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Holly M. Swisher

In this dissertation, we begin by presenting the result of F. K. C. Rankin and Swinnerton-Dyer on the location of the zeros of the Eisenstein series for the full modular group in the standard fundamental domain. Their simple but beautiful argument shows that all zeros are located on the lower boundary arc of the fundamental domain.

Then, we introduce families of certain combinations of products of Eisenstein series and explore the location of their zeros in the fundamental domain. By extending F.K.C. Rankin and Swinnerton-Dyer argument, we show that almost all modular forms in our families have property that many of their zeros in the fundamental domain lie on the boundary of the fundamental domain.

Zeros of Certain Combinations of Products of Eisenstein Series

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Major Professor, representing Mathematics

Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Jetjaroen Klangwang, Author

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Zeros of Certain Combinations of Products of Eisenstein Series

1 Introduction

Modular forms are complex-valued functions satisfying a certain analyticity and a certain type of transformation property. The theory of modular forms is directly linked with complex analysis, but the theory also arises in many applications such as algebraic topology, representation theory, mathematical physics, combinatorics, and number theory. One of the most notable examples is their role in Wiles's proof of Fermat's Last Theorem.

A modular form f of integer weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function on the complex upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ satisfying the transformation property

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and for all $\tau \in \mathbb{H}$ and f having a Fourier series expansion of the form

$$f(\tau) = \sum_{n \geq 0} a_n(f) q^n \quad \text{where } q := q(\tau) e^{2\pi i \tau}.$$

Many mathematicians have studied and determined the number of zeros of modular forms and their locations. We note that the number of zeros of a weight k modular form f for $\mathrm{SL}_2(\mathbb{Z})$ is determined by the valence formula;

$$\nu_{i\infty}(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_{e^{2\pi i/3}}(f) + \sum_{\tau \in \mathcal{F} \setminus \{i, e^{2\pi i/3}\}} \nu_\tau(f) = \frac{k}{12},$$

where $\nu_\tau(f)$ is the order of vanishing of f at τ and $\mathcal{F}$ denotes the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\mathcal{F} = \left\{|\tau| > 1 \text{ and } 0 < \mathrm{Re}(\tau) < \frac{1}{2}\right\} \cup \left\{|\tau| \geq 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq 0\right\}.$$

The simplest first examples of modular forms are the Eisenstein series. For even $k \geq 4$, the (normalized) Eisenstein series of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(c\tau + d)^k}.$$

The location of the zeros of Eisenstein series has been studied since the 1960s. Wohlfahrt [24] showed in 1963 that for $4 \leq k \leq 26$, all zeros of $E_k(\tau)$ lie on lower boundary arc of the fundamental domain $\mathcal{F}$ and conjectured that this holds for all $k \geq 4$. The range of k was extended to $4 \leq k \leq 34$, and $k = 38$ by R.A. Rankin in [19]. Eventually, Wohlfahrt's conjecture was proved by R.A. Rankin's daughter, F. K. C. Rankin, together with Swinnerton-Dyer in their famous paper [18].

Their proof, summarized in a single page in [18], is incredibly simple yet remarkable. It relies on approximations of trigonometric functions, tools from calculus such as the intermediate value theorem, and the valence formula from the theory of modular forms.

The results of F.K.C. Rankin and Swinnerton-Dyer [18] have made a huge impact concerning the zero location of modular forms. Similar results have been obtained for Eisenstein series for other congruence subgroups [9, 14], for Eisenstein series for Fricke groups [16, 22], for certain weakly holomorphic modular forms [6, 8, 13], and for other modular forms [11, 12].

In Chapter 2, we review some background of the standard facts on the theory of modular forms that will aid our study of the zeros of certain combinations of products of Eisenstein series. Then we provide an expanded argument of F.K.C. Rankin and Swinnerton-Dyer on the zeros of the Eisenstein series in Chapter 3. Their proof for the location of zeros of Eisenstein series serves as a template for the methods utilized in this dissertation. In addition to the exposition of their classical paper, we introduce some lemmas that will be used throughout this dissertation.

Chapter 4 explores the location of zeros of one-parameter families of modular forms of certain combination of products of Eisenstein series. We begin by determining the location of a zero of a weight 12 modular form of the form

$$rE_4^3(\tau) + E_6^2(\tau),$$

where $r \in \mathbb{R}$. Then we move on to locate all k zeros of the modular form of weight $12k$ defined by

$$E_{2k}(\tau)E_{10k}(\tau) + E_{4k}(\tau)E_{8k}(\tau),$$

where $k \in \mathbb{N}$.

In Chapter 5, we further extend the work of F.K.C. Rankin and Swinnerton-Dyerto two-parameter families of modular forms which are combinations of products of Eisenstein series. We begin by examining the zero location of the modular form of weight $k+l$ defined by

$$E_k(\tau)E_l(\tau) + E_{k+l}(\tau),$$

where k, l are even positive integers. This modular form is purposely defined analogously to the modular form in the recent work of Rietzes et al. in [20]. By taking $k = l$, we obtain modular form of type

$$E_k^n(\tau) + E_{nk}(\tau).$$

We only discuss the zero location for $n = 2, 3$ and leave the higher values n as an open problem. At the end of this chapter, we give a conjecture on the location of the zeros of the cusp form of weight $3k$ given by

$$E_k^3(\tau) - E_{3k}(\tau).$$

Throughout this dissertation, we use Mathematica to provide computational evidences to motivate our work and conjectures.

2 Background

This chapter introduces the fundamental objects and the most basic facts which are central objects in this dissertation. The required definitions and theorems are given, along with a proof sketch. For more information, the reader must consult the standard books in the field such as Diamond and Shurman [5], Koblitz [15], Miyake [17], or Shimura [23].

2.1 The Upper Half-Plane

The first fundamental structure of the theory of modular forms is the upper half plane.

Definition 2.1. The *upper half-plane* $\mathbb{H}$ is the set of complex numbers with positive imaginary part:

$$\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}.$$

The *general linear group* $\mathrm{GL}_2(\mathbb{R})$ consists of all 2×2 invertible matrices with real entries. It contains the subgroup $\mathrm{GL}_2^+(\mathbb{R})$ of matrices with positive determinant. This subgroup acts from the left on $\mathbb{H}$ by the *fractional linear transformations* or *Möbius transformations.*

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{H} \to \mathbb{H}, \quad \tau \mapsto \alpha(\tau) = \frac{a\tau + b}{c\tau + d}.$$

To see that this action is well-defined, we note that α maps the upper half-plane back to itself follows from the formula

$$\mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{(ad - bc)\mathrm{Im}(\tau)}{|c\tau + d|^2}.$$

We can check that the transitivity of the action follows easily.

2.2 The Full Modular Group

Definition 2.2. The *full modular group* $\mathrm{SL}_2(\mathbb{Z})$ is the set of all 2×2 integer matrices with determinant 1:

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{Z} \right\}.$$

In fact, we note that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For more details we refer the reader to [5].

2.3 Fundamental Domains

As a subgroup of $\mathrm{GL}_2^+(\mathbb{R})$, the full modular group $\mathrm{SL}_2(\mathbb{Z})$ also acts on $\mathbb{H}$ by fractional linear transformations

$$\tau \mapsto \alpha(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Under this action, we say that two points t, w in $\mathbb{H}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent if

$$w = \frac{a\tau + b}{c\tau + d},$$

for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then this gives us a complete set of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of $\mathbb{H}$ which we call the *fundamental domain*.

Definition 2.3. Let $\mathcal{F} \subset \mathbb{H}$ be a connected set. We say that $\mathcal{F}$ is a *fundamental domain* for $\mathrm{SL}_2(\mathbb{Z})$ (or the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$) if

- any $\tau \in \mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a point in $\mathcal{F}$;
- no two points of $\mathcal{F}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent.

The standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ is always given by

$$\mathcal{F} = \left\{ |\tau| \geq 1 \text{ and } -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq 0 \right\} \cup \left\{ |\tau| > 1 \text{ and } 0 < \mathrm{Re}(z) < \frac{1}{2} \right\}. \tag{2.1}$$

The detailed proof will appear in [5]. Throughout this dissertation, we write the fundamental domain for short.

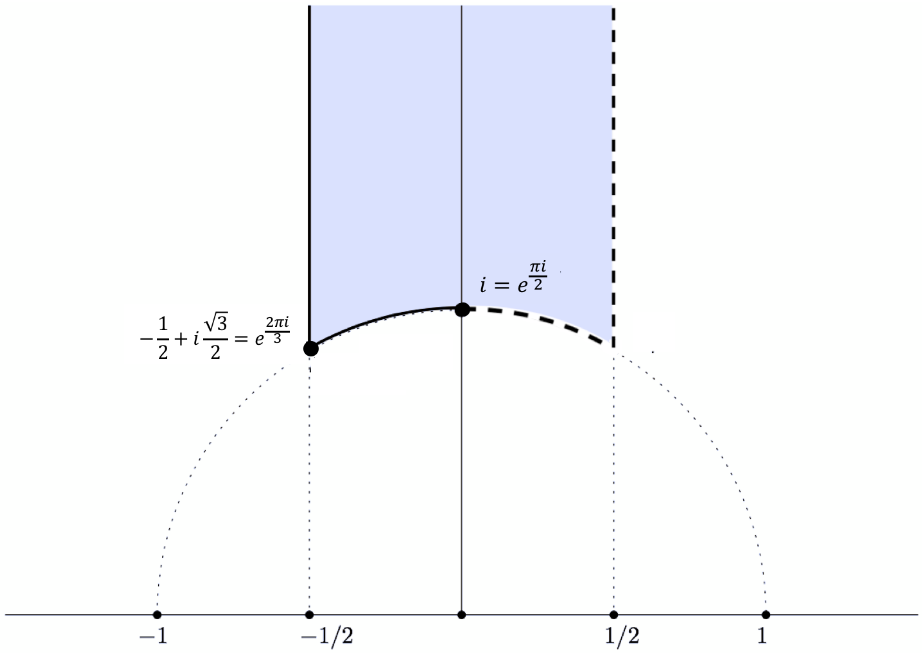

FIGURE 2.1: The fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$

2.4 Cusps

We consider the *extended upper half-plane* $\mathbb{H}^*$ defined by

$$\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$$

That is, $\mathbb{H}^*$ is defined by adjoining all the rational numbers and the point at $i\infty$. We can extend the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Q} \cup \{i\infty\}$ as follows:

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{Q} \cup \{i\infty\} \to \mathbb{Q} \cup \{i\infty\}, \quad q \mapsto \alpha(q) = \begin{cases} \frac{aq+b}{cq+d} & \text{if } q \neq i\infty \\ \frac{a}{c} & \text{otherwise} \end{cases}$$

The reader may verify directly that this is indeed an action.

Definition 2.4. The set of *cusps* of $\mathrm{SL}_2(\mathbb{Z})$ is the set of $\mathrm{SL}_2(\mathbb{Z})$-equivalent classes of $\mathbb{Q} \cup \{i\infty\}$.

We often identify cusps with a representative element from the equivalence class. For the full modular group $\mathrm{SL}_2(\mathbb{Z})$, it can easily be shown to have only one cusp at $i\infty$.

2.5 Modular Forms for $\mathrm{SL}_2(\mathbb{Z})$

Let us first define an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the set of all functions $f : \mathbb{H} \to \mathbb{C}$.

Definition 2.5. Let k be an integer. The *weight-k operator* $|_k$ is defined by

$$(f|_k\alpha)(\tau) = (ad - bc)^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right),$$

for all $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and for all $\tau \in \mathbb{H}$.

It is easy to check that the formula above defines a right action of $\mathrm{GL}_2^+(\mathbb{R})$ on the set of meromorphic functions on $\mathbb{H}$. In fact, an easy computation shows that

$$f|_k(\alpha_1\alpha_2) = (f|_k\alpha_1)|_k\alpha_2$$

for all $\alpha_1, \alpha_2 \in \mathrm{GL}_2^+(\mathbb{R})$.

Definition 2.6. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is called *weakly-modular of weight $k \in \mathbb{Z}$ for $\mathrm{SL}_2(\mathbb{Z})$* if

$$f|_k\alpha = f \quad \text{for all } \alpha \in \mathrm{SL}_2(\mathbb{Z}).$$

Since the matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we see that every weakly-modular function f for $\mathrm{SL}_2(\mathbb{Z})$ satisfies

$$f(\tau + 1) = f(\tau) \quad \text{for all } \tau \in \mathbb{H}.$$

This implies that every weakly-modular function f for $\mathrm{SL}_2(\mathbb{Z})$ is periodic and hence has a Fourier series expansion. Another way to think about this is that f has a Laurent series expansion

$$f(\tau) = \sum_{n=n_f}^{\infty} a_n(f)q^n, \quad q := q(\tau) = e^{2\pi i\tau}.$$

where $n_f \in \mathbb{Z} \cup \{-\infty\}$ depends on f. This series is often called a *q-expansion* of f or the *Fourier series expansion of f at $i\infty$.*

In order to define modular forms, we need to explain what it means for a function to be *holomorphic at $i\infty$.*

Definition 2.7. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is *meromorphic at $i\infty$* with a pole of order $|n_f|$ if $-\infty < n_f < 0$. If $n_f \geq 0$, we say that f is *holomorphic at $i\infty$*. In this case, we write $f(i\infty) = a_0(f)$. If $n_f > 0$, we say that f is *vanishes at $i\infty$* and has a zero of order n_f.

Definition 2.8. Let $k \in \mathbb{Z}$. A *weakly holomorphic modular form of weight k for $SL_2(\mathbb{Z})$* is a weakly-modular function $f : \mathbb{H} \to \mathbb{C}$ of weight k that is meromorphic on $\mathbb{H}$ and at $i\infty$.

Definition 2.9. Let $k \in \mathbb{Z}$. A *modular form of weight k for $SL_2(\mathbb{Z})$* is a weakly-modular function $f : \mathbb{H} \to \mathbb{C}$ of weight k that is holomorphic on $\mathbb{H}$ and at $i\infty$.

A special type of modular forms are called *cusp forms*.

Definition 2.10. A modular form f is called a *cusp form* if its Fourier expansion has no constant term: $a_0(f) = 0$.

One can show that if $f : \mathbb{H} \to \mathbb{C}$ is holomorphic function satisfying

$$f(\tau+1) = f(\tau) \quad \text{and} \quad f\left(-\frac{1}{\tau}\right) = z^k f(\tau),$$

for all $\tau \in \mathbb{H}$ and is holomorphic at $i\infty$, then f is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

The set of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is denoted by $M_k(\mathrm{SL}_2(\mathbb{Z}))$, and contains the set of cusp forms of weight k, which is given by $S_k(\mathrm{SL}_2(\mathbb{Z}))$. The use of the letter S for the space of cusp forms comes from the German word "Spitzenform" which means cusp form.

Later in Section 2.7, we see that both $M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $S_k(\mathrm{SL}_2(\mathbb{Z}))$ are finite dimensional $\mathbb{C}$-vector spaces. Moreover, the set of all modular forms and cusp forms

$$M(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}} M_k(\mathrm{SL}_2(\mathbb{Z})) \quad \text{and} \quad S(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}} S_k(\mathrm{SL}_2(\mathbb{Z}))$$

form a structure of a graded ring and a graded ideal.

For a nontrivial and simplest example of modular forms, let $k > 2$ and define the (*normalize*) *Eisenstein series of weight k* by

$$E_k : \mathbb{H} \to \mathbb{C}, \quad E_k(\tau) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}. \tag{2.2}$$

The series converges absolutely and uniformly and defines a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$:

$$E_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k E_k(\tau),$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and for all $\tau \in \mathbb{H}$.

We have the Fourier series expansion

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{2.3}$$

where B_k are the k^{th} *Bernoulli numbers* which are a sequence of rational numbers defined by the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{n!},$$

and the coefficient $\sigma_k(n) = \sum_{d|n} d^k$ is an extension of the sum of divisors function. Note that $E_k(\tau)$ is *normalized* so that the constant term of the Fourier series expansion is 1.

Here are some examples:

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad & E_{10}(\tau) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n, \\ E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \quad & E_{12}(\tau) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n, \\ E_8(\tau) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \quad & E_{14}(\tau) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n. \end{aligned} \tag{2.4}$$

Eisenstein series play an important role in the theory of modular form. For example, the vector space $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is a finite dimensional $\mathbb{C}$-vector space generated by $E_4^a E_6^b$ where $4a + 6b = k$ and a, b are nonnegative integers. See the discussion in [5].

Since the set of modular forms $M(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} M_k(\mathrm{SL}_2(\mathbb{Z}))$ is a graded ring, we can make modular forms out of certain sums of products of Eisenstein series. For example, the *discriminant function*

$$\Delta : \mathbb{H} \to \mathbb{C}, \quad \Delta(\tau) = \frac{1}{1728} \left(E_4^3(\tau) - E_6^2(\tau) \right). \tag{2.5}$$

The specific linear combination of E_4^3 and E_6^2 is chosen such that the constant term of the q-expansion of $\Delta(\tau)$ vanishes. Thus, $\Delta(\tau)$ is an example of a non-trivial cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. In fact, $\Delta(\tau)$ is nonzero on $\mathbb{H}$ and only vanishes at $i\infty$.

Another important example is the *modular j-function* (also called the *j-invariant*):

$$j : \mathbb{H} \to \mathbb{C}, \quad j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)}. \tag{2.6}$$

The modular j-function is not a modular form since it has a pole at $i\infty$. However, since it is a quotient of two modular forms of the same weight which implies that it is a *modular function*, i.e. it satisfies

$$j\left(\frac{a\tau+b}{c\tau+d}\right)=j(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and for all $\tau \in \mathbb{H}$ and is meromorphic on $\mathbb{H}$ and at $i\infty$.

2.6 The Valence Formula

We now come to a very important technical result on the zeros of modular forms. Let $f:\mathbb{H}\to\mathbb{C}$ be a meromorphic function in a neighborhood of a point $\tau_0 \in \mathbb{H}$. Then there is a unique integer n such that

$$\frac{f(\tau)}{(\tau-\tau_0)^n}$$

is holomorphic and nonzero in a neighborhood of τ_0. We call n the *order of vanishing of f at τ_0*, denoted by $v_{\tau_0}(f)$.

If f is a modular form of weight k, we know that f satisfies the transformation property

$$f(\tau)=(c\tau+d)^{-k}f\left(\frac{a\tau+b}{c\tau+d}\right),$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and for all $\tau \in \mathbb{H}$. This results in $\nu_{\tau_0}(f)=\nu_{\alpha(\tau_0)}(f)$ since $f(\tau_0)$ and $f(\alpha(\tau_0))$ always have the same zeros and poles. Therefore, we only need to study $\nu_{\tau_0}(f)$ for τ_0 in a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$.

For $\tau=i\infty$, we define $\nu_{i\infty}(f)=n_f$ to be the smallest value of n such that $a_n(f)\neq 0$.

Theorem 2.11. Let f be a modular form, not identically zero, of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$. Then

$$\frac{k}{12}=\nu_{i\infty}(f)+\frac{1}{2}\nu_i(f)+\frac{1}{3}\nu_{e^{2\pi i/3}}(f)+\sum_{z\in\mathcal{F}\setminus\{i,e^{2\pi i/3}\}}\nu_\tau(f). \tag{2.7}$$

Notice that (2.11) implies that the number of zeros of a modular form is roughly one-twelfth of its weight. The proof of the valence formula uses the Cauchy's argument principle from complex analysis and may be found in many standard textbooks [5, 15].

There are many applications that naturally arise about the valence formula. For example, one might be interested in determining the number of zeros of Eisenstein series which is the focus of this dissertation. Since $E_k(\tau)$ is holomorphic at $i\infty$ and the constant term in its q-expansion equals 1 for all even integer $k \geq 4$,

$$\nu_{i\infty}(E_k) = 0.$$

Thus, the valence formula for the Eisenstein series $E_k(\tau)$ is simplified to

$$\frac{k}{12} = \frac{1}{2}\nu_i(E_k) + \frac{1}{3}\nu_{e^{2\pi i/3}}(E_k) + \sum_{\tau \in \mathcal{F} \setminus \{i, e^{2\pi i/3}\}} \nu_\tau(E_k). \tag{2.8}$$

We further note that since $E_k(\tau)$ is holomorphic on $\mathbb{H}$,

$$\nu_\tau(E_k) \in \mathbb{N} \cup \{0\}, \tag{2.9}$$

for all $\tau \in \mathcal{F}$. Knowing (2.8) and (2.9), we obtain the following result.

Theorem 2.12. The Eisenstein series $E_4(\tau)$ has a simple zero at $\tau = e^{2\pi i/3}$ and no other zeroes. The Eisenstein series $E_6(\tau)$ has a simple zero at $\tau = i$ and no other zeroes.

2.7 The Space of Modular Forms

Using the valence formula on modular forms, we obtain a structure theorem for $M_k(\mathrm{SL}_2(\mathbb{Z}))$. For the proofs, we refer the reader to [5]

Theorem 2.13. The modular forms of weight 0 are $M_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}$. Let k be a nonzero even integer. If $k < 4$, then $M_k(\mathrm{SL}_2(\mathbb{Z})) = S_k(\mathrm{SL}_2(\mathbb{Z})) = 0$. If $k \geq 4$, then

$$M_k(\mathrm{SL}_2(\mathbb{Z})) = S_k(\mathrm{SL}_2(\mathbb{Z})) \oplus \mathbb{C}E_k,$$

where $E_k(\tau)$ is the normalized Eisenstein series defined in (2.2), and

$$\dim\left(S_k(\mathrm{SL}_2(\mathbb{Z}))\right) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor - 1 & \text{if } k \equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor & \text{otherwise.} \end{cases}$$

The ring of modular forms $M(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} M_k(\mathrm{SL}_2(\mathbb{Z}))$ and the ideal of cusp forms $S(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} S_k(\mathrm{SL}_2(\mathbb{Z}))$ are a polynomial ring in two variables and a principal ideal,

$$M(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6], \quad S(\mathrm{SL}_2(\mathbb{Z})) = \Delta \cdot M(\mathrm{SL}_2(\mathbb{Z})),$$

where $\Delta(\tau)$ is the discriminant function defined in (2.5).

