AN ABSTRACT OF THE THESIS OF

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Title: AUXILIARY CONDITIONS FOR THE CONVERGENCE OF SEQUENCES OF MEASURABLE FUNCTIONS

Abstract approved: Redacted for Privacy
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Let $\{f_n\}$ be a sequence of a. e. finite-valued measurable functions defined on an arbitrary measure space (X, S, μ). We consider 17 properties of (X, S, μ) and $\{f_n\}$, which include the ten auxiliary hypotheses used in reference (2), but none of the nine modes of convergence considered there. For each of the 17 conditions as a conclusion, we give a complete determination of all the subsets of the other 16 conditions whose conjunction implies the desired conclusion. Thus this study could be the starting point for other investigations of the modes of convergence more comprehensive than (2).

Fifty-two basic theorems are stated and proved, and 55 counterexamples are used to show that no additional implications of the type described are possible. Thirteen figures display the implications schematically.

Auxiliary Conditions for the Convergence of Sequences of Measurable Functions

by

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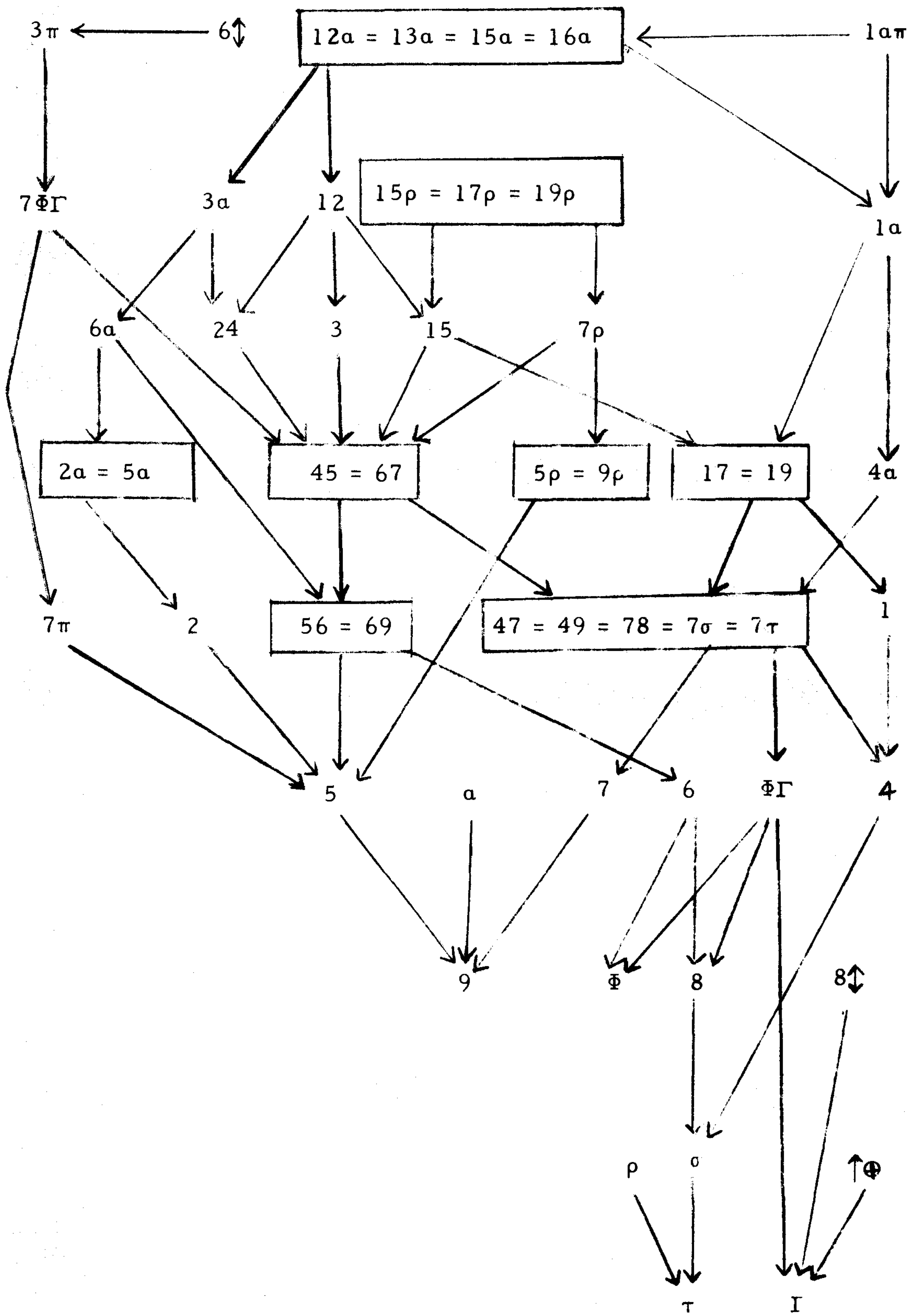


Figure 1. Implications of single and paired hypotheses.

AUXILIARY CONDITIONS FOR THE CONVERGENCE OF SEQUENCES OF MEASURABLE FUNCTIONS

I. INTRODUCTION AND DEFINITIONS

Let $\{f_n\}$ be a sequence of real almost-everywhere finite-valued measurable functions defined on an arbitrary measure space (X, S, μ). We are concerned with 17 properties that such a sequence or its underlying space may have, and in particular with the interrelationships among these various properties. The 17 hypotheses and three other related properties will be listed below together with the symbols by which they will hereafter be identified.

1. The support of the functions $E = \bigcup_{n=1}^{\infty} \{x : f_n(x) \neq 0\}$ has finite measure.
2. The sequence of functions $\{f_n\}$ is uniformly essentially bounded.
3. There is an integrable function g such that $|f_n| \leq g$ a.e., $n = 1, 2, \ldots$.
4. The sequence of functions $\{f_n\}$ is uniformly semi-finitely-supported; that is given $\epsilon > 0$, there exists a measurable set F with $\mu F < \infty$ and $\int_{X-F} |f_n| d\mu < \epsilon$ for all n.
5. The sequence of functions $\{f_n\}$ is uniformly semi-bounded; that is given $\epsilon > 0$, there exists a number K such that $\int_{|f_n| \geq K} |f_n| d\mu < \epsilon$ for all n.

6. The sequence of integrals $\{\int |f_n| d\mu\}$ is uniformly bounded.

7. The indefinite integrals of $|f_n|$ $n = 1, 2, 3, \ldots$ are equicontinuous from above at ϕ; that is for every decreasing sequence of sets $\{E_n\}$ for which $\lim_n E_n = \phi$, and for every $\epsilon > 0$, there exists n_0 such that for $n \geq n_0$ $\int_{E_n} |f_m| d\mu < \epsilon$ $m = 1, 2, 3, \ldots$.

8. Each member of the sequence of functions $\{f_n\}$ is integrable.

9. The indefinite integrals of $|f_n|$ $n = 1, 2, \ldots$ are uniformly absolutely continuous; that is given $\epsilon > 0$, there exists $\delta > 0$ such that $\int_E |f_n| d\mu < \epsilon$ $n = 1, 2, \ldots$ for measurable E for which $\mu E < \delta$.

α. The measure space (X, S, μ) is uniformly atomic; that is there exists $\delta > 0$ such that $\mu E > 0 \rightarrow \mu E > \delta$ for all measurable E.

ρ. The measure space (X, S, μ) is non-atomic

σ. The measure space (X, S, μ) is σ-finite.

τ. The measure space (X, S, μ) has no infinite atoms.

$\uparrow$. The sequence of functions $\{f_n\}$ in nondecreasing a. e.

$\downarrow$. The sequence of functions $\{f_n\}$ is nonincreasing a. e.

$\updownarrow$. The sequence of functions $\{f_n\}$ is monotonic a. e.

$\oplus$. $f_n \geq 0$ a. e. for all n.

Φ. The function f is integrable.[1]

Γ. $\lim \int f_n d\mu = \int f d\mu$ (It is assumed that $\lim_n \int f_n d\mu$ exists in the sense that it is finite, equal to ∞, or equal to $-\infty$.)[2]

π. The sequence $\{f_n\}$ converges to f a. e.

We propose to investigate the existence or nonexistence of an implication in either direction between any two subsets containing one, two, three or four hypotheses. It turns out that none of the five properties 1, ρ, α, $\updownarrow$ and $\oplus$ can be implied by any subset of the remaining 16 properties. In Chapter II we give 12 figures which summarize the results of the thesis. These results are stated as theorems and proved in detail in Chapter III. The problem arises of showing that these figures include all the cases in which one of the 17 conditions implies another. This is done by means of 55 counter-examples which appear in the last chapter. Accordingly, each figure contains all the non-redundant sets of four or fewer hypotheses which imply the given property, and each arrow indicates an implication. Following the figures, the results are stated and proved.

We record below a collection of definitions and theorems some of which are not found in standard literature. All our functions are measurable and a. e. finite.

---

[1] In this thesis, in any implication involving either Φ or Γ, the assumption π that $f_n(x) \rightarrow f(x)$ a. e. is to be understood.

[2] The same as 1.

1. Definition. A measurable set $E \subseteq X$ is said to be an atom with respect to μ if $\mu E > 0$ and if for every measurable set $A \subseteq E$ we have either $\mu A = 0$ or $\mu A = \mu E$.

2. Definition. A measure μ is said to be atomic if there exists at least one atom with respect to μ, and μ is non-atomic if there exists no atom with respect to μ.

3. Definition. A measure μ is said to be uniformly atomic if there exists a real number $\delta > 0$ such that $\mu E > 0 \rightarrow \mu E > \delta$.

4. Definition. A measurable set $E \subseteq X$ has the Darboux property with respect to μ if for every number a such that $0 \leq a \leq \mu E$ there exists a measurable set $A \subseteq E$ with $\mu A = a$.

5. Definition. μ is said to have the Darboux property if every measurable set E has the Darboux property.

6. Theorem (3, p. 28). A σ-finite non-atomic measure has the Darboux property.

7. Theorem (4, Theorem H, p. 97). The indefinite integral of an integrable function is absolutely continuous.

8. Theorem (4, p. 105). If f is an integrable function, then the set $N(f) = \{x : f(x) \neq 0\}$ has σ-finite measure.

9. Theorem (Lebesgue's bounded convergence theorem). If $\{f_n\}$ is a sequence of measurable functions which converges in measure to f [or else converges to f a.e.], and if g is an integrable function such that $|f_n| \leq g$ a.e., $n = 1, 2, \ldots,$ then f is integrable and $\{f_n\}$ converges to f in mean.

10. Theorem (4, Theorem D, p. 38). If μ is a measure on a ring R and if $\{E_n\}$ is an increasing sequence of sets in R for which $\lim_n E_n \in R$, then $\mu(\lim_n E_n) = \lim_n \mu E_n$.

11. Theorem (4, Theorem E, p. 38). If μ is a measure on a ring R and if $\{E_n\}$ is a decreasing sequence of sets in R of which at least one has finite measure and for which $\lim_n E_n \in R$, then $\mu(\lim_n E_n) = \lim_n \mu E_n$.

12. Definition. A measurable function f is said to be semi-bounded if for each $\epsilon > 0$, there exists a number K such that $\int_{|f| \geq K} |f| d\mu < \epsilon$.

13. Definition. A sequence $\{f_n\}$ of measurable functions is said to be uniformly integrable if and only if it is uniformly semi-bounded and uniformly semi-finitely-supported. Equivalently, $\{f_n\}$ is uniformly integrable if and only if the indefinite integrals $\{\int |f_n| d\mu\}$ are equicontinuous from above at ϕ and the integrals $\{\int_X |f_n| d\mu\}$ are uniformly bounded.

II. SUMMARY OF RESULTS

We summarize the thesis by considering each one of the 17 hypotheses individually. For each non-trivial case, we include a listing of implications, a table of counterexamples, a figure of logical connections among the implications and a summary of the proofs of the results. Each listing contains sets of four or fewer hypotheses which imply the hypothesis in question. An examination of the tables consisting of counterexamples and the satisfied hypotheses shows that no additional non-redundant results are possible except those involving $\uparrow\downarrow$ (i.e., $f_n(x)$ independent of n). The proofs of implications are greatly facilitated by the figures which also show the interrelationships among the results for each case.

Sequences of Functions with Finite Support (1)

Counterexamples 21 and 42 show that Hypothesis 1 is not a consequence of any combination of the 16 remaining hypotheses.

Essentially Uniformly Bounded Sequences of Functions (2)

List of Implications.

$3a$	$5a$	$6a$					
$1a\Phi$	$1a\Gamma$	$4a\Phi$	$4a\Gamma$	$7a\Phi$	$7a\Gamma$		
$1a\oplus\downarrow$	$4a\oplus\downarrow$	$7a\oplus\downarrow$	$8a\oplus\downarrow$	$8a\updownarrow\Phi$	$a\uparrow\oplus\Phi$	$a\oplus\Phi\Gamma$	$a\updownarrow\Phi\Gamma$

Counterexample	Hypotheses Satisfied																
11	1	3	4	5	6	7	8	9		ρ	σ	τ	↑	⊕	↓	Φ	Γ
22	1		4			7	8	9	a		σ	τ	↑	⊕			
37							8	9	a		σ	τ				Φ	Γ
38								9	a		σ	τ		⊕	↓	Φ	
39								9	a		σ	τ		⊕	↓		Γ
40							8	9	a		σ	τ		⊕		Φ	
41							8	9	a		σ	τ	↑	⊕			Γ

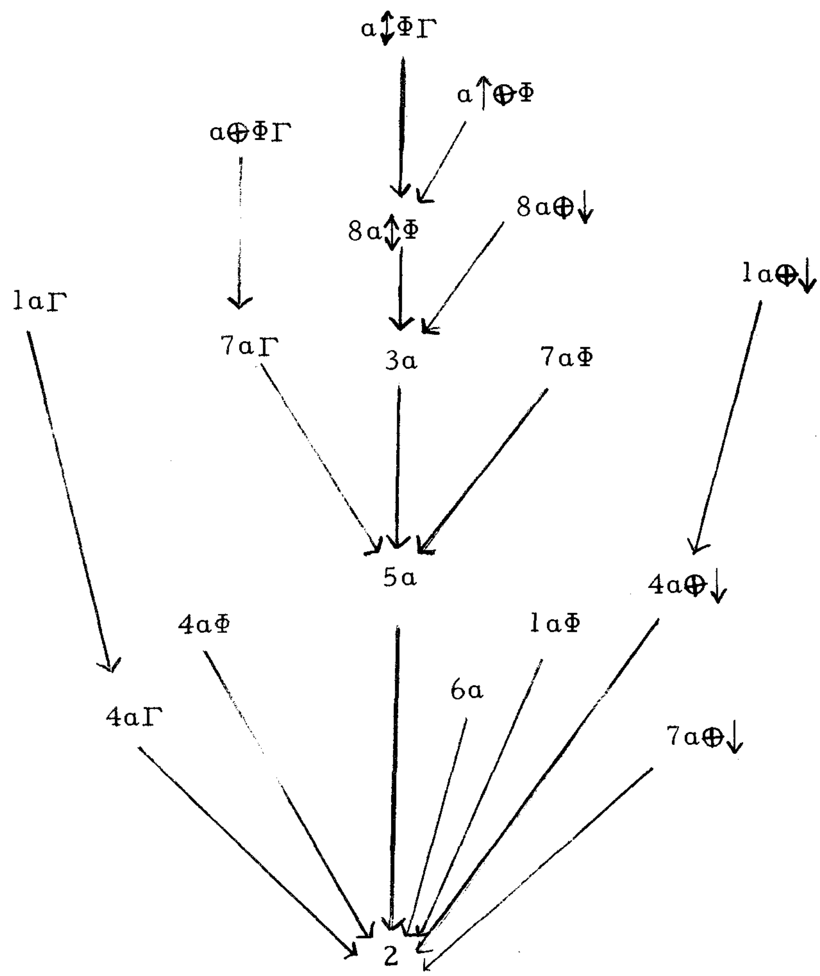


Figure 2.

Proofs of Implications in Figure 2.

$1\alpha\Phi \to 2$: Theorem 38

$1\alpha\Gamma \to 2$: Theorem 38

$6\alpha \to 2$: Theorem 37

$3\alpha \to 2$: Theorems 4, 5 and 37

$5\alpha \to 2$: Theorem 41

$7\alpha\Phi \to 2$: Theorem 42

$7\alpha\Gamma \to 2$: Theorem 42

$4\alpha\Phi \to 2$: Theorems 40 and 41

$4\alpha\Gamma \to 2$: Theorems 40 and 41

$8\alpha\updownarrow\Phi \to 2$: Theorem 37 and the fact that $8\pi\Phi \to 3$ (2, Theorem 15, p. 25)

$\alpha\updownarrow\Phi\Gamma \to 2$: The above implication and the fact that $\Phi\Gamma \to 8$

$\alpha\oplus\Phi\Gamma \to 2$: Theorems 34 and 41

$4\alpha\oplus\downarrow \to 2$: Theorem 51

$1\alpha\oplus\downarrow \to 2$: Theorems 19 and 51

$7\alpha\oplus\downarrow \to 2$: Theorem 49

$8\alpha\oplus\downarrow \to 2$: Theorem 46

$\alpha\uparrow\oplus\Phi \to 2$: Theorem 47

Dominated Sequences of Measurable Functions (3)

List of Implications.

12	$6\updownarrow$							
15α	$15\updownarrow$	16α	$1\alpha\Phi$	$1\alpha\Gamma$	$24\updownarrow$	$45\updownarrow$	$7\rho\updownarrow$	$8\updownarrow\Phi$
$8\oplus\downarrow$	$\updownarrow\Phi\Gamma$	$\uparrow\oplus\Phi$						
$17\updownarrow\Phi$	$17\updownarrow\Gamma$	$17\oplus\downarrow$	$19\rho\updownarrow$	$19\updownarrow\Phi$	$19\updownarrow\Gamma$	$19\oplus\downarrow$	$1\alpha\oplus\downarrow$	$278\updownarrow$
$27\sigma\updownarrow$	$27\tau\updownarrow$	$47\updownarrow\Phi$	$47\updownarrow\Gamma$	$47\oplus\downarrow$	$49\updownarrow\Phi$	$49\updownarrow\Gamma$	$49\rho\updownarrow$	$49\oplus\downarrow$
$4\alpha\oplus\downarrow$	$4\alpha\updownarrow\Phi$	$4\alpha\updownarrow\Gamma$	$578\updownarrow$	$57\sigma\updownarrow$	$57\updownarrow\tau$	$78\updownarrow\Gamma$	$7\sigma\updownarrow\Phi$	$7\sigma\updownarrow\Gamma$
$7\sigma\oplus\downarrow$	$7\tau\updownarrow\Phi$	$7\tau\updownarrow\Gamma$	$7\tau\oplus\downarrow$					

Counterexample	Hypotheses Satisfied																
1		2		5		7		9	a					$\oplus$	$\downarrow$	Φ	
3		2		5			8	9		ρ	σ	τ	$\uparrow$	$\oplus$			Γ
5		2		5				9		ρ	σ	τ		$\oplus$	$\downarrow$	Φ	
7		2	4	5	6	7	8	9		ρ	σ	τ		$\oplus$		Φ	Γ
9	1		4				8				σ	τ	$\uparrow$	$\oplus$			Γ
14	1		4		6		8			ρ	σ	τ		$\oplus$		Φ	
16	1		4				8			ρ	σ	τ	$\uparrow$	$\oplus$			
18		2		5			8	9	a		σ	τ	$\uparrow$	$\oplus$			Γ
22	1		4			7	8	9	a		σ	τ	$\uparrow$	$\oplus$			
23		2		5				9	a		σ	τ	$\uparrow$	$\oplus$	$\downarrow$		Γ
24	1		4	5	6	7	8	9		ρ	σ	τ		$\oplus$		Φ	Γ
25	1		4							ρ	σ	τ		$\oplus$	$\downarrow$	Φ	
43		2	4	5	6	7	8	9	a		σ	τ		$\oplus$		Φ	Γ
44	1		4							ρ	σ	τ	$\uparrow$	$\oplus$	$\downarrow$		Γ
52		2		5				9	a			τ		$\oplus$	$\downarrow$	Φ	
54		2		5		7		9	a				$\uparrow$	$\oplus$	$\downarrow$		Γ
55		2		5				9	a		σ	τ		$\oplus$	$\downarrow$	Φ	

Proofs of Implications in Figure 3.

