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Title ELASTICALLY SUPPORTED BEAM COLUMNS

Abstract approved _

A numerical method in structural analysis commonly referred to as Newmark's method is extended to determine the critical load for columns with constant moment of inertia that are supported on three supports, the center support being an elastic support. The method can readily be applied to members with variable moment of inertia. The numerical method essentially is to estimate a trial deflected configuration and calculate trial bending moments. The deflections are computed from these bending moments and compared with the trial deflections. The critical load is calculated from the ratio of the calculated deflections to the trial deflections. The numerical method is explained in detail and example calculations are included.

In order to investigate a variety of assumed shapes examples are worked out on a digital computer. Various trial deflected configurations are used to see how the successive approximations of the deflections converge and how the trial deflections affect the

critical loads. The critical loads are determined for various values of the stiffness of this one elastic support.

The results of the critical load determined by extending Newmark's numerical method are compared with the critical loads determined by a previously developed method. The comparison showed that the critical loads obtained by extending Newmark's method are nearly identical to those previously determined.

Recommendations are made with regard to the different shapes of assumed initial deflection curves used when computing the lowest possible critical load. The appropriate choice of initial deflection increases the speed in convergence to the correct shape.

ELASTICALLY SUPPORTED BEAM COLUMNS

by

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ELASTICALLY SUPPORTED BEAM COLUMNS

I. INTRODUCTION

The determination of the critical compressive axial loads for elastically supported columns can be tedious and difficult when solved by exact methods. In some cases numerical methods can be employed.

The purpose of this thesis is to show that a numerical method commonly referred to as Newmark's method (5) can be extended to determine the critical axial load for elastically supported columns.

In 1946 Newmark described a general numerical method to determine the critical loads for columns with simple and fixed end supports. This method eliminates the mathematics of either the exact differential equation approach or the finite difference approximations. But this work does not include the determination of critical loads for elastically supported columns.

Briefly, Newmark's method is to estimate a reasonable deflected configuration and calculate the corresponding bending moments due to the axial load. The deflections corresponding to these bending moments are then computed and compared with the initially estimated deflections. The process is repeated until the computed deflections are equal to the assumed deflections. A typical calculation using Newmark's numerical method is illustrated in Appendix A.

Formulas for the approximate critical load for elastically supported columns were derived by G. G. Green, G. Winter and T. R. Cuykendall (3) by use of the energy method and representing the deflected shape of the column by a Fourier series of the type

$$y = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L) \quad \text{and} \quad y_o = \sum_{n=1}^{\infty} \bar{a}_n \sin(n\pi x/L),$$

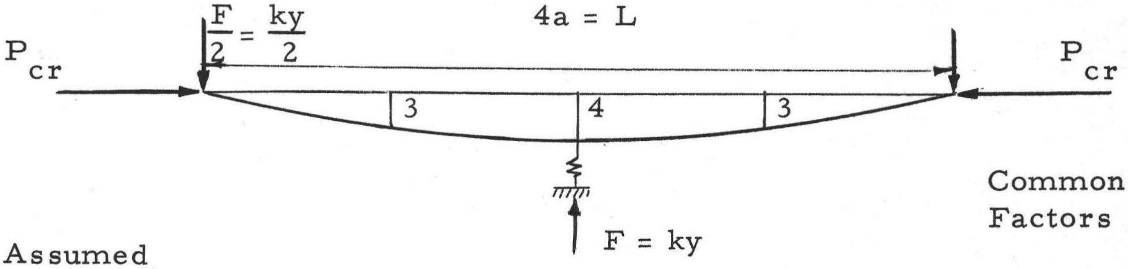
where y_o represents possible initial crookedness. The columns were loaded with concentrated axial loads and were supported elastically at various points along the column.

The approximate critical load derived by Green, Winter and Cuykendall is given as $P_{cr} = \frac{\pi^2 EI}{L^2} + \frac{3\,kL}{16}$ when the column is supported elastically at the mid-height and the spring constant of the support is equal or less than $\frac{16\,\pi^2 EI}{L^3}$. However, when the spring constant is greater than $\frac{16\,\pi^2 EI}{L^3}$, the column buckles into two half waves and the critical load is given as $P_{cr} = \frac{4\pi^2 EI}{L^2}$.

The method of Green, Winter, and Cuykendall is impractical when the moment of inertia of the column is variable along the column length. For such cases Newmark's numerical method can be used to determine the critical load.

In this thesis Newmark's method will be extended to the partiular problem of determining the critical loads for columns with one elastic support at mid-height (Figure 1). The critical load will be calculated for various values of stiffness of this one elastic support.

The procedure developed for this case can then be applied to columns with more than one elastic support.



										Common Factors
Assumed Deflection w_a	0		3		4		3		0	
Moment Due to Lateral Load	0		2		4		2		0	ka
Concentrated Angle Changes	2		12		20		12		2	$\frac{ka^2}{6EI}$
Average Slope		-22		-10		10		22		$\frac{ka^2}{6EI}$
Simple Beam Deflection w_s	0		-22		-32		-22		0	$\frac{ka^3}{6EI}$

Figure 1. Calculation of deflections due to lateral load.

The critical loads thus obtained from the extension of Newmark's procedure will be compared with the critical loads given by the formulas derived by Green, Winter and Cuykendall.

Since Newmark's procedure involves the estimation of a trial deflected configuration of the column, the writer will try various trial deflected configurations to see how the successive approximations of the deflections converge and how the trial deflections affect the buckling loads obtained.

II. DESCRIPTION OF METHOD OF ANALYSIS

Using the basic ideas as used in the procedure available for computing the critical load for simply supported columns and fixed-end supported columns (5), a procedure will be set up to determine the critical load for elastically supported columns by successive approximations.

This section will explain how the physical model of the column will be transformed into a mathematical model.

The object of analysis here is to determine the critical load of an axially loaded column in terms of the physical characteristics of the column and elastic supports. This is done by assuming an initially deflected configuration of the column, the initial configuration being described at a finite number of points along the column. From the initial deflections the procedure is developed to solve for the critical load of the column.

Because of the various investigations performed in this thesis, calculations are performed on an electronic digital computer.

Outline of Method

The following is an outline of the method of analysis to be used in the work.

1. The column is divided into a number of equal segments or

panels.

2. An initial trial deflection is assumed at the panel points.
3. The elastic reaction corresponding to the value of deflection at the elastic lateral support is computed.
4. The deflections due only to the lateral loads which are equal to the elastic reactions are computed.
5. The moments due to the axial loads on the column, corresponding to the initial trial deflections are computed.
6. The deflections of the column for the moments computed in step five are determined.
7. The deflections caused by the axial load are added to the deflections caused by the lateral loads.
8. The calculated deflections and the initial trial deflections are compared.
9. If the error between the calculated deflections and initial trial deflections is large, steps two to eight are repeated using for assumed deflections the deflections resulting from the previous cycle.
10. When the error of step nine becomes small the critical load of the column is evaluated.

Illustration of Method

To illustrate the extension of Newmark's numerical procedure to the case of elastically supported columns an example of the procedure will be worked out.

In Newmark's method it is necessary to assume an initial deflection, go through the procedure of finding the critical load, and calculate a second set of deflections. Then the second set of deflections is compared with the first set of deflections to see if the second set is equal to the first set. When these deflections are found to be equal then the first set of deflections is the correct shape of the deflected curve.

In the example shown in Figure 1, the column is divided into four segments or panels. The initial trial deflections are based upon the assumption that the deflected panel points fall on a parabolic curve.

The reaction at the elastic support is equal to the deflection of the support times the spring constant. The reactions at the ends of the columns are calculated and moments due to the lateral load are calculated.

The approach using the conjugate beam method is to load the conjugate beam with the M/EI diagram. The step-by-step procedure next involves the reduction of the variable M/EI loading on the conjugate beam into a series of equivalent concentrated loads.