3 Work of F.K.C. Rankin and Swinnerton-Dyer

In this chapter, we begin with the result of F.K.C. Rankin and Swinnerton-Dyer on the location of zeros of the classical Eisenstein series $E_k(\tau)$ for the modular group $\mathrm{SL}_2(\mathbb{Z})$. Since their technique on locating the zeros on the lower boundary arc of the fundamental domain will be served as the main template of our study, we will provide a detailed exposition of their argument in this chapter. To utilize their argument to our future functions, we also extend their argument to obtain the stronger bounds on the main and remainder terms of the related functions.

3.1 Statement of main result

Theorem 3.1 (F.K.C. Rankin and Swinnerton-Dyer)**.** For even $k \geq 4$, all zeros of $E_k(\tau)$ in the fundamental domain $\mathcal{F}$ given in (2.1) are located on the arc

$$\mathcal{A} = \left\{ e^{i\theta} : \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right] \right\}. \tag{3.1}$$

In 1960s, Wohlfahrt and R.A. Rankin gave partial results of the zeros of the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ in [24] and [19]. They proved that for even $4 \leq k \leq 34$, and $k = 38$, all zeros of $E_k(\tau)$ lie on the arc $\mathcal{A}$. To prove Theorem 3.1, F.K.C. Rankin and Swinnerton-Dyer consider $k \geq 12$. Through out this chapter, we let $k \geq 12$ be an even integer and write

$$k = 12n + s \quad \text{where } s \in \{0, 4, 6, 8, 10, 14\}. \tag{3.2}$$

F.K.C. Rankin and Swinnerton-Dyer consider the function

$$F_k(\theta) := e^{ik\theta/2} E_k(e^{i\theta}). \tag{3.3}$$

Since $e^{ik\theta/2}$ never vanishes, the zeros of $F_k(\theta)$ on the closed interval $[\pi/2, 2\pi/3]$ will correspond bijectively to the zeros of $E_k(\tau)$ on the arc $\mathcal{A}$. Moreover, we have that $F_k(\theta)$ is real on $[\pi/2, 2\pi/3]$. In fact, we prove the following lemma which will be useful for us in future sections when we adapt this method.

Lemma 3.2. For any $k \in \mathbb{Z}$, if $f(\tau)$ is a weakly holomorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with real Fourier coefficients, then the function $e^{ik\theta/2} f(e^{i\theta})$ is real for $\theta \in (0, \pi)$.

Proof. Let f be a weakly holomorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with real Fourier coefficients. Write f in terms of its Fourier series expansion as

$$f(\tau) = \sum_{n=n_f}^{\infty} a_n(f)e^{2\pi i n\tau},$$

where $a_n(f) \in \mathbb{R}$ and $n_f \in \mathbb{Z}$ is an integer depending on f. Let $\theta \in (0,\pi)$. Using the properties of complex conjugate that $\overline{e^{zw}} = e^{\bar{z}\bar{w}}$ for $z, w \in \mathbb{C}$ and the fact that f satisfies $f(S(\tau)) = \tau^k f(\tau)$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ by (2.8), we have that

$$\begin{aligned}
\overline{e^{ik\theta/2} f(e^{i\theta})} &= \overline{e^{ik\theta/2}} \sum_{n=n_f}^{\infty} a_n(f)\overline{e^{2\pi i n(e^{i\theta})}} \\
&= e^{-ik\theta/2} \sum_{n=n_f}^{\infty} a_n(f)e^{-2\pi i n(e^{-i\theta})} \\
&= e^{-ik\theta/2} f(-e^{-i\theta}) \\
&= e^{-ik\theta/2} f\left(S(e^{i\theta})\right) \\
&= e^{-ik\theta/2}\left(e^{ik\theta} f(e^{i\theta})\right) \\
&= e^{ik\theta/2} f(e^{i\theta}).
\end{aligned}$$

This proves that the function $e^{ik\theta/2} f(e^{i\theta}) \in \mathbb{R}$ for all $\theta \in (0,\pi)$. □

The remainder of this chapter is devoted to explaining the expanded method of F.K.C. Rankin and Swinnerton-Dyer to locate the zeros of $F_k(\theta)$ on $[\pi/2, 2\pi/3]$.

3.2 Extraction of main and remainder terms

Since we wish to obtain the better bound for the main and remainder terms, our notations will slightly differ from those of F.K.C. Rankin and Swinnerton-Dyer in [18].

By the definition of $E_k(\tau)$ given in (2.2), we can write

$$F_k(\theta) = e^{ik\theta/2} E_k(e^{i\theta}) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(ce^{i\theta/2} + de^{-i\theta/2})^k}. \tag{3.4}$$

We first extract the main term and then give a bound for the remainder term of $F_k(\theta)$. Let $M_k(\theta)$ denote the sum of the first six terms in the series (3.4) those with $c^2 + d^2 = 1$ and $(c, d) = \pm(1, 1)$, and denote the remainder of the series $R_k(\theta)$. Then

$$F_k(\theta) = M_k(\theta) + R_k(\theta) \tag{3.5}$$

where

$$M_k(\theta) = 2\cos\left(\frac{k\theta}{2}\right) + \left(2\cos\left(\frac{\theta}{2}\right)\right)^{-k} \tag{3.6}$$

and

$$R_k(\theta) = \left(2i\sin\left(\frac{\theta}{2}\right)\right)^{-k} + \frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c^2+d^2\geq 5}} \frac{1}{(ce^{i\theta/2} + de^{-i\theta/2})^k}. \tag{3.7}$$

using the exponential form of trigonometric functions.

In the notations of [18], F.K.C. Rankin and Swinnerton-Dyer explicitly evaluate the first 4 terms in the series (3.4) with $c^2 + d^2 = 1$ to obtain $M_k(\theta) = 2\cos(k\theta/2)$ and let the rest of the series be the remainder term. By having less terms to consider, we show that the upper bound on our remainder term is better than that in [18].

Proposition 3.3. For any even integer $k \geq 12$ and for any $\theta \in [\pi/2, 2\pi/3]$,

$$F_k(\theta) = M_k(\theta) + R_k(\theta),$$

where

$$M_k(\theta) = 2\cos\left(\frac{k\theta}{2}\right) + \left(2\cos\left(\frac{\theta}{2}\right)\right)^{-k},$$

and $|R_k|$ is monotonically decreasing as a function in k and bounded above by

$$|R_k| \leq \left(\frac{1}{2}\right)^{k/2} + 4\left(\frac{2}{5}\right)^{k/2} + \frac{93}{40}\left(2^{k/2}\right)\left(\left(\frac{1}{10}\right)^{(k-1)/2} + \frac{2}{k-3}\left(\frac{1}{10}\right)^{(k-3)/2}\right). \tag{3.8}$$

To prove Proposition 3.3, by (3.5) and (3.6), it remains to bounding $|R_k(\theta)|$. By the triangle inequality, we have from (3.7) that

$$|R_k(\theta)| \leq \left(\frac{1}{2}\right)^{k/2} + \frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c^2+d^2\geq 5}} \frac{1}{|ce^{i\theta/2} + de^{-i\theta/2}|^k} \tag{3.9}$$

using the fact that $2\sin(\theta/2) \in [\sqrt{2}, \sqrt{3}]$ on $[\pi/2, 2\pi/3]$. To bound all terms satisfying $c^2 + d^2 \geq 5$, we first consider the following lemma.

Lemma 3.4. For any $(c, d) \in \mathbb{Z} \times \mathbb{Z}$ and for any $\theta \in [\pi/2, 2\pi/3]$, we have that

$$\left|ce^{i\theta/2} + de^{-i\theta/2}\right|^2 \geq \frac{c^2 + d^2}{2}.$$

Proof. Let $(c,d) \in \mathbb{Z} \times \mathbb{Z}$ and let $\theta \in [\pi/2, 2\pi/3]$. By properties of complex numbers,

$$\begin{aligned}|ce^{i\theta/2} + de^{-i\theta/2}|^2 &= \left(ce^{i\theta/2} + de^{-i\theta/2}\right)\overline{\left(ce^{i\theta/2} + de^{-i\theta/2}\right)} \\ &= \left(ce^{i\theta/2} + de^{-i\theta/2}\right)\left(ce^{-i\theta/2} + de^{i\theta/2}\right) \\ &= c^2 + d^2 + 2cd\cos(\theta)\end{aligned}$$

Since $0 \le \cos(\theta) \le 1/2$, $cd \ge -|cd|$ and $2|cd| \le c^2 + d^2$,

$$c^2 + d^2 + 2cd\cos(\theta) \ge c^2 + d^2 - 2|cd|\cos(\theta) \ge \frac{c^2 + d^2}{2}.$$

□

When $c^2 + d^2 = 5$, we have the following eight cases:

$$(c,d) \in \{(\pm 1, \pm 2), (\pm 2, \pm 1), (\pm 1, \mp 2), (\pm 2, \mp 1)\}.$$

Since $c^2 + d^2 \ge 2$, we can use Lemma 3.4 to give us

$$\frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c^2+d^2=5}} \frac{1}{|ce^{i\theta/2} + de^{-i\theta/2}|^k} \le \frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c^2+d^2=5}} \left(\frac{2}{c^2+d^2}\right)^{k/2} = \frac{1}{2}(8)\left(\frac{2}{5}\right)^{k/2} = 4\left(\frac{2}{5}\right)^{k/2}. \tag{3.10}$$

To give upper bounds on the terms $c^2 + d^2 \ge 10$, we need the following lemma.

Lemma 3.5. For any integer $N \ge 10$, the number of integer solution (c,d) with $c^2 + d^2 = N$ is at most $2(2N^{1/2} + 1) \le \frac{93}{20}N^{1/2}$.

Proof. Let $N \ge 10$ be an integer and let $(c,d) \in \mathbb{Z} \times \mathbb{Z}$ be a solution of $c^2 + d^2 = N$. Since $|c|$ has to be less than or equal $N^{1/2}$, there are at most $2N^{1/2} + 1$ ways of selecting c (both positive and negative values and $c = 0$). For each fixed value of c, we have only one way of selecting $|d|$. Since if $d > 0$, we can select $\pm d$ and hence we conclude that there are at most $2(2N^{1/2} + 1)$ ways of selecting the integer solution of $c^2 + d^2 = N$. Finally, the second inequality follows from the fact that $\frac{40}{13} < N^{1/2}$ for $N \ge 10$. □

We now are able to prove Proposition 3.3.

Proof. By (3.6), (3.7), (3.9) and (3.10), we remain to bound all terms in the series with $c^2 + d^2 \ge 10$. Lemmas 3.4 and 3.5 yield that

$$\frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c^2+d^2\ge 10}} \frac{1}{|ce^{i\theta/2} + de^{-i\theta/2}|^k} \le \frac{1}{2}\sum_{\substack{\gcd(c,d)=1 \\ c^2+d^2\ge 10}} \left(\frac{2}{c^2+d^2}\right)^{k/2} \le \frac{1}{2}\sum_{N\ge 10} \frac{93}{20}N^{1/2}\left(\frac{2}{N}\right)^{k/2}.$$

Note that the right hand summation can be further simplified to

$$\frac{1}{2}\sum_{N\geq 10}\tfrac{93}{20}N^{1/2}\left(\frac{2}{N}\right)^{k/2}=\frac{93}{40}\left(2^{k/2}\right)\sum_{N\geq 10}N^{(1-k)/2}. \tag{3.11}$$

Since $N^{(1-k)/2}$ is decreasing as a function in N, the summation on the right hand side of (3.11) can is bounded above by

$$\frac{93}{40}\left(2^{k/2}\right)\sum_{N\geq 10}N^{(1-k)/2}\leq\frac{93}{40}\left(2^{k/2}\right)\left(10^{(1-k)/2}+\int_{10}^{\infty}N^{(1-k)/2}dN\right).$$

Evaluating the integral, we obtain

$$\begin{aligned}\frac{93}{40}&\left(2^{k/2}\right)\left(10^{(1-k)/2}+\int_{10}^{\infty}N^{(1-k)/2}dN\right)\\ &=\frac{93}{40}\left(2^{k/2}\right)\left(10^{(1-k)/2}+\frac{2}{3-k}\lim_{t\to\infty}\left(t^{(3-k)/2}-10^{(3-k)/2}\right)\right)\\ &=\frac{93}{40}\left(2^{k/2}\right)\left(\left(\frac{1}{10}\right)^{(k-1)/2}+\frac{2}{k-3}\left(\frac{1}{10}\right)^{(k-3)/2}\right).\end{aligned} \tag{3.12}$$

By (3.9),(3.10) and (3.12), the absolute value of the remainder term can be bounded above by

$$|R_k(\theta)|\leq\left(\frac{1}{2}\right)^{\frac{k}{2}}+4\left(\frac{2}{5}\right)^{\frac{k}{2}}+\frac{93}{40}\left(2^{k/2}\right)\left(\left(\frac{1}{10}\right)^{(k-1)/2}+\frac{2}{k-3}\left(\frac{1}{10}\right)^{(k-3)/2}\right). \tag{3.13}$$

Since the right hand side of (3.13) is a sum of decreasing functions on k, the remainder term $|R_k(\theta)|$ is then a decreasing function in k as desired. □

In the notations of F.K.C. Rankin and Swinnerton-Dyer in [18], for even $k\geq 12$,

$$|R_k|\leq 1+\left(\frac{1}{2}\right)^{k/2}+4\left(\frac{2}{5}\right)^{k/2}+\frac{20\sqrt{2}}{k-3}\left(\frac{2}{9}\right)^{(k-3)/2}\leq 1.03563.$$

By evaluating the right hand side of (3.13) at $k=12$, we find that for even $k\geq 12$, our remainder term satisfies

$$|R_k|\leq 0.03353$$

which gives a better bound to our future functions.

3.3 Definition of sample points

For an even integer $k\geq 12$, we define

$$\theta_m:=\theta_k(m)=\frac{2m\pi}{k}.$$

where m ranges over integers so that $\theta_m \in [\pi/2, 2\pi/3]$.

Using the parameterization $k = 12n + s$ where $s \in \{0, 4, 6, 8, 10, 14\}$ in (3.2), we observe that the number of θ_m in $[\pi/2, 2\pi/3]$ equals the number of integers in $[k/4, k/3] = [3n + s/4, 4n + s/3]$. Since $s \in \{0, 4, 6, 8, 10, 14\}$, we see that for each choice of s there are $n + 1$ integers in this interval.

Our goal for the rest of this chapter is to show that the function $F_k(\theta_m)$ is strictly positive or negative depending on the parity of an integer m in the closed interval $[k/4, k/3]$. Since $F_k(\theta) = M_k(\theta) + R_k(\theta)$ by Proposition 3.3, it suffices to show that for any integer $m \in [k/3, k/4]$, a lower bound of $(-1)^m M_k(\theta_m)$ is greater than the upper bound of $|R_k|$ given in Proposition 3.3.

Let $m \in [k/4, k/3]$ be an integer. Since $2\cos(k\theta_m/2) = 2(-1)^m$, we have by (3.6) that

$$(-1)^m M_k(\theta_m) = 2 + (-1)^m \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k}. \tag{3.14}$$

To give a bound for $(-1)^m M_k(\theta_m)$, we require some properties of the function $(2\cos(m\pi/k))^{-k}$ for $m \in [k/4, k/3]$.

3.4 Estimates for trigonometric functions

Lemma 3.6. For any positive integer $N \geq 2$, the function

$$f_N(x) = \left(2\cos\left(\frac{x\pi}{N}\right)\right)^{-N}$$

is positive and monotonically increasing on $[N/4, N/3]$.

Proof. Let $N \geq 2$ be an integer and let $x \in [N/4, N/3]$. Note that $f_N(x)$ is positive since $\cos(x\pi/N) > 0$ on $[N/4, N/3]$. To prove that $f_N(x)$ is monotonically increasing, we show that $f'_N(x) > 0$ for all $x \in [N/4, N/3]$. Since

$$f'_N(x) = \pi f_N(x) \tan\left(\frac{x\pi}{N}\right),$$

and the two functions $\cos(x\pi/N)$ and $\tan(x\pi/N)$ are positive for $x \in [N/4, N/3]$, we have that $f'_N(x) > 0$. □

The following lemma generalizes Lemma 2.2 from the work of Reitzes et al. in [20].

Lemma 3.7. For any real number $N > 0$, the function

$$g_N(x) = \left(2\cos\left(\frac{\pi}{3} - \frac{N\pi}{x}\right)\right)^{-x}$$

is positive and monotonically decreasing on $[3N/2, \infty)$.

Proof. Let N be a positive real number and let $x \in [3N/2, \infty)$. Note that $g_N(x)$ is positive since $\pi/3 - N\pi/x \in (-\pi/3, \pi/3]$ and hence $\cos(\pi/3 - N\pi/x) > 0$. To prove that $g_N(x)$ is monotonically decreasing, we show that

$$h_N(x) := -x\log(2\cos(\pi/3 - N\pi/x))$$

is monotonically decreasing on $[3N/2, \infty)$. We note that

$$h'_N(x) = -\log\left(2\cos\left(\frac{\pi}{3} - \frac{N\pi}{x}\right)\right) + \frac{N\pi}{x}\tan\left(\frac{\pi}{3} - \frac{N\pi}{x}\right).$$

By changing variables $\pi/3 - N\pi/x \mapsto z$, we thus want to show that for $z \in [-\pi/3, \pi/3]$,

$$H(z) := -\log\left(2\cos(z)\right) - \left(z - \frac{\pi}{3}\right)\tan(z) \le 0.$$

We note that

$$H\left(\frac{\pi}{3}\right) = -\log\left(2\cos\left(\frac{\pi}{3}\right)\right) - \left(\frac{\pi}{3} - \frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 0$$

and

$$H'(z) = -(z - \pi/3)\sec^2(z) \ge 0$$

on $[-\pi/3, \pi/3]$. This proves that $H(z)$ is increasing and therefore $H(z) \le H(\pi/3) = 0$ for all $z \in [-\pi/3, \pi/3]$. □

3.5 Bounding the main term

Proposition 3.8. For any even integer $k \ge 12$, and for any integer $m \in [k/4, k/3]$,

$$(-1)^m M_k(\theta_m) \ge 1.81163.$$

Proof. By (3.14), we recall that $(-1)^m M_k(\theta_m)$ is given by

$$(-1)^m M_k(\theta_m) = 2 + \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k}.$$

By Lemma 3.6, $(2\cos(m\pi/k))^{-k}$ is positive and increasing on $m \in [k/4, k/3]$. This follows that

$$(-1)^m M_k(\theta_m) \geq 2 - \left(2\cos\left(\frac{m_{odd}\pi}{k}\right)\right)^{-k} \tag{3.15}$$

where m_{odd} is the largest odd integer in $[k/4, k/3]$. Depending on the congruence classes of k modulo 3, we find that

$$m_{odd} = \frac{k}{3} - \frac{3+r}{3}$$

where we write $k \equiv r \pmod 3$ with $r \in \{0, \pm 2\}$. Combining this with (3.15), we obtain

$$(-1)^m M_k(\theta_m) \geq 2 - \left(2\cos\left(\frac{\pi}{3} - \left(\frac{3+r}{3}\right)\frac{\pi}{k}\right)\right)^{-k}.$$

Observe that the right hand side is monotonically increasing as a function in k by Lemma 3.7. Thus, for $k \geq 12$ and $k \equiv 0 \pmod 3$, we have that

$$(-1)^m M_k(\theta_m) \geq 2 - \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12}\right)\right)^{-12} \geq 1.98437. \tag{3.16}$$

Following the same reasoning, for $k \geq 16$ and $k \equiv -2 \pmod 3$, we have that

$$(-1)^m M_k(\theta_m) \geq 2 - \left(2\cos\left(\frac{\pi}{3} - \frac{5\pi}{3(16)}\right)\right)^{-16} \geq 1.99853, \tag{3.17}$$

and for $k \geq 14$ and $k \equiv 2 \pmod 3$, we have that

$$(-1)^m M_k(\theta_m) \geq 2 - \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3(14)}\right)\right)^{-14} \geq 1.81163. \tag{3.18}$$

Taking the smallest bound of (3.16), (3.16) and (3.16), we have proved Proposition 3.8. □

3.6 Proof of Theorem 3.1

Proof of Theorem 3.1. By Proposition 3.3, we find that for any even integer $k \geq 12$ and for any $\theta \in [\pi/2, 2\pi/3]$, the function $F_k(\theta) = e^{ik\theta/2}E_k(e^{i\theta})$ can be written as

$$F_k(\theta) = M_k(\theta) + R_k(\theta)$$

where $|R_k|$ is decreasing as a function in $k \geq 12$ given in Proposition 3.3. Evaluating the upper bound (3.8) of $|R_k|$ at $k = 12$, we have that for $k \geq 12$,

$$|R_k| \leq 0.03353. \tag{3.19}$$

By Proposition 3.8, for any even $k \geq 12$ and for any integer m for which $\theta_m = 2m\pi/k \in [\pi/2, 2\pi/3]$,

$$(-1)^m M_k(\theta_m) > 1.81163. \tag{3.20}$$

Comparing (3.19) with (3.20), we find that

$$(-1)^m M_k(\theta_m) > |R_k|.$$

Therefore, $F_k(\theta_m)$ is strictly positive or negative according as m is even or odd.

With the notation $k = 12n + s$ and $s \in \{0, 4, 6, 8, 10, 14\}$ in (3.2), there are $n+1$ integers m for which θ_m is in the closed interval $[\pi/2, 2\pi/3]$ as we mentioned in Section 3.3. The intermediate value theorem thus implies that the function $F_k(\theta)$, and hence $E_k(e^{i\theta})$ has at least n zeros for θ in the open interval $(\pi/2, 2\pi/3)$.