15a $\rightarrow$ 16a: Theorems 5 and 19

16a $\rightarrow$ 12: Theorem 37

1aΦ $\rightarrow$ 12: Theorem 38

1aΓ $\rightarrow$ 12: Theorem 38

12 $\rightarrow$ 3: (2, Theorem 6, p. 21)

7$\sigma\updownarrow\Phi$ $\rightarrow$ 8$\updownarrow\Phi$: (2, Theorem 15, p. 25)

8$\updownarrow\Phi$ $\rightarrow$ 3: (2, p. 18)

7$\tau\updownarrow\Gamma$ $\rightarrow$ 47$\updownarrow\Gamma$: Theorem 22

49$\updownarrow\Gamma$ $\rightarrow$ 7$\sigma\updownarrow\Gamma$: Theorems 16 and 17

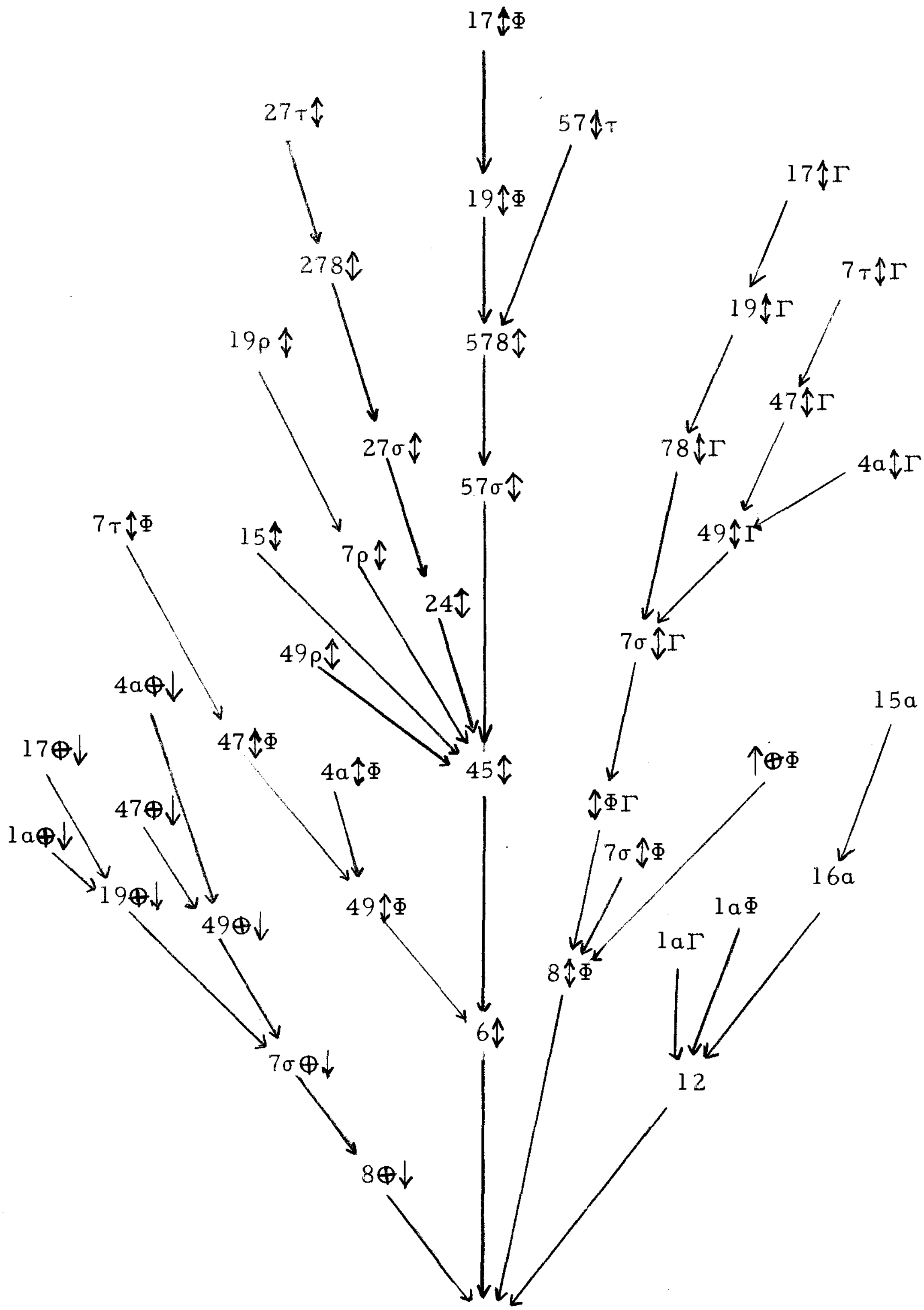

17↕Φ
27τ↕
57↕τ
19↕Φ
17↕Γ
278↕
7τ↕Γ
19↕Γ
19ρ ↕
578↕
47↕Γ
78↕Γ
27σ↕
57σ↕
4a↕Γ
7τ↕Φ
15↕
49↕Γ
7ρ↕
24↕
7σ↕Γ
49ρ↕
4a⊕↓
15a
17⊕↓
47↕Φ
↑⊕Φ
4a↕Φ
45↕
↕ΦΓ
47⊕↓
1a⊕↓
7σ↕Φ
16a
19⊕↓
49⊕↓
49↕Φ
1aΦ
1aΓ
8↕Φ
6↕
7σ⊕↓
12
8⊕↓
3

Figure 3.

$7\sigma\updownarrow\Gamma \to \updownarrow\Phi\Gamma$: (2, Theorem 15, p. 25)

$\updownarrow\Phi\Gamma \to 8\updownarrow\Phi$: Obvious

$17\updownarrow\Gamma \to 19\updownarrow\Gamma$: Theorem 18

$19\updownarrow\Gamma \to 78\updownarrow\Gamma$: (2, Theorems 7 and 15, p. 22 and 25)

$78\updownarrow\Gamma \to 7\sigma\updownarrow\Gamma$: (4, p. 105)

$57\updownarrow\tau \to 578\updownarrow$: Theorem 12

$578\updownarrow \to 57\sigma\updownarrow$: (4, p. 105)

$57\sigma\updownarrow \to 45\updownarrow$: Theorem 28

$45\updownarrow \to 6\updownarrow$: Theorem 5

$6\updownarrow \to 3$: Theorem 30

$27\tau\updownarrow \to 278\updownarrow$: Theorem 12

$278\updownarrow \to 27\sigma\updownarrow$: (4, p. 105)

$27\sigma\updownarrow \to 24\updownarrow$: Theorem 28

$24\updownarrow \to 45\updownarrow$: Theorem 15

$19\rho\updownarrow \to 7\rho\updownarrow$: (2, Theorem 7, p. 22)

$7\rho\updownarrow \to 45\updownarrow$: Theorems 14 and 22

$15\updownarrow \to 45\updownarrow$: Theorem 19

$49\rho\updownarrow \to 45\updownarrow$: Theorem 14

$7\tau\updownarrow\Phi \to 47\updownarrow\Gamma$: Theorem 22

$47\updownarrow\Phi \to 49\updownarrow\Gamma$: Theorem 18

$49\updownarrow\Phi \to 6\updownarrow$: Theorems 39 and 5

$\uparrow\oplus\Phi\pi \to 8\updownarrow\Phi$: Theorem 47

$17\oplus\downarrow \to 19\oplus\downarrow$: Theorem 18

$4\alpha\updownarrow\Phi \to 49\updownarrow\Phi$: Theorem 21

$4\alpha\updownarrow\Gamma \to 49\updownarrow\Gamma$: Theorem 21

$8\oplus\downarrow \to 3$: Theorem 46

$7\sigma\oplus\downarrow \to 8\oplus\downarrow$: (2, Theorem 15, p. 25)

$4\alpha\oplus\downarrow \to 49\oplus\downarrow$: Theorem 21

$49\oplus\downarrow \to 7\sigma\oplus\downarrow$: Theorems 16 and 17

$47\oplus\downarrow \to 49\oplus\downarrow$: Theorem 18

$19\oplus\downarrow \rightarrow 7\sigma\oplus\downarrow$: (2, Theorem 7, p. 22)

$1a\oplus\downarrow \rightarrow 19\oplus\downarrow$: Theorem 21

$7\tau\oplus\downarrow \rightarrow 8\oplus\downarrow$: Theorem 12

Uniformly Semi-Finitely-Supported Sequences of Measurable Functions (4)

List of Implications.

1	3					
67	$6\updownarrow$	78	7ρ	7σ	7τ	
$6\oplus\Gamma$	$7\Phi\Gamma$	$8\updownarrow\Phi$	$8\oplus\downarrow$	$\updownarrow\Phi\Gamma$	$\uparrow\oplus\Phi$	$\oplus\Phi\Gamma$

Counterexamples	Hypotheses Satisfied													
1	2	5		7		9	a					$\oplus$	$\downarrow$	Φ	
3	2	5			8	9		ρ	σ	τ	$\uparrow$	$\oplus$			Γ
5	2	5				9		ρ	σ	τ		$\oplus$	$\downarrow$	Φ	
10	2	5	6		8	9		ρ	σ	τ				Φ	Γ
13	2	5		7		9	a				$\uparrow$	$\oplus$	$\downarrow$		Γ
15	2	5	6		8	9		ρ	σ	τ		$\oplus$		Φ	
18	2	5			8	9	a		σ	τ	$\uparrow$	$\oplus$			Γ
23	2	5				9	a		σ	τ	$\uparrow$	$\oplus$	$\downarrow$		Γ
27	2	5				9	a		σ	τ		$\oplus$	$\downarrow$	Φ	
31	2	5			8	9	a		σ	τ				Φ	Γ
32	2	5	6		8	9	a		σ	τ		$\oplus$		Φ	
34	2	5	6		8	9	a		σ	τ				Φ	Γ

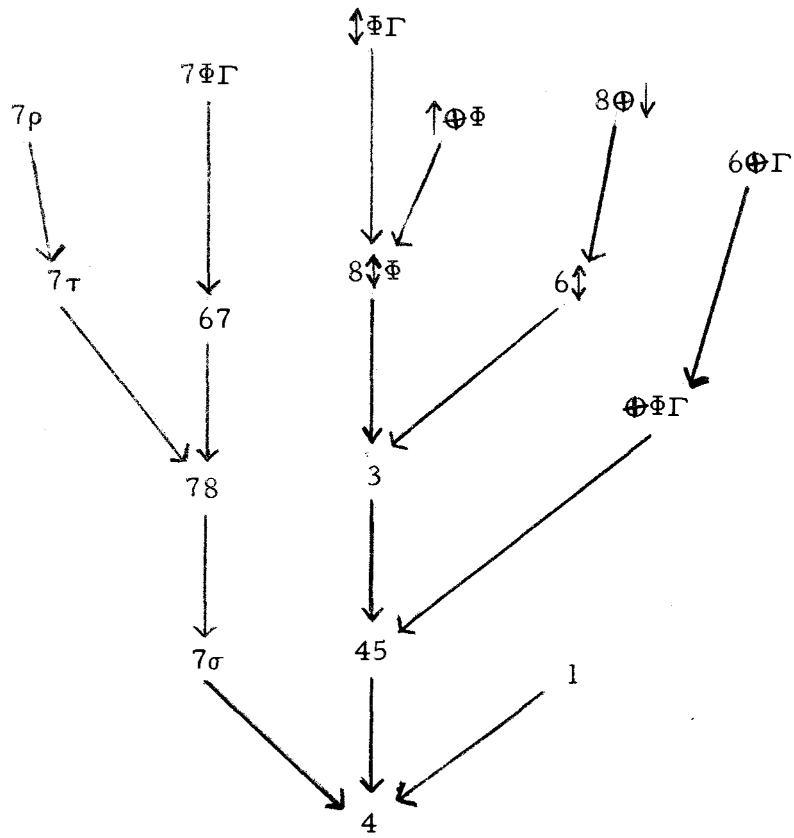


Figure 4.

Proofs of Implications in Figure 4.

1 → 4: Theorem 19

6⊕Γ → ⊕ΦΓ: Obvious

⊕ΦΓ → 4: Theorem 19

6↕ → 3: Theorem 30

3 → 4: Theorem 4

7ΦΓ → 67: Theorem 26

67 → 78: Obvious

78 → 7σ: (4, p. 105)

7σ → 4: Theorem 28

7ρ → 7τ: Obvious

$7\tau \rightarrow 78$: Theorem 12

$8\oplus\downarrow \rightarrow 6\updownarrow$: Theorems 46, 4 and 5

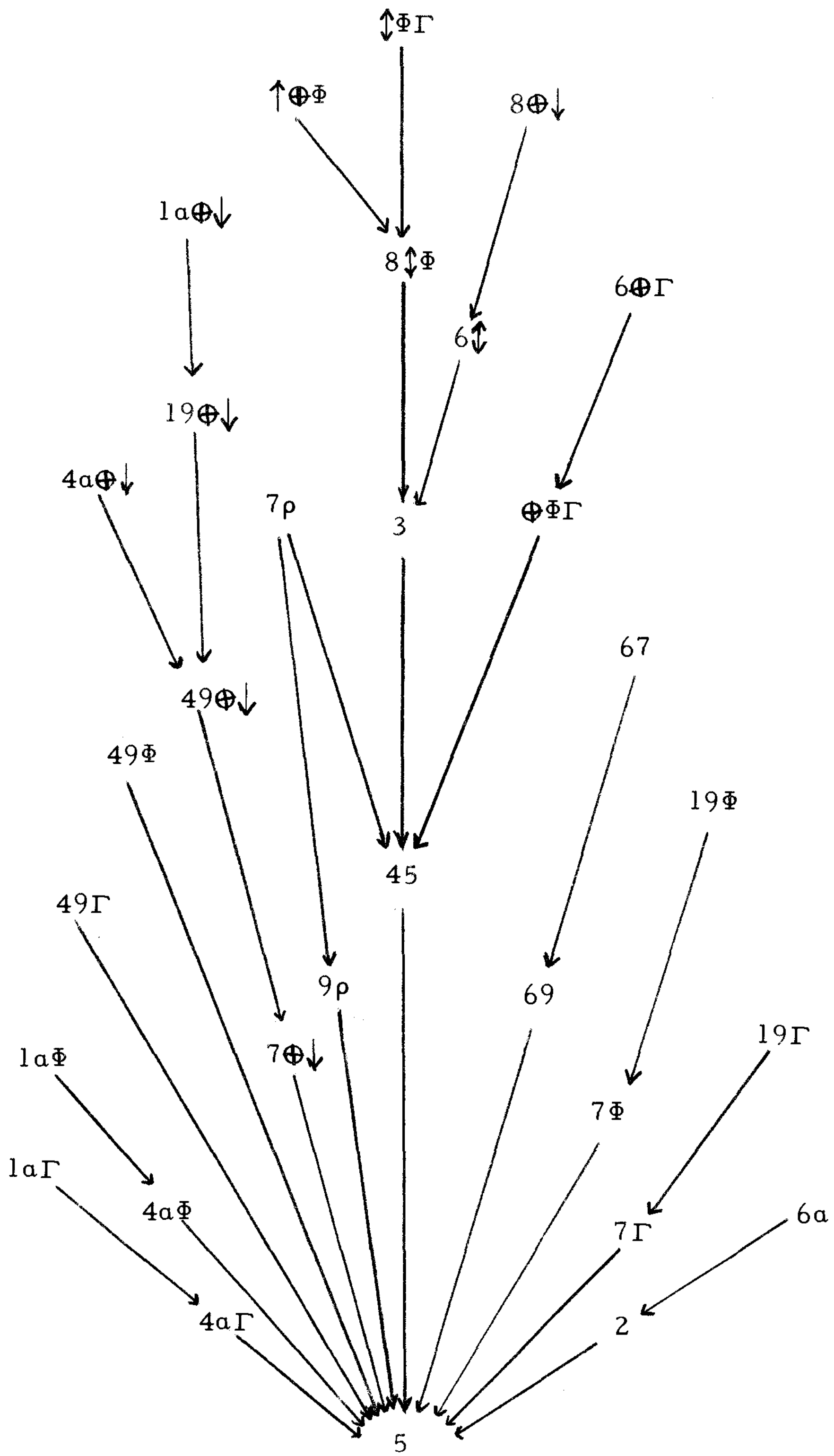
$\uparrow\oplus\Phi\pi \rightarrow 8\updownarrow\Phi$: Theorem 47

Uniformly Semi-Bounded Sequences of Measurable Functions (5)

List of Implications.

2	3							
67	69	$6a$	$6\updownarrow$	7ρ	7Φ	7Γ	9ρ	
19Φ	19Γ	$1a\Phi$	$1a\Gamma$	$4a\Phi$	$4a\Gamma$	49Φ	49Γ	$6\oplus\Gamma$
$7\oplus\downarrow$	$8\updownarrow\Phi$	$8\oplus\downarrow$	$\uparrow\oplus\Phi$	$\updownarrow\Phi\Gamma$	$\oplus\Phi\Gamma$			
$19\oplus\downarrow$	$1a\oplus\downarrow$	$49\oplus\downarrow$	$4a\oplus\downarrow$					

Counterexamples	Hypotheses Satisfied														
2	1	4	6		8			ρ	σ	τ				Φ	Γ
14	1	4	6		8			ρ	σ	τ		$\oplus$		Φ	
22	1	4		7	8	9	a		σ	τ	$\uparrow$	$\oplus$			
25	1	4						ρ	σ	τ		$\oplus$	$\downarrow$	Φ	
26	1	4			8			ρ	σ	τ	$\uparrow$	$\oplus$			Γ
37					8	9	a		σ	τ				Φ	Γ
38						9	a		σ	τ		$\oplus$	$\downarrow$	Φ	
39						9	a		σ	τ		$\oplus$	$\downarrow$		Γ
40					8	9	a		σ	τ		$\oplus$		Φ	
41					8	9	a		σ	τ	$\uparrow$	$\oplus$			Γ
44	1	4						ρ	σ	τ	$\uparrow$	$\oplus$	$\downarrow$		Γ



↕ΦΓ
↑⊕Φ
8⊕↓
1a⊕↓
8↕Φ
6⊕Γ
6↕
19⊕↓
4a⊕↓
7ρ
3
⊕ΦΓ
67
49⊕↓
49Φ
19Φ
45
49Γ
9ρ
69
19Γ
1aΦ
7⊕↓
7Φ
1aΓ
4aΦ
7Γ
6a
4aΓ
2
5

Figure 5.

