AN ABSTRACT OF THE THESIS OF

Samuel H. Peterson for the degree of Master of Science in Mathematics presented on June 8, 2012

Title: On Existence and Uniqueness of Weak Solutions to the Navier-Stokes Equations in R3

Abstract Approved: ____

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This thesis is on the existence and uniqueness of weak solutions to the Navier-Stokes equations in R3 which govern the velocity of incompressible fluid with viscosity υ. The solution is obtained in the space of tempered distributions on R3 given an initial condition and forcing data which are dominated by majorizing kernels. The solution takes the form of an expectation of functionals on a Markov process indexed by a binary branching tree.

On Existence and Uniqueness of Weak Solutions to the Navier-Stokes Equations in R3

by
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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Master of Science

Presented June 8th, 2012
Commencement June 2012

Master of Science thesis of Samuel H. Peterson presented on June 8th, 2012.

APPROVED:

Major Professor, representing Mathematics

Chair of the Department of Mathematics

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Samuel H Peterson, Author

ACKNOWLEDGEMENTS

The author expresses his sincere appreciation to his adviser, Mina Ossiander. Many thanks are also extended toward the exceptional faculty and student body of the Oregon State University Mathematics Department. Finally, the author wishes to thank Oregon State University as a whole.

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Part I

Thesis

1 Introduction

This thesis is on the existence and uniqueness of weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. The 3-dimensional Navier-Stokes equations, which govern the velocity of incompressible fluid with viscocity ν and initial velocity u_0, are given in their strong form by

$$\begin{aligned}&\frac{\partial u}{\partial t}+u\cdot\nabla u=\nu\Delta u-\nabla p+g\\&\nabla\cdot u=0\\&u_0(x)=u(x,0)\end{aligned}\tag{1}$$

where $u:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$ is the velocity field of the fluid, $p:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$is the pressure and $g:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$ is the external force.

When one considers $u,g,\vec{\nabla}g\in(C_0^\infty((0,\infty),\mathbb{R}^3)')^3$, and when one considers the derivatives in (1) in the distributional sense, then it is possible to eliminate the presure in (1) by employing the Leray projector $\mathscr{P}$, which is essentially a projection onto the space of divergence free distributional vector fields. When one incorporates this operator into the Navier-Stokes equations, one obtains the following equivalent formulation:

$$\begin{aligned}u\ &=\ e^{\nu t\Delta}u_0-\int_0^t e^{\nu(t-s)\Delta}\mathscr{P}\nabla\cdot(u\otimes u)(s)ds+\int_0^t e^{\nu(t-s)\Delta}\mathscr{P}g(s)ds\\&\quad\ \nabla\cdot u=0.\end{aligned}\tag{2}$$

Much has been written on these equations over the past decades since Leray demonstrated existence of weak solutions to the Navier-Stokes equations in 1934 Leray[5]. The background developed since then is nicely presented in Temam[7] and Lemarié-Rieusset[3].

Despite the progress made on these equations, there are still open problems associated with it. Of these, the question which is pertinent to this document is whether there exists a unique solution with arbitrarily large initial data for some time period. In matters of existence and uniqueness of the Navier-Stokes equations, the usual statements in the literature assert either: 1) the existence for all time of solutions with bounded data, or 2) the existence for some finite time of solutions with unbounded data. This document gives a statement of the second variety by extending the method used in Ossiander[1] to prove a statement of the first variety. The construction of the solution to the Navier-Stokes problem in this paper involves an expectation of a markov process indexed by a binary branching tree. The applicability of similar stochastic models to non-linear PDE's was discussed in Blömker et al.[2], and an application of this method for the Fourier-transformed Navier-Stokes equation was given in a paper by Le Jan and Sznitman[4]. In this thesis, I shall demonstrate the existence and uniqueness of weak solutions to the Navier-Stokes equations for short time periods. It is assumed that the initial data and forcing terms are dominated by respective members of a majorizing kernel pair (which will be introduced in Section 3). This is accomplished by representing the solution as the expectation of a functional on a binary branching tree. The length of time for which the solution exists depends on the 'size' of the initial data and forcing in a particular family of function spaces.

In the following sections I shall: 1) define some useful operators and derive some of their key properties, 2) define and describe properties of majorizing kernels, 3) describe the Markov processes on binary trees that will form the measure of our probability space,

4) precisely state the problem on which I am writing, 5) demonstrate the problem's solution using probabilistic methods, 6) give a calculation of some interest in a special case, and 7) conclude with some remarks on further lines of inquiry.

2 Useful Operators

In this paper, an explicit formulation of (2) will be used. It will be obtained by the use of bilinear operators which are introduced in this section.

Definition 2.1 for $z \in \mathbb{R}^3 \backslash 0$, let $P_z : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the space perpendicular to z given by the matrix equation

$$(P_z)_{i,j} = \delta_i^j - (e_z)_i (e_z)_j$$

where $e_z = \frac{z}{|z|}$. ✠

Definition 2.2 For any $y \in \mathbb{R}^3 \backslash 0$, and for $u, v \in \mathbb{R}^3$ define the bilinear forms b_1 and b_2 : $(\mathbb{R}^3 \backslash 0) \times (\mathbb{R}^3)^2 \to \mathbb{R}^3$ via

$$b_1(z; u, v) = (v \cdot e_z) P_z u + (u \cdot e_z) P_z v$$

$$b_2(z; u, v) = b_1(z; u, v) + u \cdot (I - 3 e_z e_z^t) v e_z. \text{ ✠}$$

Lemma 2.1: For any $y \in \mathbb{R}^3 \backslash 0$ and $u, v \in \mathbb{R}^3$,

$$|b_1(y; u, v)| \leq |u||v|,$$

$$|b_2(y; u, v)| \leq 2|u||v|,$$

$$|(I - 3e_y e_y^t)u| \le 2|u| \tag{3}$$

Proof: Letting $e_u \cdot e_y = \alpha$, $e_v \cdot e_y = \beta$, $e_v \cdot e_u = \gamma$, it turns out that

$$|b_1(y;u,v)| = |u||v||(\beta(e_u - \alpha e_y) + \alpha(e_v - \beta e_y))| \tag{4}$$
$$\le |u||v|(\beta|1-\alpha^2|^{1/2} + \alpha|1-\beta^2|^{1/2} \le |u||v|$$

To show (4) first observe that the two terms in the definition of b_2 given above are perpendicular to each other. It follows, then, that

$$\begin{aligned} |b_2(y;u,v)|^2 &= |b_1(y;u,v)|^2 + (u \cdot (I - 3e_y e_y^t)v)^2 \\ &= |u|^2|v|^2(\alpha^2 + \beta^2 + \alpha^2\beta^2 + (\gamma - 2\alpha\beta)^2) \end{aligned}$$

Now, $|\gamma - 2\alpha\beta| = |e_u \cdot (e_v - 2\beta e_y)| = |e_u \cdot P_y e_v| \le |P_y e_u||P_y e_v| = (1-\alpha^2)^{1/2}(1-\beta^2)^{1/2}$, so the above can be bounded by

$$\begin{aligned} |b_2(y;u,v)|^2 &\le |u|^2|v|^2(\alpha^2 + \beta^2 + \alpha^2\beta^2 + ((1-\alpha^2)^{1/2}(1-\beta^2)^{1/2} + |\alpha\beta|)^2) \\ &= |u|^2|v|^2(\alpha^2 + \beta^2 + \alpha^2\beta^2 + (1-\alpha^2)(1-\beta^2) \\ &+ |\alpha\beta|^2 + 2|\alpha\beta|(1-\alpha^2)^{1/2}(1-\beta^2)^{1/2}) \\ &= |u|^2|v|^2(1 + \alpha^2\beta^2 + 2|\alpha\beta|((1-\alpha^2)^{1/2}(1-\beta^2)^{1/2} + |\alpha\beta|)) \\ &\le |u|^2|v|^2(1 + |\alpha\beta|)^2 \le (2|u||v|)^2. \end{aligned}$$

To show (3), we calculate

$$|(I - 3e_y e_y^t)u|^2 = (u \cdot u + 9e_y^t u e_y^t u - 6e_y^t u e_y^t u) = u \cdot (I + 3e_y e_y^t)u = |u|^2(1 + 3\alpha^2) \le 4|u|^2.$$

QED.

These binary operators will be used to construct a solution to the Navier-Stokes equations. The use of bilinear operators coincides with the quadratic term of the NSE: $(u \cdot \nabla) u$.

3 Majorizing Kernels

Majorizing kernel pairs will play a central role in the framework of the problem and its solution. They provide the weight which defines the space in which the solution will exist, and they will be used to define the probability measure which will be used to construct weak solutions to the Navier-Stokes equations. These employments will reveal themselves in the following sections. For now I shall define these kernels, describe some of their properties and conclude with some examples. The kernels will be indexed by a scaler parameter $\gamma \in [3/2, 2)$. This parameter is associated with the spatial decay rate of the magnitude of initial data, force term and solutions.

Definition 3.1 For $\gamma \in [3/2, 2)$, a majorizing kernel pair with parameter γ is a pair of functions, $(h_\gamma, \tilde{h}_\gamma)$, $h_\gamma : \mathbb{R}^3 \to (0, \infty)$, $\tilde{h}_\gamma : \mathbb{R}^3 \to [0, \infty)$ both of which are lower semi-continuous, locally square-integrable, and h_γ has the property that

$$\sup_{x \in \mathbb{R}^3} \frac{\int_{\mathbb{R}^3} h_\gamma^2(x-y)|y|^{(1-2\gamma)} \mathrm{d}y}{h_\gamma(x)} \leq \lambda < \infty. \tag{5}$$

whilst $\tilde{h}_\gamma$ has the property

$$\sup_{x \in \mathbb{R}^3} \frac{\int_{\mathbb{R}^3} \tilde{h}_\gamma(x-y)|y|^{2(1-\gamma)} \mathrm{d}y}{h_\gamma(x)} \leq \tilde{\lambda} < \infty. \tag{6}$$

The pair $(\lambda, \tilde{\lambda})$ is called the constant pair associated with the majorizing kernel pair $(h_\gamma, \tilde{h}_\gamma)$. Note that if a function h_γ satisfies (5) above, then it can be augmented with the function 0 to complete a majorizing kernel pair. Also, if both elements of the constant pair associated with a majorizing kernel pair are equal to 1, then the kernel pair is said to be standard. ✠

Definition 3.2 For $\gamma \in [3/2,2)$, let $\mathscr{H}_\gamma$ be the set of majorizing kernel pairs when using the equations (5) and (6) to define their associated constants.✠

Definition 3.3 Let $(h_\gamma, \tilde{h}_\gamma)$ be a majorizing kernel pair. The pair $(u_0, g) = (u_0, \{g(\cdot,t) : t \geq 0\})$, with $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is said to be $(h_\gamma, \tilde{h}_\gamma)$-admissible if

$$\sup_{x\in\mathbb{R}^3} \frac{|u_0(x)|}{h_\gamma(x)} < \infty \; \mathit{and} \; \sup_{x\in\mathbb{R}^3, t\geq 0} \frac{|g(x,t)|}{\tilde{h}_\gamma(x)} < \infty \tag{7}$$

✠

Notice that admissibility does not depend on γ.

Proposition 3.1 For $\gamma \in [3/2,2)$ and $(h_\gamma, \tilde{h}_\gamma) \in \mathscr{H}_\gamma$ a standard majorizing kernel pair, the following hold:

(a) The following are also standard majorizing kernel pairs:

$$(h_\gamma(\cdot - \mu), \tilde{h}_\gamma(\cdot - \mu)), \;\; \mu \in \mathbb{R}^3.$$

$$(\sigma^{4-2\gamma} h_\gamma(\sigma \cdot), \sigma^{9-4\gamma} \tilde{h}_\gamma(\sigma \cdot)) \;\; \sigma > 0.$$

$$(h_\gamma(A\cdot), \tilde{h}_\gamma(A\cdot)) \;\; \mathit{where}\, \text{A is a 3 by 3 matrix with } A^t A = I.$$

(b) Let F denote a probability distribution function on $\mathbb{R}^3$. Then

$$\Big(\int_{\mathbb{R}^3} h_\gamma(\cdot - y) dF(y), \int_{\mathbb{R}^3} \tilde{h}_\gamma(\cdot - y) dF(y) \Big)$$

is also a majorizing kernel pair with parameter pair $(\lambda,\tilde{\lambda}) \in (0,1] \times (0,1]$.