By the reduced valence formula for $E_k(\tau)$ given in (2.8),

$$n + \frac{s}{12} = \frac{k}{12} = \frac{1}{2}\nu_i(E_k) + \frac{1}{3}\nu_{e^{2\pi i/3}}(E_k) + \sum_{\tau \in \mathcal{F}\setminus\{i, e^{2\pi i/3}\}} \nu_\tau(E_k).$$

By considering all possible values of $s \in \{0, 4, 6, 8, 10, 14\}$, $s/12$ determines the order of zeros at $\tau = i$ and $e^{2\pi i/3}$. Since $E_k(\tau)$ is holomorphic on $\mathbb{H}$, $\nu_\tau(E_k)$ is a nonnegative integer for all $\tau \in \mathbb{H}$. This implies thatE_k can have at most n zeros in $\mathcal{F}\setminus\{i, e^{2\pi i/3}\}$.

Since the above argument shows that there are at least n zeros on the arc $\mathcal{A}$, we thus have located all zeros of $E_k(\tau)$ in the fundamental domain and therefore this concludes the proof of Theorem 3.1.

□

4 One-parameter families of modular forms

Having given the details for the location of zeros of the Eisenstein series $E_k(\tau)$ for $\mathrm{SL}_2(\mathbb{Z})$, we now turn to determining the location of the zeros of families of certain combinations of products of Eisenstein series. In this chapter, we begin by locating one zero of the weight 12 modular form $rE_4^3(\tau) + E_6^2(\tau)$ when $r \in \mathbb{R}$. In particular, we generalize the techniques used by El Basraoui and Sebba in [7] and the expanded method of F.K.C. Rankin and Swinnerton-Dyer described in Chapter 3 to show that the zero will be either on the boundary of the fundamental domain or on the imaginary axis. Then we move on to study the modular form of weight $12k$ given by $E_{2k}(\tau)E_{10k}(\tau) + E_{4k}(\tau)E_{8k}(\tau)$. We show that all k zeros of $E_{2k}(\tau)E_{10k}(\tau) + E_{4k}(\tau)E_{8k}(\tau)$ are located on the lower boundary arc of the fundamental domain by utilizing the works of F.K.C. Rankin and Swinnerton-Dyer in Chapter 3 and Reitzes et al. in [20].

4.1 Zeros of $rE_4^3(\tau) + E_6^2(\tau)$

4.1.1 Statement of main result

Let $r \in \mathbb{R}$ and define

$$\Delta_r(\tau) := rE_4^3(\tau) + E_6^2(\tau),$$

where $E_4(\tau)$ and $E_6(\tau)$ are Eisenstein series of weight 4 and 6 defined in Chapter 2.

Clearly, $\Delta_r(\tau)$ is a modular form of weight 12. By Theorem 2.13, we have that

$$M_{12}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_{12} \bigoplus \mathbb{C}\Delta$$

where $E_{12}(\tau)$ is the Eisenstein series of weight 12 and $\Delta(\tau)$ is the discriminant function. Thus, $\Delta_r(\tau)$ can be written as a linear combination of $E_{12}(\tau)$ and $\Delta(\tau)$. Comparing the constant term and the coefficient of q, we can express $\Delta_r(\tau)$ as

$$\Delta_r(\tau) = (r+1)E_{12}(\tau) + \frac{1727}{691}(250r - 441)\Delta(\tau).$$

Since $\Delta_r(\tau)$ has a nonzero constant term in the q-expansion, by the valence formula (2.11), $\Delta_r(\tau)$ satisfies

$$1 = \frac{12}{12} = \nu_{i\infty}(\Delta_r) + \frac{1}{2}\nu_i(\Delta_r) + \frac{1}{3}\nu_{e^{2\pi i/3}}(\Delta_r) + \sum_{z\in\mathcal{F}\setminus\{i,e^{2\pi i/3}\}} \nu_\tau(\Delta_r).$$

Thus $\Delta_r(\tau)$ has only one zero (counted with multiplicity) in the fundamental domain.

To avoid well-studied cases, we may assume that $r \notin \{0, -1\}$. This is because if $r = 0$, $\Delta_0(\tau) = E_6^2(\tau)$ only has 1 zero of order 2 at $\tau = i$ by Theorem 2.12. On the other hand, $\Delta_{-1}(\tau) = -1728\Delta(\tau)$ only vanishes at $\tau = i\infty$.

For $x \in \mathbb{R}$, let us denote by $\mathcal{F}_x$ the set of complex numbers in the fundamental domain whose real part equals to x:

$$\mathcal{F}_x = \{\tau \in \mathcal{F} : \mathrm{Re}(\tau) = x\}.$$

For example, $\mathcal{F}_0$ and $\mathcal{F}_{-1/2}$ are the imaginary axis and the left-hand side boundary of the fundamental domain, respectively.

By examining the properties of the Eisenstein series on the arc $\mathcal{A} = \{e^{i\theta} : \pi/2 \le \theta \le 2\pi/2\}$ given in (3.1) and on the two vertical lines $\mathcal{F}_0$ and $\mathcal{F}_{-1/2}$, we will prove the following results on the location of τ_r.

Theorem 4.1. For any $r \in \mathbb{R}$, if τ_r is the zero of $rE_4^3(\tau) + E_6^2(z)$, then

$$\tau_r \in \begin{cases} \mathcal{A} & \text{if } r > 0, \\ \mathcal{F}_{-1/2} & \text{if } r < -1, \\ \mathcal{F}_0 & \text{if } -1 < r < 0. \end{cases}$$

4.1.2 Properties of Eisenstein series on $\mathcal{F}_0$ and $\mathcal{F}_{-1/2}$

Before moving on to the proof, we observe some properties of Eisenstein series on $\mathcal{F}_0$ and $\mathcal{F}_{-1/2}$. The first property is that $E_k(\tau)$ takes real values on $\mathcal{F}_0$ and $\mathcal{F}_{-1/2}$. In fact, we prove the following lemma.

Lemma 4.2. For any $k \in \mathbb{Z}$, if $f(\tau)$ is a weakly holomorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with real Fourier coefficients, then $f(\tau)$ is real on $\mathcal{F}_0$ and $\mathcal{F}_{-1/2}$.

Proof. Let f be a weakly holomorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with real Fourier coefficients. Write f in terms of its Fourier series expansion

$$f(\tau) = \sum_{n=n_f}^{\infty} a_n(f) e^{2\pi i n\tau},$$

where $a_n(f) \in \mathbb{R}$ and $n_f \in \mathbb{Z}$ is an integer depending on f. Substituting $\tau = iy \in \mathcal{F}_0$ into $\overline{f(\tau)}$, we find that

$$\overline{f(\tau)} = \sum_{n=n_f}^{\infty} a_n(f)\overline{e^{-2\pi yn}} = \sum_{n=n_f}^{\infty} a_n(f) e^{-2\pi yn} = f(\tau).$$

Similarly, plugging in $\tau = -1/2 + iy \in \mathcal{F}_{-1/2}$ into $\overline{f(\tau)}$ yields

$$\overline{f(\tau)} = \sum_{n=n_f}^{\infty} a_n(f)(-1)^n\overline{e^{-2\pi yn}} = \sum_{n=n_f}^{\infty} a_n(f)(-1)^n e^{-2\pi yn} = f(\tau).$$

□

Let us observe how $E_4(\tau)$ and $E_6(\tau)$ behave at $\tau = i$ and $e^{2\pi i/3}$. By Theorem 2.12, $E_4(\tau)$ only vanishes at $\tau = e^{2\pi i/3}$ and $E_6(\tau)$ only vanishes at $\tau = i$. The following lemma approximates the values of $E_4(i)$ and $E_6(e^{2\pi i/3})$

Lemma 4.3. $E_4(i)$ and $E_6(e^{2\pi i/3})$ are positive real numbers.

Proof. We first note that $E_4(\tau)$ is real on $\mathcal{F}_0$ and $E_6(\tau)$ is real on $\mathcal{F}_{-1/2}$. Since $E_4(\tau)$ and $E_6(\tau)$ have real coefficients given in (2.4), $E_4(\tau)$ is real on $\mathcal{F}_0$ and $E_6(\tau)$ is real on $\mathcal{F}_{-1/2}$ by Lemma 4.2. Let us first consider the value of $E_4(i)$.

Since the constant term of $E_4(\tau)$ equals 1, $E_4(i\infty) = 1$. Then if $E_4(i)$ were nonnegative real number then $E_4(iy)$ would have at least one zero on the open interval $[1, \infty)$ by the intermediate value theorem. However, this contradicts the fact that $E_4(\tau)$ has only one zero in the fundamental domain at $\tau = e^{2\pi i/3}$ by Theorem 2.12. Thus, $E_4(i)$ is positive real number.

Applying the same reasoning for $E_6(\tau)$ on $\mathcal{F}_{-1/2}$, we have that $E_6(e^{2\pi i/3})$ is positive real number. □

4.1.3 Proof of Theorem 4.1

We now prove Theorem 4.1.

Proof. Let $r \in \mathbb{R}\backslash\{0, -1\}$. By the definition of Eisenstein series given in (2.3), $E_k(\tau)$ has real Fourier coefficients for all $k \geq 4$. Thus Lemma 4.2 implies that $\Delta_r(\tau) = rE_4^3(\tau) + E_6^2(\tau)$ is real on the imaginary axis $\mathcal{F}_0$.

Let us first consider when $-1 < r < 0$. The following technique is used in the work of El Basraoui and Sebba in [7] to find a unique zero of $E_2(\tau)$ on the imaginary axis. Consider the following limits:

$$\lim_{y \to 1^+} \Delta_r(iy) = rE_4^3(i) + E_6^2(i) = rE_4^3(i) \tag{4.1}$$

and

$$\lim_{y \to \infty} \Delta_r(iy) = rE_4^3(i\infty) + E_6^2(i\infty) = r + 1, \tag{4.2}$$

using the fact that $E_6(i) = 0$ and $E_k(i\infty) = 1$ for all even integer $k \geq 4$.

By Lemma 4.3, we know that $E_4(i) > 0$. Thus (4.1) and (4.2) imply that

$$\Delta_r(i) < 0 \quad \text{and} \quad \Delta_r(i\infty) > 0.$$

The intermediate value theorem implies that $\Delta_r(\tau)$ has at least one zero on $\mathcal{F}_0$. Since $\Delta_r(\tau)$ has at most one zero in the fundamental domain as mentioned at the beginning of Section 4.1, $\Delta_r(\tau)$ has one zero on $\mathcal{F}_0$ for $r \in (-1, 0)$.

We now consider the case when $r < -1$. We restrict our attention to the properties of $\Delta_r(\tau)$ on the left-side boundary $\mathcal{F}_{-1/2}$ of the fundamental domain. Considering limits similar to (4.1) and (4.2), we find that

$$\lim_{y \to \sqrt{3}/2^+} \Delta_r(-1/2 + iy) = rE_4^3(e^{2\pi i/3}) + E_6^2(e^{2\pi i/3}) = E_6^2(e^{2\pi i/3}), \tag{4.3}$$

and

$$\lim_{y \to +\infty} \Delta_r(-1/2 + iy) = rE_4^3(i\infty) + E_6^2(i\infty) = r + 1, \tag{4.4}$$

using the fact that $E_4(e^{2\pi i/3}) = 0$ from Theorem 2.12 and $E_k(i\infty) = 1$ for all even $k \geq 4$. Lemma 4.3, (4.3) and (4.4) give us

$$\Delta_r(e^{2\pi i/3}) > 0 \quad \text{and} \quad \Delta(i\infty) < 0.$$

Again, the intermediate value theorem guarantees that $\Delta_r(\tau)$ has at least one zero on $\mathcal{F}_{-1/2}$ whenever $r < -1$. Since $\Delta_r(\tau)$ has at most one zero in the fundamental domain, $\Delta_r(\tau)$ has one zero on $\mathcal{F}_{-1/2}$ for $r < 1$.

We now consider the last case when $r > 0$. In this case, we want to show that $\Delta_r(\tau)$ has one zero on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$. Instead of locating the zero of $\Delta_r(\tau)$ directly, we will locate the zero of a related function

$$F_{\Delta_r}(\theta) := e^{i6\theta}\Delta_r(e^{i\theta}).$$

Since $e^{i6\theta}$ never vanishes, the zeros of $F_{\Delta_r}(\theta)$ on the closed interval $[\pi/2, 2\pi/3]$ will correspond bijectively to the zeros of $\Delta_r(e^{i\theta})$ on the arc $\mathcal{A}$. Moreover, since $\Delta_r(\tau)$ has real Fourier coefficient, we have that the function $F_{\Delta_r}(\theta)$ is real on $[\pi/2, 2\pi/3]$ by Lemma 3.2.

Following the notation $F_k(\theta) = e^{ik\theta/2}E_k(e^{i\theta})$ for $k \geq 4$ used in Section 3.3, we can write $F_{\Delta_r}(\theta)$ as

$$F_{\Delta_r}(\theta) = rF_4^3(\theta) + F_6^2(\theta).$$

Evaluating this function at $\pi/2$ and $2\pi/3$, we get

$$F_{\Delta_r}\left(\frac{\pi}{2}\right) = r\left(e^{i\pi}E_4(i)\right)^3 + \left(e^{i2\pi/3}E_6(i)\right)^2 = -rE_4^3(i),$$

and

$$F_{\Delta_r}\left(\frac{2\pi}{3}\right) = r\left(e^{i2\pi/3}E_4(e^{2\pi i/3})\right)^3 + \left(e^{i2\pi}E_6(e^{2\pi i/3})\right)^2 = E_6^2(e^{2\pi i/3})$$

using the fact that $E_6(i) = E_4(e^{2\pi i/3}) = 0$ from Theorem 2.12.

By Lemma 4.3, we obtain

$$F_{\Delta_r}\left(\frac{\pi}{2}\right) < 0 \quad \text{and} \quad F_{\Delta_r}\left(\frac{2\pi}{3}\right) > 0.$$

Therefore, the function $F_{\Delta_r}(\theta)$ and hence $\Delta_r(e^{i\theta})$ has at least one zero on the open interval $(\pi/2, 2\pi/3)$ by the intermediate value theorem. Since $\Delta_r(\tau)$ has at most one zero in the fundamental domain, $\Delta_r(\tau)$ has one zero on the arc $\mathcal{A}$ for $r > 0$.

□

4.2 Zeros of $E_{2k}(\tau)E_{10k}(\tau) + E_{4k}(\tau)E_{8k}(\tau)$

4.2.1 Statement of main result

In this section, we let $k \geq 4$ be any positive integer and define the following modular form of weight $12k$:

$$\Phi_k(\tau) := E_{2k}(\tau)E_{10k}(\tau) + E_{4k}(\tau)E_{8k}(\tau).$$

We now state our main result of this section.

Theorem 4.4. For any integer $k \geq 4$, all zeros of $E_{2k}(\tau)E_{10k}(\tau) + E_{4k}(\tau)E_{8k}(\tau)$ in the fundamental domain $\mathcal{F}$ are located on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$.

Let us first discuss how many zeros $\Phi_k(\tau)$ can have in the fundamental domain. By the valence formula (2.11), and the fact that $\Phi_k(\tau)$ is holomorphic on $\mathbb{H}$ and never vanishes at $i\infty$ (which can be checked form its q-expansion), $\Phi_k(\tau)$ satisfies

$$k = \frac{12k}{12} = \frac{1}{2}\nu_i(\Phi_k) + \frac{1}{3}\nu_{e^{2\pi i/3}}(\Phi_k) + \sum_{\tau \in \mathcal{F}\setminus\{i,\rho\}} \nu_\tau(\Phi_k),$$

which means that Φ_k can have at most k zeros in the fundamental domain.

By the observation of Gekeler in [10], we have the following proposition which determines the value of $\nu_{e^{2\pi i/3}}(\Phi_k)$.

Proposition 4.5 (Gekeler)**.** Let $k \geq 4$ be an even integer. $E_k(i) = 0$ if and only if $k \not\equiv 0 \pmod 4$ and $E_k(e^{2\pi i/3}) = 0$ if and only if $k \not\equiv 0 \pmod 3$.

It is clear that if $k \equiv 1 \pmod 3$, then $2k \equiv 2 \pmod 3$ and $10k \equiv 1 \pmod 3$. From this, Proposition 4.5, and the valence formula (2.11), $E_{2k}(\tau)$ and $E_{10k}(\tau)$ have zero of order 2 and 1 at $\tau = e^{2\pi i/3}$ which gives us that the product $E_{2k}(\tau)E_{10k}(\tau)$ has a zero of order 3 at $\tau = e^{2\pi i/3}$. Similarly, if $k \equiv 2 \pmod 3$, using the same reasoning, $E_{2k}(\tau)E_{10k}(\tau)$ also has a zero of order 3 at $\tau = e^{2\pi i/3}$. Therefore, $E_{2k}(\tau)E_{10k}(\tau)$ has a zero of order 3 at $\tau = e^{2\pi i/3}$ when $k \not\equiv 0 \pmod 3$. In the same manner we can see that, if $k \not\equiv 0 \pmod 3$, then $E_{4k}(\tau)E_{8k}(\tau)$ has a zero of order 3 at $\tau = e^{2\pi i/3}$.

The above argument implies that for $k \not\equiv 0 \pmod 3$, then $\Phi_k(\tau) = E_{2k}(\tau)E_{10k}(\tau) + E_{4k}(\tau)E_{8k}(\tau)$ has a zero of order 3 at $z = e^{2\pi i/3}$ and has at most $k - 1$ zeros in $\mathcal{F}\setminus\{e^{2\pi i/3}\}$.

By the argument of F.K.C. Rankin and Swinnerton-Dyer from Chapter 3 and the above argument on the zeros of $\Phi_k(\tau)$, proving that all k zeros of $\Phi_k(\tau)$ lie on the arc $\mathcal{A}$ is equivalent to proving that the function

$$F_{\Phi_k}(\theta) := e^{i6k\theta}\Phi_k(e^{i\theta})$$

has k zeros when $k \equiv 0 \pmod 3$ and $k-1$ zeros when $k \not\equiv 0 \pmod 3$ in the open interval $(\pi/2, 2\pi/3)$.

4.2.2 Extraction of main and remainder terms

We start with writing $F_{\Phi_k}(\theta)$ as a sum of main and remainder terms and then we give an upper bound of the remainder term.

Proposition 4.6. For any integer $k \geq 4$ and for any $\theta \in [\pi/2, 2\pi/3]$,

$$F_{\Phi_k}(\theta) = M_{\Phi_k}(\theta) + R_{\Phi_k}(\theta)$$

where $M_{\Phi_k} = M_{2k}M_{10k} + M_{4k}M_{8k}$ where M_k's are defined in (3.6) and $|R_{\Phi_k}| \leq 0.69094$.

Proof. We can write

$$F_{\Phi_k}(\theta) = e^{i6k\theta}(E_{2k}E_{10k} + E_{4k}E_{8k})(e^{i\theta}) = (F_{2k}F_{10k} + F_{4k}F_{8k})(\theta),$$

where F_k is defined as in (3.4). Expanding the right hand side using (3.5),

$$\begin{aligned} F_{\Phi_k}(\theta) &= M_{2k}(\theta)M_{10k}(\theta) + M_{4k}(\theta)M_{8k}(\theta) + M_{2k}(\theta)R_{10k}(\theta) + M_{10k}(\theta)R_{2k}(\theta) \\ &= M_{4k}(\theta)R_{8k}(\theta) + M_{8k}(\theta)R_{4k} + R_{2k}(\theta)R_{10k}(\theta) + R_{4k}(\theta)R_{8k}(\theta). \end{aligned}$$

Let $M_{\Phi_k} = M_{2k}M_{10k} + M_{4k}M_{8k}$, and let $R_{\Phi_k} = M_{2k}R_{10k} + M_{10k}R_{2k} + M_{4k}R_{8k} + M_{8k}R_{4k} + R_{2k}R_{10k} + R_{4k}R_{8k}$. To give a bound of $|R_{\Phi_k}(\theta)|$, the triangle inequality and the definition of M_k in (3.6) yield $|M_k(\theta)| \leq |2\cos(k\theta/2)| + |2\cos(\theta/2)|^{-k} \leq 3$ on $[\pi/2, 2\pi/3]$, and hence

$$|R_{\Phi_k}| \leq 3|R_{2k}| + 3|R_{10k}| + 3|R_{4k}| + 3|R_{8k}| + |R_{2k}||R_{10k}| + |R_{4k}||R_{8k}|.$$

Recall that by Proposition 3.3, $|R_k|$ is monotonically decreasing as a function in k so the term $|R_{\Phi_k}|$ is bounded above by a sum of decreasing functions. Thus, $|R_{\Phi_k}|$ is also decreasing in k. Evaluating the right hand side at $k = 4$, we get an upper bound

$$|R_{\Phi_k}| \leq 3|R_8| + 3|R_{40}| + 3|R_{16}| + 3|R_{32}| + |R_8||R_{40}| + |R_{16}||R_{32}| \leq 0.69094$$

for all $k \geq 4$. □

4.2.3 Definition of sample points

Let $k \geq 4$ be an even integer, we define the truncated interval

$$I_k := \left[\frac{\pi}{2}, \frac{2\pi}{3} - \frac{\pi}{6k}\right].$$

and define

$$\theta_m := \theta_{12k}(m) = \frac{m\pi}{6k},$$

where m ranges over integers so that $\theta_m \in I_k$. Observe that

$$\theta_m \in I_k \Leftrightarrow m \in [3k, 4k-1].$$

Then the number of θ_m values in I_k equals the number of integers in the closed interval $[3k, 4k-1]$ which equals k.

Our next goal for the rest of this chapter is to show that the function $F_{\Phi_k}(\theta)$ has at least $k-1$ zeros in the closed interval I_k. Since there are k of θ_m values in I_k as explained above, we want to show that $F_{\Phi_k}(\theta_m)$ is strictly positive or negative depending on the parity of the integer $m \in [3k, 4k-1]$. Since by Proposition 4.6, $F_{\Phi_k}(\theta) = M_{\Phi_k}(\theta) + R_{\Phi_k}(\theta)$, we aim to show that for any integer $m \in [3k, 4k-1]$, a lower bound of $(-1)^m M_{\Phi_k}(\theta_m)$ is greater than the upper bound of $|R_{\Phi_k}|$ given in Proposition 4.6.

