The Stueckelberg wave equation and the anomalous magnetic moment of the electron

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Abstract. The parametrized relativistic quantum mechanics of Stueckelberg [Helv. Phys. Acta 15, 23 (1942)] represents time as an operator, and has been shown elsewhere to yield the recently observed phenomena of quantum interference in time, quantum diffraction in time and quantum entanglement in time. The Stueckelberg wave equation as extended to a spin–1/2 particle by Horwitz and Arshansky [J. Phys. A: Math. Gen. 15, L659 (1982)] is shown here to yield the electron g–factor \( g = 2 \left(1 + \frac{\alpha}{2\pi}\right)\), to leading order in the renormalized fine structure constant \( \alpha \), in agreement with the quantum electrodynamics of Schwinger [Phys. Rev., 73, 416L (1948)].

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1. Introduction

The relativistic quantum mechanics of Dirac [1, 2] represents position as an operator and time as a parameter. The Dirac wave functions can be normalized over space with respect to a Lorentz–invariant measure of volume [2, Ch 3], but cannot be meaningfully normalized over time. Thus the Dirac formalism offers no precise meaning for the expectation of time [3, §9.5], and offers no representations for the recently–observed phenomena of quantum interference in time [4, 5], quantum diffraction in time [6, 7] and quantum entanglement in time [8]. Quantum interference patterns and diffraction patterns [9, Chs 1–3] are multi–lobed probability distribution functions for the eigenvalues of an Hermitian operator, which is typically position. Quantum Field Theory (QFT) [10, 11] also represents time as a parameter. The QFT formalism yields amplitudes for transitions between states at different times, but it does not yield normalized probability distribution functions for time or for any other quantity. It follows that, for example, the extraordinary experiments of Lindner et al. [4, 5], reported‡ to be the first evidence of quantum interference in time, cannot be analyzed

‡ http://physicsworld.com/cws/article/news/21623
as such using the quantum mechanics of Dirac or Schrödinger [12], or using QFT. The original analyses of the interference experiments [4] and diffraction experiments [7] use the conventional quantum mechanics of scattering off time–dependent potentials. The potentials are the electromagnetic traps and mirrors respectively. The conventional wave functions are normalized over space, at each time. Thus timewise plots of their squared magnitudes, at one position, are not probability distribution functions for a particle being observed there at one time over another time.

The Stueckelberg wave equation [13] represents time as an operator. The Stueckelberg wave function explicitly depends not only upon events in space–time, but also upon a real–valued parameter. The wave function is normalized over space–time, and so can yield in particular a distribution over time conditioned by position. That is, the wave function can yield a quantum interference pattern in time at a fixed position. The manifestly covariant Stueckelberg wave equation for spin–0 particles resolves the Klein paradox and explains the zitterbewegung [2, 14, 15], yields the spectrum of the relativistic Coulomb problem including the Zeeman effect [2, 16, 17] and meaningfully represents interference and diffraction in time [6, 12]. The Stueckelberg formalism is not limited to a single particle, and thus can explicitly represent both relativistic scattering [18] and entanglement in time [8, 19].

The Stueckelberg equation has been extended to a particle of spin–1/2 by Horwitz and Arshansky [20], following earlier developments by Horwitz, Piron and Reuse [21, 22]. The first–order and second–order scattering matrices are constructed here for a single spin–1/2 particle, under the influence of both an external field and the field owing to the charge current of the particle itself. The second–order vertex correction yields, to first order in the fine structure constant $\alpha$, the renormalized anomalous magnetic moment for the electron in agreement with the result originally derived by Schwinger from quantum electrodynamics [23] and again from source theory for Dirac fields [24].

The quantum mechanics of Dirac and its further development as QFT have yielded the electron anomalous magnetic moment to second order [25], third order [26] and fourth order [27] in $\alpha$. The chronological sequence of approximations is in agreement with the most precise contemporaneous measurements. The leading–order approximation reported here is only a first step towards a comparable sequence for Stueckelberg theory. However, it is a mandatory step as only the Stueckelberg formalism is capable of representing the recent measurements of quantum coherence in time.

