Microstructure Fluxes across Density Surfaces

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ABSTRACT

When averaging the equations of motion, thermodynamics, and scalar conservation over turbulent fluctuations, we perform the process in several stages. First, an average is taken over the microscopic scales of turbulence, including the centimeter-scale band in which the dissipation of kinetic energy and temperature or density variance occurs. The eddy-correlation fluxes that arise in this stage are called microstructure fluxes. Next, the equations are transformed into coordinates relative to the microscopically averaged isopycnals. Finally, an average is taken, relative to these isopycnals, over macroscopic scales of eddy variability, which may include the mesoscale band of planetary motions. Average transport terms, analogous to conventional Reynolds transports in fixed-depth averages, arise also from the macroscopic eddies. This is not so for density, for which no counterparts of macroscopic Reynolds transports exist on constant density surfaces. Only microstructure flux divergence, which is synonymous with diapycnal velocity, contributes to the density balance. Under the assumption that microstructure density variance production is in equilibrium with its molecular dissipation, the microstructure density flux has the form of the molecular flux of heat down the vertical mean gradient, amplified by the Cox number. Munk's abyssal recipe for the vertical velocity/diffusivity ratio can now be reinterpreted as the diapycnal velocity/diffusivity ratio.

1. Introduction

The use of isentropic coordinates has often been recommended for representing the equations of motion and thermodynamics in the atmosphere, when the motion conserves potential temperature or entropy (Eliassen and Kleinschmidt 1956). The use of potential density coordinates in the ocean has been suggested for similar reasons (Robinson 1965). Bleck and Smith (1990) have developed a numerical ocean model that exploits isopycnal coordinates and conserves density adiabatically. Representation of the motion in isopycnal coordinates may still have advantages if density is only approximately conserved, that is, if there is turbulent density transport. In this case one must parameterize the irreversible transport processes. A common choice is to replace Reynolds-averaged fluxes with products of eddy diffusion coefficients and mean gradients. Redi (1982) asserted that the eddy diffusivity tensor ought to be diagonal with respect to coordinate axes aligned with local isopycnal surfaces. This would be another powerful reason for preferring isopycnal coordinates.

We have reexamined the parameterization of nonconservative processes in isopycnal models. We find it useful to average the equations of motion, continuity, thermodynamics, and passive scalar concentration in

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two stages. First, an average is taken over the microscopic scales, including the Batchelor scale, typically of the order of a few centimeters but extending to scales of the order of a few meters at which overturning billows occur. The reason for this is twofold. First, we wish to average over the numerous finescale, short-lived, density inversions, which occur in a band centered at the Batchelor scale in well-developed turbulence and over the rarer but larger-scale overturning billows, and thereby ensure a monotonic average density profile. Second, we want to separate the microscopic nonhydrostatic scales from the macroscopic hydrostatic scales to which the primitive equations apply. We refer to the fluctuation-correlation terms created in this averaging process as microstructure fluxes.

Before the next stage of averaging, the equations are transformed to coordinates relative to the microscopically averaged isopycnals. This has the effect of transforming the mass-continuity equation into a prognostic equation for the specific thickness $\partial z / \partial \rho$ and transforming the thermodynamic equation for density into a diagnostic equation relating the sources and sinks of density ($\varpi \equiv D\rho / Dt$) to the microstructure density flux divergence. The isopycnal-coordinate primitive equations are further averaged over macroscopic scales, which may encompass the planetary scales of eddy variability in the ocean. This averaging produces eddy-correlation terms analogous to the usual Reynolds flux divergences in the momentum and scalar concentration equations, but not in the thermodynamic equation for density. The microstructure flux divergence (macro-

scopically averaged) remains the only source term in that equation.

We propose a parameterization for the microstructure density flux in terms of the dissipation of density variance. This has the appearance of the molecular diffusive flux of density associated with the mean gradient but amplified by the Cox number, that is, by the ratio of mean-squared density gradient to the squared mean gradient (Osborn and Cox 1972). Similar parameterizations for microstructure fluxes of momentum and scalar concentration can be given, though they must be augmented by the macroscopic turbulent transport processes for these properties.

The crucial point is that macroscopic averaging produces no new macroscopic flux divergences for density as it does for other variables. The transformation to isopycnal coordinates changes the prognostic equation for density into a diagnostic equation for the new dependent variable ϖ. This elementary fact permits a simpler, less subjective, parameterization of the mixing processes for density. For the other variables the macroscopic and microscopic fluxes are additive, and additional parameterizations must be proposed. Conventionally, these may involve eddy viscosities and diffusivities. But they have a different origin from the density microstructure flux parameterization.

We are able to conclude that Munk's (1966) abyssal recipe for the vertical advective–diffusive balance applies to the diapycnal velocity $e = \varpi/z_\rho$, rather than vertical velocity w. Specifically, we find that

$$e\frac{\partial\rho}{\partial z} = \frac{\partial}{\partial z}\left(K\frac{\partial\rho}{\partial z}\right),$$

where K is the diapycnal diffusivity due to microscopic fluctuations. No assumptions need to be made about neglecting horizontal advection or macroscopic horizontal diffusion.

Gent and McWilliams (1990) also considered averages of equations expressed in isopycnal coordinates. We have made clear the correspondences to their work in appendix C.

2. Averaging in isopycnal coordinates

In this section we first average the equations of motion, thermodynamics, and passive scalar concentration over microscopic scales. We call the fluxes that arise thereby the "microstructure fluxes." Next we transform the equations into isopycnal coordinates. Then we average the transformed equations over macroscopic scales of motion. Because the coordinate transformation makes the nonlinear rate of change of density $D\rho/Dt$ into a dependent variable ϖ, which is trivially linear (a major motivation of the procedure), the macroscopic average of the transformed density equation produces no extra Reynolds-averaged density flux. The microstructure flux is the only source term for density. For momentum and for concentrations of

scalars other than density, there are additional macroscopic eddy fluxes analogous to the conventional Reynolds-averaged fluxes. An argument for the form of the microstructure flux of density can be given that does not depend on proposing a Fickian diffusion tensor for density. Nevertheless, the form so obtained does have the appearance of an effective diapycnal diffusivity acting on the mean density gradient. This formulation circumvents Redi's (1982) procedure, in which the proposed diffusivity tensor is diagonal in local orthogonal coordinate axes aligned with mean isopycnals, thereby minimizing spurious diapycnal diffusion of density.

a. Microscopically averaged equations of motion: microstructure fluxes

The equations of motion, continuity, thermodynamics and passive scalar concentration are, in level Cartesian coordinates,

$$\frac{Du}{Dt} - fv + \frac{1}{\rho_0}p_x = \partial^T\mathbf{F}^u, \tag{1}$$

$$\frac{Dv}{dt} + fu + \frac{1}{\rho_0}p_y = \partial^T\mathbf{F}^v, \tag{2}$$

$$0 = -p_z - g\rho, \tag{3}$$

$$u_x + v_y + w_z = 0, \tag{4}$$

$$\frac{D\rho}{Dt} = \partial^T\mathbf{F}^\rho \equiv \varpi, \tag{5}$$

$$\frac{D\phi}{Dt} = \partial^T\mathbf{F}^\phi, \tag{6}$$

where

$$\frac{D}{Dt} \equiv \partial_t + u\partial_x + v\partial_y + w\partial_z, \quad \partial^T \equiv (\partial_x, \partial_y, \partial_z), \tag{7}$$

$$\mathbf{F}^\rho = -\overline{\mathbf{u}_*\rho_*}, \quad \mathbf{F}^\phi = -\overline{\mathbf{u}_*\phi_*}, \quad \text{etc.} \tag{8}$$

