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Norbert Hartmann

The problem of determining the estimability of linear parametric functions in classification models is examined. The two estimability problems which are specifically aimed at are: the problem of determining whether the design matrix is of maximal rank (i.e., all cell expectations are estimable); and in the event that the design matrix is not of maximal rank, the problem of finding a basis for a subspace of estimable parametric functions involving one particular effect.

Several procedures for conveniently obtaining spanning sets for certain subspaces of estimable parametric functions in classification models are developed; and for such spanning sets the estimability of single effect as well as the rank of the design matrix is determined.

Estimability Considerations for N-Way Classification Experimental Arrangements with Missing Observations

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Yadolah Dodge

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Chairman of Department of Statistics

Redacted for Privacy

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ESTIMABILITY CONSIDERATIONS FOR N-WAY CLASSIFICATION EXPERIMENTAL ARRANGEMENTS WITH MISSING OBSERVATIONS

I. INTRODUCTION

This dissertation is concerned with the problem of determining estimability of linear parametric functions in classification models. The two estimability problems at which it specifically aims are: the problem of determining whether the design matrix is of maximal rank (i.e., all cell expectations are estimable); and in the event that the design matrix is not of maximal rank, the problem of finding a basis for the subspace of estimable parametric functions involving any one particular effect. The contributions of this dissertation to the solution of these problems are several procedures or algorithms for conveniently obtaining spanning sets for certain subspaces of estimable parametric functions. Tools such as generalized inverses (Rao, 1962) and column-reduction (Bradley, 1968) are already available for attacking such problems, but their implementation is tedious when done by hand and is subject to round-off error when done by computer. It is hoped that the algorithms which are presented here prove to be more efficient and more accurate.

For some classification models with incomplete incidence patterns, the statistician knows beforehand that the design matrix is of maximal rank. Fractionally replicated experimental designs, Latin squares and Graeco-Latin squares are examples of such cases. However, it may happen that missing observations occur by accident, so that the statistician is unsure whether the design matrix is of maximal rank or not. The design matrix being of less than maximal rank means not all the usual parametric functions will be estimable. It is important then to find out which ones are estimable. Ignoring such considerations will lead to incorrect degrees of freedom, incorrect hypothesis tests, and attempts to invert singular matrices.

The estimability problem described briefly above has been of concern to statisticians for many years.

Bose (1949) seems to have been the first writer to rigorously attack the problem. For an additive two-way classification model (block by treatment, with arbitrary incidence) Bose introduced the notion of connectedness, and via this concept answered the question of whether every treatment contrast is estimable. In Bose's terminology a treatment α_i is said to be associated with a block β_j if the treatment is contained in the block β_j i.e., there is at least one observation in the (i, j) subclass. Two treatments, two blocks, or a treatment and a block are said to be connected if it is possible to pass from one to the other by means of a chain consisting alternately of blocks and treatments such that any two adjacent members of the chain are associated. And a design is said to be a connected design if every block and treatment of the design is connected to each other.

Bose (1949) then proved that the additive two-way model is connected if and only if every treatment contrast is estimable.

Weeks and Williams (1964) treated the additive N-way classification model. They defined the design points of such a model to be connected if all simple contrasts (i.e., differences of two levels of the same factor) are estimable, and defined two design points to be nearly identical if the N-tuples corresponding to them are equal in all except one component. Using the idea of nearly identical design points, Weeks and Williams described a procedure for determining connectedness. However, as Weeks and Williams (1965) pointed out in their errata, their condition for data to be connected is sufficient but not necessary. This is easily seen by considering an additive three-way model

$$E(Y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_k ,$$

where $i = 1,2$, $j = 1,2$, and $k = 1,2$ and with data occurring in cells $(1,1,2)$, $(2,1,1)$, $(2,2,2)$, and $(1,2,1)$. This is a 1/2 replication of a 2^3 factorial. No pair of these observations are nearly identical, but $\alpha_1 - \alpha_2$, $\beta_1 - \beta_2$, $\gamma_1 - \gamma_2$ are estimable. For instance, we can write $\gamma_1 - \gamma_2$ as:

$$\left(-\frac{1}{2}\right)[(\mu+\alpha_1+\beta_1+\gamma_2) - (\mu+\alpha_1+\beta_2+\gamma_1)$$
$$+(\mu+\alpha_2+\beta_2+\gamma_2) - (\mu+\alpha_2+\beta_1+\gamma_1)] .$$

Therefore the data in the above model is connected, but have no property of being nearly identical, so that Weeks and Williams' procedure fails to provide any information.

Srivastava and Anderson (1970) also discussed the concept of connectedness in additive N-way classification models. Their definition of connectedness is equivalent to that of Weeks and Williams (1964) but stated in a slightly different form as: "the design is said to be completely connected if and only if all the linear contrasts within each factor are estimable." They defined a chain connecting two levels of a factor to be a sequence of occupied cells such that the alternating sum of the corresponding cell expectations is a nonzero multiple of the difference of the two levels. Then they established a theorem that a simple contrast is estimable if and only if there is a chain connecting the two levels involved in the contrast. They gave no algorithm for finding such chains and it seems that in any such algorithm there would be no upper bound on the number of sequences of occupied cells that must be looked at in order to find a chain.

A graphical presentation of classification data of arbitrary incidence is contained in an unpublished paper by Mexas (1972). The possibility of extending Bose's theorem to more than two factors has been considered. By a counterexample Mexas showed that pairwise connectedness is not sufficient for maximality of rank in an additive three-way classification model.

Searle (1971) mentions that "the general problem of finding necessary conditions for main effect differences to be estimable [for an additive model having more than two factors] remains as yet unsolved."

Recently, Birkes, Dodge, Hartmann, and Seely (1972) presented general and complete results for estimability considerations in an additive two-way classification model which are easily programmed for electronic computers. They introduced an algorithm, the R-process, which determines what cell expectations are estimable. Furthermore they gave a method for determining a basis for the estimable functions involving only one effect; for determining ranks of matrices pertinent to considerations for degrees of freedom; and for determining which portions of the design are connected.

Birkes, Dodge and Seely (1972) provided results on estimability for an additive three-way classification model with arbitrary incidence. They introduced the R3-process which provides a sufficient condition for a cell expectation to be estimable. They also gave an algorithm for obtaining a spanning set for the estimable contrasts involving only a single effect. The main part of the algorithm is called the Q-process, and because of the usefulness of this process in this dissertation it is described in the Appendix.

The approach to estimability of linear parametric functions in this dissertation follows the general framework established in the last two papers mentioned above.

In Chapter VI a general classification model will be considered. However, to be more specific we will investigate in detail the additive four-way model in Chapter II. In Chapter III two algorithms for finding a spanning set for combined γ and δ-contrasts will be introduced where γ and δ denote the third and fourth effects in an additive four-way model. This is followed by Chapter IV in which some useful miscellaneous results are given. In Chapter V an attempt is made to separate γ's and δ's. Chapter VII is devoted to examples and comments related to the general model.

II. ESTIMABILITY CONSIDERATIONS FOR THE FOUR-WAY MODEL

The Model

Let $\{Y_{ijkte}\}$ be a collection of n independently distributed random variables each having a common unknown variance σ^2 and each having an expectation of the form:

$$E(Y_{ijkte}) = \mu + \alpha_i + \beta_j + \gamma_k + \delta_t ,$$

where i, j, k, t range from 1 to a, b, c, d respectively and for a given i, j, k, t the index e ranges from 1 to n_{ijkt}. As usual, $n_{ijkt} = 0$ means that no random variables with the first four subscripts i, j, k, t occur in the collection. If no random variable occurs with first subscript $i = 1$, then of course there can be no estimable linear parametric function involving α_1. Since this thesis is concerned solely with estimability, α_1 might as well be dropped from the model. For this reason it is assumed that

$$n_{i...} = \Sigma_{jkt} n_{ijkt} \neq 0 \quad \text{for} \quad i = 1, \ldots, a$$

$$n_{.j..} = \Sigma_{ikt} n_{ijkt} \neq 0 \quad \text{for} \quad j = 1, \ldots, b$$

$$n_{..k.} = \Sigma_{ijt} n_{ijkt} \neq 0 \quad \text{for} \quad k = 1, \ldots, c$$

and

$$n_{...t} = \Sigma_{ijk} n_{ijkt} \neq 0 \quad \text{for} \quad t = 1, \ldots, d.$$

Thus, we are assuming a fixed effects four-way classification model without interaction and with no restrictions on the unknown parameters occurring in the above expectations.

Definitions

A linear combination of the parameters which occur in the expectations of the Y_{ijkt}'s is called a linear parametric function. Such a function is said to be estimable if it can be written as a linear combination of the cell expectations $\mu + \alpha_i + \beta_j + \gamma_k + \delta_t$ for which $n_{ijkt} \geq 1$. A linear parametric function is said to be a contrast if the sum of the coefficients of the parameters is zero. A δ-contrast is a contrast involving only the parameters $\delta_1, \ldots, \delta_d$. An α, β and γ-contrast are defined similarly. As there are no restrictions assumed on the parameters, a fact which should be noted is that an estimable linear parametric function not involving μ must necessarily be a sum of four contrasts; these contrasts being α, β, γ, and δ-contrasts respectively. To see this let f be an estimable linear parametric function not involving μ. Then we can write

$$f = \Sigma_i \Sigma_j \Sigma_k \Sigma_t \, c_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t) \; .$$

Using the usual dot notation to denote summation over the suppressed subscripts we can write

$$f = c_{....}\mu + \Sigma_i c_{i...}\alpha_i + \Sigma_j c_{.j..}\beta_j + \Sigma_k c_{..k.}\gamma_k + \Sigma_t c_{...t}\delta_t .$$

Since f does not involve μ it follows that $c_{....} = 0$. Then $\Sigma_{i=1}^{a} c_{i...}\alpha_i$ is an α-contrast because $\Sigma_{i=1}^{a} c_{i...} = c_{....} = 0$. Similarly the other terms of f are seen to be β, γ, and δ-contrasts respectively.

Estimability

In this section we develop a procedure for obtaining a spanning set for the vector space of estimable parametric functions involving only one of the four classification effects. For convenience we concentrate on finding a spanning set for $\mathcal{H}$, the vector space of all estimable δ-contrasts. Once such a spanning set is obtained a basis for $\mathcal{H}$ can then be extracted by standard methods.

In order to obtain a spanning set for $\mathcal{H}$ we do the following steps:

1. Direct δ-differences. First apply the R-process described below to a special two-dimensional matrix. By doing this we generally find more estimable cell expectations. That is, even though a particular n_{ijkt} may be zero, it is possible that the cell expectation $\mu + \alpha_i + \beta_j + \gamma_k + \delta_t$ is estimable. After applying the R-process,

some δ-contrasts may be "directly" seen to be estimable. For example if $\mu + \alpha_1 + \beta_2 + \gamma_3 + \delta_1$ and $\mu + \alpha_1 + \beta_2 + \gamma_3 + \delta_3$ are both estimable, then their difference $\delta_1 - \delta_3$ is estimable. We collect all the "direct δ-differences" that can be obtained in this way. These direct δ-differences form a set D.

2. Direct ω-differences. We form a new two-dimensional matrix and apply the R-process again in order to find more estimable cell expectations. This time we find some estimable contrasts, called "direct ω-differences," involving γ and δ-effects. The reason that we bring γ-effects into consideration is that our procedure makes use of the Q-process of Birkes, Dodge and Seely (1972), which eliminates only μ, α, and β-effects. We collect all "direct ω-differences" that can be obtained in this way. These direct ω-differences form a set E.

3. Contrasts from the Q-process. At this step we form a special matrix $\overline{M}$. By applying the Q-process we obtain more estimable ω-contrasts. These contrasts form the set F.

4. Separation of δ-contrasts. Since the set E and F found at steps 2 and 3 contain contrasts involving γ-effects as well as δ-effects, we must take linear combinations of these ω-contrasts to obtain δ-contrasts.

Remark 2.1. Since the maximum dimension of the vector space is $d-1$, if at any point we find $d-1$ linearly independent estimable functions involving δ-effects we stop the procedure.

Let us describe the R-process.

The R-process is a procedure applied to any two-dimensional matrix, say W, to obtain a final matrix Z. The R-process is defined as follows:

1) For each pair i, j, if $w_{ij} = 0$ set $z_{ij} = 1$, otherwise, set $z_{ij} = 0$.
2) For each pair i, j, if there exist k, h such that $z_{ih} = z_{kh} = z_{kj} = 1$, then set $z_{ij} = 1$. (Pictorially, we add the fourth corner whenever three corners of a rectangle appear in the matrix.)
3) Continue step (2), using the new nonzero z_{ij}'s as corners of new rectangles, until no more entries can be changed.

Observe that the final matrix Z is a matrix of the same dimensions as the matrix W. Also, note that if we denote the columns of the matrix Z by $C_1, \ldots, C_v$, then for any two columns C_j and C_h either $C_j = C_h$, i.e., they have ones in precisely the same rows, or else $C_j' C_h = 0$.

We are now in a position to describe in detail the first step in obtaining a spanning set for $\mathcal{H}$. We transform the four-dimensional $a \times b \times c \times d$ matrix $N = (n_{ijkt})$ into a special two-dimensional abc x d matrix as shown in the diagram below, where the rows are identified by triples.

$$
\begin{array}{ccc}
 & & \begin{array}{cccccc} & & & \delta & & \\ 1 & 2 & \cdots & t & \cdots & d \end{array} \\
\begin{array}{c} \\ \\ \\ \alpha\beta\gamma \\ \\ \\ \end{array} & \begin{array}{c} (1,1,1) \\ (2,1,1) \\ \vdots \\ (i,j,k) \\ \vdots \\ (a,b,c) \end{array} & \left[\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & n_{ijkt} & & \\ & & & & & \\ & & & & & \end{array}\right]
\end{array}
$$

Apply the R-process to the above two-dimensional matrix to obtain a final matrix Z. Now we transform this two-dimensional matrix Z into a four-dimensional matrix M with entries $m_{ijkt} = z_{(i,j,k),t}$. (Note that the rows of matrix Z are identified by triples.) The matrix M is related to cell expectation as can be seen by the following proposition.

Proposition 2.1. If $m_{ijkt} = 1$, then the parametric function $\mu + \alpha_i + \beta_j + \gamma_k + \delta_t$ is estimable.

Proof. Consider the R-process as described above. At step (1), $m_{ijkt} = 1$ means $n_{ijkt} \neq 0$ so that $\mu + \alpha_i + \beta_j + \gamma_k + \delta_t$ is obviously estimable. We see that whenever one sets $m_{ijkt} = 1$ in some iteration of step (2) it is because there exists (u, v, w) and h

such that $m_{ijkh} = m_{uvwh} = m_{uvwt} = 1$. Applying an induction argument on the number of iterations of step 2, we can conclude

$$\mu + \alpha_i + \beta_j + \gamma_k + \delta_t = (\mu+\alpha_i+\beta_j+\gamma_k+\delta_h) - (\mu+\alpha_u+\beta_v+\gamma_w+\delta_h)$$
$$+ (\mu+\alpha_u+\beta_v+\gamma_w+\delta_t)$$

is estimable.