Let us first simplify $M_{\Phi_k}(\theta_m)$. With the definition of $M_k(\theta)$ given in (3.6), for $n \in \{2, 4, 8, 10\}$, we find that

$$M_{nk}(\theta_m) = 2\cos\left(\frac{nk\theta_m}{2}\right) + \left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-nk} = 2\cos\left(\frac{nm\pi}{12}\right) + \left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-nk}.$$

This implies that

$$\begin{aligned} M_{2k}(\theta_m)M_{10k}(\theta_m) &= \left(2\cos\left(\frac{m\pi}{6}\right) + \left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-6k}\right)\left(2\cos\left(\frac{5m\pi}{6}\right) + \left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k}\right) \\ &= 2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} + 2\cos\left(\frac{5m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-2k} \\ &\quad + 4\cos\left(\frac{m\pi}{6}\right)\cos\left(\frac{5m\pi}{6}\right) + \left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-12k}. \end{aligned} \tag{4.5}$$

Similarly, we have that

$$\begin{aligned} M_{4k}(\theta_m)M_{8k}(\theta_m) &= 2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} + 2\cos\left(\frac{2m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-4k} \\ &\quad + 4\cos\left(\frac{m\pi}{3}\right)\cos\left(\frac{2m\pi}{3}\right) + \left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-12k}. \end{aligned} \tag{4.6}$$

By (4.5) and (4.6) and Proposition 4.6, $M_{\Phi_k}(\theta_m)$ is simplified to

$$\begin{aligned} M_{\Phi_k}(\theta_m) &= M_{2k}(\theta_m)M_{10k}(\theta_m) + M_{4k}(\theta_m)M_{8k}(\theta_m) \\ &= 2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} + 2\cos\left(\frac{5m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-2k} \\ &\quad + 2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} + 2\cos\left(\frac{2m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-4k} \\ &\quad + 4\cos\left(\frac{m\pi}{6}\right)\cos\left(\frac{5m\pi}{6}\right) + 4\cos\left(\frac{m\pi}{3}\right)\cos\left(\frac{2m\pi}{3}\right) + 2\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-12k}. \end{aligned}$$

To help us approximate the lower bound of $(-1)^m M_{\Phi_k}(\theta_m)$, we express $M_{\Phi_k}(\theta_m)$ as

$$M_{\Phi_k}(\theta_m) = P_{\Phi_k}(\theta_m) + Q_{\Phi_k}(\theta_m),$$

where $P_{\Phi_k}(\theta_m)$ and $Q_{\Phi_k}(\theta_m)$ are defined by

$$\begin{aligned} P_{\Phi_k}(\theta_m) &:= 2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} + 2\cos\left(\frac{5m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-2k} \\ &\quad + 2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} + 2\cos\left(\frac{2m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-4k}, \end{aligned} \tag{4.7}$$

and

$$Q_{\Phi_k}(\theta_m) := 4\cos\left(\frac{m\pi}{6}\right)\cos\left(\frac{5m\pi}{6}\right) + 4\cos\left(\frac{m\pi}{3}\right)\cos\left(\frac{2m\pi}{3}\right) + 2\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-12k}. \tag{4.8}$$

4.2.4 Estimates for trigonometric functions

Since we can bound $(-1)^m M_{\Phi_k}(\theta_m)$ by bounding $P_{\Phi_k}(\theta_m)$ and $Q_{\Phi_k}(\theta_m)$, we will first start by giving a lower bound of $P_{\Phi_k}(\theta_m)$. In this subsection, we introduce lemmas that will be used to get a bound for $P_{\Phi_k}(\theta_m)$.

Lemma 4.7. For any integer $k \geq 4$, and for any integer $m \in [3k, 4k-1]$,

$$2(-1)^m\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} \geq \begin{cases} -0.00077 & \text{if } m \text{ is even,} \\ -0.02558 & \text{otherwise.} \end{cases}$$

Proof. Let $k \geq 4$ be integer and first assume that $m \in [3k, 4k-1]$ is even. By considering congruence classes modulo 12, a direct calculation shows that

$$2\cos\left(\frac{m\pi}{6}\right) = \begin{cases} 2 & \text{if } m \equiv 0 \pmod{12}, \\ 1 & \text{if } m \equiv 2, 10 \pmod{12}, \\ -1 & \text{if } m \equiv 4, 8 \pmod{12}, \\ -2 & \text{if } m \equiv 6 \pmod{12}. \end{cases} \tag{4.9}$$

Since $2\cos(m\pi/6)$ takes minimum value -2 when $m \equiv 6 \pmod{12}$ and $(2\cos(m\pi/12k))^{-10k}$ is positive and monotonically increasing on $[3k, 4k-1]$ by Lemma 3.6,

$$2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} \geq -2\left(2\cos\left(\frac{m_{even}\pi}{12k}\right)\right)^{-10k} \tag{4.10}$$

where m_{even} is the largest even integer in $[3k, 4k-1]$. Plugging in $m_{even} = 4k-2$ into (4.10) gives us

$$2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} \geq -2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{6k}\right)\right)^{-10k}. \tag{4.11}$$

Lemma 3.7 says that the right hand side of (4.11) is monotonically increasing on k. Thus, for $k \geq 4$,

$$2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} \geq -2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{6(4)}\right)\right)^{-10(4)} \geq -0.00077. \tag{4.12}$$

Now suppose $m \in [3k, 4k-1]$ is odd. By a direct calculation, we find that

$$2\cos\left(\frac{m\pi}{6}\right) = \begin{cases} \sqrt{3} & \text{if } m \equiv 1, 11 \pmod{12}, \\ 0 & \text{if } m \equiv 3, 9 \pmod{12}, \\ -\sqrt{3} & \text{if } m \equiv 5, 7 \pmod{12}. \end{cases} \tag{4.13}$$

Approaching with the similar reasoning as the proof when m is even, Lemma 3.6 and (4.13) give us

$$2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} \leq \sqrt{3}\left(2\cos\left(\frac{m_{odd}\pi}{12k}\right)\right)^{-10k} \tag{4.14}$$

where m_{odd} is the largest odd integer in $[3k, 4k-1]$. Inserting $m_{odd} = 4k-1$ in (4.14), we get

$$2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} \leq \sqrt{3}\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12k}\right)\right)^{-10k}. \tag{4.15}$$

By Lemma 3.7, the right hand side of (4.15) is monotonically decreasing on k. Therefore, for $k \geq 4$,

$$2\cos\left(\frac{m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-10k} \leq \sqrt{3}\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12(4)}\right)\right)^{-10(4)} \leq 0.02558. \tag{4.16}$$

By (4.12) and (4.16), we complete the proof. □

Now we turn to bounding $2(-1)^m\cos(5m\pi/6)(2\cos(m\pi/12k))^{-2k}$. The proof of the following lemma will proceed similarly to the proof of Lemma 4.7.

Lemma 4.8. For any integer $k \geq 4$, and for any integer $m \in [3k, 4k-1]$,

$$2(-1)^m \cos\left(\frac{5m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-2k} \geq \begin{cases} -0.41421 & \text{if } m \text{ is even,} \\ -0.74544 & \text{otherwise.} \end{cases}$$

Proof. By the difference formula from trigonometric identities, we note that

$$2\cos\left(\frac{5m\pi}{6}\right) = 2\cos\left(m\pi - \frac{m\pi}{6}\right) = 2(-1)^m \cos\left(\frac{m\pi}{6}\right).$$

From this identity, (4.9) and (4.11) in the proof of Lemma 4.7 and Lemmas 3.6 and 3.7, we have that for any integer $k \geq 4$, if $m \in [3k, 4k-1]$ is even,

$$2\cos\left(\frac{5m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-2k} \geq -2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{6(4)}\right)\right)^{-2(4)} \geq -0.41421. \tag{4.17}$$

On the other hand if $mm \in [3k, 4k-1]$ is odd, then (4.13), (4.15), Lemmas 3.6 and 3.7 give us that

$$2\cos\left(\frac{5m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-2k} \leq \sqrt{3}\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12(4)}\right)\right)^{-2(4)} \leq 0.74544. \tag{4.18}$$

By (4.17) and (4.18), we finish the proof. □

Lemma 4.9. For any integer $k \geq 4$, and for any integer $m \in [3k, 4k-1]$,

$$2(-1)^m \cos\left(\frac{2m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} \geq \begin{cases} -0.00184 & \text{if } m \text{ is even} \\ -0.03431 & \text{otherwise.} \end{cases}$$

Proof. Let $k \geq 4$ be integer and let $m \in [3k, 4k-1]$ be even. By considering congruence classes modulo 6, a direct calculation shows that

$$2\cos\left(\frac{m\pi}{3}\right) = \begin{cases} 2 & \text{if } m \equiv 0 \pmod 6, \\ -1 & \text{if } m \equiv 2, 4 \pmod 6. \end{cases} \tag{4.19}$$

So the function $2\cos(m\pi/3)$ takes minimum value -1. From this and Lemma 3.6, $(2\cos(m\pi/12k))^{-8k}$ is positive and monotonically increasing on $[3k, 4k-1]$, we obtain

$$2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} \geq -\left(2\cos\left(\frac{m_{even}\pi}{12k}\right)\right)^{-8k} \tag{4.20}$$

where m_{even} is the largest even integer in $[3k, 4k-1]$. Plugging in $m_{even} = 4k-2$ into (4.20) gives us

$$2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} \geq -\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{6k}\right)\right)^{-8k}. \tag{4.21}$$

Lemma 3.7 says that the right hand side of (4.21) is monotonically increasing on k. Thus, for $k \geq 4$,

$$2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} \geq -\left(2\cos\left(\frac{\pi}{3}-\frac{\pi}{6(4)}\right)\right)^{-8(4)} \geq -0.00184. \tag{4.22}$$

Now suppose $m \in [3k, 4k-1]$ is odd. We find that $2\cos(m\pi/3)$ has a maximum value 1 at $m \equiv 1, 5 \pmod 6$ since

$$2\cos\left(\frac{m\pi}{3}\right) = \begin{cases} 1 & \text{if } m \equiv 1, 5 \pmod 6, \\ -2 & \text{if } m \equiv 3 \pmod 6. \end{cases} \tag{4.23}$$

From this and Lemma 3.6, we have that

$$2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} \leq \left(2\cos\left(\frac{m_{odd}\pi}{12k}\right)\right)^{-8k} \tag{4.24}$$

where m_{odd} is the largest odd integer in $[3k, 4k-1]$. Inserting $m_{odd} = 4k-1$ in (4.24), we get

$$2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} \leq \left(2\cos\left(\frac{\pi}{3}-\frac{\pi}{12k}\right)\right)^{-8k}. \tag{4.25}$$

By Lemma 3.7, the right hand side of (4.25) is monotonically decreasing on k. Therefore, for $k \geq 4$,

$$2\cos\left(\frac{m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-8k} \leq \left(2\cos\left(\frac{\pi}{3}-\frac{\pi}{12(4)}\right)\right)^{-8(4)} \leq 0.03431. \tag{4.26}$$

By (4.22) and (4.26), we complete the proof. □

The proof of the following lemma is based on the proof of Lemmas 4.8 and 4.9.

Lemma 4.10. For an integer $k \geq 4$, and for integers $m \in [3k, 4k-1]$,

$$2(-1)^m\cos\left(\frac{4m\pi}{6}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-4k} \geq \begin{cases} -0.04290 & \text{if } m \text{ is even} \\ -0.37045 & \text{otherwise.} \end{cases}$$

Proof. By the difference formula from trigonometric identities, we note that

$$2\cos\left(\frac{2m\pi}{3}\right) = 2\cos\left(m\pi-\frac{m\pi}{3}\right) = 2(-1)^m\cos\left(\frac{m\pi}{3}\right).$$

From this identity, (4.19), (4.21) in the proof of Lemma 4.9 and Lemma 3.6 and 3.7, we have that for integers $k \geq 4$, if $m \in [3k, 4k-1]$ is even,

$$2\cos\left(\frac{2m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-4k} \geq -\left(2\cos\left(\frac{\pi}{3}-\frac{\pi}{6(4)}\right)\right)^{-4(4)} \geq -0.04290. \tag{4.27}$$

On the other hand if $m \in [3k, 4k-1]$ is odd, then (4.23), (4.25) and Lemma 3.6 and 3.7 give us

$$2\cos\left(\frac{2m\pi}{3}\right)\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-4k} \leq 2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12(4)}\right)\right)^{-4(4)} \leq 0.37045. \tag{4.28}$$

By (4.27) and (4.28), we finish the proof. □

4.2.5 Bounding the main term

Combining Lemmas 4.7, 4.8, 4.9, and 4.10, we obtain the following bound on $(-1)^m P_{\Phi_k}(\theta_m)$.

Proposition 4.11. For any integer $k \geq 4$ and for any integer $m \in [3k, 4k-1]$,

$$(-1)^m P_{\Phi_k}(\theta_m) \geq \begin{cases} -0.45972 & \text{if } m \text{ is even,} \\ -1.17578 & \text{otherwise.} \end{cases}$$

Now it remains to bound $(-1)^m Q_{\Phi_k}(\theta_m)$. We will prove the following result.

Proposition 4.12. For any integer $k \geq 4$ and for any integer $m \in [3k, 4k-1]$,

$$(-1)^m Q_{\Phi_k}(\theta_m) \geq \begin{cases} 2 & \text{if } m \text{ is even,} \\ 3.98729 & \text{otherwise.} \end{cases}$$

Proof. Let $k \geq 4$ be integer and let $m \in [3k, 4k-1]$ be integer. By (4.7), recall that the function $Q_{\Phi_k}(\theta_m)$ is given by

$$Q_{\Phi_k}(\theta_m) = 4\cos\left(\frac{m\pi}{6}\right)\cos\left(\frac{5m\pi}{6}\right) + 4\cos\left(\frac{m\pi}{3}\right)\cos\left(\frac{2m\pi}{3}\right) + 2\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-12k}.$$

Let us first simplify the first two terms of $Q_{\Phi_k}(\theta_m)$. By the sum and difference formulas from trigonometric identities, we find that

$$\cos(m\pi) = \cos\left(\frac{m\pi}{6} + \frac{5m\pi}{6}\right) = \cos\left(\frac{m\pi}{6}\right)\cos\left(\frac{5m\pi}{6}\right) - \sin\left(\frac{m\pi}{6}\right)\sin\left(\frac{5m\pi}{6}\right), \tag{4.29}$$

and

$$\cos\left(\frac{2m\pi}{3}\right) = \cos\left(\frac{m\pi}{6} - \frac{5m\pi}{6}\right) = \cos\left(\frac{m\pi}{6}\right)\cos\left(\frac{5m\pi}{6}\right) + \sin\left(\frac{m\pi}{6}\right)\sin\left(\frac{5m\pi}{6}\right). \tag{4.30}$$

Combining (4.29) with (4.30) , we get

$$\cos(m\pi) + \cos\left(\frac{2m\pi}{3}\right) = 2\cos\left(\frac{m\pi}{6}\right)\cos\left(\frac{5m\pi}{6}\right). \tag{4.31}$$

Similarly, one can also show by using the same argument that

$$\cos(m\pi) + \cos\left(\frac{m\pi}{3}\right) = 2\cos\left(\frac{m\pi}{3}\right)\cos\left(\frac{2m\pi}{3}\right). \tag{4.32}$$

Inserting (4.31) and (4.31) into the definition of $Q_{\Phi_k}(\theta_m)$, we derive

$$Q_{\Phi_k}(\theta_m) = 4\cos(m\pi) + 2\cos\left(\frac{m\pi}{3}\right) + 2\cos\left(\frac{2m\pi}{3}\right) + 2\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-12k}.$$

Now we are ready to bound $(-1)^m Q_{\Phi_k}(\theta_m)$. By a direct calculation, we first note that

$$4\cos(m\pi) + 2\cos\left(\frac{m\pi}{3}\right) + 2\cos\left(\frac{2m\pi}{3}\right) = \begin{cases} 2, 8 & \text{if } m \text{ is even,} \\ -4 & \text{otherwise.} \end{cases}$$

Combining this with the fact that $(2\cos(m\pi/12k))^{-12k}$ is positive on $[3k, 4k-1]$ by Lemma 3.6, we find that if m is even, then

$$Q_{\Phi_k}(\theta_m) \geq 2 + 2\left(2\cos\left(\frac{m\pi}{12k}\right)\right)^{-12k} > 2. \tag{4.33}$$

Now suppose m is odd. Since $(2\cos(m\pi/12k))^{-12k}$ is positive and increasing on $[3k, 4k-1]$ by Lemma 3.6, we find that

$$Q_{\Phi_k}(\theta_m) \leq -4 + 2\left(2\cos\left(\frac{m_{odd}\pi}{12k}\right)\right)^{-12k}, \tag{4.34}$$

where m_{odd} is the largest odd integer in $[3k, 4k-1]$. Inserting $m_{odd} = 4k-1$ in (4.34), we get

$$Q_{\Phi_k}(\theta_m) \leq -4 + 2\left(2\cos\left(\frac{(4k-1)\pi}{12k}\right)\right)^{-12k} = -4 + 2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12k}\right)\right)^{-12k}.$$

By Lemma 3.7, the right hand side is decreasing as a function in k. Therefore, for $k \geq 4$,

$$Q_{\Phi_k}(\theta_m) \leq -4 + 2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12(4)}\right)\right)^{-12(4)} \leq -3.98729. \tag{4.35}$$

By (4.33) and (4.35), we finish the proof. □

By Propositions 4.11 and 4.11 , we obtain the lower bound of $(-1)^m M_{\Phi_k}(\theta_m)$

Proposition 4.13. For any integer $k \geq 4$ and for any integer $m \in [3k, 4k-1]$,

$$(-1)^m M_{\Phi_k}(\theta_m) \geq \begin{cases} 1.54028 & \text{if } m \text{ is even,} \\ 2.81151 & \text{otherwise.} \end{cases}$$

4.2.6 Extra zero when $k \equiv 0 \pmod 3$

In the previous subsection, we have showed that for any integer $k \geq 4$, the function $F_{\Phi_k}(\theta_m)$ is strictly positive or negative for all integers m so that $\theta_m \in [\pi/2, 2\pi/3 - \pi/6k]$. We will show in Subsection 4.1.3 that this yields many zeros of $F_{\Phi_k}(\theta)$ in $[\pi/2, 2\pi/3]$.

Our goal of this subsection is to show that $F_{\Phi_k}(\theta)$ has at least one additional zero outside of $[\pi/2, 2\pi/3 - \pi/6k]$ when $k \equiv 0 \pmod 3$. To prove this, it suffices to show that $F_{\Phi_k}(\theta)$ changes sign on $(\pi/2, 2\pi/3 - \pi/6k)$.

Proposition 4.14. For any integer $k \geq 4$ with $k \equiv 0 \pmod 3$, $F_{\Phi_k}(\theta)$ has at least one zero on $(2\pi/3 - \pi/6k, 2\pi/3)$.

Proof. Let $k \geq 4$ be any integer with $k \equiv 0 \pmod 3$. Write $k = 3x$ for some $x \in \mathbb{Z}$. With the definition of $M_k(\theta)$ given in (3.6), we find that for $n = 2, 4, 8$ and 10,

$$M_{nk}\left(\frac{2\pi}{3}\right) = 2\cos\left(\frac{nk\pi}{3}\right) + \left(2\cos\left(\frac{\pi}{3}\right)\right)^{-3nx} = 2\cos(nx\pi) + 1 = 3.$$

By Proposition 4.6,

$$\begin{aligned} M_{\Phi_k}\left(\frac{2\pi}{3}\right) &= M_{2k}\left(\frac{2\pi}{3}\right)M_{10k}\left(\frac{2\pi}{3}\right) + M_{4k}\left(\frac{2\pi}{3}\right)M_{8k}\left(\frac{2\pi}{3}\right) \\ &= 3^2 + 3^2 = 18 \end{aligned}$$

which is larger than $-|R_{\Phi_k}|$ given in Proposition 3.3. Hence, $F_{\Phi_k}(2\pi/3) > 0$.

Next, we want to show that $F_{\Phi_k}(2\pi/3 - \pi/6k) < 0$. By the definition of the truncated interval I_k given in Subsection 4.2.3, $2\pi/3 - \pi/6k$ is the right end point of I_k. We start by approximating the value of $M_{2k}(2\pi/3 - \pi/6k)$. With the definition of $M_k(\theta)$ given in (3.6), and the fact that the value of $(2\cos(\pi/3 - \pi/12k))^{-nk}$ lies on $(0, 1)$ for $n = 2, 4, 8$ and 10,

$$\begin{aligned} M_{2k}\left(\frac{2\pi}{3} - \frac{\pi}{6k}\right) &= 2\cos\left(\frac{2k\pi}{3} - \frac{\pi}{6}\right) + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12k}\right)\right)^{-2k} \\ &= 2\cos\left(2x\pi - \frac{\pi}{6}\right) + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12k}\right)\right)^{-2k} \\ &= 2\cos\left(\frac{\pi}{6}\right) + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{12k}\right)\right)^{-2k} \\ &\in (\sqrt{3}, \sqrt{3} + 1). \end{aligned}$$

Using the same reasoning, we also obtain

$$M_{10k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)=2\cos\left(\frac{5\pi}{6}\right)+\left(2\cos\left(\frac{\pi}{3}-\frac{\pi}{12k}\right)\right)^{-10k}\in(-\sqrt{3},-\sqrt{3}+1),$$
$$M_{4k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)=2\cos\left(\frac{\pi}{3}\right)+\left(2\cos\left(\frac{\pi}{3}-\frac{\pi}{12k}\right)\right)^{-4k}\in(1,1+1),$$
$$M_{8k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)=2\cos\left(\frac{2\pi}{3}\right)+\left(2\cos\left(\frac{\pi}{3}-\frac{\pi}{12k}\right)\right)^{-8k}\in(-1,-1+1).$$

Thus, we find that

$$M_{2k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)M_{10k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)<-2 \quad\text{and}\quad M_{4k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)M_{8k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)<0. \qquad (4.36)$$

From (4.36) and Proposition 4.6, we have that

$$\begin{aligned}M_{\Phi_k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)&=M_{2k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)M_{10k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)+M_{4k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)M_{8k}\left(\frac{2\pi}{3}-\frac{\pi}{6k}\right)\\&\le -2.\end{aligned}$$

which is smaller than $|R_{\Phi_k}|$ given in Proposition 3.3. Therefore, $F_{\Phi_k}(2\pi/3-\pi/6k)<0$.