The shallow Boussinesq approximation has been made. We assume that Eqs. (1)–(6) have been preaveraged over microscopic fluctuations (denoted by asterisked variables) smaller than, say, several meters. This ensures that the density field has been averaged over the band of scales centered around the Batchelor scale, typically of the order of a few centimeters, at which numerous, finescale, short-lived, turbulent density inversions occur, and over rarer density inversions on larger scales—several meters—such as those associated with Kelvin–Helmholtz billows, so that the average vertical density gradient everywhere possesses the same sign. The right-hand sides of these equations contain the divergences of momentum, density, and scalar concentration fluxes, caused by these microscopic fluctuations. We call $\mathbf{F}^\rho$, $\mathbf{F}^\phi$, etc., the microstructure fluxes. For simplicity, we have taken the density

anomaly ρ to be synonymous with temperature anomaly. This implies neglect of salinity and the neglect of nonlinearities in the equation of state for seawater. (This is not essential to our analysis, nor is the Boussinesq approximation.)

b. Transformation from Cartesian coordinates to isopycnal coordinates

We now transform Eqs. (1)-(7), expressed in terms of Cartesian coordinates x, y, z, t as independent variables, into isopycnal coordinates $\tilde{x}, \tilde{y}, \rho, \tilde{t}$, where

$$x = \tilde{x}, \quad y = \tilde{y}, \quad z = z(\tilde{x}, \tilde{y}, \rho, \tilde{t}), \quad t = \tilde{t} \quad \text{(9a,b,c,d)}$$

(Eliassen and Kleinschmidt 1956). Tildes are employed to emphasize that the calculation of partial derivatives such as $\partial_{\tilde{x}}\phi$ are performed with ρ held fixed, that is, along sloping isopycnals. In contrast, $\partial_x\phi$ is calculated along level surfaces (z fixed). Notice that we are transforming to time-dependent microscopically averaged isopycnal surfaces, which are free to fluctuate in time as well as space. The partial derivatives transform according to

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_t \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & \mathbf{A} & & 0 \\ & & & 0 \\ 0 & 0 & -z_{\tilde{t}}/z_\rho & 1 \end{pmatrix} \begin{pmatrix} \partial_{\tilde{x}} \\ \partial_{\tilde{y}} \\ \partial_\rho \\ \partial_{\tilde{t}} \end{pmatrix}, \quad (10)$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -z_{\tilde{x}}/z_\rho \\ 0 & 1 & -z_{\tilde{y}}/z_\rho \\ 0 & 0 & 1/z_\rho \end{pmatrix}. \quad (11)$$

On application of the transformation (10), Eqs. (1)-(7) become

$$z_\rho\left(\frac{Du}{Dt} - fv + \frac{1}{\rho_0}\pi_{\tilde{x}}\right) = \tilde{\partial}^{\mathrm{T}}\mathbf{F}^u, \quad (12)$$

$$z_\rho\left(\frac{Dv}{Dt} + fu + \frac{1}{\rho_0}\pi_{\tilde{y}}\right) = \tilde{\partial}^{\mathrm{T}}\mathbf{F}^v, \quad (13)$$

$$0 = -\pi_\rho + gz, \quad (14)$$

$$z_{\rho\tilde{t}} + (uz_\rho)_{\tilde{x}} + (vz_\rho)_{\tilde{y}} + (\varpi z_\rho)_\rho = 0, \quad (15)$$

$$\frac{Dz}{Dt} = w, \quad (16)$$

$$z_\rho\frac{D\phi}{Dt} = \tilde{\partial}^{\mathrm{T}}\tilde{\mathbf{F}}^\phi, \quad (17)$$

where

$$\frac{D}{Dt} = \partial_{\tilde{t}} + u\partial_{\tilde{x}} + v\partial_{\tilde{y}} + \varpi\partial_\rho, \quad \tilde{\partial}^{\mathrm{T}} \equiv (\partial_{\tilde{x}}, \partial_{\tilde{y}}, \partial_\rho), \quad (18)$$

and $\pi = p + g\rho z$ is the Montgomery potential. Fluxes in isopycnal coordinates are related to fluxes in Carte-

sian coordinates by

$$\tilde{\mathbf{F}}^\phi = z_\rho\mathbf{A}^{\mathrm{T}}\mathbf{F}^\phi = (-z_\rho\overline{u_*\phi_*}, -z_\rho\overline{v_*\phi_*}, -\overline{e_*\phi_*})^{\mathrm{T}}, \quad (19)$$

where

$$e_* = w_* - u_*z_{\tilde{x}} - v_*z_{\tilde{y}} \quad (20)$$

is the fluctuating velocity normal to the macroscopic isopycnal [overlooking a factor $(1 + |\tilde{\nabla}z|^2)^{1/2}$, which is very close to 1]. The flux divergences transform according to

$$\partial^{\mathrm{T}}\mathbf{F}^\phi = \frac{1}{z_\rho}\tilde{\partial}^{\mathrm{T}}\tilde{\mathbf{F}}^\phi, \quad (21)$$

which may be verified by substituting from (10) and (19). Just as (6) transforms into (17), so the second equality in (5) transforms into

$$z_\rho\varpi = \tilde{\partial}^{\mathrm{T}}\tilde{\mathbf{F}}^\rho = \partial_{\tilde{x}}(-z_\rho\overline{u_*\rho_*})$$
$$+ \partial_{\tilde{y}}(-z_\rho\overline{v_*\rho_*}) + \partial_\rho(-\overline{e_*\rho_*}). \quad (22)$$

Comparing (1)-(7) with (12)-(18), one notes the replacement of z by ρ as independent variable, and w by ϖ as independent variable. Equation (15) is the transformed continuity equation (4). It is called the "thickness equation," because $-z_\rho$ (which is invariably positive) is the spacing of isopycnals per unit density.

The derivation and meaning of Eqs. (16), (18) deserve some comment. The application of the transformation (10) to the substantial rate-of-change operator (7) gives

$$\frac{D}{Dt} = \partial_{\tilde{t}} + u\partial_{\tilde{x}} + v\partial_{\tilde{y}} + \frac{w - z_{\tilde{t}} - uz_{\tilde{x}} - vz_{\tilde{y}}}{z_\rho}\partial_\rho.$$

Applying this to ρ and using (5), we see that

$$\varpi = \frac{w - z_{\tilde{t}} - uz_{\tilde{x}} - vz_{\tilde{y}}}{z_\rho}.$$

This relation may be used to give the final form (18) for D/Dt; it may also be rewritten as Eq. (16). This equation permits an interesting interpretation of the quantity $e \equiv z_\rho\varpi$. This is the difference between vertical velocity w and the apparent vertical motion, $z_{\tilde{t}} + uz_{\tilde{x}} + vz_{\tilde{y}}$, of an isopycnal surface. The quantity $e/(1 + |\tilde{\nabla}z|^2)^{1/2}$ is the diapycnal flux, the normal flux of volume per unit area across the isopycnal surface. Because $|\tilde{\nabla}z| \ll 1$, e itself is usually called the diapycnal flux, or diapycnal entrainment velocity. From the large-scale point of view, the area-integral of e over an entire isopycnal surface is the negative of Speer and Tziperman's (1992) water mass transformation function $F(\rho)$. The link between e and diabatic processes is made clear by (5). If there were no diabatic processes, that is, if the microstructure flux in (5) were to vanish, $D\rho/Dt = \varpi = 0$, then isopycnal surfaces $z = z(\tilde{x}, \tilde{y}, \rho, \tilde{t})$ would be material surfaces. Equation (16) restates

the kinematic consequence of $\varpi = 0$, namely, that vertical motion is given by the motion of the isopycnal surfaces, $w = z_{\tilde{t}} + u z_{\tilde{x}} + v z_{\tilde{y}}$. In general, for $\varpi \neq 0$, Eq. (16) furnishes the means to calculate w from isopycnal displacement.

c. Macroscopic averaging in isopycnal coordinates: Weighted averages

The next step is to average the momentum, continuity (or thickness), and scalar concentration equations (12)–(15), (17), and the microstructure density flux divergence ϖ [Eq. (22)], over an ensemble of macroscopic realizations of flows. Practically, we take this to be equivalent to averaging over regions of space-time (or space–density–time) containing enough fluctuations to permit the construction of reliable macroscopic averages. This requires the averaging scale to be in a spectral gap between the small-scale fluctuations and the resolved large-scale variations (Lumley and Panofsky 1964).

