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Finding new examples of compact simply connected spaces admitting a Riemannian metric of positive sectional curvature is a fundamental problem in differential geometry. Likewise, studying topological properties of families of manifolds is very interesting to topologists. The Eschenburg spaces combine both of those interests: they are positively curved Riemannian manifolds whose topological classification is known. There is a second family consisting of the Witten manifolds: they are the examples of compact simply connected spaces admitting Einstein metrics of positive Ricci curvature. Thirdly, there is a notion of generalized Witten manifold as well. Topologically, all three families share the same cohomology ring. This common ring structure motivates the definition of a manifold of type r, where r is the order of the fourth cohomology group. In 1991, M. Kreck and S. Stolz classified manifolds M of type r up to homeomorphism and diffeomorphism using invariants $\overline{s}_i(M)$ and $s_i(M)$, for $i = 1, 2, 3$. This gave rise to many new examples of nondiffeomorphic but homeomorphic manifolds. In this dissertation, new versions of the homeomorphism and diffeomorphism classification of manifolds of type r are proven. In particular, we can replace $\overline{s}_1$ and $\overline{s}_3$ by the first Pontrjagin class and the self-linking number in the homeomorphism classification of spin manifolds of type r. As the formulas of the two latter invariants are in general much easier to compute, this simplifies the classification of these manifolds up to homeomorphism significantly.

Classifying Seven Dimensional Manifolds of Fixed Cohomology Type

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1. INTRODUCTION

1.1. History

Manifolds of type r have played an important role in various different areas of differential geometry. We will describe their role in the areas of positive and non-negative sectional curvature, positive Ricci curvature and Einstein manifolds. The question of finding new examples of compact simply connected homogeneous and inhomogeneous spaces admitting a Riemannian metric with positive sectional curvature has been interesting to geometers since the 1960s. Besides the compact rank one symmetric spaces which always have positive sectional curvature, there are very few known examples. In 1961, M. Berger [Ber61] found the first spaces in dimension 7 and 13. About ten years later, three new examples of compact manifolds in dimension 6, 12, and 24 were discovered by N. Wallach [Wah72]. All of these spaces are homogeneous and admit Riemannian metrics of positive sectional curvature. In 1974, S. Aloff and N. Wallach [AlWa75] introduced an infinite family of homogeneous 7-manifolds, now called Aloff-Wallach spaces. Most of them are positively curved spaces. In the same year, D. Gromoll and W. Meyer [GrMe74] obtained a new construction of one of the exotic 7-spheres. By this construction they showed that this space admits a Riemannian metric of non-negative sectional curvature. Moreover, there is also an infinite family of homogeneous 7-manifolds admitting Einstein metrics of positive Ricci curvature. These are called Witten manifolds and were introduced by E.

Witten [Wit81] in 1981. There is a notion of generalized Witten manifolds as well but it is not known whether there exist Einstein metrics on them. One year later, in 1982 J. H. Eschenburg [Esb82] was looking for further spaces of positive sectional curvature. He introduced a new construction, called later a biquotient. This is a generalization of homogeneous spaces. Biquotients are inhomogeneous in general. In his paper, a generalization of the Aloff-Wallach spaces was introduced. This infinite family consists of compact simply connected 7-manifolds, now called Eschenburg spaces. Eschenburg spaces are examples of biquotients and most of them admit a Riemannian metric with positive sectional curvature. J. H. Eschenburg also found new biquotients in dimension six, see [Esb84]. Also using a biquotient construction, in 1996 Y. Bazaikin [Baz96] constructed an infinite family of 13-manifolds which includes the Berger 13-manifold defined in [Ber61]. These spaces are now called Bazaikin spaces and most of them are positively curved spaces.

The topology of the above spaces has been studied extensively. In this dissertation, we will focus on the topology of the Eschenburg spaces and the generalized Witten manifolds. Because both families have the same type of the cohomology ring with integer coefficients:

$$H^0 \cong H^2 \cong H^5 \cong H^7 \cong \mathbb{Z},\ H^4 \cong \mathbb{Z}_r,\ H^1 = H^3 = H^6 = 0,$$

for some $r \geq 1$, and u^2 is a generator of H^4 if u is a generator of H^2, we will call a manifold satisfying this condition a *manifold of type* r. The topology of manifolds of type r is studied by [KrSt88],[KrSt91],[AMP97],[Kru95],[Kru97],[Kru98],[Kru05],[Esc05],[CEZ07] as follows. In 1988, M. Kreck and S. Stolz [KrSt88] introduced new invariants, now called Kreck-Stolz invariants s_i, and gave a classification of manifolds of type various r up to homeomorphism and diffeomorphism. We state all theorems in the orientation preserving case; for the corresponding theorems in the orientation reversing case the linking form and the Kreck-Stolz invariants change signs.

Classification Theorem I ([KrSt88]) *Let M and M' be two smooth manifolds of type r which are both spin or both nonspin. Then M is (orientation preserving) diffeomorphic (homeomorphic) to M' if and only if*

- *there is an isomorphism $H^4(M;\mathbb{Z}) \longrightarrow H^4(M';\mathbb{Z})$ preserving the square of a generator of $H^2(M;\mathbb{Z})$, the linking form, and the first Pontrjagin class,*
- *$s_i(M) = s_i(M') \in \mathbb{Q}/\mathbb{Z}$ ($\overline{s}_i(M) = \overline{s}_i(M') \in \mathbb{Q}/\mathbb{Z}$) for $i = 1, 2, 3$.*

M. Kreck and S. Stolz also classified all Witten manifolds, see [KrSt88]. Three years later, they observed that the additional invariants like the linking form and the first Pontrjagin class can be expressed in terms of s_i. Hence, the theorem can be reformulated as:

Classification Theorem II ([KrSt91],[KrSt98]) *Let M and M' be two smooth manifolds of type r which are both spin or both nonspin. Then M is (orientation preserving) diffeomorphic (homeomorphic) to M' if and only if $s_i(M) = s_i(M') \in \mathbb{Q}/\mathbb{Z}$ ($\overline{s}_i(M) = \overline{s}_i(M') \in \mathbb{Q}/\mathbb{Z}$) for $i = 1, 2, 3$.*

M. Kreck and S. Stolz used their classification theorem to classify the Aloff-Wallach spaces up to homeomorphism and diffeomorphism, see [KrSt91]. In the process, they found the first homeomorphic but not diffeomorphic Einstein manifolds admitting positive sectional curvature. In 1997, L. Astey, E. Micha and G. Pastor [AMP97] used classification theorem II to classify a particular subfamily of Eschenburg spaces up to homeomorphism and diffeomorphism. In 1997 and 1998, B. Kruggel [Kru97],[Kru98] obtained various homotopy classifications, see section 6.1. The important homotopy classification that we will use in this dissertation is expressed as follows:

Classification Theorem III ([Kru98]) *Let M and M' be two smooth spin manifolds of type odd r with generators u_M and $u_{M'}$ of $H^2(M;\mathbb{Z})$ and $H^2(M';\mathbb{Z})$, respectively.*

- *If* $\pi_4(M) = \pi_4(M') = 0$*, M and M' are (orientation preserving) homotopy equivalent if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $2r \cdot s_2(M) = 2r \cdot s_2(M')$.
- *If* $\pi_4(M) \cong \pi_4(M') \cong \mathbb{Z}_2$*, M and M' are (orientation preserving) homotopy equivalent if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $r \cdot s_2(M) = r \cdot s_2(M')$.

In [Kru97], B. Kruggel classified all generalized Aloff-Wallach spaces and generalized Witten manifolds with odd order of the fourth cohomology group up to homotopy, see his paper for the definitions. After proving classification theorem III in [Kru98], he could classify all Eschenburg spaces up to homotopy. In 2005, generalized Witten manifolds were classified by C. Escher [Esc05] up to homeomorphism and diffeomorphism. During the same year, B. Kruggel's paper [Kru05] was published based on an earlier preprint. In this paper, he claimed a new version of the homeomorphism and diffeomorphism classification in the case of the Eschenburg spaces without proof. Fortunately, a generalization of this theorem which is the main purpose of this dissertation, will be proved in subsection 6.2.1. Moreover, B. Kruggel gave a method to compute the Kreck-Stolz invariants s_i for almost all Eschenburg spaces, namely for those Eschenburg spaces satisfying condition (C). This condition and the whole construction will be summarized in Chapter 5. Two years later, in 2007 T. Chinburg, C. Escher and W. Ziller [CEZ07] used B. Kruggel's construction and a program written in Maple and C code to classify all Eschenburg spaces satisfying condition (C) up to homotopy, homeomorphism, and diffeomorphism.

1.2. Statement of the Problem and Results

The main purpose of this dissertation is to give another version of the homeomorphism and diffeomorphism classification of most manifolds of type r. This classification theorem will be proved in section 6.2 and divided into two cases: the spin case and the nonspin case:

Classification Theorem A *Suppose that M and M' are smooth spin manifolds of type odd r with isomorphic fourth homotopy groups. Let $u_M \in H^2(M;\mathbb{Z})$ and $u_{M'} \in H^2(M';\mathbb{Z})$ be both generators.*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $s_1(M) = s_1(M')$,
  - $s_2(M) = s_2(M')$.
- *M is (orientation preserving) homeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $p_1(M) = p_1(M')$,
  - $s_2(M) = s_2(M')$.

Classification Theorem B *Suppose that M and M' are smooth nonspin manifolds of type r. Let $u_M \in H^2(M;\mathbb{Z})$ and $u_{M'} \in H^2(M';\mathbb{Z})$ be both generators.*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $s_1(M) = s_1(M')$,

– $s_2(M) = s_2(M')$.

- *M is (orientation preserving) homeomorphic to M' if and only if*

  – $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,

  – $\overline{s}_1(M) = \overline{s}_1(M')$,

  – $s_2(M) = s_2(M')$.

All of the above invariants are described in this dissertation. Firstly, the Kreck-Stolz invariants s_i were defined in [KrSt88] via a bounding manifold of a manifold of type r. They are elements in $\mathbb{Q}/\mathbb{Z}$. Also, in [KrSt91] a generalized definition of the Kreck-Stolz invariants were given without their explicit formulas. The generalized formulas will be computed in section 4.1 and can be expressed in terms of a bounding manifold W with boundary $\partial W = M$ of type r as follows:

Spin Case:

$$\begin{aligned}
S_1(W, z, c) &= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}c^2 p_1 + \frac{1}{2^7 \cdot 3}c^4, \\
S_2(W, z, c) &= -\frac{1}{2^4 \cdot 3}((zc + z^2)p_1 - (zc^3 + 3z^2c^2 + 4z^3c + 2z^4)), \\
S_3(W, z, c) &= -\frac{1}{2^3 \cdot 3}((zc + 2z^2)p_1 - (zc^3 + 6z^2c^2 + 16z^3c + 16z^4)).
\end{aligned}$$

Nonspin Case:

$$\begin{aligned}
S_1(W, z, c) &= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}(c^2 + 2zc + z^2)p_1 \\
&\quad + \frac{1}{2^7 \cdot 3}(c^4 + 4zc^3 + 6z^2c^2 + 4z^3c + z^4), \\
S_2(W, z, c) &= -\frac{1}{2^4 \cdot 3}((zc + 2z^2)p_1 - (zc^3 + 6z^2c^2 + 13z^3c + 10z^4)), \\
S_3(W, z, c) &= -\frac{1}{2^3 \cdot 3}((zc + 3z^2)p_1 - (zc^3 + 9z^2c^2 + 31z^3c + 39z^4)).
\end{aligned}$$

These $S_i(W, z, c)$ are in $\mathbb{Q}$. One can show that $S_i(W, z, c) \bmod \mathbb{Z}$ depend only on the boundary M. Therefore, we obtain the generalized Kreck-Stolz invariants of a manifold

M of type r. Note that if $c = 0$, we get back to the formulas of the original Kreck-Stolz invariants.

Secondly, $L({u_M}^2, {u_M}^2)$, where $u_M \in H^2(M; \mathbb{Z})$ is a generator, is the self-linking number of M. It is an element in $\mathbb{Q}/\mathbb{Z}$ and can be computed by using the description of the linking form L defined in [Bar65]. This definition was given in terms of the homology groups. However, in section 2.6 we will be able to give the equivalent definition in terms of the cohomology groups.

Finally, we will show that there exists a relation between the characteristic number z^4 defined on a bounding manifold W and the self-linking number defined on its boundary M. In other words, we obtain

$$L(u^2, u^2) = z^4(W) \bmod \mathbb{Z}.$$

This is one of the important facts used to prove the classifications above. This relation was stated in [KrSt88] without proof. However, we will give a proof in section 4.2. It is a pleasure to thank S. Stolz for very helpful information for this section.

1.3. Organization of this Dissertation

The mathematical background for this dissertation is described in Chapter 2. It is divided into 6 sections. The first three sections are all concerned with bundle theory. Section 2.1 gives a short overview of principal bundles and classifying spaces. Section 2.2 describes a complete picture of manifolds and bundles in each category: smooth, piecewise linear, and topological. In particular, the classification (Theorem A) will be proved in the piecewise linear category. As we will see in the proof, the piecewise linear structure will give more information than the smooth structure. A generalization of a manifold, called a Poincaré complex, will be explained in section 2.3. In this section, we will describe the Spivak normal bundle, which is equivalent to a normal bundle of a manifold, and a normal map. As seen in the previous section, the two classifications are separated by the existence of a spin structure on the manifolds. This structure is defined in section 2.4. Section 2.5 gives a definition of a biquotient. An example of a biquotient is the infinite family of Eschenburg spaces which is a manifold of type r. The last section describes the general definition of the linking form which gives rise to the notion of the self-linking number, which represents one of the homeomorphism and diffeomorphism invariants for manifolds of type r.

Chapter 3 begins with the definition of a manifold of type r and some examples (Eschenburg spaces and generalized Witten manifolds). In section 3.2, the homotopy type of a manifold of type r will be given using the homotopy invariant $s(M)$. Section 3.3 describes a relation between $s(M)$ and the self-linking number. This yields a method to compute the self-linking number. Alternatively, we also will be able to compute it by using the different method given in section 4.2.

The Kreck-Stolz invariants s_i will be described in Chapter 4. The first section, section 4.1, mentions their construction which is defined initially via a bounding manifold and describes how to obtain well-defined invariants on the boundary. The explicit generalized formulas will be computed in this section as well. Section 4.2 gives a relation between the linking form, the first Pontrjagin class, and some characteristic numbers. In particular, we will see how to obtain the self-linking number of M from a particular characteristic number of a bounding manifold W.

Chapter 5 is entirely devoted to the Eschenburg spaces. The main arguments are based on [Kru05]. The first section gives their cohomology rings together with their first Pontrjagin class which is one of the homeomorphism invariants. Bounding manifolds of most Eschenburg spaces can be explicitly constructed as seen in section 5.2 and their topology is described in section 5.3. With these specific bounding manifolds, the Kreck-

Stolz invariants can be computed and will be given in section 5.4 together with the self-linking number of all Eschenburg spaces.

The purpose of this dissertation is the proof of classification theorems A and B. The proofs will be given in section 6.2 of Chapter 6. It is divided into two subsections 6.2.1 and 6.2.2 for the spin case and the nonspin case, respectively. The first section specifies the homotopy classifications in various cases. These are based on [Kru97] and [Kru98]. Finally, combining the invariants in section 5.4 and classification theorem A yields a complete picture of the classification of the Eschenburg spaces. This is described in the last section, section 6.3.

The conclusions and future work will be described in Chapter 7, the last chapter of this dissertation. Throughout the dissertation, we always use integer coefficients for the homology and cohomology groups if we do not specify the coefficient group.

2. MATHEMATICAL BACKGROUND

2.1. Principal Bundles and Classifying Spaces

In this section we follow S. A. Mitchell's notes [Mit01] on principal bundles and classifying spaces. These notes are based on [BrDi85],[Dol63],[May99],[Mil55],[Ste51]. A principal bundle is a special case of a fiber bundle with an additional structure derived from the action of a topological group on each fiber. For example, in the case of a real vector bundle with the Euclidean metric, there are associated bundles: the disc bundle, the sphere bundle, the frame bundle, etc. All of these bundles can be described as a single object, called a principal bundle with structure group $O(n)$. Moreover, in comparing with vector bundles, principal bundles also have enough structure that one can classify them up to a homotopy equivalence, using a so-called classifying map. The definition of principal bundles can be explained as follows.

Let G be a topological group. A *right G-space* is a topological space X equipped with a continuous right G-action $X \times G \longrightarrow X$. If X and Y are right G-spaces, a *G-equivariant map* is a map $f : X \longrightarrow Y$ such that $f(x \cdot g) = f(x) \cdot g$ for all $g \in G, x \in X$. Let B a topological space. Suppose P is a right G-space equipped with a G-equivariant $\pi : P \longrightarrow B$, where G acts trivially on B. In other words, π preserves the orbit space P/G. (P, π) is called a *principal G-bundle over B* if π satisfies the *local triviality condition*: There is a covering $\{U_i\}$ of B and G-equivariant homeomorphisms $\phi_{U_i} : \pi^{-1}(U_i) \longrightarrow U_i \times G$ for each i such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi_{U_i}"] \arrow[d] & U_i \times G \arrow[dl] \\ U_i & \end{tikzcd}$$

Here for each i, $U_i \times G$ has the right G-action $(u, g)h = (u, gh)$. One can show that G

acts freely on P and the canonical map $P/G \longrightarrow B$ is a homeomorphism. In summary, a principal G-bundle over B consists of a locally trivial free G-space with orbit space B. For any principal G-bundle (Q, π_Q) over B, the equivariant maps $P \longrightarrow Q$ yield the set of isomorphism classes of principal G-bundles over B, denoted by $\mathcal{P}_G B$. A principal G-bundle is called trivial if it is isomorphic to the product principal bundle $B \times G \longrightarrow B$. One can easily show that a principal G-bundle is trivial if and only if it admits a section. If G is discrete, a principal G-bundle with connected total space P is the same thing as a regular covering map with G as a group of desk transformations.

Given a right G-space P and a left G-space Q, we define the *balanced product* $P \times_G Q$ to be the orbit space $(P \times Q)/G$ under the action $(p, q)g = (pg, g^{-1}q)$. We have the following important properties:

- If $Q = *$ consists of a single point, $P \times_G * = P/G$.
- If $Q = G$ is a topological group with the left translation action, the canonical right G-action on P induces a G-equivariant homeomorphism $P \times_G G \cong P$.
- If H is a subgroup of G, then $P \times_G (G/H) \cong P/H$ with the canonical left G-action on G/H.

We call H an *admissible subgroup* of G if the quotient map $G \longrightarrow G/H$ is a principal H-bundle. An example is given by a Lie group G and a closed subgroup H. Using the above properties, one easily obtains the following proposition.

Proposition 2.1.0.1 ([Mit01]) *If $P \longrightarrow B$ is a principal G-bundle and H is an admissible subgroup of G, then the quotient map $P \longrightarrow P/H$ is a principal H-bundle.*

Now suppose $\pi : P \longrightarrow B$ is a principal G-bundle and F is a left G-space. The trivial map $F \longrightarrow *$ induces a map $\bar{\pi} : P \times_G F \longrightarrow P \times_G * = P/G \cong B$. It is easy to check that this map satisfies the local triviality condition with fiber F, i.e. for each $b \in B$, there exist a neighborhood $U \subset B$ and a homeomorphism $\phi : \bar{\pi}^{-1}(U) \longrightarrow U \times F$ such that $p_1 \circ \phi = \bar{\pi}$, where $p_1 : U \times F \longrightarrow U$ is the natural projection. This is called a *fiber bundle with fiber F and structure group G.* For example, an n-dimensional real vector bundle is a fiber bundle with fiber $\mathbb{R}^n$ and structure group $GL_n\mathbb{R}$. In fact, there is a natural bijection between $\mathcal{P}_{GL_n\mathbb{R}}B$ and the set of isomorphism classes of real vector bundles over B given by

$$P \longmapsto P \times_{GL_n\mathbb{R}} \mathbb{R}^n.$$

If the vector bundle has a Euclidean metric, we can define an associated principal $O(n)$-bundle as well. Moreover, in the case that B is paracompact Hausdorff, there is a natural bijection between $\mathcal{P}_{O(n)}B$ and the set of isomorphism classes of real vector bundles over B with a Euclidean metric given by

$$P \longmapsto P \times_{O(n)} \mathbb{R}^n.$$

Similarly, these hold for a complex case by replacing $GL_n\mathbb{R}$ and $O(n)$ by $GL_n\mathbb{C}$ and $U(n)$, respectively.

Next, we recall that a space X is *weakly contractible* if $X \longrightarrow *$ is a *weak homotopy equivalence*, i.e. the map induces isomorphisms $\pi_n(X) \longrightarrow \pi_n(*)$ for all $n \geq 0$, or equivalently $\pi_n(X)$ is trivial for all $n \geq 0$. Note that every contractible space is weakly contractible, and by Whitehead's theorem, every weakly contractible CW-complex is contractible. For each topological group G, homotopy theory gives the following powerful theorem:

Theorem 2.1.0.1 ([Mit01]) *Suppose $P \longrightarrow B$ is a principal G-bundle and P is weakly contractible. Then for all CW-complexes X, the map $\phi : [X, B] \longrightarrow \mathcal{P}_G X$ given by $f \longmapsto f^*P$ is a bijection.*

Here f^*P is the natural pullback bundle. We call B the *classifying space* for G and P the *universal G-bundle*, denoted by BG and EG, respectively. Moreover, for a universal

G-bundle $P \longrightarrow B$, B can be taken to be a CW-complex which is unique up to homotopy and P is unique up to homotopy as well. For example, $BGL_n\mathbb{R} \cong G_n(\mathbb{R}^\infty) \cong BO(n)$ and $BGL_n\mathbb{C} \cong G_n(\mathbb{C}^\infty) \cong BU(n)$, where $G_n(\mathbb{R}^\infty), G_n(\mathbb{C}^\infty)$ are the Grassmann manifolds. Also, if G is discrete, then BG is an Eilenberg-MacLane space $K(G, 1)$.

Let $\theta : H \longrightarrow G$ be a homomorphism of topological groups. We define a left H-action on G by $hg = \theta(h)g$. If $Q \longrightarrow B$ is a principal H-bundle, then $Q \times_{H,\theta} G \longrightarrow B$ is a principal G-bundle, where $Q \times_{H,\theta} G := Q \times_H G$. Note that this total space depends on the choice of a homomorphism $\theta : H \longrightarrow G$. We omit θ in the case of θ being an inclusion. Therefore we have a natural transformation $Q_\theta : \mathcal{P}_H B \longrightarrow \mathcal{P}_G B$. Yoneda's lemma implies that there is a unique homotopy class $B\theta : BH \longrightarrow BG$ inducing Q_θ such that $B\theta$ is a classifying map for the principal G-bundle $EH \times_{H,\theta} G \longrightarrow BH$. Now suppose $P \longrightarrow B$ is a principal G-bundle. We say that P *is induced from an H-bundle* if there exist a principal H-bundle $Q \longrightarrow B$ and an isomorphism $Q \times_{H,\theta} G \cong P$. If $E \longrightarrow B$ is a fiber bundle with fiber F and structure group G, the structure group of the bundle can be reduced to H if the associated principal G-bundle is induced from a principal H-bundle. Now we assume throughout the section that the base space B is a CW-complex. The following theorem shows when we obtain this situation.