By the intermediate value theorem, the function $F_{\Phi_k}(\theta)$ must have at least one zero in the open interval $(2\pi/3-\pi/6k, 2\pi/3)$ when $k\equiv 0 \pmod 3$. □

4.2.7 Proof of Theorem 4.4

We are now able to prove our main Theorem 4.4.

Proof. By Proposition 4.6, the function $F_{\Phi_k}(\theta)=e^{i6k\theta}\Phi_k(e^{i\theta})$ can be written as

$$F_{\Phi_k}(\theta)=M_{\Phi_k}(\theta)+R_{\Phi_k}(\theta).$$

where Proposition 4.13 further gives us that

$$(-1)^m M_{\Phi_k}(\theta_m)>|R_{\Phi_k}|$$

for all integer $k\ge 4$ and for all $\theta_m=m\pi/6k\in I_k=[\pi/2, 2\pi/3-\pi/6k]$ defined in Subsection 4.2.3.

Then the intermediate value theorem guarantees that a minimum number of the zeros of $F_{\Phi_k}(\theta)$ in the open interval $(\pi/2, 2\pi/3-\pi/6k)$ equals the number of θ_m in I_k minus 1. Since the number of θ_m in I_k equals the number of integers in $[3k, 4k-1]$ which clearly has

k integers as described in Section 4.2.3, the function $F_{\Phi_k}(\theta)$ and hence the modular form $\Phi_k(e^{i\theta})$ has at least $k-1$ zeros on the open interval $(\pi/2, 2\pi/3 - \pi/6k)$.

Recall that we discussed in Subsection 4.2.1 that the modular form $\Phi_k(\tau)$ has at most k zeros in the fundamental domain and only has a zero of order 3 at $\tau = e^{2\pi i/3}$ when $k \not\equiv 0 \pmod 3$. Since the above argument gives us at least $k-1$ zeros on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$, we have showed that all zeros of Φ_k for $k \not\equiv 0 \pmod 3$ are located on the boundary arc $\mathcal{A}$ of the fundamental domain. For $k \equiv 0 \pmod 3$, Subsection 4.2.6 gives us the additional zero required from our discussion at the beginning of Subsection 4.2.1. Therefore this completes the proof of Theorem 4.4. □

5 Two-parameters families of modular forms

We are interested to know whether we can extend the argument of F.K.C. Rankin and Swinnerton-Dyer explained in 3 to study the zeros of certain modular forms of two-parameters. We begin with a result on the location of the zeros of the modular form of weight $k+l$ defined by $E_k(\tau)E_l(\tau)+E_{k+l}(\tau)$. In contrast to the results of Reitzes et al. in [20] on the zeros of the cusp form $E_k(\tau)E_l(\tau)-E_{k+l}(\tau)$, we show that all zeros of $E_k(\tau)E_l(\tau)+E_{k+l}(\tau)$ lie on the lower boundary arc of the fundamental domain. Then, we turn to investigating the zero location of the weight nk modular form defined by $E_k^n(\tau) \pm E_{nk}(\tau)$. For $n = 2, 3$, we show that all zeros of $E_k^n(\tau) + E_{nk}(\tau)$ are also located on the lower boundary arc of the fundamental domain. However, this result can not be extended to the higher values $n \geq 4$. Finally, we end this chapter by giving a conjecture on the location of the zeros of the cusp form $E_k^3(\tau) - E_{3k}(\tau)$.

5.1 Zeros of $E_k(\tau)E_l(\tau) + E_{k+l}(\tau)$

5.1.1 Statement of main result

Let $k, l \geq 10$ be even integers, and define

$$\Psi_{k,l}(\tau) := E_k(\tau)E_l(\tau) + E_{k+l}(\tau).$$

By symmetry, we may assume that $k > l$ (the case $k = l$ will be considered in Section 5.2). We note this is a modular form of weight $k + l$ defined analogously to the cusp form

$$E_k(\tau)E_l(\tau) - E_{k+l}(\tau)$$

which appeared in the work of Reitzes et al. in [20]. In their paper, they prove that if k and l are sufficiently large, then all zeros of the weight $k+l$ cusp form $E_k(\tau)E_l(\tau)-E_{k+l}(\tau)$ lie on the boundary of the standard fundamental domain.

Theorem 5.1 (Reitzes, Vulakh and Young)**.** For any even integers $k > l \geq 14$, all zeros of $E_k(\tau)E_l(\tau) - E_{k+l}(\tau)$ in the fundamental domain are either on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$ or the left side boundary $\mathcal{F}_{-1/2} = \{-0.5 + iy : y \geq \sqrt{3}/2\}$.

In contrast to their result, we prove the following theorem.

Theorem 5.2. For any even integers $k > l \geq 10$, all zeros of $E_k(\tau)E_l(\tau) + E_{k+l}(\tau)$ in the fundamental domain lie on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$.

Let $k > l \geq 10$ be even integers, and write $k + l = 12n + s$ with $s \in \{4, 6, 8, 10, 0, 14\}$. Given the Fourier series expansion of Eisenstein series in (2.3), $\Psi_{k,l}(\tau)$ has the constant coefficient equals 2 which implies that $\nu_{i\infty}(\Psi_{k,l}) = 0$. Thus , the valence formula for $\Psi_{k,l}(\tau)$ is simplified to

$$n + \frac{s}{12} = \frac{k+l}{12} = \frac{1}{2}\nu_i\left(\Psi_{k,l}\right) + \frac{1}{3}\nu_{e^{2\pi i/3}}\left(\Psi_{k,l}\right) + \sum_{\tau \in \mathcal{F} \setminus \{i, e^{2\pi i/3}\}} \nu_\tau\left(\Psi_{k,l}\right). \tag{5.1}$$

Also the fact that $\Psi_{k,l}(\tau)$ is holomorphic on $\mathbb{H}$ gives us that $\nu_\tau\left(\Psi_{k,l}\right) \in \mathbb{N} \cup \{0\}$ for all $\tau \in \mathcal{F}$. From this and (5.1), $s/12$ must determine the order of zeros at $\tau = i$ and $e^{2\pi i/3}$. Therefore, $\Psi_{k,l}(\tau)$ has a total of n zeros in $\mathcal{F} \setminus \{i, e^{2\pi i/3}\}$.

To study the zeros of $\Psi_{k,l}(\tau)$ on the arc $\mathcal{A}$, we consider the zeros of the related function

$$F_{\Psi_{k,l}}(\theta) := e^{i(k+l)\theta/2}\Psi_{k,l}(e^{i\theta}).$$

By Lemma 3.2, this function is real on the closed interval $[\pi/2, 2\pi/3]$ and clearly shares the same zero set with the function $\Psi_{k,l}(e^{i\theta})$ on that interval.

5.1.2 Extraction of the main and remainder terms

We start with analyzing the function $F_{\Psi_{k,l}}(\theta)$ on $[\pi/2, 2\pi/3]$.

Proposition 5.3. For any even integers $k > l \geq 10$ and for any $\theta \in [\pi/2, 2\pi/3]$,

$$F_{\Psi_{k,l}}(\theta) = M_{\Psi_{k,l}}(\theta) + R_{\Psi_{k,l}}(\theta) \tag{5.2}$$

where $M_{\Psi_{k,l}} = M_k M_l + M_{k+l}$ with M_k's are defined in (3.6) and $|R_{\Psi_{k,l}}| \leq 0.39018$.

Proof. We can write

$$F_{\Psi_{k,l}}(\theta) = e^{i(k+l)\theta/2}(E_k E_l + E_{k+l})(e^{i\theta}) = (F_k F_l + F_{k+l})(\theta),$$

where $F_k(\theta)$ is defined as in (3.4). Expanding the right hand side using (3.5),

$$F_{\Psi_{k,l}}(\theta) = M_k(\theta)M_l(\theta) + M_{k+l}(\theta) + M_k(\theta)R_l(\theta) + M_l(\theta)R_k(\theta) + R_k(\theta)R_l(\theta) + R_{k+l}(\theta).$$

Let $M_{\Psi_{k,l}} = M_kM_l + M_{k+l}$ and let $R_{\Psi_{k,l}} = M_kR_l + M_lR_k + R_kR_l + R_{k+l}$. To bound $|R_{\Psi_{k,l}}|$, the triangle inequality and the definition of $M_k(\theta)$ in (3.6), $|M_k(\theta)| \le |2\cos(k\theta/2)| + |(2\cos(\theta/2))^{-k}| \le 3$ on $[\pi/2, 2\pi/3]$ yield

$$|R_{\Psi_{k,l}}(\theta)| \le 3|R_l(\theta)| + 3|R_k(\theta)| + |R_k(\theta)||R_l(\theta)| + |R_{k+l}(\theta)|$$

By Proposition 3.3, $|R_{\Psi_{k,l}}(\theta)|$ is a decreasing function in terms of k, l since it is bounded above by a sum of decreasing functions. Evaluating the bound at $k = 12$ and $l = 10$, we obtain

$$|R_{\Psi_{k,l}}(\theta)| \le 3|R_{10}| + 3|R_{12}| + |R_{12}||R_{10}| + |R_{22}| < 0.34780$$

for all $k > l \ge 10$. □

5.1.3 Definition of sample points

Let $k > l \ge 10$ be even integers, and define

$$\theta_m := \theta_{k+l}(m) = \frac{2m\pi}{k+l},$$

where m ranges over integers so that $\theta_m \in [\pi/2, 2\pi/3]$. We observe that

$$\theta_m \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right] \quad \Leftrightarrow \quad m \in \left[\frac{k+l}{4}, \frac{k+l}{3}\right].$$

Our goal for the rest of this section is to show that $F_{\Psi_{k,l}}(\theta_m)$ is strictly positive or negative according to the parity of m for all $\theta_m \in [\pi/2, 2\pi/3]$. Since by Proposition 5.3, $F_{\Psi_{k,l}}(\theta) = M_{\Psi_{k,l}}(\theta) + R_{\Psi_{k,l}}(\theta)$, we show that for all $m \in [(k+l)/4, (k+l)/3]$, a lower bound of $(-1)^m M_{\Psi_{k,l}}(\theta_m)$ is greater than the upper bound of $|R_{\Psi_{k,l}}|$ given in Proposition 5.3.

We note that

$$2\cos\left(\frac{(k+l)\theta_m}{2}\right) = 2\cos(m\pi) = 2(-1)^m \tag{5.3}$$

and by the sum and difference trigonometric identities,

$$2\cos\left(\frac{k\theta_m}{2}\right) = 2\cos\left(\frac{(k+l)\theta_m}{2} - \frac{l\theta_m}{2}\right) = 2(-1)^m\cos\left(\frac{l\theta_m}{2}\right). \tag{5.4}$$

With the definition of $M_k(\theta)$ given in (3.6), the identities (5.3) and (5.4) give us

$$M_{k+l}(\theta_m) = 2\cos\left(\frac{(k+l)\theta_m}{2}\right) + \left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-(k+l)} 2(-1)^m + \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-(k+l)} \tag{5.5}$$

and

$$
\begin{aligned}
&M_k(\theta_m)M_l(\theta_m)\\
&\quad= \left(2\cos\left(\frac{k\theta_m}{2}\right) + \left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-k}\right)\left(2\cos\left(\frac{l\theta_m}{2}\right) + \left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-l}\right)\\
&\quad= \left(2(-1)^m\cos\left(\frac{l\theta_m}{2}\right) + \left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-k}\right)\left(2\cos\left(\frac{l\theta_m}{2}\right) + \left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-l}\right)\\
&\quad= 4(-1)^m\cos^2\left(\frac{l\theta_m}{2}\right) + 2\cos\left(\frac{l\theta_m}{2}\right)\left\{\left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-k} + (-1)^m\left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-l}\right\}\\
&\qquad+ \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-(k+l)}\\
&\quad= 4(-1)^m\cos^2\left(\frac{lm\pi}{k+l}\right) + 2\cos\left(\frac{lm\pi}{k+l}\right)\left\{\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-k} + (-1)^m\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l}\right\}\\
&\qquad+ \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-(k+l)}.
\end{aligned}
\tag{5.6}
$$

By Proposition 5.3, (5.5) and (5.6), we derive

$$
\begin{aligned}
M_{\Psi_{k,l}}(\theta_m) &= M_k(\theta_m)M_l(\theta_m) + M_{k+l}(\theta_m)\\
&= 4(-1)^m\cos^2\left(\frac{lm\pi}{k+l}\right) + 2\cos\left(\frac{lm\pi}{k+l}\right)\left\{\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-k} + (-1)^m\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l}\right\}\\
&\quad+ 2(-1)^m + 2\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-(k+l)}.
\end{aligned}
$$

To help us approximate the lower bound of $(-1)^m M_{\Psi_{k,l}}(\theta_m)$, we express it as

$$
(-1)^m M_{\Psi_{k,l}}(\theta_m) = P_{\Psi_{k,l}}(\theta_m) + Q_{\Psi_{k,l}}(\theta_m)
$$

where $P_{\Psi_{k,l}}(\theta_m)$ and $Q_{\Psi_{k,l}}(\theta_m)$ are defined by

$$
P_{\Psi_{k,l}}(\theta_m) := 2 + 2(-1)^m\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-(k+l)}, \tag{5.7}
$$

and

$$
Q_{\Psi_{k,l}}(\theta_m) := 4\cos^2\left(\frac{lm\pi}{k+l}\right) + 2\cos\left(\frac{lm\pi}{k+l}\right)\left\{\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l} + (-1)^m\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-k}\right\}. \tag{5.8}
$$

5.1.4 Estimates for trigonometric functions

Together with Lemma 3.6 and 3.7 in Section 3.4, we will use the following lemmas to bound the function $Q_{\Psi_{k,l}}(\theta_m)$.

Lemma 5.4. For any integer $N \geq 10$, with $N \equiv 4 \pmod 6$, the function

$$h_N(x) = 4\cos^2\left(\frac{Nx}{2}\right) + 4\cos\left(\frac{Nx}{2}\right)$$

is monotonically decreasing on $(2\pi/3 - 2\pi/3N, 2\pi/3]$.

Proof. Let $x \in (2\pi/3 - 2\pi/3N, 2\pi/3]$. It is easy to see that

$$\frac{Nx}{2} \in \left(\frac{N\pi}{3} - \frac{\pi}{3}, \frac{N\pi}{3}\right].$$

Since $N \equiv 4 \pmod 6$, we have

$$\cos\left(\frac{Nx}{2}\right) \in \left(\cos(\pi), \cos\left(\frac{4\pi}{3}\right)\right] \subseteq (-1, -0.5].$$

Then $\cos(Nx/2)$ is increasing in x. From this and writing the derivative $h'_N(x)$ as

$$h'_N(x) = \left(8\cos\left(\frac{Nx}{2}\right) + 4\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(\frac{Nx}{2}\right)\right),$$

we see that $h'_N(x)$ is nonpositive on $(2\pi/3 - 2\pi/3N, 2\pi/3]$. □

By changing the restrictions on the congruence classes of N and the domain of $h_N(x)$, we obtain a similar lemma.

Lemma 5.5. For an integer $N \geq 14$, with $N \equiv 2 \pmod 6$, the function

$$h_N(x) = 4\cos^2\left(\frac{Nx}{2}\right) + 4\cos\left(\frac{Nx}{2}\right)$$

is monotonically decreasing on $(2\pi/3 - \pi/3N, 2\pi/3]$.

Proof. Let $x \in (2\pi/3 - \pi/3N, 2\pi/3]$. It is easy to see that

$$\frac{Nx}{2} \in \left(\frac{N\pi}{3} - \frac{\pi}{6}, \frac{N\pi}{3}\right].$$

Since $N \equiv 2 \pmod 6$, we have

$$\cos\left(\frac{Nx}{2}\right) \in \left(\cos\left(\frac{2\pi}{3}\right), \cos\left(\frac{\pi}{2}\right)\right] \subseteq [-0.5, 0).$$

Then $\cos(Nx/2)$ is decreasing in x. From this and writing the derivative $h'_N(x)$ as

$$h'_N(x) = \left(8\cos\left(\frac{Nx}{2}\right) + 4\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(\frac{Nx}{2}\right)\right),$$

we see that $h'_N(x)$ is nonpositive on $(2\pi/3 - \pi/3N, 2\pi/3]$. □

5.1.5 Bounding the main term

Since we can bound $(-1)^m M_{\Psi_{k,l}}(\theta_m)$ by bounding $P_{\Psi_{k,l}}(\theta_m)$ and $Q_{\Psi_{k,l}}(\theta_m)$, we will first start by giving a lower bound of $P_{\Psi_{k,l}}(\theta_m)$.

Proposition 5.6. For any even integers $k > l \geq 10$, and for any integer $m \in [(k+l)/4, (k+l)/3]$,

$$P_{\Psi_{k,l}}(\theta_m) \geq \begin{cases} 1.98222 & \text{if } k+l \equiv 0 \pmod 6, \\ 1.99970 & \text{if } k+l \equiv 2 \pmod 6, \\ 1.64160 & \text{if } k+l \equiv 4 \pmod 6. \end{cases}$$

Proof. By (5.7), recall that the function $P_{\Psi_{k,l}}(\theta_m)$ is given by

$$P_{\Psi_{k,l}}(\theta_m) = 2 + 2(-1)^m \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-(k+l)}.$$

By Lemma 3.6, $(2\cos(m\pi/(k+l)))^{-(k+l)}$ is positive and increasing on $[(k+l)/4, (k+l)/3]$. Thus,

$$P_{\Psi_{k,l}}(\theta_m) \geq 2 - 2\left(2\cos\left(\frac{m_{odd}\pi}{k+l}\right)\right)^{-(k+l)} \tag{5.9}$$

where m_{odd} is the largest odd integer in $[(k+l)/4, (k+l)/3]$. By considering $k+l \pmod 6$,

$$m_{ood} = \frac{k+l}{3} - \frac{3+r}{3}$$

where $k+l = r \pmod 6$ with $r \in \{0, \pm 2\}$. Inserting m_{odd} in (5.9), we obtain

$$P_{\Psi_{k,l}}(\theta_m) \geq 2 - 2\left(2\cos\left(\frac{\pi}{3} - \left(\frac{3+r}{3}\right)\frac{\pi}{k+l}\right)\right)^{-(k+l)}.$$

Then Lemma 3.7 says that the right hand side is monotonically increasing on $k+l$. Hence, for $k+l \equiv 0 \pmod 6$, and $k+l \geq 24$,

$$P_{\Psi_{k,l}}(\theta_m) \geq 2 - 2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{24}\right)\right)^{-(24)} \geq 1.98222. \tag{5.10}$$

Similarly, we have that for $k+l \equiv 2 \pmod 6$, and $k+l \geq 26$,

$$P_{\Psi_{k,l}}(\theta_m) \geq 2 - 2\left(2\cos\left(\frac{\pi}{3} - \left(\frac{5}{3}\right)\frac{\pi}{26}\right)\right)^{-(26)} \geq 1.99970, \tag{5.11}$$

and for $k+l \equiv -2 \pmod 6$, and $k+l \geq 22$,

$$P_{\Psi_{k,l}}(\theta_m) \geq 2 - 2\left(2\cos\left(\frac{\pi}{3} - \left(\frac{1}{3}\right)\frac{\pi}{22}\right)\right)^{-(22)} \geq 1.64160. \tag{5.12}$$

By (5.10), (5.11), and (5.12), we finish the proof. □

We now turn to bounding $Q_{\Psi_{k,l}}(\theta_m)$.

Proposition 5.7. For any even integers $k > l \geq 10$, and for any integer $m \in [(k+l)/4, (k+l)/3]$,

$$Q_{\Psi_{k,l}}(\theta_m) \geq \begin{cases} -0.31566 & \text{if } l \equiv 0 \ (\text{mod} 6), \\ -1 & \text{otherwise.} \end{cases}$$

Proof. By (5.8), the function $Q_{\Psi_{k,l}}(\theta_m)$ is given by

$$Q_{\Psi_{k,l}}(\theta_m) = 4\cos^2\left(\frac{lm\pi}{k+l}\right) + 2\cos\left(\frac{lm\pi}{k+l}\right)\left\{\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l} + (-1)^m\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-k}\right\}. \tag{5.13}$$

Since $\theta_m \in [\pi/2, 2\pi/3]$, $2\cos(\theta_m/2) \in [1, \sqrt{2}]$ and hence for even integers $k > l \geq 10$,

$$\left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-k} = \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-k} < \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l} = \left(2\cos\left(\frac{\theta_m}{2}\right)\right)^{-l}. \tag{5.14}$$

This implies that the term in curly brackets in (5.13) is always positive. (The only way it could be zero is when $m = (k+l)/3$ is an odd integer. However, this is impossible since k and l are even and hence $(k+l)/3$ is an integer if and only if $k + l \equiv 0 \pmod 6$). This follows that $Q_{\Psi_{k,l}}(\theta_m) \geq 0$ whenever $2\cos(lm\pi/(k+l)) \geq 0$. Thus in this case, we have the desired bound.