We write

$$z = \bar{z} + z', \quad z_\rho = \bar{z}_\rho + z'_\rho, \tag{23}$$

where the bar indicates an ensemble average. However, it is convenient to introduce also an average, denoted by a circumflex or by angle brackets $\langle \ldots \rangle$, weighted by the thickness z_ρ,

$$\phi = \hat{\phi} + \phi'', \tag{24}$$

where

$$\hat{\phi} = \langle \phi \rangle = \overline{z_\rho \phi} / \bar{z}_\rho, \quad \langle \phi'' \rangle = \overline{z_\rho \phi''} / \bar{z}_\rho = 0. \tag{25}$$

[Eliassen and Kleinschmidt (1956) used similar averages, weighted by density.] Note that both weighted and unweighted averages are performed at fixed $\tilde{x}$, $\tilde{y}$, ρ, t, that is, on an isopycnal surface. Derivatives commute with the unweighted averaging operation, but not with the weighted average. Gent and McWilliams (1990) considered the density and continuity equations in isopycnal coordinates. A microscopic-scale average is implicit in their equations, although they neglected diabatic processes; that is, $\varpi = 0$. They too averaged over eddy scales in isopycnal coordinates, equivalent to what we call macroscopic averaging. However, they used unweighted-average variables. This leads to some differences that we elucidate in appendix C.

The macroscopically averaged forms of Eqs. (12)–(15), (17) are

$$\bar{z}_\rho \left(\frac{\hat{D}}{Dt} \hat{u} - f \hat{v} + \frac{1}{\rho_0} \bar{\pi}_{\tilde{x}} \right) + \frac{1}{\rho_0} \overline{z_\rho \pi'_{\tilde{x}}} = \tilde{\partial}^{\mathrm{T}} (\tilde{\bar{F}}^u + \mathbf{G}^u), \tag{26}$$

$$\bar{z}_\rho \left(\frac{\hat{D}}{Dt} \hat{v} + f \hat{u} + \frac{1}{\rho_0} \bar{\pi}_{\tilde{y}} \right) + \frac{1}{\rho_0} \overline{z_\rho \pi'_{\tilde{y}}} = \tilde{\partial}^{\mathrm{T}} (\tilde{\bar{F}}^v + \mathbf{G}^v), \tag{27}$$

$$0 = -\bar{\pi}_\rho + g \bar{z}, \tag{28}$$

$$\bar{z}_{\rho \tilde{t}} + (\bar{z}_\rho \hat{u})_{\tilde{x}} + (\bar{z}_\rho \hat{v})_{\tilde{y}} + (\bar{z}_\rho \hat{\varpi})_\rho = 0, \tag{29}$$

$$\bar{z}_\rho \frac{\hat{D}}{Dt} \hat{\phi} = \tilde{\partial}^{\mathrm{T}} (\overline{\tilde{\bar{F}}^\phi} + \mathbf{G}^\phi), \tag{30}$$

where

$$\frac{\hat{D}}{Dt} \equiv \partial_{\tilde{t}} + \hat{u} \partial_{\tilde{x}} + \hat{v} \partial_{\tilde{y}} + \hat{\varpi} \partial_\rho, \tag{31}$$

$$\mathbf{G}^{\phi \mathrm{T}} = (-\overline{z_\rho u'' \phi''}, \; -\overline{z_\rho v'' \phi''}, \; -\overline{z_\rho \varpi'' \phi''}), \tag{32}$$

etc. The use of weighted averages avoids the appearance of eddy thickness fluxes in (29) and additional eddy flux terms in (26), (27), (30). A careful comparison of our equations with the Gent and McWilliams (1990) unweighted-average equation is given in appendix C. The average microstructure fluxes $\bar{F}^\phi$, etc., can be substituted from (19). The average diapycnal flux is calculated by averaging (22):

$$\bar{e} = \overline{\bar{z}_\rho \hat{\varpi}} = \partial_{\tilde{x}} (-\bar{z}_\rho \langle \overline{u_* \rho_*} \rangle)$$
$$+ \partial_{\tilde{y}} (-\bar{z}_\rho \langle \overline{v_* \rho_*} \rangle) + \partial_\rho (-\overline{\overline{e_* \rho_*}}). \tag{33}$$

This is strongly dominated by the third term, the diapycnal gradient of density flux. The double bar indicates a macroscopic average following a microscopic average. The macroscopic momentum flux vectors $\mathbf{G}^u$, $\mathbf{G}^v$ have forms similar to (32) and are combined with the average microstructure fluxes $\bar{F}^u$, $\bar{F}^v$. Notice the appearance of thickness–pressure gradient covariance terms in (26), (27).

d. Parameterization of microstructure fluxes

To calculate the right side of (33), which defines the average diapycnal flux, we consider the balance of microscopically averaged square density variance, $\sigma \equiv 1/2 \rho_*^2$,

$$\frac{D\sigma}{Dt} = \mathbf{F}^{\rho \mathrm{T}} \partial \rho + \partial^{\mathrm{T}} \mathbf{F}^\sigma - \chi^{(\rho)}. \tag{34}$$

This is written in Cartesian coordinates, where $\mathbf{F}^\rho$ is given by (8) and

$$\mathbf{F}^\sigma = -\frac{1}{2} \overline{\mathbf{u}_* \rho_*^2}, \tag{35}$$

$$\chi^{(\rho)} = \kappa^\rho \overline{|\partial \rho_*|^2} \tag{36}$$

are the flux and the molecular dissipation, respectively, of density variance (κ^ρ is molecular diffusivity of density). Transformed to isopycnal coordinates, (34) becomes

$$z_\rho \frac{D\sigma}{Dt} = \tilde{F}^\sigma_3 + \tilde{\partial}^{\mathrm{T}} \tilde{F}^\sigma - z_\rho \chi^{(\rho)}, \tag{37}$$

where the substantial derivative is defined by (18), and $\tilde{F}_3^\sigma \equiv -\overline{e_*\rho_*}$. The macroscopic average of (37) is

$$\bar{z}_\rho \frac{\hat{D}}{Dt} \hat{\sigma} = \overline{\tilde{F}_3^\sigma} + \hat{\partial}^{\mathrm{T}}(\mathbf{G}^\sigma + \overline{\mathbf{F}^\sigma}) - \bar{z}_\rho \hat{\chi}^{(\rho)}, \quad (38)$$

where $\mathbf{G}^\sigma$ is defined similarly to (32). By assuming that the turbulence has adjusted to equilibrium and that flux divergences are negligible, the local balance of production and dissipation of density variance gives

$$\overline{\tilde{F}_3^\rho} \equiv -\overline{e_*\rho_*} = \bar{z}_\rho \hat{\chi}^{(\rho)}. \quad (39)$$

We are conservative in asserting this balance only after macroscopic averaging. It is often suggested after merely microscopic averaging. If that were done, (37) would give

$$\tilde{F}_3^\rho = z_\rho \chi^{(\rho)}. \quad (39)'$$

By introducing into (39) the Cox number,

$$\mathrm{Cx} \equiv \left\langle \overline{|\partial \rho_*|^2} \right\rangle \bar{z}_\rho^2, \quad (40)$$

the ratio of mean-square density gradient to the squared mean gradient (Osborn and Cox 1972), we can write (33), neglecting the first two terms on the right, as

$$\bar{e} = \bar{z}_\rho \hat{\omega} \approx \partial_\rho \left(\frac{K^\rho}{\bar{z}_\rho} \right), \quad \text{where} \quad K^\rho = \kappa^\rho \cdot \mathrm{Cx}. \quad (41)$$

The argument for the relation (39) between the pseudovertical density flux and dissipation has often been given, though usually with respect to fixed Cartesian coordinates, in which horizontal gradients are neglected (Osborn and Cox 1972). Our derivation serves to emphasize that the neglect of horizontal gradients is not at all necessary and that the argument furnishes the *diapycnal* component of the density flux. Moreover, it is precisely this diapycnal component that is most important in Eq. (33).