Proof: Part (a) follows from appropriate change of variables. To prove part (b) Observe that

$$\begin{aligned}
\lambda &= \sup_{x\in\mathbb{R}^3} \frac{\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} h_\gamma(x-y-z)dF(z)\right)^2 |y|^{1-2\gamma}dy}{\int_{\mathbb{R}^3} h_\gamma(x-y)dF(y)} \\
&\le \sup_{x\in\mathbb{R}^3} \frac{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} h_\gamma^2(x-y-z)|y|^{1-2\gamma}dydF(z)}{\int_{\mathbb{R}^3} h_\gamma(x-y)dF(y)} \\
&\le \sup_{x\in\mathbb{R}^3} \frac{\int_{\mathbb{R}^3} h_\gamma(x-z)dF(z)}{\int_{\mathbb{R}^3} h_\gamma(x-y)dF(y)} = 1
\end{aligned}$$

The first inequality is due to an application of the Cauchy-Schwartz inequality in the space $L_2(\mathbb{R}^3,dF)$; note that the uniform local square integrability of h_γ guarantees that $h_\gamma \in L_2(\mathbb{R}^3,dF)$ almost everywhere with respect to Lebesgue measure. The order of integration is reversed according to Tonelli's theorem. The last inequality follows from the fact that h_γ has a normalization parameter of 1. One can likewise show the same result for $\tilde{h}_\gamma$. QED

Proposition 3.2 For fixed $\gamma \in [3/2,2)$, let $\{(h_j,\tilde{h}_j) : j \ge 1\} \subset \mathscr{H}_\gamma$ be a sequence of majorizing kernel pairs with corresponding normalization parameters $\{(\lambda_j,\tilde{\lambda}_j) : j \ge 1\}$.

(a) Then $(h_1 \wedge h_2)$ is a majorizing kernel pair with normalization parameter pair $(\lambda,\tilde{\lambda}) \in (0,\lambda_1 \wedge \lambda_2] \times [0,\tilde{\lambda}_1 \wedge \tilde{\lambda}_2]$.

(b) For any $p \in (0,1)$, $(h_1^p h_2^{1-p}, \tilde{h}_1^p \tilde{h}_2^{1-p})$ is a majorizing kernel pair with normalization parameter pair $(\lambda,\tilde{\lambda}) \in (0,\lambda_1^p \lambda_2^{1-p}] \times [0,\tilde{\lambda}_1^p \tilde{\lambda}_2^{1-p}]$.

(c) For $\{p_j : j \ge 1, p_j \ge 0\}$ with $\sum_j p_j = 1$,

$$\left(\sum_j p_j h_j, \sum_j p_j \tilde{h}_j\right)$$

is a majorizing kernel pair with normalization parameter pair $(\lambda, \tilde{\lambda}) \in (0, \sum_j p_j \lambda_j] \times [0, \sum_j p_j \tilde{\lambda}_j]$.

Proof:

(a) is obvious.

(b) follows from an application of Hölder's inequality:

$$\begin{aligned}\int_{\mathbb{R}^3} h_{\gamma,1}^{2p}(x-y) h_{\gamma,2}^{2(1-p)}(x-z)|y|^{1-2\gamma} dy &= \int_{\mathbb{R}^3} \left(h_{\gamma,1}^2(x-y)|y|^{(1-2\gamma)}\right)^p \\ &\quad \left(h_{\gamma,2}^2(x-z)|y|^{(1-2\gamma)}\right)^{(1-p)} dy \\ &\leq h_{\gamma,1}^p(x) h_{\gamma,2}^{(1-p)} \lambda_1^p \lambda_2^{(1-p)},\end{aligned}$$

from which the result follows. The same method can be applied to demonstrate that $\tilde{\lambda} \leq \tilde{\lambda}_{\gamma,1}^p \tilde{\lambda}_{\gamma,2}^{1-p}$.

(c) can be demonstrated using the same method as in the proof of part (b) of proposition 3.1. QED

Proposition 3.3 Let $3/2 \leq \gamma_1 \leq \gamma_2 < 2$, $(h_{\gamma_1}, \tilde{h}_{\gamma_1}) \in \mathscr{H}_{\gamma_1}$, $(h_{\gamma_2}, \tilde{h}_{\gamma_2}) \in \mathscr{H}_{\gamma_2}$, $\gamma \in (\gamma_1, \gamma_2)$, and let $\beta \in (0,1)$ such that

$$1 - 2\gamma = \beta(1 - 2\gamma_1) + (1 - \beta)(1 - 2\gamma_2).$$

Then,

$$(h_{\gamma_1}^{\beta} h_{\gamma_2}^{1-\beta}, \tilde{h}_{\gamma_1}^{\beta} \tilde{h}_{\gamma_2}^{1-\beta}) \in \mathscr{H}_{\gamma},$$

with constant pair $(\lambda', \tilde{\lambda}') \in (0, \lambda_1^{\beta} \lambda_2^{1-\beta}] \times [0, \tilde{\lambda}_1^{\beta} \tilde{\lambda}_2^{1-\beta}]$, where $(\lambda_1, \tilde{\lambda}_1)$, $(\lambda_2, \tilde{\lambda}_2)$ are the constant pairs associated with the kernel pairs $(h_{\gamma_1}, \tilde{h}_{\gamma_1})$ and $(h_{\gamma_2}, \tilde{h}_{\gamma_2})$ respectively.

Proof: This result follows by the application of Hölder's inequality on the definition of the constants associated with $(h_{\gamma_1}^{\beta} h_{\gamma_2}^{1-\beta}, \tilde{h}_{\gamma_1}^{\tilde{\beta}} \tilde{h}_{\gamma_2}^{1-\tilde{b}}) \in \mathscr{H}_{\gamma}$.

$$\begin{aligned}\int_{\mathbb{R}^3} (h_{\gamma_1}^{\beta}(x-y) h_{\gamma_2}^{1-\beta}(x-y))^2 |y|^{1-2\gamma} dy &= \int_{\mathbb{R}^3} \left(h_{\gamma_1}^2(x-y)|y|^{1-2\gamma_1}\right)^{\beta} \\ &\quad \left(h_{\gamma_2}^2(x-y)|y|^{1-2\gamma_2}\right)^{1-\beta} dy\end{aligned}$$

$$\begin{aligned}&\leq \Big(\int_{\mathbb{R}^3} h_{\gamma_1}^2(x-y)|y|^{1-2\gamma_1} dy\Big)^{\beta} \Big(\int_{\mathbb{R}^3} h_{\gamma_2}^2(x-y)|y|^{1-2\gamma_2} dy\Big)^{1-\beta} \\ &\leq (\lambda_1 h_{\gamma_1}(x))^{\beta} (\lambda_2 h_{\gamma_2}(x))^{1-\beta}.\end{aligned}$$

Upon division by quantity $h_{\gamma_1}^{\beta}(x) h_{\gamma_2}^{1-\beta}(x)$ one obtains

$$\sup_{x \in \mathbb{R}^3} \frac{\int_{\mathbb{R}^3} (h_{\gamma_1}^{\beta}(x-y) h_{\gamma_2}^{1-\beta}(x-y))^2 |y|^{1-2\gamma} dy}{h_{\gamma_1}^{\beta}(x) h_{\gamma_2}^{1-\beta}(x)} = \lambda' \leq \lambda_1^{\beta} \lambda_2^{1-\beta}.$$

The same argument applies to the second element of the new majorizing kernel pair due to the fact that

$$2(1-\gamma) = \beta 2(1-\gamma_1) + (1-\beta) 2(1-\gamma_2).$$

QED

Scholium: The following series of propositions demonstrate some specific families of functions which are majorizing kernels. The propositions mainly involve functions which obey some form of power growth/decay, and hopefully the reader will obtain a fair intuition for what kinds of growth/decay properties majorizing kernels should have.

Specifically, the propositions below will show that if $h(x) = |x|^p$, there is only one choice of p for a given γ for which h is a majorizing kernel. However, if attention is restricted to the subset of $\mathscr{H}_\gamma$ which has a bounded power behavior equivalent to $(1+|x|)^p$, let this subset of $\mathscr{H}_\gamma$ be denoted by $\bar{\mathscr{H}}_\gamma$, then one sees that $\bar{\mathscr{H}}_{\gamma_1} \subset \bar{\mathscr{H}}_{\gamma_2}$ for $\gamma_1 < \gamma_2$.

Throughout the propositions: 3.4, 3.5, 3.6 and 3.7 the parameter $\gamma \in [3/2, 2)$ is fixed.

Proposition 3.4 The pair of functions

$$H_\gamma(x) = |x|^{2\gamma-4}, \quad \tilde{H}_\gamma(x) = |x|^{4\gamma-9}$$

form a majorizing kernel pair in $\mathscr{H}_\gamma$.

Proof: Clearly H_γ and $\tilde{H}_\gamma$ are uniformly locally square integrable and lower-semi-continuous. To show that they have finite constant pairs, one can simply use the definition of these quantities to show, using the substitution $w = y/|x|$, to obtain

$$\lambda = \sup_{x\in\mathbb{R}^3} \frac{\int_{\mathbb{R}^3} |x-y|^{2(2\gamma-4)}|y|^{1-2\gamma}dy}{|x|^{2\gamma-4}} = \int_{\mathbb{R}^3} |e_z - w|^{2(2\gamma-4)}|w|^{1-2\gamma}dw < \infty$$

and that

$$\tilde{\lambda} = \sup_{x\in\mathbb{R}^3} \frac{\int_{\mathbb{R}^3} |x-y|^{(4\gamma-9)}|y|^{2-2\gamma}dy}{|x|^{2\gamma-4}} = \int_{\mathbb{R}^3} |e_z - w|^{(4\gamma-9)}|w|^{2-2\gamma}dw < \infty.$$

QED

Note: These kernels shall be called the canonical kernels for a given γ.

Proposition 3.5 Let $f(y) = |2\pi y|^{-1}(1+|y|)^{-3}$, which is a probability distribution on $\mathbb{R}^3$. Then $(f * H_\gamma)(x)$ decays at least on the order of $|x|^{2\gamma-5+\frac{3}{p}}$ for $p \in (\frac{3}{(5-2\gamma)}, 3)$. By this I mean that for large $|x|$, there is a constant C such that $(f * H_\gamma)(x) \leq C|x|^{2\gamma-5+\frac{3}{p}}$.

Proof: Proposition 3.1 can be used to obtain a bounded kernel pair by convolving the canonical kernel pair $(H_\gamma, \tilde{H}_\gamma)$ with the probability density f. First observe that $g(x) \equiv (f * H_\gamma)(x)$ is a decreasing function in $|x|$ (See Appendix: A.2) and that $(f \star H_\gamma)(0) = K < \infty$. For the asymptotics of this kernel, apply Hölder's using the hermitian conjugates, $\frac{1}{p} + \frac{1}{q} = 1$. Only one of the hermitian conjugate values need to be restricted in order for the following inequalities to apply:

$$f \star H_\gamma(x) = \int_{\mathbb{R}^3} |2\pi y|^{-1}(1+|y|)^{-3}|x-y|^{2\gamma-4}dy \leq C_q \big(\int |x-y|^{p(2\gamma-4)}|2\pi y|^{-p}dy \big)^{1/p}$$

$$= C_q |x|^{(2\gamma-5)+3/p} \big(\int |e_z - w|^{p(2\gamma-4)}|2\pi w|^{-p}dw \big)^{1/p}, \tag{8}$$

where $C_q = \int_{\mathbb{R}^3}(1+|y|)^{-3q}dy < \infty$, $w = y/|x|$, and $p \in (\frac{3}{5-2\gamma}, 3)$. The lower bound for p ensures that the integrand in (8) decays sufficiently rapidly for the integral to be bounded:

$$\begin{aligned} \int_{[B_{(0,0,1)}(2)]^c} |e_z - w|^{p(2\gamma-4)}|2\pi w|^{-p}dw \quad &\leq \quad K_p \int_{[B_{(0,0,1)}(2)]^c} |w|^{p(2\gamma-5)}dw \\ &< \infty \ for\, p > \frac{3}{5-2\gamma}, \end{aligned}$$

whilst the upper bound for p ensures that the same integrand is locally integrable:

$$\int_{[B_{(0,0,1)}(2)]} |e_z - w|^{p(2\gamma-4)}|2\pi w|^{-p}dw \leq \tilde{K} \int_{[B_{(0,0,1)}(2)]} |w|^{-p}dw < \infty \ for\, p > 3.$$

These observations allow for the conclusion that the majorizing kernel $f * H_\gamma \in O(1+|x|)^{2\gamma-5+\frac{3}{p}}$ for $p \in (\frac{3}{(5-2\gamma)}, 3)$. QED

Proposition 3.6 For $p \in [1-2\gamma, 2\gamma-4]$, the function $(1+|x|)^p$ is a majorizing kernel. Also, if $p \in [2-2\gamma, 2\gamma-4]$, then $((1+|x|)^p, (1+|x|)^q) \in \mathscr{H}_\gamma$, for $q \leq p+2\gamma-5$.

