantor set in \( I_1 \times B^2 \times Q_4 \) and there are loops \( \{pt\} \times \partial B^2 \times \{pt\} \) in the Hilbert cube such that any contraction of these loops hits the Cantor set.

### 2.3. Modifying the Construction by Ramifying

We now modify the construction of the wild Cantor set by ramifying the manifolds. On a finite dimensional manifold of the form \( B^2 \times N \), we take a finite number of sub-discs \( D_1, D_2, ..., D_n \) on \( B^2 \). Then \( D_i \times N \) is a ramified copy of \( B^2 \times N \). Note that we have a sequence \( \{C_i\}, i \geq 3 \) in the Hilbert cube that gives us a wild Cantor set, where \( C_i = A_i \times Q_{i+1} \) and \( A_{i+1} \) is geometrically central in \( A_i \times I_{i+1} \). Given a collection \( A'_n \) of ramified copies of \( A_n \), for each component \( G_i \) of \( A'_n \) we repeat ramification on \( G_i \times I_{n+1} \) to get a collection \( \{B^2 \times X_1^i \times X_2^i \times ... \times X_{n-1}^i\} \) that is geometrically central in \( A'_n \times I_{n+1} \). Now we replace each of the components of the collection by the collections of ramified copies and...
let the union of the these ramified copies be $A'_{n+1}$. Note that this collection is geometrically central into $A'_n \times I_{n+1}$. The following figure 2.3 illustrates a two time ramification of a solid torus. Also note that if $f : \partial D^2$ is a non-trivial loop in $I^{n+1} - A'_n \times I_{n+1}$, then every contraction of this loop hits all ramified copies of some components of $A'_{n+1}$. This and Theorem 2.2.4.2 give the following result:

**Theorem 2.3.0.3.** Let $Q$, $A'_n$, $C'_n$ be as above. Then $C' = \cap_{i \geq 3} C_i$ is a wild Cantor set in the Hilbert cube $Q$.

Looking at the construction, the following observations may be made:

1. Let $\{C_1, C_2, \ldots, C_k\}$ be the family of components corresponding to the $n^{th}$ stage of construction on $I^{n+2}$. Note that if we have a loop $l_p$ in $I^{n+2}$ such that for every contraction of the loop, there is a component that will be hit by the contraction, then the contraction hits every ramified copy of that component.

2. The same idea follows in the Hilbert cube as well.
3. On each step of the construction, if $C_n$ is a component corresponding to the $n^{th}$ stage of the construction, then there is a component $C_{n-1}$ on the $(n-1)^{st}$ stage of the construction such that $C_n$ is contractible in $C_{n-1}$.

The following definitions related to the components of the Cantor set construction will be used in next chapter.

**Definition 2.3.0.4.** Let $M_1$ and $M_2$ be any two components of the $k^{th}$ stage of Cantor set construction in $Q$ with corresponding components $C_1$ and $C_2$ in $I^{k+2}$. $C_1$ and $C_2$ are said to be adjacent components if $\tau_3(M_1)$ and $\tau_3(M_2)$ are linked components in the copy of $I^3$ used in the construction corresponding to figure 3.1 or 3.2.

2.4. **Proof of Lemma 2.2.3.2**

We end this chapter by including the proof of Lemma 2.2.3.2. The proof is taken from (DE87).

**Proof.** Let $P \times [-1,1]$ be a bicollar on $B^2 \times I$. Let $W$ be a neighborhood of $F(K \times I)$ in $P$. Choose $\delta > 0$ such that $W \times [-\delta, \delta] \subset U$. By using the Borsuk Homotopy Extension Theorem, we can get a neighborhood $V$ of $K$ in $D^2$ such that $F$ extends to a map $G : V \times I \to W \times [-\delta, \delta]$ with $G_0 = f \mid_V$. Also by using Urysohn’s Lemma, there exists a map $g : D^2 \to [0,1]$ with $g^{-1}(0) = D^2 - V$ and $g^{-1}(1) = K$. Let $\pi_1$ and $\pi_2$ be the projection maps of $W \times [-\delta, \delta]$ on $W$ and $[-\delta, \delta]$ respectively.

Now we use the above maps to get a new map $\tilde{f}$ as

1. $\tilde{f}(x) = f(x), x \in D^2 - V$

2. $\tilde{f}(x) = \pi_1(G(x, g(x)), \pi_2(f(x)))$ elsewhere

Then this map satisfies all the conditions. For, by the construction, $\tilde{f} \mid_{D^2 - V} = f \mid_{D^2 - V}$. 
Next let \( x \in K \). Then, \( \tilde{f}(x) = (\pi_1(G(x, 1), 0) = (G_1(x), 0) \). By identifying \( P \) as \( P \times \{0\} \), we will have \( \tilde{f}(x) = G_1(x) \) for \( x \in K \). The third condition is straightforward. \( \square \)
3. CONSTRUCTION OF THE CELLULAR DECOMPOSITION

3.1. Introduction

For any manifold $M$, after homeomorphisms from $M$ onto itself, cellular maps on $M$ are considered to be the next simplest maps. Daverman and Garity in 1982 constructed a cellular map by using a certain cellular decomposition on $E^n$ for $n \geq 3$ (DG82) with the image nearly as complex as possible. They also constructed a most possible complicated cellular map (DG83) in 1983. The main objective of this section is to generalize this idea in order to get a certain type of cellular decomposition of the Hilbert cube that gives us a cellular map with the property that the image set is not a $Q$ manifold. We will use Daverman and Garity’s first idea along with the construction of wild Cantor sets in the Hilbert cube from Chapter 2 to get a cellular decomposition of the Hilbert cube. We will also develop methods to measure the complexity of this decomposition in Chapter 5.

3.1.1 Defining Sequence

We need the following definitions related to defining sequences for decompositions before proceeding. Finite dimensional versions of these definitions can be found in (DG82). Some of these definitions can be found in (Lay80).

**Definition 3.1.1.1.** Let $\mathcal{M} = \{\mathcal{M}_i\}$ be a sequence of collections of subsets of the Hilbert cube satisfying the following conditions:

1. For each $i, \mathcal{M}_i$ is a finite collection of compact subsets of $Q$ with disjoint interiors.

2. For every element $A$ of $\mathcal{M}_i$ and for every $j < i$ there is a unique element of $\mathcal{M}_j$ that contains $A$.

3. If $A$ is an element of $\mathcal{M}_i$ and $x, y$ is a pair of elements in $\partial A$ then there is a $j > i$ such that no element of $\{\mathcal{M}_j\}$ has both of $x$ and $y$. 

Then the sequence $\mathcal{M} = \{\mathcal{M}_i\}$ is called a defining sequence in $Q$.

**Definition 3.1.1.2.** Let $X$ be a topological space and $\mathcal{M}$ be a collection of subsets of $X$. For an arbitrary set $A$ in $X$, the star of $A$ in $\mathcal{M}$ is defined as

$$st(A, \mathcal{M}) = A \cup (\cup\{M \in \mathcal{M} : M \cap A \neq \emptyset\})$$

and for any integer $n \geq 1$, we define the $n^{th}$ star in $\mathcal{M}$ as

$$st^n(A, \mathcal{M}) = st(st^{n-1}(A, \mathcal{M}), \mathcal{M})$$

**Definition 3.1.1.3.** Let $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, ...\}$ be a defining sequence in a topological space $X$. Then the decomposition $G$ associated with the defining sequence is the relation prescribed by the rule, for any $x \in X$

$$G(x) = \cap_{i \geq 1} st^2(x, \mathcal{M}_i)$$

The following two definitions are from (Dav86).

**Definition 3.1.1.4.** A decomposition $G$ of a separable metric space $X$ is said to be upper semicontinuous (usc) if every $g \in G$ is compact and the quotient map

$$\pi : X \to X/G$$

is closed.

**Definition 3.1.1.5.** Let $X$ be a compact separable metric space, $G$ be a usc decomposition of $X$, and $\rho_G$ be a metric on $X/G$. Then the decomposition $G$ is said to be shrinkable if for every $\epsilon > 0$ there is a homeomorphism $h$ of $X$ onto itself satisfying:

1. $\rho_G(\pi h(x), \pi(x)) < \epsilon$ for every $x$ in $X$, and

2. $\text{diam} h(g) < \epsilon$ for every $g$ in $G$.

The following theorem is from (Lay80) page 25.
Theorem 3.1.1.6. (Lay80) The decomposition $G$ associated with a defining sequence $\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is upper semicontinuous.

If $\bigcup \{ \bigcup \partial(M_i) : M_i \in \mathcal{M}_i \}$ is mapped homeomorphically by the quotient map, then by (Lay80) each of the decomposition elements $g$ can be expressed as $g = \cap_{i \geq 1} (st(x, \mathcal{M}_i))$

Definition 3.1.1.7. Let $G$ be a usc cellular decomposition of the Hilbert Cube $Q$, and consider two maps $f_1, f_2 : B^2 \rightarrow Q$.

For a fixed $t$ in $I_2$, the maps $f_1$ and $f_2$ are said to be $t$-slice if every $x \in Q$ of the form $x = (0, t, x_3, x_4, \ldots)$ satisfies

$$\pi(x) \cap \pi(f_1(B_2)) \cap \pi(f_2(B_2)) \neq \emptyset,$$

where

$$\pi : Q \rightarrow Q/G$$

is the quotient map.

3.2. Construction

We write $I^k = I_1 \times I_2 \times \ldots \times I_k$ and $B^2 = I_1 \times I_3$. Similarly, we write $B^{n-1}$ as an embedded copy of $I_2 \times I_3 \times \ldots \times I_n$ in $I^n$ and $B^{n-2}$ as an embedded copy of $I_3 \times I_4 \times \ldots \times I_n$ on $I^n$. An $n$-tube is $B^{n-1} \times [0, 1]$.