Remark 2.2. Although the above proposition provides a sufficient condition for a cell expectation to be estimable, it does not in general provide a necessary and sufficient condition. The matrix M can, however, for estimability considerations be viewed as the incidence matrix for the original pattern. This is easily seen from the above proposition and from the fact that $n_{ijkt} \neq 0$ in the original incidence matrix N implies that $m_{ijkt} \neq 0$ in the matrix M. We continue now with step 1 essentially treating M as the incidence matrix for the data pattern.

For $t = 1, \ldots, d$ let M_t denote the $a \times b \times c$ matrix having entries m_{ijkt}; we call these matrices the δ-levels of M. Define an equivalence relation on $\{1, 2, \ldots, d\}$ by $t \sim h$ if $M_t = M_h$. Suppose there are s equivalence classes; by relabeling, we can assume $\{1, 2, \ldots, s\}$ is a complete set of representatives for these equivalence classes. If $t \sim h$, then $\delta_t - \delta_h$ is seen to be estimable. We refer to these estimable contrasts as direct δ-differences.

Let $D = \{\delta_t - \delta_h : 1 \leqq t \leqq s,\ s+1 \leqq h \leqq d,\ t \sim h\}$, and let $\mathcal{D}$ be the vector space spanned by D. Once we have D we reduce estimability problems to a model with fewer δ-effects by keeping δ_t for only one t in each equivalence class in $\{1, 2, \ldots, d\}$. To see intuitively why this can be done recall that a linear parametric function is estimable provided it can be expressed as a linear combination of cell expectations for which $m_{ijkt} \geqq 1$. Thus if we drop δ_h from the model while keeping δ_t where $t \sim h$, and if $m_{ijkh} \geqq 1$, then we do not lose the corresponding cell expectation, because $m_{ijkt} = 1$ and $\delta_t - \delta_h$ is in D, so that we have

$$\mu + \alpha_i + \beta_j + \gamma_k + \delta_h = (\mu+\alpha_i+\beta_j+\gamma_k+\delta_t) - (\delta_t-\delta_h) \tag{2.1}$$

To formally prove why we can drop δ_h, suppose ψ is a nonzero estimable linear parametric function. By definition, we can write

$$\psi = \Sigma_{i=1}^{a}\Sigma_{j=1}^{b}\Sigma_{k=1}^{c}\Sigma_{t=1}^{d} c_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t)$$

where $c_{ijkt} = 0$ if $m_{ijkt} = 0$. For each h, $s+1 \leq h \leq d$, find t_h, $1 \leq t_h \leq s$, such that $h \sim t_h$. Then we can write

$$\psi = \Sigma\ \Sigma\ \Sigma\ \Sigma_{t=1}^{s} c_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t)$$

$$+ \Sigma\ \Sigma\ \Sigma\ \Sigma_{h=s+1}^{d} c_{ijkh}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_h)$$

Substituting 2.1 we have:

$$\psi = \Sigma\,\Sigma\,\Sigma\,\Sigma_{t=1}^{s} c_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t)$$

$$+ \Sigma\,\Sigma\,\Sigma\,\Sigma_{h=s+1}^{d} c_{ijkh}[(\mu+\alpha_i+\beta_j+\gamma_k+\delta_{t_h})-(\delta_{t_h}-\delta_h)]$$

or

$$\psi = \Sigma\,\Sigma\,\Sigma\,\Sigma_{t=1}^{s} c_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t)$$

$$+ \Sigma\,\Sigma\,\Sigma\,\Sigma_{h=s+1}^{d} c_{ijkh}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_{t_h})$$

$$- \Sigma\,\Sigma\,\Sigma\,\Sigma_{h=s+1}^{d} c_{ijkh}(\delta_{t_h}-\delta_h)$$

where $1 \leq t_h \leq s$.

Thus, we can write

$$\psi + \Sigma_{i=1}^{a}\Sigma_{j=1}^{b}\Sigma_{k=1}^{c}\Sigma_{h=s+1}^{d} c_{ijkh}(\delta_{t_h}-\delta_h)$$

$$= \Sigma_{i=1}^{a}\Sigma_{j=1}^{b}\Sigma_{k=1}^{c}\Sigma_{t=1}^{s} c'_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t)$$

Note that $c'_{ijkt} = 0$ if $m_{ijkt} = 0$. Therefore in order to find more estimable δ-contrasts we need to consider only the estimable contrasts of the form

$$\Sigma_{i=1}^{a}\Sigma_{j=1}^{b}\Sigma_{k=1}^{c}\Sigma_{t=1}^{s} c'_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t) \qquad (2.2)$$

Remark 2.3. Similar considerations could also be made with regard to α, β and γ-effects. For instance, by applying the R-process to a special two-dimensional matrix having the columns identified by the levels of γ-effects, we may find a set, say K, which consists of direct γ-differences. In this case, if it happens that we find $c-z$ direct γ-differences then we only need to consider contrasts involving z of the γ-effects.

Remark 2.4. It is not actually necessary to do step (1) for our algorithm to work. That is, one could start immediately with step (2) described below by simply setting the matrix M' equal to N. But in many cases doing step (1) and finding the set D will facilitate the search for a spanning set for $\mathcal{H}$. The second algorithm of Chapter III bypasses the set D.

We now start the second step toward finding a spanning set for $\mathcal{H}$ by working with the $a \times b \times c \times s$ submatrix of M consisting of the first s levels $M_1, \ldots, M_s$; denote this matrix by M'.

Define ω to be the column vector whose transpose is $(\gamma_1, \ldots, \gamma_c, \delta_1, \ldots, \delta_s)$. The reason that we bring γ-effects into consideration is that we want to make use of the Q-process which eliminates only μ, α, and β-effects.

We now form the two-dimensional $ab \times cs$ matrix, as shown in the diagram below, where rows and columns are identified by pairs:

$$\begin{array}{ccc} & & \gamma\delta \\ & & \begin{array}{ccccc} (1,1) & \cdots & (k,t) & \cdots & (c,s) \end{array} \\ & \begin{array}{c} (1,1) \\ \vdots \\ (i,j) \\ \vdots \\ (a,b) \end{array} & \left[\begin{array}{ccccc} & & & & \\ & & \vdots & & \\ & \cdots & m_{ijkt} & \cdots & \\ & & \vdots & & \\ & & & & \end{array}\right] \\ \alpha\beta & & \end{array}$$

Apply the R-process to the above two-dimensional matrix to obtain a final matrix G. Now we transform this two-dimensional matrix into the four-dimensional $a \times b \times c \times s$ matrix M'' with entries $m_{ijkt} = g_{(i,j),(k,t)}$ (note that the rows and columns of G are identified by pairs).

For each pair of indices (k,t) where $1 \leq k \leq c$ and $1 \leq t \leq s$ let $M_{(k,t)}$ denote the $a \times b$ matrix with entries m_{ijkt} and let $\omega_{(k,t)} = \gamma_k + \delta_t$. Because the R-process has been applied, either $M_{(g,h)} = M_{(u,v)}$ or else, there are no i,j such that $m_{ijgh} = 1$ and $m_{ijuv} = 1$. If $M_{(g,h)} = M_{(u,v)}$, and $m_{qrgh} \neq 0$ for some q,r, then $\omega_{(g,h)} - \omega_{(u,v)}$ is directly seen to be estimable, and moreover if $m_{ijuv} = 1$, then

$$\mu + \alpha_i + \beta_j + \gamma_g + \delta_h = (\mu+\alpha_i+\beta_j+\gamma_u+\delta_v) + (\gamma_g+\delta_h-\gamma_u-\delta_v)$$

is estimable. Define an equivalence relation on $C = \{(1,1), \ldots, (c,s)\}$ by $(g,h) \sim (u,v)$ if $M_{(g,h)} = M_{(u,v)}$ and $m_{ijgh} \neq 0$ for some i,j.

Let S be a complete set of representatives for these equivalence classes. For each i, j, there is at most one pair (k, t), $(k, t) \in S$, such that $m_{ijkt} \neq 0$. Let $E = \{\omega_{(g,h)} - \omega_{(k,t)} \mid (g, h) \in S, (g, h) \sim (k, t)\}$. Let $\mathcal{E}$ be the vector space spanned by E. Then from an argument similar to the one given for Expression (2.2) it follows that in order to find more estimable δ-contrasts we need to consider only the ω-contrasts of the form

$$\Sigma_{i=1}^{a}\Sigma_{j=1}^{b}\Sigma_{(k,t)\in S} d_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t) \qquad (2.3)$$

where $d_{ijkt} = 0$ if $m_{ijkt} = 0$.

Now we start the third step toward finding a spanning set for by letting $\lambda'\omega$ be an estimable contrast which has the form of the Expression (2.3). Set $D_{ij} = \Sigma_{(k,t)\in S} d_{ijkt}$. Either $D_{ij} = 0$ or $D_{ij} = d_{ijuv}$ where u, v is the unique pair of indices such that $m_{ijuv} \neq 0$. We see that $\Sigma_i D_{ij} = 0$ for all j and $\Sigma_j D_{ij} = 0$ for all i. For some (i_1, j_1), $D_{i_1 j_1} \neq 0$. Since $\Sigma_j D_{i_1 j} = 0$, there is some $j_2, j_2 \neq j_1$, with $D_{i_1 j_2} \neq 0$. Since $\Sigma_i D_{ij_2} = 0$, there is some $i_2, i_2 \neq i_1$, with $D_{i_2, j_2} \neq 0$. In this way we get a sequence

$$(i_1, j_1)(i_1, j_2)(i_2, j_2)(i_2, j_3)(i_3, j_3)\cdots,$$

where $i_p \neq i_{p+1}$, $j_p \neq j_{p+1}$ for all p and $D_{ij} \neq 0$ for each (i, j) in the sequence.

Let p, q be the first two indices such that $j_p = j_q$ with $p < q$. In this case the sequence

$$(i_p, j_p)(i_p, j_{p+1})(i_{p+1}, j_{p+1}) \ldots (i_{q-1}, j_{q-1})(i_{q-1}, j_q)$$

yields a loop.

We define a <u>loop</u> to be a sequence of an even number $u (u > 0)$ of pairs

$$(i_1, j_1)(i_2, j_2) \ldots (i_u, j_u)$$

such that

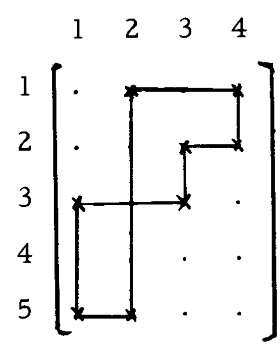
i) the pairs are distinct,

ii) $i_t = j_{t+1}$ for t odd $(t = 1, 2, \ldots, u-1)$,

iii) $j_t = j_{t+1}$ for t even $(t = 2, 4, \ldots, u-2)$,

iv) $j_1 = j_u$.

For example, $(3,3)(3,1)(5,1)(5,2)(1,2)(1,4)(2,4)(2,3)$ is a loop. If we connect the corresponding entries of a two-dimensional matrix we obtain a picture of a rectilinear loop:



Above we said the sequence

$$(i_p, j_p)(i_p, j_{p+1}) \cdots (i_{q-2}, j_{q-1})(i_{q-1}, j_{q-1})(i_{q-1}, j_q)$$

yields a loop. It is itself a loop if $i_{q-1} \neq i_p$. If $i_{q-1} = i_p$, then $(i_{q-1}, j_{q-1})(i_p, j_{p+1}) \cdots (i_{q-2}, j_{q-1})$ is a loop. For convenience let us assume $(i_p, j_p)(i_p, j_{p+1}) \cdots (i_{q-1}, j_q)$ is a loop. For each (i_u, j_v) in this sequence, let $k_{uv} t_{uv}$ be the unique index such that $d_{i_u j_v k_{uv} t_{uv}} \neq 0$. Then $m_{i_u j_v k_{uv} t_{uv}} \neq 0$, so $\mu + \alpha_{i_u} + \beta_{j_v} + \gamma_{k_{uv}} + \delta_{t_{uv}}$ is estimable. Now

$$(\mu + \alpha_{i_p} + \beta_{j_p} + \gamma_{k_{pp}} + \delta_{t_{pp}}) - (\mu + \alpha_{i_p} + \beta_{j_{p+1}} + \gamma_{k_{p,p+1}} + \delta_{t_{p,p+1}})$$

$$+ \cdots - (\mu + \alpha_{i_{q-1}} + \beta_{j_q} + \gamma_{k_{q-1,q}} + \delta_{t_{q-1,q}})$$

$$= \gamma_{k_{pp}} + \delta_{t_{pp}} - \gamma_{k_{p,p+1}} - \delta_{t_{p,p+1}} + \cdots - \gamma_{k_{q-1,q}} - \delta_{t_{q-1,q}}$$

is an estimable ω-contrast; call it $\rho'\omega$. The estimable ω-contrast $\lambda\omega - d_{i_p j_p k_{pp} t_{pp}} \rho'\omega$ can be expressed as a sum of estimable cell expectations with strictly fewer nonzero coefficients than the expression for $\lambda'\omega$.

Using induction we can argue that every estimable ω-contrast of the form $\Sigma_i^a \Sigma_j^b \Sigma_{(k,t)\in S} d_{ijkt}(\mu + \alpha_i + \beta_j + \gamma_k + \delta_t)$ can be expressed as a linear combination of estimable ω-contrasts which can be derived from the loops in the above manner.

Let $\mathcal{W}$ be the vector space of estimable ω-contrasts of the form $\Sigma\,\Sigma\,\Sigma_{(k,t)\in S}\, d_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t)$. We have just proved the following:

<u>Lemma 2.1.</u> The vector space $\mathcal{W}$ of estimable ω-contrasts of the form $\Sigma\,\Sigma\,\Sigma_{(k,t)\in S}\, d_{ijkt}(\mu+\alpha_i+\beta_j+\gamma_k+\delta_t)$ is spanned by the contrasts $(\gamma_{k_1}+\delta_{t_1}) - (\gamma_{k_2}+\delta_{t_2}) \ldots - (\gamma_{k_u}+\delta_{t_u})$ where $(i_1, j_1)\ldots(i_u, j_u)$ is a loop, and (k_r, t_r) is the unique pair in S such that $m_{i_r j_r k_r t_r} \neq 0$.

Define the matrix $\bar{M}$ such that

$$\bar{m}_{ij} = \begin{cases} (k,t) & \text{if } m_{ijkt} = 1,\ (k,t)\in S \\ 0 & \text{otherwise} \end{cases}$$

A loop $(i_1, j_1)\ldots(i_u, j_u)$ will be called a loop in $\bar{M}$ if $\bar{m}_{i_p, j_p} \neq 0$ for $p = 1, \ldots, u$. Let $\mathcal{P} = \{P \mid P = (i_1, j_1)\ldots(i_u, j_u)$ is a loop in $\bar{M}\}$ and let $\mathcal{Q} = \{P \mid P$ is a loop in $\bar{M}$ derived by the Q-process$\}$. For the sequence $P \in \mathcal{P}$ such that $P = (i_1, j_1)\ldots(i_u, j_u)$, define $\omega(P) = \Sigma_{r=1}^{u}(-1)^{r+1}(\gamma_{k_r}+\delta_{t_r})$, where (k_r, t_r) is the unique pair in S such that $m_{i_r j_r k_r t_r} \neq 0$. In Lemma 2.1 we proved that $\mathcal{W}$ is spanned by $\{\omega(P) \mid P \in \mathcal{P}\}$. Define $F = \{\omega(P) \mid P \in \mathcal{Q}\}$. Let $\gamma(P)$ be the part of $\omega(P)$ involving only γ's.