The Stueckelberg wave equation as extended in [20] to spin–1/2 particles is stated in §2, together with brief notes on the Stueckelberg parameter, free wave functions, unbounded transitions, and interference and entanglement in time. Additional notes on induced representations, the Poincaré algebra, discrete symmetries, parameter elimination and pair annihilation are included as Appendices A–E. The analysis of scattering in §3 is entirely conventional and accordingly very brief, leading directly to the vertex correction and anomalous magnetic moment.
2. The Stueckelberg wave equation

2.1. covariant formulation

For a single spin–1/2 particle the wave function is a four-spinor $\psi[x, \tau]$. The event $x$ is in $\mathbb{R}^4$, while the parameter $\tau$ is an independent variable in $\mathbb{R}$. The Lorentz–invariant inner product of two wave functions $\psi$ and $\phi$ is

$$\langle \psi|\phi \rangle_n \equiv \int d^4x \psi[x, \tau]^\dagger \circ \phi[x, \tau]$$

(1)

The invariant measure is $d^4x = dx^0dx^1dx^2dx^3$. The defining Hermitian positive–definite matrix is $\circ = -\gamma^0\delta$, where the 4–vector $n^\mu$ is timelike and there is a reference frame in which $n^\mu = \hat{n}^\mu = (+1, 0, 0, 0)$. The 4-vector $n^\mu$ is an index which, it will be seen, renders the representation reducible. The choice of ‘standard’ vector $\hat{n}^\mu = (-1, 0, 0, 0)$ is also available but for definiteness is not included here. The signature of the Lorentz metric $g_{\mu\nu}$ is $(-+++)$. The wave function $\psi$ is normalized so that $\langle \psi|\psi \rangle_n \equiv 1$. Thus $\rho \equiv \psi^\dagger \circ \psi$ is interpreted as the probability distribution function or density for detecting one event or ‘particle’, in an element of space–time for each parameter $\tau$. There may, however, be a zero probability of detecting a particle in some interval of coordinate time $x^0$ [14, Ch 13]. In particular, the creation and annihilation of particle–antiparticle pairs are represented as the vanishing of the single–particle density $\rho$ in a semi–infinite interval of $x^0$.

The Schwartz inequality applies in the usual way [14], immediately yielding for example the space–momentum and time–energy uncertainty relations $\Delta x^\mu \Delta p^\mu \geq 1/2$ (no summation), for the operators $x^\mu$ and $p^\mu$ satisfying the commutator relations

$$[x^\mu, p^\nu] = ig^{\mu\nu}$$

(2)

where e.g. $[x^\mu, p^\nu] \equiv x^\mu p^\nu - p^\nu x^\mu$, $(\Delta x^\mu)^2 \equiv \langle (x^\mu - \langle x^\mu \rangle)^2 \rangle_n$, and $\langle x^\mu \rangle_n \equiv \langle \psi|x^\mu|\psi \rangle_n \equiv \langle \psi|x^\mu\psi \rangle_n$.

In units such that $c = \hbar = 1$, the Stueckelberg equation [20] for a single spin–1/2 particle of charge $e$ is

$$i\frac{\partial}{\partial \tau} \psi = \frac{\pi^\mu}{2M} \pi^\mu \psi - \frac{e}{2M} F_{\mu\nu} \Sigma^\mu\nu_\mu \psi$$

(3)

(summation convention) where the dynamic energy–momentum, kinetic energy–momentum, Maxwell potential and electromagnetic field strength are respectively $\pi^\mu = p^\mu - eA^\mu$, $p^\mu = -i\partial / \partial x^\mu$, $A^\mu = A^\mu[x]$ and $F_{\mu\nu} = F_{\mu\nu}[x]$ where $F_{\mu\nu} = \partial^\mu A_\nu - \partial^\nu A_\mu$. The preceding definition of $F_{\mu\nu}$ is the negative of that in Bjorken and Drell [2, p282]. It is emphasized that the potential $A^\mu[x]$ is independent of the parameter $\tau$. The spin observable $\Sigma^\mu\nu_\mu$, which is Hermitian with respect to (1), is

$$\Sigma^\mu\nu_\mu \equiv \Sigma^{\mu\nu} + K^\mu n^\nu - K^\nu n^\mu$$

(4)

where $\Sigma^{\mu\nu} \equiv (i/4)[\gamma^\mu, \gamma^\nu]$ and $K^\mu \equiv \Sigma^{\mu\nu}_\mu n^\nu$. 