Theorem 2.1.0.2 ([Mit01]) *Let $P \longrightarrow B$ is a principal G-bundle and H an admissible subgroup of G. Then the following are equivalent:*

- *P is induced from an H-bundle,*
- *$P \times_G (G/H) \longrightarrow B$ admits a section,*
- *The classifying map f of P lifts to BH, up to homotopy:*

$$\begin{tikzcd} & BH \arrow[d] \\ B \arrow[ur, dotted] \arrow[r, "f"] & BG. \end{tikzcd}$$

Many applications can be derived from this theorem. For example, if $G = GL_n\mathbb{R}, H = O(n)$, then G/H is homeomorphic to the group of upper triangular matrices with positive diagonal entries which is contractible. Hence any $GL_n\mathbb{R}$-bundle is induced from an $O(n)$-bundle. In other words, every real vector bundle admits a Euclidean metric. Furthermore, a real vector bundle with a Euclidean metric is orientable if and only if the structure group can be reduced from $O(n)$ to $SO(n)$. Since $O(n)/SO(n) = \{\pm 1\}$ is the set of orientations of $\mathbb{R}^n$, $P \times_{O(n)} (O(n)/SO(n))$ is just the usual orientation bundle.

Finally, we will mention homotopical properties of classifying spaces. Let G be a topological group. One can show that $\pi_n(BG) \cong \pi_{n-1}(G)$ for $n \geq 1$. Let H be an admissible subgroup of G. We have a fiber sequence up to homotopy:

$$G/H \xrightarrow{j} BH \longrightarrow BG,$$

where j classifies the principal H-bundle $G \longrightarrow G/H$. Assume additionally that H is normal. Then there is a fiber sequence up to homotopy:

$$BH \xrightarrow{Bi} BG \xrightarrow{B\rho} B(G/H).$$

We will see an application of this fiber sequence in section 2.4.

2.2. Smooth, Piecewise Linear, and Topological Manifolds

In this section, we will give an overview of the structure of manifolds and bundles in each of the following categories: smooth(O), piecewise linear(PL), and topological(TOP). These are interesting because the existence of different structures on a manifold is an important property that may be used in classification theorems. For example, in 1956 J. Milnor [Mil56] discovered *exotic spheres*: nondiffeomorphic manifolds which are homeomorphic to S^7. Milnor's construction easily generalizes to describe the other exotic structures on S^7. There are 28 distinct diffeomorphism types on S^7: 1 standard 7-sphere and 27 exotic 7-spheres. This started the interesting search for nondiffeomorphic, homeomorphic manifolds. Comparing the objects of the three categories, the simplest definition is that of a *topological manifold* but it is the hardest one to work with. A topological manifold is a locally Euclidean Hausdorff topological space. One may additionally require the second countable property. Secondly, a *smooth manifold* is a topological manifold with a globally defined smooth structure, see [Lee06] for a detailed definition. This is the most intricate definition but it is the easiest to work with. Now we will give the definition of a piecewise linear manifold. A *piecewise linear manifold*, or simply *PL manifold*, is a topological manifold together with a piecewise linear structure on it. This means roughly that one can pass from chart to chart by piecewise linear functions, see [HiMa74],[KiSi77] for a detailed definition.

By Whitehead's theorem on triangulations [Whi40], every smooth manifold admits a PL structure. However, PL manifolds do not always have smooth structures. Also, not every topological manifold admits a PL structure. The obstruction to a PL structure is the so-called *Kirby-Siebenmann class* which lies in the fourth cohomology group of the manifold with $\mathbb{Z}_2$ coefficients. Here a PL structure on a topological manifold M is a homeomorphism from M to a PL manifold N. Similarly, a smooth structure on a PL manifold M is a PL-homeomorphism from M to a smooth manifold N. Note that a PL-homeomorphism is a piecewise linear map which is also a homeomorphism.

For a smooth n-manifold M, we have the notion of a *tangent bundle*. This is a vector bundle over M classified by a map $M \longrightarrow BO(n)$ up to homotopy. Note that one can take the Grassmann manifold $G_n(\mathbb{R}^\infty)$ as a model for $BO(n)$. Similar to the notion of a tangent bundle in the smooth category, J. Milnor [Mil64] gave the notion of a *microbundle* or a *topological tangent bundle* in the topological category. Using his work, one obtains a one-to-one correspondence between isomorphism classes of microbundles and isomorphism classes of $\mathbb{R}^n$-*bundles with a zero section.* The latter are bundles with fiber $\mathbb{R}^n$ and structure group $TOP(n)$. Here $TOP(n)$ is the group of homeomorphisms from $\mathbb{R}^n$ onto itself fixing the origin with the compact-open topology. From Theorem 2.1.0.1, for a topological n-manifold M, the topological tangent bundle is classified by a map $M \longrightarrow BTOP(n)$ up to homotopy. Finally, the same argument can be applied to the PL category. For a PL n-manifold, there exists a one-to-one correspondence between isomorphism classes of *PL tangent bundles* and isomorphism classes of *PL* $\mathbb{R}^n$*-bundles with a zero section*, and hence a PL tangent bundle is classified by a map $M \longrightarrow BPL(n)$ up to homotopy. But $PL(n)$ is defined differently since the compact-open topology is not suitable here. Instead, we let $PL(n)$ be the simplicial group whose k-simplices consist of all PL homeomorphisms from $\Delta^k \times \mathbb{R}^n$ onto itself commuting with the projection on Δ^k and preserving a zero section.

Moreover, for any PL tangent bundle, we can forget the PL structure and obtain the forgetful map $BPL(n) \longrightarrow BTOP(n)$. Since any vector bundle can be given a PL structure, there also exists the forgetful map $BO(n) \longrightarrow BPL(n)$. Regarded as stable bundles, one obtains the following conclusion:

Category	Bundle Theory	Classifying Space	Equivalence Relation
Poincaré Complexes	Spherical Fibrations	BG	Homotopy Equivalence
Topological Manifolds	$\mathbb{R}^n$-bundles with a zero section	$BTOP$	Homeomorphism
PL Manifolds	PL $\mathbb{R}^n$-bundles with a zero section	BPL	PL-homeomorphism
Smooth Manifolds	Vector Bundles	BO	Diffeomorphism

We will discuss the first line in the next section and see that there also exists a forgetful map $BTOP(n) \longrightarrow BG(n)$.

Here are some important homotopy groups of the associated homotopy fibers of the forgetful maps. For $i \geq 0$,

- $\pi_i(G/TOP) = \{\mathbb{Z}, 0, \mathbb{Z}_2, 0, ...\}$. These homotopy groups are 4-periodic.
- $\pi_i(G/PL) = \{\mathbb{Z}, 0, \mathbb{Z}_2, 0, ...\}$. These homotopy groups are 4-periodic.
- TOP/PL is an Eilenberg-MacLane space $K(\mathbb{Z}_2, 3)$.
- $\pi_i(PL/O) = \{0, 0, 0, 0, 0, 0, 0, \mathbb{Z}_{28}, \mathbb{Z}_2, (\mathbb{Z}_2)^3, \mathbb{Z}_6, ...\}$. This classifies the exotic n-spheres, see [KeMi63] for details. For example, $\pi_7(PL/O) \cong \mathbb{Z}_{28}$ implies that there are 28 distinct smooth structures on S^7 up to orientation preserving diffeomorphism.

We finish this section with the following fundamental theorems of smoothing and a remark.

Theorem 2.2.0.3 ([HiMa74],[KiSi77]) *Let τ_M be the classifying map of the topological tangent bundle of a topological n-manifold M when $n \geq 5$. Then M admits a PL structure if and only if there exists a lift $M \longrightarrow BPL$ such that the following diagram commutes.*

$$\begin{tikzcd} & BPL \arrow[d] \\ M \arrow[ur, dotted] \arrow[r, "\tau_M"] & BTOP \end{tikzcd}$$

Theorem 2.2.0.4 ([HiMa74],[KiSi77]) *Let τ_M be the classifying map of the PL tangent bundle of a PL n-manifold M. Then M admits a smooth structure if and only if there exists a lift $M \longrightarrow BO$ such that the following diagram commutes.*

$$\begin{tikzcd} & BO \arrow[d] \\ M \arrow[ur, dotted] \arrow[r, "\tau_M"] & BPL \end{tikzcd}$$

Remark 2.2.0.1 *By obstruction theory and the above theorem, the fact that PL/O is 6-connected implies that any PL manifold of dimension seven or less admits a smooth structure.*

2.3. Poincaré Complexes, Spivak Normal Bundle, and Normal Map

A Poincaré complex is a generalization of a manifold. The following definition is based on [Wal67] and [Wal99]. A finite CW-complex X with fundamental group π and a homomorphism $w : \pi \longrightarrow \{\pm 1\}$ is called a *Poincaré complex of formal dimension n* if there is a class $[X] \in H_n(X; \mathbb{Z}_w)$ such that cap product with $[X]$ induces an isomorphism:

$$[X] \frown: H^*(X; \mathbb{Z}\pi) \longrightarrow H_{n-*}(X; (\mathbb{Z}\pi)_w),$$

where $\mathbb{Z}\pi$ is the integral group ring consisting of finite formal integer linear combinations of elements g of π: $\sum n(g)g$. Here $\mathbb{Z}_w$ and $(\mathbb{Z}\pi)_w$ are the groups $\mathbb{Z}$ and $\mathbb{Z}\pi$ with right $\mathbb{Z}\pi$-module structure induced by an involution on $\mathbb{Z}\pi$: $\sum n(g)g \mapsto \sum w(g)n(g)g^{-1}$. We call w the *orientation homomorphism* and X is called orientable if w is trivial. Also, $[X]$ is called the *fundamental class*. Next, we have a notion of Poincaré pairs as well. Let (X, A) be a finite CW-pair. Assume that X is connected and equipped with a homomorphism $w : \pi \longrightarrow \{\pm 1\}$. Then (X, A) is a *Poincaré n-pair* if there is a class $[X] \in H_n(X, A; \mathbb{Z}_w)$ such that the cap product with $[X]$ induces an isomorphism:

$$[X] \frown: H^*(X; \mathbb{Z}\pi) \longrightarrow H_{n-*}(X, A; (\mathbb{Z}\pi)_w).$$

Moreover, it is required that $\partial_*([X]) \in H_{n-1}(A; \mathbb{Z}_w)$ equips A with the structure of a Poincaré complex, where the orientation homomorphism on A is the one induced by the orientation homomorphism on X. C. T. C. Wall [Wal99] showed that every closed manifold has this property and therefore has the structure of a Poincaré Complex. Also, if M is a compact orientable manifold with boundary ∂M, then $(M, \partial M)$ is a Poincaré pair. The following proposition is a simple example of building Poincaré complexes.

Proposition 2.3.0.2 ([Kle00]) *If $(M, \partial M)$ and $(N, \partial N)$ are n-manifolds with boundary or more generally, Poincaré pairs, and $h : \partial M \longrightarrow \partial N$ is a homotopy equivalence, then the amalgamated union $M \cup_h N$ is a Poincaré n-complex.*

Here this amalgamated union is the union $M \cup N$ such that x and $h(x)$ are identified for all $x \in \partial M$. From the previous section, we know that every manifold is equipped with a tangent bundle. In contrast, for a Poincaré complex there is no notion of a tangent bundle, but a Poincaré complex does have a stable normal bundle, called the Spivak normal bundle. We will discuss this later. For now, we describe the analogous bundle theory for Poincaré complexes. Let X be a CW-complex. A *spherical fibration* $p : E \longrightarrow X$ is a fibration whose fiber is homotopy equivalent to the sphere S^{n-1} for some n. Two spherical fibrations $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$ are *fiber homotopy equivalent* if there is a homotopy equivalence $h : E \longrightarrow E'$ which is fiber preserving in the sense that $p = p' \circ h$. This gives an equivalence relation. Similar to the other categories, a spherical fibration is classified by a map $X \longrightarrow BG(n)$ up to homotopy, where $G(n)$ denotes the set of self-homotopy equivalences of S^{n-1} with the compact-open topology. Suppose $p : E \longrightarrow X$ is an $\mathbb{R}^n$-bundle and $s : X \longrightarrow E$ is the zero section. Then the restricted bundle map $(E - s(X)) \longrightarrow X$ is a fibration whose fiber is homotopy equivalent to S^{n-1}. This implies that there is the forgetful map $BTOP(n) \longrightarrow BG(n)$.

Every manifold M has a *normal bundle*. It is a Whitney sum inverse for the tangent bundle. A normal bundle is not uniquely determined, but the stable class is. Here the stable bundle is classified by the composition of its classifying map and the usual inclusion $BO(n) \longrightarrow BO$, or $BPL(n) \longrightarrow BPL$, or $BTOP(n) \longrightarrow BTOP$. Then there is a well-defined homotopy class $\nu_M \longrightarrow BO$, or $\nu_M \longrightarrow BPL$, or $\nu_M \longrightarrow BTOP$. This is called the *stable normal bundle*. Now let X be a Poincaré n-complex. By the simplicial approximation theorem, X is homotopy equivalent to a finite simplicial n-complex K which embeds in $\mathbb{R}^{n+k}$ for any $k \geq n+1$. Let $N(K)$ be a regular (or tubular) neighborhood of K. Then K is a strong deformation retract of $N(K)$ and $\partial N(K)$ is a $(n+k-1)$-manifold. Note that if X is a smooth manifold and the embedding $X \subset \mathbb{R}^l$ is smooth, then $N(X)$ can be identified with the total space of the normal disc bundle to X in $\mathbb{R}^l$ and $\partial N(X)$ is the total space of the normal sphere bundle. This also holds for PL category. M. Spivak [Spi67] also showed that

$$\partial N(K) \longrightarrow N(K) \longrightarrow K \xrightarrow{\simeq} X$$

is a spherical fibration with homotopy fiber S^{k-1}. By bundle theory, there is the corresponding classifying map $\nu^k : X \longrightarrow BG(k)$. In particular, one can show that the stable class

$$\nu : X \longrightarrow BG$$

is independent of the choices of the embedding and the regular neighborhood. This bundle ν is called the *Spivak normal bundle* of X. As mentioned above in the case that X is already a smooth or PL manifold, ν is classified by a map into BO or BPL. Thus, a necessary condition for a Poincaré complex to have the homotopy type of a smooth or PL manifold is that ν admits a reduction to BO and BPL:

$$\begin{tikzcd} & BO \arrow[d, "\pi"] \\ X \arrow[ur, dotted, "\xi"] \arrow[r, "\nu"'] & BG \end{tikzcd} \qquad \begin{tikzcd} & BPL \arrow[d, "\pi"] \\ X \arrow[ur, dotted, "\xi"] \arrow[r, "\nu"'] & BG \end{tikzcd}$$

Such a reduction is often called a *normal invariant*. This is the first obstruction to a Poincaré complex having such a property. Using the fibrations

$$BO \xrightarrow{\pi} BG \longrightarrow B(G/O), \qquad BPL \xrightarrow{\pi} BG \longrightarrow B(G/PL),$$

this obstruction is an element in $[X, B(G/O)]$ or in $[X, B(G/PL)]$. Next, we will state the definition of a degree one normal map and give some results that we need in this dissertation. Note that this will lead to a second obstruction, called the surgery obstruction.

Let (X, Y) be a Poincaré n-pair. Here Y may be empty. Let $(M, \partial M)$ be a compact oriented n-manifold with boundary. One can embed $(M, \partial M)$ in (D^{n+k}, S^{n+k-1}) for large k. Then there exists the normal bundle $\nu_M^k : \nu(M) \longrightarrow M$ such that $\nu_M^k|_{\partial M}$ is the normal bundle of ∂M in S^{n+k-1}. For a $\mathbb{R}^k$-bundle $\xi^k : E \longrightarrow X$, a *degree one normal map* is a map $f : (M, \partial M) \longrightarrow (X, Y)$ of degree one together with a bundle map $\hat{f} : \nu(M) \longrightarrow E$ covering f:

$$\begin{tikzcd}
\nu(M) \arrow[r, "\hat{f}"] \arrow[d, "\nu_M^k"'] & E \arrow[d, "\xi^k"] \\
M \arrow[r, "f"] & X
\end{tikzcd}$$

Here a map is called *of degree one* if it preserves fundamental classes. In this case, $f_*([M, \partial M]) = [X, Y]$. Moreover, there is a notion of a *normal cobordism* and a *normal cobordism relative to* Y but we omit this definition here. This yields the set of cobordism classes of degree one normal maps for a bundle isomorphism ξ, where k and X are fixed. This also leads to the classical surgery problem: Given a normal map $(f, \hat{f})$, when is $(f, \hat{f})$ normally cobordant relative to Y to $(f', \hat{f}')$ where f' is a homotopy equivalence? From [Bro72] and [MaMi79], we can always make f to be $[\frac{n}{2}]$-connected and the *surgery obstruction* to the above question can be described as the following invariant.

Theorem 2.3.0.5 ([Bro72]) *Let $(f, \hat{f})$ be a normal map such that $f|_{\partial M}$ induces an isomorphism on homology. There is an invariant $\sigma(f, \hat{f})$ defined as, $\sigma = 0$ if n is odd, $\sigma = sign(M) - sign(X) \in 8\mathbb{Z}$ if $n = 4l$, and $\sigma \in \mathbb{Z}_2$ if $n = 4l + 2$. Moreover, if X is simply connected and $n \geq 5$, then $(f, \hat{f})$ is normally cobordant relative to Y to $(f', \hat{f}')$ where $f' : M' \longrightarrow X$ is a homotopy equivalence if and only if $\sigma(f, \hat{f}) = 0$.*

Now let $\xi^k : E \longrightarrow X$ be a reduction of the Spivak normal bundle to BO or BPL. By the usual transversality arguments and process of surgery [MaMi79], we have a degree one normal map $(f, \hat{f})$ where $\hat{f} : \nu(M) \longrightarrow E$ covering $f : M \longrightarrow X$ and M is a smooth or PL n-submanifold of a disc D^{n+k}, with boundary ∂M if $Y \subset X$ is not empty. Therefore, the above arguments can be applied to this bundle. For example, we can assume that f is $[\frac{n}{2}]$-connected and $\mathrm{sign}(M) - \mathrm{sign}(X)$ is divisible by 8 if $n = 4l$ and $l \geq 2$.

2.4. Spin Structure

Before giving the meaning of a spin structure, we will describe shortly the spin group and some general facts. One way to construct groups in $\mathbb{R}^n$ is to consider a finite dimensional real algebra $\mathcal{A}$ and the group given by the set of units in $\mathcal{A}$. For example, using the algebra $M_n\mathbb{R}$ consisting of all $(n \times n)$-matrices, we have the group of units $GL_n\mathbb{R}$ and the important subgroup $SO(n)$. The *spin group* denoted by $Spin(n)$ is a subgroup of the group of units in the Clifford algebra C_n. The *Clifford algebra* C_n is a real algebra of dimension 2^n generated by $e_1, e_2, ..., e_n$ such that $e_i^2 = -1$ and $e_ie_j = -e_je_i$ if $i \neq j$. For example, $C_0 = \mathbb{R},\ C_1 = \mathbb{C},\ C_2 = \mathbb{H}$. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ generated by $e_1, e_2, ..., e_n$. For each $x = a_1e_1 + a_2e_2 + ... + a_ne_n \in S^{n-1}$,

$$\begin{aligned}(x)(-x) &= (a_1e_1 + a_2e_2 + ... + a_ne_n)((-a_1)e_1 + (-a_2)e_2 + ... + (-a_n)e_n) \\ &= a_1^2 + a_2^2 + ... + a_3^2 = 1.\end{aligned}$$

This implies that S^{n-1} is a subset of the group C_n^* of units in C_n. Now let $Pin(n)$ be the subgroup of C_n^* generated by S^{n-1}. As described in [Cur84], one can construct the two-sheeted covering $\rho : Pin(n) \longrightarrow O(n)$. We then define $Spin(n) := \rho^{-1}(SO(n))$ and we obtain a short exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{i} Spin(n) \xrightarrow{\rho} SO(n) \longrightarrow 1.$$

The first few examples are

$$Spin(1) = O(1),\ Spin(2) = U(1),\ Spin(3) = Sp(1).$$

Let M be an orientable n-manifold. A vector bundle ξ over M admits a *spin structure* if and only if the associated $SO(n)$-bundle is induced from a $Spin(n)$-bundle.

We call M a spin manifold if there exists a spin structure on its tangent bundle. From section 2.1, the short exact sequence above gives rise to a fiber sequence:

$$K(\mathbb{Z}_2,1) \xrightarrow{Bi} BSpin(n) \xrightarrow{B\rho} BSO(n).$$

Let $f : M \longrightarrow BSO(n)$ be a classifying map for ξ. The obstructions to obtain a lift in the diagram:

$$\begin{tikzcd} & BSpin(n) \arrow[d, "B\rho"] \\ M \arrow[ur, dotted] \arrow[r, "f"] & BSO(n) \end{tikzcd}$$

lie in $H^j(M;\pi_{j-1}(K(\mathbb{Z}_2,1)))$ which are all zero except for $j=2$. Now we consider $BSO(n)$ as a model for M, i.e. f as the identity. Then the universal obstruction class lies in $H^2(BSO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$ which is generated by the second Stiefel-Whitney class $w_2(\gamma^n)$. If it was zero, then $BSpin(n) \longrightarrow BSO(n)$ would admit a section, and hence $K(\mathbb{Z}_2,1) \longrightarrow BSpin(n)$ would induce a monomorphism on homotopy groups. This contradicts the fact that $BSpin(n)$ is simply connected. Therefore, ξ admits a spin structure if and only if the second Stiefel-Whitney class vanishes, i.e. $w_2(\xi) = 0$. Furthermore, if $w_2(\xi) = 0$, there is a one-to-one correspondence between spin structures and elements of $H^1(M;\mathbb{Z})$, see [LaMi89].

In [LaMi89], they also show that any submanifold of a spin manifold with a spin structure on its normal bundle is canonically a spin manifold. In particular, if M is a compact manifold with boundary ∂M, then using the field of interior unit normal vectors along ∂M, one gets an embedding of the total space of oriented orthonormal frames of ∂M into the total space of oriented orthonormal frames of M. The spin structure, regarded as the two-sheeted covering of this total space, can be restricted to ∂M. Therefore, the following theorem is obtained.