Remark: Remember that if $h_\gamma(x) = (1+|x|)^p$ is a majorizing kernel for a certain γ value, then $(h_\gamma, 0) \in \mathscr{H}_\gamma$. The above states that a sufficient condition for there to be a non-zero associate of h is that $p \geq 2(1-\gamma)$. In the next proposition, it will be shown that this is a necessary condition as well.

Proof: This can be shown by bounding the integrals which define the constants associated with the majorizing kernel.

$$\frac{\int_{\mathbb{R}^3}(1+|x-y|)^{2p}|y|^{1-2\gamma}dy}{(1+|x|)^p} = \frac{|x|^{2p+4-2\gamma}\int_{\mathbb{R}^3}(\frac{1}{|x|}+|e_z-w|)^{2p}|w|^{1-2\gamma}dw}{(1+|x|)^p}. \tag{9}$$

Notice that the integral on the LHS of (9) is bounded for any particular choice of x, provided that $2p+1-2\gamma < -3$ (which is always the case if $p \in [1-2\gamma, 2\gamma-4]$), Also, the integral on the right is bounded as a function of x provided that $p > -3/2$. When this is the case, the ratio is bounded for $p+4-2\gamma \leq 0$ which gives the condition that $p \leq 2\gamma - 4$. In the case where $p \leq -3/2$, the integral on the RHS of 9 is unbounded as $|x| \to \infty$. This growth is entirely due to the singularity at the point (0,0,1). The growth of this integral can be bounded by investigating

$$\int_{B_{1/2}(0,0,1)}(\frac{1}{|x|}+|e_z-w|)^{2p}|w|^{1-2\gamma}dw \leq (1/2)^{1-2\gamma}\int_{B_{1/2}(0,0,1)}(\frac{1}{|x|}+|e_z-w|)^{2p}dw$$

$$= (1/2)^{1-2\gamma}4\pi\int_0^{1/2}(\frac{1}{|x|}+r)^{2p}r^2dr \leq (1/2)^{1-2\gamma}4\pi\int_0^{1/2}(\frac{1}{|x|}+r)^{2p+2}dr$$

$$\leq \begin{cases} K\ln(1+|x|) & if\, p = -3/2 \\ \tilde{K}|x|^{-3-2p} & if\, p < -3/2 \end{cases}, \tag{10}$$

where K and $\tilde{K}$ are finite constants. Note that the integration of this function outside

$B_{1/2}(0,0,1)$ (I will call this B^c) is bounded for all x:

$$\int_{B^c} \left(\frac{1}{|x|} + |e_z - w|\right)^{2p} |w|^{1-2\gamma} dw \leq \int_{B^c} |e_z - w|^{2p} |w|^{1-2\gamma} dw < \infty$$

By replacing the integral on the RHS of (9) with the order of growth on in (10), it becomes apparent that the ratio in (9) is bounded provided that $-p+1-2\gamma \leq 0$ which, along with the other bound which was derived, yields the requirement that $1-2\gamma \leq p \leq 2\gamma - 4$. If this is the case, then $(1+|x|)^p$ is a majorizing kernel.

Now, given that $(1+|x|)^p$, where $1-2\gamma \leq p \leq 2\gamma - 4$, is a majorizing kernel, I shall now determine suitable conditions for there to be a $q \in \mathbb{R}^3$ such that $((1+|x|)^p, (1+|x|)^q) \in \mathscr{H}_\gamma$. One can use the same method as before: start with the definition for the constant associated with the function $(1+|x|)^q$, where $(1+|x|)^p$ is being used as the majorizing kernel

$$\frac{\int_{\mathbb{R}^3} (1+|x-y|)^q |y|^{2(1-\gamma)} dy}{(1+|x|)^p} = \frac{|x|^{q+5-2\gamma} \int_{\mathbb{R}^3} (\frac{1}{|x|} + |e_z - w|)^q |w|^{2(1-\gamma)} dw}{(1+|x|)^p}. \tag{11}$$

Notice that for any particular value of x, the ratio on the LHS of (11) is bounded the integral is bounded provided that $q + 2(1-\gamma) < -3$, while the ratio on the RHS of (11) is a bounded function of $|x|$ provided that: $q + 2(1-\gamma) < -3$, $q > -3$, and that $q + 5 - 2\gamma - p \leq 0$. This gives the requirement that $q \leq p + 2\gamma - 5$. (Note: since $p < 0$ for all $\gamma \in (3/2, 2)$, the condition $q + 2(1-\gamma) < -3$ is, in fact, superfluous.) Suppose now that the integral in the RHS of (11) is not a bounded function of x, i.e. if $q \leq -3$. Then one can employ the same reasoning that led to the conclusions given in (10). Therefore, if $q = -3$ then the integral on the RHS of (11) grows as $Kln(1+|x|)$, so the inequality $2(1-\gamma) - p < 0 \Rightarrow p > 2(1-\gamma)$ must be satisfied in order for the ratio to be bounded a bounded function in $|x|$. Also, in the case where $q < -3$ the integral on the right grows as $\tilde{K}|x|^{-3-q}$. In that case, therefore, the condition $2-2\gamma-p \leq 0 \Rightarrow 2(1-\gamma) \leq p$ is necessary and sufficient for $(1+|x|)^q$ to be a valid associate of $(1+|x|)^p$. Note that if $p < 2(1-\gamma)$, then q already has to satisfy $q \leq -3$, so $p \geq 2(1-\gamma)$ is a necessary condition for a suitable q to exist. QED

A reasonable question to ask is: Can there be a non-zero associate of $(1+|x|)^p$ for $p < 2(1-\gamma)$? The answer is a resounding no.

Proposition 3.7: Let $h_\gamma(x) = (1+|x|)^p$, where $p < 2(1-\gamma)$. Then $(h_\gamma, g) \in \mathscr{H}_\gamma$ if and only if $g \equiv 0$.

Proof: The if is trivial. The only if statement can be shown by contradiction. Suppose the reverse is true, let $((1+|x|)^p, g) \in \mathscr{H}_\gamma$, for $g \neq 0$. Then due to the lower semi-continuity of g, there is a ball in $\mathbb{R}^3$ of radius δ such that $2z > \delta > 0$, with center $z \in \mathbb{R}^3$, call it B, for which $g > C > 0$. So for $|x| > 10z$,

$$\frac{\int_{\mathbb{R}^3} g(y)|x-y|^{2(1-\gamma)}dy}{(1+|x|)^p} \geq \frac{\int_B g(y)|x-y|^{2(1-\gamma)}dy}{(1+|x|)^p} \tag{12}$$

$$\geq \frac{C|B|(|x-z|+\delta)^{2(1-\gamma)}}{(1+|x|)^p} \geq \frac{C|B|(|x|+|z|+\delta)^{2(1-\gamma)}}{(1+|x|)^p} \geq \tilde{C}|B|(|x|+1)^{2(1-\gamma)-p},$$

where $|B|$ is the measure of the ball B. Since the hypothesis is that $2(1-\gamma) > p$, it appears that the RHS of (12) is unbounded, which means that g cannot be non-zero if $((1+|x|)^p, g) \in \mathscr{H}_\gamma$. Thus the hypothesis $g \neq 0$ yields a contradiction. QED

As mentioned before, the second element in a majorizing kernel pair is to be used as a bound on the forcing term in the Navier-Stokes problem. The above gives a condition for which, given a fixed γ, the force can and cannot be non-zero when the initial velocity is bounded by a function equivalent to a function of the form $(1+|x|)^p$.

4 Probabilistic Setting

In this section I shall describe the probability space which will be used to construct a solution to the Navier-Stokes equations. First I will introduce the notation for binary trees to be used for the rest of the paper, then I will introduce the Markov process which will yield the probability distribution to be used in the construction of a solution.

Definition 4.1: Binary Trees The full binary tree with root $\phi = \{0,1\}^0$is the set of all terminating binary sequences, which I shall represent as $\mathscr{V} := \bigcup_{n=0}^{\infty}\{0,1\}^n$. The boundary of $\mathscr{V}$ is given by$\partial\mathscr{V} = \{0,1\}^{\mathbb{N}}$. If $v \in \{0,1\}^n$ then $|v| = n$, $v|0 = \phi$, and $v|m = (v_1,..,v_m)$ for $m \leq n$. For $n < |v|$, $w = v|n$ is called an ancestor of v, and v is a descendant of w. The immediate ancestor of v is denoted by $\bar{v}$. i.e. if $|v| = n$ then $\bar{v} = v|(n-1)$. In anticipation of definitions in the next section, let me also define $\bar{\phi}$ to be the ancestor of ϕ. Also, Let $\star$ denote the appending operation: $v \star k = (v_1, v_2, .., v_{|v|}, k)\ for k \in \{0,1\}$. $\mathscr{W} \subseteq \mathscr{V}$ is a rooted binary sub-tree if: $\phi \in \mathscr{W}$; for any $v \in \mathscr{W}$, if $v \star j \in \mathscr{W}$ then $v \star (1-j) \in \mathscr{W}$ for $j \in \{0,1\}$; and $v|k \in \mathscr{W}, \forall k < |v|$. $\partial\mathscr{W}$ is defined to be those elements of $\mathscr{W}$ which have no descendants, i.e. $\partial\mathscr{W} = \{v \in \mathscr{W} : v \star 0 \notin \mathscr{W}\}$, and the interior of $\mathscr{W}$, is $\mathscr{W}^\circ = \mathscr{W} \backslash \partial\mathscr{W}$.

Finally, let $(B, \mathscr{B})$ be a measurable space, and let $X = \{X_v : v \in \mathscr{V}\}$ be a $\mathscr{V}$ indexed collection of B-valued random variables defined on a common probability space. Let $\mathscr{F}_v = \sigma(X_v)$, $\bar{\bar{\mathscr{F}}}_v = \sigma(\{X_{v|n} : n \leq |v|\})$ be the sigma fields generated by the random variables indexed by v and its ancestors, and let $\tilde{\bar{\mathscr{F}}}_v = \sigma(\{X_w : w\big||v| = v\})$ be the sigma field generated by the random variables indexed by v and its descendants. Finally let $\mathscr{F}_{\bar{\phi}}$ denote the trivial σ-field.

The weak solution to Navier Stokes equations will use measures induced by a $\mathscr{V}$-

indexed Markov process. In the following definitions, I will describe this underlying structure.

Definition 4.2: $\mathscr{V}$-indexed Markov Process As above, let $X = \{X_v : v \in \mathscr{V}\}$ be a collection of $\mathscr{B}$-valued random variables. X is a $\mathscr{V}$-indexed Markov process if for any $v \in \mathscr{V}$, with $|v| < \infty$, $\tilde{\bar{\mathscr{F}}}_{v\star 0}$ and $\tilde{\bar{\mathscr{F}}}_{v\star 1}$are conditionally independent given $\mathscr{F}_v$. i.e.

$$P(A_0 \bigcap A_1|\mathscr{F}_v) = P(A_0|\mathscr{F}_v)P(A_1|\mathscr{F}_v) \ a.s P \ \ for\, A_k \in \tilde{\bar{\mathscr{F}}}_{v\star k},\ k \in \{0,1\}$$

and for any $v, w \in \mathscr{V}$ and any $\tilde{\bar{\mathscr{F}}}_w$-measurable random variable Y with $E|Y| < \infty$,

$$E(Y|\bar{\bar{\mathscr{F}}}_v) = E(Y|\mathscr{F}_{v\wedge w}) \ \ a.s P. \tag{13}$$

The distribution of a $\mathscr{V}$-indexed Markov process is completely specified by the conditional distributions of the X_v given $\mathscr{F}_{\bar{v}}$ for $v \in \mathscr{V}$. ✠

For a given majorizing kernel pair $(h_\gamma, \tilde{h}_\gamma)$, I can define a $\mathscr{V}$-indexed Markov process which will be very useful in what is to follow. The Markov process will be defined by its conditional transition probabilities. The spatial transition densities are given by

$$f(y,z|x) = \frac{(5-2\gamma)|z|^{-4}|y|^{2(1-\gamma)}h_\gamma^2(x-z)1[|y| \le |z|]}{4\pi \int_{\mathbb{R}^3} h_\gamma^2(x-z)|z|^{1-2\gamma}dz}, \tag{14}$$

$$\tilde{f}(y,z|x) = \frac{(5-2\gamma)|z|^{-3}|y|^{2(1-\gamma)}\tilde{h}_\gamma(x-z)1[|y| \le |z|]}{4\pi \int_{\mathbb{R}^3} \tilde{h}_\gamma(x-z)|z|^{2(1-\gamma)}dz}, \tag{15}$$

and the temporal transition densities are

$$f_0(s|z) = C_0|z|^{2\gamma}s^{\frac{1}{2}-\gamma}K(z,2\nu s), \tag{16}$$

$$f_1(s|z) = C_1|z|^{2(\gamma-1)}s^{\frac{3}{2}-\gamma}K(z,2\nu s), \tag{17}$$

where C_0 and C_1 are the normalizing constants which make f_0 and f_1 probability densities. Also required are some branching probability weights:

$$p_1 + p_2 + p_3 = p \in (0, 1/2)$$

$$p_4 + p_5 = 1 - p$$
$$p_3 = p_2 \frac{3}{5 - 2\gamma}$$
$$p_2 = p_1 \frac{2}{\gamma - 1}$$
$$p_4 = p_5 \frac{2}{5 - 2\gamma}.$$

Now, for fixed $x \in \mathbb{R}^3$, let

$$f(y,z,s,k|x) = \begin{cases} p_1 f(y,z|x) f_0(s|z) & if\ k = 1 \\ p_2 f(y,z|x) f_1(s|z) & if\ k = 2 \\ p_3 f(y,z|x) f_1(s|y) & if\ k = 3 \\ p_4 \tilde{f}(y,z|x) f_1(s|z) & if\ k = 4 \\ p_5 \tilde{f}(y,z|x) f_1(s|y) & if\ k = 5 \end{cases} \tag{18}$$

be the joint conditional density of the quadruple $(Y, Z, \tau, \kappa) \in \mathbb{R}^3 \times \mathbb{R}^3 \times (0, \infty) \times \{1, 2, 3...$ $..., 4, 5\}$. Now that the conditional transition densities are defined, one can use them to construct a suitable $\mathscr{V}$-indexed Markov process.