The covariance of the theory with respect to the homogeneous Lorentz transformation \((x^\mu)' = \Lambda^\mu_\nu x^\nu, (n^\mu)' = \Lambda^\mu_\nu n^\nu\) and \(\psi'[x, \tau, n] = S(\Lambda)\psi[\Lambda^{-1}x, \tau, \Lambda^{-1}n]\), where \(S(\Lambda)\) is generated by \(\Sigma^{\mu\nu}\) (not \(\Sigma^{\mu\nu}_n\)) in the usual way [2, Ch 2], follows [20] from

\[ S^{-1}(\Lambda)\Sigma^{\mu\nu}_n S(\Lambda)\Lambda^\lambda_\mu \Lambda^\sigma_\nu = \Sigma^{\lambda\sigma}_n. \]  

(5)

The proof of (5) involves the identity

\[ \Lambda^\dagger_\sigma n^\mu \Lambda = \sigma^\mu (\Lambda^{-1}n)_\mu \]  

(6)

with \(n^\mu\) arbitrary, for the \(SL(2, \mathbb{C})\) representation of elements \(\Lambda\) of \(SO(3, 1)\) and using the same symbol \(\Lambda\) for the two representations. The theory is covariant with respect to the inhomogeneous Lorentz group, since the 4–momentum \(p^\mu\) commutes with \(n^\nu\) and so

\[ [p^\mu, \circ] = 0 \]  

(7)

The inner product (1) is not an inspired guess; rather, it is a construction from first principles using the theory of induced representations of the Lorentz group as outlined in Appendix A. The Poincaré algebra for the 10 linear independent quantities consisting of \(\Sigma^{\mu\nu}_n, K^\mu\) and \(p^\mu\) may be found in Appendix B. The discrete symmetries of \(\psi\) may be found in Appendix C.

If \(n^\mu = \hat{n}^\mu = (+1, 0, 0, 0)\) then \(\Sigma^{\mu\nu}_n\) reduces to \((1/2)\epsilon^{ijk}\sigma^k\) for \(\mu = i, \nu = j\) (where \(i, j, k\) all range over \(1, 2, 3\) and \(\sigma^k\) is a Pauli spin matrix), while \(\Sigma^{0ij}_n\) reduces to zero for \(j = 1, 2, 3\). The ‘standard’ 4–vector \(\hat{n}^\mu\) may be identified with the rest frame of a Stern–Gerlach apparatus that prepares the spin state, in which frame the magnetic field has a purely spacelike direction [22]. The \(g\)–factor of a particle in a weak, uniform magnetic field \(B^k = \epsilon^{ijk}\partial^j A^i = \frac{1}{2}\epsilon^{ijk}(\partial^j A^i - \partial^i A^j)\) is readily seen [2, p13] to be \(g = 2\).

The covariant and contravariant indices \(\mu, \nu, \ldots\) will be omitted or replaced wherever convenient with the centered dot notation as in \(p = p^\mu, p^2 = p \cdot p = p^\mu p_\mu\) and \(F \cdot \Sigma_n = F^{\mu\nu}_\mu \Sigma^{\mu\nu}_n\).

The single–particle Lagrangian is

\[ \mathcal{L} = -\frac{1}{4} \int d^4x F \cdot F + \int d\tau \int d^4x \psi^\dagger \circ \left( i \frac{\partial}{\partial \tau} \psi - K\psi \right) \]  

(8)

where \(K\psi\) is the right hand side of (3). Variation of \(\mathcal{L}\) with respect to \(A\) in the Lorentz gauge yields the wave equation for \(A\), with a convection current and a spin current as inhomogeneities.