Theorem 2.4.0.6 ([LaMi89]) *Let M be a compact manifold with boundary ∂M. Then any spin structure on M induces a spin structure on ∂M.*

2.5. Biquotients

A *homogeneous G-space* or simply *homogeneous space M* is a smooth manifold equipped with a transitive smooth action by a Lie group G. If p is any point of M, the homogeneous space characterization theorem [Lee06] implies that the map

$$F : G/G_p \longrightarrow M,$$

defined by $F(gG_p) = g \cdot p$ is an equivariant diffeomorphism. Here G_p is the isotropy group of the G-action which is a closed embedded Lie subgroup of G. For example, the natural action of $O(n)$ on S^{n-1} is transitive, the isotropy group at the north pole is $O(n-1)$, and hence S^{n-1} is diffeomorphic to $O(n)/O(n-1)$. Alternatively, because of this theorem, one can define a homogeneous space to be a quotient manifold of the form G/H, where G is a Lie group and H is a closed Lie subgroup of G.

In 1982, biquotients were introduced by J. H. Eschenburg. They are a generalization of homogeneous spaces which are not homogeneous in general. This leads to new interesting manifolds that have strong curvature properties. For example, Eschenburg spaces are an infinite family of 7-manifolds and most of them admit positive sectional curvature. They are a generalization of homogeneous Aloff-Wallach spaces and their explicit definition will be described in section 3.1. In [Esb84], new biquotients in dimension six were found by J. H. Eschenburg. Also, in 1996 Y. Bazaikin [Baz96] introduced a new infinite family of biquotients in dimension 13, now called Bazaikin spaces. Similarly, most of two latter families admit metrics of positive sectional curvature. Biquotients are defined as follows.

Let G be a compact Lie group and $U \subset G \times G$ a closed subgroup. Define an action of U on G by

$$(u_1, u_2) \cdot g = u_1 g u_2^{-1},$$

where $(u_1, u_2) \in U$ and $g \in G$. The orbit space of this action is called a *biquotient*, denoted by $G//U$. If the action is free, this space becomes a smooth manifold. The freeness can be described by the following lemma:

Lemma 2.5.0.1 *The biquotient action is free if and only if u_1 and u_2 are not conjugate for any $(u_1, u_2) \in U \backslash \{(e, e)\}$.*

In [Sin93], W. Singhof defined the notion of a *double coset manifold* and one can show that a biquotient can be regarded as a double coset manifold. For a compact Lie group G together with closed subgroups H, K, the following properties are equivalent:

- $H \times K$ acts freely from the left on G by $(h, k) \cdot g = hgk^{-1}$.
- H acts freely from the left on G/K by $h \cdot gK = (hg)K$.
- K acts freely from the right on $H \backslash G$ by $Hg \cdot k = H(gk)$.
- G acts freely from the left on $(H \backslash G) \times (G/K)$ by

$$g \cdot (Hg_1, g_2K) = (H(g_1 g^{-1}), (gg_2)K).$$

- $h \in H$ and $k \in K$ are conjugate, then $h = k = e$.

If one of these conditions (and hence all conditions) is satisfied, the orbit space becomes a compact smooth manifold, denoted by $H \backslash G/K$. Section (1.7) in [Sin93] implies that if U is a closed subgroup of $G \times G$ satisfying the freeness condition, then we have a diffeomorphism:

$$G//U \cong U \backslash (G \times G)/\Delta G,$$

where $\Delta G = \{(g, g) \mid g \in G\}$.

2.6. Linking Form

The *linking form* of an oriented n-manifold is a bilinear map on the torsion subgroups of $H^p(M)$ and $H^q(M)$ with values in $\mathbb{Q}/\mathbb{Z}$:

$$\mathrm{Tor}(H^p(M)) \times \mathrm{Tor}(H^q(M)) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

where $p+q=n+1$. The following description of the linking form is based on [Bar65].

Remark 2.6.0.2 *In [Bar65], Barden defined the linking form in terms of homology. However, we can obtain an equivalent definition in terms of cohomology since there is 1-1 correspondence between $H^k(M)$ and $H_{n-k}(M)$ under the Poincaré duality isomorphism.*

Let M be an n-manifold having an orientation $\mu \in H_n(M)$ so that $\mu \frown$ gives the duality isomorphism:

$$\mu \frown : H^k(M) \longrightarrow H_{n-k}(M).$$

Let $a \in H^p(M)$ and $b \in H^q(M)$ be torsion elements where $p+q=n+1$. Then $\mu \frown a$ is an element in $H_{q-1}(M)$. Note that since a is torsion, so is $\mu \frown a$. Let β be the Bockstein homomorphism which is associated to the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{j} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Consider the associated long exact sequence:

$$\cdots \longrightarrow H^{q-1}(M;\mathbb{Q}) \xrightarrow{j^*} H^{q-1}(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H^q(M;\mathbb{Z}) \xrightarrow{i^*} H^q(M;\mathbb{Q}) \longrightarrow \cdots.$$

Suppose that there exist c_1 and c_2 in $H^{q-1}(M;\mathbb{Q}/\mathbb{Z})$ such that $\beta(c_1)=\beta(c_2)=b$. The existence of c_1 and c_2 follows from the fact that b is a torsion element. Now, $c_1 - c_2 = j^*(d)$ for some $d \in H^{q-1}(M;\mathbb{Q})$ by exactness. Then

$$\langle c_1, \mu \frown a\rangle - \langle c_2, \mu \frown a\rangle = \langle j^*(d), \mu \frown a\rangle = j\,\langle d, \mu \frown a\rangle.$$

Here $\langle \varphi, \alpha \rangle$ represents evaluation of a cohomology class φ at a homology class α. Since $\mu \frown a$ is a torsion element, the above difference is zero. This implies well-definedness of

$$\left\langle \beta^{-1}(b), \mu \frown a \right\rangle \in \mathbb{Q}/\mathbb{Z},$$

which is equal to

$$\left\langle a \smile \beta^{-1}(b), \mu \right\rangle \in \mathbb{Q}/\mathbb{Z},$$

by the canonical relation between the cup and cap products. This yields the following definition.

Definition 2.6.0.1 *Let M be an n-manifold with an orientation μ, and $a \in H^p(M)$, $b \in H^q(M)$ be torsion elements where $p + q = n + 1$. Then the **linking number** of a with b is*

$$L(a, b) := \left\langle a \smile \beta^{-1}(b), \mu \right\rangle \in \mathbb{Q}/\mathbb{Z},$$

where β is the Bockstein homomorphism associated to a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Lemma D in [Bar65] translates to the cohomological definition of the linking form as follows.

Proposition 2.6.0.3 *With the above notations,*

- *L is a non-singular bilinear form on the torsion subgroups of $H^p(M)$ and $H^q(M)$.*
- *$L(a, b) + (-1)^{(p-1)(q-1)} L(b, a) = 0$ where a and b are torsion elements of $H^p(M)$ and $H^q(M)$, respectively.*

3. MANIFOLDS OF TYPE R

In the first section, we will introduce the Eschenburg spaces and the generalized Witten manifolds which have particular types of cohomology rings. As we will see these are examples of manifolds of type r. The homotopy type of a manifold M of type r is known, see [Kru97]. In section 3.2, we will describe this homotopy type and a homotopy invariant $s(M)$. In the last section we will display a connection between the self-linking number and this invariant.

3.1. Definition and Examples

After S. Aloff and N. Wallach [AlWa75] introduced an infinite family of 7-manifolds admitting a homogeneous Riemannnian metric with positive sectional curvature, J. H. Eschenburg was looking for further spaces of positive sectional curvature. In 1982, he defined a biquotient and introduced a new infinite family of 7-manifolds which consists of inhomogeneous spaces in general. These are now called *Eschenburg spaces.* They can be defined as biquotients as follows.

Let $k := (k_1, k_2, k_3)$, $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ be such that $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$. Also, let $H \subset U(3) \times U(3)$ defined by

$$H := \{(\rho_k(z), \rho_l(z)) \mid z \in S^1\},$$

where $\rho_k(z) := \mathrm{diag}(z^{k_1}, z^{k_2}, z^{k_3})$ and $\mathrm{diag} : S^1 \times S^1 \times S^1 \longrightarrow U(3)$ is the embedding defined by

$$(z, v, w) \mapsto \begin{pmatrix} z & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix}.$$

The Eschenburg space $E_{k,l}$ is the biquotient:

$$U(3)//H.$$

By Lemma 2.5.0.1, the action is free if and only if $\rho_k(z)$ is not a conjugate of $\rho_l(z)$. Hence, $\rho_k(z) \neq \sigma(\rho_l(z))$ for all permutations $\sigma \in S_3$. The last statement can be easily translated into the following conditions:

$$\begin{aligned} &\gcd(k_1 - l_1, k_2 - l_2) = 1, \;\; \gcd(k_1 - l_1, k_3 - l_2) = 1, \\ &\gcd(k_2 - l_1, k_1 - l_2) = 1, \;\; \gcd(k_2 - l_1, k_3 - l_2) = 1, \\ &\gcd(k_3 - l_1, k_1 - l_2) = 1, \;\; \gcd(k_1 - l_1, k_2 - l_2) = 1. \end{aligned} \tag{G}$$

Alternatively, by the condition that $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$, the spaces $E_{k,l}$ can be regarded as a quotient of two-sided action induced by $\rho_{k,l}$ of S^1 on $SU(3)$:

$$E_{k,l} = SU(3)/S^1,$$

where $\rho_{k,l} := (\rho_k, \rho_l)$ induces an action $S^1 \times SU(3) \longrightarrow SU(3)$ by

$$(z, A) \longmapsto \rho_k(z) A \rho_l(z^{-1}).$$

Also, the quotient is free if and only if the *condition (G)* holds. In section 5.1, we will see that these two definitions give the same space. Note that the case $l_i = 0$ for all i corresponds to the homogeneous *Aloff-Wallach spaces.*

Moreover, in [Esb82] it was shown that the spaces $E_{k,l}$ admit a metric, induced by a certain left invariant metric on $SU(3)$, of positive sectional curvature if and only if

$$k_i \notin [\min(l_1, l_2, l_3), \max(l_1, l_2, l_3)] \text{ for all } i = 1, 2, 3,$$

or

$$l_i \notin [\min(k_1, k_2, k_3), \max(k_1, k_2, k_3)] \text{ for all } i = 1, 2, 3.$$

Chapter 5 is devoted to a careful explanation of the Eschenburg spaces. At present, we only state that the cohomology ring of $E_{k,l}$ is as follows:

$$H^0(E_{k,l}) \cong H^2(E_{k,l}) \cong H^5(E_{k,l}) \cong H^7(E_{k,l}) \cong \mathbb{Z},$$

$$H^4(E_{k,l}) \cong \mathbb{Z}_r, \text{ and } H^1(E_{k,l}) = H^3(E_{k,l}) = H^6(E_{k,l}) = 0,$$

where r is the order of the fourth cohomology group and r is always odd. Furthermore, if u is a generator of $H^2(E_{k,l})$, then u^2 is a generator of $H^4(E_{k,l})$. Following [CEZ07], Maple and C code were used to completely classify most positively curved Eschenburg space $E_{k,l}$ up to homeomorphism and diffeomorphism. Their work is based on [Kru05] and the following propositions proved by [CEZ07] are very important.

Proposition 3.1.0.4 ([CEZ07]) *Each positively curved Eschenburg space $E_{k,l}$ has the following unique representation:*

$$k = (k_1, k_2, l_1 + l_2 - k_1 - k_2) \text{ and } l = (l_1, l_2, 0),$$

with $k_1 \geq k_2 > l_1 \geq l_2 \geq 0$.

Proposition 3.1.0.5 ([CEZ07]) *For each odd $r \in \mathbb{Z}$, there are only finitely many positively curved Eschenburg spaces $E_{k,l}$ with $H^4(E_{k,l}) = \mathbb{Z}_{|r|}$.*

Besides the Eschenburg spaces, there are other families of 7-manifolds that have the same type of cohomology ring as the Eschenburg spaces. One such family is described below. Moreover, M. Kreck and S. Stolz [KrSt88] give a classification theorem of 7-manifolds of this type. This motivates the following definition.

Definition 3.1.0.2 *A manifold M is called a* ***manifold of type*** *r where $r > 0$ if M is a simply connected closed 7-dimensional oriented manifold whose cohomology ring satisfies the following:*

$$H^0(M) \cong H^2(M) \cong H^5(M) \cong H^7(M) \cong \mathbb{Z},$$

$$H^4(M) \cong \mathbb{Z}_r, \text{ and } H^1(M) = H^3(M) = H^6(M) = 0,$$

such that if u is a generator of $H^2(M)$, then u^2 is a generator of $H^4(M)$.

Remark 3.1.0.3 *A manifold of type r can be considered as a smooth or topological manifold. Following [KrSt91], there exists a classification theorem for both. However, our work focuses on the smooth category.*

The other infinite family of manifolds of type r that we mentioned above is called the family of *generalized Witten manifolds*. Following [Kru97], they can be described as the quotient space of an S^1-action on $S^5 \times S^3$:

$$S^1 \times S^5 \times S^3 \longrightarrow S^5 \times S^3 \subset \mathbb{C}^3 \times \mathbb{C}^2,$$

by

$$(z, ((u_1, u_2, u_3), (v_1, v_2))) \longmapsto ((z^{k_1}u_1, z^{k_2}u_2, z^{k_3}u_3), (z^{l_1}v_1, z^{l_2}v_2)),$$

where $k := (k_1, k_2, k_3) \in \mathbb{Z}^3$, $l := (l_1, l_2) \in \mathbb{Z}^2$. The action is free if $\gcd(k_i, l_j) = 1$ for all i, j and $l_1 l_2 \neq 0$. Then the quotient becomes a manifold which is denoted by $M_{k,l}$. Using the arguments in [Sin93], one can show that $M_{k,l}$ is a manifold of type $r = |l_1 l_2|$ with a generator $u \in H^2(M_{k,l})$. Also,

$$p_1(M_{k,l}) = (k_1^2 + k_2^2 + k_3^2 + l_1^2 + l_2^2)u^2,$$

and

$$w_2(M_{k,l}) = (k_1 + k_2 + k_3 + l_1 + l_2)u.$$

More details can be found in [Kru95] and [Esc05]. We notice that if $k_1 = k_2 = k_3$ and $l_1 = l_2$, these are homogeneous spaces, called *Witten manifolds*, studied by E. Witten [Wit81] and they are classified in [KrSt88]. Furthermore, the generalized Witten manifolds are classified in [Esc05] for the case $k_1 = k_2 = k_3$.

3.2. The Invariant $s(M)$ and Homotopy Type

Let M be a smooth manifold of type r. Then M is a simply connected closed 7-dimensional oriented smooth manifold and its cohomology groups are

$$H^0(M) \cong H^2(M) \cong H^5(M) \cong H^7(M) \cong \mathbb{Z},$$

$$H^4(M) \cong \mathbb{Z}_r, \text{ and } H^1(M) = H^3(M) = H^6(M) = 0,$$

together with generators u and u^2 of $H^2(M)$ and $H^4(M)$, respectively. To obtain the homotopy type, we start with a circle bundle:

$$S^1 \longrightarrow G \xrightarrow{\pi} M,$$

with first Chern class (= the Euler class) $u \in H^2(M)$. This bundle exists because $H^2(M)$ is isomorphic to the group $[M, K(\mathbb{Z}, 2)] = [M, BS^1]$ of homotopy classes from M to BS^1. One can then regard u as a map $M \longrightarrow BS^1$. Applying the Gysin sequence, one obtains the long exact sequence:

$$\cdots \longrightarrow H^{i-2}(M) \xrightarrow{\smile u} H^i(M) \xrightarrow{\pi^*} H^i(G) \longrightarrow H^{i-1}(M) \longrightarrow \cdots ,$$

and can show that $H^*(G)$ is the exterior algebra $E(x_3, x_5)$ with two generators $x_3 \in H^3(G)$ and $x_5 \in H^5(G)$. Now consider the fiber sequence:

$$G \xrightarrow{\pi} M \xrightarrow{\eta} BS^1, \qquad (*)$$

where η is a classifying map such that η maps the universal Chern class c_1 to u. Now we will study the associated Serre spectral sequence. Starting with the E^2-term, we have

$$E^4 = E^3 = E^2 = H^*(BS^1) \otimes H^*(G) = \mathbb{Z}[c_1] \otimes E(x_3, x_5),$$

with $d^4(1 \otimes x_3) = \pm r c_1^2 \otimes 1$ since $H^4(M) \cong \mathbb{Z}_r$. Also,

$$E^6 = E^5 = \mathbb{Z}[c_1]/rc_1^2 \otimes E(x_5),$$

with $d^6(1 \otimes x_5) = s(M)c_1^3 \otimes 1$ for some integer $s(M)$ such that $\gcd(s(M), r) = 1$ since $H^6(M) = 0$ (such d^6 must be surjective). Let $(\mathbb{Z}_r)^*$ be the set of units in $\mathbb{Z}_r$. Then $s(M) \in (\mathbb{Z}_r)^*$. To take care of the choice of the generators, we have to factor out the subgroup ± 1 of $(\mathbb{Z}_r)^*$. This implies the following:

Proposition 3.2.0.6 ([Kru97]) *The number $s(M) \in (\mathbb{Z}_r)^*/(\pm 1)$ depends only on the homotopy type of M.*

By the fiber sequence $(*)$, it is easy to see that

$$\pi_i(G) \cong \pi_i(M) \text{ for } i \geq 3,$$

and by the Hurewicz theorem,

$$\pi_2(M) \cong H_2(M) \cong \mathbb{Z}.$$

Since $H^*(G) = H^*(S^3 \times S^5)$, one can show that G is equivalent to a CW-complex:

$$S^3 \cup_\alpha e^5 \cup e^8,$$

where $\alpha \in \pi_4(S^3) \cong \mathbb{Z}_2$ is the attaching map of the 5 cell to the 3-skeleton. If $\alpha \neq 0$, then $\pi_4(M) = \pi_4(G) = 0$ and G has the homotopy type of $SU(3)$. If $\alpha = 0$, then $\pi_4(M) \cong \pi_4(G) \cong \mathbb{Z}_2$ and G has the homotopy type of $S^3 \times S^5$. One can conclude that if M is a smooth manifold of type r, then the homotopy type of M can be divided into two cases:

- $\pi_4(M) = 0,\ \pi_2(M) \cong \mathbb{Z},\ \pi_i(M) \cong \pi_i(SU(3))$ for $i \geq 3$.
- $\pi_4(M) \cong \mathbb{Z}_2,\ \pi_2(M) \cong \mathbb{Z},\ \pi_i(M) \cong \pi_i(S^3 \times S^5)$ for $i \geq 3$.

Remark 3.2.0.4 *The Eschenburg spaces satisfy $\pi_4(E_{k,l}) = 0$ and the generalized Witten manifolds satisfy $\pi_4(M_{k,l}) \cong \mathbb{Z}_2$.*

In particular, B. Kruggel [Kru97] showed that M has a cell structure as follows:

Proposition 3.2.0.7 ([Kru97]) *Let M be a smooth manifold of type r. Then M is homotopy equivalent to a CW-complex:*

$$S^2 \vee S^3 \cup_{\beta_4} e^4 \cup_{\beta_5} e^5 \cup_{\beta_7} e^7,$$

where the attaching maps β_4, β_5, and β_7 are defined as follows.

$$\beta_4 = \nu_2 - r i_3 \in \pi_3(S^2 \vee S^3),$$

$$\beta_5 = \epsilon\psi + \lambda[i_2, i_3] \in \pi_4(S^2 \vee S^3 \cup_{\beta_4} e^4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_r.$$

If $\pi_4(M) = 0$, β_7 is a generator of a subgroup of index 6 of

$$\pi_6(S^2 \vee S^3 \cup_{\beta_4} e^4 \cup_{\beta_5} e^5) \cong \mathbb{Z} \oplus \mathbb{Z}_6,$$

and if $\pi_4(M) = \mathbb{Z}_2$, β_7 is a generator of a subgroup of index 24 of

$$\pi_6(S^2 \vee S^3 \cup_{\beta_4} e^4 \cup_{\beta_5} e^5) \cong \mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2,$$

where $\nu_2 : S^3 \longrightarrow S^2$ is the Hopf map, $i_2 : S^2 \longrightarrow S^2 \vee S^3, i_3 : S^3 \longrightarrow S^2 \vee S^3$ are the inclusion, $[i_2, i_3]$ is the Whitehead product, ψ is the image of the generator of $\pi_4(S^3)$ under the inclusion map $S^3 \hookrightarrow S^2 \vee S^3 \cup_{\beta_4} e^4$, $\epsilon \in \mathbb{Z}_2$, and $\lambda \in (\mathbb{Z}_r)^*$.