Definition: 4.3 Let $X = \{X_v, Y_v, Z_v, \tau_v, \kappa_v : v \in \mathscr{V}\}$ be a $\mathscr{V}$-indexed Markov process with the following transitions probabilities. Given that $X_{\bar{v}} = x$ let the quadruple $(Y_v, Z_v, ...$ $..., \tau_v, \kappa_v)$ have the density given by (18). Then let $X_v = X_{\bar{v}} - Z_v$. Assume that the distribution of $(X_v, Y_v, Z_v, \tau_v, \kappa_v)$ is independent of $X_{\bar{v}}$.

Given that the ensemble $(X_v, Y_v, Z_v, \tau_v, \kappa_v)$ only depends on the prior ensemble $(X_{\bar{v}}, \ldots$ $\ldots, Y_{\bar{v}}, Z_{\bar{v}}, \tau_{\bar{v}}, \kappa_{\bar{v}})$, via the random variable $X_{\bar{v}}$, the distribution of X depends only on the initial value $X_{\bar{\phi}}$. For the rest of this thesis, denote the probability measure corresponding to $X_{\bar{\phi}} = x$ by P_x, and the expectation with respect to this measure E_x. ✠

5 Statement of the Problem

Having defined the underlying probability space on which the solution of the problem depends, let me now state the central question. There is one last thing to do, however. I must define the space in which the weak problem lies. Due to the particular form the Navier-Stokes equation will take in this paper, the problem will actually be stated on a subspace of $(C_0^{\infty\prime}((0,\infty),\mathbb{R}^3))^3$, namely, the tempered distributions, or the dual of the Schwartz space on $\mathbb{R}^3 \times (0,\infty)$. After setting the scene adequately, I will then express the solution to the resulting integral equation in the next section.

Definition 5.1: The Schwartz Space on $\mathbb{R}^3$ is given by:

$$\mathscr{S}(\mathbb{R}^3) = \{f \in \mathscr{C}^\infty(\mathbb{R}^3) : \sup_{x\in\mathbb{R}^3} |x^\beta D^\alpha f(x)| < \infty \forall \text{multi-indices}, \alpha, and\, \beta\}. \tag{19}$$

The dual of this space is denoted by $\mathscr{S}'(\mathbb{R}^3)$

✠

Definition 5.2: A weak solution of the Navier-Stokes equations with initial velocity $u(x,0) = u_0(x) \in S'(\mathbb{R}^3)^3$, and with forcing $g(x,t) \in \mathscr{S}'(\mathbb{R}^3 \times (0,\infty))^3$ is a vector field $u(x,t) \in \mathscr{S}'(\mathbb{R}^3 \times (0,\infty))^3$ satisfying the following:

(a) u is locally square integrable on $\mathbb{R}^3 \times (0,\infty)$,

(b) $\nabla \cdot u = 0$, and

(c) there exists a $p \in \mathscr{S}'(\mathbb{R}^3 \times (0,\infty))$ with $\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + g$.

The Problem I can now state the problem on which this thesis is based: Given the data (u_0, g), determine whether there exists a unique weak solution (in the sense of Definition 5.2) to the Navier-Stokes Equations, and if so, determine the interval $[0,T)$ in the time variable on which it is defined.

Upon application of the Leray projector, $\mathscr{P}$, onto the Navier-Stokes equations, (1), one can eliminate the pressure and obtain, by way of Duhamel's principle, the integral formulation of the Navier-Stokes equations:

$$\begin{aligned} u &= e^{\nu t \Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta} \mathscr{P} \nabla \cdot (u \otimes u)(s) ds + \int_0^t e^{\nu(t-s)\Delta} \mathscr{P} g(s) ds \\ & \nabla \cdot u = 0 \end{aligned} \tag{20}$$

where $\mathscr{P} v = v - \nabla \Delta^{-1} (\nabla \cdot v)$.

By use of the Fourier transform, and appealing to some brief lemmas, the above equation can be written explicitly as follows:

Proposition 5.1 If $u : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3$ is locally square-integrable and satisfies

$$\begin{aligned} & u(x,t) = \int_{\mathbb{R}^3} u_0(x-y) K(y, 2\nu t) dy + \int_0^t \int_{\mathbb{R}^3} \Big\{ \frac{|z|}{4\nu s} K(z, 2\nu s) \\ & \times b_1(z; u(x-z, t-s), u(x-z, t-s)) + \Big(\frac{1}{|z|} K(z, 2\nu s) \\ & - \frac{3}{4\pi |z|^4} \int_{\{y : |y| \le |z|\}} K(y, 2\nu s) dy \Big) \times b_2(z; u(x-z, t-s), u(x-z, t-s)) \\ & + \Big(K(z, 2\nu s) P_z - \frac{1}{4\pi |z|^3} (I - 3 e_z e_z^t) \int_{\{|y| \le |z|\}} K(y, 2\nu s) dy \Big) \\ & \times g(x-z, t-s) \Big\} dz ds \end{aligned} \tag{21}$$

then u is a weak solution as in Definition 5.2.

Proof: See Ossiander[6].

6 Solution to the problem of Existence and Uniqueness

Before demonstrating a solution to (21), I shall define a $\mathscr{V}$-indexed random variable. It will be the expectation of this function which will yield the desired solutions.

More Definitions: The following functions will be part of a $\mathscr{V}$-indexed random variable which will be used in constructing a weak solution to the NSE. Fix $\gamma \in [3/2, 2)$ and take $(h_\gamma, \tilde{h}_\gamma) \in \mathscr{H}_\gamma$. Let the following functions be defined:

$$\varphi(x,t) = \frac{g(x,t)}{\tilde{h}_\gamma(x)} \tag{22}$$

$$m(s,x) = Cs^{\gamma-\frac{3}{2}} \frac{\int_{\mathbb{R}^3} |y|^{1-2\gamma} h_\gamma^2(x-y)dy}{h_\gamma(x)} \tag{23}$$

$$\tilde{m}(s,x) = \tilde{C}s^{\gamma-\frac{3}{2}} \frac{\int_{\mathbb{R}^3} |y|^{2(1-\gamma)} \tilde{h}_\gamma(x-y)dy}{h_\gamma(x)} \tag{24}$$

$$m_0(x,t) = \frac{\int_{\mathbb{R}^3} u_0(x-y)K(y, 2\nu s)dy}{h_\gamma(x)} \tag{25}$$

In the above equations, $C = \frac{1}{4\nu C_0 p_1}$, where C_0 comes from (16), $\tilde{C} = \frac{1}{C_1 p_4}$, where $C_1 = 4\nu(\gamma-1)C_0$.

One can now express the integral equation in (21) in terms of the functions which have been introduced above, along with the probability weights defined in section 4.

Proposition 6.1: Suppose the majorizing kernel pair $(h_\gamma, \tilde{h}_\gamma)$ is used to define f, $\tilde{f}$, f_0, f_1, φ, m, $\tilde{m}\, and\, m_0$ given in (14) to (17) and (22) to (25) respectively. If $h(x)\chi(x,t)$ is locally square integrable and χ satisfies

$$\begin{aligned}
&\chi(x,t) = m_0(x,t) + \\
&\int_0^t \int_{y\in\mathbb{R}^3} \int_{z\in\mathbb{R}^3} \Big\{ m(x,t-s)\Big(p_1 f_0(s|z) f(y,z|x) \times \\
&b_1(z;\chi(x-z,t-s),\chi(x-z,t-s)) + \big(2p_2 f_1(s|z) f(y,z|x) - \\
&3p_3 f_1(s|y) f(y,z|x)\big) \times b_2(z;\chi(x-z,t-s),\chi(x-z,t-s))\Big) \\
&+\tilde{m}(x,t-s)\Big(p_4 f_1(s|z)\tilde{f}(y,z|x) P_z \varphi(x-z,t-s) \\
&-p_5 f_1(s|y)\tilde{f}(y,z|x)(I - 3e_z e_z^t)\varphi(x-z,t-s)\Big)\Big\} dzdyzs,
\end{aligned} \tag{26}$$

then $u(x,t) = h_\gamma(x)\chi(x,t)$ is a weak solution of the Navier-Stokes equations in the sense of Definition 5.2.

Proof: Suppose that $\chi(x,t)$ satisfies (26) above and define $u(x,t) = h_\gamma(x)\chi(x,t)$. Keeping in mind that

$$\int_{y\in\mathbb{R}^3} f(y,z|x)dy = \frac{|z|^{1-2\gamma} h_\gamma^2(x-z)}{\int_{z\in\mathbb{R}^3} |z|^{1-2\gamma} h_\gamma^2(x-z)dz},$$

and that

$$\int_{y\in\mathbb{R}^3} \tilde{f}(y,z|x)dy = \frac{|z|^{2-2\gamma} \tilde{h}_\gamma(x-z)}{\int_{z\in\mathbb{R}^3} |z|^{2-2\gamma} \tilde{h}_\gamma(x-z)dz},$$

one can simply evaluate the integral in (26) along with a multiplication of a factor of $h_\gamma(x)$ to demonstrate that

$$
\begin{aligned}
h_\gamma(x)\chi(x,t) = \quad & \int_{\mathbb{R}^3} u_0(x-y)K(y,2\nu s)dy + \\
& \int_0^t \int_{z\in\mathbb{R}^3} \Big\{ \frac{h_\gamma(x)m(x,s)}{(\int_{z\in\mathbb{R}^3} |z|^{1-2\gamma} h_\gamma^2(x-z)dz)} \Big(p_1 f_0(s|z)|z|^{1-2\gamma} \times \\
& b_1(z;u(x-z,t-s),u(x-z,t-s)) + \big(\frac{p_2}{2} f_1(s|z)|z|^{1-2\gamma} - \\
& \frac{p_3}{2}\Big[\int_{|y|\le|z|} \frac{f_1(s|y)(5-2\gamma)|y|^{2(1-\gamma)}}{4\pi|z|^4} dy\Big]\big) \\
& \times b_2(z;u(x-z,t-s),u(x-z,t-s))\Big) \\
& + \frac{\tilde{m}(x,s)h_\gamma(x)}{(\int_{z\in\mathbb{R}^3} |z|^{2-2\gamma}\tilde{h}_\gamma(x-z)dz)} \Big(p_4 f_1(s|z)|z|^{2-2\gamma} P_z g(x-z,t-s) - \\
& \frac{p_5}{2}\Big[\int_{|y|\le|z|} \frac{(5-2\gamma)|y|^{2(1-\gamma)} f_1(s|y)}{|z|^3 4\pi} dy\Big] \times \\
& (I - 3e_z e_z^t)g(x-z,t-s)\Big)\Big\} dzds \\
& = \int_{\mathbb{R}^3} u_0(x-y)K(y,2\nu s)dy + \\
& \int_0^t \int_{z\in\mathbb{R}^3} \Big\{ \Big((4\nu s)^{-1} K(z,2\nu s)|z| \times \\
& b_1(z;u(x-z,t-s),u(x-z,t-s)) + \big(|z|^{-1}K(z,2\nu s) - \\
& \Big[\int_{|y|\le|z|} \frac{3K(y,2\nu s)}{4\pi|z|^4} dy\Big]\big) \times \\
& b_2(z;u(x-z,t-s),u(x-z,t-s))\Big) \\
& + \Big(K(z,2\nu s)P_z g(x-z,t-s) - \Big[\int_{|y|\le|z|} \frac{K(y,2\nu s)}{|z|^3 4\pi} dy\Big] \times \\
& (I - 3e_z e_z^t)g(x-z,t-s)\Big)\Big\} dzds.
\end{aligned}
$$