To simplify the numerical work, only the coefficients of the common factors are shown in Figure 1. The common factors to the right of the diagrams contain the quantities of force, length, EI, and relevant arithmetical coefficients. The intervals along the column are equal and all have the value of "a" in this example.

The systematic calculations follow and start by tabulating the moment due to lateral loads followed by one line of concentrated angle changes. The next line of the figure represents the average slope of each segment of the elastic curve, and from the conjugate beam analogy these are the equivalent of conjugate beam shear.

Since the deflection is equal to the moment in the conjugate beam and the area under the shear diagram between two sections is equal to the change in moment, the coefficients of deflection are obtained by starting at the left end and accumulating successive areas between intervals as coefficients of $\frac{k\ a^3}{6\ EI}$. It should be noted that as a result of the conventions followed, the downward deflection is positive.

After the deflections due to the lateral load have been calculated, the deflections due to the axial load are calculated. The calculations are shown in Figure 2.

Using the same initial deflection w_a as in Figure 1, the bending moments due to the axial load are computed at each panel point.

It should be noted that the deflected configuration being considered is curved, hence, the moment diagram is assumed to be curved between panel points. Corrections entailed by considering curved moment diagrams were developed by Newmark (5) and are included in Appendix A.

										Common Factors
Assumed Deflection w_a	0		3		4		3		0	1
Distributed Angle Changes	0		-3		-4		-3		0	$\frac{P}{EI}$
Concentrated Angle Changes	-14		-68		-92		-68		-14	$\frac{Pa}{24EI}$
Average Slope		114		46		-46		-114		$\frac{Pa}{24EI}$
Column Deflection w'_a	0		114		160		114		0	$\frac{Pa^2}{24EI}$

Figure 2. Calculation of deflections due to axial load.

The standard solution for the computed deflections is given in Figure 2 and needs no explanation since the procedure is essentially the same as in Figure 1. The computed deflections due to the axial loads are designated as w'_a.

The critical load may be calculated by different ways from the assumed deflections and calculated deflections.

The value of the critical load may be taken from the ratio of the assumed deflection to the calculated deflection at a particular point on the curve.

When the initial deflections from which the computations start are suitably chosen, the sequence of approximations of the deflections very rapidly approach the true buckled shape. However, after a limited number of cycles, the two sets of deflections, the trial deflections and the computed deflections, are not yet completely similar, and the value of P will depend on the particular point on the curve at which the ordinates, $y_{assumed}$ and $y_{computed}$, are read. This difficulty can be overcome by adding all the deflections and using the ratio of the sums of deflections in the criterion

$$\frac{\Sigma\ y_{assumed}}{\Sigma\ y_{computed}} = 1.$$

In cases where the lowest critical load corresponds to a deflection curve that has both positive and negative deflections, use of the ratio of the sums of the deflections will give incorrect results because the effect of the negative deflections will cancel the effect of the positive deflections. To correct for this possibility the method of least squares will be used to calculate the critical load. Although Newmark recommends using the ratio of the sums of deflections to calculate the critical load for the usual case, he also states that the method of least squares gives the best estimation of the critical load from the buckled shape. The derivation of the formula for finding the value for P by the least squares method is shown in Appendix B.

The computations for evaluating the critical load from the assumed deflections and calculated deflections for the example problem are as follows.

The assumed deflections denoted by w_a are from Figure 1. The calculated deflections are the combination of the simple beam deflection denoted by w_s in Figure 1 and the column deflection due to the axial load denoted by w'_a in Figure 2. The deflections for the five panel points are listed with the common factor.

						Common Factors
w_a	0	3	4	3	0	1
w'_a	0	114	160	114	0	$\frac{Pa^2}{24EI}$
w_s	0	-22	-32	-22	0	$\frac{ka^3}{6EI}$

These deflections are then placed in the formula

$$\frac{\Sigma\, w_a\, w'_a}{\Sigma\, w'_a\, w'_a + w_s\, w'_a} = 1 .$$

The terms of the numerator are

$$\Sigma\, w_a\, w'_a = ((3)(114) + (4)(160) + (3)(114))\,\frac{Pa^2}{24EI} .$$

The terms of the denominator are

$$\Sigma\, w'_a\, w'_a + w_s\, w'_a = ((114)(114) + (160)(160) + (114)(114))\left\{\frac{Pa^2}{24EI}\right\}^2$$

$$+ ((-22)(114) + (-32)(160) + (-22)(114))\ \frac{ka^3}{6EI}\ \frac{Pa^2}{24EI}$$

From this it is found that

$$P = 0.623\ \frac{EI}{a^2} + 0.784\ ka$$

In terms of L, where L = 4a,

$$P = 9.98\ \frac{EI}{L^2} + 0.196\ kL.$$

The calculated deflections are compared with the initial or assumed deflections to see how closely the calculated deflected shape is converging to the assumed deflected shape. The difference or error between the computed and assumed deflections is squared and weighted. The derivation of the formula for calculating the error is shown in Appendix C.

If the square root of the sum of the squares of the errors between the assumed deflections and the calculated deflections exceed a specified amount, a new approximation of the deflection curve will be made. The new deflections are a combination of the deflections caused by the axial load and the deflections caused by the lateral load. To the calculated deflection at the elastic support caused by the axial load, a deflection component caused by the lateral load will be added such that the total deflection at the location of the lateral support yields a force that corresponds with the lateral load.

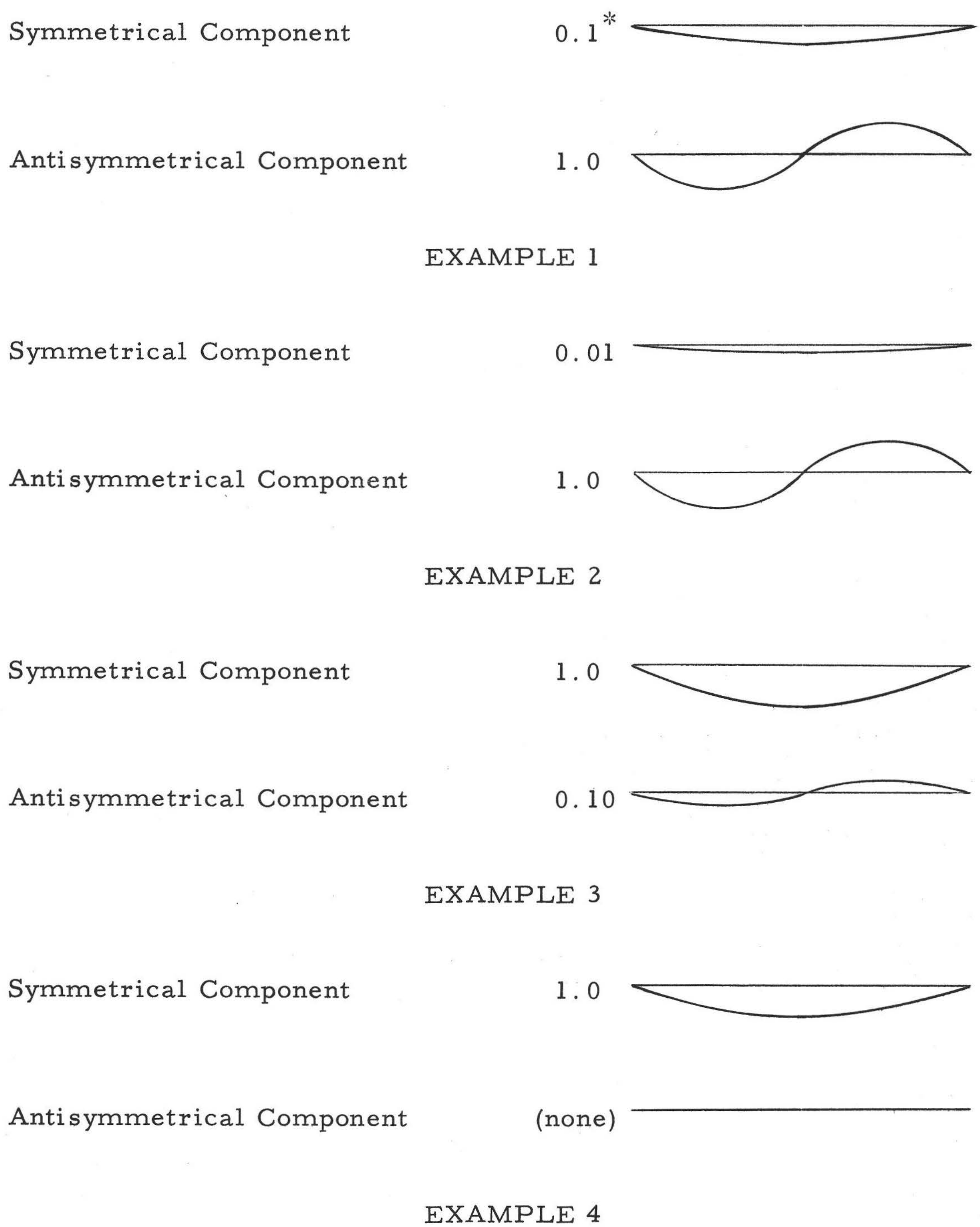
Ordinarily, convergence of several different sequences of computations involving different shapes of assumed deflection curves to the same final shape would be sufficient indication that the configuration corresponding to the lowest critical load had been reached. In some cases, however, the convergence of a sequence of computations may be very slow; this will be so when the next higher critical load differs only slightly from the lowest critical load. Methods of handling such problems can be derived (5, p. 1167).