We consider geometrically central collections for a set $B^2 \times X_1 \times X_2 \times \ldots \times X_n$, where each $X_i$ is of type $I$ or $S^1$, to be a finite collection of sets of type $B^2 \times Y_1 \times Y_2 \times \ldots \times Y_n$, where each set is of type $S^1$ or $I$. We borrow the term Parallel Interior Manifold from Daverman and Garity (DG82). Each ramified copy of a manifold is called a parallel interior manifold. We will use the following lemma about parallel interior manifolds in
many places. The proof follows from the standard example of a geometrically central set in Chapter 2.

**Lemma 3.2.0.8.** Let $M = B^2 \times N$ be an $m$-manifold. Let $D_1, D_2, \ldots, D_n$ be disjoint sub discs in $B^2$. Then each parallel interior manifold $D_i \times N$ of $M$ is geometrically central in $M = B^2 \times N$.

### 3.2.1 Zero Stage of Construction.

We start the construction viewing $Q = I^3 \times Q_4$ to get the starting element $\mathcal{M}_0$ of the defining sequence. After a suitable parametrization, we may consider $I_1 = [-3,3]$. Let us take two two-dimensional discs $D_1$ and $D_2$ with radius $r$ such that both of them are without holes and $D_1 \subset [-3,-2] \times I_3$ and $D_2 \subset [2,3] \times I_3$, as shown in the figure 3.1. Now, consider $D_1 \times I_2$ and $D_2 \times I_2$, $[-1/2,1/2] \times I_2 \times I_3$. Let us join the 3-cube $[-1/2,1/2] \times I_2 \times I_3$ to $D_1 \times I_3$ with a 3-tube and to $D_2 \times I_3$ with another 3-tube. Let the union of $D_1 \times I_3, D_2 \times I_2, [-1/2,1/2] \times I_2 \times I_3$ and the 3-tubes be $C_0$. Then the starting element of the defining sequence will be $C_0 \times Q_4$ and is denoted by $\mathcal{M}_0$. The figure 3.1 on next page illustrates this stage of construction in $I^3$. Let $D_1'$ and $D_2'$ be slightly larger discs in $I_1 \times I_3$ containing $D_1$ and $D_2$.

Let $l_1$ and $l_2$ and a fixed $\delta$ be such that $l_1 = \partial(D_1' \times \{pt\}), l_2 = \partial(D_2' \times \{pt\})$ and $\rho(l_1, \partial D_1 \times I_2) > \delta$, $\rho(l_2, \partial D_2 \times I_2) > \delta$ as shown in the figure. We may think of these two loops in the Hilbert cube $Q$ as $l_1 \times \{0\}$ and $l_2 \times \{0\}$ in $I^3 \times Q_4$ such that for every contraction $f_1$ and $f_2$ of $l_1$ and $l_2$ respectively, in general position with respect to the boundary component of $\mathcal{M}_0$, there exist discs with holes $D_1^1$ and $D_2^2$ such that $f_1 \mid_{D_1^1}$ is $I$-essential in $D_1 \times I_2$ and $f_2 \mid_{D_2^2}$ is $I$-essential in $D_2 \times I_2$. Also, for every map $g_1$ and $g_2$ from $B^2$ to $Q$ in general position with respect to $\mathcal{M}_0$ and $\rho(l_i, g_i) < \delta/2$, for $i = 1,2$, there are discs with holes $H_i$ such that $g_i \mid_{H_i}$ is $I$-essential in $D_i \times I_2$. 

3.2.2 First stage of Construction:

Now, we construct our next element of the defining sequence after $M_0$. Note that, by using lemma 2.3.1.1, for any $\epsilon > 0$, we can find a geometrically central family $W_1 = \{W_1, W_2, ..., W_{n_1}\}, n_1 > 4$ and $n_1$ odd, on $D_1 \times I_3$ such that each element is either of the form $B^2 \times I$ or of the form $B^2 \times S^1$ and the diameter of each $W_i$ is less than $\epsilon$.

By reflecting this construction about $\{(x_1, x_2, x_3) : x_1 = 0\}$, we get a similar family $X_1 = \{X_1, X_2, ..., X_{n_1}\}$ on $D_2 \times I^3$. There are $n_1^2$ ways of choosing one component of $W_1$ and one component of $X_1$. Divide the interval $I_2$ into $n_1^2$ equal subintervals and group them into $n_1$ groups such that the $j^{th}$ group is the collection of all sub-intervals $E_i^2, (j - 1)n_1 < i \leq jn_1$. Hence, we can index these subintervals as $E^2(i, j)$ for the $j^{th}$ subinterval from the $i^{th}$ group, where $1 \leq i \leq n_1$ and $1 \leq j \leq n_1$. We now divide $I_3$ into 2 equal subintervals and denote them as $E_i^3, i = 1, 2$, so that we will have $2n_1^2$ small rectangles. Let us denote
these rectangles by \( R(i, j, k) = E^2(i, j) \times E^3_k, 1 \leq i \leq n_1, 1 \leq j \leq n_1, k = 1, 2 \).

Consider the cube \([-1/4, 1/4] \times I_2 \times I_3\). Then this cube has \( 2n_1^2 \) small sub-cubes. Let us ramify each of the components of \( W_1 \) and \( X_1 \) \( 2n_1 \) times. Now we group the ramified copies of each component in to \( n_1 \) groups. Let us denote the \( k^{th} \) element from the \( j^{th} \) group of the \( i^{th} \) component by \( W_1(i, j, k) \) for \( 1 \leq i, j \leq n_1 \) and \( k = 1, 2 \) and similarly denote the ramified copies of \( X_1 \) by \( X_1(i, j, k) \).

Note that for each of \( W_1 \) and \( X_1 \) we have a total of \( 2n_1^2 \) ramified components and we have exactly \( 2n_1^2 \) small cubes on \([-1/4, 1/4] \times I_2 \times I_3\). Now we join each of the small cubes with these ramified copies in the following manner. For each odd \( i \) we join \( W(i, j, k) \) to the rectangle \( R(\frac{i+1}{2}, j, k) \) and \( X(i, j, k) \) to the rectangle \( R(j, \frac{i+1}{2}, k) \) and for each even \( i \), we join \( W(i, j, k) \) to the rectangle \( R(\frac{n_1+i+1}{2}, j, k) \) and \( X(i, j, k) \) to the rectangle \( R(j, \frac{n_1+i+1}{2}, k) \) with disjoint 3-tubes. This assures that none of the ramified copies of adjacent components of \( W_1 \), or those of \( X_1 \), are connected to rectangles corresponding to adjacent subintervals of \( I_2 \). Each component of this stage of construction is the union of a ramified copy of a component of \( W_1 \), a ramified copy of a component of \( X_1 \), the small cube corresponding to the rectangle that is joined to these ramified copies, and the tubes that join these two components. Let the collection of these components be \( C_1 \). The first element of the defining sequence will be \( \mathcal{M}_1 = C_1 \times Q_4 \). Figure 3.2 shows a representation of the construction with \( n_1 = 3 \). We may observe the following facts on these first two constructions:

1. For every contraction \( f_1 \) and \( f_2 \) of any loops with in \( \delta \) of the respective loops \( l_1 \) and \( l_2 \), there is a component \( W_1 \) on \( W_1 \) and one component \( X_1 \) on \( X_1 \) and discs \( D'_i \) with holes such that \( f_1 \mid D'_i \) is I-essential on \( i^{th} \) ramified copy of \( W_1 \). Similarly, there are discs \( D'_j \) with holes such that \( f_2 \mid D'_j \) is I-essential on \( j^{th} \) ramified copy of \( X_1 \).

2. For every \( x \) in \( Q \), there is a three cell \( B^3 \) in \( I^3 \) such that \( st(x, \mathcal{M}_1) \subset B^3 \times Q_4 \) and \( B^3 \times Q_4 \subset \mathcal{M}_0 \).
FIGURE 3.2: Representation of the first stage of construction with three components

3. For each element $M$ of $\mathcal{M}_i$, $i = 0, 1$, we can get a $\frac{1}{i+1}$-map from $M$ to a 1-complex.

4. For each contraction $f_1$ and $f_2$ of any loops within $\delta$ of $l_1$ and $l_2$, there is a subinterval $E_k^2$ of $I_2$ such that for every $E_1^3$ and $E_2^3$ of $I_3$, there are parallel components $W_1$ and $W_2$ of $\mathcal{W}_1$ and two parallel components $X_1$ and $X_2$ of $\mathcal{X}_1$ with the property that: $W_1$ and $X_1$ are connected to $[-1/4, 1/4] \times E_k^2 \times E_1^3$, $W_2$ and $X_2$ are connected to $[-1/4, 1/4] \times E_k^2 \times E_2^3$, and $f_1$ and $f_2$ are I-essential as described in the first observation.

5. For any $x$ in $\partial M_0$ with $x$ not in $[-\frac{1}{2}, \frac{1}{2}] \times Q_2$, then $x$ is not in $M_j$ for every $M_j$ in $\mathcal{M}_1$.

For the higher dimensional construction, we assume the following hypotheses to be true for the construction in $I^k \times Q_{k+1}, k \geq 3$ as in (DG82). Denote the construction on $D_1 \times I_2 \times I_4 \times \ldots \times I_k$ by $\mathcal{W}_{k-2}$ and the reflection of the construction by $\mathcal{X}_{k-2}$

**IH1:** None of the copies of adjacent components, as defined in Chapter 2, of $\mathcal{W}_{k-2}$ or $\mathcal{X}_{k-2}$
is connected to \(n-1\) cubes corresponding to adjacent subintervals of \(I_2\). Two subintervals are said to be adjacent if they share at least one boundary component.