Since the last equality above is the same as (21) from proposition 5.1, $u(x,t) = h(x)\chi(x)$ is a weak solution to the Navier-Stokes equations. QED

Now, let the following $\mathscr{V}$-indexed random variable be defined by

$$\begin{aligned} \Upsilon_v(t) &= m_0(X_{\bar{v}},t)+m(\tau_v,X_{\bar{v}})B_v(\Upsilon_{v\star 0}(t-\tau_v),\Upsilon_{v\star 1}(t-\tau_v))1[\kappa_v=1,2,3]\bigcap[\tau_v\le t] \\ &\quad +\tilde{m}(\tau_v,X_{\bar{v}})C_v\varphi(X_v,t-\tau_v)1[\kappa_v=4,5]\bigcap[\tau_v\le t], \end{aligned} \tag{27}$$

where,

$$B_v(\cdot,\cdot)=\begin{cases} b_1(Z_v;\cdot,\cdot) & if\ \ \kappa_v=1 \\ (-1)^{\kappa}b_2(Z_v;\cdot,\cdot)/2 & if\ \ \kappa_v=2,3 \end{cases}$$

and

$$C_v=\begin{cases} P_{Z_v} & if\ \ \kappa_v=4 \\ -(I-3e_ve_v^t)/2 & if\ \ \kappa_v=5 \end{cases}.$$

The solution to the NSE will be given as an expectation of this random variable.

Proposition 6.2: Let $(h_\gamma,\tilde{h}_\gamma)$ be a majorizing kernel pair. If $E_x|\Upsilon_\phi(t)|<\infty$ for all $t\le T\in(0,\infty)$, and for all $x\in\mathbb{R}^3$ where $\Upsilon_\phi$is defined as in (27) then

$$u(x,t)=h(x)E_x\Upsilon_\phi(t).$$

is a weak solution to the Navier-Stokes equations for $t\le T, and x\in\mathbb{R}^3$.

Proof: Since X, as in definition 5, is a $\mathscr{V}$-indexed Markov process, and since Υ_v is measurable $\mathscr{F}_v$, then $\Upsilon_0(t-\tau_\phi)$ and$\Upsilon_1(t-\tau_\phi)$ are conditionally independent given $\mathscr{F}_\phi$

(See A.1 in Appendix 1), which yields:

$$\begin{aligned}
E_x(\Upsilon_\phi(t)|\mathscr{F}_\phi) &= m_0(x,t) \\
&+\Big(m(\tau_\phi,x)\{b_1(Z_\phi;E_x(\Upsilon_0(t-\tau_\phi)|\mathscr{F}_\phi),E_x(\Upsilon_1(t-\tau_\phi)|\mathscr{F}_\phi))) \\
&\times 1[\kappa_\phi = 1] \\
&+\frac{1}{2}\big(1[\kappa_\phi = 2]-1[\kappa_\phi = 3]\big) \\
&\times b_2(Z_\phi;E_x(\Upsilon_0(t-\tau_\phi)|\mathscr{F}_\phi),E_x(\Upsilon_1(t-\tau_\phi)|\mathscr{F}_\phi)))\} \\
&+\tilde{m}(\tau_\phi,x) \\
&\times\{1[\kappa_\phi = 4]P_{Z_\phi}-1[\kappa_\phi = 5]\frac{1}{2}(I-3e_{Z_\phi}e^t_{Z_\phi})\}\varphi(x-Z_\phi,t-\tau_\phi)\Big) \\
&\times 1[\tau_\phi \le t]
\end{aligned}$$

After noting that $E_x(\Upsilon_0(t-\tau_\phi)|\mathscr{F}_\phi)1[Z_\phi = z,\ \tau_\phi = s] = E_{x-z}(\Upsilon_\phi(t-s))$, and that $E_x(\Upsilon_\phi(t)) = E_x(E_x(\Upsilon_\phi(t)|\mathscr{F}_\phi))$ it becomes apparent that

$$\begin{aligned}
&E_x(\Upsilon_\phi(t)) = E_x(E_x(\Upsilon_\phi(t)|\mathscr{F}_\phi)) = m_0(x,t)+ \\
&\int_{s=0}^t\int_{y\in\mathbb{R}^3}\int_{z\in\mathbb{R}^3}\Big(m(s,x)\{p_1 f_0(s|z)f(y,z|x) \\
&\times b_1(z;E_x(\Upsilon_0(t-s)|\mathscr{F}_\phi),E_x(\Upsilon_1(t-s)|\mathscr{F}_\phi))) \\
&+\frac{1}{2}\big(p_2 f_1(s|z)f(y,z|x)-p_3 f_1(s|y)f(y,z|x)\big) \\
&\times b_2(z;E_x(\Upsilon_0(t-s)|\mathscr{F}_\phi),E_x(\Upsilon_1(t-s)|\mathscr{F}_\phi)))\} \\
&+\tilde{m}(\tau_\phi,x)\{p_4 f_1(s|z)\tilde{f}(y,z|x)P_{Z_\phi} \\
&-p_5 f_1(s|y)\tilde{f}(y,z|x)\frac{1}{2}(I-3e_z e^t_z)\}\varphi(x-z,t-s)\Big)dzdyds.
\end{aligned}$$

Assuming, then, that $E_x|\Upsilon_\phi(t)| \le M < \infty$ for all x and t<T, it is clear that u(x,t) is uniformly square integrable since $|u(x,t)| = h(x)|E_x\Upsilon_\phi(t)| \le Mh(h)$. From proposition 6.1 it now follows that u is a weak solution to the N-S equations. QED.

Theorem 6.1: Let $(h_\gamma, \tilde{h}_\gamma)$ be a majorizing kernel pair with constants $(\lambda, \tilde{\lambda})$ in the normalization family $\gamma \in (3/2, 2)$. If (u_0, g) is $(h_\gamma, \tilde{h}_\gamma)$-admissible with

$$\sup_{x\in\mathbb{R}^3, t\leq T} |m_0(x,t)| \quad = \quad \sup_{x\in\mathbb{R}^3, t\leq T} \frac{|\int_{\mathbb{R}^3} u_0(x-y)K(y, 2\nu s)dy|}{h(x)} \leq M \ \ and$$

$$\sup_{x\in\mathbb{R}^3, t\geq 0} \frac{|g(x,t)|}{\tilde{h}_\gamma(x)} \leq \tilde{M}$$

for some $M, \tilde{M} \in (0,\infty)$, then there exists a $T \in (0,\infty]$ such that $u(x,t) = h(x)E_x(\Upsilon_\phi(t))$ is a weak solution to the NSE with

$$\sup_{x\in\mathbb{R}^3, 0\leq t\leq T} \frac{|u(x,t)|}{h_\gamma(x)} \leq C_{M,\tilde{M}}.$$

This is also unique in the class $\{v \in (S'(\mathbb{R}^3 \times (0,\infty)))^3 : \sup_{x\in\mathbb{R}^3, 0\leq t\leq T} \frac{|v(x,t)|}{h_\gamma(x)} \leq C_{M,\tilde{M}}\}$.

The proof of the first part of this theorem will employ a lemma which I shall give presently.

Lemma 6.1: Let $\mathscr{W} \subset \mathscr{V}$ be a finite binary sub-tree. Suppose that $\{b_v : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 | \, v \in \mathscr{W}\}$ has the property that

$$\sup_{v\in\mathscr{W}} |b_v(x,y)| \leq |x||y| \quad forall \ \ x, y \in \mathbb{R}^3,$$

$\{(y_v, z_v, \eta_v, \sigma_v) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0,\infty) \times \{0,1\} : v \in \mathscr{W}\}$ satisfies

$$\sup_{v\in\mathscr{W}} \eta_v \leq \eta < \infty$$

and if

$$\sup_{v\in\mathscr{W}} |y_v| \leq \alpha\varepsilon \; and \; \sup_{v\in\partial\mathscr{W}} |z_v| \leq \beta\varepsilon \;\; for\, some \;\; \alpha \in [0,1)$$
$$for\; \varepsilon, \beta \in (0,(1-\alpha)/\eta].$$

Then, x_v defined iteratively on $\mathscr{W}$ via

$$x_v = \begin{cases} y_v + \sigma_v \eta_v z_v & if\, v \in \partial\mathscr{W} \\ y_v + \eta_v b_v(x_{v\star 0}, x_{v\star 1}) & if\, v \in \mathscr{W}^\circ \end{cases}$$

satisfies

$$\sup_{v\in\mathscr{W}} |x_v| \leq \varepsilon.$$

Proof: If $v \in \partial\mathscr{W}$, then

$$|x_v| \leq |y_v| + \eta_v |z_v| \leq \alpha\varepsilon + \eta\beta\varepsilon \leq \varepsilon.$$

If $v \in \mathscr{W}^\circ$and $|x_{v\star k}| \leq \varepsilon$ for $k \in \{0,1\}$, then

$$|x_v| \leq |y_v| + \eta_v |b_v(x_{v\star 0}, x_{v\star 1})| \leq \alpha\varepsilon + \eta\varepsilon^2 \leq \varepsilon.$$

where the last inequality is due to the face that $\alpha + \eta\varepsilon \leq 1$. QED

Proof of Theorem: First I shall prove the existence of a solution with such a bound, then I will demonstrate uniqueness.

1) (Existence) Let

$$\mathscr{W} = \{v \in \mathscr{V} : \kappa_{v_w} = 1,2,3 \; for\, all\, w < |v|\}. \tag{28}$$

Then $\mathscr{W}$ is a random binary sub-tree corresponding to a Galton-Watson tree with each individual having either 0(with probability greater than 1/2) or 2 (with probability less than 1/2) children. As a result of these weights, $\mathscr{W}$ is, a.e-P, a finite tree; (see Athreya and Ney[1]). Let $X_{\bar{\phi}} = x \in \mathbb{R}^3$ and $S_\phi = 0$. For $v \in \mathscr{W}$ with $|v| > 0$ let $S_v = \sum_{k=0}^{|v|-1} \tau_{v|k}$. For fixed $t < \infty$, $\Upsilon_\phi(t)$ is a function of the random ensembles indexed by the nodes of the tree $\mathscr{W}(t) \subset \mathscr{W}$ defined by

$$\mathscr{W}(t) = \{v \in \mathscr{W} : S_v \le t\}. \tag{29}$$

Observe that if $v \in \partial\mathscr{W}(t)$, then

$$\begin{aligned} \Upsilon_v(t - S_v) &= m_0(X_{\bar{v}}, t - S_v) \\ &\quad + \tilde{m}(\tau_v - S_v, X_{\bar{v}}) C_v \varphi(X_v, t - S_v - \tau_v) 1[\kappa_v = 4,5] \bigcap [\tau_v \le t - S_v] \end{aligned}$$

and that if $v \in \mathscr{W}^\circ(t)$, then

$$\begin{aligned} \Upsilon_v(t - S_v) &= m_0(X_{\bar{v}}, t - S_v) + \\ &\quad m(\tau_v - S_v, X_{\bar{v}}) B_v(\Upsilon_{v\star 0}(t - S_v - \tau_v), \Upsilon_{v\star 1}(t - S_v - \tau_v)) \\ &\quad \times 1[\kappa_v = 1,2,3] \bigcap [\tau_v \le t - S_v] \end{aligned}$$

In this setting we can use Lemma 6.1 by identifying (in the notation of Lemma 6.1) $y_v = m_0(X_{\bar{v}}, t - S_v)$, $b_v = B_v$, $z_v = C_v \varphi(X_v, t - S_v - \tau_v)$,

$$\eta_v = \begin{cases} m(\tau_v - S_v, X_{\bar{v}}) & if\, v \in \mathscr{W}^\circ(t) \\ \tilde{m}(\tau_v - S_v, X_{\bar{v}}) & if\, v \in \partial\mathscr{W}(t) \end{cases},$$

and $\sigma_v = 1[\kappa_v = 4,5]$. Observe that $m(s,x) \leq \frac{\lambda}{4\nu C_0 p_1} s^{\gamma-3/2}$ and that $\tilde{m}(s,x) \leq \frac{\tilde{\lambda}}{C_1 p_4} s^{\gamma-3/2}$, where C_1 and C_0 are given in (16). Also note that due to the assumptions in the theorem, $|y_v| \leq M$, and $|z_v| \leq \tilde{M}$. By considering a small enough T and restricting $t \leq T$, the constants ε, and β which appear in lemma 6.1 can be made arbitrarily large by letting $\eta \geq \eta_v$ be arbitrarily small for all t<T (remember that $\varepsilon, \beta \in (0, \frac{1-\alpha}{\eta}])$. This will allow the reader to conclude that for all $x \in \mathbb{R}^3$, $|\Upsilon_\phi(t)| \leq \varepsilon$ a.s P_x, and, by virtue of Proposition 6.2, existence of a solution such that $|u(x,t)| \leq h(x)\varepsilon$ for t<T will have been demonstrated.