III. DEMONSTRATION OF THE METHOD

A method for determining the critical load of elastically supported columns has been presented and will be applied to various examples using a digital computer to perform the calculations. The values for the critical load derived from the method described will be compared with the values of the critical load obtained by another method.

Examples will be worked out for columns with constant moment of inertia and supported elastically at the mid-height. Since the column may buckle in a symmetrical or an antisymmetrical mode, the critical load for the two modes will be investigated. One shape of the deflected curve will be taken as a symmetric parabola. Another will be taken as a combination of components of a symmetrical shape and an antisymmetrical shape.

In the first three examples considered the initial deflected shape consists of components of a symmetrical shape and an antisymmetrical shape. The relative magnitudes of the components for each shape are varied for each example as illustrated in Figure 3. In Example 4 the initial deflected shape is a symmetrical parabolic shape. In each example the initial deflections are computed for 13 equidistant points along the length of the column. The values used for the initial deflections are listed in Figure 4.



* The numbers denote the maximum amplitude for each curve.

Figure 3. Relative maximum magnitudes of the antisymmetrical and symmetrical components of the initial deflected curve.

Example No.	Deflections												
1	.000	.586	.944	1.075	.978	.653	.100	-.459	-.800	-.925	-.834	-.526	.000
2	.000	.559	.895	1.007	.898	.566	.010	-.546	-.880	-.992	-.883	-.553	.000
3	.000	.360	.645	.850	.978	1.027	1.00	.917	.800	.650	.467	.250	.000
4	.000	.305	.556	.750	.889	.972	1.00	.972	.889	.750	.556	.305	.000

Figure 4. Initial trial deflections.

For each example the critical load is calculated for different spring constants of the elastic support ranging from $k = 10\ EI/L^3$ to $k = 1010\ EI/L^3$ in increments of $200\ EI/L^3$.

The computations to verify the method described in this thesis were run on an IBM 1620 digital computer. The computer program and printout of the results for Case 1 are shown in Appendix D. The printout of the results shows the initial deflections used, spring constant of the lateral support, permitted error, derived deflections after each cycle, final deflections, and critical load.

IV. RESULTS AND DISCUSSION

The critical loads obtained for each example are tabulated for the various spring constants in Table 1. The number of cycles that the computer goes through in approximating the deflected shape is also tabulated in Table 1.

It should be noted that in some instances the trial and computed deflected shapes do not converge within the specified limit of error in ten cycles. The specified limit used in the calculations is 0.01. When the trial and computed deflected shapes do not converge in ten cycles, the words (NO CONVERGENCE WITHIN SPECIFIED LIMIT IN 10 CYCLES) are printed on the output record of the computer. The value of the critical load P is then printed out after the ten cycles even though the deflections have not converged within the specified limit.

In Example 1 the maximum magnitude of the antisymmetrical component of the initial deflection is 1.0 while the maximum magnitude of the symmetrical component is 0.1. It is shown that the deflection shape converged to either one half wave or two half waves within ten cycles depending on the spring constant, except for the spring constant of 210 EI/L^3, where the deflections did not converge within ten cycles. An asterisk is printed in place of the number of cycles that it takes for the deflections to converge and the value of

Table 1. Tabulation of critical load, spring constant and number of cycles before convergence.

Example 1			Example 2		
Spring Constant	No. of Cycles Before Convergence	Critical Load	Spring Constant	No. of Cycles Before Convergence	Critical Load
$10\ EI/L^3$	7	$11.889\ EI/L^2$	$10\ EI/L^3$	9	$11.889\ EI/L^2$
210	*	39.505	210	2	39.476
410	7	39.469	410	3	39.470
610	6	39.468	610	3	39.468
810	5	39.469	810	3	39.467
1010	5	39.468	1010	3	39.467

Example 3			Example 4		
Spring Constant	No. of Cycles Before Convergence	Critical Load	Spring Constant	No. of Cycles Before Convergence	Critical Load
$10\ EI/L^3$	3	$11.888\ EI/L^2$	$10\ EI/L^3$	2	$11.880\ EI/L^2$
210	*	47.466	210	3	48.323
410	*	40.626	410	7	67.999
610	*	39.581	610	7	74.825
810	*	39.506	810	7	76.928
1010	*	39.488	1010	7	77.814

* No convergence within specified limit of 0.01 in ten cycles, computer prints out the critical load after the tenth cycle.

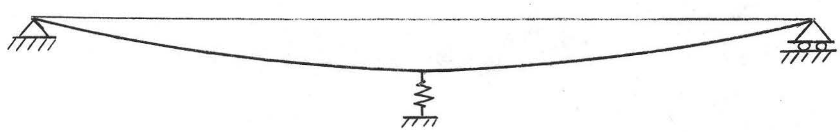
the critical load is printed. Note that the critical load when the deflections did not converge in ten cycles within the specified limit of 0.01 is 39.505 EI/L^2 while the critical load when the deflections converge within ten cycles is 39.468 EI/L^2 to 39.469 EI/L^2.

In Example 2 the maximum magnitude of the antisymmetrical component is the same as in Example 1, but the maximum magnitude of the symmetrical component is decreased to 0.01. It is shown that the deflections converged to either one half or two half waves within ten cycles depending on the spring constant.

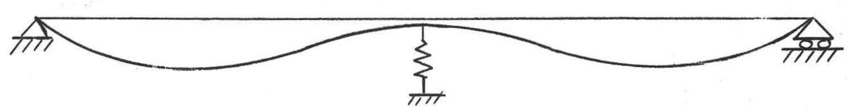
In the first two examples the maximum magnitude of the antisymmetrical component is greater than the maximum magnitude of the symmetrical component. For the third example the maximum magnitude of the antisymmetrical component is smaller than the maximum magnitude of the symmetrical component.