**IH2:** For the construction in \(I^k, k > 2\), we divide each of \(I_i\) onto \(2^k-i+1\) equal subintervals where \(i = 3, 4, ..., k\). Hence, \(B^{k-2}_i\) has \(2^{(k-2)(k-3)/2}\) small \(k-2\)-cubes. Let us denote these cubes by \(B^{k-2}_i, i = 1, 2, ..., 2^{(k-2)(k-3)/2}\).

**IH3:** (We refer to this as a special hypothesis) For any contraction \(f_1\) and \(f_2\) of any loops within \(\delta\) of loops \(l_1\) and \(l_2\) into \(I^k\), there is a subinterval \(E_2^p\) on \(I^k\) and a component, say \(W\), of \(\mathcal{M}_{k-2}\) such that every \(B^{k-2}_i \times E_2^p\) is connected to a ramified copy of \(W\) and \(f_1 | D\) is I-essential into this copy with respect to some disc \(D\) with holes and \(B^{k-2}_i \times E_2^p\) is connected to a ramified copy of \(X\) and \(f_2 | D'\) is I-essential into this copy for some \(D'\).

**IH4:** For each element \(M\) of \(\mathcal{M}_k\), there is a \(\frac{1}{k+1}\)-map from \(M\) to a 1-complex.

**IH5:** The diameter of each \(B^{k-2}_i \times E_2^p \times [-1/2^{i+1}, 1/2^{i+1}] \times Q_{k+1}\) is \(\frac{1}{2^{i+1}}\).

**IH6:** For every \(x\) in \(Q\), there is an \(k-1\) cell \(B^{k-1}\) such that

\[
st(x, \mathcal{M}_{k-2}) \subset B^{k-1} \times Q_k
\]

and \(B^{k-1} \times Q_k \subset M_j\) for some \(M_j\) in \(\mathcal{M}_{k-3}\) and \(B^{k-1} \cap \partial I^{k-1}\) has a pairwise disjoint finite number of \(k-2\) cells.

**IH7:** If \(x\) is a point in \(Q\) in \(\partial M_j\) for some \(M_j\) in \(\mathcal{M}_{k-3}\) and not in \([-1/2^{i+1}, 1/2^{i+1}] \times Q_2\), then \(x\) is not in \(M_i\) for every \(M_i\) in \(\mathcal{M}_{k-2}\).

**3.2.3 \(i^{th}\) Stage of Construction:**

This time, we consider the Hilbert Cube \(Q\) as \(I^i \times Q_{i+1}\). Assume that we have the construction \(C_k\) on \(I^k, k < i\) with the construction \(\mathcal{M}_{k-2}\) in \(Q\) that satisfies all of the above hypotheses. Let us consider the construction in \(I^{i-1}\) with the corresponding element of the defining sequence \(\mathcal{M}_{i-3}\) in \(I^{i-1} \times Q_i\). Let us take any subinterval \(E_2^N\) of \(I_2\) the corresponding to this construction. Let there be \(m\) ramified copies of a component of \(\mathcal{W}_{i-3}\)
and \(m\) copies of a component of \(X_{i-3}\) that satisfies the special hypothesis \(\text{IH3}\) with respect to this subinterval. Denote these components by \((W_1, W_2, ..., W_m)\) and \((X_1, X_2, ..., X_m)\) respectively. Note that for this subinterval \(E_N^2\), there are exactly \(m\) small \(i-1\) cubes. Divide each factor of each of these small cubes into two sub-intervals and \(I_i\) into two sub-intervals. We have \(2^{i-2}m = p\) small \(i-1\) -cubes and denote them as \((R_1, R_2, ..., R_p)\). Note that this satisfies the inductive hypothesis \(\text{IH2}\).

Let us ramify each of \(W_i\) for \(2^{i-2}\) times and denote them as \((W_1, W_2, ..., W_p)\) for \((W_{11}, ..., W_{12^{i-2}}, W_{21}, ..., W_{22^{i-2}}, ..., W_{m2^{i-2}})\). Similarly we denote \((X_1, X_2, ..., X_p)\) for the ramified copies of \((X_1, X_2, ..., X_m)\).

From the construction of the wild Cantor sets in Chapter 2, for each \(W_j \times I_i\), there is a geometrically central collection of \(n\) elements. Let us denote them as \(W(j, k)\) for the \(k^{th}\) component of the \(j^{th}\) copy. Without loss of generality, we may assume \(n > 2^i\) and \(n\) odd. We divide the subinterval \(E_N^2\) into \(n^{2p}\) subintervals and group them into \(n^p\) groups so that each of the groups has \(n^p\) elements and denote each of these subinterval as \(E(j, k)\). Ramify each of the components \(n^{2p-1}\) times. Note that we have \(n\) components corresponding to each copy of the \(p\) - ramified copies from the previous construction. Hence, there are \(n^{2p}\) ways of selecting one copy of each component from each of the copies. Note that we have a total of \(pm^{2p}\) small \(i-1\) cubes. Now let us denote each of these cubes as \(R(i, j, k)\) where \(1 \leq i, j \leq n^p, 1 \leq k \leq p\) and \(i, j\) correspond the \(i^{th}\) element of the \(j^{th}\) group of the subintervals of \(E_N^2\). For any choice \(W(1, j_1), W(2, j_2), ..., W(p, j_p)\) and \(X(1, k_1), X(2, k_2), ..., X(p, k_p)\), let us fix a subinterval \(E(j, k)\) and join each of the small cubes \(E(j, k) \times R_i\) to a ramified copy of \(W(l, j_m)\) and a ramified copy of \(X(l, j_k)\) with \(i\)-tube as before, satisfying the inductive hypotheses \(\text{IH1}\) and \(\text{IH5}\) as well, i.e. none of the adjacent components on the Cantor set construction are connected to the \(i-1\) cubes corresponding to adjacent subintervals of \(E_N^2\).

Note that we have \(n^{2p}\) such choices of selecting one component of \(W_i\) and \(X_i\) and
we have exactly $n^{2p}$ subintervals. Therefore, we can choose one subinterval to each of the choices of $W$ and $X$. Also for each subinterval and for a fixed $W(i, j)$ there are $n^{2p-1}$ choices of selecting other components, and we have exactly $n^{2p-1}$ ramified copies of this component. This gives us the exact number of ramified copies and a division of the subinterval of $E_N^2$ to satisfy the above inductive hypothesis IH3.

Taking the union of all of the components of ramified copies of $W(j, k)$, $X(l, m)$, the disjoint $i$-tubes, and small $i-1$-cubes $\times[\frac{-1}{2^i}, \frac{1}{2^i}]$ gives us the collection $C_{i-2}$. The element of the defining sequence $\mathcal{M}$ that we get from this stage of construction is $C_{i-2}\times Q_{i+1} = \mathcal{M}_{i-2}$. Note that the remaining hypotheses are satisfied by this construction. This completes the $i^{th}$ construction and then we proceed with further construction satisfying each of the inductive hypotheses in every stage.

3.2.4 Observations and Conclusion

Theorem 3.2.4.1. Let $\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ be a defining sequence that satisfies all the conditions IH1 through IH7. Then the decomposition associated with this defining sequence satisfies the following properties:

1. If $g$ is a non-degenerate decomposition element on $G$, and $U$ is any open set in $Q$ containing $g$, then there is a $n$-cell $B^n$ such that $g \subset B^n \times Q_{n+1} \subset U$

2. Each non-degenerate element of decomposition has dimension one.

3. For any contractions $f_1$ and $f_2$ of any loops within $\delta$ of $l_1$ and $l_2$ respectively, there is a $t$ in $I^2$ such that $f_1$ and $f_2$ are $t$-Slice maps.

4. Let $M$ be any element of $\mathcal{M}_k$ for some $k$ and $\mathcal{M}_k$ be an element of the defining sequence $\mathcal{M}$. Then the quotient map $\pi$ is one to one on the boundary of $M$.

5. The set $\{0\} \times Q_2$ is mapped homeomorphically by the quotient map $\pi$. 
Proof. 1. From the inductive hypothesis one {\bf IH1} if \( g \) is a single arc, then it follows obviously. Otherwise, choose a stage of the construction, say \( M_k \) such that there is an element \( S_m \) from the construction in \( I^k \) such that \( g \subset S^m \times Q^{k+1} \subset U \). By the construction, we can get a finite number of sets of the form \( B^n \times Q_{n+1} \), for some \( n \) satisfying the following properties:

a. If \( B^n_1 \times Q_{n+1}, ..., B^n_p \times Q_{n+1} \) are the sets, then \( \partial B^n_i \cap \partial B^n_j \cong B^{n-1} \) whenever \( B^n_i \cap B^n_j \neq \phi \)

b. For every \( B^n_i \), there is a \( B^n_j \) such that \( B^n_i \cap B^n_j \neq \phi \) so that \( \cup_{i=1}^{p} B^n_i = B^n \)

c. \( g \subset \cup_{i=1}^{p} B^n_i \times Q_{n+1} \subset S^m \times Q_{k+1} \subset U \).

2. The connectedness of each non-degenerate element \( g \) of the decomposition tells us that the dimension of \( g \) is \( \geq 1 \) and by one of the results in dimension theory (HW48), the dimension of \( g \leq 1 \), since from {\bf IH4} above, for every \( M \) of \( M_k \), there is a \( \frac{1}{k} \) map from \( M \) to a 1-complex. Hence, each non-degenerate element of decomposition has dimension one.