For the rest of the proof, we let $m \in [(k+l)/4, (k+l)/3]$ be any integer satisfying $2\cos(lm\pi/(k+l)) < 0$. First assume $l \equiv 0 \pmod 6$. By the definition of $Q_{\Psi_{k,l}}(\theta_m)$ defined above and $4\cos^2(lm\pi/(k+l)) > 0$, we find that

$$Q_{\Psi_{k,l}}(\theta_m) > 2\cos\left(\frac{lm\pi}{k+l}\right)\left\{\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l} + (-1)^m\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-k}\right\}. \tag{5.15}$$

By Lemma 3.6, $(2\cos(m\pi/(k+l)))^{-k}$ is a positive function in m. This results in

$$\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l} + (-1)^m\left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-k} \leq \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-l} + \left(2\cos\left(\frac{m\pi}{k+l}\right)\right)^{-(l+2)} \tag{5.16}$$

where we applied the same reasoning used in (5.14) on the second term knowing that $k \geq l+2$. Since $(2\cos(m\pi/(k+l)))^{-k}$ is increasing as a function in m by Lemma 3.6, (5.15) and (5.16) give us

$$Q_{\Psi_{k,l}}(\theta_m) \geq -2\left\{\left(2\cos\left(\frac{m'\pi}{k+l}\right)\right)^{-l} + \left(2\cos\left(\frac{m'\pi}{k+l}\right)\right)^{-(l+2)}\right\} \tag{5.17}$$

where m' is the largest value in $[(k+l)/4, (k+l)/3]$ satisfying $2\cos(lm\pi/(k+l)) \le 0$. By the aid of Mathematica,

$$m' = \frac{k+l}{3} - \frac{k+l}{2l}.$$

Inserting this into the right hand side of (5.17), we have that

$$Q_{\Psi_{k,l}}(\theta_m) \ge -2\left\{\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{2l}\right)\right)^{-l} + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{2l}\right)\right)^{-(l+2)}\right\}.$$

Lemma 3.7 gives us that the right hand side is monotonically increasing in l. Hence, for $l \equiv 0 \pmod 6$ and $l \ge 12$,

$$Q_{\Psi_{k,l}}(\theta_m) \ge -2\left\{\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{24}\right)\right)^{-12} + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{24}\right)\right)^{-14}\right\} \ge -0.31566.$$

Next we move on to the case $l \equiv 2 \pmod 6$ and first consider when $\theta_m \in [\pi/2, 2\pi/3 - \pi/3l]$ or $m \in [(k+l)/4, (k+l)/3 - (k+l)/6l]$. In this case, we bound $Q_{\Psi_{k,l}}(\theta_m)$ using the same reasoning to the previous case. In fact, Lemma 3.6 implies that

$$Q_{\Psi_{k,l}}(\theta_m) \ge -2\left\{\left(2\cos\left(\frac{m^*\pi}{k+l}\right)\right)^{-l} + \left(2\cos\left(\frac{m^*\pi}{k+l}\right)\right)^{-(l+2)}\right\} \tag{5.18}$$

where m^* is the last value in $[(k+l)/4, (k+l)/3 - (k+l)/6l]$ satisfying $2\cos(lm\pi/(k+l)) \le 0$. In this case, for $l \equiv 2 \pmod 6$, Mathematica reveals that

$$m^* = \frac{k+l}{3} - \frac{7(k+l)}{6l}.$$

Combining this with (5.18), for $l \equiv 2 \pmod 6$ and $l \ge 14$, Lemma 3.7 gives us that

$$Q_{\Psi_{k,l}}(\theta_m) \ge -2\left\{\left(2\cos\left(\frac{\pi}{3} - \frac{7\pi}{6(14)}\right)\right)^{-14} + \left(2\cos\left(\frac{\pi}{3} - \frac{7\pi}{6(14)}\right)\right)^{-16}\right\} \ge -0.02344.$$

We finish the case $l \equiv 2 \pmod 6$ by considering the θ_m values in $(2\pi/3 - \pi/3l, 2\pi/3]$. In this case, the fact that the term in curly brackets lies in $(0, 2]$ and $2\cos(m\pi/(k+l)) < 0$ make

$$Q_{\Psi_{k,l}}(\theta_m) \ge 4\cos^2\left(\frac{lm\pi}{k+l}\right) + 4\cos\left(\frac{lm\pi}{k+l}\right) = 4\cos^2\left(\frac{l\theta_m}{2}\right) + 4\cos\left(\frac{l\theta_m}{2}\right). \tag{5.19}$$

By Lemma 5.5, the right hand side is decreasing as a function in θ_m and hence takes a minimal value at $\theta_m = 2\pi/3$. Inserting $\theta_m = 2\pi/3$ into (5.19), we obtain

$$Q_{\Psi_{k,l}}(\theta_m) \ge 4\cos^2\left(\frac{l\pi}{3}\right) + 4\cos\left(\frac{l\pi}{3}\right) = 4\cos^2\left(\frac{2\pi}{3}\right) + 4\cos\left(\frac{2\pi}{3}\right) = -1.$$

using the fact that $\cos(l\pi/3) = \cos(2\pi/3)$ since $l \equiv 2 \pmod 6$.

Finally we assume that $l \equiv 4 \pmod 6$. Similarly, we start with considering all θ_m values that are not too close to $2\pi/3$. Suppose $\theta_m \in [\pi/2, 2\pi/3 - 2\pi/3l]$. Analysis similar to that in the proof of the previous case shows that for $l \equiv 4 \pmod 6$ and $l \geq 10$,

$$Q_{\Psi_{k,l}}(\theta_m) \geq -2\left\{\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3(10)}\right)\right)^{-10} + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3(10)}\right)\right)^{-12}\right\} \geq -0.68390.$$

Now suppose that $\theta_m \in (2\pi/3 - 2\pi/3l, 2\pi/3]$. Using the same reasoning as the second half of the proof of $l \equiv 2 \pmod 6$ and $\cos(l\pi/3) = \cos(4\pi/3)$,

$$Q_{\Psi_{k,l}}(\theta_m) \geq 4\cos^2\left(\frac{l\pi}{3}\right) + 4\cos\left(\frac{l\pi}{3}\right) = 4\cos^2\left(\frac{4\pi}{3}\right) + 4\cos\left(\frac{4\pi}{3}\right) = -1.$$

We finish the proof of Proposition 5.7. □

5.1.6 Proof of Theorem 5.2

Proof. By Proposition 5.3, the function $F_{\Psi_{k,l}}(\theta) = e^{i6(k+l)\theta}\Psi_{k,l}(e^{i\theta})$ can be written as

$$F_{\Psi_{k,l}}(\theta) = M_{\Psi_{k,l}}(\theta) + R_{\Psi_{k,l}}(\theta).$$

Then Propositions 5.6 and 5.7 say that for any even integers $k > l \geq 10$, and for any $\theta_m = 2m\pi/(k+l) \in [\pi/2, 2\pi/3]$,

$$(-1)^m M_{\Psi_{k,l}}(\theta_m) \geq \begin{cases} 1.32594 & \text{if } l \equiv 0 \pmod 6, \\ 0.64161 & \text{otherwise.} \end{cases}$$

Comparing this with the upper bound of $|R_{\Psi_{k,l}}|$ given in Proposition 5.3,

$$(-1)^m M_{\Psi_{k,l}}(\theta_m) > |R_{\Psi_{k,l}}|.$$

Then, the function $F_{\Psi_{k,l}}(\theta_m)$ is strictly positive or negative according as m is even or odd. Therefore, the intermediate value theorem guarantees that the number of zeros of $F_{\Psi_{k,l}}(\theta)$ in $(\pi/2, 2\pi/3)$ is at least the number of θ_m in $[\pi/2, 2\pi/3]$ minus 1. With the notation $k + l = 12n + s$ and $s \in \{0, 4, 6, 8, 10, 14\}$, considering all 6 cases, we find that there are $n + 1$ integers m for which $\theta_m \in [\pi/2, 2\pi/3]$.

Since the modular form $\Psi_{k,l}(\tau)$ has at most n zeros in the fundamental domain by the valence formula (2.11) as explained in Subsection 5.1.1, we have located all zeros of $\Psi_{k,l}(\tau)$ on the arc $\mathcal{A} = \{e^{i\theta} : \pi/2 \leq \theta \leq 2\pi/3\}$. □

5.2 Zeros of $E_k^n(\tau) + E_{nk}(\tau)$

5.2.1 Statement of main result

Let $k \geq 8$ be an even integer and let $n \in \mathbb{N}$. Define

$$\Omega_{n,k}^{+}(\tau) := E_k^n(\tau) + E_{nk}(\tau).$$

This clearly defines a family of weight nk modular forms. To study the location of their zeros, we may assume that $n \geq 2$ because $\Omega_{1,k}^{+}(\tau) = 2E_k(\tau)$ is a scalar multiple of the Eisenstein series which their zeros are already studied by F.K.C. Rankin and Swinnerton-Dyer in Chapter 3.

Our aim of this section is to study the location of the zeros of $\Omega_{n,k}^{+}$ for $n = 2$ and $n = 3$ in the fundamental domain. Let us now state our main results of this section.

Theorem 5.8. Let k be even. All zeros of $\Omega_{2,k}^{+}(\tau)$ for $k \geq 8$ and all zeros of $\Omega_{3,k}^{+}(\tau)$ for $k \geq 16$ in the fundamental domain $\mathcal{F}$ are located on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$.

Let $k \geq 8$ be even and let $n \in \{2, 3\}$. Write

$$k = \left(\frac{12}{n}\right) l_n + s_n$$

where $s_n \in \{0, 2, \ldots, (12/n) - 2\}$. By the Fourier series expansion of Eisenstein series given in (2.2), the modular form $\Omega_{n,k}^{+}(\tau)$ has the constant coefficient equals 2 which implies that $\nu_{i\infty}(\Omega_{n,k}^{+}) = 0$. Thus , the valence formula (2.11) for $\Omega_{n,k}^{+}(\tau)$ is simplified to

$$l_n + \frac{ns_n}{12} = \frac{1}{2}\nu_i\left(\Omega_{n,k}^{+}\right) + \frac{1}{3}\nu_{e^{2\pi i/3}}\left(\Omega_{n,k}^{+}\right) + \sum_{\tau \in \mathcal{F}\setminus\{i, e^{2\pi i/3}\}} \nu_\tau\left(\Omega_{n,k}^{+}\right). \tag{5.20}$$

Also the fact that $\Omega_{n,k}^{+}(\tau)$ is holomorphic on $\mathbb{H}$ gives us that $\nu_\tau\left(\Omega_{n,k}^{+}\right) \in \mathbb{N} \cup \{0\}$ for all $\tau \in \mathcal{F}$. From this and (5.20), $\Omega_{n,k}^{+}(\tau)$ has zeros at $z = i$ and $e^{2\pi i/3}$ of order at least $ns_n/12$. Therefore, $\Omega_{n,k}^{+}(\tau)$ has a total of l_n zeros in $\mathcal{F}\setminus\{i, e^{2\pi i/3}\}$.

To study the zeros of $\Omega_{n,k}^{+}(\tau)$ on the arc $\mathcal{A}$, we consider the zeros of the related function

$$F_{\Omega_{n,k}^{+}}(\theta) := e^{i(k+l)\theta/2}\Omega_{n,k}^{+}(e^{i\theta}).$$

By Lemma 3.2, this function is real on the closed interval $[\pi/2, 2\pi/3]$ and clearly shares the same zero set with the function $\Omega_{n,k}^{+}(e^{i\theta})$ on that interval.

5.2.2 Extraction of the main and remainder terms

Proposition 5.9. For any even integer $k \geq 8$, for $n \in \{2, 3\}$ and for any $\theta \in [\pi/2, 2\pi/3]$,

$$F_{\Omega^+_{n,k}}(\theta) = M_{\Omega^+_{n,k}}(\theta) + R_{\Omega^+_{n,k}}(\theta)$$

where $M_{\Omega^+_{n,k}} = M_k^n + M_{nk}$ with M_k's are defined in (3.6) and

$$\left|R_{\Omega^+_{n,k}}\right| \leq \begin{cases} 1.39237 & \text{if } n = 2 \text{ and } k \geq 8, \\ 0.17999 & \text{if } n = 3 \text{ and } k \geq 16. \end{cases}$$

Proof. We can write

$$F_{\Omega^+_{n,k}}(\theta) = e^{ink\theta/2}\left(E_k^n + E_{nk}\right)\left(e^{i\theta}\right) = \left(F_k^n + F_{nk}\right)(\theta),$$

where F_k is defined as in (3.4). Expanding the right hand side using (3.5), we derive

$$F_{\Omega^+_{n,k}}(\theta) = M_k^n(\theta) + M_{nk}(\theta) + \sum_{i=1}^{n}\binom{n}{i}M_k^i(\theta)R_k^{n-i}(\theta) + R_k^n(\theta) + R_{nk}(\theta).$$

Let $M_{\Omega^+_{n,k}} = M_k^n + M_{nk}$ and let $R_{\Omega^+_{n,k}} = \sum_{i=1}^{n}\binom{n}{i}M_k^i R_k^{n-i} + R_k^n + R_{nk}$. To bound $|R_{\Omega^+_{n,k}}|$, the triangle inequality and the definition of M_k in (3.6) yield $|M_k(\theta)| \leq |2\cos(k\theta/2)| + |(2\cos(\theta/2))^{-k}| \leq 3$ on $[\pi/2, 2\pi/3]$ and hence

$$\left|R_{\Omega^+_{n,k}}(\theta)\right| \leq \sum_{i=1}^{n}\binom{n}{i}3^i|R_k(\theta)|^{n-i} + |R_k(\theta)|^n + |R_{nk}(\theta)|.$$

By Proposition 3.3, $|R_k(\theta)|$ is monotonically decreasing in k so the term $\left|R_{\Omega^+_{n,k}}(\theta)\right|$ is also. Evaluating a bound at $k = 8$ for $n = 2$ and at $k = 16$ for $n = 3$, we easily obtain the upper bound for $|R_{\Omega^+_{n,k}}|$ given in Proposition 5.9. □

5.2.3 Definition of sample points

Let $k \geq 8$ be an even integer and let $n \in \{2, 3\}$. We define

$$\theta_m := \theta_{nk}(m) = \frac{2m\pi}{nk}$$

where m ranges over integers so that $\theta_m \in [\pi/2, 2\pi/3]$. Observe that

$$\theta_m \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right] \Leftrightarrow m \in \left[\frac{nk}{4}, \frac{nk}{3}\right].$$

Our goal for the rest of this section is to show that $F_{\Omega^+_{n,k}}(\theta_m)$ is strictly positive or negative according to the parity of m for all $\theta_m \in [\pi/2, 2\pi/3]$. Since $F_{\Omega^+_{n,k}}(\theta) = M_{\Omega^+_{n,k}}(\theta) + R_{\Omega^+_{n,k}}(\theta)$ by Proposition 5.9, we show that for all integers $m \in [nk/4, nk/3]$, a lower bound of $(-1)^m M_{\Omega^+_{n,k}}(\theta_m)$ is greater than the upper bound of $|R_{\Omega^+_{n,k}}|$ given in Proposition 5.9.

Let us first simplify $M_{\Omega^+_{n,k}}(\theta_m)$. Since $\theta_m = 2m\pi/nk$,

$$2\cos\left(\frac{k\theta_m}{2}\right) = 2\cos\left(\frac{m\pi}{n}\right) \quad \text{and} \quad 2\cos\left(\frac{nk\theta_m}{2}\right) = 2\cos(m\pi) = 2(-1)^m.$$

With the definition of $M_k(\theta)$ given in (3.6) and the above identities, we have that

$$M_k(\theta_m) = 2\cos\left(\frac{m\pi}{n}\right) + \left(2\cos\left(\frac{m\pi}{nk}\right)\right)^{-k}, \tag{5.21}$$

and

$$M_{nk}(\theta_m) = 2(-1)^m + \left(2\cos\left(\frac{m\pi}{nk}\right)\right)^{-nk}. \tag{5.22}$$

By Proposition 5.9, (5.21) and (5.22), we derive

$$M_{\Omega^+_{n,k}}(\theta_m) = \left(2\cos\left(\frac{m\pi}{n}\right) + \left(2\cos\left(\frac{m\pi}{nk}\right)\right)^{-k}\right)^n + 2(-1)^m + \left(2\cos\left(\frac{m\pi}{nk}\right)\right)^{-nk}. \tag{5.23}$$

5.2.4 Bounding the main term

We first give a lower bound on $(-1)^m M_{\Omega^+_{2,k}}(\theta_m)$.

Proposition 5.10. For any even integer $k \geq 8$ and for any integer $m \in [k/2, 2k/3]$,

$$(-1)^m M_{\Omega^+_{2,k}}(\theta_m) \geq 1.64849.$$

Proof. By (5.23), the function $M_{\Omega^+_{2,k}}(\theta_m)$ is given by

$$M_{\Omega^+_{2,k}}(\theta_m) = \left(2\cos\left(\frac{m\pi}{2}\right) + \left(2\cos\left(\frac{m\pi}{2k}\right)\right)^{-k}\right)^2 + 2(-1)^m + \left(2\cos\left(\frac{m\pi}{2k}\right)\right)^{-2k}. \tag{5.24}$$

From this, it is easy to see that $M_{\Omega^+_{2,k}}(\theta_m) \geq 2$ for all even integers $m \in [k/2, 2k/3]$. Thus in this case, we have the desired bound. Now we assume $m \in [k/2, 2k/3]$ is odd. Since $2\cos(m\pi/2) = 0$, (5.24) is simplified to

$$M_{\Omega^+_{2,k}}(\theta_m) = -2 + 2\left(2\cos\left(\frac{m\pi}{2k}\right)\right)^{-2k}.$$

By Lemma 3.6, the right hand side is monotonically increasing on odd integers in $[k/2, 2k/3]$. This follows that

$$M_{\Omega^+_{2,k}}(\theta_m) \leq -2 + 2\left(2\cos\left(\frac{m_{odd}\pi}{2k}\right)\right)^{-2k}. \tag{5.25}$$

where m_{odd} is the largest odd integer in $[k/2, 2k/3]$. By considering $k \pmod 6$,

$$m_{odd} = \frac{2k}{3} - \frac{3-r}{3},$$

where $k \equiv r \pmod 6$ with $r \in \{0, \pm 2\}$. Substituting m_{odd} into (5.25), we obtain

$$M_{\Omega^+_{2,k}}(\theta_m) \leq -2 + 2\left(2\cos\left(\frac{\pi}{3} - \left(\frac{3-r}{3}\right)\frac{\pi}{2k}\right)\right)^{-2k}$$

By Lemma 3.7, the right hand side is monotonically decreasing on k. Hence, for $k \equiv 0 \pmod 6$ and $k \geq 12$,

$$M_{\Omega^+_{2,k}}(\theta_m) \leq -2 + 2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{2(12)}\right)\right)^{-2(12)} \leq -1.98223. \tag{5.26}$$

Applying this argument to the cases $k \equiv -2 \pmod 6$, we obtain that for $k \equiv -2 \pmod 6$ and $k \geq 10$,

$$M_{\Omega^+_{2,k}}(\theta_m) \leq -2 + 2\left(2\cos\left(\frac{\pi}{3} - \frac{5\pi}{3(2(10))}\right)\right)^{-2(10)} \leq -1.99804, \tag{5.27}$$

and for $k \equiv 2 \pmod 6$ and $k \geq 8$,

$$M_{\Omega^+_{2,k}}(\theta_m) \leq -2 + 2\left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3(2(8))}\right)\right)^{-2(8)} \leq -1.62955. \tag{5.28}$$

By (5.26), (5.27), and (5.28), we have proved Proposition 5.10. □

Next, we turn to bounding $(-1)^m M_{\Omega^+_{3,k}}(\theta_m)$. The proof has the similar concept as the proof of Proposition 5.10.

Proposition 5.11. For even integer $k \geq 16$ and for integer $m \in [3k/4, k]$,

$$(-1)^m M_{\Omega^+_{3,k}}(\theta_m) \geq 0.32869.$$

Proof. By (5.23), the function $M_{\Omega^+_{3,k}}(\theta_m)$ is given by

$$M_{\Omega^+_{3,k}}(\theta_m) = \left(2\cos\left(\frac{m\pi}{3}\right) + \left(2\cos\left(\frac{m\pi}{3k}\right)\right)^{-k}\right)^3 + 2(-1)^m + \left(2\cos\left(\frac{m\pi}{3k}\right)\right)^{-3k}.$$

First assume $m \in [3k/4, k]$ is even. Since $2\cos(m\pi/3) = -1$ or 2, and $2\cos(m\pi/3k) \in [1, \sqrt{2}]$, we have

$$M_{\Omega^+_{3,k}}(\theta_m) \geq \left(-1 + \left(2\cos\left(\frac{m\pi}{3k}\right)\right)^{-k}\right)^3 + 2 + \left(2\cos\left(\frac{m\pi}{3k}\right)\right)^{-3k}.$$

By Lemma 3.6, the increasing property of the positive function $(2\cos(m\pi/3k))^{-k}$ gives us

$$M_{\Omega^+_{3,k}}(\theta_m) > (-1 + 0)^3 + 2 + 0 > 1. \tag{5.29}$$

Now assume $m \in [3k/4, k]$ is odd. Since $2\cos(m\pi/3) = -2$ or 1, and $2\cos(m\pi/3k) \in [1, \sqrt{2}]$, we have

$$M_{\Omega^+_{3,k}}(\theta_m) \leq \left(1 + \left(2\cos\left(\frac{m\pi}{3k}\right)\right)^{-k}\right)^3 - 2 + \left(2\cos\left(\frac{m\pi}{3k}\right)\right)^{-3k}. \tag{5.30}$$

By Lemma 3.6, the right hand side is monotonically increasing on odd integers in $[3k/4, k]$. This follows that

$$M_{\Omega^+_{3,k}}(\theta_m) \leq \left(1 + \left(2\cos\left(\frac{m_{odd}\pi}{3k}\right)\right)^{-k}\right)^3 - 2 + \left(2\cos\left(\frac{m_{odd}\pi}{3k}\right)\right)^{-3k}. \tag{5.31}$$

where m_{odd} is the largest odd integer in $[3k/4, k]$. Plugging in $m_{odd} = k - 1$ in (5.31) we derive

$$M_{\Omega^+_{3,k}}(\theta_m) \leq \left(1 + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3k}\right)\right)^{-k}\right)^3 - 2 + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3k}\right)\right)^{-3k}.$$

By Lemma 3.7, the right hand side is monotonically decreasing as a function in k. Evaluating a bound at $k = 16$, we have that for $k \geq 16$,if $m \in [3k/4, k]$ is odd,

$$M_{\Omega^+_{3,k}}(\theta_m) \leq \left(1 + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3(16)}\right)\right)^{-16}\right)^3 - 2 + \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3(16)}\right)\right)^{-3(16)} \leq -0.32869. \tag{5.32}$$

By (5.29) and (5.32), the proof is completed. □

5.2.5 Proof of Theorem 5.8

We are now able to prove Theorem 5.8.