In Example 3 the maximum magnitude of the symmetrical component is 1.0 while the maximum magnitude of the antisymmetrical component is 0.10. The deflections for a spring constant of 10 EI/L^3 converged in three cycles and the calculated critical load is 11.888 EI/L^2. For the other spring constants used in the example the deflections did not converge in ten cycles within the specified limits of 0.01. However, though the deflections are not converged, a value of the critical load is calculated using the assumed deflections and calculated deflections at the tenth cycle to observe what value of critical load has been obtained.

In Example 4 the initial deflection shape is taken to be a parabola, symmetrical about the mid-height of the column. When the spring constant is small, the column buckles into one half wave. However, when the spring constant is increased, the buckled shape tends toward three half waves (see Figure 5). The deflections for a spring constant of 10 EI/L^3 converged in two cycles and the critical load is 11.880 EI/L^2. But the critical load obtained when the spring constant is 210 EI/L^3 is greater than the critical load obtained when the initial deflection contained antisymmetrical components as shown in the previous examples. The critical load comparison is 48.3 EI/L^2 to 39.5 EI/L^2.



Small Spring Constant



Large Spring Constant

Figure 5. Converged shape of the deflected column beginning with parabolic symmetrical shape.

Since the primary concern is usually the lowest critical load, using only one possible mode of buckling may lead to unsafe results when other modes of buckling that will lead to a lower critical load are possible.

For small spring constants where the column will buckle into one half wave, the convergence is more rapid when the antisymetrical component of the initial deflection is relatively small compared to the symmetrical component. For a spring constant of 10 EL/L^3 the convergence took seven cycles in Example 1 while in Example 2 the convergence took nine cycles and in Example 3 the convergence took three cycles. When the antisymmetrical component is non-existent as in Example 4 the convergence took only two cycles.

For large spring constants, where the column will buckle into two half waves, the convergence is more rapid to the lowest critical load when the antisymmetrical component of the initial deflection is relatively large compared to the symmetrical component. For a spring constant of 210 EI/L^3 in Example 1 the convergence does not take place within ten cycles, while in Example 2 the convergence takes two cycles and in Example 3 the convergence does not take place within ten cycles. When the antisymmetrical component is non-existent as in Example 4, the convergence takes three cycles but not to the lowest critical load.

As shown in the previous examples where antisymmetrical components are included in the initial trial deflections, the deflected shape will converge into a single half wave when the spring constant is 10 EI/L^3 and into two half waves when the spring constant is 210 EI/L^3 or greater.

The buckling load when the spring constant of the supports equals 10 EI/L^3 is approximated by the formula $P = 9.86\ EI/L^2 + 0.20\ kL$, where k is the spring constant. When the column buckles into two half waves, the coefficient of the kL term approaches zero and the coefficient of the EI/L^2 term approaches a value of 39.46. When the value of P is calculated, the contribution from the kL term is negligible.

The values of the critical loads obtained by extending Newmark's procedure to columns with one elastic support at mid-height and a spring constant less than $16\pi^2\ EI/L^3$ compare favorably with the results obtained in the work by Green, Winter and Cuykendall (3).

By extension of Newmark's procedure

$$P = 9.87\ EI/L^2 + 0.20\ kL$$

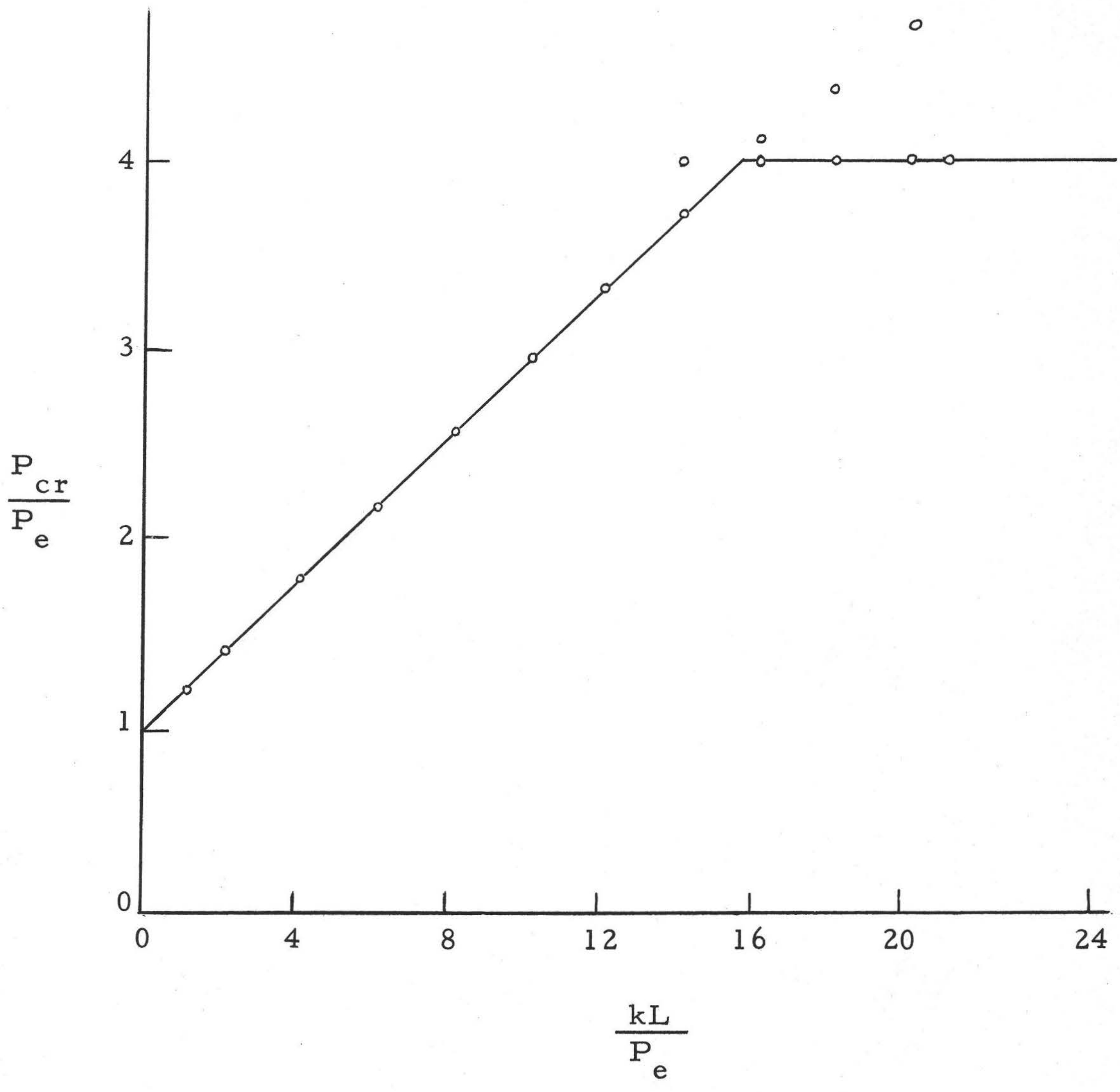
By Green, Winter and Cuykendall

$$P = 9.869\ EI/L^2 + 0.1875\ kL$$

When the spring constant exceeds $16\pi^2\ EI/L^3$, the lowest buckling load obtained by using the procedure described in this thesis is $P = 39.46\ EI/L^2$. This is comparable to $P = 4\pi^2\ EI/L^2$

for the mode when the column buckles into two half waves.

The critical buckling loads determined by the method described in this thesis are plotted with the curve for the critical buckling load from Green, Winter and Cuykendall's work (3, p. 17) in Figure 6. The curve illustrates the variation of the critical load with the rigidity of the intermediate support. For this curve the ratios $P_{cr}: \pi^2 EI/L^2 = P_{cr}: P_e$ are taken as ordinates and the ratios $kL: \pi^2 EI/L^2 = kL/P_e$ as abscissas.