A similar argument may also be advanced for the parameterization of the flux of scalar ϕ, so that

$$\hat{\partial}^{\mathrm{T}}\overline{\mathbf{F}^\phi} \approx \partial_\rho \left(K^\phi \frac{\hat{\phi}_\rho}{\bar{z}_\rho} \right), \quad (42)$$

where $K^\phi \sim K^\rho$. Similarly also for $\hat{\partial}^{\mathrm{T}}\overline{\mathbf{F}^u}$, $\hat{\partial}^{\mathrm{T}}\overline{\mathbf{F}^v}$. But this is less than half the story for the fluxes of those quantities. Parameterizations for the macroscopic fluxes $\mathbf{G}^\phi$, $\mathbf{G}^u$, $\mathbf{G}^v$ must also be given to complete the specification of Eqs. (26), (27), (30). The point here is to emphasize that the parameterization (41) of the microstructure flux of density is enough to specify the diapycnal flux. Classical random-walk diffusion models apply to the passive scalar $\hat{\phi}$ (though not necessarily to $\hat{u}$, $\hat{v}$), guaranteeing a diffusivity tensor for the parameterization of the macroscopic flux $\mathbf{G}^\phi$ (Monin and

Yaglom 1971). An important example of such a scalar is salinity! On density surfaces, salinity does not occur in the buoyancy force $g\bar{z}$ of Eq. (28), and may be considered passive.

The introduction of the Cox number in (41) does not obviate the need for a closure model for, say, K^ρ. Short of that, however, it facilitates an empirical specification of K^ρ from interpretation of measurements of microstructure variability in the ocean.

There are other processes that could contribute to $\bar{z}_\rho \hat{\omega}$, $\hat{\partial}^{\mathrm{T}}\overline{\mathbf{F}^\phi}$, etc. The differential rates of molecular diffusion of temperature and salinity provide a mechanism for transporting buoyancy (density) that is not accounted for in the discussion above. Use of the Boussinesq approximation, and linearization of the equation of state, has permitted simplification of the derivation. Non-Boussinesq effects, and nonlinearities in the equation of state, could be included in our analysis, at some cost in complexity, but without vitiating the essential results.

It has been crucial to the above formulation that the macroscopic averaging was done at fixed density, not fixed depth. The resulting microstructure fluxes and macroscopic Reynolds fluxes depend on this. It is instructive therefore to rewrite the averaged equations (26)–(30) in level coordinates in order to compare them with the conventional equations averaged at fixed depth. This has been done in appendix A. For the most part, the results are familiar, producing equations for momentum, continuity, density, and scalar concentration with forms similar to those obtained by conventionally Reynolds averaging the equations. However, it is important to emphasize that the inverse transformation of (41) shows that

$$\frac{\hat{D}}{Dt} \tilde{\rho} = \hat{\omega} = \partial_z (K^\rho \tilde{\rho}_z), \quad (43)$$

where $\tilde{\rho}$ is the macroscopically averaged density field (appendix A).

In a contrasting vein, in appendix B we display the conventional equations of motion, macroscopically averaged in fixed coordinates, transformed to macroscopically averaged isopycnal coordinates. This latter set of equations is quite distinct from Eqs. (26)–(30), obtained by the two-stage averaging process, with the macroscopic average done on microscopically averaged isopycnal coordinate surfaces. It is instructive to see, nevertheless, that a parameterization formally resembling (41) for the density flux divergence is obtained by adopting Redi's (1982) hypothesis that the Fickian diffusivity tensor for density is diagonal in isopycnal coordinates. The derivation for this form that we have presented is preferable because it identifies the microscopic scales of motion responsible for mixing of density. By enlisting Osborn and Cox's (1972) argument, it gives a stronger, more mechanistic explanation for its parameterization. It also distinguishes between the

diffusion of momentum and other properties by both macroscopic and microscopic scales, and the diffusion of density, which can be effected only by microscopic scales.

The formulation of Eq. (41) has a number of advantages over the Reynolds-averaged density equation transformed to mean isopycnal coordinates. The existence of a diffusivity tensor for mean density need not be postulated. Density influences the motion through the buoyancy force. Hence the formal derivation of a density diffusivity as the rate of change of covariance of displacement is not possible as it is for a passive scalar. The derivation assumes that fluid particle displacements are independent of both the initial scalar concentration and its source strength and that the distribution of particle displacements is normal asymptotically for large time after release (Monin and Yaglom 1971). For an active scalar like density, the statistical independence of displacements from initial concentration or source strength seems most unlikely. The particle motions may well "forget" the initial density field, but this seems like special pleading. In any event, we seek an analysis of diapycnal fluxes that is valid in general. Our analysis of the diapycnal density flux avoids this difficulty.

Even if a diffusivity tensor for density did exist, it is not obvious from rigorous arguments why the tensor should be diagonal in local orthogonal isopycnal coordinates if the distinction between microscopic and macroscopic scales is not made. While it is intuitively appealing that small-scale motions that transport properties should be constrained to lie nearly in isopycnal surfaces, a counterexample may be offered: if the averaging scale were coarse enough to include processes like baroclinic instability in which the orbits of water parcels cross mean isopycnals, then the diffusivity tensor need not be oriented with mean isopycnal surfaces. McDougall and Church's (1986) dismissal of this possibility is unsound. They argue that because water parcels conserve density approximately in baroclinic instability, there can be no diapycnal mixing, overlooking the distinction that needs to be made between *instantaneous* orbits and isopycnals on one hand and *mean* isopycnals and orbit orientations on the other. In fact, the flux of density across mean isopycnals is given by the rate of change, plus the molecular dissipation, of density variance. Perhaps the most convincing argument for an isopycnally diagonal diffusivity tensor is that it gives a form of the density diffusion law analogous to (41)! But this depends on the subtle distinction we have made between averaging over microscopic scales at fixed levels and averaging over macroscopic scales on microscopically averaged isopycnals.

Ocean data can be displayed in either isopycnally averaged or fixed-level averaged form. Unless great efforts are made to resolve high-wavenumber and high-frequency variability, raw observations of currents, density, and passive scalar concentrations are usually averages over volumes with scales of order 1 m to 10 m and times of minutes to hours. Some oceanographic atlases (e.g., Levitus 1982) display data averaged over large horizontal areas at fixed depths. In other atlases (Reid 1965), scalar concentrations and pressure (i.e., dynamic height or acceleration potential) are displayed, after some horizontal smoothing, on density surfaces, whose mean depths are also shown. When temperature–salinity relations (or other property–property relations) are shown, raw observations are plotted on scatter diagrams (Gordon et al. 1982), or histograms are prepared (Worthington 1981). The kind of averaging that such displays invite is similar to isopycnal averaging.