3. Property 3 follows from the inductive hypothesis {\bf IH3}.

4. This follows from the inductive hypothesis {\bf IH6}

5. The inductive hypotheses {\bf IH5} and {\bf IH7} imply the property 5.

\qed
4. DETECTING THE HILBERT CUBE AND CELLULARITY

4.1. Introduction

The construction of the decomposition space $Q/G$ of the Hilbert cube associated with the defining sequence $\mathcal{M}$ was done in the previous chapter. In this chapter we investigate further properties of the decomposition space $Q/G$. There are examples of cell-like totally noncellular decomposition of the Hilbert cube (Lay80). Here we will also try to find out some similarities and some dissimilarities between these examples and our example. For this, some results related to characterizations of Hilbert cube manifolds and of absolute neighborhood retract (ANR) theory are used. It is also shown that each non-degenerate decomposition element associated with the defining sequence $\mathcal{M}$ is cellular. The main objective of this chapter is to prove the following three properties of the decomposition space $Q/G$:

1. $Q/G$ is an ANR

2. $Q/G$ does not satisfy the Disjoint discs Property

3. $Q/G \times I \cong Q$

For the details of previous results in the literature that are used, see (DW81b),(Zvo80), (Tor80)(LJ80),(Wal83)(Dav81),(CD81),(McM64) and (Koz81).

4.1.1 Cellularity

One of the main dissimilarities between our example and the examples in (Lay80) is the cellularity of the decomposition element. In (Lay80), none of the non-degenerate elements are cellular. In our construction, every decomposition element is cellular. The cellularity of a subset of a manifold depends on how the subset is embedded in the manifold. Cellular sets have the property of being point-like (Zvo80). Here, we will prove that every
non-degenerate element of the decomposition $G$ is cellular. Although the definitions used here are given in other places also, readers are encouraged to see (Cha76) and (Zvo80) for further references.

**Definition 4.1.1.1.** (Cha76) A closed set $A$ in a space $X$ is said to be a $Z$-set in $X$ if for every open cover $\mathcal{U}$ of $X$, there is a map from $X$ to $X - A$ which is $\mathcal{U}$-close to the identity.

The following lemma from (Cha76) gives a condition for a subset of the pseudo interior of $Q$ to be a $Z$-set.

**Lemma 4.1.1.2.** (Cha76) Any compact subset $A \subset Q$ with $A \subset s$, where $s$ is the pseudo interior of $Q$, is a $Z$-set.

Recall the definition of cell-like from the introduction. Cellular sets in $n$ dimensional manifolds are nested intersections of $n$ cells. In the Hilbert cube, we replace $n$ cells with normal cubes.

**Definition 4.1.1.3.** (Zvo80) A closed subset $A$ of $Q$ is said to be cellular if $A = \cap_{i > 0} K_i$, where $A \subset int K_i, K_i \cong Q$, $Bd K_i \cong Q$ and $Bd K_i$ is a $Z$-set in $K_i$. For each $i$, the $K_i$ are called normal cubes.

**Definition 4.1.1.4.** (Cha76) Let $X$ be a compact space and $Y$ be an ANR containing $X$. Then $X$ is said to have trivial shape if $X$ is contractible in $U$ for every neighborhood $U$ of $X$ in $Y$.

**Definition 4.1.1.5.** A metric space $Y$ is said to be an absolute neighborhood retract (ANR) if for every map $f : A \to Y$ and for every closed $A$ of a metric space $X$ there is a neighborhood $U$ of $A$ such that $f$ is continuously extendable over $U$.

**Definition 4.1.1.6.** Let $X$ be a metric space. Let $f$ and $g$ be two maps from a 2-disc $B^2$ to $X$, and let $\epsilon > 0$ be given. Then the space $X$ is said to satisfy the disjoint discs property (DDP) if there exist maps $f'$ and $g'$ satisfying

$$\rho(f, f') < \epsilon, \rho(g, g') < \epsilon$$
and

\[ f'(B^2) \cap g'(B^2) = \phi \]

The following lemma follows from combining all the definitions.

**Lemma 4.1.1.7.** Let \( X \) be a closed subset of \( Q \). \( X \) has trivial shape if for every open neighborhood \( U \) of \( X \), there is an \( n > 1 \) and \( B^n \subseteq I^n \) such that \( B^n \times Q_{n+1} \) contains \( X \) and is contained in \( U \).

**Proof.** Since \( B^n \times Q_{n+1} \) is contractible in \( U \) and \( U \) is an arbitrary open set, \( X \) is also contractible in \( U \).

**Definition 4.1.1.8.** Let \( X \) be a subset of the Hilbert cube \( Q \). \( Q - X \) is said to be \( \{S^1\}\)-trivial at \( \infty \) if for every open neighborhood \( U \) of \( X \), there is an open neighborhood \( V \) of \( X \) such that, every map \( f : S^1 \to V - X \) can be extended to a map from \( B^2 \) to \( U - X \).

Also note that every cell-like set has trivial shape. A list of equivalent conditions for a subset of \( Q \) to be cellular are given in (Zvo80). The following subset of these conditions will be used to show that every non-degenerate decomposition element of our construction is cellular.

**Lemma 4.1.1.9.** (Zvo80) Let \( A \) be a finite dimensional compactum in \( Q \). Then the following statements are equivalent.

1. \( A \) is cellular in \( Q \).
2. \( A \) is a Z-set in \( Q \) and has trivial shape.
3. \( A \) has trivial shape and \( Q/A \) is a \( Q \) manifold.
4. \( M = Q - A \) is \( \{S^1\} \)-trivial at \( \infty \).

**Theorem 4.1.1.10.** Every decomposition element associated with the defining sequence \( \mathcal{M} \) from the previous Chapter 3 is cellular.
Proof. Since every non-degenerate element that does not intersect any face of the Hilbert cube is a Z set by the Lemma 4.1.1.2, then by the above Lemma 4.1.1.9, it is cellular. Let $g$ be a non-degenerate element of the decomposition $G$. We may assume that $g$ intersects the pseudo boundary of the Hilbert cube $Q$. Let $U$ be any open neighborhood of $g$, and let $V = U$ and $f : S^1 \to V - g$ be a map. Since there is a distance $\epsilon > 0$ between $f(S^1)$ and $g$, by property 1 of Theorem 3.2.4.1, there is an $n$-cell $B^n$ such that $g \subset B^n \times Q_{n+1} \subset U$ and $f(S^1) \subset U - B^n \times Q_{n+1}$. Let $F$ be any contraction of $f$. The $B^n$ intersects the $\partial I^n$ at a finite number of disjoint components. Now removing these components from $\partial I^n \cong S^{n-1}, n \geq 4$ leaves a simply connected component. Let us write the map $F$ as $F = (F_n, F_{Q_n})$. Note that $F_n^{-1}(\partial B^n)$ is a collection of a finite number of closed curves. Let $J$ be one of the innermost curves among these closed curves. Now $J$ bounds a disc $H_J$. We can extend the map $F_n |_J$ to a map $F_n' |_{H_J}$ on $\partial B^n$, and similarly, taking one innermost component at a time we can adjust the map $F_n$ as a map $G_n$ on $\partial B^n$, that is, we can extend the map $f$ to a map $G$ from $B^2$ to $U - g$. This shows that $Q - g$ is $\{S^\infty\}$-trivial at $\infty$. Then by Lemma 4.1.1.9, $g$ is cellular in $Q$. \qed

4.2. Absolute neighborhood retract (ANR)

We first write some definitions. See (Dav86) for further information on ANRs.

Definition 4.2.0.11. (Lay80) Let $f : X \to Y$ be a map from a compact metric space $X$ onto a compact metric space $Y$. Then $f$ is said to be Approximately Right Invertible (ARI) if for each $\epsilon$, there is a map $g : Y \to X$ such that $\rho(fog, I_Y) < \epsilon$

Using the definition of the defining sequence from Chapter 3, we write the following definition:

Definition 4.2.0.12. (Lay) A defining sequence $\mathcal{M} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is said to be sharp if $\bigcup_{i \geq 1} \{\partial M : M \in \mathcal{M}_i\}$ is embedded in the decomposition space by the quotient map.
The following result is from (Koz81).

**Theorem 4.2.0.13.** (Koz81) Let \( f : X \to Y \) be a map from a compact metric space \( X \) to a compact metric space \( Y \). Then \( Y \) is ANR if \( f \) is ARI.

**Lemma 4.2.0.14.** The defining sequence \( M \) constructed in Chapter 3 is sharp.

*Proof.* By properties 4 and 5 from Theorem 3.2.4.1 in Chapter 3, the defining sequence for the decomposition \( G \) is sharp. \( \square \)

The following theorem is from (Lay).

**Theorem 4.2.0.15.** (Lay) If \( M \) is a sharp defining sequence for a decomposition of the Hilbert cube, then the quotient map is an ARI and hence the space \( Q/G \) is an ANR.

**Theorem 4.2.0.16.** The decomposition space \( Q/G \) constructed in Chapter 3 is an ANR.

*Proof.* The proof follows by using lemmas 4.2.0.13, 4.2.0.14 and 4.2.0.15 \( \square \)

### 4.3. Infinite Codimension and Disjoint Discs Property

**Definition 4.3.0.17.** Let \( X \) be a metric space. Let \( f : B^2 \to X \) and \( g : I \to X \) be two maps. Then the space \( X \) is said to satisfy the disjoint arc disc property (DADP) if there exist maps \( f' : B^2 \to X \) and \( g' : I \to X \) satisfying

\[
\rho(f, f') < \epsilon, \rho(g, g') < \epsilon
\]

and

\[
f'(B^2) \cap g'(I) = \phi
\]

**Definition 4.3.0.18.** A metric space \( X \) is said to satisfy the disjoint \( n \)-disc property if every pair of maps

\[
f, g : B^n \to X
\]
can be arbitrarily closely approximated by maps $f'$ and $g'$ such that

$$f'(B^n) \cap g'(B^n) = \phi$$

The following is Torunczyk’s characterization of Hilbert cube manifolds.

**Theorem 4.3.0.19.** (Tor80) A locally compact ANR $X$ satisfies the disjoint $n$-disc property if and only if $X$ is a $Q$ manifold.