Legend

———— Green, Winter and Cuykendall

∘ Method described in this thesis

Figure 6. Comparison of the critical buckling load.

V. CONCLUSIONS

The critical load for columns with one elastic support can be determined by extending Newmark's numerical procedure as demonstrated in this thesis.

For small spring constants, the initial deflections will converge more rapidly to the final deflected shape when the symmetrical component of the initial deflections is large compared to the antisymmetrical component of the initial deflections than when the symmetrical component of the initial deflections is small compared to the antisymmetrical component of the initial deflections.

For large spring constants, the initial deflections will converge more rapidly to the final deflected shape when the antisymmetrical component of the initial deflections is large compared to the symmetrical component of the initial deflections than when the antisymmetrical component of the initial deflections is small compared to the symmetrical component of the initial deflections.

A computation involving one shape of an assumed deflection curve may lead to a final shape that does not correspond to the lowest critical load.

VI. RECOMMENDATIONS

It is recommended that in using the method described herein different shapes of assumed deflection curves be used as the initial assumed deflected shape when computing the critical load of an elastically supported column. Each shape of the assumed deflection shape should correspond to a possible mode of buckling of the column.

For columns with one elastic support at mid-height one shape should correspond to the first mode of buckling and be a shape symmetrical about the mid-height of the column. To increase the speed in convergence to the symmetrical shape the relative magnitude of the symmetrical component of the assumed deflection curve should be large compared to the antisymmetrical component. The other shape should correspond to the mode of buckling when the column buckles into two half waves and be a shape that is essentially antisymmetrical about the mid-height of the column. To increase the speed in convergence to the antisymmetrical shape, the relative magnitude of the antisymmetrical component of the assumed deflection curve should be large compared to the symmetrical component.

It is recommended that the method described herein be developed further to determine the critical buckling loads for columns with more than one elastic support. It is seen that the method developed can readily be applied to members with variable moment of inertia.

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APPENDICES

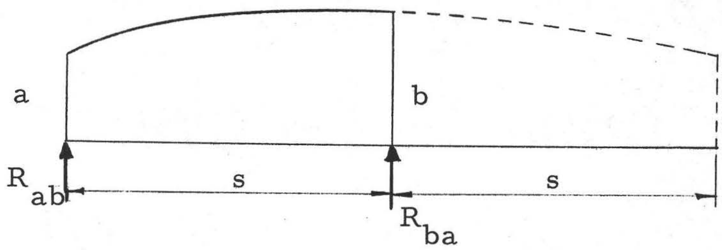
APPENDIX A

ILLUSTRATIVE PROBLEM OF NEWMARK'S NUMERICAL METHOD

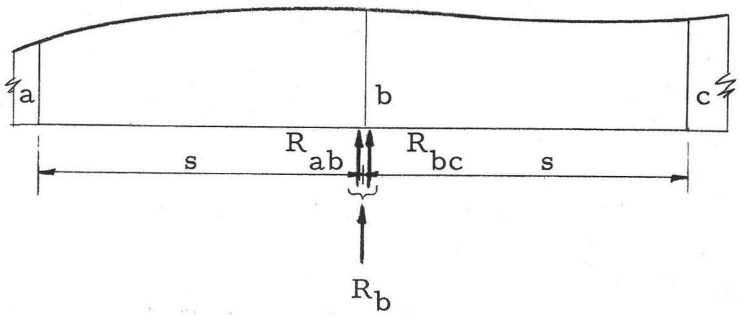
A typical calculation of Newmark's numerical method is illustrated in this section.

The formulas for equivalent concentrated loads are listed for parabolic loading curves (5, p. 1167).

Parabolic Curves



$$R_{ab} = \frac{s}{24}(7a + 6b - c) \qquad R_{ba} = \frac{s}{24}(3a + 10b - c)$$



$$R_b = R_{ba} + R_{bc} = \frac{s}{12}(a + 10b + c)$$

Figure 7 illustrates a loaded column and a set of assumed deflections. Figure 8 illustrates the corresponding moment diagram. Figure 9 is the M/EI diagram. The deflection curve for this example is assumed to be a parabolic curve and is denoted by w_a in Figure 10.

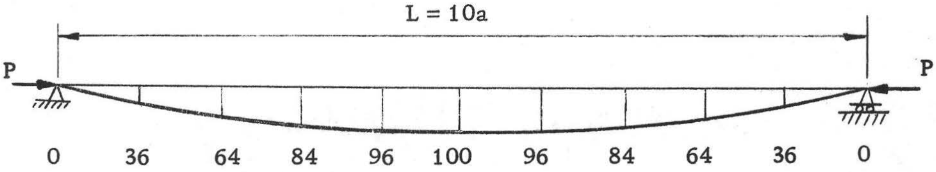


Figure 7. Loaded real column with deflections shown at panel points.

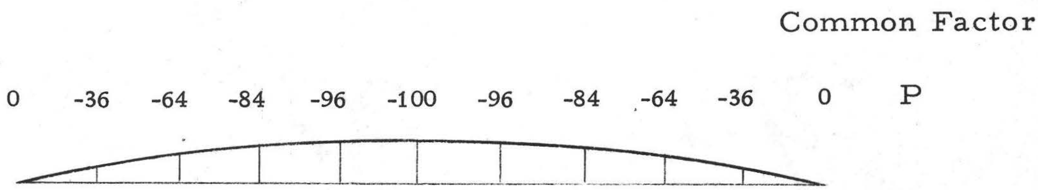


Figure 8. Moment diagram of the moment due to axial load.

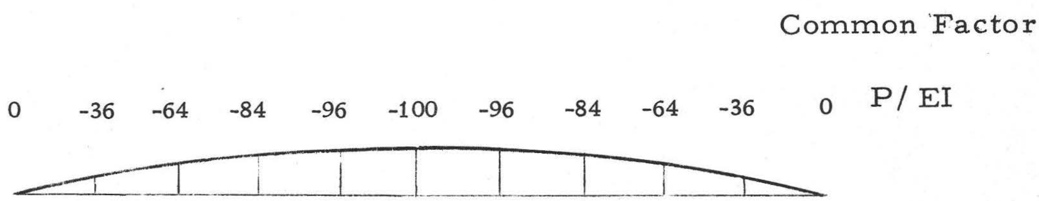
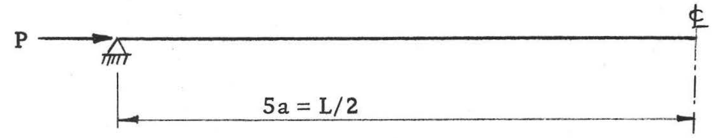


Figure 9. M/EI diagram.



							Common Factors
Assumed Deflections, w_a	0	36	64	84	96	100	
Distributed Angle Changes	0	-36	-64	-84	-96	-100	P/EI
Concentrated Angle Changes	-6.33	-35.33	-63.33	-83.33	-95.33	-99.33	Pa/EI
Average Slope		327	291.67	228.33	145	49.67	Pa/EI
Resultant Deflections, w'_a	0	327	618.7	847	992	1041.7	Pa^2/EI

Ratio of sums of deflections $\dfrac{\Sigma\, w_a}{\Sigma\, w'_a} = \dfrac{0.0998\ EI}{Pa^2}$ $\quad P_{cr} = \dfrac{9.98\ EI}{L^2}$

Best ratio by least squares $\dfrac{\Sigma\, w_a\, w'_a}{\Sigma\, w'_a\, w'_a} = \dfrac{0.987\ EI}{Pa^2}$ $\quad P_{cr} = \dfrac{9.87\ EI}{L^2}$

Figure 10. Critical buckling load for column of constant cross section, starting with assumed parabolic deflection curve.