3. Munk's abyssal recipe

The mean diapycnal flux, or entrainment velocity, Eq. (41), may be written

$$\bar{e} = \bar{z}_\rho \hat{\varpi} = \partial_\rho \left(K^\rho \frac{1}{z_\rho} \right) \qquad (44)$$

or

$$\bar{e} \frac{\partial \tilde{\rho}}{\partial z} = \frac{\partial}{\partial z} \left(K^\rho \frac{\partial \tilde{\rho}}{\partial z} \right). \qquad (45)$$

The similarity of (45) to Munk's (1966) vertical advective–diffusive balance

$$w \frac{\partial \rho}{\partial z} = \frac{\partial}{\partial z} \left(K_v \frac{\partial \rho}{\partial z} \right), \qquad (46)$$

where w is vertical velocity and K_v is vertical diffusivity, is apparent. Munk (1966) took pains to emphasize the arbitrariness of neglecting horizontal advection and diffusion in adopting (46). He used (46) to estimate w/K_v, assumed constant, by fitting a curve to the observed midocean pycnocline. (Actually, he did this separately for the thermocline and halocline using temperature and salinity balances like (46), with T and S replacing ρ. He also fitted w/K_v to nonconservative scalars like dissolved oxygen and ^{14}C concentrations by allowing for their source and destruction functions.)

Equation (45) was derived with no mixing length hypothesis nor any assumptions about the magnitude of horizontal (meaning along-isopycnal) advection and diffusion. Only rather mild assumptions about (i) density variance production and dissipation being in equilibrium, and (ii) neglect of horizontal density flux divergence [Eq. (33)], are made in the argument leading to (41). If, as in section 2, the average $\bar{e}$ is calculated in isopycnal coordinates, it is not even necessary to posit formally a diffusivity tensor for density, nor its diagonality in isopycnal coordinates. Hence we can give

a far stronger interpretation of Munk's calculations, merely by reinterpreting the ratio w/K_v as $\bar{e}/K^\rho$.

4. Discussion and summary

We have carefully examined the issues involved in averaging the equations of motion, continuity, thermodynamics, and scalar concentration, and transforming them into isopycnal coordinates. We recommend first averaging at fixed levels over microscopic scales—meaning the band of scales containing the fluctuations that dissipate kinetic energy and density variance, and extending up to the scales of density-overturning billows. This guarantees a monotonic mean vertical density gradient so that unique isopycnal surfaces can be defined. Next we would average on isopycnal surfaces over the macroscopic scales of eddies, which may significantly transport properties laterally. This is conveniently done after transforming to isopycnal coordinates. Both averaging processes produce turbulent transports due to the microscopic and macroscopic scales. We distinguish these by the terms "microstructure flux," and "macroscopic" or "Reynolds flux."

The microstructure flux divergence of density is most important because it is the only source of density. One of the consequences of the use of isopycnal coordinates is to transform the rate of density change $D\rho/Dt$, nonlinear in the dependent variables, into the density source ϖ, which is a new dependent variable in place of vertical velocity. The macroscopic averaging of this variable produces no eddy correlation fluxes in isopycnal coordinates. Averaging the total rate of change of any other variable, such as momentum or scalar concentration, produces familiar eddy fluxes. An argument for parameterizing microstructure flux of density in terms of Cox number amplification of the molecular flux operating on the mean gradients follows immediately (Osborn and Cox 1972).

The mean diapycnal velocity $\bar{e} = \bar{z}_\rho \varpi$ is the net flux of volume per unit area across an isopycnal surface. This flux is necessary to supply the density diffused across the convoluted, instantaneous, random realizations of the isopycnal surface. The form we obtained for the diapycnal velocity resembles the vertical velocity that Munk (1966) proposed from the vertical advective–diffusive heat and salinity balances. With the crucial reinterpretation of Munk's vertical velocity as mean diapycnal velocity $\bar{e}$, we find that Munk's calculations and inferences apply, without the necessity of any assumptions about the dominant directions of advection or diffusion. Thus, a much stronger statement of Munk's (1966) abyssal recipe is possible. The integral of microstructure density flux over the whole submerged area of an isopycnal ρ_1 ought to be the same as the area-integrated surface flux of density (heat flux minus freshwater flux, each appropriately scaled) into surface waters lighter than ρ_1, given that there are no long-term trends. If the latter is thought of as a function

of ρ_1, its derivative with respect to density is Speer and Tziperman's (1992) water mass transformation function, the incremental surface density flux, per unit density, into waters of density ρ_1. The negative of the water-mass transformation function is the same as the integral of the diapycnal volume flux $\bar{e}$ integrated over the whole submerged area of the isopycnal ρ_1.

Redi (1982) asserted that the Fickian diffusivity tensor for eddy density flux, averaged with respect to fixed coordinates, ought to be diagonal in orthogonal coordinate frames oriented with the local isopycnal surfaces. While this assertion may be intuitively appealing, it does not bear close scrutiny. Its consequence is that density diffusion is entirely diapycnal. But this condition comes about quite naturally by microscopically averaging the equations first, then transforming to isopycnal coordinates, and finally taking macroscopic averages. It is only necessary to assume that microstructure density variance production is in equilibrium with molecular dissipation of density variance.

It is a slight weakness of our formulation that we have treated density as synonymous with temperature, possessing a well-defined molecular diffusivity. Of course, in the ocean salinity is an important influence on density. The mechanism of double diffusion, which depends on the differential rates of molecular diffusion of heat and salt, can cause net density flux. It is difficult to know what an acceptable parameterization of this process might be, but it must involve mean temperature and salinity gradients separately, not a lumped density gradient. Apart from double-diffusive effects, salinity on density surfaces can be treated as a passive scalar to which classical random-walk diffusion models apply, guaranteeing a well-defined macroscopic salinity diffusivity tensor.

Nevertheless, a firm basis has been established for the parameterization of diapycnal velocity and density flux divergence as a process like diapycnal diffusion. This is important for the development of models that properly handle diapycnal processes. In models that resolve the larger, planetary-scale, macroscopic eddies, we would advocate using parameterizations like those used in (41), (42) for representing microstructure fluxes of density, scalar concentration, and momentum. Although our point of view (involving conditional averaging on isopycnal surfaces) is different from Redi's (1982) (conventional fixed-level averaging), these parameterizations are functionally equivalent to her suggestion of using diffusivity tensors diagonal with respect to isopycnal surfaces. Although such parameterizations may be implemented in fixed-level coordinate models, their natural form in isopycnal coordinates is compelling.

APPENDIX A

Isopycnally Averaged Equations in Level Coordinates

To transform Eqs. (26)–(30) from isopycnal coordinates $\tilde{x}$, $\tilde{y}$, ρ, $\tilde{t}$ to level coordinates x, y, z, t, set

$$\tilde{x} = x, \quad \tilde{y} = y, \quad \rho = \tilde{\rho}(x, y, z, t), \quad \tilde{t} = t. \quad \text{(A1a,b,c,d)}$$

At fixed x, y, t, the function $\tilde{\rho}(x, y, z, t)$ is the inverse of $\bar{z}(x, y, \rho, t)$, the mean vertical height of isopycnal ρ; that is,

$$\tilde{\rho}(x, y, \bar{z}(x, y, \rho, t), t) = \rho. \quad \text{(A2)}$$

This function differs from the mean density $\Sigma(x, y, z, t)$ at the fixed space–time point x, y, z, t (appendix B). The partial derivatives transform according to

$$\begin{pmatrix} \partial_{\tilde{x}} \\ \partial_{\tilde{y}} \\ \partial_{\rho} \\ \partial_{\tilde{t}} \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & \mathbf{B} & & 0 \\ & & & 0 \\ 0 & 0 & -\tilde{\rho}_t/\tilde{\rho}_z & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_t \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & -\tilde{\rho}_x/\tilde{\rho}_z \\ 0 & 1 & -\tilde{\rho}_y/\tilde{\rho}_z \\ 0 & 0 & 1/\tilde{\rho}_z \end{pmatrix}. \quad \text{(A3)}$$