Proofs of Implications in Figure 5.

19Γ → 7Γ: (2, Theorem 7, p. 22)

7Γ → 5: Theorem 33

19Φ → 7Φ: (2, Theorem 7, p. 22)

7Φ → 5: Theorem 33

67 → 69: Theorem 18

69 → 5: Theorem 36

6⊕Γ → ⊕ΦΓ: Obvious

⊕ΦΓ → 5: Theorem 34

↕ΦΓ → 8Φ↕: Obvious

8Φ↕ → 3: (2, Theorem 15, p. 25)

3 → 5: Theorem 4

7ρ → 9ρ: Theorem 18

9ρ → 5: Theorem 14

6α → 5: Theorem 34

49Φ → 5: Theorem 39

49Γ → 5: Theorem 39

1αΦ → 5: Theorem 38

4αΓ → 5: Theorem 40

4αΦ → 5: Theorem 40

1αΓ → 5: Theorem 38

2 → 5: Theorem 15

7⊕↓ → 5: Theorem 49

49⊕↓ → 7⊕↓: Theorem 17

19⊕↓ → 49⊕↓: Theorem 19

1α⊕↓ → 19⊕↓: Theorems 52, 15 and 13

4α⊕↓ → 49⊕↓: Theorem 21

8⊕↓ → 6↕: Theorems 46, 4 and 5

↑⊕Φπ → 8↕Φ: Theorem 47

Uniformly Bounded Integrals of Measurable Functions (6)

List of Implications

3

12   15   24   45   7ρ

17Φ   17Γ   19ρ   19Φ   19Γ   1αΦ   1αΓ   278   27σ

27τ   47Φ   47Γ   49Φ   49Γ   4αΦ   4αΓ   49ρ   578

57σ   57τ   78Φ   78Γ   7σΦ   7τΦ   7τΓ   7ΦΓ   7σΓ

8⊕↓   8↕Φ   ↑⊕Φ   ↕ΦΓ   ⊕ΦΓ

19⊕↓   1α⊕↓   4α⊕↓   47⊕↓   49⊕↓   7σ⊕↓   7τ⊕↓   17⊕↓

Counterexamples	Hypotheses Satisfied															
1		2		5	7		9	α					⊕	↓	Φ	
3		2		5		8	9		ρ	σ	τ	↑	⊕			Γ
5		2		5			9		ρ	σ	τ		⊕	↓	Φ	
12		2		5		8	9		ρ	σ	τ				Φ	Γ
13		2		5	7		9	α				↑	⊕	↓		Γ
17		2		5		8	9		ρ	σ	τ		⊕		Φ	
1[illegible]		2		5		8	9	α		σ	τ	↑	⊕			Γ
22	1		4		7	8	9	α		σ	τ	↑	⊕			
23		2		5			9	α		σ	τ	↑	⊕	↓		Γ
25	1		4						ρ	σ	τ		⊕	↓	Φ	
26	1		4			8			ρ	σ	τ	↑	⊕			Γ
27		2		5			9	α		σ	τ		⊕	↓	Φ	
31		2		5		8	9	α		σ	τ				Φ	Γ
33		2		5		8	9	α		σ	τ		⊕		Φ	
35	1		4			8			ρ	σ	τ		⊕		Φ	
36	1		4			8			ρ	σ	τ				Φ	Γ
44	1		4						ρ	σ	τ	↑	⊕	↓		Γ
53		2		5			9	α			τ		⊕	↓		Γ

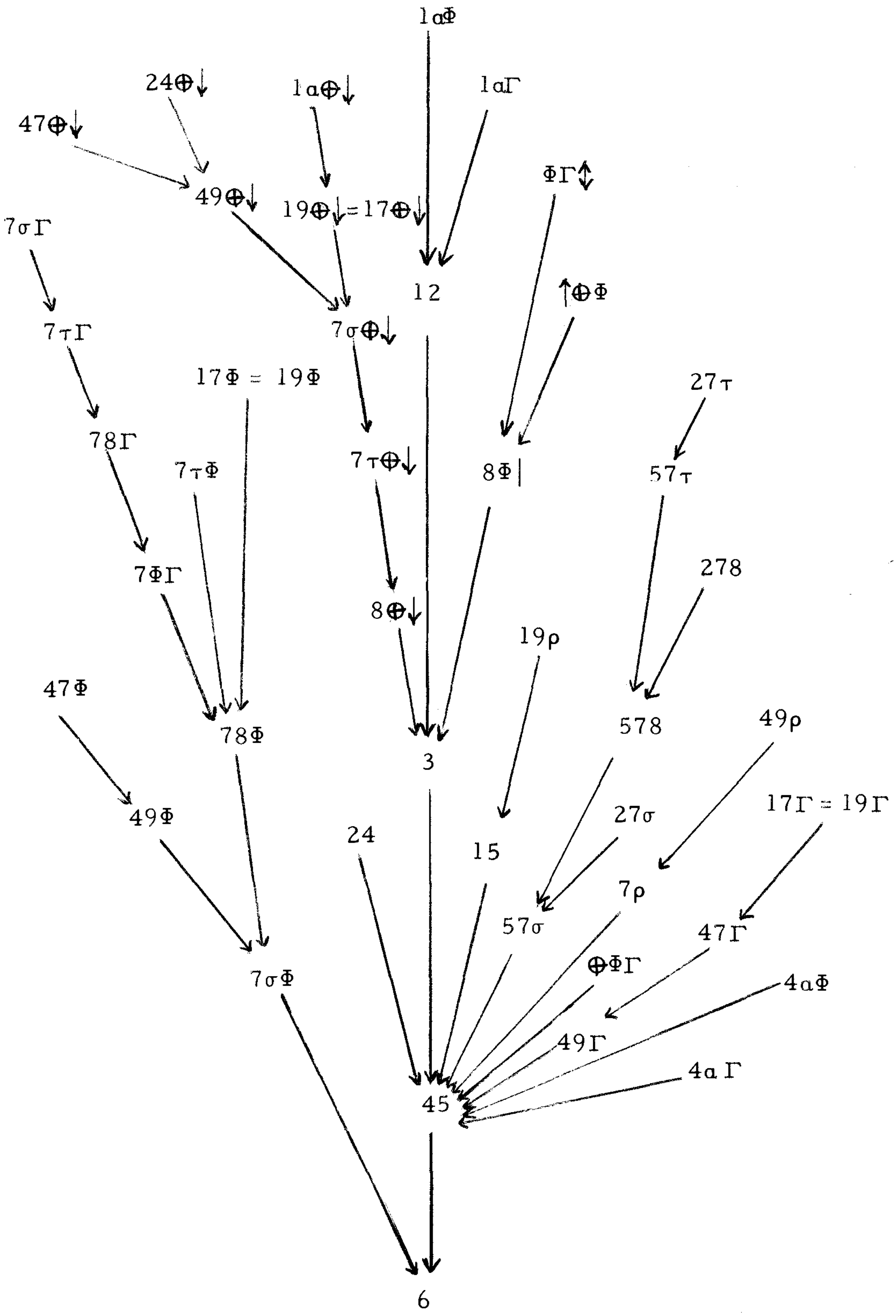

1αΦ
24⊕↓
1α⊕↓
1αΓ
47⊕↓
49⊕↓
19⊕↓=17⊕↓
ΦΓ↕
7σΓ
12
↑⊕Φ
7τΓ
7σ⊕↓
17Φ = 19Φ
27τ
78Γ
7τΦ
7τ⊕↓
8Φ|
57τ
278
7ΦΓ
8⊕↓
19ρ
47Φ
78Φ
3
578
49ρ
49Φ
24
15
27σ
17Γ = 19Γ
7ρ
57σ
47Γ
⊕ΦΓ
4αΦ
7σΦ
49Γ
4α Γ
45
6

Figure 6.

Proofs of Implications in Figure 6.

$27\tau \to 57\tau$: Theorem 15

$57\tau \to 578$: Theorem 12

$278 \to 578$: Theorem 15

$578 \to 57\sigma$: (4, p. 105)

$27\sigma \to 57\sigma$: Theorem 15

$57\sigma \to 45$: Theorem 28

$19\rho \to 15$: Theorem 14

$15 \to 45$: Theorem 19

$\Phi\Gamma\updownarrow \to 8\Phi|$: Obvious

$8\Phi\updownarrow \to 3$: (2, Theorem 15, p. 25)

$3 \to 45$: Theorem 4

$1\alpha\Gamma \to 12$: Theorem 38

$12 \to 3$: (2, Theorem 8, p. 23)

$19\Phi \to 17\Phi$: (2, Theorem 7, p. 22)

$17\Phi \to 78\Phi$: (2, Theorem 15, p. 25)

$78\Phi \to 7\sigma\Phi$: (4, p. 105)

$7\sigma\Phi \to 6$: Theorem 27

$7\tau\Phi \to 78\Phi$: Theorem 12

$78\Gamma \to 7\Gamma\Phi$: Theorem 44

$7\sigma\Gamma \to 7\tau\Gamma$: Obvious

$7\tau\Gamma \to 78\Gamma$: Theorem 12

$7\Gamma\Phi \to 78\Phi$: Obvious

$47\Phi \to 49\Phi$: Theorem 18

$49\Phi \to 7\sigma\Phi$: Theorems 16 and 17

$4\alpha\Gamma \to 45$: Theorem 40

$4\alpha\Phi \to 45$: Theorem 40

$19\Gamma \to 17\Gamma$: (2, Theorem 7, p. 22)

$17\Gamma \to 49\Gamma$: Theorems 18 and 19

$47\Gamma \to 49\Gamma$: Theorem 18

49Γ	$\rightarrow 45$:	Theorems 39 and 5
$\oplus\Phi\Gamma$	$\rightarrow 45$:	Theorems 29 and 34
49ρ	$\rightarrow 7\rho$:	Theorem 17
7ρ	$\rightarrow 45$:	Theorems 14 and 22
45	$\rightarrow 6$:	Theorem 5
24	$\rightarrow 45$:	Theorem 15
$\uparrow\oplus\Phi$	$\rightarrow 8\Phi\updownarrow$:	Obvious
$8\oplus\downarrow$	$\rightarrow 3$:	Theorem 46
$7\tau\oplus\downarrow$	$\rightarrow 8\oplus\downarrow$:	Theorem 12
$7\sigma\oplus\downarrow$	$\rightarrow 7\tau\oplus\downarrow$:	Obvious
$17\oplus\downarrow$	$\rightarrow 19\oplus\downarrow \rightarrow 7\sigma\oplus\downarrow$:	(2, Theorem 7, p. 22) and Theorem 18
$1a\oplus\downarrow$	$\rightarrow 19\oplus\downarrow$:	Theorem 21
$49\oplus\downarrow$	$\rightarrow 7\sigma\oplus\downarrow$:	Theorems 17 and 16
$47\oplus\downarrow$	$\rightarrow 49\oplus\downarrow$:	Theorem 18
$24\oplus\downarrow$	$\rightarrow 49\oplus\downarrow$:	Theorems 15 and 13

Sequences of Measurable Functions Whose Indefinite Integrals are Equicontinuous From Above at ϕ (7)

List of Implications.

3

12	15	19	1a	24	45	49	4a	$6\updownarrow$
$6\oplus\Gamma$	$8\updownarrow\Phi$	$8\oplus\downarrow$	$\uparrow\oplus\Phi$	$\updownarrow\Phi\Gamma$	$\oplus\Phi\Gamma$			

Counterexamples	Hypotheses Satisfied															
2	1		4		6	8			ρ	σ	τ				Φ	Γ
3		2		5		8	9		ρ	σ	τ	↑	⊕			Γ
5		2		5			9		ρ	σ	τ		⊕	↓	Φ	
10		2		5	6	8	9		ρ	σ	τ				Φ	Γ
14	1		4		6	8			ρ	σ	τ		⊕		Φ	
15		2		5	6	8	9		ρ	σ	τ		⊕		Φ	
18		2		5		8	9	a		σ	τ	↑	⊕			Γ
23		2		5			9	a		σ	τ	↑	⊕	↓		Γ
25	1		4						ρ	σ	τ		⊕	↓	Φ	
26	1		4			8			ρ	σ	τ	↑	⊕			Γ
27		2		5			9	a		σ	τ		⊕	↓	Φ	
32		2		5	6	8	9	a		σ	τ		⊕		Φ	
34		2		5	6	8	9	a		σ	τ				Φ	Γ
44	1		4						ρ	σ	τ	↑	⊕	↓		Γ

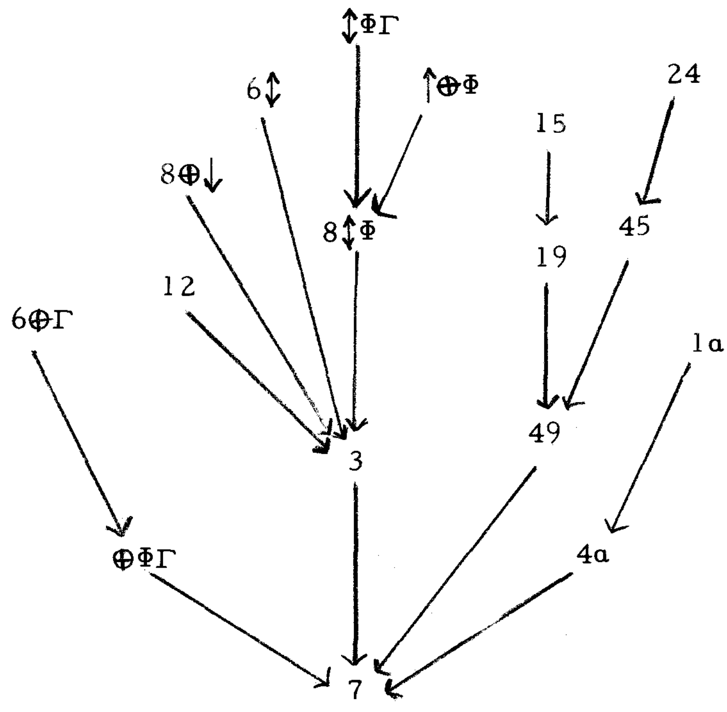


Figure 7.

Proofs of Implications in Figure 7.

3 → 7: (2, Theorem 5, p. 21)

12 → 7: (2, Theorem 6, p. 21)

19 → 7: (2, Theorem 7, p. 22)

8Φ⇕ → 7: (2, Theorem 15, p. 25 and Theorem 4, p. 20)

ΦΓ⇕ → 7: Obvious

6↕ → 7: Theorem 30

ΦΓ⊕ → 7: Theorem 31

49 → 7: Theorem 17

15 → 7: Theorem 13

1a → 7: Obvious

45 → 7: Theorems 13 and 17

4a → 7: Theorems 21 and 17

6Γ⊕ → 7: Theorem 31 plus the fact that 6Γ⊕ → ΦΓ⊕

8⊕↓ → 3: Theorem 46

↑⊕Γπ → 8↕Γ: Obvious

Integrability of Each Member of the Sequence $\{f_n\}$ (8)

List of Implications.

3	6							
12	15	17	19	1a	24	45	47	49
4a	7ρ	7σ	7τ	ΦΓ				
↑⊕Φ								

Counterexamples	Hypotheses Satisfied														
1		2		5	7	9	α					⊕	↓	Φ	
5		2		5		9		ρ	σ	τ		⊕	↓	Φ	
20						9	α		σ	τ		⊕		Φ	
25	1		4					ρ	σ	τ		⊕	↓	Φ	
27		2		5		9	α		σ	τ		⊕	↓	Φ	
44	1		4					ρ	σ	τ	↑	⊕	↓		Γ
45		2		5		9	α		σ	τ	↑	⊕	↓		Γ
46		2		5		9		ρ	σ	τ	↑	⊕	↓		Γ
53		2		5		9	α			τ		⊕	↓		Γ
54		2		5	7	9	α				↑	⊕	↓		Γ

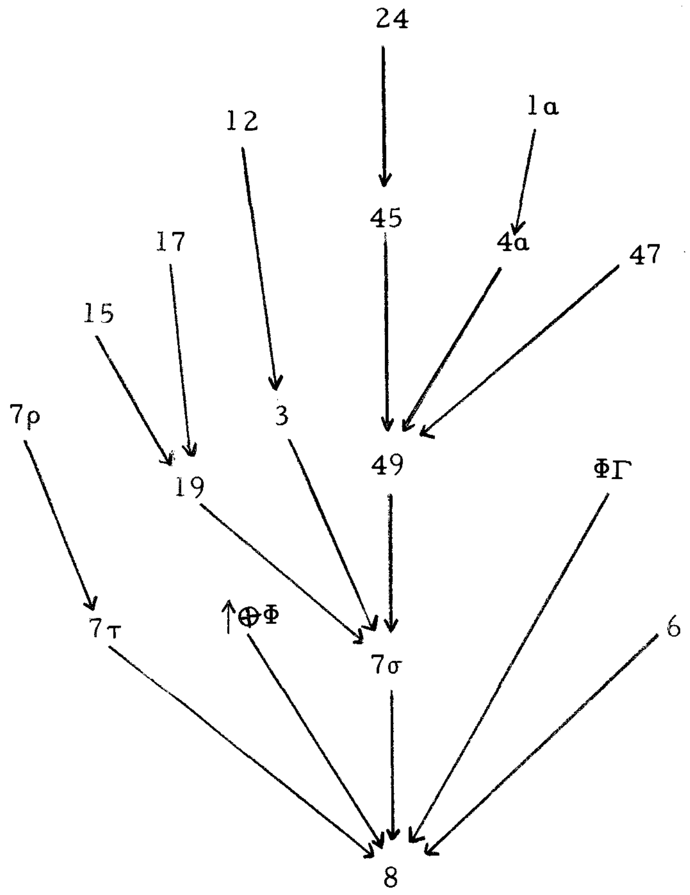


Figure 8.

Proofs of Implications in Figure 8.