2.2. Stueckelberg parameter

The constant \(M\) in (3) is a mass, but has no physical significance as it may be absorbed into the parameter \(\tau\). The parameter \(\lambda\) in Stueckelberg [13] is identified with \(\tau/M\) here. Introduction of the ‘target mass’ \(M\) is becoming conventional, as it renders intermediate formulae recognizable and it aids dimensional checks. Scaling \(\tau\)
as time suggests a natural ‘τ–clock’ [28, 29]. In bound states, with Stueckelberg wave function \( \psi \propto \exp\{-i\varpi \tau\} \), the parameter \( \tau \) need not be considered explicitly and the Stueckelberg ‘frequency’ \( \varpi \) is simply an eigenvalue of the Hamiltonian \( K \) in (3). The terminology ‘bound’ is somewhat misleading, as the support for states of lowest energy is a subregion of the exterior of the light cone in Minkowski space and the distance in the subregion is hyperbolic [32, 16].

The Newton–Wigner position operator [30], that can only localize position \( x \) within a distance of \( O(m^{-1}) \) where \( m \) is the conventional rest mass, is extendable in the Stueckelberg formalism to an \( x^\mu \) event operator with a footprint of the same order. However, the Stueckelberg formalism does not restrict the state to a single rest mass. Consider again a Stueckelberg wave function \( \psi \propto \exp\{-i\varpi \tau\} \), for arbitrary \( \varpi \). In effect, \( m = \pm \sqrt{-2M\varpi} \) which has infinite range and so perfect localization of \( x^\mu \) and of \( p^\mu \) is attainable [31, 29].

The scattering matrix is formally defined in terms of the asymptotic limits as \( \tau \to -\infty \) (incident) and as \( \tau \to \infty \) (final). In scattering calculations, the incident and final particles have Stueckelberg frequencies \( \varpi_i, \varpi_f \) respectively. In the final state the scattered wavefunction is a sum of free waves and, owing to the orthogonality of the free waves with respect to \( x^\mu \), the scattering matrix is independent of \( \tau \).

For an examination of the relationship between the parameter \( \tau \), proper time and variable mass see [33]. For extensive phenomenological discussions of \( \tau \) see [34, 28]. Finally, suppose that for one value \( \zeta \) of the parameter \( \tau \) there is a frame in which the density \( \rho = \psi^\dagger \circ \psi \) is restricted to the hypersurface of one coordinate time \( t = s \), that is, \( \rho[t, x, \zeta] = \delta(t - s)\chi[x] \), with \( \int \chi d^3x = 1 \), and suppose further that for \( \tau = \zeta \) the phase of the wave function \( \psi \) is independent of \( t \). If the Maxwell potential \( A^0 \) in (3) is not vanishing, the support of \( \rho \) will extend off the hypersurface of \( t = s \) as \( \tau \) increases. That is, quantum coherence in time \( t \) can develop with the passage of \( \tau \) owing to the action of the electric potential.

2.3. free wavefunctions

Let \( \phi \) denote a free particle wave function of the form

\[
\phi[x, \tau, n, p, s] = (2\pi)^{-2} u[n, p, s] \exp\{i(p \cdot x - \Omega[p] \tau)\}
\]

in the continuum normalization [2], where \( n^\mu \) is the defining timelike unit vector in the inner product (1), while \( s^\mu \) is the spin polarization. In the reference frame, \( s^\mu = \hat{s}^\mu = (0, \hat{s}) \), with \( \hat{s} \cdot \hat{s} = 1 \). The dispersion relation for the Stueckelberg frequency is \( \Omega[p] = p^2/2M \). Only subluminal free particles will be admitted, that is, \( \Omega < 0 \). The 4–spinor amplitude \( u \) may be expressed in terms of eigen spinors of the projection operators \((1 \mp \gamma^5)/2\) and \((1 \mp \gamma^5\gamma^0)/2\) in the standard way. For a given timelike standard 4–vector, say \( \hat{n}^\mu = (+1, 0, 0, 0) \), there are four positive–energy and four negative–energy solutions. Four solutions combine the upper (lower) sign in the projections for the positive (negative) energy solution, polarized (antipolarized) with respect to \( s^\mu \). The other four resolve the helicity [20].
Most of the arguments of \( \phi \) will be suppressed in the following. Let \( \phi_i \) and \( \phi_f \) denote incident and final free wave functions, respectively, with parameters \( n, p_i, s_i \) and \( n, p_f, s_f \). The spinor amplitudes will be denoted \( u_i = u[n, p_i, s_i] \) and \( u_f = u[n, p_f, s_f] \), thus \( \phi_i = (2\pi)^{-2}u_i \exp\{i(p_i \cdot x - \varpi_i \tau)\} \) and \( \phi_f = (2\pi)^{-2}u_f \exp\{i(p_f \cdot x - \varpi_f \tau)\} \).