[If $\tilde{\rho}(x, y, z, t)$ were replaced by the actual random density $\rho(x, y, z, t)$, which is the inverse function of (9c), then the transformation (A3) would be the inverse of the transformation (10).] By defining

$$\tilde{w} = \frac{\hat{D}}{Dt} \bar{z} = \bar{z}_{\tilde{t}} + \hat{u}\bar{z}_{\tilde{x}} + \hat{v}\bar{z}_{\tilde{y}} + \hat{\varpi}\bar{z}_{\rho}, \quad \text{(A4)}$$

the derivative following the macroscopic motion of a macroscopically averaged isopycnal surface, we can show that the substantial derivative (31) following the weighted-average motion transforms into

$$\frac{\hat{D}}{Dt} = \partial_t + \hat{u}\partial_x + \hat{v}\partial_y + \tilde{w}\partial_z \quad \text{(A5)}$$

in level coordinates. Applying (31) and (A5) to $\tilde{\rho}$, we see that

$$\frac{\hat{D}}{Dt} \tilde{\rho} = \hat{\varpi}, \quad \text{(A6)}$$

where, from (41),

$$\hat{\varpi} \approx \partial_z(K^\rho \tilde{\rho}_z). \quad \text{(A7)}$$

The macroscopically averaged momentum, thickness, and scalar concentration equations (26)–(30) transform into

$$\frac{\hat{D}}{Dt} \hat{u} - f\hat{v} + \frac{1}{\rho_0} \bar{p}_x + \tilde{\rho}_z \overline{z'_\rho \pi'_{\tilde{x}}}/\rho_0 = \partial^T(\mathbf{G}_1^u + \mathbf{F}_1^u), \quad \text{(A8)}$$

$$\frac{\hat{D}}{Dt} \hat{v} + f\hat{u} + \frac{1}{\rho_0} \bar{p}_y + \tilde{\rho}_z \overline{z'_\rho \pi'_{\tilde{y}}}/\rho_0 = \partial^T(\mathbf{G}_1^u + \mathbf{F}_1^u), \quad \text{(A9)}$$

$$0 = -\bar{p}_z - g\tilde{\rho}, \quad \text{(A10)}$$

$$\hat{u}_x + \hat{v}_y + \tilde{w}_z = 0, \quad \text{(A11)}$$

$$\frac{\hat{D}}{Dt} \hat{\phi} = \partial^T(\mathbf{G}_1^\phi + \mathbf{F}_1^\phi), \quad \text{(A12)}$$

where $\partial^T = (\partial_x, \partial_y, \partial_z)$ and $\bar{p} = \bar{\pi} - g\tilde{\rho}z$ is the averaged pressure on isopycnal surfaces. Note the reappearance of the familiar form of the continuity equation (A11) with the macroscopic vertical velocity defined by (A4). Analogous to (19), the relation between $\mathbf{G}_1^\phi$ and $\mathbf{G}^\phi$ [Eq. (32)] is

$$\mathbf{G}_1^\phi = \tilde{\rho}_z \mathbf{B}^T \mathbf{G}^\phi$$
$$= (-\tilde{\rho}_z \overline{z_\rho u''\phi''}, -\tilde{\rho}_z \overline{z_\rho v''\phi''}, -\tilde{\rho}_z \overline{z_\rho w''\phi''})^T, \quad \text{(A13)}$$

where $w'' = u''\bar{z}_{\tilde{x}} + v''\bar{z}_{\tilde{y}} + \varpi''\bar{z}_\rho$ is an equivalent vertical fluctuating velocity; similarly for $\mathbf{G}_1^u$, $\mathbf{G}_1^v$. Analogous to (21), the microstructure flux divergences transform according to

$$\partial^T \mathbf{F}_1^\phi = \tilde{\rho}_z \tilde{\partial}^T \overline{\mathbf{F}^\phi} \approx \partial_z(K^\phi \partial_z \hat{\phi}), \quad \text{(A14)}$$

where the approximation (42) has been used. The momentum flux divergences transform similarly. The divergence of macroscopic flux $\partial^T \mathbf{G}_1^\phi$ is added to (A14) on the right side of (A12). The parameterization of the macroscopic flux (A13) is still an open question, beyond the scope of this paper. Macroscopic flux divergences do *not* occur in the density equation (A6).

Because $|\tilde{\rho}_z z'_\rho| \ll 1$, the pressure gradient-thickness covariance terms in (A8), (A9) are negligible compared to the horizontal pressure gradient.

APPENDIX B

Averaging at Fixed Depth: Average-Isopycnal Coordinates

We show that the equations of motion, first Reynolds-averaged in Cartesian coordinates, then transformed into coordinates fixed in the macroscopically averaged isopycnal surfaces, assume forms similar to the equations obtained above by macroscopically averaging *after* transformation to isopycnal surfaces. We show how in the former approach the analog of the diapycnal flux, forced by the average microstructure flux in section 2, must be represented by a parameterization of eddy density flux as Fickian diffusion with a diffusivity tensor diagonal in local isopycnal axes.

a. Conventional Reynolds-averaged equations

The equations of motion, continuity, thermodynamics, and scalar concentration (A8)–(A12), averaged macroscopically on isopycnals, and expressed in level coordinates, are not the same as the conventional equations Reynolds-averaged at fixed depth. The latter are

$$\mathcal{D}_t U - fV = -\Pi_x + \partial^T \mathbf{G}^U, \quad \text{(B1)}$$

$$\mathcal{D}_t V - fU = -\Pi_y + \partial^T \mathbf{G}^V, \qquad (\text{B2})$$

$$0 = -\Pi_z - g\Sigma, \qquad (\text{B3})$$

$$U_x + V_y + W_z = 0, \qquad (\text{B4})$$

$$\mathcal{D}_t \Sigma = \partial^T \mathbf{G}^\Sigma, \qquad (\text{B5})$$

$$\mathcal{D}_t \Phi = \partial^T \mathbf{G}^\Phi, \qquad (\text{B6})$$

where

$$\mathcal{D}_t \equiv \partial_t + U\partial_x + V\partial_y + W\partial_z, \qquad (\text{B7})$$

and uppercase symbols denote the forms of the dependent variables Reynolds-averaged at fixed depth. The terms $\mathbf{G}^U$, $\mathbf{G}^V$, $\mathbf{G}^\Sigma$, $\mathbf{G}^\Phi$ on the right are the usual eddy fluxes; for example,

$$\mathbf{G}^\Sigma = -\overline{\mathbf{u}^+ \rho^+}. \qquad (\text{B8})$$

Variables like U, V, $\ldots$, averaged at fixed depth, differ from $\hat{u}$, $\hat{v}$, $\ldots$, averaged at fixed density. If $\hat{u}$ and U differ, then so do the fluctuations u'', u^+,

$$u = \hat{u} + u'' = U + u^+. \qquad (\text{B9})$$

Practically, however, the clear distinction between isopycnal averaging and averaging at fixed depth may be lost when the theoretical ensemble is replaced by an ergodic average over space–time. If the averaging volume centered on a level z_1, say, is always large enough to include an isopycnal ρ_1, then there may be little distinction between the ergodic average at z_1 and that at ρ_1. Necessary conditions for this to be true are that there be no secular trend in the isopycnals and that the scale of the averaging volume be larger than the root-mean-square isopycnal displacement. If these conditions are met, we are justified in blurring the distinction between (B1)–(B6) and (A8)–(A12).