<u>Lemma 2.2.</u> The vector space $\mathcal{W}$ of estimable ω-contrasts is spanned by F.

The proof of this lemma is a direct analogy to the lemma on page 12 of Birkes, Dodge and Seely (1972).

<u>Theorem 2.1.</u> The vector space $\mathcal{S}$ of estimable ω-contrasts may be written as

$$\mathcal{S} = \mathcal{D} \oplus \{\mathcal{E} + \mathcal{W}\}$$

where $\oplus$ denotes direct sum.

The direct sum in Theorem 2.1 follows from the fact that after finding a direct δ-difference $\delta_t - \delta_h$ for the set D, we can eliminate δ_h, $s+1 \leq h \leq d$, and only keep δ_t, $1 \leq t \leq s$, in the model for the remainder of the process. Therefore no linear combinations of the elements in $\mathcal{D}$ will occur in $\{\mathcal{E} + \mathcal{W}\}$ i.e.,

$$\mathcal{D} \cap \{\mathcal{E} + \mathcal{W}\} = \{\emptyset\} .$$

As the fourth and final step for finding a spanning set for $\mathcal{H}$, let $\mathcal{S}$ be the set of sequences P such that $P \in \mathcal{Q}$ or P is the sequence of two pairs $(k_1, t_1)(k_2, t_2)$ such that $m_{ijk_1t_1} \neq 0$ and $m_{ijk_2t_2} \neq 0$ for some i, j. Let

$$\mathcal{J} = \{a_1\omega(P_1) + \ldots + a_n\omega(P_n) \mid P_1, \ldots, P_n \in \mathcal{S}, a_1\gamma(P_1) + \ldots + a_n\gamma(P_n) = 0\} .$$

Recalling that $\mathcal{H}$ is the vector space of all estimable δ-contrasts we get the following theorem.

<u>Theorem 2.2.</u> $\mathcal{H} = \mathcal{D} \oplus \mathcal{J}$, where $\mathcal{D}$ is the vector space spanned by the set D of direct δ-differences and $\mathcal{J}$ is the vector space defined as above, i.e., $\mathcal{J} = \mathcal{H} \cap \{\mathcal{E} + \mathcal{W}\}$.

The proof of this theorem follows from the fact that $\mathcal{D} \subset \mathcal{H}$.

It should be noted that the δ-differences found directly after applying the R-process at the first step yield a linearly independent set D of d-s differences. The problem then becomes to extract a basis for estimable δ-contrasts from the sets E and F. Fortunately, for small experiments such a basis is often easily obtained by hand.

III. TWO ALGORITHMS FOR FINDING A SPANNING SET FOR $\mathcal{G}$

In this chapter we give two algorithms for finding a spanning set for ω-contrasts. We assume the same model, assumptions and notations as introduced in Chapter II.

The first algorithm follows approximately the proof of Theorem 2.1. The second algorithm is different from the first one in that the R-process is only implicitly used and Remark 2.4 is taken into account. It will be seen that each of these two algorithms has some advantages over the other depending upon the structure of the incidence matrix. We will demonstrate both these algorithms via some examples.

Algorithm 3.1

This algorithm consists of the following procedure:

Step (1). Transform the four-dimensional $a \times b \times c \times d$ matrix $N = (n_{ijkt})$ into a two-dimensional $abc \times d$ matrix whose columns are correspondent to δ-effects. (See Chapter II for the exact transformation.)

Step (2). Apply the R-process to this two-dimensional matrix to obtain a final matrix.

Step (3). Compare the columns of the final matrix. If two columns, say t and h, have nonzero entries in the same row, then $\delta_t - \delta_h$ is estimable. Keep one column and eliminate the other one.

Collect all the direct δ-differences that can be obtained in this way. These direct δ-differences form the set D.

Step (4). Suppose we are left with s of these columns. Transform this $abc \times d$ matrix into an $a \times b \times c \times s$ matrix M'. (Note that this is a submatrix of the matrix M of Chapter II.)

Step (5). Form a two-dimensional $ab \times cs$ matrix whose columns corresponded to pair (γ_k, δ_t).

Step (6). Apply the R-process to this two-dimensional matrix to obtain a final matrix.

Step (7). Transform this final matrix into a four-dimensional matrix M''.

Step (8). For $(g,h) \neq (u,v)$ if $M''_{(g,h)} = M''_{(u,v)}$, and $m''_{ijgh} \neq 0$, for some i,j then $(\gamma_g+\delta_h) - (\gamma_u+\delta_v)$ is estimable. Keep either $M''_{(g,h)}$ or $M''_{(u,v)}$ and ignore the other one. Collect all direct ω-differences that can be obtained at this step. These direct ω-differences form the set E.

Step (9). Form a matrix $\bar{M}$ as described in Chapter II.

Step (10). Apply the Q-process to this matrix. Collect the ω-contrasts that we can obtain by this step into our spanning set for $\mathcal{G}$ and stop.

Example 3.1. For an $a \times b \times c \times d$ incidence matrix N let $N_1, \ldots, N_d$ denote the δ-levels, i.e., N_t is an $a \times b \times c$ matrix with entries n_{ijkt}. Consider the following $4 \times 3 \times 2 \times 5$ incidence matrix N:

$N_1 =$

δ_1	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	3				1	
α_2			2			
α_3						
α_4						

$N_2 =$

δ_2	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1						
α_2						
α_3						
α_4	1					1

N_3 =

	δ_3					
	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1					1	
α_2						
α_3						
α_4		4			2	

N_4 =

	δ_4					
	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1						
α_2	1					
α_3					6	
α_4			1			

N_5 =

	δ_5					
	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	1					
α_2						
α_3				1		
α_4						

Steps (1)-(2). Transform the four-dimensional $4 \times 3 \times 2 \times 5$ matrix $N = (n_{ijkt})$ into a two-dimensional 24×5 matrix. The final matrix Z obtained from N by the R-process is

$$Z = \begin{array}{c} \alpha\beta\gamma \\ \\ \begin{array}{c} (1,1,1) \\ (2,1,1) \\ (3,1,1) \\ (4,1,1) \\ (1,2,1) \\ (2,2,1) \\ (3,2,1) \\ (4,2,1) \\ (1,3,1) \\ (2,3,1) \\ (3,3,1) \\ (4,3,1) \\ (1,1,2) \\ (2,1,2) \\ (3,1,2) \\ (4,1,2) \\ (1,2,2) \\ (2,2,2) \\ (3,2,2) \\ (4,2,2) \\ (1,3,2) \\ (2,3,2) \\ (3,3,2) \\ (4,3,2) \end{array} \end{array} \begin{array}{c} \begin{array}{ccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{array} \\ \begin{bmatrix} 1 & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ x & 0 & 1 & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

where x indicates a cell that has been filled by applying the R-process.

Step (3). We see that columns 1, 3 and 5 have nonzero entries in the first row. Thus $\delta_1 - \delta_3$ and $\delta_1 - \delta_5$ are estimable, and they form the set D. Keep columns 1, 2 and 4 and eliminate columns 3 and 5. Thus we are left with a 24×3 matrix.

Step (4). Transform the 24×3 matrix into the four-dimensional $4 \times 3 \times 2 \times 3$ matrix M' with entries $m'_{ijkt} = z_{(i,j,k),t}$. Thus we have:

$$M'_1 = \begin{array}{|c|c|c|}\hline 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

$$M'_2 = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array}$$

$$M'_4 = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

<u>Steps (5)-(6).</u> Form a two-dimensional 12×6 matrix with rows and columns identified by pairs, and apply the R-process to this two-dimensional matrix to obtain a final matrix G. After applying the R-process we have the following matrix where, as before, x indicates a cell filled in by the R-process.

$$
\begin{array}{ccc}
 & & (\gamma, \delta) \\
 & & \begin{array}{cccccc} (1,1) & (2,1) & (1,2) & (2,2) & (1,4) & (2,4) \end{array} \\
\begin{array}{c} (\alpha, \beta) \\ G = \end{array} & \begin{array}{c} (1,1) \\ (2,1) \\ (3,1) \\ (4,1) \\ (1,2) \\ (2,2) \\ (3,2) \\ (4,2) \\ (1,3) \\ (2,3) \\ (3,3) \\ (4,3) \end{array} & \begin{bmatrix}
1 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x & 1 & 0 \\
x & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

<u>Step (7).</u> Transform this two-dimensional matrix into the four-dimensional $4 \times 3 \times 2 \times 3$ matrix M'' with entries

$m''_{ijkt} = g_{(i,j),(k,t)}$ we have:

$$M''_{(1,1)} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M''_{(2,1)} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M''_{(1,2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad M''_{(2,2)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M''_{(1,4)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M''_{(2,4)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step (8). We see that $M''_{(1,1)} = M''_{(2,1)}$ and $M''_{(2,2)} = M''_{(1,4)}$ which implies $(\gamma_1+\delta_1) - (\gamma_2+\delta_1)$ and $(\gamma_2+\delta_2) - (\gamma_1+\delta_4)$ are estimable. Keep $M''_{(1,1)}$, $M''_{(1,2)}$, $M''_{(2,2)}$ and $M''_{(2,4)}$. Thus the set $E = \{\gamma_1-\gamma_2, -\gamma_1+\gamma_2+\delta_2-\delta_4\}$.

Step (9). Form a matrix $\bar{M}$:

$\overline{M}$ =

(1, 1)	(1, 1)	-
(2, 2)	-	(1, 1)
(1, 1)	(2, 4)	-
(1, 2)	(1, 1)	(2, 2)

Step (10). Apply the Q-process:

i)

(1, 1)	(1, 1)	-
(2, 2)	-	(1, 1)
(1, 1)	(2, 4)	-
(1, 2)	(1, 1)	(2, 2)

$$(\gamma_1+\delta_1) - (\gamma_1+\delta_1) + (\gamma_2+\delta_4) - (\gamma_1+\delta_1)$$

ii)

	(1, 1)	-
(2, 2)	-	(1, 1)
(1, 1)	(2, 4)	-
(1, 2)	(1, 1)	(2, 2)

$$(\gamma_2+\delta_2) - (\gamma_1+\delta_1) + (\gamma_2+\delta_2) - (\gamma_1+\delta_2)$$

iii)

	(1, 1)	-
	-	(1, 1)
(1, 1)	(2, 4)	-
(1, 2)	(1, 1)	(2, 2)

$$(\gamma_1+\delta_1) - (\gamma_2+\delta_4) + (\gamma_1+\delta_1) - (\gamma_1+\delta_2)$$

(iv)

	(1, 1)	-
	-	(1, 1)
	(2,4)	-
(1, 2)	(1, 1)	(2, 2)

At this point there are no further loops which can be obtained and hence the Q-process and step (10) are concluded. From this step the Q-process has given us the following three estimable contrasts:

$$\left.\begin{array}{ccccc} \gamma_1 & \gamma_2 & \delta_1 & \delta_2 & \delta_4 \\ \hline -1 & 1 & -1 & 0 & 1 \\ -2 & 2 & -1 & 1 & 0 \\ 1 & -1 & 2 & -1 & -1 \end{array}\right\} \text{ Set F}$$

Note: There are many different ways we could have done the Q-process above and each way would give a different set of three estimable contrasts. But they would always span the same vector space $\mathcal{W}$.

Thus from step (4) we found the set $D = \{\delta_1-\delta_3, \delta_1-\delta_5\}$ and from step (8) we found the set $E = \{\gamma_1-\gamma_2, -\gamma_1+\gamma_2+\delta_2-\delta_4\}$ and from step (10) we found $F = \{-\gamma_1+\gamma_2-\delta_1+\delta_4, -2\gamma_1+2\gamma_2-\delta_1+\delta_2, \gamma_1-\gamma_2+2\delta_1-\delta_2-\delta_4\}$.

By Theorem 2.1 the vector space $\mathcal{G}$ of estimable ω-contrasts is spanned by D, E, and F. After taking linear combinations, we see that $\mathcal{G}$ is spanned by $\{\gamma_1-\gamma_2, \delta_1-\delta_2, \delta_1-\delta_3, \delta_1-\delta_4, \delta_1-\delta_5\}$. Thus $\dim \mathcal{G} = 5$.

Let us point out the difference between Algorithm 3.1 above and Algorithm 3.2 which is presented below. Steps (1)-(4) of the first algorithm are bypassed. Steps (5)-(8) are replaced by another method of finding direct ω-differences. Steps (9)-(10) of the first algorithm are the same as steps (5)-(6) of the second algorithm.

Algorithm 3.2

This algorithm consists of the following procedure:

Step (1). Begin to form the matrix M by changing every nonzero entry of N to 1.

Step (2). For $(k,t) \neq (u,v)$, if $m_{fgkt} = 1$ and $m_{fguv} = 1$ for some f,g then $\omega_{(k,t)} - \omega_{(u,v)}$ is estimable (recall $\omega_{(k,t)} = \gamma_k + \delta_t$) and should be put into the set E' of direct ω-differences.

Step (3). For (k,t) and (u,v) as in step (2), redefine the submatrix $M_{(k,t)}$ and $M_{(u,v)}$ of M by

$$m_{ijkt} = \begin{cases} 1 & \text{if} \quad m_{ijkt} = 1 \quad \text{or} \quad m_{ijuv} = 1 \\ 0 & \text{otherwise} . \end{cases}$$

Eliminate the submatrix $M_{(u,v)}$.

Step (4). Repeat steps (2) and (3) until no more changes can be made in the matrix M.

Step (5). Form the matrix $\bar{M}$ as defined in Chapter II.

Step (6). Apply the Q-process to the matrix $\bar{M}$. Add the ω-contrasts that we can obtain by this step to those found by step (2), and stop. We now have a spanning set for $\mathcal{G}$.

Remark 3.1. A method similar to steps (1)-(4) of Algorithm 3.2 can be used for finding direct ω-differences. This is usually preferable to the method in steps (1)-(4) of Algorithm 3.1 when calculations are being done by hand. See Chapter IV for a complete description.

Example 3.2. Consider a $4 \times 4 \times 3 \times 3$ factorial experiment with the following incidence matrix:

$N_1 =$

	δ_1											
	γ_1				γ_2				γ_3			
	β_1	β_2	β_3	β_4								
α_1	2											
α_2						1						
α_3			3									
α_4										1		

$$\delta_2$$

$$N_2 = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & 4 \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & 1 & \\ \hline & & & \\ \hline & & & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & 2 \\ \hline & & & \\ \hline \end{array}$$

$$\delta_3$$

$$N_3 = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & 5 & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & 2 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & 3 & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

<u>Step (1).</u> Change every nonzero entry of N to 1. We can assume such a change without drawing the above pattern again.