6	$\rightarrow$ 8:	Obvious
$\Phi\Gamma$	$\rightarrow$ 8:	Obvious
47	$\rightarrow$ 49:	Theorem 18
49	$\rightarrow 7\sigma$:	Theorems 16 and 17
7σ	$\rightarrow$ 8:	(2, Theorem 15, p. 25)
1α	$\rightarrow 4\alpha$:	Theorem 19
4α	$\rightarrow$ 49:	Theorem 21
24	$\rightarrow$ 45:	Theorem 15
45	$\rightarrow$ 49:	Theorem 13
12	$\rightarrow 3 \rightarrow 7\sigma$:	(2, Theorems 5 and 7, p. 21-22)
17	$\rightarrow 19 \rightarrow 7\sigma$:	Theorem 18
15	$\rightarrow$ 19:	Theorem 13
7ρ	$\rightarrow 7\tau$:	Obvious
7τ	$\rightarrow$ 8:	Theorem 12
$\uparrow\oplus\Phi$	$\rightarrow$ 8:	Theorem 46

Sequences of Functions with Uniformly Absolutely Continuous Indefinite Integrals (9)

List of Implications.

2	3	5	7	α	
$6\updownarrow$					
$6\oplus\Gamma$	$8\updownarrow\Phi$	$8\oplus\downarrow$	$\uparrow\oplus\Phi$	$\updownarrow\Phi\Gamma$	$\oplus\Phi\Gamma$

Counterexamples	Hypotheses Satisfied										
2	1	4	6	8	ρ	σ	τ				Φ	Γ
14	1	4	6	8	ρ	σ	τ		$\oplus$		Φ	
25	1	4			ρ	σ	τ		$\oplus$	$\downarrow$	Φ	
26	1	4		8	ρ	σ	τ	$\uparrow$	$\oplus$			Γ
44	1	4			ρ	σ	τ	$\uparrow$	$\oplus$	$\downarrow$		Γ

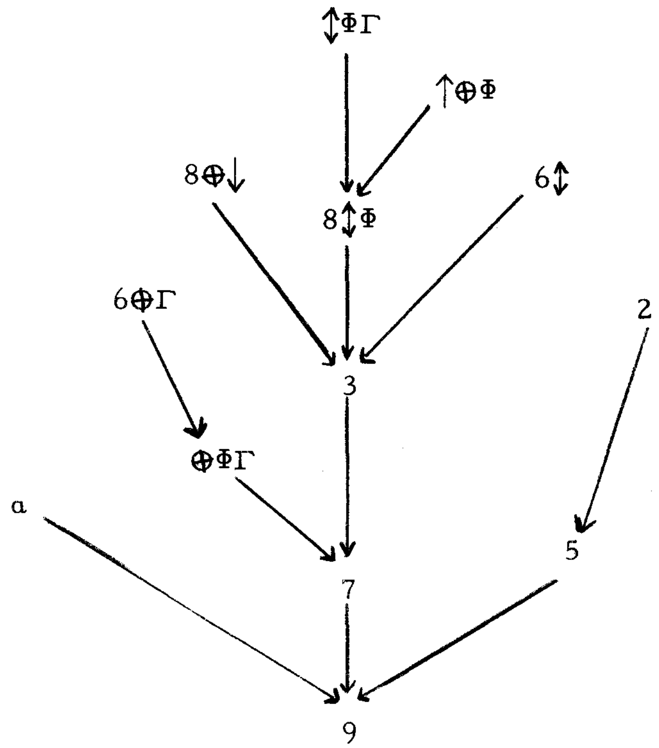


Figure 9.

Proofs of Implications in Figure 9.

$\updownarrow\Phi\Gamma \rightarrow 8\updownarrow\Phi \rightarrow 3 \rightarrow 7 \rightarrow 9$: (2, Theorem 5, p. 21) and Theorem 18

$6\updownarrow \rightarrow 3$: Theorem 30

$6\oplus\Gamma \rightarrow \Phi\oplus\Gamma \rightarrow 7$: Theorem 31

$2 \rightarrow 5 \rightarrow 9$: Theorems 15 and 13

$a \rightarrow 9$: Theorem 21

$8\oplus\downarrow \rightarrow 3$: Theorem 46

$\uparrow\oplus\Phi\pi \rightarrow 8\updownarrow\Phi$: Theorem 47

Uniformly Atomic Measure Spaces(α)

No subset of the set of the remaining 16 hypotheses can imply α as shown by counterexample 6.

Non-Atomic Measure Spaces(ρ)

Counterexample 19 shows that there are no possible implications in this case.

Sequences of Functions Whose Support has σ-Finite Measure (σ)

List of Implications.

1	3	4	6	8
7ρ	7τ	$\Phi\Gamma$		
$\uparrow\oplus\Phi$				

Counterexamples	Hypotheses Satisfied										
1	2	5	7	9	α				$\oplus$	$\downarrow$	Φ	
4	2	5	7	9	α			$\uparrow$	$\oplus$	$\downarrow$		Γ
29	2	5		9		ρ	τ	$\uparrow$	$\oplus$	$\downarrow$		Γ
30	2	5		9	α		τ	$\uparrow$	$\oplus$	$\downarrow$		Γ
51	2	5		9		ρ	τ		$\oplus$	$\downarrow$	Φ	
52	2	5		9	α		τ		$\oplus$	$\downarrow$	Φ	

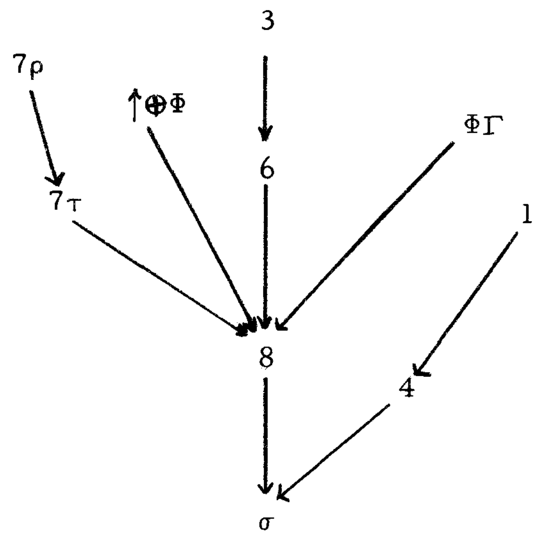


Figure 10.

Proofs of Implications in Figure 10.

1	→ 4:	Theorem 19
4	→ σ:	Theorem 16
ΦΓ	→ 8:	Obvious
3	→ 6:	Theorems 4 and 5
6	→ 8:	Obvious
7ρ	→ 7τ:	Obvious
7τ	→ 8:	Theorem 12
8	→ σ:	(4, p. 105)
↑⊕Φπ	→ 8:	Theorem 47

Sequences of Functions Whose Support has no Infinite Atoms (τ)

List of Implications.

1 3 4 6 8 ρ σ

ΦΓ

↑⊕Φ

Counterexamples	Hypotheses Satisfied									
1	2	5	7	9	a		⊕	↓	Φ	
54	2	5	7	9	a	↑	⊕	↓		Γ

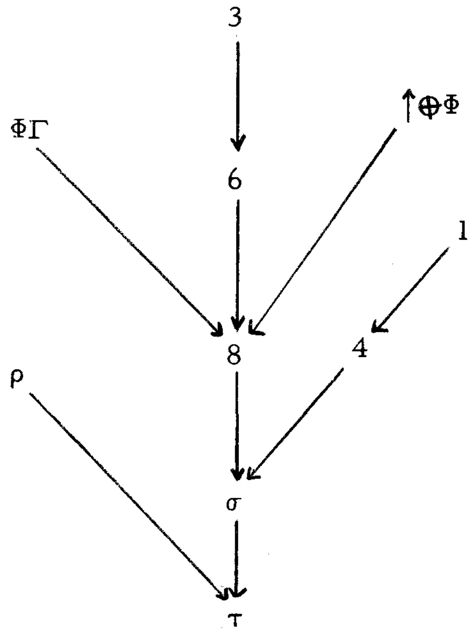


Figure 11.

Proofs of Implications in Figure 11.

$\rho \to \tau$: Obvious

$1 \to 4$: Theorem 19

$4 \to \sigma$: Theorem 16

$\sigma \to \tau$: Obvious

$3 \to 6$: Theorems 4 and 5

$6 \to 8$: Obvious

$8 \to \sigma$: (4, p. 105)

$\Phi\Gamma \to 8$: Obvious

$\uparrow \oplus \Phi \pi \to 8$: Theorem 47 a.e. Monotonic Sequences of Measurable Functions (↕)

Counterexamples 47 and 48 show that no subset of the remaining 16 hypotheses can imply ↕.

Sequences of Non-Negative Measurable Functions (⊕)

Counterexamples 49 and 50 show that no implications are possible for the property ⊕.

Integrability of the Limit Function (Φ)

List of Implications.

3	6							
12	15	17	19	1a	24	45	47	49
4a	78	7ρ	7σ	7τ				
8↓⊕								

Counterexamples	Hypotheses Satisfied														
3		2		5		8	9		ρ	σ	τ	↑	⊕		Γ
18		2		5		8	9	a		σ	τ	↑	⊕		Γ
26	1		4			8			ρ	σ	τ	↑	⊕		Γ
44	1		4						ρ	σ	τ	↑	⊕	↓	Γ
45		2		5			9	a		σ	τ	↑	⊕	↓	Γ
46		2		5			9		ρ	σ	τ	↑	⊕	↓	Γ
54		2		5	7		9	a				↑	⊕	↓	Γ

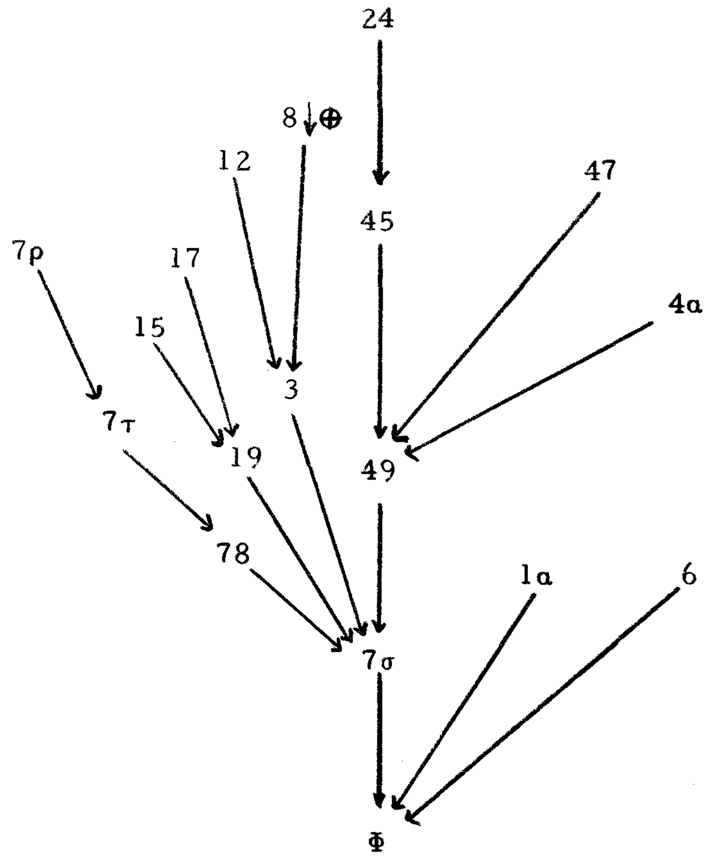


Figure 12.

Proofs of Implications in Figure 12.

6π → Φ: Theorem 43

1aπ → Φ: Theorem 38

4a → 49: Theorem 21

47 → 49: Theorem 18

24 → 45: Theorem 15

45 → 49: Theorem 13

49 → 7σ: Theorems 16 and 17

7σπ → Φ: Theorem 44

12 → 3 → 7σ: (2, Theorems 8 and 9, p. 21-22)

17 → 19 → 7σ: Theorem 18 and (2, Theorem 7, p. 22)

15 $\rightarrow$ 19: Theorem 13

7ρ $\rightarrow$ 7τ: Obvious

7τ $\rightarrow$ 78: Theorem 12

78 $\rightarrow$ 7σ: (4, p. 105)

$8\downarrow\oplus$ $\rightarrow$ 3: Theorem 46

Convergence of the Integrals (Γ)

List of Implications.

3

12	15	17	19	1a	24	45	47	49
4a	67	$6\updownarrow$	78	7ρ	7σ	7τ	$8\updownarrow$	$\uparrow\oplus$

Counterexamples	Hypotheses Satisfied														
1		2		5		7		9	a				$\oplus$	$\downarrow$	Φ
5		2		5				9		ρ	σ	τ	$\oplus$	$\downarrow$	Φ
14	1		4		6		8			ρ	σ	τ	$\oplus$		Φ
15		2		5	6		8	9		ρ	σ	τ	$\oplus$		Φ
25	1		4							ρ	σ	τ	$\oplus$	$\downarrow$	Φ
27		2		5				9	a		σ	τ	$\oplus$	$\downarrow$	Φ
32		2		5	6		8	9	a		σ	τ	$\oplus$		Φ

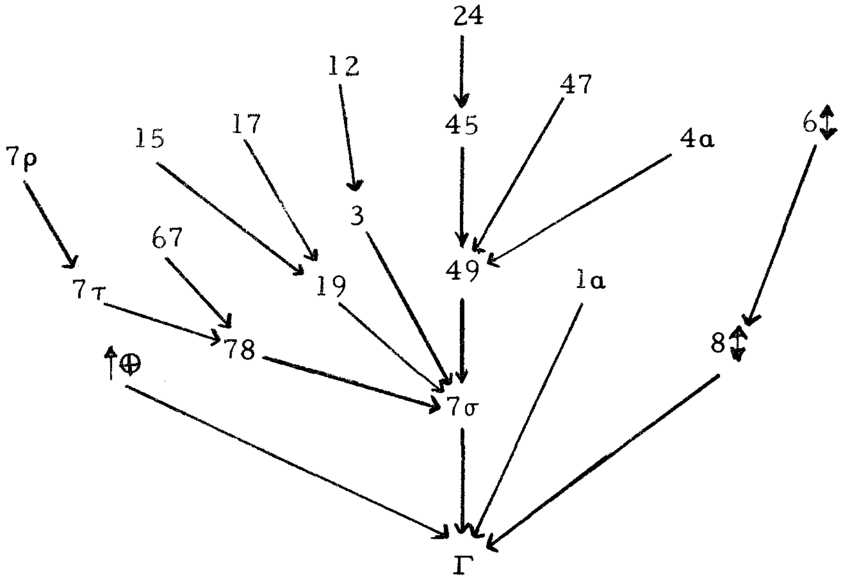


Figure 13.

Proofs of Implications in Figure 13.

$6\updownarrow$	$\rightarrow 8\updownarrow$: Obvious
$8\updownarrow$	$\rightarrow \Gamma$: Theorem 45
$1a$	$\rightarrow \Gamma$: Theorem 38
$4a$	$\rightarrow 49$: Theorem 21
47	$\rightarrow 49$: Theorem 18
24	$\rightarrow 45$: Theorem 15
45	$\rightarrow 49$: Theorem 13
49	$\rightarrow 7\sigma$: Theorems 16 and 17
12	$\rightarrow 3 \rightarrow 7\sigma$: (2, Theorems 8 and 9, p. 21-22)
17	$\rightarrow 19 \rightarrow 7\sigma$: Theorem 18 and (2, Theorem 7, p. 22)
15	$\rightarrow 19$: Theorem 13
67	$\rightarrow 78$: Obvious
78	$\rightarrow 7\sigma$: (4, p. 105)
7ρ	$\rightarrow 7\tau$: Obvious
7τ	$\rightarrow 78$: Theorem 12
$\uparrow\oplus\pi$	$\rightarrow \Gamma$: Theorem 50

III. PROOFS OF THEOREMS

In this chapter we prove 52 theorems which establish various implications among the 17 properties of the sequence $\{f_n\}$ of measurable functions and their support. We begin by studying these properties for the case of one function.

Equivalent Definitions for $f(x)$ to be Semi-Bounded $(\bar{I})$

$(\bar{a})$ For each $\epsilon > 0$ there exists a real number K such that

$$\int_{|f| \geq K} |f| d\mu < \epsilon ,$$

$(\bar{b})$ There exist two positive numbers L and L' such that

$$\int_{|f| \geq L} |f| d\mu < L', \quad \text{and}$$

$(\bar{c})$ $f = f_I + \bar{f}$ where f_I is integrable and $|\bar{f}(x)| < M < \infty$ for all x.

Proofs. Obviously $(\bar{a}) \rightarrow (\bar{b})$. To show $(\bar{b}) \rightarrow (\bar{a})$ apply the monotone convergence theorem to the sequence $\{f_n\}$ where $f_n = |f| \chi_{E_n}$ and $E_n = \{x : L \leq |f(x)| < n\}$. $(\bar{b}) \rightarrow \mu E_n < \infty$ for all n. Choose n so that $\int_{|f| \geq L} (|f| - |f_n|) d\mu < \epsilon$. Then $K = n$ satisfies $(\bar{a})$, since $f_n(x) = 0$ when $|f(x)| \geq n$. Hence $(\bar{a}) \leftrightarrow (\bar{b})$.

To show $(\bar{b}) \rightarrow (\bar{c})$ take $M = L$; $\bar{f}$ vanishes on $\{x : |f(x)| \geq L\}$ and f_I vanishes on its complement. $(\bar{c}) \rightarrow (\bar{b})$: Take $L = 2M$. Adding $2M \leq |f(x)|$, $|\bar{f}(x)| < M$, $|f(x)| \leq |f_I(x)| + |\bar{f}(x)|$ we get $M < |f_I(x)|$, which defines a set E with $\mu E < \infty$. Thus $(\bar{b})$ is bounded by $\int |f_I| d\mu + M\mu E$.

Theorem 1. $(\bar{d})$ + hypothesis $\rho \rightarrow (\bar{I}) \rightarrow \bar{d}$ where $(\bar{d})$ is $\mu E < \infty \rightarrow \int_E |f| d\mu < \infty$ for all measurable E.

Proofs. Obviously $(\bar{c}) \rightarrow (\bar{d})$. It remains to show that $\sim(\bar{b})$ and μ nonatomic $\rightarrow \sim(\bar{d})$. $\sim(\bar{b})$ is the statement that for any positive number L we have $\int_{|f| \geq L} |f| d\mu = \infty$. If there is a number L such that $\mu\{x : |f(x)| \geq L\} < \infty$ then this set E_L disproves $(\bar{d})$. If $\mu E_L = \infty$ for each positive number L, then the Darboux property of nonatomic measures (see Theorem I. 6) permits one to find disjoint sets $E_1, E_2, \ldots$ such that $\mu E_n = 2^{-n}$ and $|f(x)| \geq 2^n$ on E_n. Then $E = \cup E_n$ disproves $(\bar{d})$.

The example $f(x) = x$ with μ the counting measure on the integers satisfies $(\bar{d})$ but not $(\bar{I})$. Moreover, if μ is Lebesgue measure, counterexample 26 shows that neither is satified.