b. Eddy flux parameterizations: Diffusivity tensor

It is conventional to parameterize the eddy fluxes that occur, for example, in (B8) by

$$\mathbf{G}^\Sigma = -\overline{\mathbf{u}^+ \rho^+} = \mathbf{K}\partial\Sigma, \qquad (\text{B10})$$

where $\mathbf{K}$ is a diffusivity tensor. In general $\mathbf{K}$ should be positive definite and symmetric. The positive definiteness follows from the requirement that in the density variance balance the term $(-\overline{\mathbf{u}^+ \rho^+})^T\partial\Sigma$ should be invariably positive; that is, variance tends to be created by density flux (B10) occurring down the mean density

gradient $\partial\Sigma$. These properties imply that at every point there is an orthogonal transformation that diagonalizes $\mathbf{K}$. Without loss of generality, the transformation can be considered a rotation $\mathbf{R}$:

$$\mathbf{R}^T \mathbf{K} \mathbf{R} = \mathbf{K}^*, \qquad (\text{B11a})$$

where

$$\mathbf{K}^* = \text{diag}(K_1, K_2, K_3). \qquad (\text{B11b})$$

Hence there is a local coordinate frame in which the tensor $\mathbf{K}$ becomes diagonal. Redi (1982) asserted that this coordinate frame is spanned by two basis vectors $\mathbf{i}_1^*$, $\mathbf{i}_2^*$ tangent to the local isopycnal surface and one basis vector $\mathbf{i}_3^*$ normal to it, such that

$$K_1 = K_2 \gg K_3, \qquad (\text{B11c})$$

with the inequality spanning six or seven orders of magnitude. McDougall and Church (1986) concurred in this assertion.

c. Transformation to average-isopycnal coordinates: Redi's diffusivity tensor

Suppose (B1)–(B6) were transformed into *mean* isopycnal coordinates X, Y, Σ, τ, according to

$$x = X, \quad y = Y, \quad z = Z(X, Y, \Sigma, \tau), \quad t = \tau,$$
$$(\text{B12a,b,c,d})$$

where $Z(X, Y, \Sigma, \tau)$ is, at fixed X, Y, τ, the inverse function of $\Sigma(X, Y, Z, \tau)$, the density averaged at fixed depth; that is,

$$Z(X, Y, \Sigma(X, Y, z, \tau), \tau) = z. \qquad (\text{B13})$$

Then the partial derivatives transform according to

$$\begin{pmatrix} \partial \\ \partial_t \end{pmatrix} = \begin{pmatrix} & & & 0 \\ & \mathbf{C} & & 0 \\ & & & 0 \\ 0 & 0 & -\dfrac{Z_\tau}{Z_\Sigma} & 1 \end{pmatrix} \begin{pmatrix} \partial_X \\ \partial_\tau \end{pmatrix}, \qquad (\text{B14a})$$

where $\partial_X^T = (\partial_X, \partial_Y, \partial_\Sigma)$, and

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -Z_X/Z_\Sigma \\ 0 & 1 & -Z_Y/Z_\Sigma \\ 0 & 0 & 1/Z_\Sigma \end{pmatrix}. \qquad (\text{B14b})$$

The tensor that satisfies Redi's (1982) requirements is, in level coordinates,

$$\mathbf{K} = \frac{K_1}{1 + Z_X^2 + Z_Y^2} \begin{pmatrix} 1 + Z_Y^2 + \epsilon Z_X^2 & -Z_X Z_Y(1 - \epsilon) & Z_X(1 - \epsilon) \\ -Z_X Z_Y(1 - \epsilon) & 1 + Z_X^2 + \epsilon Z_Y^2 & Z_Y(1 - \epsilon) \\ Z_X(1 - \epsilon) & Z_Y(1 - \epsilon) & Z_X^2 + Z_Y^2 + \epsilon \end{pmatrix}, \qquad (\text{B15})$$

where $\epsilon = K_3/K_1$. Equations (B5), (B10) can be expressed as

$$\mathcal{D}_t \Sigma = \partial^T(\mathbf{K}\partial\Sigma) = \frac{1}{Z_\Sigma} \partial_X^T(Z_\Sigma \mathbf{K}_1 \partial_X P), \qquad (\text{B16})$$

where

$$\mathbf{K}_1 = \mathbf{C}^T \mathbf{KC} = K_1 \begin{pmatrix} 1 - \dfrac{Z_X^2(1-\epsilon)}{1+Z_X^2+Z_Y^2} & \dfrac{-Z_X Z_Y(1-\epsilon)}{1+Z_X^2+Z_Y^2} & -\epsilon \dfrac{Z_X}{Z_\Sigma} \\ \dfrac{-Z_X Z_Y(1-\epsilon)}{1+Z_X^2+Z_Y^2} & 1 - \dfrac{Z_Y^2(1-\epsilon)}{1+Z_X^2+Z_Y^2} & -\epsilon \dfrac{Z_Y}{Z_\Sigma} \\ -\epsilon \dfrac{Z_X}{Z_\Sigma} & -\epsilon \dfrac{Z_Y}{Z_\Sigma} & \dfrac{\epsilon}{Z_\Sigma^2}(1+Z_X^2+Z_Y^2) \end{pmatrix}. \quad (B17)$$

Suppose typical "horizontal" and "vertical" length scales are L and D, so that

$$\partial_X, \partial_Y \sim L^{-1}, \quad (1/Z_\Sigma)\partial_\Sigma \sim D^{-1}, \quad (B18)$$

and

$$|\nabla_X Z| \sim \delta = D/L \ll 1. \quad (B19)$$

Then $\mathbf{K}_1$ may be approximated by

$$\mathbf{K}_1 \approx \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_1 & 0 \\ 0 & 0 & K_3/Z_\Sigma^2 \end{pmatrix} \quad (B20)$$

to $O(\delta^2 + \epsilon)$. Hence, Eq. (B16) simplifies to

$$\mathcal{D}_t \Sigma \approx \frac{1}{Z_\Sigma} \partial_\Sigma \left(K_3 \frac{1}{Z_\Sigma} \right), \quad (B21)$$

which we may compare to Eq. (41).

If we use the same diffusivity tensor as in (B10) to parameterize the flux of mean scalar concentration Φ; that is, we set

$$\mathbf{G}^\Phi = -\overline{\mathbf{u}^+ \phi^+} = \mathbf{K}\partial\Phi, \quad (B22)$$

then Eq. (B6) transforms into mean isopycnal coordinates as

$$\mathcal{D}_t \Phi = \partial^T (\mathbf{K}\partial\Phi) = \frac{1}{Z_\Sigma} \partial_X^T (Z_\Sigma \mathcal{H} \partial_X \Phi) \quad (B23)$$

[cf. (B16)]. Using (B20), this can be written

$$\mathcal{D}_t \Phi \approx \frac{1}{Z_\Sigma} \left\{ \partial_X (Z_\Sigma K_1 \partial_X \Phi) + \partial_Y (Z_\Sigma K_1 \partial_Y \Phi) \right.$$

$$\left. + \partial_\Sigma \left(\frac{K_3}{Z_\Sigma} \partial_\Sigma \Phi \right) \right\}. \quad (B24)$$

Comparing this to (30), (42), we note the similarity in form of the third term on the right to the approximation (42). An eddy diffusivity parameterization of $\mathbf{G}^\Phi$ in (32), similar to (B22), will also produce terms like those on the right of (B24).

APPENDIX C

Weighted versus Unweighted Averages

Introduction of the thickness-weighted ensemble averages (24), (25) permits a compact form for the momentum, thickness, and scalar concentration equa-

tions (26)–(30). Gent and McWilliams (1990) employed unweighted ensemble averages in isopycnal coordinates. This leads to differences in the appearance of the equations, which we shall now describe.