2.4. unbounded transitions

Bound states of two particles may be constructed [16] from two–particle Hamiltonians that include an additive \( \tau \)–independent potential \( U[y, z] \), where the 4–vectors \( y \) and \( z \) are the spacetime coordinates of the two particles. In the case of a potential that depends only upon the pair separation \( x = y - z \), that is \( U = U[x] \), it is as usual convenient to introduce also a centre–of–mass coordinate \( X \). The metric is Lorentzian, so square–integrability of the bound–state wave function over \( x \) does not imply that large values of separation time \( |x^0| \) are improbable.

Consider perturbations to the bound state induced by a \( \tau \)–independent perturbation potential \( \Delta U[y, z] \), resulting in a transition to another bound state. Assume that \( \Delta U \) is insignificant for either \( y \) or \( z \) outside a neighbourhood of the origin bounded in \( \mathbb{E}^4 \). The asymptotic states \( (\tau \to \pm \infty) \) are further assumed to occupy regions that do not significantly overlap the support of \( \Delta U \). The Hamiltonian \( K \) is a constant of the evolution in \( \tau \), and so there can be no transitions between levels of different \( K \). Hence, while there may be transitions between positive and negative values of energy \( \pm|p^0| \) for the relative motion, with the same value for \( |p^0| \), there can be no transitions to unbounded negative values for \( p^0 \). There is accordingly no need for a Stueckelberg hole theory [20], [16, p73].

2.5. interference and entanglement in time

Distinguishing the observer’s time and space coordinates as \( x^\mu = [t, \mathbf{x}] \), the density is then of the form \( \rho[t, \mathbf{x}, \tau] \) with \( t \) and \( \mathbf{x} \) regarded as jointly distributed random variables. For clarity, the dependence upon \( n^\mu \) is not shown. The conditional distribution of \( t \) given \( \mathbf{x} \) may then be derived in the usual way. That is, defining the marginal distribution of position as

\[
\rho[\mathbf{x}, \tau] \equiv \int \rho[t, \mathbf{x}, \tau]dt
\]

then the conditional distribution of \( t \) given \( \mathbf{x} \), that is

\[
\rho[t|\mathbf{x}, \tau] \equiv \rho[t, \mathbf{x}, \tau]/\rho[\mathbf{x}, \tau]
\]

may be identified as the quantum interference pattern in time \( t \), at position \( \mathbf{x} \) and for parameter \( \tau \). Again, \( \rho[x^\mu, \tau] \) and hence \( \rho[t|\mathbf{x}, \tau] \) are independent of \( \tau \) in free or bound states.

Two–particle states that are entangled at times \( t, s \) or at positions \( \mathbf{x}, \mathbf{y} \) may be represented in the usual way, with antisymmetric combinations of tensor products of
single–particle, 4–spinor wave functions such as

\[ \Psi[x, y, \tau] = \frac{1}{\sqrt{2}} (\psi[x, \tau] \otimes \phi[y, \tau] - \psi[y, \tau] \otimes \phi[x, \tau]) \]  

(12)

where \( x^\mu = [t, x] \), \( y^\mu = [s, y] \). Such a representation is simpler and more general than the Quantum Field Theoretic representation for entanglement in the massless vacuum state [35]. The single–particle, spin–1/2 Stueckelberg inner product (1), wave equation (3) and Lagrangian (8) are readily extended to two–particle, spin–1/2 wave functions in the space of the abovementioned tensor products. The straightforward details are omitted here. The electromagnetic fields remain defined over a single copy of Minkowski space, that is, \( A^\mu = A^\mu[x] \). The sources for the fields acting on a particle are constructed by integrating over the dependences of the two–particle currents upon one of the particles. In particular, a particle may be acted on by its own field. The spin–1/2 development is very similar to that for spin–0 in [18].

3. Scattering

3.1. scattering matrix

The Stueckelberg equation (3) is often described as the ‘covariant Schrödinger equation’. Scattering theory for the nonrelativistic Schrödinger is well known [36], and the theory for (3) mimics the nonrelativistic theory very closely. Accordingly, few details will be given here as the development is so conventional [2, 18].