Remark 3.2.0.5 *If $\pi_4(M) = 0$, the exact sequence of the pair (M^5, M^4) between its 5-skeleton and 4-skeleton shows that $\pi_4(M^4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_r$ is cyclic, see [Kru97] for details. Therefore, r must be odd. By the previous remark, the Eschenburg spaces satisfy $H^4(E_{k,l}) \cong \mathbb{Z}_r$, where r is odd. Alternatively, the fact that the Eschenburg spaces are the manifolds of odd type can also be proved in Proposition 5.1.0.12 using an elementary method.*

3.3. The Self-Linking Number and $s(M)$

Let M be a smooth manifold of type r and $u \in H^2(M)$ a generator. Then u^2 is torsion since $H^4(M) \cong \mathbb{Z}_r$. By Definition 2.6.0.1, there exists the linking number of u^2 with itself:

$$L(u^2, u^2) = \left\langle u^2 \smile \beta^{-1}(u^2), [M] \right\rangle,$$

where β is the Bockstein homomorphism associated to a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

In this situation,

$$\beta : H^3(M; \mathbb{Q}/\mathbb{Z}) \longrightarrow H^4(M; \mathbb{Z}).$$

This gives rise to the following definition:

Definition 3.3.0.3 *For a smooth manifold M of type r with a generator u of $H^2(M)$,*

$$L(u^2, u^2) = \left\langle u^2 \smile \beta^{-1}(u^2), [M] \right\rangle \in \mathbb{Q}/\mathbb{Z}$$

*is called the **self-linking number** of u^2.*

In this section, we will see how $s(M)$ determines the self-linking number. Starting with a circle bundle

$$S^1 \longrightarrow G \xrightarrow{\pi} M,$$

with first Chern class (= the Euler class) $u \in H^2(M)$, we have the fiber sequence:

$$G \xrightarrow{\pi} M \xrightarrow{\eta} BS^1, \qquad (*)$$

where η is a classifying map such that η^* maps the universal Chern class c_1 to u. First, we will show how the transgression homomorphism of $(*)$ is defined. We identify M as the pullback under η of the universal disc bundle over BS^1:

$$D^2 \longrightarrow ES^1 \times_{S^1} D^2 \longrightarrow BS^1.$$

Then M is a manifold with boundary G and we can consider the composition:

$$H^q(G) \xrightarrow{\delta} H^{q+1}(M,G) \xleftarrow{\eta^*} H^{q+1}(BS^1),$$

where δ is the coboundary of a pair (M,G). Then the transgression homomorphism τ is a map from a subgroup of $H^q(G)$ to a quotient group of $H^{q+1}(BS^1)$:

$$\tau : \delta^{-1}(\mathrm{Im}(\eta^*)) \longrightarrow H^{q+1}(BS^1)/\mathrm{Ker}(\eta^*).$$

Next, we will see that τ is closely related to the integration along the fiber:

$$\pi_! : H^q(G) \longrightarrow H^{q-1}(M),$$

of the circle bundle $S^1 \longrightarrow G \xrightarrow{\pi} M$. Here the integration along the fiber is the same map as the corresponding one in the Gysin sequence. Consider an orientation disk bundle:

$$(D^2, S^1) \longrightarrow (M,G) \longrightarrow M,$$

see [Hat02] for a detailed definition. There exist the Thom class $U_\eta \in H^2(M,G)$ and the Thom isomorphism:

$$H^{q-1}(M) \xrightarrow{\Phi_\eta} H^{q+1}(M,G).$$

Hence, the commutative diagram

$$\begin{tikzcd}
H^q(M,G) \arrow[r, "j^*"] & H^q(M) \arrow[r, "i^*"] & H^q(G) \arrow[r, "\delta"] & H^{q+1}(M,G) \\
H^{q-2}(M) \arrow[u, "\Phi_\eta"'] \arrow[r, "\smile u"] & H^q(M) \arrow[u, "="'] \arrow[r, "\pi^*"] & H^q(G) \arrow[u, "="'] \arrow[r, "\pi_!"] & H^{q-1}(M) \arrow[u, "\Phi_\eta"']
\end{tikzcd}$$

where the the upper and lower lines come from the long exact sequence of a pair (M,G) and the Gysin sequence of the circle bundle $S^1 \longrightarrow G \xrightarrow{\pi} M$, respectively, implies that $\delta = \Phi_\eta \circ \pi_!$. For $q = 3, 5$, using the Serre spectral sequence of $(*)$, one can show that $\delta^{-1}(\mathrm{Im}(\eta^*)) = H^q(G)$ and hence the transgression homomorphism τ is given by the composition:

$$H^q(G) \xrightarrow{\pi_!} H^{q-1}(M) \xrightarrow{\Phi_\eta} H^{q+1}(M,G) \xleftarrow{\eta^*} H^{q+1}(BS^1)/\mathrm{Ker}(\eta^*).$$

Recall, $H^*(G)$ is the exterior algebra $E(x_3, x_5)$. Then we have

$$\tau(x_3) = (\eta^*)^{-1}((\Phi_\eta \circ \pi_!)(x_3)) = (\eta^*)^{-1}(\pi_!(x_3) \smile U_\eta),$$

and

$$\tau(x_5) = (\eta^*)^{-1}((\Phi_\eta \circ \pi_!)(x_5)) = (\eta^*)^{-1}(\pi_!(x_5) \smile U_\eta).$$

By the diagram above, the map $j^* : H^2(M, G) \longrightarrow H^2(M)$ is an isomorphism and sends U_η to the first Chern class u. This implies that

$$\tau(x_3) = (\eta^*)^{-1}(\pi_!(x_3)) \smile c_1 \text{ and } \tau(x_5) = (\eta^*)^{-1}(\pi_!(x_5)) \smile c_1.$$

On the other hand, τ is identical with the differential $d^q : H^q(G) \longrightarrow H^{q+1}(BS^1)$ in the Serre spectral sequence of $(*)$. From the previous section, we have

$$\tau(x_3) = rc_1^2 \text{ and } \tau(x_5) = s(M)c_1^3.$$

Therefore,

$$\pi_!(x_3) = ru \text{ and } \pi_!(x_5) = s(M)u^2.$$

Now using the above properties of $\pi_!$, a relation between the self-linking number and the invariant s can be expressed as follows:

Lemma 3.3.0.2 ([Kru05]) *The self-linking number of $u^2 \in H^4(M)$ is given by*

$$L(u^2, u^2) = -\frac{s^{-1}}{r} \in \mathbb{Q}/\mathbb{Z},$$

where s^{-1} is some integer such that $s^{-1} \cdot s(M) \equiv 1$ mod r.

Proof From the circle bundle $S^1 \longrightarrow G \xrightarrow{\pi} M$, we obtain the commutative diagram:

$$\begin{tikzcd} H_3(G) \arrow[r, "\pi_*"] & H_3(M) \\ H^5(G) \arrow[u, "\cong"] \arrow[r, "\pi_!"] & H^4(M) \arrow[u, "\cong"] \end{tikzcd}$$

where the vertical arrows are the Poincaré duality isomorphisms. Since $H^3(G) \cong \mathrm{Hom}(H_3(G), \mathbb{Z})$, let $x_3^* \in H_3(G)$ be the dual homology class of x_3. Then $x_3^* = [G] \frown x_5$, where [G] is an orientation class. Since $\pi_!(x_5) = s(M)u^2$, we obtain that

$$\begin{aligned} L(u^2, u^2) &= \left\langle u^2 \smile \beta^{-1}(u^2), [M] \right\rangle \\ &= \left\langle \beta^{-1}(u^2), [M] \frown u^2 \right\rangle \\ &= \left\langle \beta^{-1}(u^2), s^{-1} \cdot \pi_*(x_3^*) \right\rangle . \end{aligned}$$

In this situation, β depends only on the 4-skeleton of M and we will show that it can be explicitly constructed. Let

$$\alpha : S^2 \longrightarrow M$$

be such that $\alpha^*(u^2) = s^2$ is the canonical generator of $H^2(S^2)$. Then the pullback of $S^1 \longrightarrow G \xrightarrow{\pi} M$ by α gives the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{\nu} S^2$ and a map $\hat{\alpha} : S^3 \longrightarrow G$ that makes the following diagram commute.

$$\begin{tikzcd} S^3 \arrow[r, "\hat{\alpha}"] \arrow[d, "\nu"] & G \arrow[d, "\pi"] \\ S^2 \arrow[r, "\alpha"] & M \end{tikzcd}$$

By integration along the fiber, one obtains

$$\begin{tikzcd} H^3(S^3) \arrow[d, "\nu_!"] & H^3(G) \arrow[l, "\hat{\alpha}^*"'] \arrow[d, "\pi_!"] \\ H^2(S^2) & H^2(M). \arrow[l, "\alpha^*"'] \end{tikzcd}$$

Since $\pi_!(x_3) = ru$ and $\nu_!$ is an isomorphism, we have $\hat{\alpha}^*(x_3) = rs^3$ where s^3 is the canonical generator of $H^3(S^3)$. The Hurewicz theorem implies that $\pi_3(G) \cong H_3(G)$ since G is 2-connected. Let $\gamma \in \pi_3(G)$ be the preimage of $x_3^* \in H_3(G)$ under the Hurewicz homomorphism. Note that

$$\hat{\alpha}_*(i_3) = r\gamma,$$

where $i_3 \in \pi_3(S^3)$ is the canonical generator. We can see that

$$\alpha \vee (\pi\gamma) : S^2 \vee S^3 \longrightarrow M$$

is a 3-connected (a map $f : X \longrightarrow Y$ is called *n-connected* if it induces isomorphisms $f_* : \pi_i(X) \longrightarrow \pi_i(Y)$ for $i < n$, and a surjection $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$), and

$$\alpha_*(\nu) = \alpha\nu = \pi\hat{\alpha} = (\pi\hat{\alpha})_*(i_3) = \pi_*\hat{\alpha}_*(i_3) = \pi_*(r\gamma) = \pi_*\gamma_*(ri_3) = (\pi\gamma)_*(ri_3).$$

Then the kernel of $(\alpha \vee (\pi\gamma))_* : \pi_3(S^2 \vee S^3) \longrightarrow \pi_3(M)$ is generated by $\nu - ri_3$, and hence we have a 4-connected map by attaching a 4-cell e^4 with attaching map $\beta_4 = \nu - ri_3$

$$S^2 \vee S^3 \cup_{\beta_4} e^4 \longrightarrow M.$$

Now consider the cellular chain complex:

$$0 \longrightarrow C_4 \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2 \longrightarrow 0$$

associated to the cell complex $S^2 \vee S^3 \cup_{\beta_4} e^4$ and $e^4 \in C_4,\ i_3 \in C_3,\ i_2 \in C_2$ as the canonical generators. Note that $e^4 \mapsto -ri_3$ and $i_3 \mapsto 0$. Let

$$\xi : C_3 \longrightarrow \mathbb{Q}/\mathbb{Z}$$

be the homomorphism defined by $\xi(i_3) = \frac{1}{r}$. Since i_3 is a generator of C_3 (hence, $H_3(M)$) and $H_3(M) \cong \mathbb{Z}_r$, ξ represents a generator of $\mathrm{Hom}(H_3(M), \mathbb{Q}/\mathbb{Z}) = H^3(M; \mathbb{Q}/\mathbb{Z})$. Also, the composition

$$\delta\xi = \xi\partial : C_4 \xrightarrow{\partial} C_3 \xrightarrow{\xi} \mathbb{Q}/\mathbb{Z}$$

mapping $e^4 \mapsto -ri_3 \mapsto -1 \in \mathbb{Z}$ represents the cohomology class $\beta(\xi) = -u^2 \in H^4(M; \mathbb{Z})$. Therefore,

$$\begin{aligned} L(u^2, u^2) &= \left\langle \beta^{-1}(u^2), s^{-1} \cdot \pi_*(x_3^*) \right\rangle \\ &= \left\langle -\xi, s^{-1} \cdot \pi_*(x_3^*) \right\rangle \\ &= -s^{-1} \left\langle \xi, i_3 \right\rangle \\ &= -\frac{s^{-1}}{r}. \end{aligned}$$

$\square$

Moreover, we can easily apply this lemma to obtain the following equivalent statement between the invariants $L(u^2, u^2)$ and $s(M)$.

Corollary 3.3.0.1 *For manifolds M and M' of type r together with generators $u_M \in H^2(M)$ and $u_{M'} \in H^2(M')$, respectively,*

$$L(u_M^2, u_M^2) = L(u_{M'}^2, u_{M'}^2) \ \in \mathbb{Q}/\mathbb{Z} \Longleftrightarrow s(M) \equiv s(M') \ \mathit{mod}\ r.$$

Proof For the corresponding inverses $s^{-1}(M)$ and $s^{-1}(M')$, we have

$$\begin{aligned} L(u_M^2, u_M^2) = L(u_{M'}^2, u_{M'}^2) &\Longleftrightarrow -\frac{s^{-1}(M)}{r} \equiv -\frac{s^{-1}(M')}{r} \bmod \mathbb{Z} \\ &\Longleftrightarrow s^{-1}(M) \equiv s^{-1}(M') \bmod r \\ &\Longleftrightarrow 1 \equiv s(M) \cdot s^{-1}(M') \bmod r \\ &\Longleftrightarrow s(M') \equiv s(M) \bmod r. \end{aligned}$$

$\square$

4. KRECK-STOLZ INVARIANTS

In 1988, M. Kreck and S. Stolz were interested in a class of homeomorphic homogeneous spaces which are not diffeomorphic. They considered the Witten manifolds and classified them up to homeomorphism and diffeomorphism by new invariants s_i for $i = 1, 2, 3$, which are a generalization of the Eells-Kuiper invariant defined in [EeKu62]. The classification will be described in Chapter 6. In the first section, we will give a definition of the Kreck-Stolz invariants. Note that they are defined via a bounding manifold W of a closed smooth or topological 7-manifold M with $H^4(M;\mathbb{Q}) = 0$ in general. In other words, they can be expressed as linear combinations of some characteristic numbers of W. Now we assume throughout the chapter that a manifold can be considered as a smooth or topological manifold if we do not explicitly specify the type. We will need the most general form of the Kreck-Stolz invariants which will be expressed at the end of this section. Finally, we will see a relation between the linking form of M, the first Pontrjagin class of M, and those characteristic numbers of W in section 4.2.

4.1. Definition and Generalized Formulas

In [KrSt88], M. Kreck and S. Stolz defined new invariants $s_i(M)$, $i = 1, 2, 3$ for a closed 7-manifold M with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u \bmod 2$ (nonspin case). Now let us consider a smooth manifold M of type r and u a generator of $H^2(M)$. In this case, $w_2(M) = u \bmod 2$ if M is nonspin and as always trivial if M is spin. Also, $H^4(M;\mathbb{Q}) = 0$ since $H^4(M) \cong \mathbb{Z}_r$. Therefore, a smooth manifold of type r satisfies the Kreck-Stolz conditions, and hence we can use the invariants. The construction of these new invariants can be described as follows.

Definition 4.1.0.4 *Let M be a closed 7-manifold with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case). (M, u) is the* ***boundary of a pair*** *(W, z) if W is an 8-manifold with $\partial W = M$, and $z \in H^2(W)$ restricts to u on the boundary such that $w_2(W) = 0$ (spin case) or $w_2(W) = z$ mod 2 (nonspin case).*

M. Kreck and S. Stolz showed the existence of a bounding pair (W, z) of (M, u), see [KrSt91] for the proof.

Proposition 4.1.0.8 ([KrSt91]) *For any closed 7-manifold M with $u \in H^2(M; \mathbb{Z})$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case), such a pair (W, z) exists.*

Remark 4.1.0.6 *By the proposition above, for a closed nonspin 7-manifold M with $u \in H^2(M; \mathbb{Z})$, there is a pair (W, z) such that $w_2(W) = z$ mod 2. Consider the following commutative diagram:*

$$\begin{array}{ccc} H^2(W;\mathbb{Z}) & \xrightarrow{mod\ 2} & H^2(W;\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ H^2(M;\mathbb{Z}) & \xrightarrow{mod\ 2} & H^2(M;\mathbb{Z}_2) \end{array}$$

We have $w_2(W) = z$ mod 2 is non-trivial since $w_2(M) = u$ mod 2 is non-trivial. Hence, the above proposition implies that there exist a bounding spin manifold of the spin boundary and a bounding nonspin manifold of the nonspin boundary.

Now let M be a closed 7-manifold with $H^4(M; \mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case). Then there exists a bounding pair (W, z). For such a pair (W, z), one defines characteristic numbers

$S_i(W, z) \in \mathbb{Q}$ as follows:

$$
\begin{aligned}
S_1(W, z) &:= \left\langle e^{d/2}\widehat{A}(W), [W, M] \right\rangle, \\
S_2(W, z) &:= \left\langle \mathrm{ch}(\lambda(z) - 1)e^{d/2}\widehat{A}(W), [W, M] \right\rangle, \\
S_3(W, z) &:= \left\langle \mathrm{ch}(\lambda^2(z) - 1)e^{d/2}\widehat{A}(W), [W, M] \right\rangle,
\end{aligned}
$$

where

- $d = 0$ in the spin case, $d = z$ in the nonspin case.
- $\lambda(z)$ is the complex line bundle over W with first Chern class $c_1(W) = z$.
- ch is the Chern character, i.e. for a line bundle V, $\mathrm{ch}(V) = \exp(c_1(V))$.
- $\widehat{A}(W)$ is the $\widehat{A}$-polynomial of W.
- $[W, M]$ is the relative fundamental class of a pair (W, M).

Here $\langle \varphi, \alpha \rangle$ represents the evaluation of a cohomology class φ at a homology class α. In particular, $\mathrm{ch}(\lambda(z)-1)$ and $\mathrm{ch}(\lambda^2(z)-1)$ represent $\exp(z)-1$ and $\exp(2z)-1$, respectively, and the first few terms of the $\widehat{A}$-polynomial are

$$
\widehat{A_0} = 1, \quad \widehat{A_1} = -\frac{p_1}{2^3 \cdot 3}, \quad \text{and} \quad \widehat{A_2} = \frac{-4p_2 + 7{p_1}^2}{2^7 \cdot 3^2 \cdot 5}.
$$

Note that the cohomology classes

$$
e^{d/2}\widehat{A}(W), \quad \mathrm{ch}(\lambda(z) - 1)e^{d/2}\widehat{A}(W), \quad \text{and} \quad \mathrm{ch}(\lambda^2(z) - 1)e^{d/2}\widehat{A}(W)
$$

are elements in $H^8(W)$ which can be viewed as elements in $H^8(W; \mathbb{Q})$. However, they can not be evaluated on the relative fundamental class $[W, M]$. In order to make sense of the above definition, we need some explanation. First, $\mathrm{ch}(\lambda(z) - 1)e^{d/2}\widehat{A}(W)$ and $\mathrm{ch}(\lambda^2(z) - 1)e^{d/2}\widehat{A}(W)$ are rational linear combinations of $z^2 p_1(W)$ and z^4. Consider the long exact sequence for the pair (W, M) with rational coefficients:

$$
\cdots \longrightarrow H^4(W, M; \mathbb{Q}) \xrightarrow{j^*} H^4(W; \mathbb{Q}) \xrightarrow{i^*} H^4(M; \mathbb{Q}) \longrightarrow \cdots.
$$

Then $j^* : H^4(W, M; \mathbb{Q}) \longrightarrow H^4(W; \mathbb{Q})$ is surjective since $H^4(M; \mathbb{Q}) = 0$. Hence, z^2 can be regarded as an element in $H^4(W, M; \mathbb{Q})$ under the pullback $(j^*)^{-1}$. Therefore, the cup products $z^2 p_1(W)$ and z^4 should be interpreted as $(j^*)^{-1}(z^2) \smile p_1(W)$ and $(j^*)^{-1}(z^2) \smile z^2$, and hence as elements in $H^8(W, M; \mathbb{Q})$. They can then be evaluated on the relative fundamental class $[W, M]$. Since $e^{d/2}\widehat{A}(W)$ involves the term $p_2(W)$, the same argument can not be used. It is not possible to regard $p_2(W)$ as an element in $H^8(W, M; \mathbb{Q})$. However, we can use the Hirzebruch signature theorem to eliminate $p_2(W)$, and $e^{d/2}\widehat{A}(W)$ eventually is a rational linear combination of ${p_1}^2(W)$, $z^2 p_1(W)$, z^4, and $\mathrm{sign}(W)$, the signature of W. Similarly, ${p_1}^2(W)$ can be regarded as $(j^*)^{-1}(p_1(W)) \smile p_1(W)$.

Now one can write down the explicit formulas for S_i as follows:

Spin Case:

$$\begin{aligned}
S_1(W, z) &= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2, \\
S_2(W, z) &= -\frac{1}{2^4 \cdot 3}z^2 p_1 + \frac{1}{2^3 \cdot 3}z^4, \\
S_3(W, z) &= -\frac{1}{2^2 \cdot 3}z^2 p_1 + \frac{2}{3}z^4,
\end{aligned}$$

Nonspin Case:

$$\begin{aligned}
S_1(W, z) &= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}z^2 p_1 + \frac{1}{2^7 \cdot 3}z^4, \\
S_2(W, z) &= -\frac{1}{2^3 \cdot 3}z^2 p_1 + \frac{5}{2^3 \cdot 3}z^4, \\
S_3(W, z) &= -\frac{1}{2^3}z^2 p_1 + \frac{13}{2^3}z^4,
\end{aligned}$$

where

- $\mathrm{sign}(W)$ is the signature of W.
- $z^2 p_1 := \big\langle (j^*)^{-1}(z^2) \smile p_1(W), [W, M] \big\rangle$.
- $z^4 := \big\langle (j^*)^{-1}(z^2) \smile z^2, [W, M] \big\rangle$.

- $p_1^2 := \langle (j^*)^{-1}(p_1(W)) \smile p_1(W), [W, M] \rangle$.

Here $j^* : H^4(W, M; \mathbb{Q}) \longrightarrow H^4(W; \mathbb{Q})$ is the canonical homomorphism. Note that the signature of W is always an integer. It is defined to be the number of positive diagonal entries minus the number of negative ones of the diagonal symmetric matrix $[\langle a_i \smile a_j, [W] \rangle]$ for some basis $\{a_1, ...a_r\}$ for $H^4(W; \mathbb{Q})$, see [MiSt74] for details.

To obtain the invariants for the boundary M of W, we need the following proposition. The original version in [KrSt91] contains a mistake for the homeomorphism classification of smooth spin manifold as was pointed out by [AMP97]. Following [KrSt98], they fixed this problem and the correct version could be obtained as follows. However, this version only works for the case when M is smooth.

Proposition 4.1.0.9 ([KrSt91]) *Let* $S(W, z) = (S_1(W, z), S_2(W, z), S_3(W, z)) \in \mathbb{Q}^3$. *Then*

$$\{S(W, z) \mid W \textit{ is a closed smooth manifold}, \ w_2(W) = 0\} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

$$\{S(W, z) \mid W \textit{ is a closed smooth manifold}, \ w_2(W) = z \textit{ mod } 2\} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

$$\{S(W, z) \mid W \textit{ is a closed topological manifold}, \ w_2(W) = 0\} \cong (\frac{1}{28}\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z},$$

$$\{S(W, z) \mid W \textit{ is a closed topological manifold}, \ w_2(W) = z \textit{ mod } 2\} \cong (\frac{1}{28}\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Using this proposition and the fact that

$$S_i(W, z) = S_i(\partial W, z|_{\partial W}) + S_i(W - \partial W, z|_{W - \partial W})$$

one obtains the following:

Smooth case:

$S_i(W, z)$ mod $\mathbb{Z}$ depend only on the boundary of (W, z), and one defines

$$s_i(M, u) := S_i(W, z) \text{ mod } \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}.$$

Topological case:

Let

$$\overline{S}_1(W,z) = 28 \cdot S_1(W,z),\ \overline{S}_2(W,z) = S_2(W,z),\ \overline{S}_3(W,z) = S_3(W,z).$$

Then we define

$$\overline{s}_i(M,u) := \overline{S}_i(W,z) \bmod \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}.$$

Since u is a element in $H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case), it turns out that these $s_i(M,u)$ and $\overline{s}_i(M,u)$ do not change when we replace u by $-u$. Therefore, if M is a manifold of type r with a generator $u \in H^2(M)$, one has the new well-defined invariants $s_i(M)$ and $\overline{s}_i(M) \in \mathbb{Q}/\mathbb{Z}$ for $i = 1,2,3$, called *Kreck-Stolz invariants*. Note that s_i is a generalization of the *Eells-Kuiper invariant* defined in [EeKu62].

In some situations we may not be able to find an explicit pair (W,z) with the required properties. This means that we may not always be able to find an explicit bounding spin manifold W if the boundary M is spin. In this case we modify the invariants as follows. Let M be a closed 7-manifold with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ as above, W an 8-manifold with $\partial W = M$, $z, c \in H^2(W)$ such that $z|_M = u, c|_M = 0$, and $w_2(W) = c$ mod 2 (spin case) or $w_2(W) = z + c$ mod 2 (nonspin case). We define characteristic numbers:

$$\begin{aligned}
S_1(W,z,c) &:= \left\langle e^{(c+d)/2}\widehat{A}(W), [W,M] \right\rangle,\\
S_2(W,z,c) &:= \left\langle \mathrm{ch}(\lambda(z)-1)e^{(c+d)/2}\widehat{A}(W), [W,M] \right\rangle,\\
S_3(W,z,c) &:= \left\langle \mathrm{ch}(\lambda^2(z)-1)e^{(c+d)/2}\widehat{A}(W), [W,M] \right\rangle,
\end{aligned}$$

where $d = 0$ in the spin case and $d = z$ in the nonspin case. With the same argument, these characteristic numbers for closed manifolds depend only on (M,u), in particular we can define $s_i(M,u) = S_i(W,z,c)$ mod $\mathbb{Z}$, $\overline{s}_i(M,u) = \overline{S}_i(W,z,c)$ mod $\mathbb{Z}$, and call them *generalized Kreck-Stolz invariants*.