The following lemma from (Dav81) gives a relation between DDP and DADP.

**Lemma 4.3.0.20.** (Dav81) Let $X$ be a metric space that satisfies DADP. Then $X \times I$ satisfies DDP.

The following theorem is from (Dav86).

**Theorem 4.3.0.21.** (Dav86) Let $Q$ be a cell-like decomposition of the Hilbert cube $Q$, $K$ be a simplicial $n$-complex and $L$ be a subcomplex of $K$. Let

$$f : K \to Q/G$$

and

$$F_L : L \to Q$$

be maps such that $\pi F_L = f \mid_L$ and let $\epsilon > 0$ be given. Then there is a map $F : K \to Q$ such that $\rho(f, \pi F) < \epsilon$ and $F \mid_L = F_L$

**Lemma 4.3.0.22.** The decomposition space $Q/G$ constructed in the previous chapter satisfies DADP.

**Proof.** Let $\epsilon > 0$ be given and let

$$f : B^2 \to Q/G, h : I \to Q/G$$
be two given maps. Then by using DADP of \( Q \) and Theorem 4.3.0.21 (taking \( L = \phi \)), there exist maps
\[
\bar{f} : B^2 \to Q, \bar{h} : I \to Q
\]
such that
\[
\bar{f}(B^2) \cap \bar{h}(I) = \phi
\]
and
\[
\rho(f, \pi\bar{f}) < \epsilon/4, \rho(g, \pi\bar{h}) < \epsilon/4
\]
Let us write \( f' = \pi\bar{f} \) and \( h' = \pi\bar{h} \). Performing a careful adjustment, there is a stage of defining sequence, say \( M_k \), such that \( \bar{h}(I) \cap M_k \subset [-1/2^{k+1}, 1/2^{k+1}] \times Q_2 \) with the property that if \( p \) is the adjusted map of \( \bar{h} \), then \( \rho(\pi\bar{f}, \pi p) < \epsilon/4 \). Continuing this inductively, we can get an approximation \( p \) of \( \bar{h} \) such that \( p \) can be decomposed into a finite number of arcs with the property that each of these arcs intersects the non-degenerate elements in at most one point and \( \rho(\pi\bar{f}, \pi h) < \epsilon/4 \). Let \( g \) be such an element of \( G \). Note that by Theorem 4.1.1.10, each decomposition element \( g \) is cellular and \( Q/g \cong Q \), hence we can find an approximation \( q \) of the map \( \bar{f} \) that misses the \( g \) and \( \rho(\pi q, \pi\bar{f}) < \epsilon/4 \), \( \pi q(B^2) \cap \pi p(B^1) = \phi \), \( \rho(f, \pi q) < \epsilon \) and \( \rho(h, \pi p) < \epsilon \).

We will first write the following theorem from (DW81b) for an alternate characterization of the Hilbert cube.

**Theorem 4.3.0.23.** (DW81b) Let \( G \) be a decomposition of \( Q \) such that \( Q/G \) is an ANR and \( Q/G \times I \) satisfy the disjoint discs property (DDP). Then \( Q/G \times I \cong Q \) if any if the following conditions are satisfied:

1. \( Q/G \times I \) satisfies the Disjoint Čech Carrier Property.

2. \( Q/G \times I \) has Čech carriers of infinite codimension.
3. $Q/G \times I$ contains closed subsets $F_1, F_2...$ with each $F_i$ having infinite co-dimension and each closed subset of $Q/G \times I - \bigcup_{i \geq 1} F_i$ having infinite dimension.

4. Points in $Q/G \times I$ have infinite codimension and $Q/G \times I$ has finite dimensional Čech Carriers.

In order to show $Q/G \times I \cong Q$, we use the idea of infinite codimension. We write the following definitions from (DW81b). More about the finite codimension in the Hilbert cube will be discussed in the next chapter. $H_\ast, \check{H}_\ast$ denote the singular and Čech homology respectively with respect to integral coefficients.

**Definition 4.3.0.24.** A closed subset $F$ of an ANR $X$ is said to have infinite codimension (in $X$) provided $H_q(U, U - F) = 0$ for all integers $q \geq 0$ and for all open subsets $U$ of $X$. A set $A$ in $X$ is said to have infinite codimension if every closed set contained in $A$ has infinite codimension.

**Definition 4.3.0.25.** A Čech carrier for an element $z \in H_q(U, V)$ for $V \subset U$, subsets of a space $X$, and for an integer $q \geq 0$ is a pair of compact sets $\partial P \subset P$ with $(P, \partial P) \subset (U, V)$ such that

$$\text{im}\{i_* : (\check{H}_q(P, \partial P)) \to H_q(U, V)\}$$

where $i_*$ is the inclusion induced homomorphism.

**Theorem 4.3.0.26.** (DW81b) If points in an ANR have infinite codimension, then so does every finite dimensional subset.

**Corollary 4.3.0.27.** For the decomposition $G$ associated with the defining sequence from the previous chapter, $\pi^{-1}(x)$ has infinite codimension in $Q$ for every $x$ in $Q/G$.

**Proof.** It follows from property 2 of Theorem 3.2.4.1 that non-degenerate elements of the decomposition $G$ of the Hilbert cube $Q$ associated with the defining sequence $\mathcal{M}$ are one dimensional. Then by 4.3.0.26, $\pi^{-1}(x)$ has infinite codimension in $Q$. □
We write the Vietoris-Begle mapping theorem (Beg52).

**Theorem 4.3.0.28.** (Beg52) Let $X$ and $Y$ be compact metric spaces, and let $f : X \to Y$ be surjective and continuous. Suppose

$$H_i^\#(f^{-1}(x)) \cong 0, i = 0, 1, 2, \ldots, n - 1, x \in Y.$$ 

Then,

$$H_i^\#(X) \cong H_i^\#(Y)$$

Note that this also applies to the homology of pairs.

**Corollary 4.3.0.29.** Points in $Q/G$ in our example have infinite codimension in $Q/G$.

**Proof.** Note that, by (Lac77), the quotient map

$$\pi : Q \to Q/G$$

satisfies the conditions of Theorem 4.3.0.28. Also note that $\pi^{-1}(x)$ has infinite codimension by corollary 4.3.0.27. Let us consider the following exact sequence:

$$\to H_k^\#(Q - \pi^{-1}(x)) \to H_k^\#(Q) \to H_k(Q, Q - \pi^{-1}(x)) \to$$

Then,

$$H_k(Q, Q - \pi^{-1}(x)) \cong H_k^\#(Q) \cong H_k^\#(Q/G) \cong H_k(Q/G, Q/G - x) \cong 0.$$ 

This shows that points in $Q/G$ have infinite codimension.

We write some results directly from (DW81b) to prove the decomposition space $Q/G \cong Q$.

**Lemma 4.3.0.30.** (DW81b) Let $A$ be a closed subset of an ANR $X$ with infinite codimension and $B$ be a closed subset of $A$. Then $B$ has infinite codimension.
Lemma 4.3.0.31. (DW81b) Let a subset $A$ of an ANR $X$ have infinite codimension. Then $F \times I$ has infinite codimension in $X \times I$.

Lemma 4.3.0.32. (DW81b) If $G$ is a cell-like decomposition of the Hilbert cube $Q$, $Q/G \times I$ satisfies DDP, and $\pi^{-1}(x)$ has infinite codimension in $Q$, then, $X \times I \cong Q$.

Lemma 4.3.0.33. Let points in an ANR $X$ have infinite codimension. Then points in $X \times I$ have infinite codimension.

Proof. Let $p = (x, t)$ in $X \times I$. Then $x$ has infinite codimension in $X$. By Lemma 4.3.0.31, $\{x\} \times I$ is of infinite codimension. Lemma 4.3.0.30 implies that $p$ is of infinite codimension. In particular,
\[ H_\ast(Q/G \times I, Q/G \times I - \{x\}) \cong 0 \]

\[ \square \]

Theorem 4.3.0.34. For the decomposition space $Q/G$ in our example, $Q/G \times I \cong Q$.

Proof. By Lemma 4.2.0.16, the space $Q/G$ is an ANR. By Lemma 4.3.0.29, points in the decomposition space $Q/G$ have infinite codimension. By Lemma 4.3.0.22, $Q/G$ satisfies DADP. Hence, by Lemma 4.3.0.20, $Q/G \times I$ satisfies DDP. Then by Lemma 4.3.0.32, $Q/G \times I \cong Q$.

\[ \square \]

Theorem 4.3.0.35. The decomposition space $Q/G$ in our example does not satisfy the DDP.

Proof. By the induction hypothesis IH3 and the zero stage of construction, if $f_1$ and $f_2$ are two maps in general position with respect to each $\mathcal{M}_i$ from $B^2$ to $Q$ with $\rho(l_i, f_i \mid_{\partial B^2}) < \delta/2$, then $f_1$ and $f_2$ are $t$-slice maps for some $t$ in $I_2$. Let us take two maps $g_1$ and $g_2$ from $B^2$ to $Q$ with $\rho(l_i, g_i \mid_{\partial B^2}) < \delta/2$ that satisfy the IH3, then $g_1$ and $g_2$ will be $t$ slice maps. This implies that the decomposition space does not satisfy the DDP.

\[ \square \]
One of the properties that \( Q/G \) has is it does not satisfy DDP but \( Q/G \times I \) satisfies not only the DDP, but also Disjoint \( n \)-disc Property (Tor80).
5. FINITE CODIMENSION THEORY IN THE HILBERT CUBE

5.1. Introduction

In the example of a cellular decomposition of $\mathbb{R}^n$ in (DG82), the complexity of a cellular map was measured by measuring the dimension of the non-manifold part of the image under the cellular map. That is, by measuring the intrinsic dimension of the image of the union of non-degenerate decomposition elements. Finding a technique to measure complexity of such maps in infinite dimensional manifolds like the Hilbert cube is the main problem in generalizing the example in (DG82) to $Q$.