The first–order matrix for the scattering of a spin–1/2 charged particle by an external potential \( A[x] \) is

\[ S^{(1)}_{fi} = -i \int d\tau \int d^4x \phi^+_j[x, \tau] \circ V[x] \phi_i[x, \tau] \]  

(13)

with first–order interaction Hamiltonian

\[ V = -\frac{e}{M} A \cdot p - \frac{e}{2M} \mathcal{F} \cdot \Sigma_n \]  

(14)

where \( \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). It suffices here to consider only the external spin Hamiltonian. That is, \( V \) in (13) is reduced to

\[ V = V_s = -\frac{e}{2M} \mathcal{F} \cdot \Sigma_n \]  

(15)

Inserting the free wave function forms (9) into (13) leads to

\[ S^{(1)}_{fi} = -i (2\pi)^{-3} \delta[\Delta p] u^+_f \circ V_s[\Delta p] u_i \]  

(16)

where in general \( V_s[p] = \int d^4x V_s[x] \exp\{-ip \cdot x\} \) denotes the four–dimensional Fourier transform of \( V_s[x] \). The energy–momentum argument \( \Delta p \) of the spin Hamiltonian \( V_s \) in (16) is the change in that of the particle, as a result of scattering off the external
potential. In other words, energy–momentum is conserved. Finally, \( \Delta \omega = \omega_f - \omega_i = (p_f^2 - p_i^2)/2M \), and thus scattering off the \( \tau \)-independent external potential does not change the rest mass of the free particle.

The objective here is the calculation of the next additive correction to (16). It suffices to consider only

\[
\mathcal{S}_{fi}^{(2)} = -i \int d\tau \int d^4x \int d\sigma \int d^4y \phi_f^\dagger[x, \tau] \circ \mathcal{V}_s[x] G_0[x - y, \tau - \sigma] V_c[y] \phi_i[y, \sigma] \tag{17}
\]

where \( G_0 \) is the single–particle free propagator

\[
G_0[x' - x, \tau' - \tau] = (2\pi)^{-5} \int d\theta \int d^4q \frac{\exp\{i(q \cdot (x' - x) - \theta(\tau' - \tau))\}}{\theta - q^2/2M + i\epsilon} \tag{18}
\]

with \( 0 < \epsilon \ll 1 \). The free propagator \( G_0[x' - x, \tau' - \tau] \) vanishes for \( \tau' < \tau \). Closing the contour in the complex \( \theta \)–plane, below the origin for \( \tau' > \tau \), yields

\[
G_0[x' - x, \tau' - \tau] = -i(2\pi)^{-4} \int d^4q \exp\{i(q \cdot (x' - x) - \Omega[q](\tau' - \tau))\} \tag{19}
\]

where again \( \Omega[q] = q^2/2M \).

The convective Hamiltonian \( V_c \) appearing in (17), that is

\[
V_c = -\frac{e}{M} A \cdot p \tag{20}
\]

owes to the potential \( A \) of the particle itself. The potential obeys

\[
\Box A[x] = -\partial_\mu \partial^\mu A[x] = e \int (j_c[x, \tau] + j_s[x, \tau]) d\tau \tag{21}
\]

where \( j_c \) is the convection current

\[
j_c[x, \tau] = \phi_f^\dagger[x, \tau] \circ \left( \frac{p}{M} \right) \phi_i[x, \tau] \tag{22}
\]

and \( j_s \) is the spin current

\[
j_s^{\mu}[x, \tau] = -\left( \frac{i\Delta p^\mu}{M} \right) \phi_f^\dagger[x, \tau] \circ \Sigma_\mu^\nu \phi_i[x, \tau] \tag{23}
\]

where \( \overline{p} = (p_f + p_i)/2 \) and \( \Delta p = p_f - p_i \). Neither the spin current \( j_s \) nor the spin Hamiltonian \( V_s \) need be considered for weak scattering (\( \Delta p \to 0 \)). To proceed, (21) and (22) yield

\[
A[x] = e \int d\zeta \int d^4z \ D_F[x - z] \phi_f^\dagger[z, \zeta] \circ \left( \frac{\overline{p}}{M} \right) \phi_i[z, \zeta] \tag{24}
\]

where \( D_F \) is the forward–in–time photon propagator [2, Ch 7.4].
3.2. vertex correction