Remark 4.1.0.7 *If M is spin, the bounding manifold could be spin or nonspin. On the other hand, if M is nonspin, a similar argument in Remark 4.1.0.6 implies that $w_2(W) = z + c \mod 2$ is non-trivial, equivalently W must always be nonspin. Alternatively, there cannot be any spin structure on W if M is nonspin because Theorem 2.4.0.6 implies that any spin structure on a compact manifold with boundary induces a spin structure on its boundary.*

As before, the below proposition is the result of deriving the characteristic numbers in the above definition in terms of the signature of W and suitable characteristic numbers.

Proposition 4.1.0.10 *Let M be a closed 7-manifold with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u \mod 2$ (nonspin case). Suppose that W is an 8-manifold with $\partial W = M$, $z, c \in H^2(W)$ such that $z|_M = u, c|_M = 0$, and $w_2(W) = c \mod 2$ (spin case) or $w_2(W) = z + c \mod 2$ (nonspin case). The following are the explicit formulas of S_i.*
Spin Case:

$$\begin{aligned}
S_1(W, z, c) &= -\frac{1}{2^5 \cdot 7} sign(W) + \frac{1}{2^7 \cdot 7} p_1^2 - \frac{1}{2^6 \cdot 3} c^2 p_1 + \frac{1}{2^7 \cdot 3} c^4, \\
S_2(W, z, c) &= -\frac{1}{2^4 \cdot 3}((zc + z^2)p_1 - (zc^3 + 3z^2c^2 + 4z^3c + 2z^4)), \\
S_3(W, z, c) &= -\frac{1}{2^3 \cdot 3}((zc + 2z^2)p_1 - (zc^3 + 6z^2c^2 + 16z^3c + 16z^4)),
\end{aligned}$$

Nonspin Case:

$$\begin{aligned}
S_1(W, z, c) &= -\frac{1}{2^5 \cdot 7} sign(W) + \frac{1}{2^7 \cdot 7} p_1^2 - \frac{1}{2^6 \cdot 3} (c^2 + 2zc + z^2) p_1 \\
&\quad + \frac{1}{2^7 \cdot 3}(c^4 + 4zc^3 + 6z^2c^2 + 4z^3c + z^4), \\
S_2(W, z, c) &= -\frac{1}{2^4 \cdot 3}((zc + 2z^2)p_1 - (zc^3 + 6z^2c^2 + 13z^3c + 10z^4)), \\
S_3(W, z, c) &= -\frac{1}{2^3 \cdot 3}((zc + 3z^2)p_1 - (zc^3 + 9z^2c^2 + 31z^3c + 39z^4)),
\end{aligned}$$

where p_1, z^2, c^2, zc *can be regarded as elements in* $H^4(W, M; \mathbb{Q})$ *under the pullback* $(j^*)^{-1}$ *and the above classes* $p_1^2, c^2p_1, c^4, zcp_1$, *etc. are abbreviations for the characteristic numbers*

$$p_1^2 := \left\langle (j^*)^{-1}(p_1(W)) \smile p_1(W), [W, M] \right\rangle,$$

$$zcp_1 := \left\langle (j^*)^{-1}(zc) \smile p_1(W), [W, M] \right\rangle, etc. \ .$$

Proof As the calculations are very similar, we will only show how to obtain the above $S_1(W, z, c)$ for the spin case and $S_3(W, z, c)$ for the nonspin case. First, we can see that

$$e^{(c+d)/2}\widehat{A} = (1 + \frac{c}{2} + \frac{c^2}{8} + \frac{c^3}{48} + \frac{c^4}{384} + ...)(\widehat{A_0} + \widehat{A_1} + \widehat{A_2} + ...),$$

for the spin case, and

$$\begin{aligned}
\mathrm{ch}(\lambda^2(z) - 1)e^{(c+d)/2}\widehat{A} &= \mathrm{ch}(\lambda^2(z) - 1)e^{(z+c)/2}\widehat{A} \\
&= (2z + 2z^2 + \frac{4z^3}{3} + \frac{2z^4}{3} + ...)(1 + \frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{48} + ...) \\
&\quad (1 + \frac{c}{2} + \frac{c^2}{8} + \frac{c^3}{48} + ...)(\widehat{A_0} + \widehat{A_1} + ...) \\
&= (2z + 3z^2 + \frac{31z^3}{12} + \frac{39z^4}{24} + ...)(1 + \frac{c}{2} + \frac{c^2}{8} + \frac{c^3}{48} + ...) \\
&\quad (\widehat{A_0} + \widehat{A_1} + ...),
\end{aligned}$$

for the nonspin case. Hence, by the definitions of $S_1(W, z, c)$ and $S_3(W, z, c)$,

$$\begin{aligned}
S_1(W, z, c) &= \frac{c^4}{384} \cdot \widehat{A_0}(W) + \frac{c^2}{8} \cdot \widehat{A_1}(W) + \widehat{A_2}(W) \\
&= \frac{1}{2^7 \cdot 3}c^4 - \frac{1}{2^6 \cdot 3}c^2p_1 - \frac{4}{2^7 \cdot 3^2 \cdot 5}p_2 + \frac{7}{2^7 \cdot 3^2 \cdot 5}p_1^2 \\
&= \frac{1}{2^7 \cdot 3}c^4 - \frac{1}{2^6 \cdot 3}c^2p_1 - \frac{4}{2^7 \cdot 3^2 \cdot 5} \cdot \frac{(45 \cdot \mathrm{sign}(W) + p_1^2)}{7} + \frac{7}{2^7 \cdot 3^2 \cdot 5}p_1^2 \\
&= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}c^2p_1 + \frac{1}{2^7 \cdot 3}c^4,
\end{aligned}$$

for the spin case, and

$$\begin{aligned}
S_3(W, z, c) &= (\frac{zc^3}{24} + \frac{3z^2c^2}{8} + \frac{31z^3c}{24} + \frac{39z^4}{24})\widehat{A_0}(W) + (zc + 3z^2)\widehat{A_1}(W) \\
&= (\frac{zc^3}{24} + \frac{3z^2c^2}{8} + \frac{31z^3c}{24} + \frac{39z^4}{24}) - \frac{(zc + 3z^2)p_1}{24} \\
&= -\frac{1}{2^3 \cdot 3}((zc + 3z^2)p_1 - (zc^3 + 9z^2c^2 + 31z^3c + 39z^4)),
\end{aligned}$$

for the nonspin case. $\square$

Alternatively, the invariants $s_i(M)$ can be expressed analytically in terms of the η-invariant of certain self-adjoint operators on M and integrals over M, see [KrSt88] for details.

Theorem 4.1.0.7 ([KrSt88]) *Let M be a closed 7-manifold with $H^4(M;\mathbb{Q}) = 0$ together with a class $u \in H^2(M)$ such that $w_2(M) = 0$ (spin case) or $w_2(M) = u$ mod 2 (nonspin case).*

- *If M is a spin manifold, then*

$$s_1(M) = \frac{1}{2^5 \cdot 7}\eta(B,0) + \xi(D) - \frac{1}{2^7 \cdot 7}\int_M p_1(M,g) \wedge h \ mod\ \mathbb{Z},$$
$$s_2(M) = \xi(D_{\lambda(z)}) - \xi(D) - \frac{1}{2^4 \cdot 3}\int_M v \wedge (-p_1(M,g) + 2u \wedge u) \ mod\ \mathbb{Z},$$
$$s_3(M) = \xi(D_{\lambda^2(z)}) - \xi(D) - \frac{1}{2^2 \cdot 3}\int_M v \wedge (-p_1(M,g) + 8u \wedge u) \ mod\ \mathbb{Z}.$$

- *If M is a nonspin manifold, then*

$$s_1(M) = \frac{1}{2^5 \cdot 7}\eta(B,0) + \xi(D) - \frac{1}{2^7 \cdot 3 \cdot 7}\int_M [3p_1(M,g) \wedge h$$
$$= +v \wedge (-14p_1(M,g) + 7u \wedge u)] \ mod\ \mathbb{Z},$$
$$s_2(M) = \xi(D_{\lambda(z)}) - \xi(D) - \frac{1}{2^3 \cdot 3}\int_M v \wedge (-p_1(M,g) + 5u \wedge u) \ mod\ \mathbb{Z},$$
$$s_3(M) = \xi(D_{\lambda^2(z)}) - \xi(D) - \frac{1}{2^3}\int_M v \wedge (-p_1(M,g) + 13u \wedge u) \ mod\ \mathbb{Z}.$$

Here

- *If P is an elliptic self-adjoint operator, then $\xi(D) = \frac{dim(KerP) + \eta(P,0)}{2}$ where $\eta(P,0)$ is the η-invariant.*

- *$\eta(B,0)$ is the η-invariant of the signature operator B which depends on the Riemannian metric g.*

- *D is the Dirac operator and $D_{\lambda(z)}$, $D_{\lambda^2(z)}$ are the Dirac operators with coefficients in $\lambda(z)$, $\lambda^2(z)$, respectively.*

- *$p_1(M,g) \in \Omega^4(M)$ is the Pontrjagin form with respect to the Levi-Civita connection on M.*

- *u is interpreted as the harmonic 2-form representing the cohomology class u.*

- *h and v are 3-forms such that $dh = p_1(M,g)$ and $dv = u \wedge u$.*

4.2. Linking Form, first Pontrjagin Class, and Characteristic Numbers

In this section, we will describe a different approach to the linking number. In particular, we can obtain the linking number between two arbitrary elements in $H^n(M)$ of a $(2n-1)$-manifold M by computing an appropriate characteristic number of a bounding compact oriented $2n$-manifold W. This gives rise to the special fact that

$$L(u^2, u^2) = z^4(W) \bmod \mathbb{Z},$$

where $L(u^2, u^2)$ is the self-linking number of a manifold M of type r with a bounding pair (W, z). Note that this fact was stated in [KrSt88] without proof. Moreover, we will see a relation between the first Pontrjagin class, the linking form, and some characteristic numbers.

Let W be compact oriented $2n$-manifold with boundary M. Let $a, b \in H^n(M;\mathbb{Z})$ be torsion elements such that $a = \bar{a}|_{\partial W}$ and $b = \bar{b}|_{\partial W}$ for some $\bar{a}, \bar{b} \in H^n(W;\mathbb{Z})$. Consider the following commutative diagram of two long exact sequences for a pair (W, M) with

$\mathbb{Z}, \mathbb{Q}$ - coefficients:

$$\begin{tikzcd}
H^n(W,M;\mathbb{Z}) \arrow[r] \arrow[d] & H^n(W;\mathbb{Z}) \arrow[r, "f"] \arrow[d, "i^*"] & H^n(M;\mathbb{Z}) \arrow[d, "g"] \\
H^n(W,M;\mathbb{Q}) \arrow[r, "j^*"] & H^n(W;\mathbb{Q}) \arrow[r, "h"] & H^n(M;\mathbb{Q})
\end{tikzcd}$$

Since a is torsion, $h(i^*(\bar{a})) = g(f(\bar{a})) = g(a) = 0$. Similarly, $h(i^*(\bar{b})) = 0$. Then there exist $x, y \in H^n(W,M;\mathbb{Q})$ such that

$$j^*(x) = i^*(\bar{a}) \text{ and } j^*(y) = i^*(\bar{b}),$$

where $j^* : H^n(W,M;\mathbb{Q}) \longrightarrow H^n(W;\mathbb{Q})$ and $i^* : H^n(W;\mathbb{Z}) \longrightarrow H^n(W;\mathbb{Q})$. First, we define a relation $l : \mathrm{Tor}(H^n(M;\mathbb{Z})) \times \mathrm{Tor}(H^n(M;\mathbb{Z})) \longrightarrow \mathbb{Q}$ as follows:

Definition 4.2.0.5 *With the above notations, let*

$$l : \mathit{Tor}(H^n(M;\mathbb{Z})) \times \mathit{Tor}(H^n(M;\mathbb{Z})) \longrightarrow \mathbb{Q}$$

be the relation defined by

$$l(a,b) = (-1)^n \langle x \smile y, [W,M] \rangle,$$

where $[W,M]$ *is the relative fundamental class of the pair* (W,M)*.*

Note that

$$l(a,b) = \begin{cases} \langle x \smile y, [W,M] \rangle & \text{if dimension of } a \text{ is even,} \\ -\langle x \smile y, [W,M] \rangle & \text{if dimension of } a \text{ is odd,} \end{cases}$$

and l is not well-defined in general. However, after reducing mod $\mathbb{Z}$, we are able to show that

$$\widetilde{l}(a,b) := l(a,b) \bmod \mathbb{Z} \ \in \mathbb{Q}/\mathbb{Z}$$

is independent of the choice of $\bar{a}$, $\bar{b}$, x, and y. That is, $\widetilde{l}(a,b)$ is well defined. In particular, as we will see it is indeed the same as the linking form:

$$L(a,b) = \big\langle a \smile \beta^{-1}(b), [M] \big\rangle \ \in \mathbb{Q}/\mathbb{Z},$$

where $\beta : H^{n-1}(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow H^n(M;\mathbb{Z})$ is the Bockstein homomorphism which is associated to the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

We notice that the following equation

$$\begin{aligned} \langle x \smile y, [W,M] \rangle &= \big\langle (j^*)^{-1}(i^*(\bar{a})) \smile y, [W,M] \big\rangle \\ &= \langle i^*(\bar{a}) \smile y, [W,M] \rangle \\ &= \langle \bar{a} \smile y, [W,M] \rangle \ \in \mathbb{Q} \end{aligned}$$

implies that $l(a,b)$ is independent of x. Similarly, it is independent of y as well. To show independence of the choice of $\bar{a}$ and $\bar{b}$, it is convenient to use the following commutative diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
H^{n-1}(M;\mathbb{Z}) \arrow[r] \arrow[d] & H^n(W,M;\mathbb{Z}) \arrow[r, "k"] \arrow[d, "p"] & H^n(W;\mathbb{Z}) \arrow[r, "f"] \arrow[d, "i^*"] & H^n(M;\mathbb{Z}) \arrow[d, "g"] \\
H^{n-1}(M;\mathbb{Q}) \arrow[r] \arrow[d] & H^n(W,M;\mathbb{Q}) \arrow[r, "j^*"] \arrow[d, "q"] & H^n(W;\mathbb{Q}) \arrow[r, "h"] \arrow[d] & H^n(M;\mathbb{Q}) \arrow[d] \\
H^{n-1}(M;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "\beta"] & H^n(W,M;\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & H^n(W;\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & H^n(M;\mathbb{Q}/\mathbb{Z}) \arrow[d] \\
H^n(M;\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(W,M;\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(W;\mathbb{Z}) \arrow[r] \arrow[d] & H^{n+1}(M;\mathbb{Z}) \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

where the horizontal and vertical lines come from the long exact sequences for the pair (W, M) with various coefficients and the ones for various spaces associated to the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

respectively. Suppose that b is the image of $\bar{b_1}$ and $\bar{b_2}$ under the map f. Then $\bar{b_1} - \bar{b_2} \in \mathrm{Ker}(f) = \mathrm{Im}(k)$. So, we have

$$y_1 - y_2 = (j^*)^{-1}(i^*(\bar{b_1})) - (j^*)^{-1}(i^*(\bar{b_2})) = (j^*)^{-1}(i^*(\bar{b_1} - \bar{b_2})) \in \mathrm{Im}(p),$$

and hence

$$\bar{y_1} = \bar{y_2} \ \in H^n(W, M; \mathbb{Q}/\mathbb{Z}),$$

where $q(y_1) = \bar{y_1}$ and $q(y_2) = \bar{y_2}$. This implies that

$$\langle \bar{a} \smile \bar{y_1}, [W, M] \rangle = \langle \bar{a} \smile \bar{y_2}, [W, M] \rangle \ \in \mathbb{Q}/\mathbb{Z}.$$

Hence, $\widetilde{l}(a,b)$ is independent of the choice of $\bar{b}$. Similarly, it is independent of $\bar{a}$. Therefore, $\widetilde{l}(a,b)$ depends only on a, b. Now it remains to show that

$$\widetilde{l}(a,b) = L(a,b).$$

Here $L(a,b)$ is the linking number of a with b. By the commutative diagram above, the construction of $\bar{y}$ implies that there is an element $\hat{y} \in H^{n-1}(M;\mathbb{Q}/\mathbb{Z})$ that maps to $\bar{y}$ under the boundary map:

$$\delta : H^{n-1}(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow H^n(W,M;\mathbb{Q}/\mathbb{Z}),$$

and $\hat{y}$ maps to b under the Bockstein homomorphism:

$$\beta : H^{n-1}(M;\mathbb{Q}/\mathbb{Z}) \longrightarrow H^n(M;\mathbb{Z}).$$

That is,

$$\delta(\hat{y}) = \bar{y} \text{ and } \beta(\hat{y}) = b.$$

By exactness and $a = \bar{a}|_{\partial W}$, $\delta(a)$ is trivial. Using the definition of a coboundary δ and its relation to the cup product, we have

$$\delta(a \smile \hat{y}) = (-1)^n(\bar{a} \smile \delta(\hat{y})).$$

Hence,

$$\begin{aligned}
\widetilde{l}(a,b) &= l(a,b) \bmod \mathbb{Z} \\
&= (-1)^n \langle \bar{a} \smile y, [W,M] \rangle \bmod \mathbb{Z} \\
&= (-1)^n \langle \bar{a} \smile \bar{y}, [W,M] \rangle \\
&= \langle (-1)^n(\bar{a} \smile \delta(\hat{y})), [W,M] \rangle \\
&= \langle \delta(a \smile \hat{y}), [W,M] \rangle \\
&= \langle a \smile \hat{y}, [M] \rangle \\
&= \left\langle a \smile \beta^{-1}(b), [M] \right\rangle \in \mathbb{Q}/\mathbb{Z}.
\end{aligned}$$

This gives rise to the following proposition:

Proposition 4.2.0.11 *Let W be compact oriented $2n$-manifold with boundary M and $a, b \in H^n(M;\mathbb{Z})$ torsion elements such that $a = \bar{a}|_{\partial W}$ and $b = \bar{b}|_{\partial W}$ for some $\bar{a}, \bar{b} \in H^n(W;\mathbb{Z})$. Then*

$$L(a,b) = l(a,b) \ mod\ \mathbb{Z} \ \in \mathbb{Q}/\mathbb{Z},$$

where $L(a,b)$ is the linking number of a with b and l is defined as above.

Now let M be a smooth manifold of type r with u a generator of $H^2(M)$ and (W,z) a bounding pair of (M,u) in the sense of Definition 4.1.0.4. Using the same argument as above, we observe that

$$z^4 = \left\langle (j^*)^{-1}(z^2) \smile z^2, [W,M] \right\rangle = l(u^2, u^2).$$

Moreover, if the first Pontrjagin of W, $p_1(W)$, restricts to $p_1(M)$ on the boundary, then we obtain the equations:

$$z^2 p_1 = \left\langle (j^*)^{-1}(z^2) \smile p_1(W), [W,M] \right\rangle = l(u^2, p_1(M)),$$

and

$$p_1^2 = \left\langle (j^*)^{-1}(p_1(W)) \smile p_1(W), [W,M] \right\rangle = l(p_1(M), p_1(M)).$$

Therefore, these equations imply the following corollary:

Corollary 4.2.0.2 *For a manifold M of type r with a generator u of $H^2(M)$ and a bounding pair (W,z) of (M,u), the self-linking number is*

$$L(u^2, u^2) = z^4 \ mod\ \mathbb{Z}.$$

In particular, if $p_1(W)$ restricts to $p_1(M)$ on the boundary, the following linking numbers hold:

$$L(u^2, p_1(M)) = z^2 p_1 \ mod\ \mathbb{Z},$$

and

$$L(p_1(M), p_1(M)) = p_1^2 \ mod\ \mathbb{Z}.$$

5. ESCHENBURG SPACES

Recall, the Eschenburg space $E_{k,l}$, where $k := (k_1, k_2, k_3)$, $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ satisfying $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$ and the gcd condition (G), is alternatively defined to be a quotient of two-sided action induced by $\rho_{k,l}$ of S^1 on $SU(3)$:

$$E_{k,l} := SU(3)/S^1,$$

where $\rho_{k,l} := (\rho_k, \rho_l)$ induces an action $S^1 \times SU(3) \longrightarrow SU(3)$ by

$$(z, A) \longmapsto \rho_k(z) A \rho_l(z^{-1}).$$

Here $\rho_k(z) = \mathrm{diag}(z^{k_1}, z^{k_2}, z^{k_3})$, $\rho_l(z^{-1}) = \mathrm{diag}(z^{-l_1}, z^{-l_2}, z^{-l_3})$ and $\mathrm{diag} : S^1 \times S^1 \times S^1 \longrightarrow U(3)$ is the embedding defined by

$$(z, v, w) \mapsto \begin{pmatrix} z & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix}.$$

In this chapter, we will review important results based on [Kru05] about the Eschenburg spaces. The cohomology rings of Eschenburg spaces will be briefly discussed in the first section. In section 5.2, we will see how to construct a particular cobordism of most of the $E_{k,l}$. In section 5.3, we describe the topology of this cobordism and in section 5.4, we use this topology and the fact that the Kreck-Stolz invariants are additive to compute the invariants s_i. Some of these invariants, the first Pontrjagin class and the self-linking number, give rise to a diffeomorphism and homeomorphism classification which was claimed without proof in [Kru05]. We will give a proof in Chapter 6.