In this chapter, an idea of homological approach to codimension theory is used to address the problem. This is used as a tool to measure the complexity of the decomposition space. As a justification, we prove that the two approaches of covering codimension and homological codimension are equivalent in finite dimensional settings. There is some previous work regarding infinite codimension (Wal83), zero codimension (Lay80) and finite codimension (Gar83). Here, we will discuss subsets of the Hilbert cube with finite codimension. As usual, $\check{H}, H$ and $H^\#$ denote the Čech, singular and reduced homology respectively.

5.2. Codimension Theory

5.2.1 Cohomological Dimension

To develop the homological approach of finite codimension, we use cohomological dimension as a tool. The main objective of this section is to detect the intrinsic codimension (to be defined later in 5.2) of the decomposition space $Q/G$. For this, it is convenient to give definitions of a few terms. See (Wal83),(DG82),(Eng77),(Dra88),(Dow50)(LJ80) for additional details.
Definition 5.2.1.1. Let $X$ be a topological space. The cohomological dimension of the space $X$ with respect to an abelian group $G$ (here we consider $G = \mathbb{Z}$) denoted by $c\text{-}dim X$, is an integer $\geq -1$ or $\infty$ defined as follows.

1. $c\text{-}dim X = -1$ if and only if $X = \emptyset$

2. $c\text{-}dim X \leq n$, where $n = 0, 1, 2, ...$, if $\check{H}^{n+i}(X, A) = 0$ for every closed set $A \subset X$ and for $i = 1, 2, ...$,

3. $c\text{-}dim X = n$ if and only if $c\text{-}dim \leq n$ but not $c\text{-}dim \leq (n - 1)$

4. $c\text{-}dim = \infty$ if $c\text{-}dim > n$ for $n = -1, 0, 1, ...$.

The following theorem from (Wal80) gives the relation between the covering and cohomological dimension.

Theorem 5.2.1.2. (Wal80) Let $X$ be a finite dimensional space. Then the cohomological and covering dimensions are equal.

5.2.2 Codimension in Finite Dimensions

Definition 5.2.2.1. Let $A$ be an $m$ dimensional closed subset of an $n$ dimensional space. Then the covering codimension of $A$ in $X$ is $n - m$.

This approach to codimension can not be generalized to infinite dimensional spaces. In order to define codimension in infinite dimensional manifolds like the Hilbert cube, we consider a different approach based on homology and we define this as the homological approach to codimension.

Definition 5.2.2.2. A closed set $A \subset X$ has homological codimension $\geq k$ if $H_i(U, U - A) = 0$ for every open set $U$ of $X$ and for $i < k$. The subset is said to have codimension equal to $k$ if it has codimension $\geq k$ but it does not have codimension $\geq (k + 1)$.

The following definition is from (Lay80).
**Definition 5.2.2.3.** Let $U, V$ be a pair of open subsets of an ANR $X$ with $V \subset U$, and let $z$ be an element of $H_n(U, V)$ for some integer $n \geq 0$. A compact pair $(C, \partial C) \subset (U, V)$ is called a compact carrier if

$$z \in \text{im}\{i_* : H_n(C, \partial C) \to H_n(U, V)\},$$

where $i_*$ is the homomorphism induced by the inclusion map.

The following theorem gives the relation between the homological approach and the covering dimension approach of codimension in finite dimensional manifolds.

**Theorem 5.2.2.4.** If $X$ is an $n$-dimensional manifold and $A$ is an $m$-dimensional subset of $X$, then the codimension of $A$ in $X$ is $(n-m)$ under both definitions. In other words, in finite dimensions, the two definitions of codimension are equivalent.

Before proving this theorem, we write and prove some lemmas we need to prove the theorem. The following lemma is from (Dav81). For completeness, we include its proof.

**Lemma 5.2.2.5.** (Dav81) Suppose $X$ is locally compact metric space such that for some integer $r > 0$ and for every $x$ in $X$, $H_i(X, X - \{x\}) \cong 0, (i = 0, 1, 2, ..., r)$. Then for each $k$-dimensional closed subset $A$ of $X$, where $k \leq r, H_j(X, X - A) \cong 0$ whenever $j \in \{0, 1, 2, ..., r - k\}$

*Proof.* We use induction over the dimension of $A$. If $\dim A = -1$, then the lemma is valid. Assume it is true for all closed subsets of dimension $< k$. Let $A$ be a $k$-dimensional closed subset of $X$, and let $z \in H_j(X, X - A)$, where $0 \leq j \leq r - k$. We will show $z = 0$.

Let $A'$ and $A''$ be two closed subsets of $X$ with $\dim (A' \cap A'') < k$. Now considering the Mayer Vietories sequence for excisive couple

$$\{(X, X - A'), (X, X - A'')\}$$
we will get an inclusion induced isomorphism $\alpha$

$$\rightarrow H_{j+1}(X, X-(A'\cap A'')) \rightarrow H_{j}(X, X-(A'\cup A'')) \ni H_{j}(X, X-A') \oplus H_{j}(X, X-A'') \rightarrow ...$$

The inductive assumption implies that

$$H_{p}(X, X-(A'\cap A'')) \cong 0, p = j, j + 1$$

fix a compact pair $(C, C') \subset (X, X-A)$ carrying a representative of $z$. Since

$$H_{j}(X, X-\{x\}) \cong 0$$

for every $x \epsilon X$, each $a \epsilon A$ has a neighborhood $N_{a}$ in $A$ for which the image of $z$ in $H_{j}(X, X-N_{a})$ is trivial. We can find a cover $\{C_{i} : i = 1, 2, ...m\}$ of $C \cap A$ by closed sets such that $\{C_{i}\}$ refines the cover $\{N_{a} : a \epsilon C \cap A\}$, the interior of $\cup C_{i}$ contains $C \cap A$ and the boundary of each $C_{i}$ has dimension $\leq k - 1$. Define $A_{i}$ as $Cl(A-\cup_{i+1}^{m}C_{i}), i = 0, 1, 2, ...m$. Since $A_{0}$ does not intersect $C$, the image of $z$ in $H_{j}(X-A_{0}) \cong 0$. Inductively, for $A' = A_{i-1}$ and $A'' = C_{i}$, we presume that the image of $z$ in $H_{j}(X, X-A')$ is trivial, and we know that it is trivial in $H_{j}(X, X-A'')$ by construction

$$dim(A'\cap A'') \leq dim \partial C_{i} \leq k - 1.$$ 

Then the Mayer Vietoris argument above reveals that the image of $z$ in

$$H_{j}(X-(A' \cup A'')) = H_{j}(X, X-A_{i})$$

is trivial. Of course, when $i = m$, this proves that $z$ itself is trivial. \hfill \Box

The above Lemma 5.2.2.5 is used to prove the following lemma in finite dimensional manifolds.

**Lemma 5.2.2.6.** If $X$ is an $n$-dimensional manifold and $A$ is an $m$-dimensional closed subset of $X$, then $H_{i}(X, X - A) \cong 0$ for $i = 0, 1, 2, ..., n - m - 1.$
Proof. Note that for every point \( x \in X \), we have

\[
H_n(X, X - \{x\}) \not\equiv 0
\]

and for \( i < n \),

\[
H_i(X, X - \{x\}) \cong 0
\]

Let \( x \in X \) be any element and \( U \) be an open neighborhood \( x \) which is homeomorphic to an open ball in \( \mathbb{R}^n \). Now taking the excision between \((X - U)\) and \((X - \{x\})\), we will have

\[
H_i(X - (X - U), (X - \{x\}) - (X - U)) \cong H_i(X, X - \{x\}).
\]

Equivalently,

\[
H_i(U, U - \{x\}) \cong H_i(X, X - \{x\})
\]

Let us consider the following exact homology sequence from excision and also note that \( U \) is contractible.

\[
\ldots \to H_k^\#(U - \{x\}) \to H_k^\#(U) \to H_k(U, U - \{x\}) \to H_{k-1}^\#(U - \{x\}) \to \ldots
\]

Then,

\[
H_i(U, U - \{x\}) \cong H_{i-1}^\#(U - \{x\}).
\]

Note that \((U - \{x\})\) is homeomorphic to \( S^{n-1} \). Hence,

\[
H_n(X, X - \{x\}) \not\equiv 0
\]

and \( H_i(X, X - \{x\}) \cong 0 \) for \( i = 0, 1, 2, \ldots, n - 1 \). Then, by above Lemma 5.2.2.5, we have

\[
H_i(X, X - A) \cong 0, i = 0, 1, \ldots, n - m - 1
\]

\[\square\]

Proof of theorem 5.2.2.4
As one step showing that the codimension of \( A \) is \( n - m \), we show that there exists an open set \( U \) such that,

\[
H_{n-m}(U, U - A) \not\equiv 0
\]
Since $A$ is an $m$-dimensional subset, by Theorem 5.2.1.2 $c - \dim A = m$. Then by the definition of $c - \dim$, there is a closed set $B \subset A$ such that

$$\tilde{H}^m(A, B) \not\approx 0.$$ 

By the Alexander Duality theorem (GJ81), we have

$$\tilde{H}^m(A, B) \cong H_{n-m}(X - B, X - B - A) \not\approx 0.$$ 

Take $U = X - B$. Then we will have

$$H_{n-m}(U, U - A) \not\approx 0.$$ 

To show that codimension of $A = n - m$, we still need to show that the codimension is

$$\geq (n - m - 1).$$

For this, let $U$ be any open subset of $X$ and $A$ be any $m$ dimensional closed subset of $X$. We can consider the open set as an $n$-manifold and use the above argument. Note that $\dim(U \cap A) \leq \dim A$. Then by the above Lemma 5.2.2.6 we can write,

$$H_i(X, X - A) \cong H_i(U, U - A) \cong H_i(U, U - (U \cap A)) = 0, i = 0, 1, 2, \ldots, (n - \dim(A \cap U) - 1)$$

Note that,

$$n - \dim(A \cap U) - 1 \geq n - m - 1.$$ 

This proves part of Theorem 5.2.2.4, that is, if $A$ is an $m$ dimensional closed subset of an $n$-manifold $X$, then the homological codimension of $A$ in $X$ is $n - m$.