<u>Steps (2)-(4)</u>. a. $m_{2221} \neq 0$ and $m_{2213} \neq 0$. Thus $\omega_{(2,1)} - \omega_{(1,3)}$ is estimable. Keep m_{2221} and zero out m_{2213}. Similarly, from m_{2221} and m_{2233}, we find that $\omega_{(2,1)} - \omega_{(3,3)}$ is estimable. Again we keep one of the cells and zero out the other one. We keep m_{2221}.

b. $m_{3412} \neq 0$ and $m_{3432} \neq 0$, which implies $\omega_{(1,2)} - \omega_{(3,2)}$ is estimable. Keep m_{3412} and zero out m_{3432}.

c. Finally, because $m_{5231} \neq 0$ and $m_{5223} \neq 0$, $\omega_{(3,1)} - \omega_{(2,3)}$ is estimable. Keep m_{5231} and zero out m_{5223}.

Note that (a-c) are steps (2)-(4) of the algorithm.

<u>Step (5).</u> Form the matrix $\bar{M}$.

$$\bar{M} = \begin{array}{|c|c|c|c|} \hline (1,1) & - & - & - \\ \hline - & (2,1) & (2,2) & - \\ \hline - & - & (1,1) & (1,2) \\ \hline - & (3,1) & - & (2,2) \\ \hline \end{array}$$

<u>Step (6).</u> By applying the Q-process to the matrix $\bar{M}$, we find that

$$\omega_{(2,1)} - \omega_{(2,2)} + \omega_{(1,1)} - \omega_{(1,2)} + \omega_{(2,2)} - \omega_{(3,1)}$$

is estimable. Thus we get sets

$$E' = \{\omega_{(2,1)} - \omega_{(1,3)}, \omega_{(2,1)} - \omega_{(3,3)}, \omega_{(1,2)} - \omega_{(3,2)}, \omega_{(3,1)} - \omega_{(2,3)}\}$$

from steps (2)-(4) and

$$F = \{\omega_{(2,1)} + \omega_{(1,1)} - \omega_{(1,2)} - \omega_{(3,1)}\}$$

from step (6).

$\mathcal{G}$ is spanned by E' and F. This is true by Theorem 2.1 and the fact that all direct δ-differences in D will occur in the set E' if Algorithm 3.2 is used. Thus we have the following 5 estimable contrasts:

γ_1	γ_2	γ_3	δ_1	δ_2	δ_3
-1	1	0	1	0	-1
0	1	-1	1	0	-1
1	0	-1	0	0	0
0	-1	1	1	0	-1
0	1	-1	1	-1	0

If we row-reduce the above 5 x 6 matrix we see that $\dim \mathcal{G} = 4$.

IV. SOME USEFUL PROCEDURES

In this chapter we assume the same model, assumptions, and notation as introduced in Chapter II. The purposes of this chapter are: to describe the method for finding direct δ-differences which was referred to in Remark 3.1; to show how estimability problems can be reduced to models with fewer effects and sometimes even fewer factors; and to illustrate some miscellaneous shortcuts.

Notation. In the following examples we will use X to denote the design matrix. We can write $X = (\underline{1}, A, B, C, D)$, where $\underline{1}, A, B, C, D$ are the submatrices corresponding respectively to μ, α, β, γ and δ-effects. The notation $\underline{r}(X)$ denotes the rank of X.

Definition. The degrees of freedom (d.f.) for any effect is defined to be the dimension of the subspace of estimable linear parametric functions involving that effect.

An Alternative Method for Finding Direct δ-Differences

Suppose we have an additive four-way classification model with incidence matrix N. For $t = 1, \ldots, d$, let N_t be the $a \times b \times c$ matrix having entries n_{ijkt}. To find direct δ-differences we do the following steps:

Step (1). Begin to form the matrix M' (the same M' as in step (4) of Algorithm 3.1) by changing every nonzero entry of N to 1.

Step (2). For $t \neq h$, if $m'_{efgt} = 1$ and $m'_{efgh} = 1$ for some e, f, g, then $\delta_t - \delta_h$ is estimable and should be put into the set D of direct δ-differences.

Step (3). For t and h as in step (2), redefine the submatrix M'_t of M' by

$$m'_{ijkt} = \begin{cases} 1 & \text{if } m'_{ijkt} = 1 \text{ or } m'_{ijkh} = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Eliminate the submatrix M'_h.

Step (4). Repeat steps (2) and (3) until no more changes can be made in the matrix M'.

Example 4.1. Consider again Example 3.1. After step (1) we have the following pattern:

$M'_1 =$

	δ_1					
	γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3
α_1	1				1	
α_2			1			
α_3						
α_4						

$$M'_2 = \overbrace{\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline 1 & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & & 1 \\ \hline \end{array}}^{\delta_2}$$

$$M'_3 = \overbrace{\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & 1 & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & 1 & \\ \hline & & \\ \hline & & \\ \hline & 1 & \\ \hline \end{array}}^{\delta_3} \longleftarrow m'_{4223}$$

$$M'_4 = \overbrace{\begin{array}{|c|c|c|}\hline & & \\ \hline 1 & & \\ \hline & & \\ \hline & & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & 1 & \\ \hline & & \\ \hline \end{array}}^{\delta_4}$$

$$M'_5 = \overbrace{\begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline 1 & & \\ \hline & & \\ \hline \end{array}}^{\delta_5}$$

We see that $m'_{1221} = 1$ and $m'_{1223} = 1$; thus $\delta_1 - \delta_3$ is estimable. We now eliminate the submatrix M'_3 and replace 1's in the appropriate cells in M'_1. Therefore we have the following pattern:

$$M'_1 = \quad \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & 1 \\ \hline & & \\ \hline & 1 & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & 1 & \\ \hline & & \\ \hline & & \\ \hline & 1 & \\ \hline \end{array}$$

$$M'_2 = \quad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline 1 & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & & 1 \\ \hline \end{array}$$

$$M'_4 = \quad \begin{array}{|c|c|c|}\hline & & \\ \hline 1 & & \\ \hline & & \\ \hline & & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & 1 & \\ \hline & & \\ \hline \end{array}$$

$$M'_5 = \quad \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline 1 & & \\ \hline & & \\ \hline \end{array}$$

We see that $m'_{1111} = 1$ and $m'_{1115} = 1$; thus $\delta_1 - \delta_5$ is estimable. By the same procedure as above we eliminate M_5' and keep M_1'. This leads to the following pattern:

$$M_1' = \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & 1 \\ \hline & & \\ \hline & 1 & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & 1 & \\ \hline & & \\ \hline 1 & & \\ \hline & 1 & \\ \hline \end{array}$$

$$M_2' = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline 1 & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline & & 1 \\ \hline \end{array}$$

$$M_4' = \begin{array}{|c|c|c|}\hline & & \\ \hline 1 & & \\ \hline & & \\ \hline & & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & 1 & \\ \hline & & \\ \hline \end{array}$$

We see that no more changes can be made in the matrix M', and the set D is $\{\delta_1 - \delta_3, \delta_1 - \delta_5\}$.

Reduced Model

In order to show how a problem can be reduced to a smaller one let us consider the $4 \times 4 \times 3 \times 3$ factorial experiment of Example 3.2. Using Algorithm 3.2, we found that $\dim \mathcal{G} = 4$, which implies that we have full degrees of freedom for γ and δ-effects; in other words, all the linear contrasts of γ-effects and of δ-effects are estimable. In particular we know that $\delta_1 - \delta_2$ is estimable. Consider any occupied cell in N_2, such as cell $(2, 3, 2, 2)$. Recalling Equation (2.1), we see that

$$\mu + \alpha_2 + \beta_3 + \gamma_2 + \delta_1 = (\mu+\alpha_2+\beta_3+\gamma_2+\delta_2) - (\delta_1-\delta_2)$$

is estimable; we can indicate this by placing a 1 in cell $(2, 3, 2, 1)$. Now we can place a 0 in cell $(2, 3, 2, 2)$ without losing any information about estimability, because

$$\mu + \alpha_2 + \beta_3 + \gamma_2 + \delta_2 = (\mu+\alpha_2+\beta_3+\gamma_2+\delta_1) - (\delta_1-\delta_2)$$

In general, knowing that $\delta_1 - \delta_2$ is estimable, we can change the entries in all occupied cells of N_2 to 0 if we place a 1 in the corresponding cells in N_1. Thus we can eliminate N_2 and reduce the problem to a new model with only two δ-effects. The new incidence matrix $N^{(1)}$, obtained from N as just described, is shown below.

$N_1^{(1)} =$

	δ_1											
	γ_1				γ_2				γ_3			
	β_1	β_2	β_3	β_4								
α_1	1											
α_2						1	1					
α_3			1	1								1
α_4								1		1		

$N_3^{(1)} =$

δ_2											
	1										
									1		
					1						

We also know $\delta_1 - \delta_3$ is estimable. By the same procedure as above we can eliminate $N_3^{(1)}$ after placing 1's in the appropriate cells in $N_1^{(1)}$. Now the problem is reduced to a model with incidence matrix $N^{(2)}$ with only one δ-level:

$$
N_1^{(2)} = \begin{array}{c|cccc|cccc|cccc|}
 & \multicolumn{4}{c}{\gamma_1} & \multicolumn{4}{c}{\gamma_2} & \multicolumn{4}{c}{\gamma_3} \\
 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & & & & & & & & \\
\hline
\alpha_1 & 1 & & & & & & & & & & & \\
\hline
\alpha_2 & & 1 & & & & 1 & 1 & & & 1 & & \\
\hline
\alpha_3 & & & 1 & 1 & & & & & & & & 1 \\
\hline
\alpha_4 & & & & & & 1 & & 1 & & 1 & & \\
\hline
\end{array}
$$

But of course this is just the incidence matrix of a $4 \times 4 \times 3$ factorial design. Thus, knowing that all δ-contrasts are estimable, we have been able to reduce the problem from a four-way model to a three-way model.

Because all the γ-contrasts are estimable, by the same procedure the problem reduces from a three-way to a two-way model with the following incidence matrix:

$$
\begin{array}{c|c|c|c|c|}
 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\hline
\alpha_1 & 1 & & & \\
\hline
\alpha_2 & & 1 & 1 & \\
\hline
\alpha_3 & & & 1 & 1 \\
\hline
\alpha_4 & & 1 & & 1 \\
\hline
\end{array}
$$

As we see, it turns out that this incidence matrix has the same problem as the matrix $\bar{M}$, i.e., they both have nonzero entries in the same positions. The reason for this is that by applying the

R-process at (any) step (2) and step (6) of Algorithm 3.1 no cell (i, j, h, t) will be filled unless there is already some cell (i, j, u, v) which is filled.

In order to find the degrees of freedom for α and β-effects we use the method developed in Birkes, Dodge and Seely (1972). Apply the R-process to the above incidence matrix. We get the following final matrix

	β_1	β_2	β_3	β_4
α_1	1			
α_2		1	1	1
α_3		1	1	1
α_4		1	1	1

From the final matrix we see that $\alpha_2 - \alpha_3$ and $\alpha_2 - \alpha_4$ form a basis for the space of estimable α-contrasts so the degrees of freedom for α-effects is 2. (See Remark (4.1).) We can find $\underline{r}(A, B) = b + (\text{d.f. for } \alpha) = 4 + 2 = 6$, and the degrees of freedom for β-effects is $\underline{r}(A, B) - 2 = 6 - 4 = 2$.

<u>Remark 4.1.</u> The above procedure for finding the degrees of freedom for α and β-effects using a two-way model is valid only because we know there are full degrees of freedom for γ and δ-effects. In case there are not full degrees of freedom for γ and δ-effects, we can switch the roles of α and β with the roles of γ

and δ in order to find the degrees of freedom for α and β-effects. However, the above procedure is always valid for finding $\underline{r}(A, B)$.

We are now in a position to write the table of degrees of freedom. Note that $\underline{r}(X) = \underline{r}(A, B) + \dim \mathcal{Y} = 6 + 4 = 10$ which implies the design matrix is not of maximal rank (i.e., is not 11).

S.V.	d.f.
Mean	1
α	2
β	2
γ	2
δ	2
Confounded	1
Residual	$n - \underline{r}(X) = 11 - 10 = 1$
Total	11

We now introduce via examples some shortcuts that are very useful.

Some Miscellaneous Shortcuts

It may happen that the incidence matrix is such that we can obtain all estimable contrasts by direct differences. In such a case one can use the method developed by Weeks and Williams (1964).

Example 4.2. Consider the following $3 \times 3 \times 2 \times 2$ incidence matrix:

$N_1 =$

	δ_1					
	γ_1			γ_2		
	β_1	β_2	β_3			
α_1	1					
α_2		1				
α_3			1			1

$N_2 =$

	δ_2					
				1		
	1				1	1
			1			

Notice that cells $(3, 3, 1, 1)$ and $(3, 3, 2, 1)$ lead to the estimability of $\gamma_1 - \gamma_2$, and cells $(3, 3, 1, 1)$ and $(3, 3, 1, 2)$ lead to the estimability of $\delta_1 - \delta_2$. Now the problem is to find estimable contrasts for α and β effects.

Because $\gamma_1 - \gamma_2$ and $\delta_1 - \delta_2$ are estimable, we can form a two-way table as follows:

	β_1	β_2	β_3
α_1	1		
α_2	1	1	1
α_3			1

From cells $(1,1)$ and $(2,1)$ we find $\alpha_1 - \alpha_2$ to be estimable, and from cells $(2,3),(3,3)$, $\alpha_2 - \alpha_3$ is estimable, similarly from $(2,1),(2,2)$ and $(2,2),(2,3)$ we find that $\beta_1 - \beta_2$ and $\beta_2 - \beta_3$ are estimable. Note that all estimable functions have been found simply by direct differences. The design matrix for the above problem is of maximal rank, i.e., $r(X) = 7$, and table of degrees of freedom is as follows:

S.V.	d.f.
Mean	1
α	2
β	2
γ	1
δ	1
Residual	2
Total	7

It is important to notice that each N_t can be considered as the design matrix of a three-way classification model, so it is sometimes wise and efficient to see if we can find all γ-contrasts via applying the method described in Birkes, Dodge and Seely (1972). If it happens that we can obtain all γ-contrasts by this method, the problem can be considered as a three-way model by dropping the γ-effects.

Example 4.3. Consider a 2^4 design with incidence matrix as follows:

	δ_1			
	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	1			1
α_2		1	1	

δ_2			
γ_1		γ_2	
β_1	β_2	β_1	β_2
	1		

Consider the N_1 matrix corresponding to the first level of δ. N_1 is a $2 \times 2 \times 2$ matrix and applying the Q-process, it follows immediately that $\gamma_1 - \gamma_2$ is estimable. We see this by forming the matrix $\bar{N}_1 = \Sigma_{k=1}^{c} kN_1$.

$$\bar{N}_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

Now applying the Q-process to the matrix $\bar{N}_1$ we get $2\gamma_1 - 2\gamma_2$ to be estimable. Therefore we can now work with the following pattern by noting that γ-levels have been dropped from the model.

$$N_1' = \begin{array}{c|c|c|} & \multicolumn{2}{c}{\delta_1} \\ & \beta_1 & \beta_2 \\ \hline \alpha_1 & 1 & 1 \\ \hline \alpha_2 & 1 & 1 \\ \hline \end{array}$$

and

$$N_2' = \begin{array}{c|c|c|} & \multicolumn{2}{c}{\delta_2} \\ & \beta_1 & \beta_2 \\ \hline \alpha_1 & & 1 \\ \hline \alpha_2 & & \\ \hline \end{array}$$

Again this problem can be considered as a three-way additive model, from which it immediately follows that all estimable differences of α's, β's, and δ's exist.