5.1. Cohomology Ring

Let $k := (k_1, k_2, k_3)$, $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ be such that $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$ and the gcd condition (G) holds. Consider the Eschenburg space $E_{k,l}$. It is defined as a biquotient and the cohomology ring of a biquotient can be computed by using the *Borel construction* and the pullback of a well known bundle, see [Esb82] for details. Later W. Singhof [Sin93] used another equivalent description of a biquotient to compute its homotopy type and tangent bundle as well. This description is called a double coset space. Note that $E_{k,l} = SU(3)/S^1$ can be regarded as the double coset space:

$$\rho_{k,l}(S^1) \backslash G / \Delta(U(3)),$$

where

$$G := \{(A, B) \in U(3) \times U(3) \mid \det(A) = \det(B)\},$$

and

$$\Delta : U(3) \longrightarrow G$$

is the diagonal mapping. This holds because of the following. First, it is easy to show that a map $SU(3) \longrightarrow G/\Delta(U(3))$ given by $A \longmapsto [A, I_3]$ is a diffeomorphism. Next, passing through the $\rho_{k,l}$ action, we notice that for any $A, B \in SU(3)$,

$$[A, I_3] \sim_r [B, I_3] \Longleftrightarrow [A, I_3] = (\rho_k(z), \rho_l(z))[B, I_3],$$

for $z \in S^1$. Then

$$[A, I_3] = [\rho_k(z)B, \rho_l(z)],$$

and so,

$$(A^{-1}\rho_k(z)B, \rho_l(z)) = (A, I_3)^{-1}(\rho_k(z)B, \rho_l(z)) \in \Delta(U(3)).$$

This implies that

$$A^{-1}\rho_k(z)B = \rho_l(z).$$

The last equation holds if and only if $A \sim_\rho B$. Here $\sim_r$ is the equivalence relation on the left coset space $G/\Delta(U(3))$ and $\sim_\rho$ is the relation on $SU(3)$ induced by the $\rho_{k,l}$ action. Therefore $SU(3) \longrightarrow G/\Delta(U(3))$ pushes down to some diffeomorphism $E_{k,l} \longrightarrow \rho_{k,l}(S^1)\backslash G/\Delta(U(3))$. Following [Sin93], there exists a pullback diagram:

$$\begin{tikzcd}
SU(3) \arrow[r] & E_{k,l} \arrow[r] \arrow[d] & BS^1 \arrow[d, "B\rho_{k,l}"] \\
SU(3) \arrow[r] & BU(3) \arrow[r, "\Delta"] & BG
\end{tikzcd}$$

Using the Serre spectral sequence for the top horizontal fibration, the E^2-term is given by

$$E^2 = H^*(BS^1) \otimes H^*(SU(3)) = \mathbb{Z}[c_1] \otimes E(x_3, x_5),$$

where c_1 is the universal Chern class and $E(x_3, x_5)$ is the exterior algebra with two generators $x_3 \in H^3(G)$ and $x_5 \in H^5(G)$. Also, this sequence converges to $H^*(E_{k,l})$. Moreover, by the Serre spectral sequence of the bottom horizontal fibration as described in [Esb82] and by naturality, we obtain that all differential maps d^n are trivial except:

$$d^4(x_3) = r(k,l)c_1^2 \text{ and } d^6(x_5) = s(k,l)c_2^3,$$

where

$$r := r(k,l) = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3),$$

and

$$s := s(k,l) = \sigma_3(k_1, k_2, k_3) - \sigma_3(l_1, l_2, l_3).$$

Here σ_i is the *i-th elementary symmetric function*, and hence

$$\sigma_1(k_1, k_2, k_3) = k_1 + k_2 + k_3,$$

$$\sigma_2(k_1, k_2, k_3) = k_1k_2 + k_1k_3 + k_2k_3,$$

$$\sigma_3(k_1, k_2, k_3) = k_1k_2k_3.$$

By Remark 3.2.0.5, we know that $\gcd(r, s) = 1$. Alternatively, this fact was also proved by R. Hepworth [Hep05]. His elementary proof can be given as follows:

Proposition 5.1.0.12 ([Hep05]) *Let $k := (k_1, k_2, k_3)$, $l := (l_1, l_2, l_3) \in \mathbb{Z}^3$ be such that $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$. The gcd condition (G) holds if and only if $\gcd(r, s) = 1$. Moreover, the gcd condition (G) implies that r must be odd.*

Proof Define a polynomial

$$f(x) = \prod_i (x + k_i) - \prod_i (x + l_i)$$

in $\mathbb{Z}[x]$. We observe that a prime p divides each of $k_1 - l_{\sigma(1)}, k_2 - l_{\sigma(2)}, k_3 - l_{\sigma(3)}$ for some $\sigma \in S_3$ if and only if

$$\prod_i (x + k_i) = \prod_i (x + l_{\sigma(i)}) \ \in \mathbb{Z}_p[x].$$

The last equation holds if and only if

$$f(x) = 0 \ \in \mathbb{Z}_p[x].$$

However, one can easily show that

$$f(x) = rx + s.$$

Since $rx + s = 0 \ \in \mathbb{Z}_p[x]$ if and only if p divides both r and s, the first statement is proven. For the second one, we assume that $\gcd(r, s) = 1$ and that r is even. Then s must be odd. Hence, there are exactly two possible cases. Either the k_i are all odd and at least one of l_i is even, or vice versa. But $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$ implies that either the k_i are all odd and exactly two of l_i are even, or vice versa. In both cases r is odd which is a contradiction. $\square$

Since the Serre spectral sequence converges to $H^*(E_{k,l})$, one can show that $E_{k,l}$ is a manifold of type $|r|$. Moreover, R. J. Milgram [Mig00] computed the tangent bundle of $E_{k,l}$ which can be written as the sum of a trivial bundle and complex lines bundles as follows:

Proposition 5.1.0.13 ([Mig00]) *Let $E_{k,l}$ be the Eschenburg space and u a generator of $H^2(E_{k,l})$. Then*

$$T(E_{k,l}) \oplus \mathbb{R} \cong \mathbb{R}^2 \oplus \lambda_1 \oplus \lambda_2 \oplus \lambda_3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the complex line bundles with first Chern classes $(k_1 - k_2)u, (k_1 - k_3)u, (k_2 - k_3)u$, respectively.

This immediately implies that

$$w_2(E_{k,l}) = (k_1 - k_2)u + (k_1 - k_3)u + (k_2 - k_3)u \bmod 2 \ \in \ \mathbb{Z}_2$$

which vanishes and

$$\begin{aligned} p_1(E_{k,l}) &= (k_1 - k_2)^2 u^2 + (k_1 - k_3)^2 u^2 + (k_2 - k_3)^2 u^2 \\ &= (2(k_1^2 + k_2^2 + k_3^2) - 2(k_1 k_2 + k_1 k_3 + k_2 k_3))u^2 \\ &= (2(k_1 + k_2 + k_3)^2 - 6(k_1 k_2 + k_1 k_3 + k_2 k_3))u^2 \\ &= (2\sigma_1(k)^2 - 6\sigma_2(k))u^2 \ \in \mathbb{Z}_{|r|}. \end{aligned}$$

Remark 5.1.0.8 *For the formula of $p_1(E_{k,l})$, we can replace k by l since $\sigma_1(k) = \sigma_1(l)$ and $\sigma_2(k) \equiv \sigma_2(l)$ mod r. Moreover, $p_1(E_{k,l})$ can also be computed using a particular cobordism as we will see in section 5.4.*

5.2. Construction of Cobordism

Recall that the Eschenburg space $E_{k,l}$ is the quotient of a two-sided action induced by $\rho_{k,l}$ of S^1 on $SU(3)$:

$$E_{k,l} = \rho_k(z) \backslash SU(3) / \rho_l(z).$$

To obtain a cobordism of $E_{k,l}$, we first construct a bounding manifold Q of $SU(3)$. Secondly, the action $\rho_{k,l}$ on $SU(3)$ can be extended to Q but this action is not free in general.

However, there is another gcd condition depending on k, l, called condition (C), which guarantees that there are at most three exceptional orbits. Next, we will remove small equivariant neighborhoods of these orbits. Dividing by the free action, one obtains a smooth 8-manifold $W_{k,l}$ which bounds $E_{k,l}$ and three other well-known manifolds. Since the Kreck-Stolz invariants s_i are additive, $s_i(E_{k,l})$ can be computed in terms of $s_i(W_{k,l})$ and the known s_i of the three other components.

We will now construct a bounding manifold of $SU(3)$. The 3×3 matrices in $SU(3)$ are completely determined by two arbitrary rows or columns. This shows that the projection to the first two columns gives a diffeomorphism of $SU(3)$ to the *complex Stiefel manifold*:

$$V_2(\mathbb{C}^3) = \{(U,V) \in S^5 \times S^5 \mid U \cdot V = 0\},$$

where

$$(U,V) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \text{ and } U \cdot V = u_1\overline{v}_1 + u_2\overline{v}_2 + u_3\overline{v}_3.$$

In order to construct a bounding manifold of $SU(3)$, we will construct one of $V_2(\mathbb{C}^3)$. Define the maps

$$f_1 : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{R},\ (U,V) \longmapsto U \cdot U + V \cdot V - 2,$$

$$f_2 : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C},\ (U,V) \longmapsto U \cdot V,$$

$$f_3 : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{R},\ (U,V) \longmapsto V \cdot V.$$

Note that

$$V_2(\mathbb{C}^3) = \{(U,V) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid f_1(U,V) = 0,\ f_2(U,V) = 0,\ f_3(U,V) = 1\}.$$

Definition 5.2.0.6 *Define Q and P as follows:*

$$Q := \{(U,V) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid f_1(U,V) = 0,\ f_2(U,V) = 0,\ f_3(U,V) \leq 1\}.$$

$$P := \{(U,V) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid f_1(U,V) = 0,\ f_2(U,V) = 0\}.$$

Following [Kru05], one can show that P is a compact submanifold of $\mathbb{C}^3 \times \mathbb{C}^3$ and Q is a bounding manifold of $V_2(\mathbb{C}^3)$. The action $\rho_{k,l}$ of S^1 on $SU(3)$ can be extended to Q in the canonical way. That is, for $(U,V) \in V_2(\mathbb{C}^3)$ and $z \in S^1$,

$$z \cdot (U,V) \longmapsto \rho_k(z)(U,V)\rho_l(z^{-1}) = \begin{pmatrix} z^{k_1-l_1}u_1 & z^{k_1-l_2}v_1 \\ z^{k_2-l_1}u_2 & z^{k_2-l_2}v_2 \\ z^{k_3-l_1}u_3 & z^{k_3-l_2}v_3 \end{pmatrix}.$$

But this action is not necessarily free. To deal with this problem, we need a gcd condition on k, l: (k,l) satisfy *condition (C)* if the matrix

$$(k_i - l_j) = \begin{pmatrix} k_1 - l_1 & k_1 - l_2 & k_1 - l_3 \\ k_2 - l_1 & k_2 - l_2 & k_2 - l_3 \\ k_3 - l_1 & k_3 - l_2 & k_3 - l_3 \end{pmatrix} \tag{C}$$

contains a row or a column whose $k_i - l_j$ are pairwise relatively prime. To see how to construct a cobordism of $E_{k,l}$ satisfying condition (C), it is enough to assume that the elements in the first column are pairwise relative prime. It is easy to check that the conditions (C) and (G) imply that there are at most three exceptional orbits depending on whether $k_1 - l_1, k_2 - l_1, k_3 - l_1 \neq \pm 1$:

$$B_1 := \{(u,0,0,0,0,0) \mid u \in S^1\},$$

$$B_2 := \{(0,v,0,0,0,0) \mid v \in S^1\},$$

$$B_3 := \{(0,0,w,0,0,0) \mid w \in S^1\},$$

with the isotropy groups:

$$\mathbb{Z}_{|k_1-l_1|},\ \mathbb{Z}_{|k_2-l_1|},\ \mathbb{Z}_{|k_3-l_1|},$$

respectively. Here we regard Q as a 2-codimensional submanifold of the sphere S^{11} of radius $\sqrt{2}$ and the inclusion map $Q \hookrightarrow S^{11}$ is obviously equivariant relative to the representation $\alpha : S^1 \longrightarrow U(6)$ defined by

$$\alpha : z \longmapsto \mathrm{diag}(z^{k_1-l_1}, z^{k_2-l_1}, z^{k_3-l_1}, z^{k_1-l_2}, z^{k_2-l_2}, z^{k_3-l_2}).$$

Next, we claim that the total spaces of the normal bundles of these orbits are as follows:

$$\nu(B_1) := \{(u, v_1, v_2, 0, v_3, v_4) \mid u \in S^1,\ v_i \in \mathbb{C}\},$$
$$\nu(B_2) := \{(v_1, v, v_2, v_3, 0, v_4) \mid v \in S^1,\ v_i \in \mathbb{C}\},$$
$$\nu(B_3) := \{(v_1, v_2, w, v_3, v_4, 0) \mid w \in S^1,\ v_i \in \mathbb{C}\}.$$

To obtain the claim, it is enough to consider B_1 only. For a fixed $p = (u, 0, 0, 0, 0, 0) \in B_1$, since $(U, V) \in Q \subset \mathbb{C}^3 \times \mathbb{C}^3$ implies $U \cdot V = 0$, the normal vectors to p in the first copy of C^3 have the form $(0, v_1, v_2, 0, 0, 0)$ and the normal vectors to p in the second copy of C^3 have the form $(0, 0, 0, 0, v_3, v_4)$. Hence, the set of the normal vectors to p consists of the vectors of the form $(0, v_1, v_2, 0, 0, 0) + (0, 0, 0, 0, v_3, v_4) = (0, v_1, v_2, 0, v_3, v_4)$ and so the normal space at p can be identified as $\{(u, v_1, v_2, 0, v_3, v_4) \mid v_i \in \mathbb{C}\}$. Then the claim is proved.

One has the canonical extended action of S^1 on $\nu(B_i)$ for $i = 1, 2, 3$. By the tubular neighborhood theorem [MiSt74], there are equivariant diffeomorphisms:

$$\varphi_i : \nu(B_i) \longrightarrow Q$$

on disjoint tubular neighborhoods of B_i for $i = 1, 2, 3$, respectively. Let U_i be disjoint small equivariant neighborhoods of B_i for $i = 1, 2, 3$. Then for each i, we have

$$\nu(B_i) \cong V_i \cong U_i \sim B_i \sim S^1,$$

where

$$V_1 := \{(u, v_1, v_2, 0, v_3, v_4) \mid u \in S^1,\ v_i \in \mathbb{C} \text{ such that } \sum |v_i|^2 < 1\},$$
$$V_2 := \{(v_1, v, v_2, v_3, 0, v_4) \mid v \in S^1,\ v_i \in \mathbb{C} \text{ such that } \sum |v_i|^2 < 1\},$$
$$V_3 := \{(v_1, v_2, w, v_3, v_4, 0) \mid w \in S^1,\ v_i \in \mathbb{C} \text{ such that } \sum |v_i|^2 < 1\}.$$

Therefore, if we remove $U := \bigcup_i U_i$ from Q, then the action $\rho_{k,l}$ becomes free and the quotient under the action $\rho_{k,l}$ of S^1 will be a smooth 8-manifold.

Definition 5.2.0.7 *Define Q_0 to be the complement of U. The quotient under the action $\rho_{k,l}$ of S^1 on Q_0 is denoted by $W_{k,l}$. That is,*

$$Q_0 := Q - U \text{ and } W_{k,l} := Q_0/S^1.$$

Since $U_i \cong V_i$, the boundary of Q_0 can be regarded as $SU(3)+K_1+K_2+K_3$, where $K_i \cong S^1 \times S^7$ are defined as follows:

$$K_1 := \{(u, v_1, v_2, 0, v_3, v_4) \mid u \in S^1,\ v_i \in \mathbb{C} \text{ such that } \sum |v_i|^2 = 1\},$$
$$K_2 := \{(v_1, v, v_2, v_3, 0, v_4) \mid v \in S^1,\ v_i \in \mathbb{C} \text{ such that } \sum |v_i|^2 = 1\},$$
$$K_3 := \{(v_1, v_2, w, v_3, v_4, 0) \mid w \in S^1,\ v_i \in \mathbb{C} \text{ such that } \sum |v_i|^2 = 1\}.$$

Here the positive sign represents the disjoint union. Passing through the action $\rho_{k,l}$ of S^1 on Q_0, we see that $W_{k,l}$ bounds the four components:

$$E_{k,l} + L_1 + L_2 + L_3,$$

where L_i are the *7-dimensional lens spaces*:

$$L_1 := L(k_1 - l_1; k_2 - l_1, k_3 - l_1, k_2 - l_2, k_3 - l_2),$$
$$L_2 := L(k_2 - l_1; k_1 - l_1, k_3 - l_1, k_1 - l_2, k_3 - l_2),$$
$$L_3 := L(k_3 - l_1; k_1 - l_1, k_2 - l_1, k_1 - l_2, k_2 - l_2).$$

5.3. Topology of Cobordism

The homotopy type of Q and the cohomology ring of Q_0 and $W_{k,l}$ can be described by the following three lemmas:

Lemma 5.3.0.3 *Consider $f_3 : P \longrightarrow \mathbb{R}$. The subspace $f_3^{-1}(0)$ is diffeomorphic to S^5 and the inclusion $f_3^{-1}(0) \hookrightarrow Q$ is a homotopy equivalence.*

Proof By construction, $f_3^{-1}(0) \cong S^5$. For each $A = [a_1\ a_2\ a_3] \in SU(3)$ where a_i is a column vector in $\mathbb{C}^3$ and $(U, V) \in \mathbb{C}^3 \times \mathbb{C}^3$, we have

$$\begin{aligned} f_3(A(U,V)) &= f_3(AU, AV) \\ &= f_3(u_1a_1 + u_2a_2 + u_3a_3, v_1a_1 + v_2a_2 + v_3a_3) \\ &= (v_1a_1 + v_2a_2 + v_3a_3) \cdot (v_1a_1 + v_2a_2 + v_3a_3) \\ &= v_1\overline{v}_1 + v_2\overline{v}_2 + v_3\overline{v}_3 \\ &= f_3(U,V). \end{aligned}$$

This implies that $f_3 : P \longrightarrow \mathbb{R}$ is invariant under the canonical action of $SU(3)$ on P. Then the orbits of this action are $f_3^{-1}(s) \cong SU(3)$ if $s \neq 0$ and $f_3^{-1}(0) \cong S^5$ if $s = 0$. Therefore, 0 is only the critical value of f_3 and hence Q can be deformed by the gradient flow into $f_3^{-1}(0)$. Then Q and $f_3^{-1}(0)$ are homotopy equivalent.

□

Lemma 5.3.0.4 *The cohomology ring of Q_0 is given as follows:*

i	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$H^i(Q_0)$	$\mathbb{Z}$	*0*	*0*	*0*	*0*	$\mathbb{Z}$	*0*	$\mathbb{Z}^3$	$\mathbb{Z}^3$	*0*

Proof Since $B_i \sim S^1$ for all i, applying Lefschetz duality implies that

$$H^j(Q_0) = H^j(Q \backslash (SU(3) + 3 \cdot S^1)) \cong H_{9-j}(Q, SU(3) + 3 \cdot S^1).$$

Using the long exact sequence for a pair $(Q, SU(3) + 3 \cdot S^1)$ and $Q \sim S^5$, one obtains

$$H^j(Q_0) \cong H_{9-j}(Q, SU(3) + 3 \cdot S^1) \cong H_{8-j}(SU(3) + 3 \cdot S^1),$$

for $j \leq 7$ and $j \neq 3, 4$. Then

$$H^j(Q_0) \cong \begin{cases} 0 & \text{if } j = 1, 2, 6, \\ \mathbb{Z} & \text{if } j = 0, 5, \\ \mathbb{Z}^3 & \text{if } j = 7. \end{cases}$$

Similarly,

$$\begin{aligned}
H^3(Q_0) &\cong H_6(Q, SU(3) + 3 \cdot S^1) \\
&\cong \mathrm{Ker}(H_5(SU(3) + 3 \cdot S^1) \to H_5(Q)) = 0, \\
H^4(Q_0) &\cong H_5(Q, SU(3) + 3 \cdot S^1) \\
&\cong H_5(Q)/\mathrm{Im}(H_5(SU(3) + 3 \cdot S^1) \to H_5(Q)) = 0, \\
H^8(Q_0) &\cong H_1(Q, SU(3) + 3 \cdot S^1) \\
&\cong \mathrm{Ker}(H_0(SU(3) + 3 \cdot S^1) \to H_0(Q)) = \mathbb{Z}^3, \\
H^9(Q_0) &\cong H_0(Q, SU(3) + 3 \cdot S^1) \\
&\cong H_0(Q)/\mathrm{Im}(H_0(SU(3) + 3 \cdot S^1) \to H_0(Q)) = 0.
\end{aligned}$$

□

Now one easily applies the Gysin sequence to the fibration

$$S^1 \longrightarrow Q_0 \longrightarrow W_{k,l}$$

to yield the following lemma:

Lemma 5.3.0.5 *The cohomology ring of $W_{k,l}$ is given as follows:*

i	0	1	2	3	4	5	6	7	8
$H^i(W_{k,l})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0

Moreover, $H^2(W_{k,l})$ is generated by the first Chern class z of the circle bundle

$$S^1 \longrightarrow Q_0 \longrightarrow W_{k,l},$$

and z^2 is a generator of $H^4(W_{k,l})$.

5.4. The Invariants

In this section, we will compute the self-linking number, the first Pontrjagin class, and the Kreck-Stolz invariants of the Eschenburg spaces. As we will see in the next chapter, these invariants play a role in the homotopy, homeomorphism, and diffeomorphism classification of the Eschenburg spaces, which are manifolds of odd type. Now let $E_{k,l}$ be an Eschenburg space and $u \in H^2(E_{k,l})$ be a generator.

From the first section of this chapter, we know that $E_{k,l}$ is a manifold of odd type,

$$r = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3),$$

and

$$s(E_{k,l}) = \sigma_3(k_1, k_2, k_3) - \sigma_3(l_1, l_2, l_3)$$

is a homotopy invariant. Here σ_i is the i-th elementary symmetric function. In particular, Lemma 3.3.0.2 shows that the self-linking number is given by

$$L(u^2, u^2) = -\frac{s^{-1}}{r} \ \in \mathbb{Q}/\mathbb{Z},$$

where s^{-1} is the integer such that $s^{-1} \cdot s(E_{k,l}) \equiv 1 \bmod r$.