Given a particular value for $M, \tilde{M}, \lambda, and\, \tilde{\lambda}$, I shall determine a sufficiently small T to guarantee existence. To do this, I shall work backwards. Let such a T be determined. Then let

$$\eta = \max\left(\frac{\lambda}{4\nu C_0 p_1} T^{\gamma-3/2}, \frac{\tilde{\lambda}}{C_1 p_4} T^{\gamma-3/2}\right) \geq \eta_v. \tag{30}$$

By looking at the bounds for suitable ε and β values in Lemma 6.1, one sees that T is suitably small if $M \leq \alpha \frac{1-\alpha}{\eta}$, and $\tilde{M} \leq (\frac{(1-\alpha)}{\eta})^2$ where $\alpha \in (0,1)$. This is the case if

$$M \leq \alpha(1-\alpha) \frac{4\nu C_0 p_1}{\lambda} T^{3/2-\gamma},$$

and if

$$\tilde{M} \leq (1-\alpha)^2 \left(\frac{4\nu C_0 p_1}{\lambda}\right)^2 T^{3-2\gamma}.$$

Therefore, if

$$\left(\frac{M\lambda}{4\nu C_0 p_1}\right)^{\gamma-3/2} \geq T,$$

and if

$$\left(\frac{\tilde{M}\lambda}{4\nu C_0 p_1}\right)^{2\gamma-3} \geq T$$

then, by lemma 6.1, $|\Upsilon_\phi(t)| \leq \varepsilon$ for all $t < T$.

As a result then, for any $M, \tilde{M} \in (0,\infty)$, if

$$\sup_{x\in\mathbb{R}^3, s\leq T} \frac{|\int_{\mathbb{R}^3} u_0(x-y)K(y,2\nu s)dy|}{h_\gamma(x)} < M \;\; and \quad \sup_{x\in\mathbb{R}^3, T\geq t\geq 0} \frac{|g(x,t)|}{\tilde{h}_\gamma(x)} \leq \tilde{M}$$

then there is a solution, u, to the Navier Stokes Equations for which

$$\sup_{x\in\mathbb{R}^3, 0\leq t\leq T} \frac{|u(x,t)|}{h_\gamma(x)} < C_{M,\tilde{M}} \;\; fort < T,$$

where $C_{M,\tilde{M}} \leq \frac{1-\alpha}{\eta}$, and η is obtained from (30).

2) (Uniqueness) Suppose v is a solution to (2) with

$$\sup_{x\in\mathbb{R}^3, T\geq t\geq 0} \frac{|v(x,t)|}{h_\gamma(x)} \leq C_{M,\tilde{M}}.$$

Let $\rho(x,t) = v(x,t)/h_\gamma(x)$. Take $X_{\bar{\phi}} = x \in \mathbb{R}^3$and let $0 < t \leq T$ be fixed. Let

$$\mathscr{W}^{(n)}(t) = \{v \in \mathscr{W}(t) : |v| \leq n\}$$

Where $\mathscr{W}(t)$ is as defined in (29). For $n \geq 0$ and $v \in \mathscr{W}(t)$, define the random functions $\Psi_v^{(n)}$ via

$$\Psi_v^{(0)}(t) = m_0(X_{\bar{v}},t)$$
$$+m(\tau_v, X_{\bar{v}})B_v(\rho(X_v, t-\tau_v), \rho(X_v, t-\tau_v))1[\kappa_v = 1,2,3]\bigcap[\tau_v \leq t]$$
$$+\tilde{m}(\tau_v, X_{\bar{v}})C_v\varphi(X_v, t-\tau_v)1[\kappa_v = 4,5]\bigcap[\tau_v \leq t]$$

and, for $n \geq 1$,

$$\begin{aligned}
&\Psi_v^{(n)}(t) = m_0(X_{\bar{v}}, t) \\
&+m(\tau_v, X_{\bar{v}}) B_v(\Psi_v^{(n-1)}(t-\tau_v), \Psi_v^{(n-1)}(t-\tau_v)) 1[\kappa_v = 1,2,3] \bigcap [\tau_v \leq t] \\
&+\tilde{m}(\tau_v, X_{\bar{v}}) C_v \varphi(X_v, t-\tau_v) 1[\kappa_v = 4,5] \bigcap [\tau_v \leq t].
\end{aligned}$$

Observe that for each n, $\Psi_\phi^{(n)}$ depends only on the ensembles in X for which $v \in \mathscr{W}^{(n)}(t)$, also observe that if $\mathscr{W}^{(n)}(t) = \mathscr{W}(t)$, then $\Psi_\phi^{(n)}(t) = \Upsilon_\phi(t)$. Since $v(x,t)$ is a solution to (2),

$$v(x,t) = h(x)\rho(x,t) = h(x) E_x \Psi_\phi^{(0)}(t),$$

and for all v,

$$\begin{aligned}
E_x(\Psi_v^{(0)}(t-S_v) | \bar{\mathscr{F}}_{\bar{v}}) 1[t-S_v &\geq 0] \\
&= E_x(\Psi_v^{(0)}(t-S_v) | \mathscr{F}_{\bar{v}}) 1[t-S_v \geq 0] \\
&= \rho(X_{\bar{v}}, t-S_v) 1[t-S_v \geq 0],
\end{aligned}$$

Where the first equality follows from (13). Now, if for some $n \geq 1$ and for all $v \in \mathscr{V}$,

$E_x(\Psi_v^{(n-1)}(t-S_v)|\bar{\bar{\mathscr{F}}}_{\bar{v}}) = \rho(X_{\bar{v}}, t-S_v)$ on the set $[t-S_v \geq 0]$, then

$$
\begin{aligned}
&E_x(\Psi_v^{(n)}(t-S_v)|\bar{\bar{\mathscr{F}}}_v)1[t-S_v \geq 0] \\
&= m_0(X_{\bar{v}}, t-S_v) \\
&+ \Big\{ m(\tau_v, X_{\bar{v}}) B_v(E_x(\Psi_{v\star 0}^{(n-1)}(t-S_v-\tau_v)|\bar{\bar{\mathscr{F}}}_v), \Psi_{v\star 1}^{(n-1)}(t-S_v-\tau_v)|\mathscr{G}_v)) \\
&\times 1[\kappa_v = 1,2,3] \\
&+ \tilde{m}(\tau_v, X_{\bar{v}}) C_v \varphi(X_v, t-S_v-\tau_v) 1[\kappa_v = 4,5] \Big\} 1[\tau_v \leq t-S_v] \\
&= m_0(X_{\bar{v}}, t-S_v) + \Big\{ m(\tau_v, X_{\bar{v}}) B_v(\rho(X_v, t-\tau_v-S_v), \rho(X_v, t-\tau_v-S_v)) \\
&\times 1[\kappa_v = 1,2,3] \\
&+ \tilde{m}(\tau_v, X_{\bar{v}}) C_v \varphi(X_v, t-S_v-\tau_v) 1[\kappa_v = 4,5] \Big\} 1[\tau_v \leq t-S_v].
\end{aligned}
$$

So, on the set $[\tau_v \leq t-S_v]$,

$$
\begin{aligned}
&E_x(\Psi_v^{(n)}(t-S_v)|\bar{\bar{\mathscr{F}}}_{\bar{v}}) \\
&= E_x(E_x(\Psi_v^{(n)}(t-S_v)|\bar{\bar{\mathscr{F}}}_v)|\bar{\bar{\mathscr{F}}}_{\bar{v}}) \\
&= m_0(X_{\bar{v}}, t) \\
&+ \int_{s=0}^{t} \int_{z\in\mathbb{R}^3} \int_{y\in\mathbb{R}^3} \Big\{ m(s, X_{\bar{v}}) \Big(p_1 f_0(s|z) f(y,z|X_{\bar{v}}) \\
&\times b_1(z; \rho(X_{\bar{v}}-z, t-S_v-s), \rho(X_{\bar{v}}-z, t-S_v-s))
\end{aligned}
$$

$$
\begin{aligned}
&+ \big(p_2 f_1(s|z) f(y,z|X_{\bar{v}}) - p_3 f_1(s|y) f(y,z|X_{\bar{v}}) \big) \\
&b_2(z; \rho(X_{\bar{v}}-z, t-S_v-s), \rho(X_{\bar{v}}-z, t-S_v-s)) \Big) \\
&+ \tilde{m}(s, X_{\bar{v}}) \Big(p_4 f_1(s|z) \tilde{f}(y,z|X_{\bar{v}}) P_z \\
&- p_5 f_1(s|y) \tilde{f}(y,z|X_{\bar{v}})(I - 3e_z e_z^t) \Big) \varphi(X_{\bar{v}}-z, t-S_v-s) \Big\} dz dy ds = \rho(X_{\bar{v}}, t-S_v).
\end{aligned}
$$

Now by induction it follows that $E_x(\Psi_v^{(n)}(t-S_v)|\bar{\mathscr{F}}_{\bar{v}}) = \rho(X_{\bar{v}}, t-S_v)$ on the set $[t - S_v \geq 0]$, for all n, v. It follows that $E_x(\Psi_\phi^{(n)}(t)) = \rho(x,t)$, since $\bar{\mathscr{F}}_{\bar{\phi}}$ is the trivial σ-field. Since $\mathscr{W}(t)$ is finite a.s-P_x, then with probability 1, there exists and N>0 such that $\mathscr{W}^{(n)}(t) = \mathscr{W}(t)$ and that $\Upsilon_\phi(t) = \Psi_\phi^{(n)}(t)$ for all n>N. Since both functions are bounded in magnitude by $C_{M\tilde{M}}$, it is clear that

$$\begin{aligned} |\chi(x,t) - \rho(x,t)| &= |E_x(\Upsilon_\phi(t) - \Psi_\phi^{(n)}(t))| \\ &\leq 2C_{M\tilde{M}} P_x(\mathscr{W}(t) \neq \mathscr{W}^{(n)}(t)) \to 0 \ as n \to \infty. \end{aligned}$$

Therefore, $\rho = \chi \implies u(x,t) = v(x,t)$ QED.

Scholium So much for the existence and uniqueness in the case where $\gamma > 3/2$. In this case, the treatment of the Navier Stokes Equations is just a generalization to that which was given by Ossiander[6]. In her paper, conditions for existence and uniqueness were given for the case in which $\gamma = 3/2$. Notice that in this case, the proof above does not work because the functions m and $\tilde{m}$ cannot be made arbitrarily small for some non-zero window of time by making T sufficiently small. Rather, in her paper, she gave a condition on the data to ensure existence and uniqueness of a solution for all time– basically, it involved making the data small enough for the contraction principle in lemma 6.1 to work. What follows now is a line of inquiry on how one can demonstrate existence and uniqueness for a short time, given arbitrarily large data and given that $\gamma = 3/2$.

7 The case of $\gamma = 3/2$

In finding a suitable existence and uniqueness result for the case where $\gamma = 3/2$, the use of a contraction argument as given in Theorem 6.1 will not work if one is to be given arbitrarily large data. So, I shall start with a lemma that attempts to estimate $|\Upsilon_\phi|$ given that a certain number of branching events happen on the $\mathscr{V}$-indexed random variable X.

Lemma 7.1: Using the notation from Lemma 6.1, let $\eta N \geq 2$, and let $b < \infty$ be the number of births in a given tree. Let $\frac{|m_0(x,t)|}{h}, \frac{|g|}{\tilde{h}} \leq N$. Then $|\Upsilon_\phi| \leq \frac{1}{\eta}((1+\eta)\eta N)^{1+2b}$.