For uniform semi-boundedness $(U\bar{I})$ of a sequence of functions $\{f_n\}$, we require K or L, L', M to be independent of n. $(U\bar{a}) \rightarrow (U\bar{b})$ but not conversely. Consider $f_n(x) = n$ for

$0 \leq x \leq \frac{1}{n}$, $f_n(x) = 1$ elsewhere, Lebesgue measure on the real line. This satisfies $(U\bar{b})$ but not $(U\bar{a})$.

Equivalent Definitions for $f(x)$ to be Semi-Finitely-Supported (I_0)

(a_0) For each $\epsilon > 0$ there exists a measurable set E with $\mu E < \infty$ such that $\int_{X-E} |f| d\mu < \epsilon$.

(b_0) There exist a measurable set F with $\mu F < \infty$ and a positive number L such that $\int_{X-F} |f| d\mu < L$.

(c_0) $f = f_I + f_0$ where f_I is integrable and f_0 vanishes outside of a set F of finite measure.

(d_0) There are positive numbers K and K' such that $\int \min(K, |f|) d\mu < K'$.

Proofs. $(a_0) \to (b_0)$: obvious. $(b_0) \to (a_0)$: apply the monotone convergence theorem to the sequence $\{f_n\}$ where $f_n = \max(|f| - \frac{1}{n}, 0)$ on $X-F$. (b_0) implies $\mu\{x : x \notin F$ and $|f(x)| \geq \frac{1}{n}\} = \mu E_n < \infty$ for all n. Choose n so that $\int_{X-F} (|f| - f_n) d\mu < \epsilon$. Then $E = F \cup E_n$ satisfies (a_0), since $f_n = 0$ on $X-E$. $(b_0) \to (c_0)$: define $f_I = f\chi_{X-F}$ and $f_0 = f\chi_F$. Since f_I is integrable and $f_0 = 0$ on $X-F$ then

$$\int_X |f_I| d\mu = \int_{X-F} |f| d\mu < \infty.$$

$(c_0) \to (d_0)$:

$$\int \min(K, |f|)d\mu \leq \int f_I d\mu + \int_F K d\mu \leq \int f_I d\mu + K\mu(F)$$

which is a bound for d_0. $(d_0) \to (b_0)$: Let $F = \{x : |f(x)| \geq K\}$; then F has finite measure and satisfies (b_0).

Theorem 2. A measurable function f is integrable (I) iff it is semi-bounded $(\bar{I})$ and semi-finitely supported (I_0), or equivalently, satisfies $(\bar{d})$ and I_0.

Proof. Obviously $(I) \to (\bar{c})$ and (c_0). Conversely, if $(\bar{d})$ and (b_0) hold, then

$$\int |f| d\mu = \int_{X-F} |f| d\mu + \int_F |f| d\mu < \infty \quad \text{with} \quad \mu(F) < \infty.$$

A still broader class of functions f are those representable as $f = f_I + \bar{f} + f_0$. The previous examples with $f(x) = x$ do not belong to this class.

(a_0), (b_0), (d_0) give rise to three possible definitions for $f(x)$ to be uniformly semi-finitely-supported (UI_0); obviously $(Ua_0) \to (Ub_0)$. $f_n = \frac{1}{n}\chi_{[0,n]}$ satisfies (Ub_0) with F empty and $L = 2$, and (Ud_0) with $K = K' = 2$, but not (Ua_0) nor the conclusion of Theorem 6 below. $f_n = n\chi_{[n,n+1]}$ satisfies (Ud_0)

but not (Ub_0) or (Ua_0).

Theorem 3. $(Ua_0) \rightarrow (Ub_0) \rightarrow (Ud_0)$.

Proof. The first implication is obvious; let $L = \epsilon$. To prove the second, for any $K > 0$, we have

$$\int \min(K, |f|)d\mu = \left[\int_F + \int_{X-F}\right] \min(K, |f|)d\mu < K\mu F + L = K'.$$

Definition. $\{f_n\}$ is said to be uniformly integrable (UI) iff it is $(U\bar{I})$ and (UI_0), meaning $(U\bar{a})$ and (Ua_0). Equivalently, for each $\epsilon > 0$, there exist a measurable set E with $\mu E < \infty$ and a positive number K such that $\int_{X-F_n} |f_n| d\mu < \epsilon$ uniformly in n where $F_n = E \cap \{x : |f_n(x)| < K\}$.

Theorem 4. $(3 \rightarrow 45)$. If there exists an integrable function $g \geq |f_n|$ for all n, then $\{f_n\}$ is uniformly integrable.

Proof. Apply the monotone convergence theorem to g as above in proving $(\bar{b}) \rightarrow (\bar{a})$ and $(b_0) \rightarrow (a_0)$.

The converse is not true. Counterexamples are: $f_n = \chi_{[n,n+1]}$ and $f_n = 2^m \chi_{[(k-1)4^{-m}, k4^{-m}]}$ for $k = 1, 2, \ldots, 4^m$; $m = 1, 2, 3, \ldots$; $n = 4 + 4^2 + \ldots + 4^{m-1} + k$; $X = [0, 1]$. We assume Lebesgue measure in both cases.

Theorem 5. (45 → 6). If $\{f_n\}$ is uniformly integrable then, there exists $M < \infty$ such that all $\int |f_n| d\mu < M$.

Proof. With arbitrary $\epsilon > 0$, $\int |f_n| d\mu \leq K\mu E < \infty$ for all n.

Theorem 6. If $\{f_n\}$ is uniformly integrable and $f_n \rightarrow f$ a. e. then $\int f_n d\mu \rightarrow \int f d\mu$ and f is integrable.

Proof. Since $\mu E < \infty$, $f_n \rightarrow f$ a. e. on E and all f_n's and f are finite a. e. we can apply Egorov's theorem (5, p. 158) to extract a set $G \subset E$ with $\mu(E-G) < \epsilon'$ and $f_n \rightarrow f$ uniformly on G. Then

$$\left| \int f_n d\mu - \int f d\mu \right| \leq \int_G |f_n - f| + \int_{X-G} (|f_n| + |f|) d\mu.$$

Applying Fatou's Lemma (5, p. 172) to $\int_{X-G} |f| d\mu$ we get

$$\int_{X-G} |f| d\mu \leq \underline{\lim} \int_{X-G} |f_n| d\mu$$

so that for sufficiently large n,

$$\left| \int f_n d\mu - \int f d\mu \right| \leq \int_G |f_n - f| d\mu + 2 \int_{X-G} |f_n| d\mu$$

$$\leq \epsilon'' \mu(E) + 3 \int_{X-F_n} |f_n| d\mu + 3 \int_{F_n - G} |f_n| d\mu$$

$$\leq \epsilon'' \mu(E) + 3\epsilon + 3\epsilon' K \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty .$$

To prove the second assertion, apply Fatou's lemma again to get

$$\int |f| d\mu \leq \underline{\lim} \int |f_n| d\mu = \underline{\lim} \left[\int_{F_n} + \int_{X-F_n} \right] |f_n| d\mu$$

$$< K\mu E + \epsilon$$

so that f is integrable.

The preceding counterexamples show that this is a significant generalization of the standard Lebesgue convergence theorem that assumes the existence of an integrable function g such that $g \geq f_n$ for all n.

In Theorems 7-12 it is shown that if $\{\int f_n d\mu\}$ is equicontinuous from above at ϕ and if (X, S, μ) has no infinite atoms then f_n is integrable for all n.

<u>Theorem 7</u>. The indefinite integral of an integrable function is continuous from above at ϕ.

<u>Proof</u>. $\int f d\mu$ is continuous from above at ϕ if and only if $\int |f| d\mu$ has the same property. We can therefore assume f to be nonnegative. Let $\{E_n\}$ be a decreasing sequence of measurable sets such that $\lim_n E_n = \phi$. Let $\epsilon > 0$ be given. By Theorem 2 there exists a measurable set F with $\mu(F) < \infty$ and $\int_{X-F} f d\mu < \epsilon$. Put

$$A_n = E_n \cap F, \qquad B_n = E_n - F$$

So that $A_n \cup B_n = E_n$. Hence we have

$$\int_{B_n} f d\mu < \epsilon .$$

Since $\lim_n A_n = \phi$, then by I. 11 $\mu A_n \to 0$. Now, for all sufficiently large n, by Theorem I. 7, we have

$$\int_{A_n} f d\mu < \epsilon .$$

Therefore,

$$\int_{E_n} f d\mu = \int_{A_n} f d\mu + \int_{B_n} f d\mu < 2\epsilon$$

i. e.,

$$\int_{E_n} f d\mu \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem 8. If f is a semi-finitely-supported function and its indefinite integral is absolutely continuous, then f is integrable.

Proof. $\int f$ is absolutely continuous and f is semi-finitely-supported if and only if the same if true of $|f|$. Hence assume $f > 0$. Let $\epsilon > 0$ be given. There exists a set F of finite measure such that

$$\int_{X-F} fd\mu < \epsilon .$$

Since

$$\int fd\mu = \int_{X-F} fd\mu + \int_F fd\mu \leq \int_F fd\mu + \epsilon ,$$

then there is no loss in generality in assuming $\mu X < \infty$. Since we are only considering functions which are finite a. e. then the set

$$E = \{x : f(x) = \infty\}$$

has measure zero. Define

$$E_n = \{x : f(x) \geq n\}.$$

$\{E_n\}$ is a decreasing sequence with $\lim_n E_n = E$. Hence $\mu E_n \to 0$ as $n \to \infty$. By the absolute continuity

$$\lim_n \int_{E_n} fd\mu < \infty .$$

Hence there exists an N such that $n \geq N \to$

$$\int_{E_n} fd\mu < \infty .$$

Consequently,

$$\int fd\mu = \int_{E_0} fd\mu = \int_{E_0 - E_N} fd\mu + \int_{E_N} fd\mu < \infty .$$

Theorem 9. Suppose (X, S, μ) is a sigma-finite measure space. If the indefinite integral of f is continuous from above at ϕ then f is semi-finitely-supported (I_0).

Proof. Write

$$X = \bigcup_{n=1}^{\infty} E_n, \quad \mu E_n < \infty \quad \text{for} \quad n = 1, 2, 3, \ldots$$

and

$$F_n = X - (E_1 \cup E_2 \cup \ldots \cup E_n).$$

The sequence $\{F_n\}$ is decreasing and $\lim_n F_n = \phi$. Let $\epsilon > 0$ be given. By the property of continuity from above at ϕ, there exists an N_ϵ such that $n \geq N_\epsilon \rightarrow$

$$\int_{F_n} f d\mu < \epsilon$$

i.e.,

$$\int_{X-(E_1 \cup E_2 \cup \ldots \cup E_n)} f d\mu < \epsilon$$

and f is (I_0).

Theorem 10. Suppose (X, S, μ) has no atoms of infinite measure. If the indefinite integral of f is continuous from above at ϕ then f is semi-finitely supported (I_0).

Proof. f is semi-finitely-supported if an only if the same is true for $|f|$. If $f = 0$ then the theorem holds trivially. So, we can suppose $f > 0$ since the point of interest here is the support of f. Suppose (I_0) is false. Then $\mu X = \infty$. Put

$$\alpha = \sup_F \{\int_F f d\mu : \mu F < \infty\}.$$

The assumption $\alpha < \infty$ is impossible: Select a sequence $\{F_k\}$ of sets of finite measure such that

$$\int_{F_k} f d\mu \to \alpha \quad \text{and} \quad \int_F f d\mu = \alpha \quad \text{where} \quad F = \bigcup_k F_k.$$

Now, consider $X-F$. Let $A \subseteq X-F$ be any subset of finite measure. Since $F \cap A = \phi$ then

$$\int_A f d\mu = 0$$

for otherwise the number

$$\sup \{\int_{F_k \cup A} f d\mu : F_k \text{ is one of the } F_k\text{'s defined above}\}$$

would contradict the definition of α. Since $f > 0$, $\mu A = 0$. Now, since every subset of $X-F$ is either of infinite measure or of measure zero and since μ has no atoms of infinite measure, then

$$\mu(X-F) = 0.$$

Consequently,

$$\int f d\mu = \int_F f d\mu + \int_{X-F} f d\mu = \int_F f d\mu = \alpha$$

which cannot be true if (I_0) is false as we assumed.

Since $\alpha = \infty$, we can find a disjoint sequence $\{S_k\}$ of sets of finite measure such that

$$\int_{S_k} f d\mu \geq 1 \quad \text{for} \quad k = 1, 2, 3, \ldots$$

Consider the new sequence $\{E_n\}$ where

$$E_n = \bigcup_{k \geq n} S_k.$$

$\{E_n\}$ is a decreasing sequence and $\mu(\lim_n E_n) = 0$. But

$$\int_{E_n} f d\mu \geq \int_{S_n} f d\mu \geq 1$$

for all n, which is a contradiction to the hypothesis of continuity from above at ϕ.

Theorem 11. Suppose (X, S, μ) has no atoms of infinite measure. A measurable function f is integrable over X if and only if the indefinite integral is continuous from above at ϕ.

Proof. Suppose f is integrable. By Theorem 7, $\int f d\mu$ is continuous from above at ϕ. Conversely, if $\int |f| d\mu$ is continuous from above at ϕ, then by Theorem 10 f is semi-finitely supported. Moreover, $\int |f| d\mu$ is absolutely continuous (2, Theorem 3.1, p. 19) and thus the proof is complete by Theorem 8.

Theorem 12. ($7\rho \rightarrow 7\tau \rightarrow 8$). Let $\{f_n\}$ be a sequence of functions defined on a measure space having no infinite atoms. If the indefinite integrals $\int |f_n| d\mu$ are equicontinuous from above at ϕ, then f_n is integrable for each n.

Proof. First, we observe that a non-atomic measure space cannot have an atom of infinite measure. Hence, $7\rho \rightarrow 7\tau$. The rest of the proof follows from Theorem 11.

Theorem 13. ($5 \rightarrow 9$). If $\{f_n\}$ is a uniformly semi-bounded sequence of measurable functions then $\{\int f_n\}$ is uniformly absolutely continuous.

Proof. Let B be a measurable subset of X and write

$$B_{nk} = \{x : |f_n(x)| \geq k\}.$$

then

$$\int_B |f_n| d\mu = \int_{B \cap B_{nk}} |f_n| d\mu + \int_{B - B_{nk}} |f_n| d\mu$$

$$\leq \int_{B_{nk}} |f_n| + k\mu B.$$

Let $\epsilon > 0$ be given. By 5, there exists k_ϵ such that $k \geq k_\epsilon$ implies

$$\sup_n \int_{B_{nk}} |f_n| < \frac{\epsilon}{2}.$$

Now, Hypothesis 9 follows if we set $\delta = \frac{\epsilon}{2k_\epsilon}$ in the definition of uniform absolute continuity.

<u>Theorem 14.</u> ($9\rho \rightarrow 5$). Let $\{f_n\}$ be a sequence of measurable functions defined on a non-atomic measure space (X, S, μ). If the indefinite integrals $\int |f_n| d\mu$ are uniformly absolutely continuous, then $\{f_n\}$ is uniformly semi-bounded.

<u>Proof.</u> Write $B_{n,k} = \{x : |f_n(x)| \geq k\}$ and assume Hypothesis 5 is false. There exists ϵ_0 such that for all k there is n_k such that

$$\int_{B_{n_k, k}} |f_{n_k}| d\mu \geq \epsilon_0 .$$

Let δ be any positive number. Let A_k be a measurable set either containing or contained in $B_{n_k, k}$ such that

$$\frac{\delta}{2} < \mu A_k < \delta,$$

which is possible by Hypothesis ρ. Then

$$\int_{A_k} |f_{n_k}| d\mu \geq \min \left[\int_{B_{n_k, k}} |f_{n_k}| d\mu, \int_{A_k} k d\mu \right] \geq \min (\epsilon_0, \frac{k\delta}{2}) .$$

By choosing $k > \frac{2\epsilon_0}{\delta}$ we see that

$$\int_{A_k} |f_{n_k}| \geq \epsilon_0$$

and thus Hypothesis 4 is violated.

<u>Theorem 15.</u> ($2 \rightarrow 5$). If the sequence of functions $\{f_n\}$ is essentially uniformly bounded by a constant, then $\{f_n\}$ is uniformly semi-bounded.

<u>Proof.</u> Obvious.

<u>Theorem 16.</u> ($4 \rightarrow \sigma$). If the sequence of functions $\{f_n\}$ is uniformly semi-finitely supported, then the support of the functions f_n's has σ-finite measure.

<u>Proof.</u> For each $\epsilon > 0$ there exists a measurable set $F \subseteq X$

with $\mu F < \infty$ such that

$$\int_{X-F} |f_n| d\mu < \epsilon$$

for all n; i.e., $|f_n|$ is integrable on X-F and hence the support of the functions on X-F is of σ-finite measure (4, p. 105). Consequently, since F is of finite measure it follows that the support of the f_n's has σ-finite measure.

<u>Theorem 17.</u> (49 → 7). If $\{f_n\}$ is uniformly semi-finitely supported and the indefinite integrals are uniformly absolutely continuous, then they are equicontinuous from above at ϕ.

<u>Proof.</u> Let $\{E_m\}$ be a decreasing sequence of measurable sets with $\lim_m E_m = \phi$ and let $\epsilon > 0$ be given. We must find M such that $m \geq M \rightarrow \int_{E_m} |f_n| d\mu < \epsilon$ for all n. By Hypothesis 4, there exists a set E with $\mu E < \infty$ and $\int_{X-E} |f_n| d\mu < \frac{\epsilon}{2}$ for all n. Then $\mu(E \cap E_m) \rightarrow 0$ and by Hypothesis 9, there exists $\delta > 0$ such that $\mu F < \delta \rightarrow \int_F |f_n| d\mu < \frac{\epsilon}{2}$ for all n. Find M such that $m \geq M \rightarrow \mu(E \cap E_m) < \delta$. Then $m \geq M \rightarrow$

$$\int_{E_m} |f_n| d\mu \leq \left[\int_{E \cap E_m} + \int_{E_m - E}\right] |f_n| d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

<u>Theorem 18.</u> (7 → 9). If a sequence of functions $\{f_n\}$ is such

that the indefinite integrals of $|f_n|$ are equicontinuous from above at ϕ, then the indefinite integrals of $|f_n|$ $n = 1, 2, 3, \ldots,$ are uniformly absolutely continuous.

Proof. By virtue of Lemma 32.2 (6, p. 230) we only need to show that each $\int |f_n| d\mu$ is absolutely continuous. Note first that if $\{E_n\}$ is a decreasing sequence of measurable sets with intersection F having measure zero then we may replace E_n by $E_n - F$ without affecting $\int_{E_n} |f| d\mu$ where f is any one of the f_n's. Now suppose that Hypothesis 9 fails and select $\epsilon_0 > 0$ and F_k for $k = 1, 2, \ldots$ such that $\mu F_k < \frac{1}{2^k}$ while $\int_{F_k} |f| d\mu \geq \epsilon_0$. Then the sequence

$$E_n = \bigcup_{k \geq n} F_k$$

provides a contradiction, for it is decreasing with $\mu E_n \leq \frac{1}{2^{n-1}}$ while, contrary to Hypothesis 7,

$$\int_{E_n} |f| d\mu \geq \int_{F_n} |f| d\mu \geq \epsilon_0$$

for every n.