Remark 5.4.0.9 *By Corollary 3.3.0.1, for Eschenburg spaces $E_{k,l}$ and $E_{k',l'}$ of type r together with generators $u \in H^2(E_{k,l})$ and $u' \in H^2(E_{k',l'})$, respectively, we obtain*

$$L(u^2, u^2) = L(u'^2, u'^2) \ \in \mathbb{Q}/\mathbb{Z} \Longleftrightarrow s(E_{k,l}) \equiv s(E_{k',l'}) \ mod\ r.$$

Proposition 5.1.0.13 implies that $E_{k,l}$ is a spin manifold and its first Pontrjagin class is given by

$$p_1(E_{k,l}) = (2\sigma_1(k)^2 - 6\sigma_2(k))u^2 \ \in \mathbb{Z}_{|r|}.$$

Next, our goal is to find the Kreck-Stolz invariants of $E_{k,l}$. They can be computed via the characteristic classes p_1^2, $z^2 p_1$, z^4, and the signature of a bounding manifold. Using condition (C), we constructed a cobordism $W_{k,l}$ of the Eschenburg spaces $E_{k,l}$. Therefore, we obtain the Kreck-Stolz invariants only for Eschenburg spaces $E_{k,l}$ satisfying condition (C). Following [CEZ07], for $r < 5,000$, a program written in Maple and C code shows that there are 54 Eschenburg spaces of positive sectional curvature for which condition (C) fails. For example, the Eschenburg space when $k = (35, 21, -34)$ and $l = (12, 10, 0)$:

$$(k_i - l_j) = \begin{pmatrix} 23 & 5^2 & 5 \cdot 7 \\ 3^2 & 11 & 3 \cdot 7 \\ -2 \cdot 23 & -2^2 \cdot 11 & -2^2 \cdot 17 \end{pmatrix}$$

Now suppose that condition (C) holds. By the same argument as in section 5.2, we may assume that the elements in the first column are pairwise relative prime, see [CEZ07] for the general situation. We begin by computing the first Pontrjagin class of $W_{k,l}$. Recall from the previous section, we have the circle bundle

$$\eta : S^1 \longrightarrow Q_0 \longrightarrow W_{k,l},$$

and $W_{k,l}$ is the cobordism of $E_{k,l}$ and the lens spaces L_i for $i = 1, 2, 3$. We regard Q_0 as a 2-codimensional submanifold of the sphere S^{11} of radius $\sqrt{2}$. Then the inclusion map $Q_0 \hookrightarrow S^{11}$ is equivariant relative to the representation $\alpha : S^1 \longrightarrow U(6)$ defined by

$$\alpha : z \longmapsto \mathrm{diag}(z^{k_1-l_1}, z^{k_2-l_1}, z^{k_3-l_1}, z^{k_1-l_2}, z^{k_2-l_2}, z^{k_3-l_2}).$$

Lemma 5.4.0.6 *The first Pontrjagin class of $W_{k,l}$ is given by*

$$\begin{aligned} p_1(W_{k,l}) =& ((k_1 - l_1)^2 + (k_2 - l_1)^2 + (k_3 - l_1)^2 + (k_1 - l_2)^2 \\ & + (k_2 - l_2)^2 + (k_3 - l_2)^2 - (l_1 - l_2)^2)z^2. \end{aligned}$$

Proof We can form the following vector bundle over $W_{k,l}$:

$$\xi_\eta := Q_0 \times_\alpha \mathbb{C}^6.$$

One can show that this bundle is isomorphic to the sum of complex line bundles with first Chern classes up to sign as follows:

$$(k_1 - l_1)z, (k_2 - l_1)z, (k_3 - l_1)z, (k_1 - l_2)z, (k_2 - l_2)z, \text{ and } (k_3 - l_2)z,$$

where z is a generator of $H^2(W_{k,l})$. Let ν be the normal bundle of the 9-dimensional submanifold Q_0 of S^{11}. From the equivariance property

$$\begin{aligned} f_2(z^{k_1-l_1}u_1, z^{k_2-l_1}u_2, z^{k_3-l_1}u_3, z^{k_1-l_2}v_1, z^{k_2-l_2}v_2, z^{k_3-l_2}v_3) \\ = z^{l_2-l_1}u_1\overline{v}_1 + z^{l_2-l_1}u_2\overline{v}_2 + z^{l_2-l_1}u_3\overline{v}_3 \\ = z^{l_2-l_1}f_2(u_1, u_2, u_3, \overline{v}_1, \overline{v}_2, \overline{v}_3), \end{aligned}$$

we have ν/S^1 is a complex line bundle with first Chern class $\pm(l_2-l_1)z$. Following [Szc64], we obtain

$$\tau(W_{k,l}) \oplus \tau_\eta/S^1 \oplus \nu/S^1 \cong \xi_\eta,$$

where τ_η is a trivial 1-dimensional bundle. Hence,

$$\begin{aligned} p_1(W_{k,l}) =& p_1(\xi_\eta) - p_1(\nu/S^1) \\ =& ((k_1-l_1)^2 + (k_2-l_1)^2 + (k_3-l_1)^2 + (k_1-l_2)^2 \\ &+ (k_2-l_2)^2 + (k_3-l_2)^2 - (l_1-l_2)^2)z^2, \end{aligned}$$

since the first Pontrjagin class of any complex line bundle is equal to the square of its first Chern class.

□

Secondly, we will compute the Kreck-Stolz invariants of the cobordism $W_{k,l}$ and the ones of the lens spaces L_i for $i = 1, 2, 3$. Eventually, since these invariants are additive and

$$\partial(W_{k,l}) = E_{k,l} + L_1 + L_2 + L_3,$$

this implies that

$$\begin{aligned} s_i(E_{k,l}) =& S_i(W_{k,l}) - s_i(L(k_1-l_1; k_2-l_1, k_3-l_1, k_2-l_2, k_3-l_2)) \\ &- s_i(L(k_2-l_1; k_1-l_1, k_3-l_1, k_1-l_2, k_3-l_2)) \\ &- s_i(L(k_3-l_1; k_1-l_1, k_2-l_1, k_1-l_2, k_2-l_2)) \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

Following [Kru05], there is a unique spin structure on $W_{k,l}$ inducing the spin structure on boundary. Also, one can choose the orientation on $W_{k,l}$ to be compatible with the orientation of $E_{k,l}$. Recall that

$$\begin{aligned}
S_1(W_{k,l},z) &= -\frac{1}{2^5\cdot 7}\mathrm{sign}(W_{k,l}) + \frac{1}{2^7\cdot 7}p_1^2(W_{k,l}),\\
S_2(W_{k,l},z) &= -\frac{1}{2^4\cdot 3}z^2p_1(W_{k,l}) + \frac{1}{2^3\cdot 3}z^4,\\
S_3(W_{k,l},z) &= -\frac{1}{2^2\cdot 3}z^2p_1(W_{k,l}) + \frac{2}{3}z^4.
\end{aligned}$$

Theorem 5.4.0.8 ([Kru05],[CEZ07]) *The Kreck-Stolz invariants of $W_{k,l}$ are given as follows:*

$$\begin{aligned}
S_1(W_{k,l},z) &= \frac{4\cdot|r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)| - (q(k,l))^2}{2^7\cdot 7\cdot r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)},\\
S_2(W_{k,l},z) &= \frac{q(k,l)-2}{2^4\cdot 3\cdot r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)},\\
S_3(W_{k,l},z) &= \frac{q(k,l)-8}{2^2\cdot 3\cdot r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)},
\end{aligned}$$

where

$$\begin{aligned}
q(k,l) :=& (k_1-l_1)^2 + (k_2-l_1)^2 + (k_3-l_1)^2 + (k_1-l_2)^2\\
&+ (k_2-l_2)^2 + (k_3-l_2)^2 - (l_1-l_2)^2.
\end{aligned}$$

Proof From Lemma 5.3.0.4 and Lemma 5.3.0.5, we obtain

$$H^5(W_{k,l},\partial W_{k,l}) = 0,\ H^4(W_{k,l},\partial W_{k,l}) \cong \mathbb{Z},\ H^4(W_{k,l}) \cong \mathbb{Z},\ \text{and,}$$

$$\begin{aligned}
H^4(\partial W_{k,l}) &\cong H^4(E_{k,l}) \oplus H^4(L_1) \oplus H^4(L_2) \oplus H^4(L_3)\\
&\cong \mathbb{Z}_{|r(k,l)|} \oplus \mathbb{Z}_{|k_1-l_1|} \oplus \mathbb{Z}_{|k_2-l_1|} \oplus \mathbb{Z}_{|k_3-l_1|}.
\end{aligned}$$

Because of $H^5(\partial W_{k,l}) = 0$, we have the short exact sequence:

$$0 \longrightarrow H^4(W_{k,l},\partial W_{k,l}) \xrightarrow{j^*} H^4(W_{k,l}) \xrightarrow{i^*} H^4(\partial W_{k,l}) \longrightarrow 0.$$

Since $r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)\cdot z^2$ is mapped to 0 under i^*, there exists a unique generator $v \in H^4(W_{k,l}, \partial W_{k,l})$ such that

$$j^*(v) = -r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)\cdot z^2.$$

Then $vz^2 = +1$ since the self-linking number in Lemma 3.3.0.2 and Corollary 4.2.0.2 must have compatible signs. The signature of $W_{k,l}$ is

$$\operatorname{sign}(W_{k,l}) = |-r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)|.$$

Moreover,

$$\begin{aligned} p_1^2(W_{k,l}) &= (j^*)^{-1}(p_1(W_{k,l})) \smile p_1(W_{k,l}) \\ &= \frac{-q(k,l)}{r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)} v(q(k,l)z^2) \\ &= \frac{-(q(k,l))^2}{r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)}, \end{aligned}$$

$$\begin{aligned} z^2 p_1(W_{k,l}) &= (j^*)^{-1}(z^2) \smile p_1(W_{k,l}) \\ &= \frac{-1}{r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)} v(q(k,l)z^2) \\ &= \frac{-q(k,l)}{r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)}, \end{aligned}$$

and

$$\begin{aligned} z^4 &= (j^*)^{-1}(z^2) \smile z^2 \\ &= \frac{-1}{r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)} vz^2 \\ &= \frac{-1}{r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)}. \end{aligned}$$

Using the formulas of S_i, an easy calculation gives rise to the Kreck-Stolz invariants of $W_{k,l}$.

$\square$

It remains to compute the Kreck-Stolz invariants of the lens spaces. Refer to [Kru05], he used the fact that the Kreck-Stolz invariants can also be expressed analytically, see Theorem 4.1.0.7. By analytic calculation, one obtains the following theorem:

Theorem 5.4.0.9 ([Kru05],[CEZ07]) *Let $L(p; p_1, p_2, p_3, p_4)$ be the lens space where p_1, p_2, p_3, p_4 are relatively prime to p. Then*

$$\begin{aligned}
s_1(L(p; p_1, p_2, p_3, p_4)) =& \frac{1}{2^5 \cdot 7 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=i}^{4} cot(k\pi p_j/p) \\
&+ \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} \prod_{j=i}^{4} csc(k\pi p_j/p), \\
s_2(L(p; p_1, p_2, p_3, p_4)) =& \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} (e^{\frac{2\pi i k}{|p|}} - 1) \prod_{j=i}^{4} csc(k\pi p_j/p), \\
s_3(L(p; p_1, p_2, p_3, p_4)) =& \frac{1}{2^4 \cdot p} \sum_{k=1}^{|p|-1} (e^{\frac{4\pi i k}{|p|}} - 1) \prod_{j=i}^{4} csc(k\pi p_j/p).
\end{aligned}$$

6. CLASSIFICATION THEOREM

The classification of manifolds of type r up to homeomorphism and diffeomorphism was originally provided by M. Kreck and S. Stolz [KrSt88] in 1988. In 1997 and 1998, B. Kruggel [Kru97],[Kru98] obtained various homotopy classifications for particular subfamilies of manifolds of type r, which we will see in the first section of this chapter. In section 6.2, we will prove the main result of this dissertation. This result is divided into two cases: the spin case and the nonspin case. For the spin case, we use Kreck-Stolz's and Kruggel's classifications to obtain a new homeomorphism and diffeomorphism classification. This will be a generalization of the classification of the Eschenburg spaces stated without proof in [Kru05]. For the nonspin case, using the relation proved in section 4.2 between the self-linking number and the characteristic number z^4, the invariants $s_3, \overline{s}_3$ in Kreck-Stolz's classification can be replaced by the self-linking number. Moreover, a complete picture of the classification in [Kru05] of the Eschenburg spaces satisfying condition (C) will be restated in the last section.

6.1. Homotopy Classification

Following [Kru97] and [Kru98], B. Kruggel focused on the homotopy type of a manifold M of type r. He used the cell decomposition described in section 3.2:

$$S^2 \vee S^3 \cup_{\beta_4} e^4 \cup_{\beta_5} e^5 \cup_{\beta_7} e^7,$$

with the attaching maps $\beta_4, \beta_5, \beta_7$ and determined which invariants can detect each cell. It turns out that the invariants r and $s(M) \in (\mathbb{Z}_r)^*/(\pm 1)$ detect the 5-skeleton. Equivalently, since $s(M)$ determines the self-linking number, one can say that r and $L(u^2, u^2)$ detect the 5-skeleton, where $u \in H^2(M)$ is a generator. The most difficult part is the detection of the top cell. This gives rise to various homotopy classifications for particular subfamilies of manifolds of type r. In the first paper [Kru97], B. Kruggel obtained the homotopy classification of nonspin manifolds of type r. But the classification only holds in the case that r is divisible by 24. In that case, it turns out that $p_1(M)$ mod 24 completely detects the attaching map β_7. B. Kruggel also classified the generalized Witten manifolds up to homotopy in the case that r is odd and found that $p_1(M)$ mod 3 detects the attaching map β_7. These two classifications can be expressed as follows:

Theorem 6.1.0.10 ([Kru97]) *Let M and M' be two smooth nonspin manifolds of type r such that r is divisible by 24. Let u_M and $u_{M'}$ be generators of $H^2(M)$ and $H^2(M')$. Then M and M' are (orientation preserving) homotopy equivalent if and only if*

- $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
- $p_1(M) = p_1(M')$ *mod* 24.

Theorem 6.1.0.11 ([Kru97]) *Let $M_{k,l}$ and $M'_{k',l'}$ be two generalized Witten manifolds of type odd r which are both spin or both nonspin. Let u and u' be generators of $H^2(M_{k,l})$ and $H^2(M'_{k',l'})$. Then $M_{k,l}$ and $M'_{k',l'}$ are (orientation preserving) homotopy equivalent if and only if*

- $L(u^2, u^2) = L(u'^2, u'^2)$,
- $p_1(M) = p_1(M')$ *mod* 3.

Moreover, the homotopy classification of a subfamily of the Eschenburg spaces was originally proven in this paper. However, B. Kruggel [Kru98] could later classify all Eschenburg spaces up to homotopy. Note that the Eschenburg spaces all are spin manifolds of type odd r with trivial fourth homotopy group. In the second paper [Kru98], he concentrated only on the spin case. He obtained a homotopy classification theorem for spin manifolds of odd type. Hence, this classification works for all Eschenburg spaces.

One of the Kreck-Stolz invariants plays a role in this classification. It means roughly that some multiple of $s_2(M)$ and the self-linking number determine the homotopy type of a spin manifold M of type odd r. The proof of his classification theorem depends in an essential way on the spin structure of M and the parity of r. Indeed, these conditions guarantee the existence of a bounding manifold which is necessary to compute the Kreck-Stolz invariant s_2. The homotopy classification can be described as follows:

Theorem 6.1.0.12 ([Kru98]) *Let M and M' be two smooth spin manifolds of type odd r with generators u_M and $u_{M'}$ of $H^2(M)$ and $H^2(M')$, respectively.*

- *If $\pi_4(M) = \pi_4(M') = 0$, M and M' are (orientation preserving) homotopy equivalent if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $2r \cdot s_2(M) = 2r \cdot s_2(M')$.
- *If $\pi_4(M) \cong \pi_4(M') \cong \mathbb{Z}_2$, M and M' are (orientation preserving) homotopy equivalent if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $r \cdot s_2(M) = r \cdot s_2(M')$.

Remark 6.1.0.10 *Since any Eschenburg space $E_{k,l}$ is a manifold of odd type and $\pi_4(E_{k,l}) = 0$, the above theorem is a generalization of the homotopy classification of the Eschenburg spaces.*

6.2. Homeomorphism and Diffeomorphism Classification

Recall, M. Kreck and S. Stolz gave two versions of a homeomorphism and diffeomorphism classification in [KrSt88] and [KrSt91], and a corrected version in [KrSt98]. One of the classifications that we will use in this dissertation can be described as follows:

Theorem 6.2.0.13 ([KrSt91],[KrSt98]) *Let M and M' be two smooth manifolds of type r which are both spin or both nonspin. Then M is (orientation preserving) diffeomorphic (homeomorphic) to M' if and only if $s_i(M) = s_i(M') \in \mathbb{Q}/\mathbb{Z}$ ($\bar{s}_i(M) = \bar{s}_i(M') \in \mathbb{Q}/\mathbb{Z}$) for $i = 1, 2, 3$.*

In this section, a new version of this homeomorphism and diffeomorphism classification will be proved. The proof is divided into two cases: the spin case and the nonspin case. As we will see, the classification of spin manifolds is more interesting than the one for nonspin manifolds. This is because the spin structure plays a role in the proof. However, the classification of nonspin manifolds can be slightly improved using the relation between the self-linking number and the characteristic number z^4.

6.2.1 Spin Case

Let M be a smooth spin manifold of type r, u a generator of $H^2(M)$, and (W, z) a bounding pair. The following are the Kreck-Stolz invariants for the spin case:

$$\begin{aligned}
s_1(M) &= -\frac{1}{2^5 \cdot 7}\operatorname{sign}(W) + \frac{1}{2^7 \cdot 7} p_1^2 \bmod \mathbb{Z}, \\
s_2(M) &= -\frac{1}{2^4 \cdot 3} z^2 p_1 + \frac{1}{2^3 \cdot 3} z^4 \bmod \mathbb{Z}, \\
s_3(M) &= -\frac{1}{2^2 \cdot 3} z^2 p_1 + \frac{2}{3} z^4 \bmod \mathbb{Z}.
\end{aligned}$$

One observes that the invariants $s_2(M)$ and $s_3(M)$ both involve the terms $z^2 p_1$ and z^4. The only difference comes from the rational coefficients. We believe that there is a strong relation between $s_2(M)$ and $s_3(M)$. Note that this argument will also be utilized for the nonspin case, but by using a completely different method to obtain the relation. For the spin case, we will see how the first Pontrjagin class affects the invariant $s_1(M)$ as well. In order to obtain a new version of the classification theorem, we need appropriate bounding manifolds W and W' of M and M' so that we can compare the terms $z^2 p_1$, z^4, p_1^2, and the signature. Fortunately, the construction of the bounding manifolds in Theorem

2.1 [Kru98] perfectly works in this situation. But we have to start with two spin manifolds that are homotopy equivalent. This gives us a special version of our classification theorem.

Theorem 6.2.1.1 *Let M and M' be two smooth spin manifolds of type r. Then*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - *they are (orientation preserving) homotopy equivalent,*
  - *$s_1(M) = s_1(M')$,*
  - *$s_2(M) = s_2(M')$.*
- *M is (orientation preserving) homeomorphic to M' if and only if*
  - *they are (orientation preserving) homotopy equivalent,*
  - *$p_1(M) = p_1(M')$,*
  - *$s_2(M) = s_2(M')$.*

Using the homotopy classification, Theorem 6.1.0.12, for spin manifolds of type odd r, one can replace the homotopy equivalence statement by the self-linking number and s_2. Hence, combining this fact with the above classification theorem gives the desired classification. Note that this proof only works for smooth spin manifolds of odd type.

Classification Theorem A *Suppose that M and M' are smooth spin manifolds of type odd r with isomorphic fourth homotopy groups. Let $u_M \in H^2(M;\mathbb{Z})$ and $u_{M'} \in H^2(M';\mathbb{Z})$ be both generators.*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - *$L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,*
  - *$s_1(M) = s_1(M')$,*
  - *$s_2(M) = s_2(M')$.*

- *M is (orientation preserving) homeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $p_1(M) = p_1(M')$,
  - $s_2(M) = s_2(M')$.

Before proving Classification Theorem 6.2.1.1, we will show how to construct each bounding manifold of two (orientation preserving) homotopy equivalent spin manifolds and describe some general results.

Let M and M' be two smooth spin manifolds of the same type. Suppose that they are (orientation preserving) homotopy equivalent. Let

$$h : M' \longrightarrow M,$$

be a homotopy equivalence and u_M a generator of $H^2(M)$. The construction is done in the PL category. Note that every smooth manifold admits a PL structure by Whitehead's theorem on triangulations [Whi40]. By Proposition 4.1.0.8, there exists a pair (W, z_w) such that W is a PL 8-manifold with a spin structure, M is its boundary, and $z_w \in H^2(W)$ restricts to u_M on the boundary. Following [KrSt91], $BPL\langle 4\rangle$ classifies PL bundles with a spin structure and is equivalent to $BSpin$ in low dimensions. By surgery theory [Bro72] and [MaMi79], we can assume that the induced map

$$W \longrightarrow BS^1 \times BSpin$$

given by z_w and the spin structure is a 4-connected. Define

$$Q := W \cup_h M' \times I,$$

where $(x, 0) \in M' \times \{0\}$ is identified with $h(x) \in M \subset W$ and M' is considered as $M' \times \{1\}$, a subspace of Q. Proposition 2.3.0.2 shows that (Q, M') is a Poincaré pair. By construction, W and Q are homotopy equivalent. Let

$$\nu : Q \longrightarrow BSG$$

be the Spivak normal bundle where BSG is the classifying space of oriented spherical fibrations. Consider M' as a PL manifold. The restriction of ν has a lift to $BSPL$, the classifying space of oriented PL bundles. Then there is a commutative diagram:

$$\begin{array}{ccc} M' & \xrightarrow{\nu_{M'}} & BSPL \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\nu} & BSG \end{array}$$

The existence of a PL structure on Q depends on obstructions which lie in the cohomology groups:

$$H^{i+1}(Q, M'; \pi_i(G/PL)) \cong H^{i+1}(Q, M') \otimes \pi_i(G/PL)$$

since $H^*(Q, M')$ is free abelian. One knows that $\pi_i(G/PL)$ is trivial for odd integers i, see section 2.2. Also, since $W \longrightarrow BS^1 \times BSpin$ is a 4-connected, $\pi_i(W) = 0$ for $i = 1, 2, 3$. Using the Hurewicz theorem and the universal coefficient theorem, $H^i(W) = H_i(W) = 0$ for $i = 1, 2, 3$. By Lefschetz duality, $H^{i+1}(Q, M') = H^{i+1}(W, M') = H_{8-i-1}(W) = 0$ for even integers i. Therefore, all obstructions vanish. This implies that there exists a lift:

$$\nu_Q : Q \longrightarrow BSPL$$

of ν relative to M'. By the usual transversality arguments and the process of surgery [MaMi79], there is a 4-connected degree one normal map:

$$f : (W', M') \longrightarrow (Q, M')$$

relative to M' and the difference of the signatures of W' and Q is divisible by 8, where W' is a bounding PL 8-manifold of M' with the canonical spin structure. Let $z_Q \in H^2(Q)$ be a generator. Then $H^2(W') \cong H^2(Q) \cong \mathbb{Z}$ has a generator $z_{W'} = f^*(z_Q)$ with $z_{W'}|_{M'} = u_{M'}$, a generator of $H^2(M')$. Define the first Pontrjagin class of Q as

$$p_1(Q) := p_1(-\nu_Q).$$

Hence, we have

$$p_1(W') = f^*(p_1(Q)).$$

Now consider the long exact sequences for the pairs (W, M), (Q, M'), and (W', M'):

$$\begin{tikzcd}
\cdots \arrow[r] & H^4(W,M) \arrow[r, "j^*"] \arrow[d, "\cong"] & H^4(W) \arrow[r, "i^*"] \arrow[d, "\cong"] & H^4(M) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H^4(Q,M') \arrow[r, "j^*"] \arrow[d, "f^*"] & H^4(Q) \arrow[r, "i^*"] \arrow[d, "f^*"] & H^4(M') \arrow[r] \arrow[d, "f^*"] & \cdots \\
\cdots \arrow[r] & H^4(W',M') \arrow[r, "j^*"] & H^4(W') \arrow[r, "i^*"] & H^4(M') \arrow[r] & \cdots
\end{tikzcd}$$