To prove the other part of the Theorem 5.2.2.4, let $A$ be a closed set of an $n$-manifold $X$ with homological codimension $k$. If possible, let the dimension of $A = m \neq n - k$ so $k \neq n - m$. Then from the above part of the theorem, the homological codimension is $n - m \neq k$. This contradiction proves Theorem 5.2.2.4. $\square$
5.2.3 Codimension in the Hilbert cube

Theorem 5.2.2.4 justifies using homological codimension of a closed set in finite dimensional manifolds in place of covering codimension. Infinite codimension is defined in Definition 4.3.0.24 in Chapter 4. We are now in a position to generalize definition 5.2.2.2 for finite codimension subsets of an infinite dimensional manifold as follows:

**Definition 5.2.3.1.** Let $A$ be a closed subset of an infinite dimensional manifold $X$. $A$ is said to have codimension $\geq n$ in $X$ if for every open subset $U$ and for every $i < n$

$$H_i(U, U - A) \cong 0$$

The subset $A$ is said to have codimension equal to $n$ if it has codimension $\geq n$ but does not have codimension $\geq (n + 1)$. In other words, a closed subset $A$ of $X$ is said to have codimension $n$ if $H_i(U, U - A) \cong 0$ for all open sets $U$ and $i < n$, and $H_n(V, V - A) \not\cong 0$ for some open set $V$ of $X$. An arbitrary subset $X$ of the Hilbert Cube $Q$ is said to be of codimension $n$ if any closed $C \subset X$ in $Q$ has codimension $n$.

Recall that if $G$ is a decomposition of the Hilbert Cube $Q$, then we let $N_G$ denote the union of all non-degenerate elements of $G$. We generalize the definition of secret dimension from the finite dimensional case [see (DG82)] to the infinite dimensional cases as:

**Definition 5.2.3.2.** Let $G$ be a cell-like usc decomposition of the Hilbert Cube $Q$. Then the decomposition space $Q/G$ is said to be secretly of codimension $d$ if $\pi$ is arbitrarily closely approximable by a CE map

$$p : Q \to Q/G$$

such that $p(N_p)$ has codimension $\geq d$.

**Definition 5.2.3.3.** A cell-like usc decomposition $G$ of the Hilbert cube $Q$ is said to be of intrinsic codimension $d$ if it is secretly of codimension $d$ but not secretly of codimension $d + 1$. 
The above two definitions imply that a usc decomposition $G$ of $Q$ has intrinsic codimension $m$ if

1. For every $\epsilon > 0$ there is an $\epsilon$ cell-like approximation $f$ of $\pi$ with codimension of $f(N_j) \geq m$.

2. There exists an $\epsilon > 0$ such that for every $\epsilon$ cell-like approximation $f$ of $\pi$ has the property that codimension of $f(N_j) \leq m$.

We will write the following infinite dimensional version of a result from (Dav86) page 175.

**Lemma 5.2.3.4.** Let $G$ denote a cellular decomposition of the Hilbert Cube $Q$, and for $j = 1, 2, \ldots$ let $Z_j$ denote a closed subset of $Q/G$ such that the decomposition induced over $Z_j$ is shrinkable. Then the quotient map

$$\pi : Q \to Q/G$$

can be approximated, arbitrarily closely, by a cell-like map

$$F : G \to Q/G$$

that is one to one over $\bigcup_j Z_j$.

**Proof.** The proof of this lemma follows from (Dav86) and (Zvo80).

5.2.4 Finite Dimension and Finite Codimension

Some results analogous to finite dimensional results are established. Mazurkiewicz’s theorem gives a condition for separation (Eng77) in finite dimensional separable metric spaces. Here, we will find a similar result in the Hilbert cube by using the notion of codimension.

**Theorem 5.2.4.1.** Let $A$ be a closed set in the Hilbert cube with codimension $\geq 2$. Then for every pair of points $x, y$ in $Q - A$, there is a continuum in $Q - A$ containing $x, y$. 
Proof. It suffices to show that \( A \) is path connected, that is \( H^\#_0(Q - A) \cong 0 \). We consider the following section of the homology sequence of the pair \((Q, Q - A)\):

\[
H_1(Q, Q - A) \rightarrow H^\#_0(Q - A) \rightarrow H^\#_0(Q) \rightarrow H_0(Q, Q - A)
\]

Note that \( A \) has codimension \( \geq 2 \). Then, \( H_1(Q, Q - A) \cong H_0(Q, Q - A) \cong 0 \).

Also, \( H^\#_0(Q) \cong 0 \). This shows that \( H^\#_0(Q - A) \cong 0 \) and \( Q - A \) has only one path component.

\[\square\]

**Theorem 5.2.4.2.** Let \( A \subset \{0\} \times Q_2 \) be a closed subset of \( Q \) with codimension \( \geq n + 1 \) in \( Q \). Then \( A \) is of codimension \( \geq n \) in \( Q^2 = \{0\} \times Q_2 \).

**Proof.** We adopt the idea of this proof from (DW81b). We prove this by ruling out the possibilities of \( A \) being of codimension 0, 1, 2, ..., \( n - 1 \). Let \( U \) be an open set in \( Q^2 \). We will show that \( H_i(U, U - A) \cong 0 \) for \( i = 0, 1, 2, ..., n - 1 \). Let us take

\[
U_+ = U \times [0, 1], U_- = U \times [-1, 0], V = U - A
\]

\[
V_+ = V \times [0, 1], V_- = V \times [-1, 0],
\]

and

\[
V_1 = V_+ \cup (U \times (1/2, 1]), V_2 = V_- \cup (U \times [-1, -1/2))
\]

Note that \( V_1, V_2 \) are homotopy equivalent to \( U_+ \) and \( U_- \) respectively. Therefore

\[
H_i(U_+, V_1) \cong H_i(U_-, V_2) \cong 0
\]

for all \( i \). Also note the following from the fact that \( A \) is of codimension \( n + 1 \) in \( Q \).

\[
H_i(U \times I, U \times I - A) \cong 0
\]

for \( i = 1, 2, ..., n - 1 \) and,

\[
U_+ \cup U_- = U \times I,
\]
\[ V_1 \cap V_2 = V. \]

Taking the excisive couple of pairs,

\[ \{(U_+, V_1), (U_-, V_2)\}, \]

the Relative Mayer-Vietoris sequence (Spa89) gives us

\[ \rightarrow H_i(U_+ \cap U_-, V_1 \cap V_2) \rightarrow H_i(U_+, V_1) \oplus H_i(U_-, V_2) \rightarrow H_i(U_+ \cup U_-, V_1 \cup V_2) \rightarrow \]

Since \( V_1 \cup V_2 \) is homotopy equivalent to \( U \times I - A \), this gives,

\[ H_i(U \times I, U \times I - A) \cong H_{i-1}(U, U - A), \]

for every \( i = 0, 1, 2, ..., n \). This shows that \( A \) has codimension \( \geq n \) in \( Q^2 \).

The following result gives the codimension of a closed subset of \( Q^2 \) in \( Q \).

**Theorem 5.2.4.3.** Let \( A \subset \{0\} \times Q_2 \) be closed and have codimension \( \geq n \) in \( \{0\} \times Q_2 \). Then \( A \) is of codimension \( \geq n + 1 \) in \( Q \).

**Proof.** We adopt the idea from (DW81b) to prove this lemma. For every open set \( W \) of \( Q \), we need to show that,

\[ H_i(W, W - A) \cong 0, \quad i = 0, 1, 2, ..., n \]

Let us fix an open subset \( W \) of \( Q \) and also let us fix an integer \( i \). Let us consider the collection

\[ B = \{U \times J\}, \]

where \( U \) is open in \( Q_2 \) and \( J \) is convex and open in \( I \). Then the collection \( B \) is a basis for \( Q \).

Take an element \( \alpha \) of \( H_i(W, W - A) \). Then \( \alpha \) has a compact carrier \( (C, \partial C) \subset (W, W - A) \) with \( C \subset \bigcup_{l=1}^p (U \times J_l) \subset W \) for some finite number of basis elements. Then it suffices to show

\[ H_i(\bigcup_{l=1}^p (U_l \times J_l), \bigcup_{l=1}^p (U_l \times J_l) - A) \cong 0 \]
for $i = 0, 1, 2, ..., n$.