In a four-way model with incidence matrix N, if one two-dimensional submatrix $N_{(k,t)}$ has all its cell expectations estimable (which is equivalent to the condition that all its cells can be filled by applying the R-process), then the problem can be considered as a two-way classification model.

<u>Example 4.4.</u> Consider a $4 \times 3 \times 2 \times 2$ factorial design with the following pattern:

$$
N_1 = \begin{array}{c|ccc|ccc|}
 & \multicolumn{6}{c}{\delta_1} \\
 & \multicolumn{3}{c}{\gamma_1} & \multicolumn{3}{c}{\gamma_2} \\
 & \beta_1 & \beta_2 & \beta_3 & & & \\
\hline
\alpha_1 & 1 & 1 & & & & \\
\alpha_2 & & 1 & 1 & & & \\
\alpha_3 & & & 1 & & 1 & \\
\alpha_4 & & & 1 & & & \\
\hline
\end{array}
$$

$$
N_2 = \begin{array}{c|ccc|ccc|}
 & \multicolumn{6}{c}{\delta_2} \\
\hline
 & & & & & & \\
 & 1 & & & & & \\
 & 1 & & & & & \\
 & & 1 & & & & \\
\hline
\end{array}
$$

After applying the R-process to the two-dimensional matrix $N_{(1,1)}$ all cells will be filled. This means that we have full degrees of freedom for α and β-effects. The problem of finding degrees of freedom for γ and δ-effects reduces to considering a two-way model with the following incidence matrix:

$$
\begin{array}{c|cc|}
 & \delta_1 & \delta_2 \\
\hline
\gamma_1 & 1 & 1 \\
\gamma_2 & 1 & \\
\hline
\end{array}
$$

We see that $\gamma_1 - \gamma_2$ and $\delta_1 - \delta_2$ are estimable. The design matrix is of maximal rank, and the table of degrees of freedom is as follows:

S.V.	d.f.
Mean	1
α	3
β	2
γ	1
δ	1
Residual	2
Total	10

A Comparison

In this section we try to compare the effectiveness of the R-process with the method developed by Weeks and Williams (1964).

Consider a $3 \times 2 \times 3 \times 2$ factorial having the following pattern:

	δ_1					
	γ_1		γ_2		γ_3	
	β_1	β_2	β_1	β_2	β_1	β_2
α_1				1		
α_2					1	
α_3						1

	δ_2					
	γ_1		γ_2		γ_3	
	β_1	β_2	β_1	β_2	β_1	β_2
α_1		1				
α_2	1					
α_3				1	1	

By applying the method introduced by Weeks and Williams (1964) (see Chapter I), we see that no pair of these observations is nearly identical. However, the design matrix is of maximal rank as we will see below.

Let us investigate the effect of the R-process on the same problem. Form a two-dimensional matrix, as shown in the diagram below, where the rows and the columns are identified by pairs

$$
\begin{array}{cc}
 & (\gamma, \delta) \\
 & \begin{array}{cccccc} (1,1) & (2,1) & (3,1) & (2,1) & (2,2) & (3,2) \end{array} \\
(\alpha, \beta)\ \begin{array}{c} (1,1) \\ (2,1) \\ (3,1) \\ (1,2) \\ (2,2) \\ (3,2) \end{array} &
\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

Applying the R-process to the above matrix, we get

$$
\begin{array}{ccc}
 & & (\gamma, \delta) \\
 & & \begin{array}{cccccc} (1,1) & (2,1) & (3,1) & (2,1) & (2,2) & (3,2) \end{array} \\
(\alpha, \beta) & \begin{array}{c} (1,1) \\ (2,1) \\ (3,1) \\ (1,2) \\ (2,2) \\ (3,2) \end{array} & \left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & 1 & 1 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & \boxed{1} & 1 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & 1 & 1 & \boxed{1} & 0
\end{array} \right]
\end{array}
$$

where $\boxed{1}$ is in cells filled after applying the R-process. Now form another two-dimensional matrix with columns corresponding to δ-effects and apply the R-process again. We get the two-dimensional matrix shown on the left of the following page, where 1 is in cells filled after applying the R-process. Now form a two-dimensional matrix, with columns corresponding to α-effects and apply the R-process again. We get the two-dimensional matrix shown on the right of the following page, where 1 is in cells filled after applying the R-process.

$$
\begin{array}{cc}
 & \delta \\
 & \begin{array}{cc} 1 & 2 \end{array} \\
\alpha\ \beta\ \gamma \quad \begin{array}{c}
(1,1,1) \\ (2,1,1) \\ (3,1,1) \\ (1,2,1) \\ (2,2,1) \\ (3,2,1) \\ (1,1,2) \\ (2,1,2) \\ (3,1,2) \\ (1,2,2) \\ (2,2,2) \\ (3,2,2) \\ (1,1,3) \\ (2,1,3) \\ (3,1,3) \\ (1,2,3) \\ (2,2,3) \\ (3,2,3)
\end{array} &
\left[\begin{array}{cc}
0 & 0 \\
1 & 1 \\
0 & 0 \\
1 & 1 \\
0 & 0 \\
1 & \boxed{1} \\
0 & 0 \\
\boxed{1} & \boxed{1} \\
0 & 0 \\
1 & \boxed{1} \\
0 & 0 \\
\boxed{1} & 1 \\
0 & 0 \\
1 & 1 \\
1 & 1 \\
\boxed{1} & 1 \\
0 & 0 \\
1 & 1
\end{array}\right]
\end{array}
$$

$$
\begin{array}{cc}
 & \alpha \\
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\beta\ \gamma\ \delta \quad \begin{array}{c}
(1,1,1) \\ (2,1,1) \\ (1,2,1) \\ (2,2,1) \\ (1,3,1) \\ (2,3,1) \\ (1,1,2) \\ (2,1,2) \\ (1,2,2) \\ (2,2,2) \\ (1,3,2) \\ (2,3,2)
\end{array} &
\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right]
\end{array}
$$

Now transform this matrix back to a four-dimensional matrix. We have

	δ_1					
	γ_1		γ_2		γ_3	
	β_1	β_2				
α_1	1	1	1	1	1	1
α_2	1	1	1	1	1	1
α_3	1	1	1	1	1	1

	δ_2					
	γ_1		γ_2		γ_3	
	1	1	1	1	1	1
	1	1	1	1	1	1
	1	1	1	1	1	1

We see that all cells are filled, i.e., all cell expectations are estimable, and so the design matrix is of maximal rank. Hence, in general, the R-process is more effective than the method developed by Weeks and Williams (1964).

V. A TECHNIQUE FOR SEPARATING γ-CONTRASTS FROM δ-CONTRASTS

In this chapter we assume the same model, assumptions, and notation as introduced in Chapter II. The basic purpose of the present chapter is to present an algorithm, which is essentially a row-reduction algorithm for obtaining a spanning set for $\mathcal{H}$, the vector space of estimable δ-contrasts. Recall from Chapters II and III that several means are available for obtaining a spanning set for $\mathcal{G}$, the vector space of estimable ω-contrasts, and that the algorithms in Chapter III conveniently provide a decomposition of $\mathcal{G}$ of the form

$$\mathcal{G} = \mathcal{D} \oplus (\mathcal{E} + \mathcal{W}) .$$

Since $\mathcal{D} \subset \mathcal{H}$ it then follows that

$$\mathcal{H} = \mathcal{D} \oplus \mathcal{J}, \quad \mathcal{J} = \mathcal{H} \cap (\mathcal{E} + \mathcal{W}) .$$

And thus we need only concentrate on finding a spanning set for $\mathcal{J}$ which is the vector space of all estimable δ-contrasts in $\mathcal{E} + \mathcal{W}$.

<u>Definition.</u> For a loop $P = (i_1, j_1) \ldots (i_u, j_u)$ in $\bar{Q}$ (see step (7) below), define $\gamma(P) = \gamma_{k_1} - \gamma_{k_2} + \ldots - \gamma_{k_u}$ and $\delta(P) = \delta_{t_1} - \delta_{t_2} + \ldots - \delta_{t_u}$ where k_r, t_r for $r = 1, \ldots, u$ are unique indices such that $n_{i_r j_r k_r t_r} = 1$. Furthermore, for a sequence of pairs $P = (k_1, t_1)(k_2, t_2)$ where (k_1, t_1) and (k_2, t_2) are such

that $n_{ijk_1t_1} = 1$ and $n_{ijk_2t_2} = 1$ let $\gamma(P) = \gamma_{k_1} - \gamma_{k_2}$ and $\delta(P) = \delta_{t_1} - \delta_{t_2}$ for some i, j.

We assume that the set D of direct δ-differences has already been found. Thus we are left with the matrix M' which is the incidence matrix of a four-way model which has no direct δ-differences. Therefore, for the purposes of this chapter we can assume that we have an $a \times b \times c \times s$ incidence matrix N with all entries either 0 or 1 and with no direct δ-differences.

Our technique involves finding a spanning set for γ-contrasts which should be estimable if δ-effects were not in the model. We refer to such a γ-contrast as a <u>γ-path</u>.

The technique consists of the following steps:

<u>Step (1).</u> Form the matrix $\overline{N} = \Sigma_{t=1}^{s} tN_t$, where N_t is an $a \times b \times c$ matrix with entries n_{ijkt}.

<u>Step (2).</u> Form the matrix Q with entries

$$q_{ijk} = \begin{cases} 1 & \text{if } \overline{n}_{ijk} \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Let Q_k be the $a \times b$ matrix with entries q_{ijk}.

<u>Step (3).</u> If $q_{ijk} = 1$ and $q_{ijh} = 1$ for some i, j, then we have $\gamma(P_1) = \gamma_k - \gamma_h$ where $P_1 = (k, t_k)(h, t_h)$ (we do not care what t_k and t_h are until step (5)). We emphasize again that this

$\gamma_k - \gamma_h$ is not necessarily estimable in the four-way model. Set either q_{ijk} or q_{ijh} equal to zero and keep the other. Continue in this way until there are no nonzero cells in common between Q_k's for $k = 1, \ldots, c$. Collect all $\gamma(P_i)$'s that can be obtained by this step, and also keep track of the P_i's .

<u>Step (4).</u> In the collection of γ-paths, find, if possible, $\gamma_1(P_1), \ldots, \gamma(P_m)$ such that $a_1\gamma(P) + \ldots + a_m\gamma(P_m) = 0$ for some numbers $a_1, \ldots, a_m$ which are not all zero. If this is not possible go to step (7).

<u>Step (5).</u> Search for $\delta(P_1), \ldots, \delta(P_m)$ in $\bar{N}$. Form $a_1\delta(P_1) + \ldots + a_m\delta(P_m)$ and keep it as an estimable δ-contrast.

<u>Step (6).</u> Take one $a_i, a_i \neq 0$, and eliminate $\gamma(P_i)$ from the collection of γ-paths. (The reason we can do this is shown in Lemma 5.1 below.) (Go to step (4).)

<u>Step (7).</u> Form the matrix $\bar{Q} = \Sigma_{k=1}^{c} kQ_k$.

<u>Step (8).</u> Apply the Q-process to the matrix $\bar{Q}$. Add all $\gamma(P)$ (where P is a loop in $\bar{Q}$) that can be obtained by this step to the collection of γ-paths.

<u>Step (9).</u> Repeat steps (4)-(6) with "go to step (7)" replaced by "go to step (10)".

<u>Step (10).</u> Stop. The δ-contrasts found in step (5) form a spanning set for $\mathcal{J}$.

If at any step we find $d-1$ linearly independent estimable contrasts we stop.

Remark 5.1. At step (3) sometimes it is efficient whenever we find a γ-path to search in the matrix $\bar{N}$ to see whether a corresponding δ-contrast is equal to zero. If it is, then depending on the degrees of freedom for γ-effects, the problem could possibly be reduced to a three way model.

Lemma 5.1. Consider steps (6)-(8) above. Suppose we find $a_1\gamma(P_1) + \ldots + a_m\gamma(P_m) = 0$, $a_1 \neq 0$, in step (6) and we eliminate $\gamma(P_1)$ from the collection of γ-paths in step (8). If $b_1\gamma(P_1) + \ldots + b_m\gamma(P_m) = 0$, then we will be able to tell that $b_1\delta(P_1) + \ldots + b_m\delta(P_m)$ is estimable.

Proof. From the algorithm in step (6) we find $a_1\gamma(P_1) + \ldots + a_m\gamma(P_m) = 0$. Then we know $a_1\delta(P_1) + \ldots + a_m\delta(P_m)$ is estimable. Suppose $a_1 \neq 0$ and at step (8) we eliminate $\gamma(P_1)$. Thus we cannot find the zero combination $b_1\gamma(P_1) + \ldots + b_m\gamma(P_m) = 0$ directly from the algorithm because $\gamma(P_1)$ has been eliminated. From step (6) we can write:

$$\gamma(P_1) = \left(\frac{1}{a_1}\right)[-a_2\gamma(P_2) - \ldots - a_m\gamma(P_m)], \quad a_1 \neq 0 .$$

By substitution we can write:

$$b_1\{(\frac{1}{a_1})[-a_2\gamma(P_2)-\dots-a_m\gamma(P_m)]\} + b_2\gamma(P_2)+\dots+b_m\gamma(P_m) = 0,$$

or

$$(b_2-\frac{b_1}{a_1}a_2)\gamma(P_2)+\dots+(b_m-\frac{b_1}{a_1}a_m)\gamma(P_m) = 0 . \qquad (5.1)$$

It is possible to get (5.1) directly from the remaining γ-paths $\gamma(P_2),\dots,\gamma(P_m)$. Moreover we see that

$$b_1\gamma(P_1)+\dots+b_m\gamma(P_m) = (\frac{b_1}{a_1})[a_1\gamma(P_1)+\dots+a_m\gamma(P_m)]$$

$$+ [(b_2-\frac{b_1}{a_1}a_2)\gamma(P_2)$$

$$+\dots+(b_m-\frac{b_1}{a_1}a_m)\gamma(P_m)] .$$

Thus eliminating $\gamma(P_1)$ in step (8) would not lead to losing $b_1\gamma(P_1)+\dots+b_m\gamma(P_m) = 0$. Corresponding to the zero combination of γ-paths in (5.1), we know

$$(b_2-\frac{b_1}{a_1}a_2)\delta(P_2) + \dots + (b_m-\frac{b_1}{a_1}a_m)\delta(P_m)$$

is estimable. Thus we can write

$$b_1\delta(P_1)+\ldots+b_m\delta(P_m) = \left(\frac{b_1}{a_1}\right)[a_1\delta(P_1)+\ldots+a_m\delta(P_m)]$$

$$+\left[\left(b_2-\frac{b_1}{a_1}a_2\right)\delta(P_2)+\ldots+\left(b_m-\frac{b_1}{a_1}a_m\right)\delta(P_m)\right].$$

Since the right hand side is estimable, the left hand side is also estimable.