There exist elements $v_W \in H^4(W, M)$, $v_Q \in H^4(Q, M')$, and $v_{W'} \in H^4(W', M')$ such that

$$j^*(v_W) = r \cdot {z_W}^2,\ j^*(v_Q) = r \cdot {z_Q}^2, \text{and } j^*(v_{W'}) = r \cdot {z_{W'}}^2$$

since these image elements are trivial in $H^4(M) \cong H^4(M') \cong \mathbb{Z}_r$. Similarly, there exist elements $u_W \in H^4(W, M)$, $u_Q \in H^4(Q, M')$, and $u_{W'} \in H^4(W', M')$ such that

$$j^*(u_W) = r \cdot p_1(W),\ j^*(u_Q) = r \cdot p_1(Q), \text{and } j^*(u_{W'}) = r \cdot p_1(W').$$

This construction yields the following lemma:

Lemma 6.2.1.1 *Suppose that M and M' are (orientation preserving) homotopy equivalent smooth spin manifolds of type r. With the above notations, the following hold:*

$$r \cdot {z_W}^2 p_1(W) = r \cdot {z_{W'}}^2 p_1(W') \ \mathit{mod}\ 24,$$

and

$$r \cdot {z_W}^4 = r \cdot {z_{W'}}^4 \ \in \mathbb{Z}.$$

Moreover, if $p_1(M) = p_1(M') \in \mathbb{Z}_r$*, then*

$$r \cdot {p_1}^2(W) = r \cdot {p_1}^2(W') \ \in \mathbb{Z}.$$

Proof By the construction of Q, the restriction of the PL bundle, induced by ν_Q over Q, to W is fiber homotopy equivalent to the PL bundle induced by the classifying map $\nu_W : W \longrightarrow BSPL$ over W. It follows by [Mig87] that

$$p_1(W) = p_1(Q) \in H^4(W; \mathbb{Z}_{24}),$$

after the canonical identification of $H^4(W)$ and $H^4(Q)$. This also implies that

$$v_W p_1(W) = v_Q p_1(Q) \text{ mod } 24,$$

and

$$v_W {z_W}^2 = v_Q {z_Q}^2 \ \in \mathbb{Z},$$

since v_W and v_Q are identical. The properties of a degree one normal map f show that

$$v_Q p_1(Q) = v_{W'} p_1(W') \ \in \mathbb{Z},$$

and

$$v_Q {z_Q}^2 = v_{W'} {z_{W'}}^2 \ \in \mathbb{Z}.$$

Hence,

$$\begin{aligned}
r \cdot {z_W}^2 p_1(W) &= \big\langle (j^*)^{-1}(r \cdot {z_W}^2) \smile p_1(W), [W, M] \big\rangle \\
&= v_W p_1(W) \\
&= v_Q p_1(Q) \text{ mod } 24 \\
&= v_{W'} p_1(W') \text{ mod } 24 \\
&= \big\langle (j^*)^{-1}(r \cdot {z_{W'}}^2) \smile p_1(W'), [W', M'] \big\rangle \text{ mod } 24 \\
&= r \cdot {z_{W'}}^2 p_1(W') \text{ mod } 24,
\end{aligned}$$

and

$$
\begin{aligned}
r \cdot {z_W}^4 &= \left\langle (j^*)^{-1}(r \cdot {z_W}^2) \smile {z_W}^2, [W, M] \right\rangle \\
&= v_W {z_W}^2 \\
&= v_Q {z_Q}^2 \\
&= v_{W'} {z_{W'}}^2 \\
&= \left\langle (j^*)^{-1}(r \cdot {z_{W'}}^2) \smile {z_{W'}}^2, [W', M'] \right\rangle \\
&= r \cdot {z_{W'}}^4 \ \in \mathbb{Z}.
\end{aligned}
$$

Now we will show the last statement. The definition of $p_1(Q)$ implies that

$$
p_1(Q)|_{M'} = i^*(p_1(Q)) = p_1(-\nu_Q|_{M'}) = p_1(-\nu_{M'}) = p_1(M').
$$

This is equivalent to say that the first Pontrjagin class $p_1(Q)$ restricts to $p_1(M')$ on M'. Since (W, M) and (Q, M') are homotopy equivalent, $p_1(W)$ restricts to $p_1(M)$ on the boundary M as well. With the same argument as above, if $p_1(M) = p_1(M')$ under the identification $H^4(M) \cong H^4(M')$, then

$$
\begin{aligned}
r \cdot {p_1}^2(W) &= \left\langle (j^*)^{-1}(r \cdot p_1(W)) \smile p_1(W), [W, M] \right\rangle \\
&= u_W p_1(W) \\
&= u_Q p_1(Q) \\
&= u_{W'} p_1(W') \\
&= \left\langle (j^*)^{-1}(r \cdot p_1(W')) \smile p_1(W'), [W', M'] \right\rangle \\
&= r \cdot {p_1}^2(W') \ \in \mathbb{Z}.
\end{aligned}
$$

□

Applying this lemma introduces homotopy invariants $2r \cdot s_2(M)$ and $r \cdot s_3(M)$ for a smooth spin manifold of type r as in the following corollary. Note that the first invariant was proved by B. Kruggel [Kru98], and we can use the similar argument for the second one.

Corollary 6.2.1.1 *If M is a smooth spin manifold of the type r, then $2r \cdot s_2(M)$ and $r \cdot s_3(M)$ are (oriented) homotopy invariants.*

Proof Let M and M' be (orientation preserving) homotopy equivalent smooth spin manifolds of the same type. By the above construction, we have (M, u_M) and $(M', u_{M'})$ are the boundary of the pairs (W, z_W) and $(W', z_{W'})$ such that

$$r \cdot {z_W}^2 p_1(W) = r \cdot {z_{W'}}^2 p_1(W') \bmod 24,$$

and

$$r \cdot {z_W}^4 = r \cdot {z_{W'}}^4 \ \in \mathbb{Z}.$$

These two equations imply that

$$\begin{aligned} 2r \cdot s_2(M) &= 2r \cdot S_2(W, z_W) \\ &= -\frac{r}{2^3 \cdot 3} \cdot {z_W}^2 p_1(W) + \frac{r}{2^2 \cdot 3} \cdot {z_W}^4 \\ &= -\frac{r}{2^3 \cdot 3} \cdot {z_{W'}}^2 p_1(W') + \frac{r}{2^2 \cdot 3} \cdot {z_{W'}}^4 \\ &= 2r \cdot S_2(W', z_{W'}) \\ &= 2r \cdot s_2(M') \ \in \mathbb{Q}/\mathbb{Z}, \end{aligned}$$

and similarly

$$\begin{aligned} r \cdot s_3(M) &= -\frac{r}{2^2 \cdot 3} \cdot {z_W}^2 p_1(W) + \frac{2r}{3} \cdot {z_W}^4 \\ &= -\frac{r}{2^2 \cdot 3} \cdot {z_{W'}}^2 p_1(W') + \frac{2r}{3} \cdot {z_{W'}}^4 \\ &= r \cdot s_3(M') \ \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

□

For any smooth spin manifold of type r, combining the above lemma and Corollary 4.2.0.2 determines its self-linking number as follows:

Corollary 6.2.1.2 *Suppose that M is a smooth spin manifold of type r and u_M is a generator of $H^2(M)$. Then the self-linking number of M can be written as*

$$L({u_M}^2, {u_M}^2) = \frac{N}{r} \ \in \mathbb{Q}/\mathbb{Z},$$

where N is some integer.

Proof With the previous manifold W, we have $r \cdot {z_W}^4 \ \in \mathbb{Z}$, and so ${z_W}^4$ is some integer over r. By Corollary 4.2.0.2, $L({u_M}^2, {u_M}^2) = {z_W}^4$ mod $\mathbb{Z}$. We are done. □

Now we are ready to prove Classification Theorem 6.2.1.1 which gives rise to one of the main classifications in this dissertation.

Proof of Classification Theorem 6.2.1.1 ($\Longrightarrow$) It is obvious by using the homeomorphism and diffeomorphism classification, Theorem 6.2.0.13, and the fact that the first Pontrjagin class is a homeomorphism invariant.

($\Longleftarrow$) Using the above notations, the pairs (W, z_W) and $(W', z_{W'})$ are bounding manifolds of (M, u_M) and $(M', u_{M'})$. First, suppose that M and M' are (orientation preserving) homotopy equivalent and $s_2(M) = s_2(M') \in \mathbb{Q}/\mathbb{Z}$. By Lemma 6.2.1.1, we have the fact that

$$r \cdot {z_W}^4 = r \cdot {z_{W'}}^4 \ \in \mathbb{Z}$$

which is equivalent to the equation:

$${z_W}^4 = {z_{W'}}^4 \ \in \mathbb{Q}.$$

By the assumption that $s_2(M) = s_2(M') \in \mathbb{Q}/\mathbb{Z}$, we have

$$-\frac{1}{2^4 \cdot 3}{z_W}^2 p_1(W) + \frac{1}{2^3 \cdot 3}{z_W}^4 = -\frac{1}{2^4 \cdot 3}{z_{W'}}^2 p_1(W') + \frac{1}{2^3 \cdot 3}{z_{W'}}^4$$

as elements in $\mathbb{Q}/\mathbb{Z}$. This implies that

$${z_W}^2 p_1(W) = {z_{W'}}^2 p_1(W') \text{ mod } (2^4 \cdot 3).$$

Therefore,

$$\begin{aligned} s_3(M) &= -\frac{1}{2^2 \cdot 3} {z_W}^2 p_1(W) + \frac{2}{3} {z_W}^4 \\ &= -\frac{1}{2^2 \cdot 3} {z_{W'}}^2 p_1(W') + \frac{2}{3} {z_{W'}}^4 \\ &= s_3(M') \ \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

Now the condition $s_1(M) = s_1(M') \in \mathbb{Q}/\mathbb{Z}$ gives the complete proof for the diffeomorphism case. For the homeomorphism case, we need to assume further that $p_1(M) = p_1(M') \in \mathbb{Z}_r$. Using Lemma 6.2.1.1, it follows that

$$r \cdot {p_1}^2(W) = r \cdot {p_1}^2(W') \ \in \mathbb{Z}$$

which is equivalent to the equation:

$${p_1}^2(W) = {p_1}^2(W') \ \in \mathbb{Q}.$$

By the process of surgery [MaMi79], the construction of the normal map f and the PL manifold W' yields that

$$\mathrm{sign}(W) = \mathrm{sign}(Q) = \mathrm{sign}(W') \bmod 8,$$

where the first equation holds because W and Q are homotopy equivalent. Therefore,

$$\begin{aligned} 28 \cdot s_1(M) &= -\frac{1}{2^3} \mathrm{sign}(W) + \frac{1}{2^5} {p_1}^2(W) \\ &= -\frac{1}{2^3} \mathrm{sign}(W') + \frac{1}{2^5} {p_1}^2(W') \\ &= 28 \cdot s_1(M') \ \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

Thus, we finish the proof of the homeomorphism case. $\square$

6.2.2 Nonspin Case

Let M be a nonspin manifold of type r together with a generator $u \in H^2(M)$. Proposition 4.1.0.8 ensures that there exists a nonspin bounding pair (W, z). The Kreck-

Stolz invariants can be described as follows:

$$\begin{aligned}
S_1(W,z) &= -\frac{1}{2^5 \cdot 7}\mathrm{sign}(W) + \frac{1}{2^7 \cdot 7}p_1^2 - \frac{1}{2^6 \cdot 3}z^2 p_1 + \frac{1}{2^7 \cdot 3}z^4,\\
S_2(W,z) &= -\frac{1}{2^3 \cdot 3}z^2 p_1 + \frac{5}{2^3 \cdot 3}z^4,\\
S_3(W,z) &= -\frac{1}{2^3}z^2 p_1 + \frac{13}{2^3}z^4.
\end{aligned}$$

In general, one can not construct any bounding manifold W of M with a spin structure. If we did, then the spin structure on W would induce a spin structure on its boundary. This is a contradiction, see Theorem 2.4.0.6. Now we loose the spin structure on the bounding manifold. Note that the spin structure of the bounding manifold was an essential ingredient in the proof for the spin case. Hence, the same method can not be applied to obtain the diffeomorphism and homeomorphism classification in the nonspin case. However, we can use the fact that $L(u^2, u^2) = z^4 \in \mathbb{Q}/\mathbb{Z}$ and an elementary calculation to show that s_2 and the self-linking number determine s_3. We note that this elementary argument can not be used for the spin case.

Lemma 6.2.2.1 *If M is a smooth nonspin manifolds of type r and u is a generator of $H^2(M)$. Then*

$$s_3(M) = 3s_2(M) + L(u^2, u^2).$$

Proof Let (W, z) be a bounding pair of (M, u). Then

$$\begin{aligned}
s_3(M) &= -\frac{1}{2^3}z^2 p_1 + \frac{13}{2^3}z^4\\
&= -\frac{1}{2^3}z^2 p_1 + \frac{5}{2^3}z^4 + z^4\\
&= 3(-\frac{1}{2^3 \cdot 3}z^2 p_1 + \frac{5}{2^3 \cdot 3}z^4) + z^4\\
&= 3s_2(M) + z^4 \ \in \mathbb{Q}/\mathbb{Z}.
\end{aligned}$$

By Corollary 4.2.0.2, we know that $L(u^2, u^2)$ and z^4 are the same in $\mathbb{Q}/\mathbb{Z}$. Hence,

$$s_3(M) = 3s_2(M) + L(u^2, u^2).$$

□

Applying Lemma 6.2.2.1 to the homeomorphism and diffeomorphism classification, Theorem 6.2.0.13, we can replace s_3 by the self-linking number. Therefore, we obtain a new version of the homeomorphism and diffeomorphism classification of nonspin manifolds of type r as follows:

Classification Theorem B *Suppose that M and M' are smooth nonspin manifolds of type r. Let $u_M \in H^2(M;\mathbb{Z})$ and $u_{M'} \in H^2(M';\mathbb{Z})$ be both generators.*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $s_1(M) = s_1(M')$,
  - $s_2(M) = s_2(M')$.
- *M is (orientation preserving) homeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $\overline{s}_1(M) = \overline{s}_1(M')$,
  - $s_2(M) = s_2(M')$.

6.3. A Complete Picture of Eschenburg Classification

Since the Eschenburg spaces are spin manifolds of odd type with trivial fourth homotopy group, they are classified by Classification Theorem A. Hence, the following classification theorem claimed by B. Kruggel in [Kru05] is now proven.

Theorem 6.3.0.1 *Let $E_{k,l}$ and $E_{k',l'}$ be two Eschenburg spaces with the same order of the fourth cohomology group. Let $u \in H^2(E_{k,l})$ and $u' \in H^2(E_{k',l'})$ be both generators. Then*

- *$E_{k,l}$ is (orientation preserving) diffeomorphic to $E_{k',l'}$ if and only if*
  - $L(u^2, u^2) = L(u'^2, u'^2)$,
  - $s_1(E_{k,l}) = s_1(E_{k',l'})$,
  - $s_2(E_{k,l}) = s_2(E_{k',l'})$.
- *$E_{k,l}$ is (orientation preserving) homeomorphic to $E_{k',l'}$ if and only if*
  - $L(u^2, u^2) = L(u'^2, u'^2)$,
  - $p_1(E_{k,l}) = p_1(E_{k',l'})$,
  - $s_2(E_{k,l}) = s_2(E_{k',l'})$.

In section 5.4, we describe the above invariants of those Eschenburg spaces satisfying condition (C). B. Kruggel did not know whether or not this condition holds for all Eschenburg spaces. Unfortunately, from [CEZ07], we know that condition (C) is not always satisfied. The homeomorphism and diffeomorphism classification of the Eschenburg spaces not satisfying condition (C) is still an open problem. The following is a complete picture of the classification of the Eschenburg spaces satisfying condition (C).

Theorem 6.3.0.2 *For the Eschenburg spaces $E_{k,l}$ satisfying condition (C), the following is a complete set of invariants:*

- *For (orientation preserving) diffeomorphism type,*
  - $|r(k,l)| \in \mathbb{Z}$,
  - $s(k,l)/r(k,l) \in \mathbb{Q}/\mathbb{Z}$,
  - $s_1(k,l) \in \mathbb{Q}/\mathbb{Z}$,
  - $s_2(k,l) \in \mathbb{Q}/\mathbb{Z}$.
- *For (orientation preserving) homeomorphism type,*

- $|r(k,l)| \;\in \mathbb{Z}$,
- $s(k,l)/r(k,l) \;\in \mathbb{Q}/\mathbb{Z}$,
- $p_1(k,l)/r(k,l) \;\in \mathbb{Q}/\mathbb{Z}$,
- $s_2(k,l) \;\in \mathbb{Q}/\mathbb{Z}$.

Here

$$
\begin{aligned}
r(k,l) =& \sigma_2(k_1,k_2,k_3) - \sigma_2(l_1,l_2,l_3),\\
s(k,l) =& \sigma_3(k_1,k_2,k_3) - \sigma_3(l_1,l_2,l_3),\\
p_1(k,l) =& 2\sigma_1(k)^2 - 6\sigma_2(k),\\
s_1(k,l) =& \frac{4\cdot|r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)| - (q(k,l))^2}{2^7\cdot 7\cdot r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)}\\
& - s_1(L(k_1-l_1;k_2-l_1,k_3-l_1,k_2-l_2,k_3-l_2))\\
& - s_1(L(k_2-l_1;k_1-l_1,k_3-l_1,k_1-l_2,k_3-l_2))\\
& - s_1(L(k_3-l_1;k_1-l_1,k_2-l_1,k_1-l_2,k_2-l_2)),\\
s_2(k,l) =& \frac{q(k,l)-2}{2^4\cdot 3\cdot r(k,l)(k_1-l_1)(k_2-l_1)(k_3-l_1)}\\
& - s_2(L(k_1-l_1;k_2-l_1,k_3-l_1,k_2-l_2,k_3-l_2))\\
& - s_2(L(k_2-l_1;k_1-l_1,k_3-l_1,k_1-l_2,k_3-l_2))\\
& - s_2(L(k_3-l_1;k_1-l_1,k_2-l_1,k_1-l_2,k_2-l_2)),
\end{aligned}
$$

where

$$
\begin{aligned}
q(k,l) =&(k_1-l_1)^2+(k_2-l_1)^2+(k_3-l_1)^2+(k_1-l_2)^2\\
&+(k_2-l_2)^2+(k_3-l_2)^2-(l_1-l_2)^2,\\
s_1(L(p;p_1,p_2,p_3,p_4)) =&\frac{1}{2^5\cdot 7\cdot p}\sum_{k=1}^{|p|-1}\prod_{j=i}^{4} cot(k\pi p_j/p)\\
&+\frac{1}{2^4\cdot p}\sum_{k=1}^{|p|-1}\prod_{j=i}^{4} csc(k\pi p_j/p),\\
s_2(L(p;p_1,p_2,p_3,p_4)) =&\frac{1}{2^4\cdot p}\sum_{k=1}^{|p|-1}(e^{\frac{2\pi ik}{|p|}}-1)\prod_{j=i}^{4} csc(k\pi p_j/p).
\end{aligned}
$$

7. CONCLUSIONS AND FUTURE WORK

In my dissertation we obtain a new version of the homeomorphism and diffeomorphism classification of most manifolds of type r. This classification can be divided into two cases: the spin case and the nonspin case. First of all, the classification for the spin case can be expressed as follows:

Classification Theorem A *Suppose that M and M' are smooth spin manifolds of type odd r with isomorphic fourth homotopy groups. Let $u_M \in H^2(M;\mathbb{Z})$ and $u_{M'} \in H^2(M';\mathbb{Z})$ be both generators.*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $s_1(M) = s_1(M')$,
  - $s_2(M) = s_2(M')$.
- *M is (orientation preserving) homeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $p_1(M) = p_1(M')$,
  - $s_2(M) = s_2(M')$.

To obtain this classification we first use the homeomorphism and diffeomorphism classification by M. Kreck and S. Stolz, Theorem 6.2.0.13, to prove the following theorem:

Theorem *Let M and M' be two smooth spin manifolds of type r. Then*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - *they are (orientation preserving) homotopy equivalent,*

- $s_1(M) = s_1(M')$,
- $s_2(M) = s_2(M')$.

- *M is (orientation preserving) homeomorphic to M' if and only if*
  - *they are (orientation preserving) homotopy equivalent,*
  - $p_1(M) = p_1(M')$,
  - $s_2(M) = s_2(M')$.

In the proof we construct bounding manifolds for M and M' and compare their invariants. Our construction was built using the spin structure on the bounding manifolds. Note that there can not be a spin structure on any bounding manifold of a nonspin manifold, see Theorem 2.4.0.6. This is why we have to assume that M and M' are spin. A similar classification of smooth nonspin manifolds of type r is still an open problem. For the proof of Classification Theorem A, we combine the homotopy classification by B. Kruggel, Theorem 6.1.0.12, and the above theorem. Since the homotopy classification assumes that r is odd and the fourth homotopy groups are isomorphic, our theorem classifies all smooth spin manifolds of odd type with isomorphic fourth homotopy groups. The homotopy classification of smooth spin manifolds of even type is still an open problem. Note that some of the generalized Witten manifolds are examples of manifolds of even type.

Secondly, a new version of the homeomorphism and diffeomorphism classification of nonspin manifolds of type r can be expressed as follows:

Classification Theorem B *Suppose that M and M' are smooth nonspin manifolds of type r. Let $u_M \in H^2(M;\mathbb{Z})$ and $u_{M'} \in H^2(M';\mathbb{Z})$ be both generators.*

- *M is (orientation preserving) diffeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,

- $s_1(M) = s_1(M')$,
- $s_2(M) = s_2(M')$.

- *M is (orientation preserving) homeomorphic to M' if and only if*
  - $L({u_M}^2, {u_M}^2) = L({u_{M'}}^2, {u_{M'}}^2)$,
  - $\overline{s}_1(M) = \overline{s}_1(M')$,
  - $s_2(M) = s_2(M')$.

A totally different method is applied to prove this classification. This is because we do not have a homotopy classification of all smooth nonspin manifolds of type r. This also is still an open problem. For the proof of our classification, we basically only use the homeomorphism and diffeomorphism classification by M. Kreck and S. Stolz, and combine it with the relation between the self-linking number and the characteristic number z^4 in section 4.2 to replace the Kreck-Stolz invariants $s_3, \overline{s}_3$ by the self-linking number $L(u^2, u^2)$.

Finally, since the Eschenburg spaces are smooth spin manifolds of odd type, they all are classified by Classification Theorem A. In Chapter 5, following [Kru05] we construct particular cobordisms of the Eschenburg spaces satisfying condition (C) and the homeomorphism and diffeomorphism invariants are computed. This gives a complete picture of the homeomorphism and diffeomorphism classification of the Eschenburg spaces at least satisfying condition (C) as seen in section 6.3. The complete picture of the classification of all Eschenburg spaces is not yet known.

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