Proof. Recall that by Proposition (5.9), the function $F_{\Omega^+_{n,k}}(\theta) = e^{ink\theta/2}\Omega^+_{n,k}(e^{i\theta})$ can be written as

$$F_{\Omega^+_{n,k}}(\theta) = M_{\Omega^+_{n,k}}(\theta) + R_{\Omega^+_{n,k}}(\theta).$$

Then Propositions 5.10 and 5.11 further give us that

$$(-1)^m M_{\Omega^+_{n,k}}(\theta_m) > \left| R_{\Omega^+_{n,k}} \right|$$

for large enough even integers k and for $\theta_m = 2m\pi/nk$ in the closed interval $[\pi/2, 2\pi/3]$.

This implies that $F_{\Omega^+_{n,k}}(\theta_m)$ is strictly positive or negative according as m is even or odd. Then the intermediate value theorem guarantees that a minimum number of the zeros of $F_{\Omega^+_{n,k}}(\theta)$ in the open interval $(\pi/2, 2\pi/3)$ equals the number of θ_m in $[\pi/2, 2\pi/3]$ minus 1.

Since the number of θ_m in $[\pi/2, 2\pi/3]$ equals the number of integers in $[nk/4, nk/3]$ which can be shown easily that there are $l_n + 1$ of them where we recall $k = (12/n)l_n + s_n$ with $s_n \in \{0, 2, \ldots, (12/n) - 2\}$, we conclude that $F_{\Omega^+_{n,k}}(\theta)$ has at least l_n zeros on $(\pi/2, 2\pi/3)$.

As $\Omega^+_{n,k}(\tau)$ has at most l_n zeros in the fundamental domain as described at the beginning of Section 5.2.1 and the above argument shows that there are at least l_n zeros on the arc $\mathcal{A}$, we finish the proof of Theorem 5.8. □

5.2.6 Higher values of n

Computational evidence shows that the result in Theorem 5.8 does not extend to higher values $n \geq 4$ (please see Figures 5.1 and 5.2). The main difficulty in carrying out the method of F.K.C. Rankin and Swinnerton-Dyer is that when $n \geq 4$, the bound of the remainder term $|R_{\Omega^+_{n,k}}(\theta)|$ is getting bigger than the main term $(-1)^m M_{\Omega^+_{n,k}}(\theta_m)$ as the values of θ_m get closer and closer $2\pi/3$. It would be very interesting to see what result holds for higher n. We leave this an open problem.

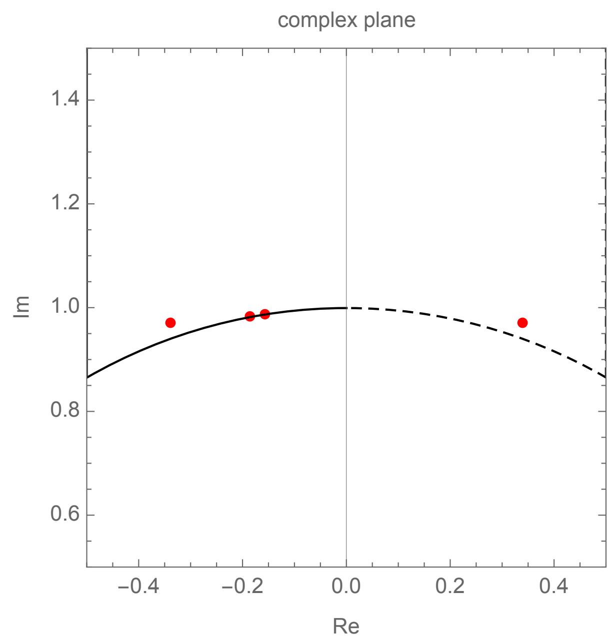

FIGURE 5.1: Zeros of $E_k^4(\tau) + E_{4k}(\tau)$ when $k = 12$

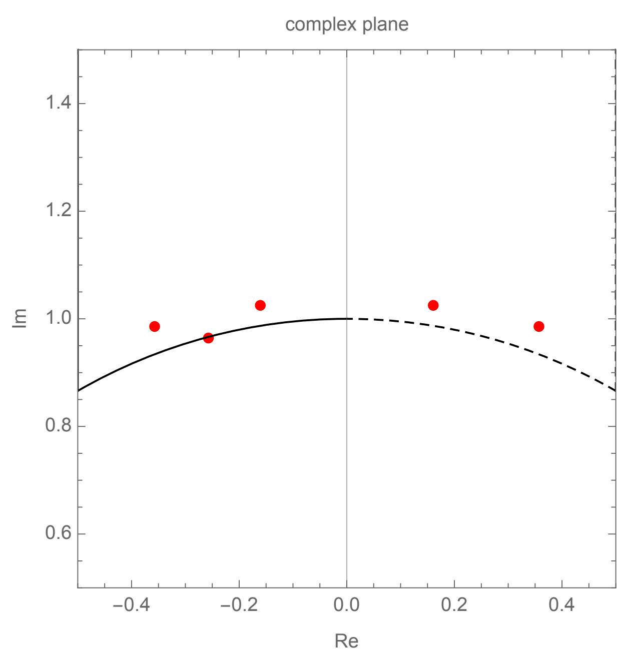

FIGURE 5.2: Zeros of $E_k^5(\tau) + E_{5k}(\tau)$ when $k = 12$

5.3 Zeros of $E_k^n(\tau) - E_{nk}(\tau)$

Let $k \geq 10$ be an even integer and let $n \in \mathbb{N}$. Define

$$\Omega_{n,k}^-(\tau) := E_k^n(\tau) - E_{nk}(\tau).$$

It is easy to check that the constant term of its q-expansion is zero. Then this defines a family of cusp forms of weight nk. Since $\Omega_{1,k}^-(\tau) = 0$, we may assume that $n \geq 2$. For $n = 2$, Reitzes et al. proved in [20] that all zeros of $\Omega_{2,k}^-(\tau)$ are located on the left side boundary of the fundamental domain.

Theorem 5.12 (Reitzes, Vulakh and Young)**.** For any even integer $k \geq 14$, all zeros of $E_k^2(\tau) - E_{2k}(\tau)$ in the fundamental domain $\mathcal{F}$ are on the left side boundary $\mathcal{F}_{-1/2} = \{-0.5 + iy : y \geq \sqrt{3}/2\}$.

The purpose of this section is to explore the zero location of the function $\Omega_{3,k}^-(\tau)$ by extending the method of F.K.C. Rankin and Swinnerton-Dyer explained in Chapter 3.

Let $k \geq 10$ be even integer and write $k = 4n + s$ with $s \in \{0, 2\}$. By this notation and the valence formula (2.11), we find that $\Omega_{3,k}^-(\tau)$ satisfies

$$n + \frac{s}{4} - 1 = \frac{3k}{12} = \frac{1}{2}\nu_i\left(\Omega_{3,k}^-\right) + \frac{1}{3}\nu_{e^{2\pi i/3}}\left(\Omega_{3,k}^-\right) + \sum_{\tau \in \mathcal{F}\setminus\{i, e^{2\pi i/3}\}} \nu_\tau\left(\Omega_{3,k}^-\right).$$

where -1 represents the zero of order one at $\tau = i\infty$ (which can be checked from the constant term of the Fourier series expansion of $\Omega_{3,k}^-(\tau)$.

Since $s/4$ determines the value of $\nu_i\left(\Omega_{3,k}^-\right)$, $\Omega_{3,k}^+(\tau)$ has at most $n-1$ zeros in the fundamental domain $\mathcal{F}\setminus\{i, i\infty\}$. One may expect that all zeros might locate in the lower boundary arc of the fundamental domain as the zeros of Eisenstein series do. In contrast to this guess, we conjecture the following result.

Conjecture 5.13. For any even integer $k \geq 10$, all zeros of $E_k^3(\tau) - E_{3k}(\tau)$ in the fundamental domain $\mathcal{F}$ are either on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$ or the left side boundary $\mathcal{F}_{-1/2} = \{-0.5 + iy : y \geq \sqrt{3}/2\}$.

5.3.1 Statement of main result

Our goal of this subsection is to obtain the lower bound of the number of the zeros on the arc $\mathcal{A}$ of the fundamental domain. In particular, we want to prove that at least one third of zeros of $\Omega^-_{3,k}(\tau)$ are on the arc $\mathcal{A}$.

Theorem 5.14. For any even integer $k \geq 10$, $E_k^3(\tau) - E_{3k}(\tau)$ has at least $A(k)$ zeros on the arc $\mathcal{A} = \{e^{i\theta} : \theta \in [\pi/2, 2\pi/3]\}$ where

$$A(k) = \begin{cases} q+1 & \text{if } k \equiv 2 \pmod 6, \\ q & \text{otherwise} \end{cases}$$

where $k = 12q + r$ with $r \in \{0, 4, 6, 8, 10, 14\}$.

To study the zeros of $\Omega^-_{3,k}(\tau)$ on the arc $\mathcal{A}$, we consider the zeros of the related function

$$F_{\Omega^-_{3,k}}(\theta) := e^{i3k\theta}\Omega^-_{3,k}(e^{i\theta}).$$

By Lemma 3.2, this function is real on the closed interval $[\pi/2, 2\pi/3]$ and clearly shares the same zero set with the function $\Omega^-_{3,k}(e^{i\theta})$ on that interval.

5.3.2 Extraction of main and remainder terms

Proposition 5.15. For any even integer $k \geq 10$, and for any $\theta \in [\pi/2, 2\pi/3]$,

$$F_{\Omega^-_{3,k}}(\theta) = M_{\Omega^-_{3,k}}(\theta) + R_{\Omega^-_{3,k}}(\theta),$$

where $M_{\Omega^-_{3,k}} = M_k^3 - M_{3k}$ with M_k's are defined in (3.6) and $|R_{\Omega^-_{3,k}}| \leq 2.25473$.

Proof. We can write

$$F_{\Omega^-_{3,k}}(\theta) = e^{i3k\theta/2}(E_k^3 - E_{3k})(e^{i\theta}) = (F_k^3 - F_{3k})(\theta).$$

where F_k is defined as in (3.4). Expanding the right hand side using (3.5), we derive

$$F_{\Omega^-_{3,k}}(\theta) = M_k^3(\theta) - M_{3k}(\theta) + 3M_k^2(\theta)R_k(\theta) + 3M_k(\theta)R_k^2(\theta) + R_k^3(\theta) - R_{3k}(\theta).$$

Let $M_{\Omega^-_{3,k}} = M_k^3 - M_{3k}$, and let $R_{\Omega^-_{3,k}} = 3M_k^2R_k + 3M_kR_k^2 + R_k^3 - R_{3k}$. To bound $|R_{\Omega^-_{3,k}}|$, the triangle inequality and the definition of M_k in (3.6) yield $|M_k(\theta)| \leq |2\cos(k\theta/2)| +$

$|(2\cos(\theta/2))^{-k}| \le 3$ on $[\pi/2, 2\pi/3]$ and hence

$$|R_{\Omega^-_{3,k}}| \le 27|R_k| + 9|R_k|^2 + |R_k|^3 + |R_{3k}|.$$

Recall that by Proposition 3.3, $|R_k|$ is monotonically decreasing as a function in k so the term $|R_{\Omega^-_{3,k}}|$ is also. Evaluating the right hand side at $k = 10$, we get an upper bound

$$|R_{\Omega^-_{3,k}}| \le 27|R_{10}| + 9|R_{10}|^2 + |R_{10}|^3 + |R_{30}| \le 2.25473$$

for all $k \ge 10$. □

5.3.3 Definition of sample points

Let $k \ge 10$ be an even integer and write $k \equiv x \pmod 3$ with $x = 0, 1, 2$. Define the truncated interval

$$I_k := \left[\frac{\pi}{2}, \frac{2\pi}{3} - \frac{2\pi x}{3k}\right],$$

and define

$$\alpha_m := \frac{2m\pi}{k},$$

where m ranges over integers so that $\alpha_m \in I_k$. Observe that

$$\alpha_m \in I_k \quad \text{and} \quad m \in \left[\frac{k}{4}, \frac{k-x}{3}\right].$$

Our goal for the rest of this section is to show that $F_{\Omega^-_{3,k}}(\alpha_m)$ is strictly positive or negative depending on the parity of the integer $m \in [k/4, (k-x)/3]$. Since $F_{\Omega^-_{3,k}}(\theta) = M_{\Omega^-_{3,k}}(\theta) + R_{\Omega^-_{3,k}}(\theta)$ by Proposition 5.15, we show that for all integer $m \in [k/4, (k-x)/3]$, a lower bound of $(-1)^m M_{\Omega^-_{3,k}}(\alpha_m)$ is greater than the upper bound of $|R_{\Omega^-_{3,k}}|$ given in Proposition 5.15.

Let us first simplify $M_{\Omega^-_{3,k}}(\alpha_m)$. Since $\alpha_m = 2m\pi/k$,

$$2\cos\left(\frac{k\alpha_m}{2}\right) = 2\cos(m\pi) = 2(-1)^m \quad \text{and} \quad 2\cos\left(\frac{3k\alpha_m}{2}\right) = 2\cos(3m\pi) = 2(-1)^m. \tag{5.33}$$

With the definition of $M_k(\theta)$ given in (3.6) and (5.33), we have that

$$M_k(\alpha_m) = 2(-1)^m + \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k} \quad \text{and} \quad M_{3k}(\alpha_m) = 2(-1)^m + \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-3k}. \tag{5.34}$$

By Proposition 5.15 and (5.34), we derive

$$M_{\Omega^+_{3,k}}(\alpha_m) = \left(2(-1)^m + \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k}\right)^3 - 2(-1)^m - \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-3k}$$

Further simplifying the right hand side, we obtain

$$(-1)^m M_{\Omega^-_{3,k}}(\alpha_m) = 6\left[1 + (-1)^m \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k}\right]^2. \tag{5.35}$$

5.3.4 Bounding the main term

Proposition 5.16. For any even integer $k \geq 12$ and for any integer $m \in [k/2, (k-x)/3]$ where $k \equiv x \pmod 3$,

$$(-1)^m M_{\Omega^-_{3,k}}(\alpha_m) > 3.85550.$$

Proof. By (5.35), we recall that $(-1)^m M_{\Omega^-_{3,k}}(\alpha_m)$ is given by

$$(-1)^m M_{\Omega^-_{3,k}}(\alpha_m) = 6\left[1 + (-1)^m \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k}\right]^2.$$

By Lemma 3.6, $(2\cos(m\pi/k))^{-k}$ is positive and increasing on $m \in [k/2, (k-x)/3]$. This implies that

$$6\left[1 + (-1)^m \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k}\right]^2 \geq 6\left[1 - \left(2\cos\left(\frac{m_{odd}\pi}{k}\right)\right)^{-k}\right]^2 \tag{5.36}$$

where m_{odd} is the largest odd integer in $[k/4, (k-x)/3]$. A direct computation reveals

$$m_{odd} = \frac{k}{3} - \frac{3+r}{3}$$

where $k \equiv r \pmod 6$ with $r \in \{0, \pm 2\}$. Inserting m_{odd} in (5.36), we obtain

$$6\left[1 + (-1)^m \left(2\cos\left(\frac{m\pi}{k}\right)\right)^{-k}\right]^2 \geq 6\left[1 - \left(2\cos\left(\frac{\pi}{3} - \left(\frac{3+r}{3}\right)\frac{\pi}{k}\right)\right)^{-k}\right]^2.$$

Lemma 3.7 shows that the right hand side is monotonically increasing as a function in k. Hence, for $k \equiv 0 \pmod 6$, and $k \geq 12$,

$$(-1)^m M_{\Omega^-_{3,k}}(\alpha_m) \geq 6\left[1 - \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{(12)}\right)\right)^{-12}\right]^2 \geq 5.81396. \tag{5.37}$$

Similarly, we have that for $k \equiv 2 \pmod 6$, and $k \geq 14$,

$$(-1)^m M_{\Omega^-_{3,k}}(\alpha_m) \geq 6\left[1 - \left(2\cos\left(\frac{\pi}{3} - \frac{5\pi}{3(14)}\right)\right)^{-14}\right]^2 \geq 5.97705, \tag{5.38}$$

and for $k \equiv -2 \pmod 6$, and $k \geq 10$,

$$(-1)^m M_{\Omega^-_{3,k}}(\alpha_m) \geq 6\left[1 - \left(2\cos\left(\frac{\pi}{3} - \frac{\pi}{3(10)}\right)\right)^{-10}\right]^2 \geq 3.85550. \tag{5.39}$$

By (5.37), (5.38), and (5.39), we finish the proof.

□

5.3.5 Extra zero when $k \equiv 2 \pmod 6$

This subsection is intended to show that the function $F_{\Omega^-_{3,k}}(\theta)$ has at least one zero on $(2\pi/3 - 4\pi/3k, 2\pi/3)$ for $k \equiv 2 \pmod 6$. Note that this extra zero is outside the truncated interval I_k. In fact, $k \equiv 2 \pmod 6$ implies that $k \equiv 2 \pmod 3$ and hence by the definition of I_k given in Section 5.3.3,

$$I_k = \left[\frac{\pi}{2}, \frac{2\pi}{3} - \frac{4\pi}{3k}\right]. \tag{5.40}$$

We want to prove the following result.

Proposition 5.17. For any even integer $k \geq 12$ with $k \equiv 2 \pmod 6$, $F_{\Omega^-_{3,k}}(\theta)$ has at least one zero on $(2\pi/2 - 4\pi/3k, 2\pi/3)$.

Proof. With the definition of $M_k(\theta)$ in (3.6), by Proposition 5.15,

$$\begin{aligned} M_{\Omega^-_{3,k}}\left(\frac{2\pi}{3}\right) &= M_k^3\left(\frac{2\pi}{3}\right) + M_{3k}\left(\frac{2\pi}{3}\right) \\ &= \left(2\cos\left(\frac{k\pi}{3}\right) + \left(2\cos\left(\frac{\pi}{3}\right)\right)^{-k}\right)^3 - 2\cos(k\pi) - \left(2\cos\left(\frac{\pi}{3}\right)\right)^{-3k}. \end{aligned}$$

Since $k \equiv 2 \pmod 6$ is even, $2\cos(k\pi) = 2$ and $2\cos(k\pi/3) = 2\cos(2\pi/3) = -1$ and then

$$M_{\Omega^-_{3,k}}(2\pi/3) = (-1+1)^3 - 2 - 1 = -3. \tag{5.41}$$

Now we turn to computing $M_{\Omega^-_{3,k}}(2\pi/3 - 4\pi/3k)$. By definition of I_k in (5.40), $2\pi/3 - 4\pi/3k$ is the right endpoint of I_k. In fact, we claim that

$$\frac{2\pi}{3} - \frac{4\pi}{3k} = \alpha_m$$

for some integer m. By (5.40), $\alpha_m \in I_k$ if and only if

$$m \in \left[\frac{k}{4}, \frac{k-2}{3}\right].$$

Since $k \equiv 2 \pmod 6$, $(k-2)/3$ is the largest even integer in that interval. By the definition of sample points given in Section 5.3.3, we find that

$$\alpha_{(k-2)/3} = \frac{2\pi}{k}\left(\frac{(k-2)}{3}\right) = \frac{2\pi}{3} - \frac{4\pi}{3k}.$$

This proves our claim. Now from this and Proposition 5.16, we obtain the lower bound

$$M_{\Omega^-_{3,k}}(\alpha_m) = M_{\Omega^-_{3,k}}(2\pi/3 - 4\pi/3k) > 3.85550. \tag{5.42}$$

Comparing (5.41) and (5.42) with the upper bound of $|R_{\Omega^-_{3,k}}|$ given in Proposition 5.15, we get

$$F_{\Omega^-_{3,k}}\left(\frac{2\pi}{3}\right) < 0 \quad \text{and} \quad F_{\Omega^-_{3,k}}\left(\frac{2\pi}{3} - \frac{4\pi}{3k}\right) > 0.$$

Then the intermediate value theorem guarantees that $F_{\Omega^-_{3,k}}(\theta)$ has at least one zero on $(2\pi/2 - 4\pi/3k, 2\pi/3)$. □

5.3.6 Proof of Theorem 5.14

Proof. By Proposition 5.15, the function $F_{\Omega^-_{3,k}}(\theta) = e^{i3k\theta/2}\Omega^-_{3,k}(e^{i\theta})$ can be written as

$$F_{\Omega^-_{3,k}}(\theta) = M_{\Omega^-_{3,k}}(\theta) + R_{\Omega^-_{3,k}}(\theta). \tag{5.43}$$

where Propositions 5.15 and 5.16 give us

$$(-1)^m M_{\Omega^-_{3,k}}(\alpha_m) > \left|R_{\Omega^-_{3,k}}\right|,$$

for all even integer $k \geq 12$, and for all $\alpha_m = 2m\pi/k \in I_k$ where I_k is defined in Section 5.3.3.

Then the intermediate value theorem guarantees that a minimum number of the zeros of $F_{\Omega^-_{3,k}}(\theta)$ in the interior of I_k is at least the number of θ_m in I_k minus 1. A straightforward counting argument shows that there are $q+1$ of θ_m in that interval where we write $k = 12q+r$ with $s \in \{0, 4, 6, 8, 10, 14\}$. This implies that $F_{\Omega^-_{3,k}}(\theta)$ has at least q zeros in the interior of I_k. Hence, we have proved Theorem 5.14 for the case $k \not\equiv 2 \pmod 6$.

For $k \equiv 2 \pmod 6$, Section 5.3.5 gives us the additional zero required from our discussion at the beginning of Section 5.3.1. Therefore this completes the proof of Theorem 5.14.