Example 5.1. Consider the following $3 \times 3 \times 2 \times 2$ incidence matrix N:

$$\begin{array}{cc}
 & \overbrace{\hspace{16em}}^{\delta_1} \\
 & \begin{array}{c} \overbrace{\hspace{6em}}^{\gamma_1} \\ \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \end{array}\hspace{10em} \\
N_1 = & \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array}\begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline & & 1 \\ \hline 1 & & \\ \hline & & \\ \hline \end{array}
\end{array}$$

$$\begin{array}{cc}
 & \overbrace{\hspace{16em}}^{\delta_2} \\
N_2 = & \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline 1 & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline & 1 & \\ \hline & & \\ \hline & & 1 \\ \hline \end{array}
\end{array}$$

Step (1). Form $\bar{N} = N_1 + 2N_2$

$\bar{N} =$

1		
	1	2
2		1

	2	1
1		
		2

Step (2). Form the matrix Q as follows:

$Q =$

1		
	1	1
1		1

	1	1
1		
		1

Step (3). From Q it is seen that we can estimate $\gamma_1 - \gamma_2$ (ignoring δ-effects), because $q_{331} = 1$ and $q_{332} = 1$. Save the γ-path $\gamma(P_1) = \gamma_1 - \gamma_2$ and $P_1 = (1, 1)(2, 2)$ set $q_{331} = 0$. This is the only γ-path we get from step (3). Thus steps (4)-(6) will be eliminated.

Step (7). Form the matrix $\bar{Q}$

$\bar{Q} = Q_1 + 2Q_2 =$

1	2	2
2	1	1
1		2

Step (8). Applying the Q-process to the matrix $\bar{Q}$, we see that $P_2 = (1, 1)(1, 2)(2, 2)(2, 1)$ forms a loop in $\bar{Q}$, which leads to the γ-path $\gamma(P_2) = 2\gamma_1 - 2\gamma_2$.

Step (9). From step (3) we have $\gamma(P_1) = \gamma_1 - \gamma_2$ and from step (8) we have $\gamma(P_2) = 2\gamma_1 - 2\gamma_2$. We see that $2\gamma(P_1) - \gamma(P_2) = 0$.

Step (10). Corresponding to these $\gamma(P)$'s in $\overline{N}$ are $\delta(P_1) = \delta_1 - \delta_2$ and $\delta(P_2) = \delta_1 - \delta_2 + \delta_1 - \delta_1 = \delta_1 - \delta_2$. Therefore, $2\delta(P_1) - \delta(P_2) = \delta_1 - \delta_2$ is estimable.

We can stop because there can only be one degree of freedom for δ-effects.

Example 5.2. Consider the following $4 \times 5 \times 2 \times 4$ incidence matrix N:

N_1 = (δ_1)

	γ_1: β_1	β_2	β_3	β_4	β_5	γ_2: β_1	β_2	β_3	β_4	β_5
α_1						1				
α_2	1									
α_3										
α_4										

N_2 = (δ_2)

	γ_1: β_1	β_2	β_3	β_4	β_5	γ_2: β_1	β_2	β_3	β_4	β_5
α_1		1								
α_2										
α_3									1	
α_4								1		

δ_3

$N_3 =$

	1			
				1

	1			

δ_4

$N_4 =$

			1	
1				

		1		

Step (1). Form $\bar{N} = N_1 + 2N_2 + 3N_3 + 4N_4$:

$\bar{N} =$

	2			
1			4	
	3			
4				3

1				
	3			
		4	2	
		2		

Step (2). Form the matrix Q:

Q =

	1			
1			1	
	1			
1				1

1				
	1			
		1	1	
		1		

Step (3). No γ-path can be found by this step. Therefore step (4)-(6) will be eliminated.

Step (7). Form the matrix $\bar{Q} = Q_1 + 2Q_2$:

$\bar{Q}$ =

2	1			
1	2		1	
	1	2	2	
1		2		1

Step (8). Apply the Q-process to the matrix $\bar{Q}$. We see that $P_1 = (1,1)(1,2)(2,2)(2,1)$ and $P_2 = (3,3)(3,2)(1,2)(1,1)(4,1)(4,3)$ and $P_3 = (3,4)(3,2)(1,2)(1,1)(2,1)(2,4)$ are loops in $\bar{Q}$. The first and the second loops lead to $\gamma(P_1) = -2\gamma_1 + 2\gamma_2$ and $\gamma(P_2) = \gamma_1 - \gamma_2$ and we have $\gamma(P_1) + 2\gamma(P_2) = 0$. Corresponding to these $\gamma(P)$'s are $\delta(P_1) = \delta_1 - \delta_2 + \delta_3 - \delta_1 = -\delta_2 + \delta_3$ and $\delta(P_2) = \delta_4 - \delta_3 + \delta_2 - \delta_1 + \delta_4 - \delta_2 = -\delta_1 - \delta_3 + 2\delta_4$. Therefore $\delta(P_1) + \delta(P_2) = -2\delta_1 - \delta_2 - \delta_3 + 4\delta_4$. The third loop leads to

$\gamma(P_3) = \gamma_2 - \gamma_1 + \gamma_1 - \gamma_2 + \gamma_1 - \gamma_1 = 0$, and corresponding to this γ-path is $\delta(P_3) = \delta_2 - \delta_3 + \delta_2 - \delta_1 + \delta_1 - \delta_4 = 2\delta_2 - \delta_3 - \delta_4$.

Therefore $\{-2\delta_1 - \delta_2 - \delta_3 + 4\delta_4, 2\delta_2 - \delta_3 - \delta_4\}$ is a basis for .

VI. ESTIMABILITY CONSIDERATIONS FOR THE GENERAL MODEL

In this chapter we extract the essential features of Chapter II which are applicable to very general classification models. It is the purpose of this chapter to present a complete and more general solution to the problem of estimability in classification models. In Chapter II we provided a spanning set for estimable functions involving γ and δ-effects. In Chapter IV we introduced an algorithm to separate γ from δ-effects within this spanning set. The general model which will be presented here is faced with the same difficulties. That is, it only provides a spanning set for estimable functions not involving μ, α and β-effects. However, as will be seen via (several) examples, sometimes the structure of the incidence matrix is such that the separation of effects does not require too much effort.

The Model

Let $\{Y_{ijk}\}$ be a collection of n independently distributed random variables each having a common unknown variance σ^2 and each having an expectation of the form:

$$E(Y_{ijt}) = \mu + \alpha_i + \beta_j + k_{ijt} \cdot \eta$$

where η is a column vector of parameters not including μ,

$\alpha_1, \ldots, \alpha_a$ and $\beta_1, \ldots, \beta_b$ and k_{ijt} is a vector of real numbers, so that the dot product $k_{ijt} \cdot \eta$ is a linear combination of parameters in η. The indices i and j range from 1 to a and b respectively, and for each i, j the index t ranges in a set T_{ij} where T_{ij} is a finite (possible empty) index set.

Definition

A linear parametric function is estimable if it can be expressed as a linear combination of the expectations $\mu + \alpha_i + \beta_j + k_{ijt} \cdot \eta$ for $i = 1, \ldots, a$, $j = 1, \ldots, b$, and $t \in T_{ij}$. An η-contrast is defined to be any linear parametric function involving only parameters in η.

Notation and Purpose

Let $\mathcal{J}$ denote the vector space of all estimable η-contrasts. Then it is the purpose of this chapter to obtain a spanning set for the vector space $\mathcal{J}$ of estimable η-contrasts.

Estimability

Let $K = \{k_{ijt} \mid (i,j,t) \in I\}$, where I denotes the index set (i,j,t) such that i, j range from 1 to a, b respectively and $t \in T_{ij}$. Note since K is written in set form that K consists of the set of distinct vectors k_{ijt}. Now for each $k \in K$ define an $a \times b$ matrix $N(k) = (n_{ij}(k))$ such that:

$$n_{ij}(k) = \begin{cases} 1 & \text{if} \quad k = k_{ijt} \quad \text{for some} \quad t \in T_{ij} \\ 0 & \text{otherwise} . \end{cases}$$

Lemma 6.1. An extimable η-contrast can be written in the form $\Sigma_{k \in K} c_k (k \cdot \eta)$, where the c_k are real numbers.

The proof of this lemma follows directly from the definition of estimability and the fact that the coefficients of μ and α_i and β_j are zero.

Direct η-Differences

Suppose there are two indices u, v such that T_{uv} has more than one element, and let t_1 and t_2 be two elements of T_{uv}. We see that if we set $k_1 = k_{uvt_1}$ and $k_2 = k_{uvt_2}$, it follows that:

$$(k_1 - k_2) \cdot \eta = (\mu + \alpha_u + \beta_v + k_1 \cdot \eta) - (\mu + \alpha_u + \beta_v + k_2 \cdot \eta)$$

is estimable. This is equivalent to saying that if $N(k_1)$ and $N(k_2)$ have a non-zero entry in common, i.e., for some u, v both $n_{uv}(k_1) \neq 0$ and $n_{uv}(k_2) \neq 0$, then $(k_1 - k_2) \cdot \eta$ is directly seen to be estimable. We refer to these direct differences as direct η-differences. Moreover if $(k_1 - k_2) \cdot \eta$ is estimable, and $n_{ij}(k_1) \neq 0$ for some i, j then

$$\mu + \alpha_i + \beta_j + k_2 \cdot \eta = (\mu + \alpha_i + \beta_j + k_1 \cdot \eta) - (k_1 - k_2) \cdot \eta$$

is estimable.

Define a two-dimensional matrix $Z = (Z_{(i,j),k})$ with rows identified by the pairs (i,j) where i,j range from 1 to a,b respectively, with columns identified by the vectors $k \epsilon K$ and with entries defined by $Z_{(i,j),k} = n_{ij}(k)$. That is, the matrix $N(k)$ is transformed to column k of Z. Now apply the R-process to the matrix Z to obtain a final matrix W with entries $w_{(i,j),k}$. For each $k \epsilon K$ define an $a \times b$ matrix $M(k) = (m_{ij}(k))$ by $m_{ij}(k) = w_{(i,j),k}$, i.e., take column k and put it back into an $a \times b$ matrix.

<u>Proposition 6.1</u>. If $m_{ij}(k) = 1$, then $\mu + \alpha_i + \beta_j + k \cdot \eta$ is estimable. The proof of the proposition follows exactly from Proposition 2.1.

If $M(k_1) = M(k_2)$, then $(k_1 - k_2) \cdot \eta$ is directly seen to be estimable.

Now define an equivalence relation on K, by $k_1 \sim k_2$ if $M(k_1) = M(k_2)$. Let S be a complete set of representatives of these equivalence classes in K. For each pair (i,j) there is at most one $s \epsilon S$ such that $m_{ij}(s) = 1$. Let J be the set of triples (i,j,s) such that $m_{ij}(s) = 1$, $i = 1, \ldots, a$, $j = 1, \ldots, b$, $s \epsilon S$.

Let $D^* = \{(k-s) \cdot \eta \mid k \epsilon K \text{ and } s \epsilon S,\ k \sim s\}$ and recall from above that the elements of D^* are estimable. Now let $\mathcal{D}^*$ be the vector space spanned by D^*. Let $\mathcal{W}$ be the vector space of estimable η-contrasts of the form $\Sigma_{(i,j,s) \in J}\, c_{ijs}(\mu+\alpha_i+\beta_j+s \cdot \eta)$. Then similar to Theorem 2.1 we get:

Lemma 6.2.

$$\mathcal{G} = \mathcal{D}^* + \mathcal{W}$$

For each $s \in S$ select a symbol θ_s which is not the numeral 0. Define a matrix $\bar{M}$ such that

$$m_{ij} = \begin{cases} \theta_s & \text{if } m_{ij}(s) = 1 \text{ for some } s \in S \\ 0 & \text{if } m_{ij}(s) = 0 \text{ for all } s \in S \end{cases}$$

The symbol θ_s is a device which can help us remember which vector S is associated with which pair (i, j).

As in Chapter II, let $\mathcal{P} = \{P \mid P = (i_1, j_1) \ldots (i_u, j_u)$ is a loop in $\bar{M}\}$ and let $\mathcal{Q} = \{P \mid P$ is a loop in $\bar{M}$ derived by the Q-process$\}$. For $p \in \mathcal{P}$, $P = (i_1, j_1) \ldots (i_u, j_u)$, define

$$\eta(P) = s_1 \cdot \eta - s_2 \cdot \eta + \ldots - s_u \cdot \eta = (s_1 - s_2 + \ldots - s_u) \cdot \eta\ ,$$

where s_r is a unique element of S such that $m_{i_r j_r}(s_r) = 1$ for each $r = 1, 2, \ldots, u$.

Lemma 6.3. $\mathcal{W}$ is spanned by $\{\eta(P) \mid P \in \mathcal{P}\}$.

The proof of this lemma is directly analogous to the proof of Lemma 2.1.

Lemma 6.4. Let $F = \{\eta(P) \mid P \in \mathcal{Q}\}$, then $\mathcal{W}$ is spanned by F.

The proof is directly analogous to the proof of the lemma on page 12 of Birkes, Dodge and Seely (1972).

Lemma 6.5. The vector space $\mathcal{J}$ of estimable η-contrasts is spanned by $D^* \cup F$.

VII. APPLICATIONS

In Chapters III, IV, and V we presented several examples to demonstrate techniques for finding spanning sets for vector spaces of certain estimable parametric functions. We now provide some more examples which illustrate these techniques as well as the general theory of Chapter VI. With regard to the notation used in the following examples, several comments seem appropriate. In all of the examples we will use the notation introduced in Chapter VI with η consisting of all parameters in the model except μ, $\alpha_1, \ldots, \alpha_a$ and $\beta_1, \ldots, \beta_b$. Thus, $\mathcal{G}$, $\mathcal{D}^*$ and $\mathcal{W}$ will denote, respectively, the vector space of estimable η-contrasts, the vector space spanned by D^*, and the vector space spanned by F, where D^* is the set of direct η-difference and F is the set of estimable contrasts obtained from the Q-process. For the examples in which we have an additive four-way model we use the notation $\mathcal{D}$ to denote the vector space spanned by estimable direct δ-differences and $\mathcal{E}$ to denote the vector space spanned by estimable ω-differences found after obtaining $\mathcal{D}$. It should be noted that the notation $\mathcal{G}$ and $\mathcal{W}$ of the general model is consistent with the notation $\mathcal{G}$ and $\mathcal{W}$ as used in an additive four-way model; (also, our usage of $\mathcal{D}$ and $\mathcal{E}$ in four-way models is consistent with the definitions in Chapter II). Also note that when the additive four-way model is viewed in the general setting, the vector η is the vector ω. We begin with a 2^4 factorial design to illustrate how the results of Chapter VI can be compared with those of Chapter II.