The moment diagram loads or angle change loads are transmitted directly to the beam in the form of concentrated loads. Figure 11 illustrates the concentrated loads. Since the loads are derived from the M/EI loading, they physically represent concentrated angle changes, or abrupt changes in the slope of the elastic curve.

Common Factor

Pa/ EI

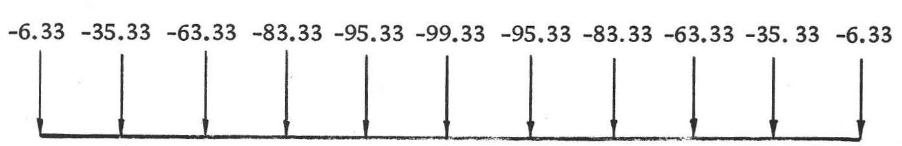


Figure 11. Equivalent concentrated angle changes.

APPENDIX B

METHOD OF LEAST SQUARES

The method of least squares (6, p. 512-513) asserts that the curve of best fit is that one for which the sum of squares of the deviations is a minimum.

The method of least squares will be derived here to find a value for the parameter P of the computed deflections so that the deviations of the computed deflections from the trial deflections is a minimum.

Let the given set of trial deflections be represented by w_{oi}, $i = 1, \ldots, 13$ and the set of calculated deflections be represented by $w_{ai} P + w_{si}$, $i = 1, \ldots, 13$.

Let $d_i = w_{oi} - (w_{ai} P + w_{si})$ be the residuals or deviations of the trial deflections from the calculated deflections.

Since $f = \sum_{i=1}^{13} d_i^{\,2}$ (2, p. 308) is a function of the unknown P, it follows that the problem is to find a value of P where the sum of the squares of the deviations is a minimum.

To do this, the equation

$$\frac{\partial f}{\partial P} = 0 \quad \text{must be solved.}$$

$$f = \sum_{i=1}^{13} (w_{oi} - w_{si})^2 + (w_{ai} P)^2 - 2(w_{ai} P)(w_{oi} - w_{si})$$

$$\frac{\partial f}{\partial P} = \sum_{i=1}^{13} (0 + 2 P w_{ai}^2 - 2 w_{ai} (w_{oi} - w_{si})) = 0$$

$$2 \sum_{i=1}^{13} P w_{ai} w_{ai} = 2 \sum_{i=1}^{13} (w_{ai} w_{oi} - w_{ai} w_{si})$$

Divide both sides by $\sum_{i=1}^{13} w_{ai} w_{ai}$. This division is permissible because the sum of squares of the deflections is non-zero, or must be positive.

The value of P will then be

$$P = \frac{\sum_{i=1}^{13} w_{ai} w_{oi}}{\sum_{i=1}^{13} w_{ai} w_{ai}} - \frac{\sum_{i=1}^{13} w_{ai} w_{si}}{\sum_{i=1}^{13} w_{ai} w_{ai}} .$$

APPENDIX C

DERIVATION OF CONVERGENCE CRITERIA

In order that the criteria for convergence of the calculated deflections to the assumed deflection be made uniform for the different cases that will be studied, a method will be derived to sum the error between the calculated and assumed deflections of the column.

The amount of error between the assumed and calculated deflections will be based on the root mean square method.

Let

d_i = difference between the calculated and trial deflections

y_i = deflection of column

$x_i = d_i / y_i$ error

$f_i = y_i^2$ weight function

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$ arithmetic mean of the error

The root mean square is obtained by squaring the error and taking a positive square root.

$$E_{rms} = \sqrt{\frac{\sum_{i=1}^{n} f_i x_i^2}{\sum_{i=1}^{n} f_i}} \qquad (2, \text{ p. } 306)$$

$$= \sqrt{\frac{\sum_{i=1}^{n} y_i^2 (d_i / y_i)^2}{\sum_{i=1}^{n} y_i^2}} = \sqrt{\frac{\sum_{i=1}^{n} d_i^2}{\sum_{i=1}^{n} y_i^2}}$$

APPENDIX D

COMPUTER PROGRAM AND PRINTOUT OF RESULTS

The computer program for the computations is in the Fortran II language (4). The initial deflections and the permitted error are read into the computer. The computer will calculate the critical load for various spring constants in fixed increments. The magnitude of the first spring constant, the magnitude of the increment of the spring constant, and the magnitude of the maximum spring constant that the computer will execute are also read into the computer.

The input terms are:

WO(I) - the array name that represents the deflection for 13 equidistant points along the length of the column. The units are length.

ER - the permitted error.

ISC1 - the first value of spring constant of the elastic support that the computer will use.

ISC2 - the maximum value of the spring constant that the computer will use.

ISC3 - the value that the spring constant is increased by for each succeeding spring constant used.

The printout of the results consists of the initial deflections, spring constant, permitted error, sequence of calculated deflections, final deflections, the formula for the critical load, and the critical load.