□

5.3.7 Zeros on the left side boundary

Recall that at the beginning of Section 5.3, we explain that for any even integer $k \geq 10$, $\Omega^+_{3,k}(\tau)$ has at most $n-1$ zeros in the fundamental domain $\mathcal{F}\backslash\{i, i\infty\}$ where $k = 4n+s$ with $s \in \{0,2\}$. Computational results for $k \leq 24$ shows that $\Omega^-_{3,k}(\tau)$ only has zeros on the boundary of the fundamental domain (see Figures 5.3,5.4, 5.5, 5.6 as examples).

k	Number of zeros
10	1
12	1
14	1
16	2
18	2
20	2
22	3
24	3

TABLE 5.1: Number of zeros of $\Omega^-_{3,k}(\tau)$ for $k \leq 24$

In particular, they reveal some trends of the number of the zeros $\Omega^-_{3,k}$ on the left side boundary $\mathcal{F}_{-1/2}$ (see Table 5.1). Based on the above interesting patterns, and the lower bound of the number of zeros on the arc $\mathcal{A}$ given in Theorem 5.14, we leave the following conjecture as an open problem.

Conjecture 5.18. For any even integer $k \geq 10$, $E^3_k(\tau) - E_{3k}(\tau)$ has at least $B(k)$ zeros on the left side boundary $\mathcal{F}_{-1/2} = \{-0.5 + iy : y \geq \sqrt{3}/2\}$ where

$$B(k) = \begin{cases} n-q-2 & \text{if } n \equiv 2 \pmod 6, \\ n-q-1 & \text{otherwise} \end{cases}$$

where $k = 12q + r$ with $r \in \{0,4,6,8,10,14\}$.

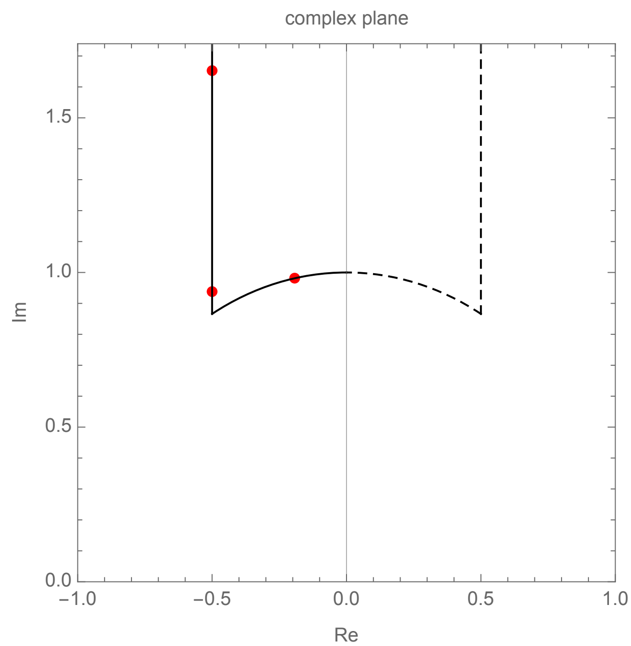

FIGURE 5.3: Zeros of $E_k^3(\tau) - E_{3k}(\tau)$ when $k = 16$

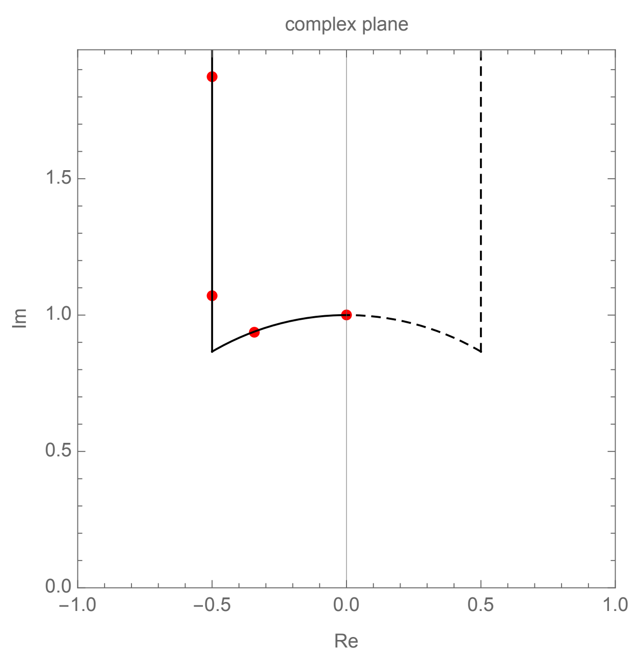

FIGURE 5.4: Zeros of $E_k^3(\tau) - E_{3k}(\tau)$ when $k = 18$

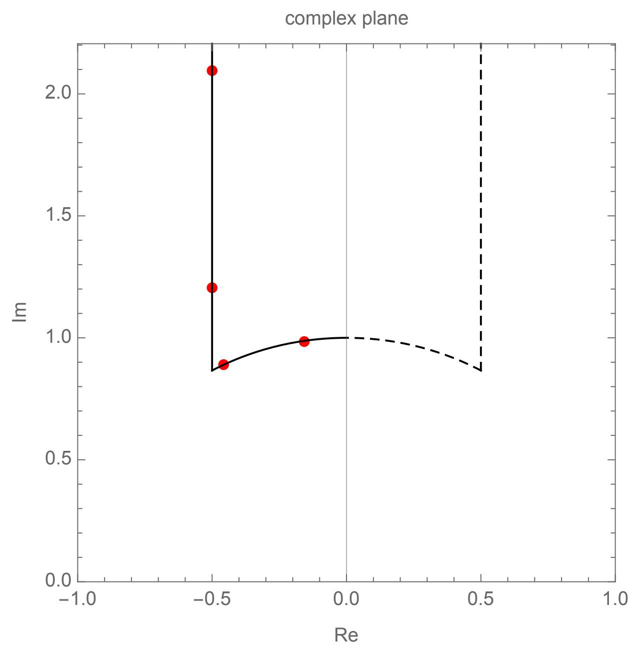

FIGURE 5.5: Zeros of $E_k^3(\tau) - E_{3k}(\tau)$ when $k = 20$

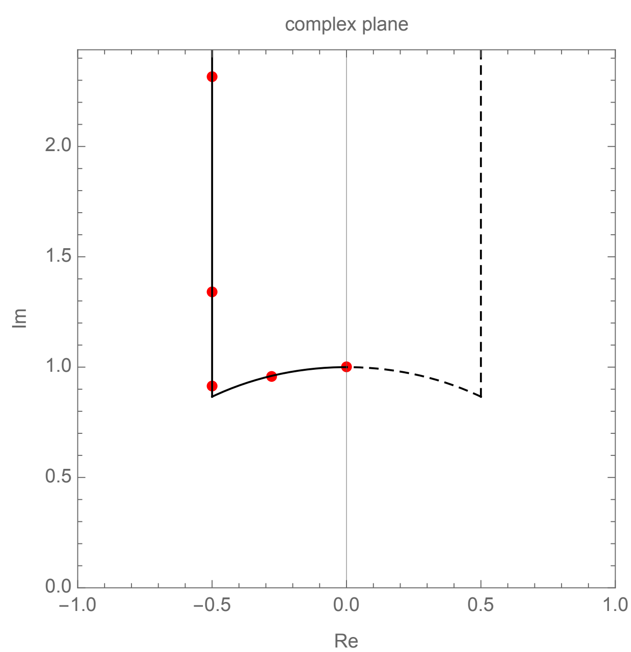

FIGURE 5.6: Zeros of $E_k^3(\tau) - E_{3k}(\tau)$ when $k = 22$

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APPENDICES

A Techniques for finding the zeros of Eisenstein series

In this chapter, we explain the zero-finding algorithm that will be used to not only approximate the zeros in the fundamental domain of higher weight Eisenstein series but also can be generalized to approximate the zeros of our family of certain combination of products of Eisenstein series defined in Chapter 4 and 5.

Since our technique involves some basic knowledge from the theory of elliptic functions, we first introduce the notion of the hypergeometric functions.

Definition A.1. Let a, b and c be arbitrary complex numbers, except that c cannot be a nonpositive integer. Then for $|z| < 1$, the *Gaussian* or *ordinary hypergeometric function* ${}_2F_1(a, b; c; z)$ is defined by

$$ {}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

where $(a)_0 = 1$ and for $n \geq 1$, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$.

In Ramanujan's notation, the most relevant Eisenstein series to our study are defined for $|q| < 1$ by

$$ Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, $$

and

$$ R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. $$

By taking $q := q(\tau) = e^{2\pi i \tau}$ with τ in the upper half plane $\mathbb{H}$, we have that

$$ Q(q) = E_4(\tau) \quad \text{and} \quad R(q) = E_6(\tau). $$

where $E_4(\tau)$ and $E_6(\tau)$ are normalized Eisenstein series of weight 4 and 6 defined in Section 2.

Ramanujan recorded numerous identities connecting certain cases of hypergeometric series with Eisenstein series $Q(q)$ and $R(q)$. These identities can be obtained from his lost notebook [1, 2, 3] which can also be found in [4]. In particular, he derived the following identities

Theorem A.2. If $|q| < 1$, then

$$Q(q) = z^4(1 + 14x + x^2) \quad \text{and} \quad R(q) = z^6(1 + x)(1 - 34x + x^2), \tag{A.1}$$

with

$$z :=_2 F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \varphi^2(q), \quad x := \lambda(2\tau), \tag{A.2}$$

and

$$q := \exp\left(-\pi \frac{{}_2F_1\,(1/2, 1/2; 1; 1 - x)}{{}_2F_1\,(1/2, 1/2; 1; x)}\right) \tag{A.3}$$

where φ and λ are the classical theta and lambda functions

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \lambda(\tau) := \frac{16\eta^8(\tau/2)\eta^{16}(2\tau)}{\eta^{24}(\tau)}. \tag{A.4}$$

Since a set of modular forms is a $\mathbb{C}$-algebra generated by two Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, the higher weight Eisenstein series $E_k(\tau)$ can be written as a polynomial in $E_4(\tau)$ and $E_6(\tau)$ through the following recurrence relation.

Theorem A.3. Let $b(k) := 2(2k+1)\zeta(2k+2)E_{2k+2}(\tau)$. Then for $n \geq 3$, the $b(k)$ satisfy the relation

$$b(k) = \frac{3}{(2k+3)(k-2)} \sum_{n=0}^{k-2} \binom{k-2}{n} b(n)b(k-1-n). \tag{A.5}$$

This recurrence relation is one of may interesting relations between the Weierstrass function and Eisenstein series. For more details, we refer the reader to the Ramanujan's notebooks Part II [1].

From this recurrence relation (A.5) and two identities on $E_4(\tau)$ and $E_6(\tau)$ provided in (A.1), we find that any higher weight Eisenstein series $E_k(\tau)$ can be written as

$$E_k(\tau) = z^k P_k(x)$$

where $P_k(x)$ is a polynomial with rational coefficients, x and z are defined as above (A.2).

Since the theta function $\varphi^2(q)$ is nonvanishing on the upper half plane, finding the zeros of $E_k(\tau)$ is corresponding to finding the zeros of $P_k(x)$.

Let $\{x_\alpha\}$ denote the set of all zeros of the associated polynomial $P_k(x)$. If $|x_\alpha| < 1$, we are able to approximate the corresponding zero of the Eisenstein series using the following *Jacobi Inversion formula.*

Theorem A.4. If $x = \lambda(2\tau)$ with $|x| < 1$, and $q = \exp\left(-\pi \frac{{}_2F_1(1/2,1/2;1;1-x)}{{}_2F_1(1/2,1/2;1;x)}\right)$, then we have that $\tau = T(x)$ where T is defined by

$$T(x) = \frac{i}{2} \frac{{}_2F_1\left(1/2, 1/2; 1; 1-x\right)}{{}_2F_1\left(1/2, 1/2; 1; x\right)}. \tag{A.6}$$

On the other hand, if $|x_\alpha| \geq 1$, we may use the standard analytic continuation of $T(x_\alpha)$.

Applying the function T on the zero set $\{x_\alpha\}$, we obtain the set of zeros $\{\tau_\alpha := T(x_\alpha)\}$ of Eisenstein series $E_k(\tau)$ in the upper half plane. To pick only those lying on the standard fundamental domain $\mathcal{F}$, we will use the $\mathrm{SL}_2(\mathbb{Z})$-invariant property of the modular j-function. Recall that the modular j-function is a modular function defined by

$$j(\tau) := 1728 \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)}.$$

where $\tau \in \mathbb{H}$ and $E_4(\tau), E_6(\tau)$ are Eisenstein series of weight 4 and 6 respectively. Using the Ramanujan's identities for $E_4(\tau)$ and $E_6(\tau)$ in (A.1), we define

$$J(x) := j(x) = 1728 \frac{(1 + 14x + x^2)^3}{(1 + 14x + x^2)^3 - (1 + x)^2(1 - 34x + x^2)^2}. \tag{A.7}$$

It is a property of the modular j-function that it maps the boundary $\partial\mathcal{F}$ of the fundamental domain $\mathcal{F}$ and the imaginary axis $\mathcal{F}_{-1/2}$ to the real line. In fact, one remarkable notice is that j maps the boundary arc $\mathcal{A}$ to the real interval $[0, 1729]$. We also note that the function $J = j \circ x$ maps all equivalent points up to a transformation by elements of $\mathrm{SL}_2(\mathbb{Z})$ to a single point.

This implies that for each output y of the $J(x)$ function, we can pick a unique zero $x_y \in \{x_\alpha\}$ of $P_k(x)$ such that $J(x_y) = y$ and $T(x_y)$ is a zero of the Eisenstein series in the fundamental domain $\mathcal{F}$. Hence, we obtain a set of all zeros $\{T(x_y)\}$ in the fundamental domain for the Eisenstein series $E_k(\tau)$.

Not only this method can be used to find closed expressions for the zeros of higher weight Eisenstein series but also allows us to approximate the zeros of any certain combination of products of Eisenstein series. To reduce the complication, we will use Mathematica to compute the recursions and apply the identities on Eisenstein series. The outline of the zero-finding algorithm for certain combination of products of Eisenstein series can be obtained in the next Appendix B.

B Outline for zero-finding algorithm of $E_k^3(\tau) - E_{3k}(\tau)$ and $E_k^4(\tau) + E_{4k}(\tau)$ when $k = 12$

This chapter will be devoted to finding zeros in the fundamental domain $\mathcal{F}$ of $E_k^3(\tau) - E_{3k(\tau)}$ and $E_k^4(\tau) + E_{4k}(\tau)$ when $k = 12$. We use the aid of Mathematica commands and Ramanujan identities mentoined in Appendices A to approximate the zeros in the fundamental domain. This technique was presented in the work of SD Sandragorsian in his master thesis [21].

As explained in Appendices A, any Eisenstein series of weight $k \geq 4$ can be written as

$$E_k(\tau) = z^k P_k(x)$$

where $P_k(x)$ is a polynomial with all coefficients in $\mathbb{Q}$, x and z are given in terms of lambda and theta functions defined in (A.2). These identities along with the recurrence relation given in (A.5) allow us to express any combination of products of Eisenstein series in terms of variables x and z. In particular, we find that

$$(E_k^n \pm E_{nk})(\tau) = z^{nk} (P_k^n \pm P_{nk})(x). \tag{B.1}$$

To locate the zeros of $E_{12}^3 - E_{36}$, we begin by solving $(P_{12}^3 - P_{36})(x) = 0$. With the aid of Mathematica, we find that $(P_{12}^3 - P_{36})(x) = 0$ if and only if

$$\begin{aligned} x \in \{&0, 1, -121.982, -3.27024, -0.305788, -0.00819793, -0.996013 \pm 26.9738i, \\ &- 0.929399 \pm 5.1138i, -0.0344034 \pm 0.189297i, -0.00136707 \pm 0.0370226i\}. \end{aligned} \tag{B.2}$$

Computing the image of these zeros under the J function (defined in A.7) using the RootApproximant command in Mathematica, we obtain

$$J(x) \in \left\{\infty, \frac{1}{5}\left(-3282 - \sqrt{10702249}\right), \frac{1}{2}\left(623 + 3\sqrt{32729}\right)\right\}. \tag{B.3}$$

This means that the above zero set (B.2) is divided into 3 disjoint subsets depending on its image under J function, say

$$S_1 := \{0, 1\}$$

$$S_2 := \{-121.982, -0.00819793, -0.929399 \pm 5.1138i, , -0.0344034 \pm 0.189297i\},$$

$$S_3 := \{-3.27024, -0.305788, -0.996013 \pm 26.9738i, -0.00136707 \pm 0.0370226i\}.$$

Since the function $J(x) = (j \circ x)(\tau)$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, for $i = 1, 2, 3$, all elements in S_i are mapped to the $\mathrm{SL}_2(\mathbb{Z})$- equivalence zeros. This implies that $E_{12}^3 - E_{36}$ has exactly 3 zeros in the fundamental domain $\mathcal{F}$ including $i\infty$ which is consistent to the valence formula (2.11).

To reveal the zeros in the fundamental domain of $E_{12}^3 - E_{36}$, we apply the *Jacobi Inversion formula* given in Theorem A.4 on each subset S_i and only select the unique $x \in S_i$ so that the corresponding zero $T(x) \in \mathcal{F} \cup \{i\infty\}$;

$$T(0) = i\infty,$$

$$T(-0.00819793) \approx -0.5 + 1.20648i,$$

$$T(-0.00136707 + 0.0370226i) \approx -0.258816 + 0.965927i.$$

Therefore, $E_{12}^3 - E_{36}$ has one zero at $\tau = i\infty$, one zero on the left side boundary $\mathcal{F}_{-1/2}$ at $\tau = -0.5 + 1.20648i$ and one zero on the arc $\mathcal{A}$ at $\tau = -0.258816 + 0.965927i$ since

$$|-0.258816 + 0.965927i| \approx 1.$$

Note that by (B.3), the J function maps all zeros of $(P_{12}^3 - P_{36})(x)$ to real numbers. From this, we can guess that all zeros of $E_{12}^3(\tau) - E_{36}(\tau)$ might be on the boundary of the fundamental domain $\mathcal{F}$.

Now, we turn to locating the zeros of $E_{12}^4(\tau) + E_{48}(\tau)$ in the fundamental domain. Using the identities in (B.1), we have that

$$(E_{12}^4 + E_{48})(\tau) = z^{48}\left(P_{12}^4 + P_{48}\right)(x).$$

Computing the zeros of the associated polynomial $(P_{12}^4 + P_{48})(x)$, we obtain

$$\begin{aligned} x \in \{&-2.30694, -2.00167, -0.499583, -0.433474, -14.0723 \pm 23.4622i, \\ &-8.20688 \pm 18.1996i, -4.67885 \pm 0.641554i, -0.209784 \pm 0.0287651i, \\ &-0.0205903 \pm 0.0456613i, -0.0188005 \pm 0.0313454i, 0.013318 \pm 0.030358i, \\ &0.0181556 \pm 0.0263272i, 12.1186 \pm 27.6239i, 17.7518 \pm 25.7416i\}. \end{aligned} \tag{B.4}$$

Mapping these zeros under the J function (defined in A.7) using the RootApproximant command in Mathematica, we obtain

$$J(x) \in \left\{ \frac{1}{4}(2031 + \sqrt{4139165}), 601 + \sqrt{359543}, \frac{794077 \pm i2\sqrt{24989316879}}{3355} \right\}. \tag{B.5}$$

Hence, the above zero set (B.4) is divided into 4 disjoint subsets depending on its image under J function, say

$$S_1 := \{-2.30694, -0.433474, 0.013318 \pm 0.030358i, 12.1186 \pm 27.6239i\}$$

$$S_2 := \{-2.00167, -0.499583, 0.0181556 \pm 0.0263272i, 17.7518 \pm 25.7416i\},$$

$$\begin{aligned} S_3 := \{&-14.0723 - 23.4622i, -8.20688 + 18.1996i, -4.67885 - 0.641554i, \\ &-0.209784 + 0.0287651i, -0.0205903 - 0.0456613i, -0.0188005 + 0.0313454i\}. \end{aligned}$$

and

$$\begin{aligned} S_4 := \{&-14.0723 + 23.4622i, -8.20688 - 18.1996i, -4.67885 + 0.641554i, \\ &-0.209784 - 0.0287651i, -0.0205903 + 0.0456613i, -0.0188005 - 0.0313454i\}. \end{aligned}$$

Since the function $J(x) = (j \circ x)(\tau)$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, for $i = 1, 2, 3, 4$, all elements in S_i are mapped to the $\mathrm{SL}_2(\mathbb{Z})$- equivalence zeros. This implies that $E_{12}^4(\tau) + E_{48}(\tau)$ has exactly 4 zeros in the fundamental domain $\mathcal{F}$ which is consistent to the valence formula (2.11).

To reveal the zeros in the fundamental domain, we apply the *Jacobi Inversion formula* given in Proposition A.4 on each subset S_i and select the unique $x \in S_i$ so that the corresponding zero $T(x) \in \mathcal{F}$;

$$T(0.013318 + 0.030358i) \approx -0.186645 + 0.982428i,$$

$$T(0.0181556 + 0.0263272i) \approx -0.156041 + 0.987751i,$$

$$T(-0.0188005 + 0.0313454i) \approx -0.338442 + 0.969437i.$$

$$T(-0.0188005 - 0.0313454i) \approx \ 0.338442 + 0.969437i.$$

Therefore, $E_{12}^4(\tau) + E_{48}(\tau)$ has two zeros on the lower boundary arc $\mathcal{A}$ at $\tau = -0.186645 + 0.982428i, -0.156041 + 0.987751i$ since

$$|-0.186645 + 0.982428i| = |-0.156041 + 0.987751i| \approx 1.$$

and two zeros inside the interior of the fundamental domain at $\tau = \pm 0.338442 + 0.969437i$ since

$$|\text{Re}(\pm 0.338442 + 0.969437i)| \approx 0.338442 < 0.5,$$

and

$$| \pm 0.338442 + 0.969437i| \approx 1.02682 > 1.$$

Note that by (B.5), in this case the J function maps two groups of the zeros of $(P_{12}^4 + P_{48})(x)$ to 2 complex numbers. From this, we can guess that two zeros of $E_{12}^4(\tau) + E_{48}(\tau)$ might be inside the interior of the fundamental domain $\mathcal{F}$.