Theorem 19. $(1 \rightarrow 4)$. If the sequence of functions $\{f_n\}$ has finite support, then $\{f_n\}$ is uniformly semi-finitely-supported.

Proof. The result follows if we set $F = \mu$ (support of the f_n's) in the definition of Hypothesis 4.

Theorem 20. $(1\alpha\pi \rightarrow 6)$. Let $\{f_n\}$ be a sequence of functions converging a. e. to a function f on a uniformly atomic measure space of finite measure. Then the integrals $\int |f_n| d\mu$ are uniformly bounded.

Proof. Write $X = X_1 \cup X_2 \cup \ldots \cup X_p$ where each X_i is an atom and $\mu X_i > \delta > 0$ for $i = 1, 2, \ldots, p$. Every measurable function is a constant on each atom. By the a. e. convergence we may assume Hypothesis 2. Consequently, we have Hypothesis 5. Since $1 \rightarrow 4$ by Theorem 19 we see that $45 \rightarrow 6$ by Theorem 5.

Theorem 21. $(\alpha \rightarrow 9)$. Let (X, S, μ) be a uniformly atomic measure space. Then for every sequence $\{f_n\}$ of measurable functions the sequence $\{\int |f_n| d\mu\}$ is uniformly absolutely continuous.

Proof. Given $\epsilon > 0$, choose $\delta < \delta_0$ in the definition of uniform atomicity (I. 3).

Theorem 22. $(7\rho \rightarrow 7\tau \rightarrow 4)$. Suppose (X, S, μ) is a measure space having no infinite atoms. If $\{f_n\}$ is equicontinuous from above at ϕ, then it is uniformly semi-finitely-supported.

Proof. By Theorem 10, each f_n is semi-finitely-supported.

Assume the theorem is false. There exists $\epsilon_0 > 0$ such that, for any $F \subseteq X$ with $\mu F < \infty$, there is at least one n satisfying

$$\int_{X-F} |f_n| d\mu \geq \epsilon_0 .$$

Starting with $F = \phi$, there is an n_1 such that

$$\int_{X-\phi} |f_{n_1}| d\mu \geq \epsilon_0$$

and there is an F_1 with $\mu F_1 < \infty$ such that

$$\int_{F_1} |f_{n_1}| d\mu \geq \frac{\epsilon_0}{2} .$$

Next we choose n_2 so that

$$\int_{X-F_1} |f_{n_2}| d\mu \geq \epsilon_0, \quad F_2 \subset X-F_1$$

and

$$\int_{F_2} |f_{n_2}| d\mu \geq \frac{\epsilon_0}{2} .$$

Continue by induction. The set $F_1 \cup \ldots \cup F_k$ has finite measure. So, for some n

$$\int_{X-(F_1 \cup \ldots \cup F_k)} |f_n| d\mu \geq \epsilon_0 .$$

In this way, we obtain a disjoint sequence $\{F_k\}$ of sets and a sequence $\{n_k\}$ of positive integers such that

$$\int_{F_k} |f_{n_k}| d\mu \geq \frac{\epsilon_0}{2} \quad \text{for} \quad k = 1, 2, 3, \ldots .$$

Let

$$E_m = \bigcup_{k \geq m} F_k .$$

Then

$$\int_{E_m} |f_{n_m}| d\mu \geq \frac{\epsilon_0}{2} \quad \text{for all} \quad m$$

thus contradicting Hypothesis 7.

Theorem 23. ($1\alpha\Phi\pi \to 2$). Let (X, S, μ) be a uniformly atomic and finite measure space. If $f_n \to f$ a.e. and f is integrable, then the sequence $\{f_n\}$ is uniformly bounded.

Proof. $1\alpha \to X$ can have at most a finite number of atoms. f is a constant on each atom and since $f_n \to f$ a.e. there exists N such that $n \geq N \to |f_n| \leq |f| + \epsilon$ and 2 is satisfied.

Theorem 24. ($1\alpha\Phi\pi \to 8$). Let (X, S, μ) be a uniformly atomic and finite measure space. If $f_n \to f$ a.e. and f is integrable, then f_n is integrable for all $n \geq N$.

Proof. Follows from Theorem 23.

<u>Theorem 25.</u> ($\oplus\Phi\Gamma\pi \to 6$). Suppose $f_n \to f$ a.e. and $\int f_n d\mu \to \int f d\mu$. If $f_n \geq 0$ for all n, then the integrals $\int f_n d\mu$ are uniformly bounded.

<u>Proof.</u> By $\Phi\Gamma$, $\int f_n d\mu < \int f d\mu + \epsilon$ and by $\oplus$, $\int |f_n| d\mu = \int f_n d\mu < \int f d\mu + \epsilon < \infty$ for all n.

<u>Theorem 26.</u> ($7\Phi\Gamma\pi \to 6$). If $f_n \to f$ a.e., $\int f_n \to \int f$ and the sequence $\{f_n\}$ is equicontinuous from above at ϕ, then the integrals $\int |f_n| d\mu$ are uniformly bounded.

<u>Proof.</u> Since we are assuming that $f_n \to f$ a.e., then (2, p. 78) $f_n \to f$ in mean. We have

$$\int |f_n| d\mu = \int |f_n - f + f| d\mu \leq \int |f_n - f| d\mu + \int |f| d\mu$$

from which it follows that

$$\lim_n \int |f_n| d\mu = \int |f| d\mu.$$

Given $\epsilon > 0$, there exists N such that $n \geq N \to$

$$\int |f_n| d\mu \leq \int |f| d\mu + \epsilon$$

Since $\Phi\Gamma \to$ the integrability of each f_n, then setting

$$c = \max \{ \int |f_1| d\mu, \ldots, \int |f_N| d\mu \},$$

we get

$$\int |f_n| d\mu \leq c + \int |f| d\mu + \epsilon$$

and consequently 6 is satisfied.

Theorem 27. ($7\sigma\Phi\pi \rightarrow 6$). Suppose $f_n \rightarrow f$ a. e. on a σ-finite measure space (X, S, μ). If $\{\int |f_n| d\mu\}$ is equicontinuous from above at ϕ, then the integrals $\int |f_n| d\mu$ are uniformly bounded.

Proof. Convergence a. e. + hypotheses $7\sigma \rightarrow$ convergence in mean (2, Theorem 7, p. 77). We have

$$\int |f_n| d\mu \leq \int |f_n - f| d\mu + \int |f| d\mu < \epsilon + \int |f| d\mu$$

Consequently, 6 follows by Theorem 26.

Theorem 28. ($7\sigma \rightarrow 4$). Suppose (X, S, μ) is a σ-finite measure space. If the sequence $\{\int |f_n| d\mu\}$ is equicontinuous from above at ϕ, then it is uniformly semi-finitely-supported.

Proof. By Hypothesis σ we may assume $X = \bigcup_{k=1}^{\infty} F_k$ where $\mu F_k < \infty$ for all k. Put $G_m = \bigcup_{k=1}^{\infty} F_k$. Then $\{X - G_m\}$ is a decreasing sequence and $\lim_m (X - G_m) = \phi$. By Hypothesis 7, given

$\epsilon > 0$, there is M such that $m \geq M \rightarrow \int_{X-G_m} |f_n| d\mu < \epsilon$ for all n, and 4 is satisfied.

Theorem 29. ($\oplus\Phi\Gamma\pi \rightarrow 4$). Suppose $f_n \rightarrow f$ a.e., $\int f_n \rightarrow \int f < \infty$, and $f_n \geq 0$ for all n. Then $\{f_n\}$ is uniformly semi-finitely supported.

Proof. Put $g_n = f_n - f$. $f_n \rightarrow f$ a.e. $\rightarrow g_n \rightarrow 0$ a.e. and $g_n^- \rightarrow 0$ a.e. From the inequality

$$|g_n| = |f_n - f| \leq f_n + f$$

we see that

$$0 \leq g_n^- \leq f.$$

Hence by the Lebesgue bounded convergence theorem we have

$$\lim_n \int g_n^- d\mu = 0$$

i.e.,

$$\lim_n \int (f_n - f)^- d\mu = 0.$$

By Hypotheses $\Phi\Gamma$

$$\lim_n \int (f_n - f) d\mu = 0$$

from which we conclude that

$$\lim_n \int (f_n - f)^+ d\mu = 0$$

Therefore,

$$\lim_n \int |f_n - f| d\mu = 0.$$

Now, for any measurable $E \subseteq X$, we have

$$\left| \int_E (f_n - f) d\mu \right| \leq \int_E |f_n - f| d\mu \leq \int_X |f_n - f| d\mu$$

and

$$\lim_n \int_E f_n d\mu = \int_E f d\mu .$$

Since f is integrable, then f is semi-finitely-supported. Given $\epsilon > 0$, there exists F with $\mu F < \infty$ such that $\int_{X-F} f < \frac{\epsilon}{3}$, and there exists N such that $n > N \rightarrow$

$$\int_{X-F} f_n d\mu \leq \int_{X-F} f d\mu + \frac{\epsilon}{3} .$$

Moreover, we can find $F_1, F_2, \ldots, F_N$, $\mu F_i < \infty$ such that

$$\int_{X-F_1} f_1 < \frac{\epsilon}{3N}, \ldots, \int_{X-F_N} f_N < \frac{\epsilon}{3N} .$$

If $E = F \cup F_1 \cup \ldots \cup F_N$, then $\mu E < \infty$ and

$$\int_{X-E} f_n < \frac{\epsilon}{3} + \frac{\epsilon}{3} + N\frac{\epsilon}{3N}$$

for all n.

Theorem 30. ($6\updownarrow \to 3$). If $\{f_n\}$ is a monotonic sequence of measurable functions and if the integrals $\int |f_n| d\mu$ are uniformly bounded, then there exists an integrable function g such that $g \geq |f_n|$ for all n.

Proof. We consider the case when $|f_1| \leq |f_2| \leq \cdots$. All other cases are trivial. Define $g(x) = \lim_n |f_n(x)|$ (the existence of the limit is guaranteed by the Beppo Levi Theorem (1, p. 67)) and note that $|f_n| \leq g$ for all n. Moreover, g is integrable: By Fatou's Lemma and Hypothesis 6

$$\int g d\mu = \int \lim |f_n| d\mu = \int \underline{\lim} |f_n| d\mu \leq \underline{\lim} \int |f_n| d\mu \leq M.$$

Theorem 31. ($\ominus\Phi\Gamma\pi \to 7$). Suppose $f_n \to f$ a.e., $\int f_n d\mu \to \int f d\mu < \infty$, and $f_n \geq 0$ for all n. Then the sequence $\{\int |f_n| d\mu\}$ is equicontinuous from above at ϕ.

Proof. From the proof of Theorem 29 we see that

$$\lim_n \int |f_n - f| d\mu = 0.$$

Let $\{E_k\}$ be a decreasing sequence of measurable subsets of X such that $\lim_k E_k = \phi$. Consider $\epsilon > 0$ and a positive integer n_0 such that $\int |f_n - f| d\mu < \epsilon$ for all $n \geq n_0$. Then for $n \geq n_0$, we have

$$\int_{E_k} f_n d\mu \leq \int_{E_k} f d\mu + \int_{E_k} |f_n - f| d\mu$$

$$< \int_{E_k} f d\mu + \epsilon \ . \qquad \text{(i)}$$

Now, consider the sequence $\{f \chi_{E_k}\}$. Clearly $f\chi_{E_k}(x) \leq f(x)$ a. e. on X. By the Lebesgue dominated convergence theorem,

$$\lim_k \int_{E_k} f d\mu = \lim_k \int_X f \chi_{E_k} d\mu = \int_X \lim_k f \chi_{E_k} d\mu$$

$$= \int_X f \chi_\phi d\mu = 0.$$

Consequently, there is k_0 such that $k \geq k_0 \rightarrow$

$$\int_{E_k} f d\mu < \epsilon$$

and moreover, by (i) we see that for all $n \geq n_0$ and $k \geq k_0$

$$\int_{E_k} f_n d\mu < 2\epsilon \ . \qquad \text{(ii)}$$

Finally, if $n \in \{1, 2, \ldots, n_0\}$ and $g = \max\{f_1, f_2, \ldots, f_{n_0}\}$, then by the dominated convergence theorem (and the fact that $\Phi\Gamma \to 8 \to g$ integrable)

$$\lim_k \int_{E_k} f_n d\mu \leq \lim_k \int_{E_k} g d\mu = \lim_k \int g \chi_{E_k} d\mu = \int \lim_k g \chi_{E_k} d\mu = 0.$$

Hence there exists k_1 such that $k \geq k_1$ and $n \in \{1, 2, \ldots, n_0\} \to \int_{E_k} |f_n| d\mu < \epsilon$ so that by (ii) there exists $k^* = \max\{k_0, k_1\}$ such that $k \geq k^* \to$

$$\int_{E_k} f_n d\mu < \epsilon$$

uniformly in n.

<u>Theorem 32.</u> If the indefinite integral of a function f is continuous from above at ϕ, then f is semi-bounded.

<u>Proof.</u> Put $B_k = \{x : |f(x)| \geq k\}$. The sequence $\{B_k\}_{k=1}^{\infty}$ is decreasing and $\mu(\bigcap_{k=1}^{\infty} B_k) = 0$ since f is finite a.e. Let $\epsilon > 0$ be given. There is k^* such that $k \geq k^*$ implies

$$\int_{B_k} |f| d\mu < \epsilon.$$

<u>Theorem 33.</u> $(7\pi \to 5)$. Suppose $f_n \to f$ a.e. and the

indefinite integrals $\int |f_n| d\mu$ are equicontinuous from above at ϕ. Then the sequence $\{f_n\}$ is uniformly semi-bounded.

<u>Proof</u>. Write $B_{n,k} = \{x : |f_n(x)| \geq k\}$ and $E_k = \bigcup_{n=k}^{\infty} B_{n,k}$. Clearly $E_1 \supset E_2 \supset \ldots,$ and $\overline{\lim_{n\to\infty}} |f_n(x)| = \infty$ on $\bigcap_{k=1}^{\infty} E_k$. Hence $\mu(\bigcap_{k=1}^{\infty} E_k) = 0$. Write $F_k = E_k - \bigcap_{k=1}^{\infty} E_k$. Use the equicontinuity of $\int |f_n| d\mu$ to choose k_0 such that for $k \geq k_0$ and for all n,

$$\int_{F_k} |f_n| d\mu < \epsilon. \tag{i}$$

For $n \geq k_0$, we have $B_{n,k_0} \subset E_{k_0}$ and so for $n \geq k_0$ and $k \geq k_0$, it follows that

$$\int_{B_{n,k}} |f_n| d\mu \leq \int_{B_{n,k_0}} |f_n| d\mu \leq \int_{E_{k_0}} |f_n| d\mu = \int_{F_{k_0}} |f_n| d\mu < \epsilon. \tag{ii}$$

For $n \epsilon \{1, \ldots, k_0 - 1\}$, f_n is semi-bounded by the preceding theorem. Consequently, there are integers k_n such that

$$\int_{|f_n| \geq k_n} |f_n| d\mu < \epsilon.$$

If we choose $k^* = \max\{k_0, k_1, \ldots, k_{k_0 - 1}\}$ we see that (ii) holds for $k \geq k^*$ and all n.

Theorem 34. ($\oplus\Phi\Gamma\pi \to 5$). Suppose $f_n \to f$ a.e., $\int f_n d\mu \to \int f d\mu < \infty$, and $f_n \geq 0$ for all n. Then the sequence $\{f_n\}$ is uniformly semi-bounded.

Proof. $\Phi\Gamma\oplus\pi \to 7\pi \to 5$ by Theorems 31 and 33.

Theorem 35. ($6\oplus\Gamma\pi \to 45$). Suppose $f_n \to f$ a.e., $f_n \geq 0$ for all n, $\int f_n d\mu \to \int f d\mu$ and the integrals $\int |f_n| d\mu \leq M$ for all n. Then the sequence $\{f_n\}$ is uniformly semi-bounded and uniformly semi-finitely-supported.

Proof. Obviously, if $M \geq \int f_n d\mu$ then $M \geq \int f d\mu$ and Hypothesis Φ is satisfied. Therefore, $6\oplus\Gamma\pi \to \Phi\Gamma\oplus\pi \to 45$ by Theorems 29 and 34.

Theorem 36. ($69 \to 5$). If the indefinite integrals of the sequence of functions $\{f_n\}$ are uniformly absolutely continuous and uniformly bounded, then $\{f_n\}$ is uniformly semi-bounded.

Proof. $B_{nk} = \{x: |f_n(x)| \geq k\}$. Hypothesis 6 asserts the existence of a positive number M such that $\int |f_n| d\mu < M$ for all n.

$$\mu(B_{nk})k = \int_{B_{nk}} k \leq \sup_n \int_{B_{nk}} |f_n| d\mu \leq \sup_n \int |f_n| d\mu \leq M.$$

Consequently, $\mu B_{nk} \to 0$ as $k \to \infty$ uniformly in n. Let $\epsilon > 0$

be given. By 9, there is $\delta > 0$ (which in turn implies the existence of k_0 such that $k \geq k_0 \rightarrow \mu B_{nk} < \delta$) such that $\mu B_{nk} < \delta \rightarrow$

$$\int_{B_{nk}} |f_n| d\mu < \epsilon$$

uniformly in n. Hence 5 is satisfied.

Theorem 37. ($6a \rightarrow 2 \rightarrow 5$). Let $\{f_n\}$ be a sequence of functions defined on a uniformly atomic measure space (X, S, μ) such that the integrals $\int |f_n| d\mu$ are uniformly bounded. Then the sequence $\{f_n\}$ is uniformly bounded and therefore uniformly semi-bounded.

Proof. It suffices to show that $(\sim 2, a) \rightarrow \sim 6$. $a \rightarrow \bigcup_{a \in I} X_a$ where the X_a's are disjoint atoms and $\mu X_a > \delta_0 > 0$ for all a. Let k be any positive integer and use (~ 2) to find a function f_{n_k} such that $|f_{n_k}| > k$ on a set E_k with $\mu E_k > \delta_0$. Consequently,

$$\sup_n \int |f_n| d\mu \geq \int_{E_k} |f_{n_k}| d\mu > \int_{E_k} k d\mu \geq k\delta_0 \rightarrow \infty$$

as $k \rightarrow \infty$ and ~ 6 is proved. Theorem 15 completes the proof.

Theorem 38. $1a\pi \rightarrow (1, 2, 3, 4, 5, 6, 7, 8, 9, \Phi, \Gamma, \sigma, a)$.