Example 7.1. Consider a collection of random variables Y_{ijuve} such that

$$E(Y_{ijuve}) = \mu + \alpha_i + \beta_j + \gamma_u + \delta_v ,$$

where i, j, u and v range from 1 to 2, respectively, and $e = 1, \ldots, n_{ijuv}$. Let n_{ijuv}'s be arranged in incidence matrices N_{11}, N_{21}, N_{12} and N_{22} with entries (i, j) of N_{uv} being n_{ijuv}. Suppose that we have the following pattern:

$$N_{11} = \begin{array}{|c|c|} \hline 3 & \\ \hline & 2 \\ \hline \end{array} \qquad N_{21} = \begin{array}{|c|c|} \hline & \\ \hline 1 & \\ \hline \end{array}$$

$$N_{12} = \begin{array}{|c|c|} \hline & \\ \hline 2 & \\ \hline \end{array} \qquad N_{22} = \begin{array}{|c|c|} \hline & 4 \\ \hline & \\ \hline \end{array}$$

Let $\eta' = (\gamma_1, \gamma_2, \delta_1, \delta_2)$. The set K consists of the vectors $k = k_{uv}$ for $u = 1, 2$ and $v = 1, 2$ where $k_{uv} \cdot \eta = \gamma_u + \delta_v$. The matrix $N(k_{uv})$ is obtained from N_{uv} by changing all nonzero entries to 1's. We see that $k_{21} \sim k_{12}$. Hence, let

$S = \{k_{11}, k_{21}, k_{22}\}$ so that $D^* = \{\gamma_1-\gamma_2+\delta_2-\delta_1\}$ and let $\theta_{k_{uv}} = (u,v)$ so that

$$\bar{M} = \begin{array}{|c|c|} \hline (1, 1) & (2, 2) \\ \hline (2, 1) & (1, 1) \\ \hline \end{array}$$

Apply the Q-process to the above matrix $\bar{M}$. We find that $F = \{2\gamma_1-2\gamma_2+\delta_1-\delta_2\}$. Here the set D^* is the set E of Chapter II. By Theorem 2.1 $\mathcal{G} = \mathcal{E} + \mathcal{W}$ and by Theorem 6.1 $\mathcal{G} = \mathcal{D}^* + \mathcal{W}$; in other words, by Theorem 2.1 $\mathcal{G}$ is spanned by E and F and by Theorem 6.1. $\mathcal{G}$ is spanned by $D^* \cup F$. After reduction we see that $\mathcal{G}$ is spanned by $\{\gamma_1-\gamma_2, \delta_1-\delta_2\}$.

Since $\gamma_1 - \gamma_2$ and $\delta_1 - \delta_2$ are estimable, then $\dim \mathcal{G} = 2$ and $\underline{r}(X) = \underline{r}(A, B) + \dim \mathcal{G} = 3 + 2 = 5$, which implies X is of maximal rank. This means that the table of degrees of freedom is as follows:

Source	d.f.
Mean	1
α	1
β	1
γ	1
δ	1
Residual	7
Total	12

Example 7.2. Consider the following 4×4 Graeco-Latin square with three missing observations:

	c_1	c_2	c_3	c_4
r_1	-	$B\beta$	$C\gamma$	$D\delta$
r_2	$B\gamma$	$A\delta$	-	$C\beta$
r_3	$C\delta$	$D\gamma$	$A\beta$	$B\alpha$
r_4	$D\beta$	-	$B\delta$	$A\gamma$

This design could be viewed as a 4^4 factorial design with a $4 \times 4 \times 4 \times 4$ incidence matrix. This incidence matrix would be presented as sixteen 4×4 submatrices, one corresponding to each pair of levels (e.g, (B, γ)) of the third and fourth factors. There would be no direct ω-differences because of the way in which a Graeco-Latin square is designed. Thus $\mathcal{G} = \mathcal{W}$. Now apply the Q-process.

1)

-	$B\beta$	$C\gamma$	$D\delta$
$B\gamma$	$A\delta$	-	$C\beta$
$C\delta$	$D\gamma$	$A\beta$	$B\alpha$
$D\beta$	-	$B\delta$	$A\gamma$

$B\gamma - A\delta + D\gamma - C\delta$

2)

-	$B\beta$	$C\gamma$	$D\delta$
	$A\delta$	-	$C\beta$
$C\delta$	$D\gamma$	$A\beta$	$B\alpha$
$D\beta$	-	$B\delta$	$A\gamma$

$B\beta - D\delta + C\beta - A\delta$

3)

-		$C\gamma$	$D\delta$
	$A\delta$	-	$C\beta$
$C\delta$	$D\gamma$	$A\beta$	$B\alpha$
$D\beta$	-	$B\delta$	$A\gamma$

$A\delta - C\beta + B\alpha - D\gamma$

4)

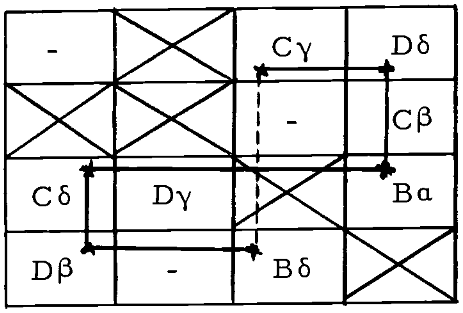
-		$C\gamma$	$D\delta$
		-	$C\beta$
$C\delta$	$D\gamma$	$A\beta$	$B\alpha$
$D\beta$	-	$B\delta$	$A\gamma$

$A\gamma - B\alpha + A\beta - B\delta$

5)

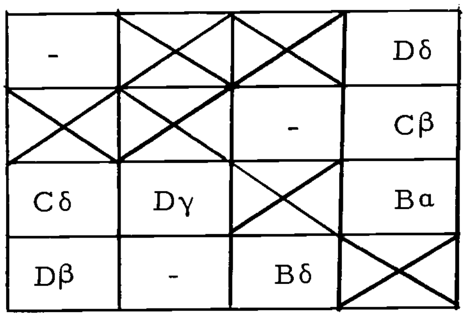
-		$C\gamma$	$D\delta$
		-	$C\beta$
$C\delta$	$D\gamma$	$A\beta$	$B\alpha$
$D\beta$	-	$B\delta$	

$A\beta - C\delta + D\beta - B\delta$

6) 

$C\gamma - D\delta + B\alpha - C\delta + D\beta - B\delta$

We are left with no more loops:



The Q-process above has given us the following 6 estimable contrasts:

	A	B	C	D	α	β	γ	δ
1)	-1	1	-1	1	0	0	2	-2
2)	-1	1	1	-1	0	2	0	-2
3)	1	1	-1	-1	1	-1	-1	1
4)	2	-2	0	0	-1	1	1	-1
5)	1	-1	-1	1	0	2	0	-2
6)	0	0	0	0	1	1	1	-3

NOTE: There are many different ways we could have done the Q-process above and each way might give a different set of 6 estimable contrasts. But they would always span the same vector space

$\mathcal{W}$. If we row-reduce the above 6×8 matrix we see that $\dim \mathcal{W} = 6$. Since $\mathcal{G} = \mathcal{W}$ it follows that $\underline{r}(X) = \underline{r}(A, B) + \dim \mathcal{W} = 7 + 6 = 13$, so the design has maximal rank.

We could see that $\underline{r}(X) = 13$ by row-reducing the 13×17 matrix X. But clearly the 6×8 matrix is easier to row-reduce. Now we get the following table of degrees of freedom:

Source	d.f.
Mean	1
Rows	3
Columns	3
Greek	3
Latin	3
Residual	0
Total	13

Example 7.3. Consider another 4×4 Graeco-Latin square, this time with two missing observations.

-	$B\beta$	$C\gamma$	$D\delta$
$B\gamma$	$A\delta$	$D\alpha$	$C\beta$
$C\delta$	-	$A\beta$	$B\alpha$
$D\beta$	$C\alpha$	$B\delta$	$A\gamma$

From the Q-process we get 7 estimable contrasts involving $A, B, C, D, \alpha, \beta, \gamma, \delta$. By row-reducing the corresponding 7×8 matrix we see $\dim \mathcal{G} = 5$. In fact, we get

Greek letter contrasts: $\beta - \delta, \ \alpha - \beta + \gamma - \delta$

Latin letter contrasts: $B - C, \ A - B - C + D$

Inseparable contrasts: $(D-A) + (\alpha-\gamma)$

$\underline{r}(X) = \underline{r}(A, B) + \dim \mathcal{G} = 7 + 5 = 12$, which is not maximal rank. We see that there are 2 degrees of freedom for Greek letter effects and 2 degrees of freedom for Latin letter effects.

If we switch the role of the Greek and Latin letter effects with the role of the row and column effects, we can apply the Q-process again to get the corresponding results for row and column effects. The table of degrees of freedom is as follows:

Source	d.f.
Mean	1
Rows	2
Columns	2
Greek	2
Latin	2
Confounded	3
Residual	2
Total	14

Note that in the above example we had two missing cells and we found that the design matrix is not of maximal rank, while in Example 7.2 we had three missing cells and we found that the design matrix is of maximal rank. This very well may be due to the kind of cells that are missing. Notice that in Example 7.2 all missing cells had α-effects in common, but in Example 7.3 the two missing cells had no Greek or Latin letter effects in common.

Example 7.4. Consider a 4×4 hyper-Graeco-Latin square with four missing observations according to the following pattern:

	c_1	c_2	c_3	c_4
r_1	-	$B\beta b$	$C\gamma c$	$D\delta d$
r_2	$D\gamma b$	$C\delta a$	$B\alpha d$	-
r_3	$C\beta d$	$D\alpha c$	-	$B\gamma a$
r_4	$B\delta c$	-	$D\beta a$	$C\alpha b$

In order to illustrate how the results of Chapter VI can be applied to the above additive five-way classification model, let T_{11}, T_{24}, T_{33}, and T_{42} be empty and for each occupied (i, j) cell let $T_{ij} = \{t\}$ where t consists of the letter combination capital Latin, Greek, and small Latin in that cell; $\eta' = (A, B, C, D, \alpha, \beta, \gamma, \delta, a, b, c, d)$, the set K consists of twelve distinct 12×1 vectors which are collection of k_{ijt}'s, where, for example, $k_{12t} = (0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0)'$. Thus, there are twelve $N(k)$ matrices. Note that hyper-Graeco-Latin square are fractional factorial designs, i.e., they are $(1/p^3)$ fraction of p^5 design. Therefore, for each $k = k_{ijt} \in K$, $N(k)$ has a 1 in cell (i, j) and zero elsewhere. Thus, as we saw in Example 7.2 there are no direct differences, so that $S = K$, $M(k) = N(k)$ for all $k \in K$ and the vector space of estimable η-contrasts $\mathcal{G}$ is equal to $\mathcal{W}$. We now form the matrix $\bar{M}$. For the symbol θ_s we use t where $s = k_{ijt}$. Thus $\bar{M}$ is the original hyper-Graeco-Latin square having triples symbols (e.g., (Bαc)) in cell (i, j). Applying the Q-process to the above incidence matrix leads to the following 5 estimable contrasts:

1) $D\gamma b - C\delta a + D\alpha c - C\beta d$

2) $C\gamma c - B\beta b + C\delta a - B\alpha d$

3) $C\gamma c - B\beta b + C\delta a - D\gamma b + B\delta c - D\beta a$

4) $D\delta d - B\beta b + D\alpha c - B\gamma a$

5) $D\delta d - B\beta b + C\delta a - D\gamma b + B\delta c - C\alpha b$

The above five estimable contrasts are all linearly independent and thus $\dim \mathcal{G} = 5$, $\underline{r}(X) = \underline{r}(A, B) + \dim \mathcal{G} = 5 + 7 = 12$, so the design matrix is not of maximal rank. Moreover, after row reduction on the 5 x 12 matrix which is obtained from the above five estimable contrasts, we find out that there is only 1 degree of freedom for capital letters, 1 degree of freedom for Greek letters and 3 degrees of freedom are confounded. This fact is rather strange in the sense that in all four missing observations the letter A is common and thus one might expect some degree of freedom for the small letters.

The above example and the previous examples related to Graeco-Latin squares bring up an open problem: Is there a relation between the degrees of freedom and the pattern of the design? One may well construct tables of degrees of freedom for given patterns of Graeco or hyper-Graeco-Latin squares.

<u>Example 7.5.</u> Consider a $3^2 \times 2^3$ factorial design with the following incidence pattern:

	τ_1											
	δ_1						δ_2					
	γ_1			γ_2			γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3	β_1	β_2	β_3	β_1	β_2	β_3
α_1	1											
$N_1 =$ α_2					1							1
α_3								1				

	τ_2											
	δ_1						δ_2					
	γ_1			γ_2			γ_1			γ_2		
	β_1	β_2	β_3	β_1	β_2	β_3	β_1	β_2	β_3	β_1	β_2	β_3
α_1												1
$N_2 =$ α_2									1			
α_3			1	1								

We see that

$$(\mu+\alpha_2+\beta_3+\gamma_2+\delta_2+\tau_1) - (\mu+\alpha_2+\beta_3+\gamma_1+\delta_2+\tau_2) = \gamma_2 - \gamma_1 + \tau_1 - \tau_2$$

is estimable. This is the set D^* of Chapter VI. Now we form the matrix $\bar{M}$:

$\bar{M} =$

(1, 1, 1)	-	(2, 2, 2)
-	(2, 1, 1)	(2, 2, 1)
(2, 1, 2)	(1, 2, 1)	(1, 1, 2)

Applying the Q-process to the above matrix leads to:

(1, 1, 1)	-	(2, 2, 2)
-	(2, 1, 1)	(2, 2, 1)
(2, 1, 2)	(1, 2, 1)	(1, 1, 2)

$$-2\gamma_1 + 2\gamma_2 - \delta_1 + \delta_2 - \tau_1 + \tau_2$$

$\times$	-	(2, 2, 2)
-	(2, 1, 1)	(2, 2, 1)
(2, 1, 2)	(1, 2, 1)	(1, 1, 2)

$$-2\delta_1 + 2\delta_2 + \tau_1 - \tau_2$$

Thus $F = \{-2\gamma_1+2\gamma_2-\delta_1+\delta_2-\tau_1+\tau_2,\ -2\delta_1+2\delta_2+\tau_1-\tau_2\}$, and it is clear that we are left with no loops.

From the Q-process and direct η-differences we found the following 3 estimable contrasts:

$$
\begin{array}{cccccc l}
\gamma_1 & \gamma_2 & \delta_1 & \delta_2 & \tau_1 & \tau_2 & \\
\hline
-1 & 1 & 0 & 0 & 1 & -1 & \} \text{ From the set } D^* \\
-2 & 2 & -1 & 1 & -1 & 1 & \multirow{2}{*}{$\Big\}$ From the set F} \\
0 & 0 & -2 & 2 & 1 & -1 &
\end{array}
$$

If we row reduce the above 3×6 matrix we see that $\gamma_1 - \gamma_2$, $\delta_1 - \delta_2$, $\tau_1 - \tau_2$ are estimable and thus $\dim \mathcal{G} = 3$. Thus, $\underline{r}(X) = \underline{r}(A, B) + \dim \mathcal{G} = 8$, so the design has maximal rank.