Proof:

In the case where b=1, one sees that $|\Upsilon_\phi| \leq N + \eta(N+\eta N)^2 = \frac{1}{\eta}(\eta N) + \frac{1}{\eta}((1+\eta)(\eta N))^2 \leq \frac{1}{\eta}((1+\eta)(\eta N))^3$. Suppose now that the theorem holds for b=p, and then consider the case where b=p+1. Then $|\Upsilon_\phi| \leq N + \eta|\Upsilon_0||\Upsilon_1|$. Now the combined number of births on the trees starting at $v = 0, 1$ is equal to p, so our inductive hypothesis can be applied to $|\Upsilon_0|$ and $|\Upsilon_1|$. Therefore, $|\Upsilon_\phi| \leq N + \eta(\frac{1}{\eta})^2((1+\eta)\eta N)^{2(p+1)} \leq \frac{1}{\eta}((1+\eta)\eta N)^{1+2(p+1)}$. Therefore, by induction of this result, our lemma is proven. QED

Now that we have established some sort of bounding behavior of $|\Upsilon_\phi|$ with respect to a particular tree, I shall now describe the particular case where h lacks what I shall call “traps”. This will hold for all bounded h.

7.1 The Case where no traps exist.

Here I shall write about the case where there are no traps,

Definition 7.1.1 The majorizing kernel, h, which generates the conditional probability distribution for the Markov process introduced in section 4, is said to have no traps if

$$\forall \varepsilon > 0, \exists \delta > 0\, s.t\; P_x[|Z_\phi| < \delta | \kappa_\phi = 1, 2, or\, 3] \leq \varepsilon, \tag{31}$$
$$and\; P_x[|Y_\phi| < \delta | \kappa_\phi = 1, 2, or\, 3] \leq \varepsilon \quad \forall x.$$

In this case, I shall show that given any bounds on the data: $\frac{u_0}{h}, \frac{g}{h} \leq N$, a weak solution will exist for some time T>0. The constraints in (31) will allow one to make the probability that any node will produce sterile children as close to 1 as one wishes. I will then show that such a property allows for the demonstration of a non-zero window of existence.

Note: By "sterile children" of the node $v \in \mathscr{V}$, I mean that the the value of $\Upsilon_\phi(t)$, for a given element ω in the underlying probability space, does not depend on the value of the $X_{v\star k\star j}$ for $k, j \in \{0, 1\}$. This happens when either $\kappa_v = 4, 5$ (death), or $\tau_v > t$ (expiration).

Proposition 7.1.1: For t>0, $z, y \in (0, \infty)$

$$\begin{aligned}
P_x[\tau_\phi \leq t |\, |Z_\phi| \geq z,\, \kappa_\phi = 2] &\leq \int_0^{t/(|z|^2)} Ce^{-1/4\nu w} dw, \\
P_x[\tau_\phi \leq t |\, |Y_\phi| \geq y,\, \kappa_\phi = 3] &\leq \int_0^{t/(|y|^2)} Ce^{-1/4\nu w} dw, \\
P_x[\tau_\phi \leq t |\, |Z_\phi| \geq z,\, \kappa_\phi = 1] &\leq \int_0^{t/(|z|^2)} Ke^{-1/4\nu w} dw,
\end{aligned} \tag{32}$$

where C and K above are constants.

Proof: This follows from the change of variable $s = |z|^2 w$ in the case where $\kappa_\phi = 1, 2$ or $s = |y|^2 w$ in the case where $\kappa_\phi = 3$ on the conditional waiting times (16) for $\kappa = 1$

and (17) for $\kappa_\phi = 2,3$. QED

Scholium: All three of these probabilities can be made arbitrarily small by making t small. Furthermore, when h has no traps, we can strengthen the above proposition as follows.

Corollary 7.1.1: For $\varepsilon > 0$ there exists a $t > 0$ such that $P_x[\tau_\phi \le t|\, \kappa_\phi = 1,2,3] \le \varepsilon$.

Proof: let δ be such that $P_x[|Z_\phi| < \delta|\kappa_\phi = 1,2] \le \varepsilon/2,$ *and* $P_x[|Y_\phi| < \delta|\kappa_\phi = 3] \le \varepsilon/2$. Let t be such that $P_x[\tau_\phi \le t|\, \kappa_\phi = 1,2\,,\, |Z_\phi| \ge \delta] \le \varepsilon/2$, and $P_x[\tau_\phi \le t|\, \kappa_\phi = 3\,,\, |Y_\phi| \ge \delta] \le \varepsilon/2$. The corollary now follows, since (for $W^{(j)} = Z$ for $j = 1,2$, $W^{(j)} = Y$ for $j = 3$)

$$\begin{aligned}
P_x[\tau_\phi \quad \le t|\kappa_\phi = 1,2,3] = \quad & \sum_{j=1}^{3} p_j P_x[\tau_\phi \le t|k_\phi = j] \\
& \le \sum_{j=1}^{3} p_j (P_x[\tau_\phi \le t||W_\phi^j| \ge \delta, \kappa_\phi = j] \\
& + \frac{\varepsilon}{2} \times P_x[\tau_\phi \le t||W_\phi^j| < \delta, \kappa_\phi = j]) \\
& \le \sum_{j=1}^{j=3} p_j (\frac{\varepsilon}{2} + \frac{\varepsilon}{2}) \le \varepsilon.
\end{aligned}$$

QED.

Scholium: Observe that in (27), $|\Upsilon_v(t)|$ does not depend on the successors of nodes on the binary tree that are past the time t. In other words, if a node at the point $v \in \mathscr{V}$ is such that $\tau_v > t$, then the children of that node will be sterile. By the corollary above, one has a way of making the probability of sterile children being born arbitrarily close to 1. This sterility rate will suppress the probability weight that long trees will exist, resulting in a finite expectation: $E_x|\Upsilon_\phi|$.

Theorem 7.1.1: If h has no traps as in Definition 7.1.1, and if $\frac{u_0}{h}, \frac{g}{\tilde{h}} \leq N$, then there is a T>0, s.t $E_x|\Upsilon_\phi(t)| < \infty$ for all t<T, and for all $x \in \mathbb{R}^3$.

Proof: Without loss of generality, the number of birthing events on a particular tree can be assumed finite. This is because the Markov process in question is a sub-critical branching process, therefore the trees are of finite length a.s.–(see Athreya and Ney[1]). Observe that, from lemma 7.1, on a given tree, $|\Upsilon_\phi(t)| \leq (KN)^{1+2b}$ where b is the number of birthing events, or child-bearing nodes. For any given b, the number of parent nodes, v, such that $[\tau_v \leq t] \bigcap [\kappa_v \in \{1,2,3\}]$ is at least:

$$p = \min\{\lfloor \frac{b-1}{2} \rfloor, 0\}.$$

This is due to the fact that any parent can only produce two children, and the -1 term is due to the fact that the first node has no ancestor.

Let $A_b = \{\omega : number\,of\,births\, = b\}$, and let $\varepsilon > 0$ be the uniform (in x) lower bound for the probability that $\tau_\phi \leq T$. Then $P_x[A_b] \leq \varepsilon^p$. This last estimate is due to the fact that there must be p births for which the time elapse is less than T; also, these events are independent. Now observe that

$$E_x|\Upsilon_\phi(t)| \leq \sum_{b=0}^{\infty}(KN)^{1+2b}P_x[A_b] \leq \sum_{b=0}^{\infty}(KN)^{1+2b}\varepsilon^p \leq \tag{33}$$

$$\sum_{b=0}^{9}(KN)^{1+2b} + KN\sum_{b=10}^{\infty}(K^2N^2\varepsilon^{\frac{1}{4}})^b. \tag{34}$$

By choosing an ε which is small enough, then, (33) implies that $E_x|\Upsilon_\phi(t)| < \infty$ for all t<T.QED

If h is a bounded majorizing kernel, then h does not have traps. However, this is not the case for all unbounded kernels. The Appendix B will cover one such case.

Part II

Appendix A: Some Useful Lemmas.

A.1: Let $Y(x,t)$ be a measurable function on $\mathbb{R}^{m+n}$, where $m,n \in \mathbb{N}$, and let X and T be $\mathbb{R}^m$- and $\mathbb{R}^n$-valued random variables respectively on some probability space (Ω,$\mathscr{O}$,P). then Y(X,T) is a random variable. Suppose $\mathscr{G}$ is a $sub-\sigma-field$ on the probability space for which the random variable T is measurable– $\sigma(T) \subset \mathscr{G}$. Then $E[Y(X,T)|\mathscr{G}]_\omega = E[Y(X,T(\omega))|\mathscr{G}]_\omega$.

Proof: First, suppose $Y(x,t) = 1_{B\times C}$, where B and C are Borel sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then, certainly $E[Y(X,T)|\mathscr{G}]_\omega = E[1_{[(X,T)\in B\times C]}|\mathscr{G}]_\omega =$
$E[1_{[X\in B]}1_{[Y\in C]}(\omega)|\mathscr{G}]_\omega$. I shall now demonstrate that the sets $B \in Q \subset \mathscr{B}^{m+n}$ for which $E[1_{(X,T)\in B}|\mathscr{G}]_\omega = E[1_{(X,T(\omega))\in B}|\mathscr{G}]_\omega$ form a lambda system. First, since $\mathbb{R}^2$ is a measurable rectangle, it is in Q, as is the empty set. It is also obvious that complements are in Q. Finally, if $\{B_i\}$ is a disjoint sequence of sets in Q, then

$$\begin{aligned} 1_{(X,T)\in\bigcup B_i} &= \sum_i 1_{B_i} \implies E[1_{(X,T)\in\bigcup B_i}|\mathscr{G}]_\omega = \sum_i E[1_{(X,T(\omega))\in B_i}|\mathscr{G}]_\omega \\ &= E[\sum_i 1_{(X,T(\omega))\in B_i}|\mathscr{G}]_\omega \end{aligned}$$

P- almost surely.

Therefore, the collection of all sets B for which $E[1_{(X,T)\in B}|\mathscr{G}]_\omega = E[1_{(X,T(\omega))\in B}|\mathscr{G}]_\omega$ holds forms a λ system which contains the π-system of the measurable rectangles of $\mathscr{B}^2$. Therefore, due to Dynkin's $\pi-\lambda$ theorem, for any Borel set $B \in \mathscr{B}^2$, the property that $E[1_{(X,T)\in B}|\mathscr{G}]_\omega = E[1_{(X,T(\omega))\in B}|\mathscr{G}]_\omega$ holds true. Consequently, for simple functions φ, $E[\varphi(X,T)|\mathscr{G}]_\omega = E[\varphi(X,T(\omega))|\mathscr{G}]_\omega$, and the same goes for measurable functions

Y(x,t). QED

A.2: Let $h(x), g(x) : \mathbb{R}^n \to [0,\infty)$ be spherically symmetric functions which are decreasing functions of $|x|$. Then $(h*g)(x)$ decreases with $|x|$.

Proof: Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the cartesian projection map. Spherical symmetry of the functions ensures that $(h*g)(x) = (h*g)(e_1|x|)$, where e_1 is the unit vector such that $\pi_1(e_1) = 1$. Let $|x_1| < |x_2|$ then

$$(h*g)(e_1|x_1|) - (h*g)(e_1|x_2|) = \int_{\mathbb{R}^3} (g(e_1|x_1| - y) - g(e_1|x_2| - y))h(y)dy.$$

Now on the hyperplane $\pi_1(x) = (|x_2| - |x_1|)/2$, the function $(g(e_1|x_1| - y) - g(e_1|x_2| - y))$ is 0, and it is anti-symmetric with respect to reflections about this hyperplane. However, when reflecting about this same hyperplane (call the reflection R), one sees that $h(y) \geq h(Ry)$ for $y \in [\pi_1(x) < (|x_2| - |x_1|)/2]$. Thinking of hdy as a measure on $\mathbb{R}^n$, it is clear that for any interval (a,b), $a,b > 0$, the hdy measure of the set $[y : (g(e_1|x_2| - y) - g(e_1|x_1| - y)) \in (a,b)]$ is less than or equal to the hdy measure of the set $[(g(e_1|x_2| - y) - g(e_1|x_1| - y)) \in (-b,-a)]$. One can therefore conclude that $(h*g)(e_1|x_2|) - (h*g)(e_1|x_1|) \leq 0$. QED

Part III

Appendix B: The Case where $\gamma = 3/2$ and traps exist

Let $\gamma = 3/2$. The majorizing kernel $h(x) = |x|^{-1}$ does have a trap at x=0, as is shown in proposition B.2. This makes a situation where we cannot employ Theorem 7.1.1 to demonstrate existence for a solution in the case where the initial conditions are dominated by such an h. In this appendix-I attempt (but fail) to demonstrate a counterexample– an initial condition for which the solution does not exist for any time after 0. However, the following propositions will suggest certain properties of the bilinear operators b_1 and b_2 which will be necessary in order for a solution to exist for some finite time. I start by trying to demonstrate that, in the case where $h = |x|^{-1}$, there is some lower bound for the initial data:

$$\frac{m_0(x,t)}{h} \leq N, \ \ \frac{g}{\tilde{h}} \leq \tilde{N}$$

such that there is no non-zero time T for which a solution is guaranteed to exist. I shall do this by underestimation of $E_x[\Upsilon_\phi(t)]$ for $|x| \leq |x_0| \in (0,\infty)$. My first step is to demonstrate, given T>0, a lower bound on the probability that a particular lineage along the binary tree survives and stays below time T which is exponential and uniform in the starting points $|x| \leq |x_0|$. I shall do this in the next two propositions.