Proof. We can write $X = x_1 \cup x_2 \cup \ldots \cup x_p$, where the x_i's are disjoint atoms and $\mu x_i > \delta_0$ $(i = 1, 2, \ldots, p)$. Every measurable function defined on an atom must be a constant. Let c_i $(i = 0, 1, 2, \ldots)$ be a constant such that $c_0 > \max |f(x)|$ and $c_i > \max |f_i(x)|$. By π there are p positve integers $N_1, N_2, \ldots, N_p$ such that $n \geq N_i \rightarrow$

$$|f_n(x_i) - f(x_i)| < 1 \qquad (*)$$

$(i = 1, 2, \ldots, p)$. Let $N = \max \{N_1, \ldots, N_p\}$. If $n \geq N$, then $(*) \rightarrow$

$$|f_n(x)| \leq c_0 + 1$$

for all $x \in X$. Now, if $k = \max \{c_0 + 1, c_1, c_2, \ldots, c_{N-1}\}$, then

$$|f_n(x)| \leq k$$

for all x in X and all n. Hence 2 is satisfied. Now, it is easy to see (2, p. 18-27) that $12 \rightarrow 3789\Phi\sigma$. Moreover, $1 \rightarrow 4$, $2 \rightarrow 5$ and $45 \rightarrow 6$ by Theorems 19, 15 and 18 respectively. It remains to show Γ: $f_n \rightarrow f$ a. e. $+ 3 \rightarrow f_n \rightarrow f$ in mean (2, Theorem 2, p. 76). Hence

$$\left| \int (f_n - f) d\mu \right| \leq \int |f_n - f| d\mu$$

and thus

$$\lim_n \int f_n d\mu = \int f d\mu.$$

Theorem 39. ($49\pi \to 5$). Suppose $\{f_n\}$ is a uniformly semi-finitely-supported sequence converging a.e. to a function f. If the indefinite integrals are uniformly absolutely continuous, then $\{f_n\}$ is uniformly semi-bounded.

Proof. $49 \to 7$ by Theorem 17. The rest follows from Theorem 33.

Theorem 40. ($4\alpha\pi \to 5$). Let $\{f_n\}$ be a uniformly semi-finitely-supported sequence of functions defined on a uniformly atomic measure space (X, S, μ). If $f_n \to f$ a.e. then the sequence $\{f_n\}$ is uniformly semi-bounded.

Proof. By Theorem 21, $\alpha \to 9$. The proof is clear by Theorem 39.

Theorem 41. ($5\alpha \to 2$). If $\{f_n\}$ is a uniformly semi-bounded sequence of functions defined on a uniformly atomic measure space (X, S, μ), then the sequence $\{f_n\}$ is uniformly essentially bounded.

Proof. $X = \bigcup_{\alpha \in I} X_\alpha$ where the X_α's are disjoint atoms and $\mu X_\alpha > \delta_0 > 0$ for all α. $5 \to \exists k_0$ such that for all $k \geq k_0$ and $\forall_n$ we have

$$\int_{|f_n| \geq k} |f_n| d\mu < 1. \tag{i}$$

If $k^* = \max \{k_0, \frac{1}{\delta_0}\}$, then in order for (i) to hold we must have

$$\mu\{x : |f_n(x)| \geq k^*\} < \delta_0 .$$

By the uniform atomicity of the space we see that

$$\mu\{x : |f_n(x)| \geq k^*\} = 0,$$

i.e., $|f_n(x)| < k^*$ a.e. uniformly in n.

Theorem 42. ($7\alpha\pi \rightarrow 2$). Suppose $f_n \rightarrow f$ a.e., the indefinite integrals $\int |f_n| d\mu$ are equicontinuous from above at ϕ and the measure space (X, S, μ) is uniformly atomic. Then the sequence $\{f_n\}$ is essentially uniformly bounded.

Proof. $7\pi \rightarrow 5$ by Theorem 33, and $5\alpha \rightarrow 2$ by Theorem 41.

Theorem 43. ($6\pi \rightarrow \Phi$). If the sequence of functions $\{f_n\}$ converges a.e. to a function f and the integrals $\int |f_n| d\mu$ are uniformly bounded, then f is integrable.

Proof. Since $f_n \rightarrow f$ a.e. then $|f_n| \rightarrow |f|$ a.e. and by Fatou's Lemma (7, p. 72), we have

$$\int |f| d\mu = \int \underline{\lim} |f_n| d\mu \leq \underline{\lim} \int |f_n| d\mu \leq M;$$

and thus $|f|$ is integrable. Consequently, f is integrable.

Theorem 44. ($7\sigma\pi \to \Phi\Gamma$). If $f_n \to f$ a.e., the support of the f_n's is σ-finite and the indefinite integrals $\int |f_n| d\mu$ are equicontinuous from above at ϕ, then f is integrable and $\lim_n \int f_n d\mu = \int f d\mu$.

Proof. Let $\epsilon > 0$ be given. By Theorem 28, there exists a measurable set A with $\mu A < \infty$ such that for all n

$$\int_{A'} |f_n| d\mu < \epsilon. \qquad \text{(i)}$$

By Theorem 18, there is $\delta > 0$ such that

$$\mu E < \delta \to \int_E |f_n| d\mu < \epsilon \qquad \text{(ii)}$$

for all n. By Fatou's Lemma,

$$\int_{A'} |f| d\mu = \int_{A'} \underline{\lim} |f_n| d\mu \leq \underline{\lim} \int_{A'} |f_n| d\mu < \epsilon$$

and

$$\int_{A'} |f_n - f| d\mu \leq \int_{A'} |f| d\mu + \int_{A'} |f_n| d\mu < 2\epsilon;$$

thus, it is sufficient to prove the theorem for the totally finite case.

For a given $\epsilon > 0$, use (ii) to find $\delta > 0$. By Egoroff's Theorem (4, p. 88), there exists a measurable set B such that $\mu B < \delta$ and $f_n \to f$ uniformly on B'. By Fatou's Lemma and (ii) we have

$$\int_B |f| d\mu = \int_B \underline{\lim} |f_n| d\mu \leq \underline{\lim} \int_B |f_n| d\mu < \epsilon \qquad \text{(iii)}$$

and

$$\int_B |f_n - f| d\mu \leq \int_B |f_n| d\mu + \int_B |f| d\mu < 2\epsilon \qquad \text{(iv)}$$

for all large n. By the uniform convergence on B', $\exists N$ such that $n \geq N \to$

$$\int_{B'} |f_n - f| d\mu < \epsilon \mu B'. \qquad \text{(v)}$$

By Theorem 3.15 (2, p. 25-27), we may assume that each f_n is integrable. Hence, for some large n

$$\int_{B'} |f| d\mu \leq \int_{B'} |f - f_n| d\mu + \int_{B'} |f_n| d\mu < \epsilon \mu B' + M < \infty \qquad \text{(vi)}$$

since $\mu B' < \mu X < \infty$. Finally, by (iii) and (vi) we see that f is integrable:

$$\int |f| d\mu = \int_B |f| d\mu + \int_{B'} |f| d\mu < \infty,$$

and by (iv) and (v) and for large n, we have

$$\left| \int (f_n - f) d\mu \right| \leq \int |f_n - f| d\mu = \int_B |f_n - f| d\mu + \int_{B'} |f_n - f| d\mu < 2\epsilon + \epsilon \mu B'$$

and thus

$$\lim_n \int f_n d\mu = \int f d\mu .$$

Theorem 45. ($8 \updownarrow \pi \rightarrow \Gamma$). Let $\{f_n\}$ be a monotomic sequence of integrable functions converging a. e. to a function f, then

$$\lim_n \int f_n d\mu = \int f d\mu .$$

Proof. Let $\{f_n\}$ be a nondecreasing sequence of integrable functions converging a. e. to a function f. If we put

$$g_k = f_{k+1} - f_k$$

then $g_k \geq 0$ for all k and

$$\lim_n f_n = \lim_n \left[f_1 + \sum_{k=1}^{n-1} g_k \right] = f_1 + \sum_{k=1}^{\infty} g_k .$$

Consequently (7, Corollary 16, p. 201),

$$\int f d\mu = \int \lim_n f_n d\mu = \int f_1 d\mu + \int \sum_{k=1}^{\infty} g_k = \int f_1 d\mu + \sum_{k=1}^{\infty} \int g_k d\mu$$

$$= \int f_1 d\mu + \sum_{k=1}^{\infty} \int (f_{k+1} - f_k) d\mu = \lim_n \int f_n d\mu.$$

On the other hand, if the sequence $\{f_n\}$ is nonincreasing and $f_n \to f$ a.e., then the sequence $\{-f_n\}$ is nondecreasing and $-f_n \to -f$ a.e.. By the above argument, we have

$$\lim_n \int -f_n d\mu \to \int -f d\mu$$

which implies that

$$\lim_n \int f_n d\mu = \int f d\mu.$$

Note: The theorem remains true if we delete Hypothesis 8 and assume that only f_1 is integrable.

<u>Theorem 46.</u> ($8 \oplus \downarrow \to 3$). If $\{f_n\}$ is a nonincreasing sequence of non-negative integrable functions, then there exists an integrable function $g \geq 0$ such that $f_n \leq g$ a.e. for all n.

<u>Proof.</u> Let $g = f_1$.

<u>Theorem 47.</u> ($\uparrow \oplus \Phi\pi \to 8$). If $\{f_n\}$ is a nondecreasing

sequence of non-negative functions converging a. e. to an integrable function f, then f_n is integrable for each n.

Theorem 48. ($7\oplus\downarrow \rightarrow 5$). If $\{f_n\}$ is a nonincreasing sequence of non-negative functions such that the indefinite integrals $\int |f_n| d\mu$ are equicontinuous from above at ϕ, then $\{f_n\}$ is uniformly semi-bounded.

Proof. By Theorem 32, f_1 is semi-bounded. Since $f_1 \geq f_2 \geq f_3 \geq \ldots$ then $\{f_n\}$ is uniformly semi-bounded.

Theorem 49. ($7a\oplus\downarrow \rightarrow 2$). Suppose (X, S, μ) is a uniformly atomic measure space. If $\{f_n\}$ is a non-increasing sequence of non-negative functions such that the indefinite integrals are equicontinuous from above at ϕ, then $\{f_n\}$ is uniformly essentially bounded.

Proof. By Theorem 48, $\{f_n\}$ is uniformly semi-bounded, and by Theorem 41, $\{f_n\}$ is uniformly essentially bounded.

Theorem 50. ($\uparrow\oplus\pi \rightarrow \Gamma$). If $\{f_n\}$ is a nondecreasing sequence of non-negative functions converging a. e. to a function f, then $\lim_n \int f_n d\mu = \int f d\mu$.

Proof. Follows from the monotone convergence theorem.

Theorem 51. $(4\alpha\oplus\downarrow \to 2)$. Let $\{f_n\}$ be a nonincreasing sequence of non-negative functions defined on a uniformly atomic measure space (X, S, μ). If $\{f_n\}$ is uniformly semi-finitely-supported, then $\{f_n\}$ is uniformly essentially bounded.

Proof. Follows from Theorems 21, 17 and 49.

Theorem 52. $(1\alpha\oplus\downarrow \to 12 \to 3)$. Let $\{f_n\}$ be a nonincreasing sequence of non-negative functions defined on a uniformly atomic measure space (X, S, μ). If f_n is integrable for each n, then $\{f_n\}$ is uniformly essentially bounded.

Proof. By Theorem 19. $1 \to 4$. By Theorem 51, we may assume Hypothesis 2. Theorem 3.9 (2, p. 23) asserts that $12 \to 3$.

IV. LIST OF COUNTEREXAMPLES

Each counterexample listed below involves a sequence of functions $\{f_n\}$ defined on a measure space (X, S, μ). The symbol R will denote the real line with Lebesgue measure on all Lebesgue-measurable subsets, and the symbol I will denote the set of all integers with counting measure $\mu\{x\} = 1$ for all x on all subsets. Other measure spaces are required only occasionally and will be described in detail. Some of the 25 examples in (2, p. 28-42) have been used. The hypotheses that are satisfied are indicated in each case.

1. X : any non-empty set; S : $\{X, \phi\}$; $\mu X = \infty$, $\mu\phi = 0$; $f_n(x) = \frac{1}{n}$ for all x; $f(x) = 0$ for all x. Satisfies 2579α⊕↓Φπ.

2. R; $f_n = n\chi_{(0,\frac{1}{n})} - n\chi_{[-\frac{1}{n}, 0]}$; $f(x) = 0$ for all x. Satisfies 1468ρστΦΓπ.

3. R; $f(x) = \frac{1}{x+1}$ if $x > 0$, 0 otherwise; $f_n = f\chi_{[0, n]}$. Satisfies 2589ρστ↑⊕Γπ.

4. X : any non-empty set; S : $\{X, \phi\}$; $\mu X = \infty$, $\mu\phi = 0$; $f_n(x) = 1 - \frac{1}{n}$ for all x; $f(x) = 1$ for all x. Satisfies 2579α↑⊕Γπ.

5. R; $f_n(x) = \frac{1}{n}$ for all x; $f(x) = 0$ for all x. Satisfies 259ρστ O↓Φπ.

6. R; $f_n(x) = x^n$ if $0 \leq x \leq 1$, 0 elsewhere; $f(x) = 0$ for all x. Satisfies 123456789ρστ⊕↓ΦΓπ.

7. R; $f_n(x) = 1$ if $n - \frac{1}{n} \le x \le n$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies 2456789$\rho\sigma\tau\oplus\Phi\Gamma\pi$.

8. X : the set of positive integers; S : all subsets of X; $\mu(x) = 2^{-x}$,

$\mu E = \sum_{x \in E} \mu(x)$; $f_n(x) = 2^x$ if $x \ge n$, 0 otherwise; $f(x) = 0$ for all x. Satisfies 14$\sigma\tau\oplus\downarrow\Phi\pi$.

9. (X, S, μ) is the same as in Example 8; $f_n(x) = 2^x$ if $x \le n$,

0 otherwise; $f(x) = 2^x$ for all x. Satisfies 148$\sigma\tau\uparrow\oplus\Gamma\pi$.

10. R; $f_n(x) = 1$ if $n \le x \le n+1$, -1 if $-n-1 \le x \le -n$, 0 otherwise;

$f(x) = 0$ for all x. Satisfies 25689$\rho\sigma\tau\Phi\Gamma\pi$.

11. R; $f_n(x) = x^{-\frac{1}{2}} = f(x)$ if $0 < x \le 1$, 0 otherwise.

Satisfies 13456789$\rho\sigma\tau\uparrow\oplus\downarrow\Phi\Gamma\pi$.

12. R; $f_n(x) = \frac{1}{n}$ if $0 < x \le n^2$, $= -\frac{1}{n}$ if $-n^2 \le x \le 0$, 0 otherwise;

$f(x) = 0$ for all x. Satisfies 2589$\rho\sigma\tau\Phi\Gamma\pi$.

13. X : any non-empty set; S : $\{X, \phi\}$; $\mu X = \infty$, $\mu\phi = 0$; $f_n(x) = 1 = f(x)$

for all x. Satisfies 2579$\alpha\uparrow\oplus\downarrow\Gamma\pi$.

14. R; $f_n(x) = n$ if $0 < x \le \frac{1}{n}$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies 1468$\rho\sigma\tau\oplus\Phi\pi$.

15. R; $f_n(x) = \frac{1}{n}$ if $-n \le x \le n$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies 25689$\rho\sigma\tau\oplus\Phi\pi$.

16. R; $f_n(x) = n$ if $0 \le x \le 1$, 0 otherwise. Satisfies 148$\rho\sigma\tau\uparrow\oplus$.

17. R; $f_n(x) = \frac{1}{n}$ if $0 < x \le n^2$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies 2589$\rho\sigma\tau\oplus\Phi\pi$.

18. I; $f(x) = \frac{1}{x+1}$ if $x > 1$, 0 otherwise; $f_n = f\chi_{\{1, 2, 3, \ldots, n\}}$.

Satisfies $2589\alpha\sigma\tau\uparrow\oplus\Gamma\pi$

19. X : any non-empty set; S : $\{X, \phi\}$; $\mu X = 1$, $\mu\phi = 0$; $f_n(x) = \frac{1}{n}$ for all x; $f(x) = 0$ for all x. Satisfies $123456789\alpha\sigma\tau\oplus\downarrow\Phi\Gamma\pi$.

20. I; $f_n(x) = 0$ if $x \leq n$, n if $x > n$; $f(x) = 0$ for all x.

Satisfies $9\alpha\sigma\tau\oplus\Phi\pi$.

21. R; $f_n(x) = \frac{1}{n} e^{-x^2}$ for all x; $f(x) = 0$ for all x. Satisfies 2345 $6789\rho\sigma\tau\oplus\downarrow\Phi\Gamma\pi$.

22. X : any non-empty set; S : $\{X, \phi\}$; $\mu X = 1$, $\mu\phi = 0$; $f_n(x) = n$ for all x. Satisfies $14789\alpha\sigma\tau\uparrow\oplus$.

23. I; $f_n(x) = 1 = f(x)$ if $x \geq 1$, 0 otherwise. Satisfies $259\alpha\sigma\tau\uparrow\oplus\downarrow\Gamma\pi$.

24. R; $f_n(x) = n+1$ if $\frac{1}{n+1} \leq x \leq \frac{1}{n}$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies $1456789\rho\sigma\tau\oplus\Phi\Gamma\pi$.

25. R; $f_n = \sum_{i=1}^{\infty} \chi_{\left(\frac{1}{i+1}, \frac{1}{i}\right]}$ where $i > n + 1$; $f(x) = 0$ for all x.

Satisfies $14\rho\sigma\tau\oplus\Phi\Gamma\pi$.

26. R; $f_n(x) = 2$ if $\frac{1}{2} < x \leq 1$, 3 if $\frac{1}{3} < x \leq \frac{1}{2}$, . . . , n+1 if $\frac{1}{n+1} < x \leq \frac{1}{n}$, 0 otherwise; $f(x) = 2$ if $\frac{1}{2} < x \leq 1$, 3 if $\frac{1}{3} < x \leq \frac{1}{2}$, 4 if $\frac{1}{4} < x \leq \frac{1}{3}$, . . . , 0 otherwise. Satisfies $148\rho\sigma\tau\uparrow\oplus\pi$.

27. I; $f_n(x) = \frac{1}{n}$ if $x \geq 1$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies $259\alpha\sigma\tau\oplus\downarrow\Phi\pi$.

28. X : [0, 1] x [0, 1]; S : the sigma-algebra generated by all measurable rectangles E = A x B where A and B are measurable

subsets of $[0,1]$; $\mu E = \mu(A \times B) = \sum_{x \in B} m(A)$ where m is taken to be Lebesgue measure on $[0,1]$; $f_n(x,y) = 1 - \frac{1}{n}$ for all (x,y); $f(x,y) = 1$ for all (x,y). Satisfies $259\rho\tau\uparrow\oplus\Gamma\pi$.