We use induction over $l$. We use a similar argument using a relative Mayer Vietoris sequence as in 5.2.4.2. For every $l = 1, 2, ..., p$

$$H_i(U_l \times J_l, U_l \times J_l - A) \cong H_{i-1}(U_l, U_l - A)$$

That is,

$$H_i(U_l \times J_l, U_l \times J_l - A) \cong 0, i = 0, 1, 2, ..., n$$

Note that

$$(U_1 \times J_1) \cap (U_2 \times J_2) = V \times J$$

for some open set $V = U_1 \cap U_2$ and an interval

$$J = J_1 \cap J_2.$$

Now, using the Mayer Vietoris sequence for the excisive couple of pairs

$$(U_1 \times J_1, U_1 \times J_1 - A), (U_2 \times J_2, U_2 \times J_2 - A),$$

we will have

$$\longrightarrow H_i(V \times J, V \times J - A) \longrightarrow H_i(U_1 \times J_1, U_1 \times J_1) \oplus H_i(U_2 \times J_2, U_1 \times J_2)$$

$$\longrightarrow H_i(U_1 \times J_1 \cup U_2 \times J_2, U_1 \times J_1 \cup U_2 \times J_2 - A) \rightarrow$$

That gives us

$$H_i(U_1 \times J_1 \cup U_2 \times J_2, U_1 \times J_1 \cup U_2 \times J_2 - A) \cong \{0\}, i = 0, 1, 2, ..., n$$

since

$$H_i(V \times J, V \times J - A) \cong 0, i = 0, 1, 2, ..., n$$

Continuing inductively we will get

$$H_i(\bigcup_{l=1}^{k} (U_l \times I_l) \cup \bigcup_{l=1}^{k} (U_l \times I_l - A) \cong 0$$
for every $k < n$. Using the same argument with the inductive excisive couple of pairs, we will have

$$H_i(\bigcup_{i=1}^{p} (U_i \times J_i), \bigcup_{i=1}^{p} (U_i \times J_i) - A) \cong 0$$

for $i = 0, 1, 2, ..., n$. This proves that the codimension of $A$ in $Q$ is $\geq n + 1$.

The following corollary can be written as the combination of the above two theorems.

**Corollary 5.2.4.4.** Let $A \subset \{0\} \times Q_2$ be a closed subset. Then $A$ has codimension $\geq n$ in $\{0\} \times Q_2$ if and only if $A$ has codimension $\geq n + 1$ in $Q$.

**Lemma 5.2.4.5.** Let $A$ be a subset of $Q^2$ of codimension $\geq n$. Then $A \times I$ has codimension $\geq n$ in $Q$.

*Proof.* The proof follows from (DW81b)

The following lemma gives a result similar to that in finite dimensional manifolds in (DW81b).

**Lemma 5.2.4.6.** Let $A$ and $B$ be closed subsets of $Q$ such that $B \subset A$ and the codimension of $A$ in $Q$ is $\geq n$. Then the codimension of $B$ in $Q$ is also $\geq n$.

*Proof.* Let $U$ be any open set in $Q$. Then, $U - A = (U - B) - A$ and $U - B$ is also open. Hence,

$$H_i(U - B, (U - B) - A) = H_i(U - B, U - A) \cong 0, i < n.$$

Let us consider the following exact sequence for the triple $(U - A, U - B, U)$

$$... \to H_i(U - B, U - A) \to H_i(U, U - A) \to H_i(U, U - B) \to H_{i-1}(U - B, U - A) \to ...$$

Then we will have

$$H_i(U, U - A) \cong H_i(U, U - B) \cong 0, i < n.$$

This proves that the codimension of $B$ is $\geq n$. 

\[\Box\]
5.2.5 Measuring Codimension in $Q/G$

We prove the infinite dimensional version of lemma 1 of (DG82) to show that the intrinsic codimension of the decomposition space is exactly two.

Theorem 5.2.5.1. Let $G$ be a cellular decomposition of the Hilbert cube $Q$ that satisfies all the induction hypotheses on the construction (Chapter 3) and assume

1. $\pi |_{\{0\} \times Q_2}$ is a homeomorphism,
2. $\pi(N_{\pi}) \subset \pi(\{0\} \times Q_2),$
3. If $f_1$ and $f_2$ are maps from $B^2$ into $Q$ with $\rho(f_e | \partial B^2, l_e | \partial B^2) < \delta$ for $e = 1, 2$ and for the $\delta$ that was fixed at the beginning of the construction, then for some $t$ in $[-1, 1], f_1$ and $f_2$ are $t$-slice maps.

Then the decomposition space has intrinsic codimension exactly equal to two.

Proof. We use the infinite dimensional technique of the proof given in (DG82). In order to show the intrinsic codimension is $\geq 3$, let us assume the intrinsic codimension is $\geq 3$. Then by definition, the map $\pi : Q \to Q/G$ can be approximated by a cell like map $p$ such that $p(N_p)$ has codimension $\geq 3$. Note that $L_1(B^2) \cap L_2(B^2) = \phi$, where $L_1$ and $L_2$ are contractions of $l_1$ and $l_2$ respectively. Therefore, $h_1(B^2) \cap h_2(B^2)$ has codimension $\geq 3$ and is a subset of $\{0\} \times Q_2$, where $h_1 = p \circ L_1$ and $h_2 = p \circ L_2$. Then, by Theorem 5.2.4.2, $h_1(B^2) \cap h_2(B^2)$ has codimension $\geq 2$ in $\{0\} \times Q_2$. Then by Theorem 5.2.4.1, there is a continuum $\alpha$ on $Q_2$ from $Q_3 \times \{1\}$ to $Q_3 \times \{-1\}$ such that

$$\pi(\alpha) \cap h_1(B^2) \cap h_2(B^2) = \phi$$

We can approximate $p$ closely enough to $\pi$ to have approximation lifting $f_1$ and $f_2$ of $h_1$ and $h_2$ respectively such that

$$f_1(B^2) \cap f_2(B^2) \cap \alpha = \phi$$
and
\[ \rho(f_e \mid \partial B^2), l_e) < \delta/2, e = 1, 2. \]

This contradicts the t-slice condition of 3 above. Hence the intrinsic codimension is \( \leq 2 \).

Next we will prove that the intrinsic codimension \( \geq 2 \). Since
\[ \pi(N) \subset \{0\} \times Q_2, \]
the intrinsic codimension is \( \geq 1 \) in \( Q \). Note that
\[ D = \{0\} \times I_2 \times Q_3 \cong \pi(\{0\} \times I_2 \times Q_3). \]

Also, \( \pi \) is one to one over \( \{0\} \times Q_2 \). Let us consider a countable dense subset \( \{x_i\} \) of \( \pi(D) \). Note that the decomposition induced by each of \( x_i \) is shrinkable. Hence, by the Lemma 5.2.3.4, there is a cell like map
\[ p : Q \to Q/G \]
which is 1-1 over \( \{x_i\} \). Therefore,
\[ p(N_p) \subset (\pi(D) - \bigcup_{i \geq 1} \{x_i\}). \]

By Lemma 5.2.4.6, it suffices to prove that the codimension of \( D - \bigcup_{i \geq 1} \{x_i\} \) has codimension \( \geq 1 \) in \( \{0\} \times Q_2 \). Let \( U \) be any open subset of \( \{0\} \times Q_2 \). Since \( \{x_i\} \) is dense, and \( U - (D - \bigcup_{i \geq 1} \{x_i\}) \neq \emptyset \), then by proposition 13.9 and 13.10 of (GJ81),
\[ H_0(U, U - (D - \bigcup_{i \geq 1} \{x_i\})) \cong 0 \]
This proves that the codimension of \( p(N_p) \) is \( \geq 1 \) in \( \{0\} \times Q_2 \). By the Theorem 5.2.4.3, the codimension of \( p(N_p) \) is \( \geq 2 \). This proves that the intrinsic codimension of the decomposition space is exactly two. \( \square \)
5.3. Main Result

The following theorem gives the main result of this thesis.

**Theorem 5.3.0.2.** There is an intrinsic codimension two cellular decomposition $G$ of the Hilbert cube $Q$ with the property that the decomposition space $Q/G$ is not a $Q$-manifold but $Q/G \times I \cong Q$

*Proof.* The proof of this theorem follows from the construction of the cellular decomposition of the Hilbert and from Chapter 3 and Lemma 5.2.5.1
6. CONCLUSION AND SOME OPEN QUESTIONS

6.1. Conclusion

The main result of this work is the existence of a cellular decomposition $G$ of the Hilbert cube $Q$ such that the decomposition space $Q/G$ is not a $Q$ manifold but $Q/G \times I \cong Q$. In other words, it is shown that the Hilbert cube can be factored into a non-manifold and a very simple manifold, i.e. a closed interval, by using a cellular decomposition. Looking at the characterization of the Hilbert cube [Theorem 4.3.0.23], the decomposition space $Q/G$ does not satisfy the Disjoint Disc Property (DDP), but the space $Q/G \times I$ satisfies not only the DDP but also satisfies Disjoint $n$-Disc Property for every integer $n$. For the construction of such a cellular construction, we generalize the idea of (DG82) by using a non-standard embedding of Cantor set in $Q$, the cellularity criteria of a subset of the Hilbert cube. In addition, this cellular decomposition has intrinsic codimension nearly as low as possible.

The finite codimension theory in Chapter 5 is another main result of this work. Although there is some literature related to either zero codimension or infinite codimension, this work successfully established some analogous results of finite dimension into the Hilbert cube. The finite codimension theory can be also extended to other infinite dimensional manifolds. Theorem 5.2.4.1 generalizes Mazurkiewics’s theorem to the Hilbert cube. Similarly, other results are also established. In this example, the theory of finite codimension is used to measure the complexity of the non-manifold factor of the Hilbert cube. The complexity is measured in terms of the intrinsic codimension of the decomposition space, and it is shown that the intrinsic codimension of this decomposition space is exactly two. The following theorem is the main result from this thesis.

**Theorem 6.1.0.3.** There is an intrinsic codimension two cellular decomposition $G$ of the
Hilbert cube $Q$ with the property that the decomposition space $Q/G$ is not a $Q$-manifold but $Q/\times I \cong Q$
6.2. Question

In $\mathbb{R}^n$, there is an example of a cellular map from $\mathbb{R}^n \to X$ such that the image of the union of a non-degenerate pre image set has intrinsic dimension $n - 1$, that is, the map $f$ has intrinsic codimension one. Furthermore, $X$ is not a manifold and $X \times I \cong \mathbb{R}^n$ (DG83). Is it possible to generalize this result in the Hilbert cube?
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