When the deflections do not converge within the specified limits, the words (NO CONVERGENCE WITHIN SPECIFIED LIMIT IN 10 CYCLES) are printed out.

```
C       WO(N)=ASSUMED DEFLECTIONS
C       WA(N)=DERIVED DEFLECTIONS DUE TO AXIAL LOAD
C       WS(N)=DEFLECTIONS DUE TO LATERAL SUPPORT
C       WT(N)=TOTAL DEFLECTIONS                    SC=SPRING CONSTANT
C       EI=FLEXURAL RIGIDITY
        DIMENSION WO(13), WA(13), WS(13), WAS(13), ZMOM(13), CAC(13), AVSLP(13)
        DIMENSION WOCL(13), DACHG(13), WT(13), WA1(13), TRWA(13), EI(13)
   20   FORMAT (23HWO= INITIAL DEFLECTIONS/2X  7E11.4/13X  6E11.4)
   40   FORMAT (3HSC=, F9.1, 10H  EI/(L**3))
   50   FORMAT (3HP= F10.5, 10HEI/(L**2)+, F8.5, 5H  SC*L, 10X  2H= F10.5, 10HEI/
       1(L**2)  //)
   70   FORMAT (2X  7E11.4/13X  6E11.4)
   80   FORMAT (3HWT=, 18H  FINAL DEFLECTIONS/2X  7E11.4/13X  6E11.4)
  100   FORMAT (3X, F6.4)
  110   FORMAT (3HER=, F6.4)
  120   FORMAT (53H   NO CONVERGENCE WITHIN SPECIFIED LIMIT   IN 10 CYCLES)
  200   FORMAT (3X, 3I5)
    1   READ 70, (WO(N), N=1, 13)
        READ 100, ER
        READ 200, ISC1, ISC2, ISC3
    2   PUNCH 20, (WO(N), N=1, 13)
    3   DO  52  ISC=ISC1, ISC2, ISC3
        SC=ISC
C       SCA=SC*EI/(L**3)=SC*EI/(1728*LAMBDA**3)      L=12*LAMBDA
        SCA=SC/(1728.)
    4   PUNCH  40, SC
        PUNCH  110, ER
        DO  5  N=1, 7
        Z=N
    5   ZMOM(N)=-WO(7)*SCA/(2.)*(Z-1.)
        DO  6  N=8, 13
        N1=14-N
    6   ZMOM(N)=ZMOM(N1)
        CAC(1)=-(2.*ZMOM(1)+ZMOM(2))/6.
        DO  7  N=2, 7
    7   CAC(N)=-(ZMOM(N-1)+4.*ZMOM(N)+ZMOM(N+1))/6.
        DO  8  N=8, 13
        N1=14-N
    8   CAC(N)=CAC(N1)
        AVSLP(7)=CAC(7)/2.
        DO  9  N=1, 6
        N1=7-N
    9   AVSLP(N1)=AVSLP(N1+1)-CAC(N1+1)
        DO  11  N=8, 13
   11   AVSLP(N)=AVSLP(N-1)+CAC(N)
        WS(1)=0.
        DO  12  N=2, 13
   12   WS(N)=(WS(N-1)+AVSLP(N-1))
        DO  14  N=1, 13
   14   WOCL(N)=WO(N)
        DO  29  NC=1, 10
   15   DO  16  N=1, 13
```

```
   16  DACHG(N)=-WOCL(N)
       CAC(1)=(7.*DACHG(1)+6.*DACHG(2)-DACHG(3))/24.
       DO 17 N=2,12
   17  CAC(N)=(DACHG(N-1)+10.*DACHG(N)+DACHG(N+1))/12.
       CAC(13)=(7.*DACHG(13)+6.*DACHG(12)-DACHG(11))/24.
       AVSLP(1)=0.
       DO 18 N=2,13
   18  AVSLP(N)=AVSLP(N-1)+CAC(N)
       TRWA(1)=0.
       DO 19 N=2,13
   19  TRWA(N)=TRWA(N-1)+AVSLP(N-1)
C      TRWA=TRIAL DEFLECTIONS
       WA(1)=0.
       DO 21 N=2,13
       Z=N
   21  WA(N)=TRWA(N)-TRWA(13)*((Z-1.)/12.)
       SWOWA=0.
       SWSWA=0.
       SWAWA=0.
       DO 22 N=1,13
       SWOWA=SWOWA+WOCL(N)*WA(N)
       SWSWA=SWSWA+WS(N)*WA(N)
   22  SWAWA=SWAWA+WA(N)*WA(N)
       P1=SWOWA/SWAWA
       P2=-(SWSWA/SWAWA)*(WOCL (7)/WO(7))
       P=P1+P2
       DO 23 N=1,13
   23  WA1(N)=WA(N)*P
       DO 24 N=1,13
   24  WT(N)=WA1(N)+WS(N)*WOCL(7)/WO(7)
       SWOCL=0.
       SDIFF=0.
       DO 28 N=2,12
       SDIFF=SDIFF+(WT(N)-WOCL(N))*(WT(N)-WOCL(N))
   28  SWOCL=SWOCL+WOCL(N)*WOCL(N)
       ERROR=SQRTF(SDIFF/SWOCL)
       IF(ERROR-ER)32,32,27
   27  C1=WA1(7)/(WO(7)-WS(7))
       DO 273 N=1,13
  273  WOCL(N)=(WA1(N)+WS(N)*C1)
  311  PUNCH 70,(WT(N),N=1,13)
   29  CONTINUE
       PUNCH 120
   32  P3=P1*144.
       P4=P2/(SCA*12.)
       PC=P1*144.+P2*144.
  321  PUNCH 80,(WT(N),N=1,13)
   49  PUNCH 50,P3,P4,PC
   52  CONTINUE
       GO TO 1
       END
```

```
        CASE 1

WO=  INITIAL DEFLECTIONS
     0.0000E-99 5.8600E-01 9.4400E-01 1.0750E-00 9.7800E-01 6.5300E-01 1.0000E-01
               -4.5900E-01-8.0000E-01-9.2500E-01-8.3400E-01-5.2600E-01 0.0000E-99
SC =     10.0 EI/(L**3)
ER=  .0100
     0.0000E-99 5.6266E-01 9.8516E-01 1.1790E-00 1.1122E-00 8.0901E-01 3.5017E-01
               -1.3167E-01-5.0287E-01-6.7919E-01-6.2996E-01-3.7802E-01-4.0000E-10
     0.0000E-99 4.2084E-01 7.6046E-01 9.5821E-01 9.8724E-01 8.5938E-01 6.2126E-01
                3.4176E-01 9.1577E-02-7.5560E-02-1.3520E-01-9.6775E-02-1.2281E-09
     0.0000E-99 2.5538E-01 4.7557E-01 6.3258E-01 7.1111E-01 7.1085E-01 6.4537E-01
                5.3734E-01 4.1061E-01 2.8561E-01 1.7507E-01 8.1875E-02-2.2683E-09
     0.0000E-99 1.9381E-01 3.6866E-01 5.0845E-01 6.0226E-01 6.4554E-01 6.4017E-01
                5.9271E-01 5.1076E-01 4.0280E-01 2.7716E-01 1.4098E-01-2.5275E-09
     0.0000E-99 1.7520E-01 3.3639E-01 4.7111E-01 5.6975E-01 6.2646E-01 6.3950E-01
                6.1053E-01 5.4216E-01 4.3925E-01 3.0880E-01 1.5927E-01-2.5549E-09
     0.0000E-99 1.6962E-01 3.2672E-01 4.5993E-01 5.6006E-01 6.2083E-01 6.3944E-01
                6.1603E-01 5.5174E-01 4.5034E-01 3.1841E-01 1.6482E-01-2.5575E-09
WT=  FINAL DEFLECTIONS
     0.0000E-99 1.6794E-01 3.2381E-01 4.5658E-01 5.5715E-01 6.1915E-01 6.3944E-01
                6.1770E-01 5.5464E-01 4.5368E-01 3.2131E-01 1.6649E-01-2.5577E-09
P =      9.86952EI/(L**2) +   .20194 SC*L          =    11.88898EI/(L**2)

SC=      210.0 EI/(L**3)
ER=  .0100
     0.0000E-99 5.4554E-01 9.3540E-01 1.0735E-00 9.2860E-01 5.3442E-01-1.0249E-02
               -5.4887E-01-9.3135E-01-1.0664E-00-9.2455E-01-5.3775E-01 0.0000E-99
     0.0000E-99 5.5671E-01 9.6274E-01 1.1100E-00 9.6027E-01 5.5401E-01 1.5750E-03
               -5.4575E-01-9.4272E-01-1.0863E-00-9.4025E-01-5.4305E-01 0.0000E-99
     0.0000E-99 5.6425E-01 9.7683E-01 1.1271E-00 9.7559E-01 5.6383E-01 4.3491E-03
               -5.5083E-01-9.5486E-01-1.1017E-00-9.5362E-01-5.5041E-01 0.0000E-99
     0.0000E-99 5.6879E-01 9.8486E-01 1.1366E-00 9.8404E-01 5.6895E-01 4.5441E-03
               -5.5624E-01-9.6483E-01-1.1136E-00-9.6401E-01-5.5640E-01 0.0000E-99
     0.0000E-99 5.7152E-01 9.8964E-01 1.1423E-00 9.8899E-01 5.7175E-01 4.0020E-03
               -5.6070E-01-9.7248E-01-1.1226E-00-9.7182E-01-5.6093E-01 0.0000E-99
     0.0000E-99 5.7316E-01 9.9253E-01 1.1457E-00 9.9196E-01 5.7333E-01 3.3272E-03
               -5.6413E-01-9.7817E-01-1.1292E-00-9.7761E-01-5.6429E-01 0.0000E-99
     0.0000E-99 5.7414E-01 9.9425E-01 1.1477E-00 9.9375E-01 5.7423E-01 2.7122E-03
               -5.6668E-01-9.8236E-01-1.1340E-00-9.8186E-01-5.6677E-01 0.0000E-99
     0.0000E-99 5.7469E-01 9.9524E-01 1.1489E-00 9.9480E-01 5.7473E-01 2.1999E-03
               -5.6856E-01-9.8543E-01-1.1376E-00-9.8499E-01-5.6859E-01 0.0000E-99
     0.0000E-99 5.7499E-01 9.9577E-01 1.1495E-00 9.9539E-01 5.7498E-01 1.7861E-03
               -5.6994E-01-9.8768E-01-1.1402E-00-9.8730E-01-5.6994E-01 0.0000E-99
     0.0000E-99 5.7511E-01 9.9602E-01 1.1499E-00 9.9569E-01 5.7509E-01 1.4557E-03
               -5.7096E-01-9.8933E-01-1.1422E-00-9.8901E-01-5.7094E-01 0.0000E-99
  NO CONVERGENCE WITHIN SPECIFIED LIMIT  IN 10 CYCLES
WT=  FINAL DEFLECTIONS
     0.0000E-99 5.7511E-01 9.9602E-01 1.1499E-00 9.9569E-01 5.7509E-01 1.4557E-03
               -5.7096E-01-9.8933E-01-1.1422E-00-9.8901E-01-5.7094E-01 0.0000E-99
```

```
P =      39.32491EI/(L**2) +   .00085 SC*L        =   39.50484EI/(L**2)

SC=        410.0 EI/(L**3)
ER=    .0100
      0.0000E-99 5.2841E-01 8.8564E-01 9.6796E-01 7.4496E-01 2.5982E-01-3.7067E-01
                -9.6606E-01-1.3598E-00-1.4536E-00-1.2191E-00-6.9747E-01-4.0000E-08
      0.0000E-99 5.8981E-01 1.0097E-00 1.1387E-00 9.3554E-01 4.4899E-01-1.9248E-01
                -8.1437E-01-1.2505E-00-1.3844E-00-1.1763E-00-6.7355E-01-2.0268E-08
      0.0000E-99 6.1862E-01 1.0661E-00 1.2182E-00 1.0307E-00-5.5203E-01-8.9884E-02
                -7.2189E-01-1.1754E-00-1.3292E-00-1.1401E-00-6.5531E-01-1.0074E-08
      0.0000E-99 6.3010E-01 1.0887E-00 1.2507E-00 1.0711E-00 5.9722E-01-4.3857E-02
                -6.7976E-01-1.1406E-00-1.3031E-00-1.1230E-00-6.4688E-01-5.2509E-09
      0.0000E-99 6.3483E-01 1.0981E-00 1.2646E-00 1.0889E-00 6.1763E-01-2.2734E-02
                -6.6028E-01-1.1245E-00-1.2911E-00-1.1152E-00-6.4308E-01-2.8620E-09
      0.0000E-99 6.3695E-01 1.1024E-00 1.2711E-00 1.0973E-00 6.2754E-01-1.2363E-02
                -6.5068E-01-1.1165E-00-1.2852E-00-1.1114E-00-6.4126E-01-1.6090E-09
WT=  FINAL DEFLECTIONS
      0.0000E-99 6.3797E-01 1.1045E-00 1.2743E-00 1.1016E-00 6.3267E-01-6.9409E-03
                -6.4564E-01-1.1124E-00-1.2822E-00-1.1095E-00-6.4034E-01-9.2216E-10
P =      39.46279EI/(L**2) +   .00001 SC*L        =   39.46896EI/(L**2)

SC=        610.0 EI/(L**3)
ER=    .0100
      0.0000E-99 5.1128E-01 8.3588E-01 8.6240E-01 5.6131E-01-1.4777E-02-7.3110E-01
                -1.3832E-00-1.7883E-00-1.8409E-00-1.5137E-00-8.5720E-01-2.0000E-08
      0.0000E-99 6.3423E-01 1.0807E-00 1.2060E-00 9.6550E-01 4.1456E-01-3.0300E-01
                -9.9244E-01-1.4691E-00-1.6039E-00-1.3538E-00-7.7277E-01-7.8744E-09
      0.0000E-99 6.8471E-01 1.1800E-00 1.3482E-00 1.1405E-00 6.1005E-01-1.0152E-01
                -8.0273E-01-1.3062E-00-1.4768E-00-1.2667E-00-7.2808E-01-2.8793E-09
      0.0000E-99 6.9950E-01 1.2092E-00 1.3908E-00 1.1941E-00 6.7113E-01-3.7950E-02
                -7.4276E-01-1.2548E-00-1.4369E-00-1.2396E-00-7.1440E-01-1.1881E-09
      0.0000E-99 7.0413E-01 1.2185E-00 1.4046E-00 1.2119E-00 6.9185E-01-1.6207E-02
                -7.2230E-01-1.2374E-00-1.4236E-00-1.2308E-00-7.1002E-01-5.4782E-10
WT=  FINAL DEFLECTIONS
      0.0000E-99 7.0580E-01 1.2219E-00 1.4098E-00 1.2188E-00 6.9994E-01-7.6517E-03
                -7.1428E-01-1.2306E-00-1.4185E-00-1.2275E-00-7.0842E-01-2.7139E-10
P =      39.46442EI/(L**2) +  0.00000 SC*L        =   39.46780EI/(L**2)

SC=        810.0 EI/(L**3)
ER=    .0100
      0.0000E-99 4.9416E-01 7.8613E-01 7.5685E-01 3.7767E-01-2.8937E-01-1.0915E-00
                -1.8004E-00-2.2167E-00-2.2281E-00-1.8083E-00-1.0169E-00 0.0000E-99
      0.0000E-99 6.8440E-01 1.1630E-00 1.2899E-00 1.0166E-00 4.0488E-01-3.8672E-01
                -1.1435E-00-1.6627E-00-1.8025E-00-1.5163E-00-8.6407E-01 0.0000E-99
      0.0000E-99 7.5404E-01 1.3000E-00 1.4867E-00 1.2602E-00 6.7885E-01-1.0230E-01
                -8.7319E-01-1.4276E-00-1.6168E-00-1.3878E-00-7.9800E-01 0.0000E-99
      0.0000E-99 7.7002E-01 1.3317E-00 1.5329E-00 1.3185E-00 7.4554E-01-3.2617E-02
                -8.0706E-01-1.3705E-00-1.5722E-00-1.3574E-00-7.8258E-01 0.0000E-99
WT=  FINAL DEFLECTIONS
      0.0000E-99 7.7413E-01 1.3400E-00 1.5453E-00 1.3346E-00 7.6432E-01-1.2835E-02
                -7.8839E-01-1.3546E-00-1.5600E-00-1.3493E-00-7.7859E-01 0.0000E-99
```