29. (X, S, μ) is the same as in Example 28; $f_n(x,y) = f(x,y) = 1$ for all (x,y). Satisfies $259\rho\tau\uparrow\oplus\downarrow\Gamma\pi$.

30. (X, S, μ) is any uncountable set with counting measure; $f_n(x) = f(x) = 1$ for all x. Satisfies $259\alpha\tau\uparrow\oplus\downarrow\Gamma\pi$.

31. I; $f_n(x) = \frac{1}{n}$ if $0 < x \leq n^2$, $-\frac{1}{n}$ if $-n^2 \leq x \leq 0$, 0 otherwise; $f(x) = 0$ for all x. Satisfies $2589\alpha\sigma\tau\Phi\Gamma\pi$.

32. I; $f_n(x) = \frac{1}{n}$ if $0 < x \leq n$, 0 otherwise; $f(x) = 0$ for all x. Satisfies $25689\alpha\sigma\tau\oplus\Phi\pi$.

33. I. $f_n(x) = \frac{1}{n}$ if $0 < x \leq n^2$, 0 otherwise; $f(x) = 0$ for all x. Satisfies $2589\alpha\sigma\tau\oplus\Phi\pi$.

34. I; $f_n(x) = 1$ if $x = 2n$, $= -1$ if $x = 2n+1$, 0 otherwise; $f(x) = 0$ for all x. Satisfies $25689\alpha\sigma\Phi\Gamma\pi$.

35. R; $f_n = n^2 \chi_{[0, \frac{1}{n}]}$; $f(x) = 0$ for all x. Satisfies $148\rho\sigma\tau\oplus\Phi\pi$.

36. R; $f_n(x) = n^2$ if $0 \leq x \leq \frac{1}{n}$, $-n^2$ if $-\frac{1}{n} \leq x \leq 0$, 0 otherwise; $f(x) = 0$ for all x. Satisfies $148\rho\sigma\tau\Phi\Gamma\pi$.

37. I; $f_n(x) = 0$ if $x \in [0,n] \cup [-n,0]$, n if $x \in [n, 2^n]$, $-n$ if $x \in [-2^n, -n]$, 0 otherwise; $f(x) = 0$ for all x. Satisfies $89\alpha\sigma\tau\Phi\Gamma\pi$.

38. I; $f_n(x) = 2^{x-n}$ if $x \geq 1$, 0 otherwise; $f(x) = 0$ for all x. Satisfies $9\alpha\sigma\tau\oplus\downarrow\Phi\pi$.

39. I; $f_n(x) = 2^{x-n} + 1$ if $x \geq 1$, 0 otherwise; $f(x) = 1$ for all x.

Satisfies $9\alpha\sigma\tau\oplus\downarrow\Gamma\pi$.

40. (X, S, μ) is the same as in Example 37 with the exception of re-defining $f_n(x) = f(x) = 0$ for all negative integers x.

Satisfies $89\alpha\sigma\tau\oplus\Phi\pi$.

41. I; $f_n(x) = x$ if $0 < x \leq n$, 0 otherwise; $f(x) = x$ for all x.

Satisfies $89\alpha\sigma\tau\uparrow\oplus\Gamma\pi$.

42. I; $f(x) = f_n(x) = 2^{-x}$ if $x \geq 1$, 0 otherwise for all n.

Satisfies $23456789\alpha\sigma\tau\uparrow\oplus\downarrow\Phi\Gamma\pi$.

43. I; $f_n(x) = \frac{1}{n}$ if $x = n$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies $2456789\alpha\sigma\tau\oplus\Phi\Gamma\pi$.

44. R; $f_n(x) = f(x) = \frac{1}{x}$ if $0 < x \leq 1$, 0 otherwise for all n.

Satisfies $14\rho\sigma\tau\uparrow\oplus\downarrow\Gamma\pi$.

45. I; $f_n(x) = f(x) = 1$ for all $x \geq 1$, 0 otherwise. Satisfies $259\alpha\sigma\tau\uparrow\oplus\downarrow\Gamma\pi$.

46. R; $f_n(x) = f(x) = 1$ for all x and all n. Satisfies $259\rho\sigma\tau\uparrow\oplus\downarrow\Gamma\pi$.

47. R; $f_{2n-1}(x) = 1 + \frac{1}{n}$, $f_{2n}(x) = 1 - \frac{1}{n}$ if $0 \leq x \leq 1$, 0 otherwise; $f(x) = 1$ for all x. Satisfies $123456789\rho\sigma\tau\oplus\Phi\Gamma\pi$.

48. X : any non-empty set; S : $\{X, \phi\}$; $\mu X = 1$, $\mu\phi = 0$; $f_{2n-1}(x) = 1 + \frac{1}{n}$, $f_{2n}(x) = 1 - \frac{1}{n}$ for all x; $f(x) = 1$ for all x.

Satisfies $123456789\alpha\sigma\tau\oplus\Phi\Gamma\pi$.

49. R; $f_n(x) = \frac{1}{n}$ if $0 \leq x \leq 1$, 0 otherwise; $f(x) = 0$ for all x.

Satisfies $123456789\rho\sigma\tau\uparrow\Phi\Gamma\pi$.

50. X : any non-empty set; S : $\{X, \phi\}$; $\mu X = 1$, $\mu\phi = 0$; $f_n(x) = -\frac{1}{n}$ for all x; $f(x) = 0$ for all x. Satisfies 123456789$\alpha\sigma\tau\uparrow\Phi\Gamma\pi$.

51. (X, S, μ) is the non-sigma-finite non-atomic space of Example 28; $f_n(x) = \frac{1}{n}$ for all x; $f(x) = 0$ for all x. Satisfies 259$\rho\tau\oplus\downarrow\Phi\pi$.

52. (X, S, μ) is any uncountable set with counting measure; $f_n(x) = \frac{1}{n}$ for all x; $f(x) = 0$ for all x. Satisfies 259$\alpha\tau\oplus\downarrow\Phi\pi$.

53. (X, S, μ) is any uncountable with counting measure; $f_n(x) = 1 + \frac{1}{n}$ for all x; $f(x) = 1$ for all x. Satisfies 259$\alpha\tau\oplus\downarrow\Gamma\pi$.

54. X : any uncountable set; S : $\{X, \phi\}$; $\mu X = \infty$, $\mu\phi = 0$; $f_n(x) = f(x) = 1$ for all x. Satisfies 2579$\alpha\uparrow\oplus\downarrow\Gamma\pi$.

55. I; $f_n = \chi_{[n,\infty)}$; $f(x) = 0$ for all x. Satisfies 259$\alpha\sigma\tau\oplus\downarrow\Phi\pi$.

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APPENDIX

The body of this thesis has determined all the minimal subsets of the 17 hypotheses of size 4 or less, not including the combination $\uparrow\downarrow$, that imply another one of the conditions. To check and complete the work, it was reviewed in a manner that should yield all such minimal subsets without any restriction. Not one such subset of larger size was found, but the inclusion of the combination $\uparrow\downarrow$ did of course yield new results and require additional counterexamples. The condition $\uparrow\downarrow$ ($f_n(x)$ monotone nondecreasing and nonincreasing in n) evidently requires that $f_n(x)$ be independent of n. It is convenient to give an independent summary of this relatively trivial but not wholly uninteresting case (see Figure 14). The table on page 80 lists the additional data. The indicated deletions reflect harmless redundancies and apply only to the particular conclusion indicated.

The following counterexamples have not been previously defined:

39 modified. I; $f_n(x) = f(x) = |x|$. Satisfies $9\alpha\sigma\tau\uparrow\oplus\downarrow\Gamma\pi$, adding $\uparrow$ to the list for the original example 39.

56. R; $f_n(x) = f(x) = \frac{1}{x}$ for $|x| \leq 1$, 0 otherwise. Satisfies $14\rho\sigma\tau\uparrow\downarrow\pi$.

57. R; $f_n(x) = f(x) = 1$ for $x \geq 0$, $= -1$ for $x < 0$. Satisfies $259\rho\sigma\tau\uparrow\downarrow\pi$.

58. I; otherwise like the preceding. Satisfies $259\alpha\sigma\tau\uparrow\downarrow\pi$.

59. $X = \{-1, 1\}$; $\mu\{-1\} = \mu\{1\} = \infty$; $f_n(x) = f(x) = x$. Satisfies $2579\alpha\uparrow\downarrow\pi$.

Conclusion	New counterexamples	New implications, given $\uparrow\downarrow$
2	56	1α, 4α, 7α, 8α, $\alpha\Phi$
3	26, 46; delete 14	8, Φ, 17, 19, 1α, 47, 49, 4α, 7σ, 7τ
4	34, 46; delete 31, 44	8, Φ
5	39 modified	8, Φ
6	29, 46; delete 53	8, Φ, 17, 19, 1α, 47, 49, 4α, 7σ, 7τ
7	46	8, Φ
8	delete 20	Φ
9	none	8, Φ
σ	13; delete 4	Φ
τ	none	Φ
Φ	none	8
Γ	56-59	Φ

Proofs of Implications.

$1\alpha \to 4\alpha$: Theorem 19

$3\alpha = 6\alpha = 8\alpha = \alpha\Phi$: Obvious since $3 = 6 = 8 = \Phi$

$3\alpha \to 4\alpha$: Theorem 4

$4\alpha \to 3\alpha$: Theorems 21, 17, 16 and 12

$3\alpha \to 7\alpha$: (2, Theorem 5, p. 21)

$7\alpha \to 2\alpha$: Theorem 42

$2\alpha \to 5\alpha$: Theorem 15

$5\alpha \to 2\alpha$: Theorem 41

$\alpha \to 9$: Theorem 21

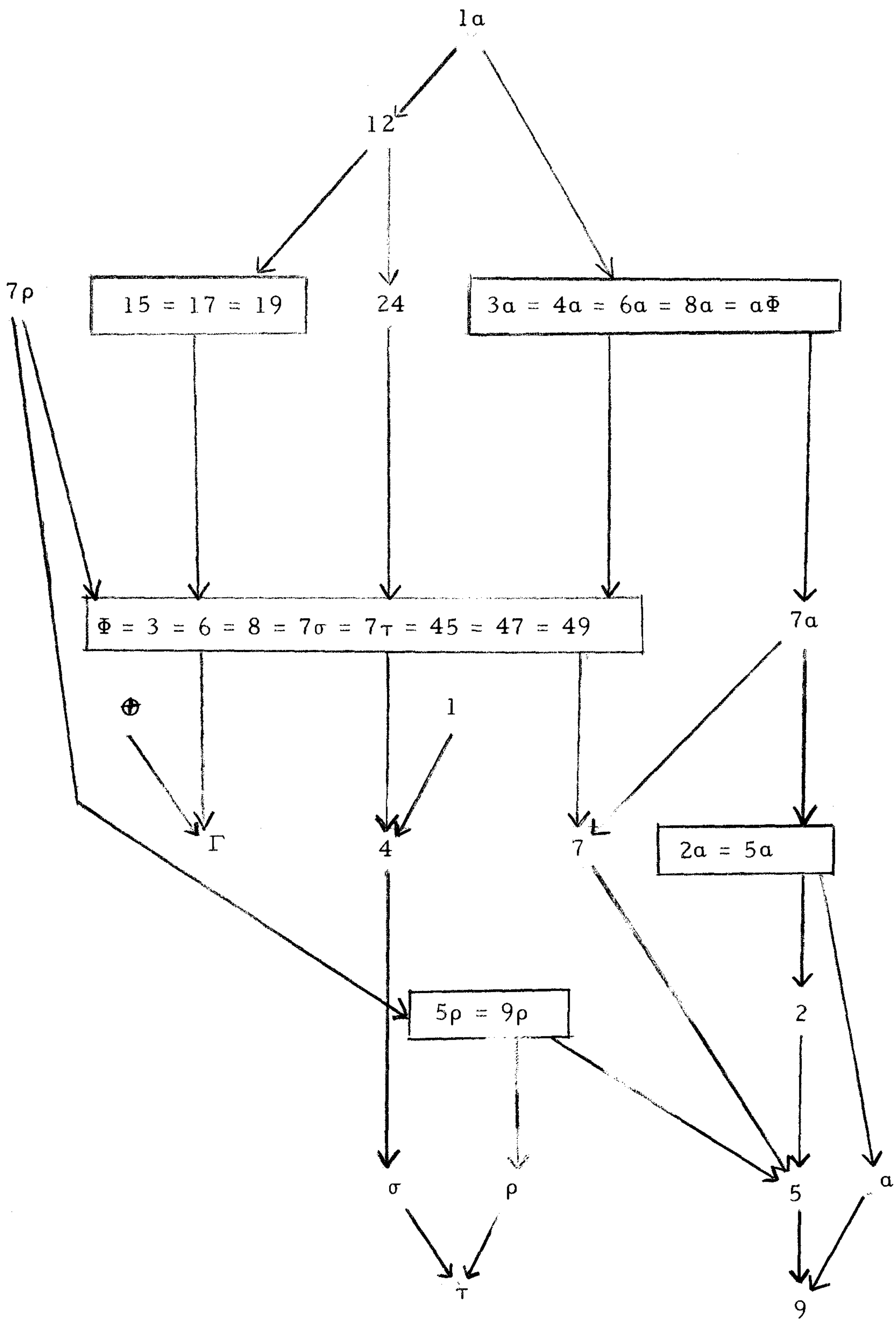

Figure 14. Implications when $f_1 = f_2 = f_3 = \ldots$ (↑↓).

$2 \rightarrow 5 \rightarrow 9$: Theorems 15 and 13

$7 \rightarrow 5$: Theorem 32

$1\alpha \rightarrow 12 \rightarrow 24$: Theorems 38 and 19

$12 \rightarrow 15$: Theorem 15

$15 = 19$: Theorems 13, 19 and 39

$19 = 17$: Theorem 18 and (2, Theorem 7, p. 22)

$17 \rightarrow 49$: Theorem 19 and 18

$\Phi = 3 = 6 = 8 = 7\sigma = 7\tau$: Theorem 12

$7\tau \rightarrow 45$: Theorems 22 and 33

$45 \rightarrow 7\tau$: Theorems 5 and 16

$45 = 47 = 49$: Theorems 13, 17, 18, and 39

$1 \rightarrow 4$: Theorem 19

$4 \rightarrow \sigma$: Theorem 16

$\sigma \rightarrow \tau$: Obvious

$7\rho \rightarrow 7\tau$: Obvious

$\boldsymbol{\Phi} \rightarrow \Gamma$: Obvious

$7\rho \rightarrow 9\rho$: Theorem 18

$5\rho = 9\rho$: Theorems 13 and 14

$9\rho \rightarrow \rho \rightarrow \tau$: Obvious

The number of implications yielding conditions 3 and 6 (pages 8 and 17) substantially reduced by introducing the symbol $\not 7$ for any of the equivalent pairs of hypotheses $47 = 49 = 78 = 7\sigma = 7\tau$. The implications that yield $\not 7$ itself can be obtained by combining those for any one of the five defining pairs. All such implications are contained in Figure 15. It should be noted that $5\not 7 = 45 = 67$ and $7\Phi\Gamma = 45\pi = 67\pi = 5\not 7\pi = \not 7\pi = \not 7\Phi = \not 7\Gamma$.

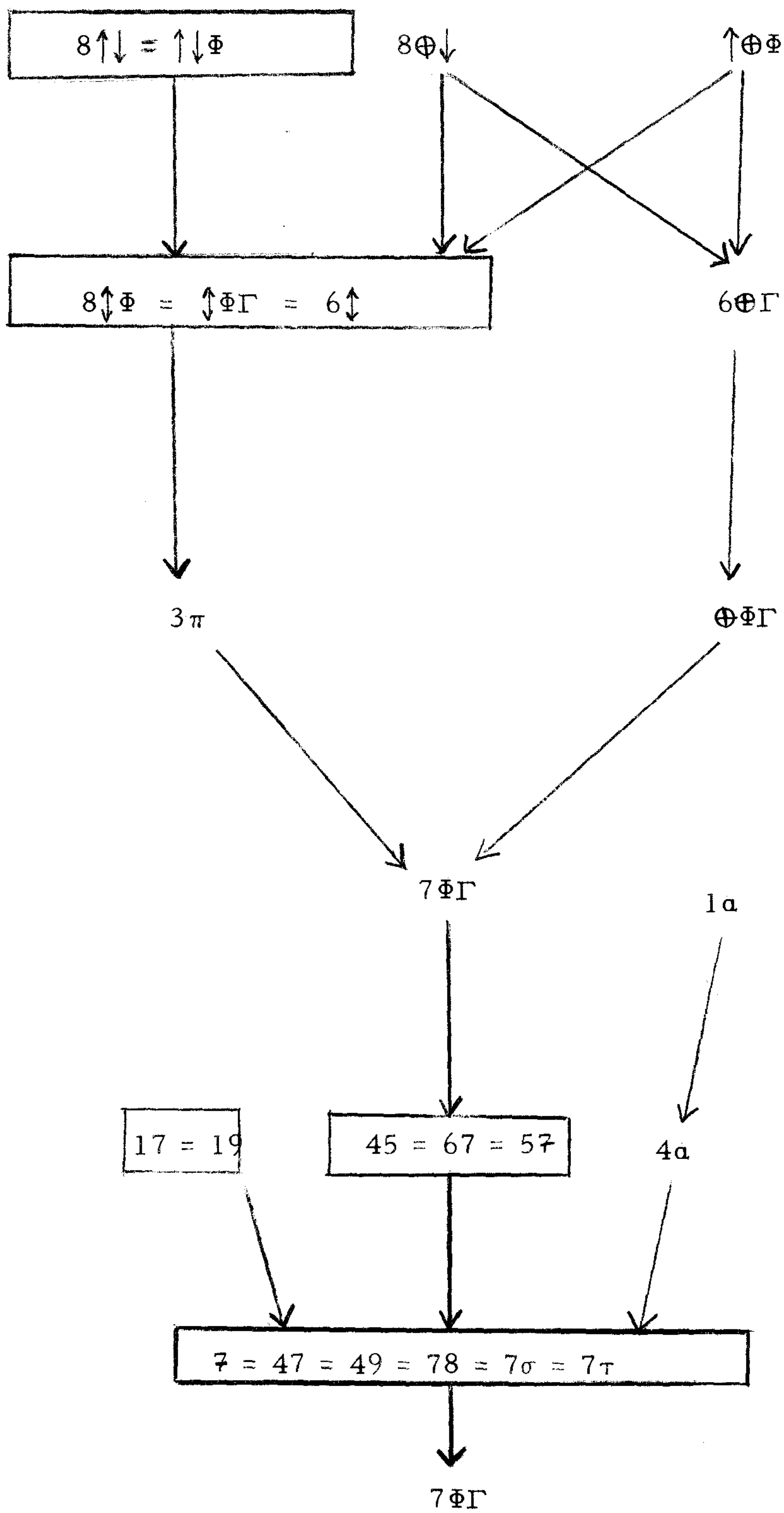

Figure 15.