The relation between the weighted average and the unweighted average of a variable $\phi = \hat{\phi} + \phi'' = \bar{\phi} + \phi'$ is

$$\bar{z}_\rho \hat{\phi} = \bar{z}_\rho \bar{\phi} + \overline{z_\rho' \phi'}. \quad (C1)$$

Hence Eq. (29) may be written

$$\bar{z}_{\rho t} + (\bar{z}_\rho \bar{u})_{\hat{x}} + (\bar{z}_\rho \bar{v})_{\hat{y}} + (\bar{z}_\rho Q)_\rho = 0, \quad (C2)$$

where

$$\bar{z}_\rho Q = \int^\rho \{ \partial_{\hat{x}} (\overline{z_\rho' u'}) + \partial_{\hat{y}} (\overline{z_\rho' v'}) \} d\rho + \bar{z}_\rho \hat{\omega}. \quad (C3)$$

Gent and McWilliams (1990) derived this form [their Eqs. (6), (8)] except that they neglected the diapycnal flux $\bar{z}_\rho \hat{\omega}$. They also prescribed an average scalar concentration equation in terms of unweighted-average variables. If we perform the required operations, we obtain

$$\bar{z}_\rho \frac{\bar{D}}{Dt} \hat{\phi} = \partial_{\hat{x}} (-\overline{z_\rho' u' \phi}) + \partial_{\hat{y}} (-\overline{z_\rho' v' \phi}) + \partial_\rho (\bar{z}_\rho Q \hat{\phi})$$

$$- \partial_{\hat{i}} (\overline{z_\rho' \phi'}) + \hat{\partial}^T \mathbf{G}_2^\phi + \hat{\partial}^T \overline{\mathbf{F}}^\phi, \quad (C4)$$

where

$$\frac{\bar{D}}{Dt} \equiv \partial_{\hat{i}} + \bar{u}\partial_{\hat{x}} + \bar{v}\partial_{\hat{y}} + Q\partial_\rho, \quad (C5)$$

$$\mathbf{G}_2^{\phi T} = (-\overline{(z_\rho u)' \phi'}, \ -\overline{(z_\rho v)' \phi'}, \ -\bar{z}_\rho \hat{\omega} \bar{\phi} - \overline{(z_\rho \omega)' \phi'}), \quad (C6)$$

and $(z_\rho u)' = \bar{z}_\rho u' + z_\rho' \bar{u}$, etc. If we compare this to Eqs. (10), (11), (13) of Gent and McWilliams, they show the first three terms on the right of (C4), with the remaining terms either included in their alongisopycnal diffusion term $R(\phi)$ or neglected (by setting $\omega = 0$). Contrast (C4), (C5), (C6) with (30), (31), (32). If the transformation (A3) to level coordinates is made, the rate of change operator (C5) becomes

$$\frac{\bar{D}}{Dt} \tilde{\rho} = Q \quad (C9)$$

where

$$\bar{w} \equiv \frac{\bar{D}}{Dt}\bar{z}. \qquad (C8)$$

It is easy to show that

$$\frac{\bar{D}}{Dt}\tilde{p} = Q \qquad (C9)$$

and that (C2) transforms into

$$\bar{u}_x + \bar{v}_y + \bar{w}_z = 0. \qquad (C10)$$

Gent and McWilliams (1990) propose a parameterization for the eddy thickness flux $\overline{z'_p \mathbf{u}'}$, in addition to an implicit parameterization of $\mathbf{G}_2^{\phi}$. Whatever the parameterization, they remark that its effect in an arbitrary scalar concentration equation is to provide an additional horizontal advection by the thickness flux; that is, the total effective along-isopycnal advective velocity is the thickness-weighted average

$$\hat{\mathbf{u}} = \bar{\mathbf{u}} + \overline{z'_p \mathbf{u}'}/\bar{z}_p \qquad (C11)$$

that we have used in the average substantial rate of change operator (31). The parameterization is to set

$$\overline{z'_p \mathbf{u}'} = -\partial_p(K_1 \tilde{\nabla}\bar{z}) \qquad (C12)$$

in isopycnal coordinates. Substituted in (C3), this is equivalent to setting

$$\bar{z}_p Q = -\tilde{\nabla} \cdot (K_1 \tilde{\nabla}\bar{z}), \qquad (C13)$$

with $\bar{z}_p \tilde{w}$ neglected. Transforming (C13) into level coordinates according to (A3), we obtain

$$Q = \nabla \cdot (K_1 \nabla \tilde{p}) + \partial_z(-K_1 \delta^2 \tilde{p}_z) \qquad (C14)$$

with

$$\delta = |\nabla \tilde{p}/\tilde{p}_z|, \qquad (C15)$$

∇ being the horizontal gradient operator in level coordinates. [Compare (C14) to Gent and McWilliams's (1990) Eq. (23).] Now (C14) resembles a Fickian diffusion law but with a nonpositive diffusivity tensor. This means that downgradient flux of density cannot be guaranteed. For example, in a local density configuration where $K_1 \nabla \tilde{p}$ is independent of horizontal po-

sition, density would flow locally against its vertical gradient. For this reason we do not believe that (C12) or (C14) represents by itself an acceptable parameterization of turbulent flux. If $\bar{z}_p \tilde{w}$ were not neglected in (C13) and a term like (A7) were added to the right of (C14), then K^p might exceed $-K_1 \delta^2$ and the positiveness of the diffusivity would be preserved.

In any case, we consider the parameterization of thickness flux a side issue, because if thickness-weighted average $\hat{\mathbf{u}}$ is used instead of $\bar{\mathbf{u}}$, then the thickness flux need never be explicitly considered.

REFERENCES

Bleck, R., and L. T. Smith, 1990: A wind-driven isopycnic model of the North and Equatorial Atlantic Ocean. 1. Model development and supporting experiments. *J. Geophys. Res.,* **95,** 3273–3285.

Eliassen, A., and E. Kleinschmidt, 1956: Dynamic meteorology. *Handb. Phys.,* **48,** 1–154.

Gent, P. R., and J. C. McWilliams, 1990: Isopycnal mixing in ocean circulation models. *J. Phys. Oceanogr.,* **20,** 150–155.

Gordon, A. L., and E. J. Molinelli, 1982: Thermohaline and chemical distributions and the atlas data set. *Southern Ocean Atlas,* A. L. Gordon, Ed., Columbia University Press, 1–11. (plates 1–233).

Levitus, S., 1982: *Climatological Atlas of the World Ocean.* NOAA Prof. Paper No. 13, U.S. Department of Commerce, Rockville, MD, 173 pp.

Lumley, J. L., and H. A. Panofsky, 1964: *The Structure of Atmospheric Turbulence.* Wiley-Interscience, 239 pp.

McDougall, T. J., and J. A. Church, 1986: Pitfalls with the numerical representation of isopycnal and diapycnal mixing. *J. Phys. Oceanogr.,* **16,** 196–199.

Monin, A. S., and A. M. Yaglom, 1971: *Statistical Fluid Mechanics.* Vol. 1. The MIT Press, 769 pp. [English translation by J. L. Lumley.]

Munk, W. H., 1966: Abyssal recipes. *Deep-Sea Res.,* **13,** 707–730.

Osborn, T. R., and C. S. Cox, 1972: Oceanic fine structure. *Geophys. Fluid Dyn.,* **3,** 321–345.

Redi, M. H., 1982: Oceanic isopycnal mixing by coordinate rotation. *J. Phys. Oceanogr.,* **12,** 1154–1158.

Reid, J. L., 1965: *Intermediate Waters of the Pacific Ocean. The Johns Hopkins Oceanographic Studies, No. 2.* The Johns Hopkins Press, 85 pp.

Robinson, A. R., 1965: A three-dimensional model of inertial currents in a variable-density ocean. *J. Fluid Mech.,* **21,** 211–223.

Speer, K., and E. Tziperman, 1992: Rates of water mass formation in the North Atlantic Ocean. *J. Phys. Oceanogr.,* **22,** 93–104.

Worthington, L. V., 1981: The water masses of the world ocean: Some results of a finescale census. *Evolution of Physical Oceanography,* B. A. Warren and C. Wunsch, Eds., The MIT Press, 42–69.