At this point we would again like to compare this method with doing row reduction on the original design matrix. We could see that $\underline{r}(X) = 8$ by row-reducing the 8×13 matrix X. But clearly the 3×6 matrix is easier to row reduce. Now we get the following table:

Source	d.f.
Mean	1
α	2
β	2
γ	1
δ	1
τ	1
Residual	0
Total	8

Let us at this point use the obvious generalization of the technique developed in Chapter IV for finding a basis for each individual effect. This example is for illustration purposes only. We do the following steps:

1) $\bar{N} = N_1 + 2N_2$

=

1		
		2

	1	
2		

		2
	1	

		2
		1

2) Let

Q_1

1		
		1

	1	
1		

Q_2

		1
	1	

		1
		1

3) Let $\bar{Q} = Q_1 + 2Q_2$

$\bar{Q}$ =

1		
		2
	2	1

		2
	1	2
1		

4) Let

Q_* =

1		
		1
	1	1

		1
	1	1
1		

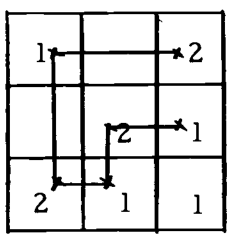
We see that $q^*_{231} = 1$ and $q^*_{232} = 1$, therefore $\gamma(P_1) = \gamma_1 - \gamma_2$ is estimable ignoring δ and τ-effects. We call this a γ-path. Set either q^*_{231} or q^*_{232} to zero, say $q^*_{232} = 0$, and form a matrix Q_{**} as below:

$$Q_{**} = \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & & 1 \\ \hline & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & 0 \\ \hline 1 & & \\ \hline \end{array} \leftarrow q^*_{232}$$

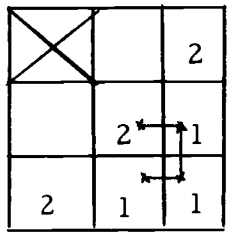
5) Let $\bar{Q}_{**} = Q_{**} + 2Q_{**}$

$$\begin{array}{|c|c|c|}\hline 1 & & 2 \\ \hline & 2 & 1 \\ \hline 2 & 1 & 1 \\ \hline \end{array}$$

Now we look for γ-paths using the Q-process.



leads to $\gamma(P_2) = -3(\gamma_1 - \gamma_2)$. Another path is of the form:



which leads to $\gamma(P_3) = -\gamma_1 + \gamma_2$. Now combination of these γ-paths lead to $5\gamma(P_1) + 2\gamma(P_2) - \gamma(P_3) = 0$.

6) We should see whether this zero combination of γ-paths leads to a zero combination of δ-paths (refer to $\bar{Q}$). We get

$$5(\delta_2-\delta_2) + 2(\delta_2-\delta_1+\delta_1-\delta_2+\delta_1-\delta_2) - (\delta_1-\delta_2+\delta_1-\delta_2) = 0 .$$

7) Now look in $\bar{N}$ for τ-contrasts corresponding to the above zero combination of δ-paths. We get

$$5(\tau_2-\tau_1) + 2(\tau_2-\tau_1+\tau_2-\tau_1+\tau_1-\tau_1) - (\tau_2-\tau_1+\tau_2-\tau_2)$$

$$= 8(\tau_2-\tau_1)$$

to be estimable. From this point on the problem should be considered as a four-way classification.

Example 7.6. In a 2^n factorial experiment with n factors each at two levels we need $n+1$ cells filled for the design matrix to be of maximal rank given that these filled cells have some certain positions. We consider a 2^6 factorial experiment with 57 missing cells in order to demonstrate the following facts, keeping in mind that we are only concerned about the main effects. These facts are as follows:

1) The effect of changing the position of a missing cell on the rank of the design matrix, and consequently the estimability of the main effects, may be dramatic.

2) After finding some estimable contrasts, the problem of estimability may be reduced to considering a design with fewer factors.

3) The determination of $\underline{r}(X)$ in 2^n factorial experiment is relatively simple compared to other factorial designs that have been discussed before. The model is of the form

$$E(Y_{ijktuve}) = \mu + \alpha_i + \beta_j + \gamma_k + \delta_t + \tau_u + \xi_v .$$

See Pattern 7.1.

Pattern 7.1.

			δ_1				δ_2				
			γ_1		γ_2		γ_1		γ_2		
			β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2	
ξ_1	τ_1	α_1	1								
		α_2		1							
	τ_2	α_1								0	← n_{122221}
		α_2									
ξ_2	τ_1	α_1	1				1				← n_{111212}
		α_2			1						
	τ_2	α_1									
		α_2		1				1			

Since $n_{111111} \neq 0$ and $n_{111112} \neq 0$, $\xi_1 - \xi_2$ is directly estimable. Now the problem collapses to a 2^5 factorial having the following pattern:

	τ_1							
	δ_1				δ_2			
	γ_1		γ_2		γ_1		γ_2	
	β_1	β_2						
α_1	1				1			
α_2		1	1					

	τ_2							
	δ_1				δ_2			
	γ_1		γ_2		γ_1		γ_2	
		1				1		

Since $n_{22111} \neq 0$ and $n_{22112} \neq 0$, $\tau_1 - \tau_2$ is directly seen to be estimable. Now the problem collapses to a 2^4 factorial having the following pattern:

$$
\begin{array}{cccc}
\multicolumn{2}{c}{\delta_1} & \multicolumn{2}{c}{\delta_2} \\
\gamma_1 & \gamma_2 & \gamma_1 & \gamma_2 \\
\begin{array}{|c|c|}\hline 1 & \\ \hline & 1 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline & \\ \hline 1 & \\ \hline\end{array} &
\begin{array}{|c|c|}\hline 1 & \\ \hline & 1 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}
\end{array}
$$

We see that $n_{1111} \neq 0$ and $n_{1112} \neq 0$ which implies that $\delta_1 - \delta_2$ is directly estimable and therefore the problem can be considered as a 2^3 factorial having the following pattern:

$$
\begin{array}{ccc}
 & \gamma_1 & \gamma_2 \\
 & \beta_1 \quad \beta_2 & \\
\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} &
\begin{array}{|c|c|}\hline 1 & \\ \hline & 1 \\ \hline\end{array} &
\begin{array}{|c|c|}\hline & \\ \hline 1 & \\ \hline\end{array}
\end{array}
$$

We are left with no more direct differences. Now we form the matrix $\overline{M}$.

$$
\overline{M} = \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & 1 \\ \hline\end{array}
$$

Applying the Q-process to the above matrix $\overline{M}$, we see that there is no loop, which implies there is no estimable γ-contrast.

In order to find degrees of freedom for α we switch the role of γ with α. In doing this we have the following pattern:

$$\begin{array}{c|cc|cc|}
 & \multicolumn{2}{c}{\alpha_1} & \multicolumn{2}{c}{\alpha_2} \\
 & \beta_1 & \beta_2 & & \\
\gamma_1 & 1 & & & 1 \\
\gamma_2 & & & 1 & \\
\end{array}$$

Because there is no direct difference we form the matrix $\bar{M}$.

$$\bar{M} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$$

Applying the Q-process to the above matrix $\bar{M}$, we see that there is no loop, which implies there is no estimable α-contrast. By the same procedure, we find that no β-contrast. Therefore the $r(X) = r(A, B) + \dim \mathcal{J} = 3 + 3 = 6$, which is not of maximal rank and the table of degrees of freedom is as follows:

S.V.	d.f.
Mean	1
α	0
β	0
γ	0
δ	1
τ	1
ξ	1
Confounded	2
Residual	1
Total	7

Pattern 7.2.

			τ_1							
			δ_1				δ_2			
			γ_1		γ_2		γ_1		γ_2	
			β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
ξ_1	τ_1	α_1	1							
		α_2		1						
	τ_2	α_1								1
		α_2								
ξ_2	τ_1	α_1	1				0			
		α_2				1				
	τ_2	α_1								
		α_2		1				1		

Now let us change the position of the cell $(1,1,1,2,1,2)$ to $(1,2,2,2,2,1)$. By this we mean having <u>no</u> observation in the position $(1,1,1,2,1,2)$ and one or more observations in the position $(1,2,2,2,2,1)$. Therefore $n_{111212} = 0$ and $n_{122221} \neq 0$. See Pattern 7.2.

Because $n_{111111} \neq 0$ and $n_{111112} \neq 0$, then $\xi_1 - \xi_2$ is directly estimable and the problem collapses to a 2^5 factorial having the following pattern:

	τ_1							
	δ_1				δ_2			
	γ_1		γ_2		γ_1		γ_2	
	β_1	β_2						
α_1	1							
α_2		1	1					

	τ_2							
	δ_1				δ_2			
	γ_1		γ_2		γ_1		γ_2	
								1
		1				1		

Since $n_{22111} \neq 0$ and $n_{22112} \neq 0$, $\tau_1 - \tau_2$ is directly estimable and the problem collapses to a 2^4 factorial having the following pattern:

δ_1				δ_2			
γ_1		γ_2		γ_1		γ_2	
1							1
	1	1			1		

We see that $n_{2211} \neq 0$ and $n_{2212} \neq 0$, and therefore $\delta_1 - \delta_2$ is seen to be directly estimable. Now the problem collapses to a 2^3 factorial with the following pattern:

γ_1		γ_2	
1			1
	1	1	

We are left with no direct differences. We now form the matrix $\bar{M}$.

$$\bar{M} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

By applying the Q-process we find that $2\gamma_1 - 2\gamma_2$ is estimable and thus the problem collapses to a 2^2 factorial with the following pattern:

	β_1	β_2
α_1	1	1
α_2	1	1

We see that $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$ are directly estimable. The design matrix X is of maximal rank 7 and the table of degrees of freedom is as follows:

S.V.	d.f.
Mean	1
α	1
β	1
γ	1
δ	1
τ	1
ξ	1
Residual	0
Total	7

Again note how the position of a filled cell changes the estimability of each effect and consequently the rank of the design matrix. Moreover, only with 7 filled cells and 57 missing cells, all factors are estimable, and this is the minimum number of filled cells needed for the maximality of $\underline{r}(X)$.

BIBLIOGRAPHY

Birkes, D., Dodge, Y., Hartmann, N., and Seely, J. (1972). Estimability and analysis for additive two-way classification models with missing observations. Oregon State University. Dept. of Statistics, Technical Report No. 30.

Birkes, D., Dodge, Y., and Seely, J. (1972). Estimability and analysis for additive three-way classification models with missing observations. Oregon State University. Dept. of Statistics, Technical Report No. 33.

Bose, R.C. (1949). Least square aspects of analysis of variance. Inst. Stat., Mimeo Series 9, Chapel Hill, NC.

Bradley, E.H. (1968). Multiple classification analysis for arbitrary experimental arrangements. <u>Technometrics</u>, 10, 13-27.

Mexas, A.G. (1972). Analysis of classification data of arbitrarily imcomplete data. Universidad Autonoma de Guerrero. Chilpancingo, Mexico. (Mimeo)

Rao, C.R. (1962). A note on a generalized inverse of a matrix with applications to problems in mathematical statistics. <u>J. Royal Statist. Soc.</u>, Series B, 24, 152-158.

Searle, S.R. (1971). <u>Linear Models</u>. John Wiley & Sons, New York. 532 p.

Srivastava, J.N. and Anderson, D.A. (1970). Some basic properties of multidimensional partially balanced designs. Ann. Math. Statist. 41, 1438-1445.

Weeks, D.L. and Williams, D.R. (1964). A note on the determination of connectedness in an N-way cross classification. <u>Technometrics</u>, 6, 319-324. Errata, <u>Technometrics</u>, 7, 281.

APPENDIX

APPENDIX

It is the purpose of this appendix to describe the Q-process which is used for finding loops in the matrix $\bar{M}$. This description is for the four-way model. The generalization of the Q-process follows immediately. For the proof of the validity of the Q-process see Birkes, Dodge and Seely (1972).

For convenience we isolate a part of the Q-process and call it the X-process. The X-process applied to a two-dimensional matrix W consists of the following procedure:

1) Find every entry of W which is the only nonzero entry in its row or column and change it to 0.

2) Continue this until each row and column either is all 0's or has at least two nonzero entries.

We now describe the Q-process. At the first stage put the submatrix of $\bar{M}$ consisting of the first two columns into a temporary working area W. Apply the X-process to W. If the entire second column of W is 0 (and hence $W = 0$), then proceed to the second stage. Suppose there is a nonzero entry (g_1, h_1) in position $(i_1, 2)$. Since the X-process has been applied, there must be a nonzero entry (g_2, h_2) in position $(i_1, 1)$. Similarly, there must be a nonzero entry (g_3, h_3) in position $(i_2, 1)$, $i_2 \neq i_1$, and also a nonzero entry (g_4, h_4) in position $(i_2, 2)$. These entries form a loop in $\bar{M}$, so $\omega_{(g_1,h_1)} - \omega_{(g_2,h_2)} + \omega_{(g_3,h_3)} - \omega_{(g_4,h_4)}$ is estimable. We include it to our spanning set.

Now change $\bar{m}_{i_2,2} = (g_4,h_4)$ to 0 both in W and permanently in $\bar{M}$. If there was another loop in $\bar{M}$ which involved $(i_2, 2)$, the corresponding ω-contrast will not be lost.

Again apply the X-process to W. If there are any nonzero entries left, find a loop and derive an ω-contrast as was done above. Then change the last entry in the loop to 0 in both W and $\bar{M}$. Apply the X-process again and continue as above. Eventually a point is reached where $W = 0$. This finishes the first stage.

To begin the second stage put the submatrix of $\bar{M}$ consisting of the first three columns into a working area W.

Let us suppose that we have proceeded through the first p-1 stages of the Q-process and have thus obtained a spanning set for all ω-contrasts which can be derived from loops in the first p columns of $\bar{M}$. If $p = b$, the Q-process has been completed. If $p < b$, begin the pth stage by putting the first p+1 columns of $\bar{M}$ into a temporary area W. Apply the X-process to W. If the entire (p+1)th column of W is 0, then $W = 0$, because every loop in W must involve column p+1 since there are no loops in the first p columns of $\bar{M}$. In this case proceed to the next stage. Suppose there is a nonzero entry (g_{10},h_{10}) in position $(i_1, p+1)$. Since the X-process has been applied, there must be another nonzero entry

(g_{11}, h_{11}) in row i_1, say in position (i_1, j_1). Similarly, there must be another nonzero entry (g_{21}, h_{21}) in column j_1, say in position (i_2, j_1). Alternately look along rows and columns for the next nonzero entry until one is found in column $p+1$, say in position (i_u, j_u), $j_u = p+1$. Then $(i_1, p+1)(i_2, j_1) \ldots (i_u, j_{u-1})(i_u, p+1)$ is a loop and $\omega_{(g_{10}, h_{10})} - \omega_{(g_{11}, h_{11})} + \omega_{(g_{21}, h_{21})} - \ldots + \omega_{(g_{uu}, h_{uu-1})}$ $- \omega_{(g_{uu}, h_{uu})}$ is estimable. We include it in our spanning set. Now change $\bar{m}_{i_u, p+1} = (g_{uu}, h_{uu})$ to 0 both in W and permanently in $\bar{M}$.

Apply the X-process to the current matrix in W. If any nonzero entries remain, find a loop and derive a ω-contrast to include in our spanning set. Change the last entry in the loop to 0 in both W and $\bar{M}$. Apply the X-process to W again. This completes the $(p+1)$th stage. After $b-1$ stages the Q-process has been completed.