Proposition B1: If one considers the case where $\kappa_1 = 2$, then for any T>0:

$$P_x[\tau_1 \leq T/2|Z_1 \leq y] \leq \int_0^{T/(2|y|^2)} Ce^{-1/4\nu w} dw \quad \text{where C is a constant.}$$

Proof: This result follows from the definition of the conditional waiting time given in (17) (this is the $f_1(s|z)$ term) and the use of the substitution $s = |y|w$. QED

Corollary B1: Given a time T>0, a sufficient bound on $|Z_v|$ such that,

$$P_x[\tau_v \leq \frac{T-\tau_{\bar{v}}}{2}|\kappa_v = 2, |Z_v| \leq \beta \in (0,\infty)] \geq K_{time} > 0, \ \text{where } K_{time} \text{ is a constant}$$

is the following:

$$|Z_v| \leq \sqrt{\frac{T-\tau_{\bar{v}}}{2\alpha}}, \text{ for some } \alpha > 0$$

Scholium Please notice that if we make $|Z_v|$ smaller, then the probability of the event above only grows.

Now that we know know that for sufficiently small displacements (remember how Z_v is related to X_v. See section 4) the time elapsed from one generation to the next can be made arbitrarily small. The next proposition will also give us a non-zero probability of the displacements decaying exponentially.

Proposition B2: Given that $h = |x|^{-1}$,

$$P_x[|X_\phi| \leq |x|/2, \tau_\phi \leq T/2|\, \kappa_\phi = 2] \geq K_{space} K_{time}^{(x)} p_2 > 0$$

where $K_{space}, K_{time}^{(x)} > 0$ are constants, K_{space}being independent of x, and $K_{time}^{(\tilde{x})} \geq K_{time}^{(x)}$ $\forall |\tilde{x}| \leq |x|$.

Proof: Fix $x \in \mathbb{R}^3$. I begin with a calculation involving the conditional transition density given in (14). Namely,

$$f(y,z|x) = \frac{|y|^{-1}|z|^{-4}|x-z|^{-2}1[|z|>|y|]}{2\pi \int_{\mathbb{R}^3} |z|^{-2}|x-z|^{-2}\mathrm{d}z}.$$

Letting U be the region in $\mathbb{R}^3$ such that $\{x-U\} = B_0(|x|/2)$ (note that $X_v = X_{\bar{v}} - Z_v$). We see that

$$\int_U \int_{\mathbb{R}^3} f(y,z|x)\mathrm{d}y\mathrm{d}z = \frac{\int_U |z|^{-2}|x-z|^{-2}\mathrm{d}z}{\pi^3|x|^{-1}}$$
$$\geq \frac{|x|^{-1}\int_{R=\{(3|x|/4)\leq|z|\leq 4|x|/5\}\bigcap U}|z|^{-2}\mathrm{d}z}{2\pi^3} \geq K_{space} > 0.$$

Now all that remains is to use the probability density for the quadruple $(Y_v, Z_v, \tau_v, \kappa_v)$ in the case where $\kappa_v = 2$, given in (5) to demonstrate the desired result:

$$\begin{aligned} &P_x[|X_\phi| \leq |x|/2,\, \tau_\phi \leq T/2|\, \kappa_\phi = 2] = \int_U \int_{\mathbb{R}^3} \int_0^{T/2} p_2 f_1(s|z) f(y,z|x)\mathrm{d}s\mathrm{d}y\mathrm{d}z \\ &\geq \int_U \int_{\mathbb{R}^3} f(y,z|x)\left(\int_0^{T/2} p_2 f_1(s|\,|x|)\mathrm{d}s\right)\mathrm{d}y\mathrm{d}z \\ &\geq K_{time}^{(x)} p_2 \int_U \int_{\mathbb{R}^3} f(y,z|x)\mathrm{d}y\mathrm{d}z \\ &\geq K_{time}^{(x)} K_{space} p_2 > 0. \end{aligned} \tag{35}$$

Finally, $K_{time}^{(\tilde{x})} \geq K_{time}^{(x)}\ \forall|\tilde{x}| \leq |x|$, follows from the fact that $\int_0^{T/2} f_1(s|\,|x|)\mathrm{d}s$ is monotonically decreasing in $|x|$.QED

Scholium The last two propositions demonstrate that the probability of a single lineage on the binary branching tree to take on an exponential decay in the step size for both time and space has an exponential lower bound, independent of the starting point $|x| \leq |x_0|$. A consequence of this result is that the probability that the n^{th}node of a strand lying in the set $A_n = \{(x,t) : |x| \leq |x_0|2^{-n}, t \leq T(1-2^{-2})\}$ is bounded below by $(K_{time}^{(x_0)} K_{space} p_2)^n$. Suppose now that all other branches have children and that the entire n^{th} generation of a tree lies in the set A_n. Then, as there are $2^n$of these nodes, the probability of this occurring is bounded below by $(K_{time}^{(x)} K_{space} p_2)^{2^n}$.

Let me now return to my description of the growth of $|\Upsilon_\phi|$. In the case where

$$\frac{|u_0|}{h} \leq \tilde{N}, \quad \frac{g}{\tilde{h}} \leq \tilde{N},$$

one can conclude that $|\Upsilon_\phi(t)| \leq ((1+\eta)\tilde{N})^{2|\partial V|-1}$, where ∂V is the boundary of the particular instance of the binary tree for which $|\Upsilon_\phi|$ is being approximated(cf. lemma 7.1)(note that $|\partial V| = 1+b$). If one can also find a set of non-zero probability, measurable $\tilde{\mathscr{F}}_\phi$ such that, for some $\tilde{\tilde{N}} > 0$, $\tilde{\tilde{N}}^{|\partial V|} \leq |\Upsilon_\phi(t)|$ with probability $p_d^{|\partial V|}$ for all starting points $|X_{\bar{\phi}}| \leq |x_0|$ and all times t>0, then one can then use the remarks above to conclude that $E_x|\Upsilon_\phi(t)| \geq (K_{time}^{(x)} K_{space} p_d p_2 \tilde{\tilde{N}})^{2^n} \to \infty \; as \; n \to \infty$ if $K_{time}^{(x)} K_{space} p_2 p_d \tilde{\tilde{N}} > 1$. Determining such a condition, and such a $\tilde{\tilde{N}}$ will require a more detailed look at the definition of Υ_v and the geometry governing it.

Recall from (27), that

$$\begin{aligned}
&\Upsilon(t) = \chi_0\big(V_v(t)\big) \\
&+\frac{11m(X_{\bar{v}})}{\rho} B_v(\Upsilon_{v\star 0}(t-\tau_v), \Upsilon_{v\star 1}(t-\tau_v))1[\kappa_v = 1,2,3]\bigcap[\tau_v \leq t] \\
&+\frac{4\tilde{m}(X_{\bar{v}})}{1-p} C_v \varphi(X_v, t-\tau_v)1[\kappa_v = 4,5]\bigcap[\tau_v \leq t] \; .
\end{aligned}$$

Now B_v is of the form:

$$B_v(\cdot,\cdot) = \begin{cases} b_1(Z_v;\cdot,\cdot) & if \;\; \kappa_v = 1 \\ \\ (-1)^\kappa b_2(Z_v;\cdot,\cdot)/2 & if \;\; \kappa_v = 2,3 \end{cases} \tag{36}$$

where

$$
\begin{aligned}
b_1(y;u,v) &= (u\cdot e_y)P_y v + (v\cdot e_y)P_y u \\
b_2(y;u,v) &= b_1(y;u,v) + u\cdot(I-3e_y e_y^t)ve_y. \\
P_y v &= v-(v\cdot e_y)e_y \perp y
\end{aligned}
\tag{37}
$$

Since I am dealing, for the sake of simplicity, with the process in which $\kappa_v = 2$ for all $v \in \mathscr{V}$, I shall be focused on showing some key features of the operator b_2. First notice that it is broken into two terms, the first of which lies perpendicular to the parameter vector, the second lies parallel to the same. For now let us consider the second term:

$$|b_2(y;u,v)| \leq |u||v||\cos\theta_{uv} - 3\cos\theta_{yv}\cos\theta_{yu}|.$$

$$
\begin{aligned}
|b_2(y;u,v)|^2 \quad \leq \quad & |u|^2|v|^2((\cos\theta_{uv} - 3\cos\theta_{yv}\cos\theta_{yu})^2 \\
& +(\cos\theta_{uy}\sin\theta_{vy} + \cos\theta_{vy}\sin\theta_{uy})^2) \\
& |u|^2|v|^2((\cos\theta_{uv} - 3\cos\theta_{yv}\cos\theta_{yu})^2 + (\sin(\theta_{uy}+\theta_{vy}))^2)
\end{aligned}
\tag{38}
$$

Proposition B.3 For any $\delta > 0$, and for any given pair of vectors u and v, there exists a $K^{(\delta)} > 0$ such that, for e_y on the unit sphere, given by the spherical coordinates (θ,ϕ), $|b_2(y;u,v)| \geq |\cos\theta_{uv} - 3\cos\theta_{yv}\cos\theta_{yu}| > K^{(\delta)}$ whenever $(\theta,\phi) \in A^{\delta}_{uv}$ where $d(A^{(\delta)}_{uv}, \gamma_{uv}) > \delta$ and γ_{uv} is some curve on the (θ,ϕ)-plane determined by $e_u\, and\, e_v$.

Proof:

Fix u and v. the points on the θ,ϕ-plane which describe e_y such that $|\cos\theta_{uv} - 3\cos\theta_{yv}\cos\theta_{yu}| = 0$ forms a curve on the θ,ϕ-plane which shall be called γ_{uv}. Fixing some $\delta > 0$, for e_y described by any point on the θ,ϕ-plane whose distance from γ_{uv} is greater than δ, $|\cos\theta_{uv} - 3\cos\theta_{yv}\cos\theta_{yu}| > K_{uv}^{(\delta)} > 0$. Since the unit sphere is compact, and $K_{uv}^{(\delta)}$ can be constructed to be continuous in e_u and e_v there exists some $K^{(\delta)}$ for which $|\cos\theta_{uv} - 3\cos\theta_{yv}\cos\theta_{yu}| > K^{(\delta)} > 0$ for all (θ, ϕ) outside some band of width 2δ. QED

Remark: This result is made stronger by the better bound on $|b_2|$given in (38). Here, one observes that $|b_2| = 0$ only on a single point of the unit sphere. The result is still the same, make $|b_2(y,u,v)| \geq K^{(\delta)}|u||v|$ by excluding e_y from some small (in some sense) subset of the unit sphere.

As a result of this proposition above, in order to bound $b_2(Z_v; \Upsilon_{v\star 0}, \Upsilon_{v\star 1}) \geq K^{(\delta)}|\Upsilon_{v\star 0}||\Upsilon_{v\star 1}|$, $\frac{Z_v}{|Z_v|}$must be outside of some neighborhood of the unit sphere, where the particular neighborhood depends on $\Upsilon_{v\star 0}$ and $\Upsilon_{v\star 1}$ which in general is not independent of Z_v. This is not the only difficulty in determining a condition such that $|\Upsilon_\phi| \geq \tilde{\tilde{N}}^{\partial V}$, for there are also the terms $\chi_0(V_v(t-\tau_v))$ showing up in each node on the tree. These difficulties have convinced me to abandon my search for a counter-example demonstrating the non-existence of a time $T > 0$ such that a weak solution in the space of interest exists in the case where $h = |x|^{-1}$. Suffice it to say that due to proposition B.2 and the remark following, bounding $|\Upsilon_\phi|$ by means applying the triangle inequality to the terms on the nodes (such as arguments like lemma 7.1) without reference to the geometry of the problem will not be enough to demonstrate existence of a solution for any non-zero time. Conversely, if one demonstrates that a solution does exist for some non-zero time, we can infer some interesting results about the nature in which the geometry of the problem retards the growth of $|\Upsilon_\phi|$ for small time in large trees.

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