P = 39. 46352EI/(L**2) + 0. 00000 SC*L 39. 46941EI/(L**2)

SC= 1010. 0 EI/(L**3)
ER= . 0100

```
0. 0000E-99 4. 7703E-01 7. 3637E-01 6. 5130E-01 1. 9402E-01 -5. 6397E-01 -1. 4519E-00
            -2. 2176E-00-2. 6452E-00-2. 6154E-00-2. 1029E-00-1. 1766E-00-3. 0000E-08
0. 0000E-99 7. 3755E-01 1. 2511E-00 1. 3821E-00 1. 0779E-00 4. 0651E-01 -4. 5868E-01
            -1. 2833E-00-1. 8461E-00-1. 9928E-00-1. 6729E-00-9. 5232E-01 -8. 8769E-09
0. 0000E-99 8. 2440E-01 1. 4219E-00 1. 6275E-00 1. 3824E-00 7. 4984E-01 -1. 0136E-01
            -9. 4239E-01 -1. 5482E-00-1. 7563E-00-1. 5087E-00-8. 6783E-01 -2. 2313E-09
0. 0000E-99 8. 4081E-01 1. 4545E-00 1. 6751E-00 1. 4426E-00 8. 1874E-01 -2. 9247E-02
            -8. 7382E-01 -1. 4889E-00-1. 7099E-00-1. 4770E-00-8. 5175E-01 -7. 5043E-10
```

WT= FINAL DEFLECTIONS

```
0. 0000E-99 8. 4446E-01 1. 4619E-00 1. 6863E-00 1. 4572E-00 8. 3585E-01 -1. 1194E-02
            -8. 5679E-01 -1. 4745E-00-1. 6989E-00-1. 4698E-00-8. 4818E-01 -3. 1831E-10
```

P = 39. 46446EI/(L**2) + 0. 00000 SC*L = 39. 46808EI/(L**2)