A rearrangement of (17) is now made, exactly following [2, Ch 7.5] and precisely
equivalent to the substitution

$$\phi_i[z, \zeta] \phi^\dagger_f[x, \tau] \circ \Rightarrow i G_0[z - x, \zeta - \tau]$$

(25)
suggested by (9) and (19). The substitution is justified, since the incident and final
monochromatic plane wave functions $\phi_i$ and $\phi_f$ are in practice normalized beams of
finite bandwidth. Assuming random spin polarizations, and taking into account the
orthonormality of the spinor basis, (25) becomes exact as $\Delta p \to 0$. The quantum–
mechanical substitution (25) has an analog in QFT [10, 11], where the free fermion
propagator is a two–event correlation.

Combining (17), (24) and (25) yields

$$S^{(2)}_{fi} = + \frac{e^2 \mathcal{P}^2}{M^2} \int d\tau \int d^4 x \int d\sigma \int d^4 y \int d\zeta \int d^4 z \times \phi^\dagger_f[z, \zeta] G_0[z - x, \zeta - \tau] \circ V_s[x] G_0[x - y, \tau - \sigma] \phi_i[y, \sigma] D_F[y - z]$$

(26)

which is a vertex correction, the vertex being at $x$. The plus sign in (26) replaces the
minus sign that is correct in the case of the sources for both $V_s$ and $V_c$ in (17) being
the currents of a second electron, and combining a pair of incident and final lines for
that second electron analogously to (25). The box diagram for that case converts to a
vertex correction by an exchange of vertices for two electron lines, one external and one
internal. As argued in [2, Ch 8.1], the Fermi–Dirac statistics for electrons then require
a sign change so that the additive contribution of the two diagrams to the scattering
is antisymmetric. The need for the sign replacement emphasizes the absence of a spin–
statistics theorem in relativistic quantum mechanics.

Fourier–transforming reduces (26) to

$$S^{(2)}_{fi} = S^{(1)}_{fi} \times (i e^2 \mathcal{P}^2 / M^2) \times I_{VC}$$

(27)
as $\Delta p \to 0$, where the vertex correction integral is

$$I_{VC}[\mathcal{P}, M] = (2\pi)^{-4} \int d^4 k \frac{4M^2}{(2M \mathcal{P} - (k + \mathcal{P})^2 + i \epsilon)^2 k^2 - i \epsilon}$$

(28)

again as $\Delta p \to 0$. The Fourier transforms of the fermion propagators in (26) with respect
to their Stueckelberg parameters are only for subluminal $\mathcal{P} = \mathcal{P}_i < 0$ and subluminal
$\mathcal{P} = \mathcal{P}_f < 0$. The integrand in (28) is $O(k^3 dk/k^6)$ at high wavenumber and, since
$\mathcal{P} = \mathcal{P}^2 / 2M$, is $O(k^3 dk/k^4)$ at low wave number. The integral is therefore convergent
at high wavenumber, but logarithmically divergent at low wavenumber. Evaluating in
the usual way [2, 11] yields $I_{VC} = -i(1 - \log \kappa^2) M^2 / 8\pi^2 \mathcal{P}^2$, where $0 < \kappa \ll 1$ is a
dimensionless cutoff. Hence

$$S^{(2)}_{fi} = S^{(1)}_{fi} \frac{\alpha_R}{2\pi}$$

(29)
where the renormalized fine–structure constant is

$$\alpha_R \equiv \frac{e_R^2}{4\pi} = \frac{e^2}{4\pi} (1 - \log \kappa^2)$$  \hspace{1cm} (30)$$

Recalling the spin Hamiltonian (15) owing to the external potential $A^\mu[x]$, it is inferred that the $g$–factor of the spin–1/2 particle is

$$g = 2 \left(1 + \frac{\alpha_R}{2\pi}\right)$$  \hspace{1cm} (31)$$

The unspecified target mass $M$ is not involved, and nor are the incident and final squared rest masses $-p_i^2$ and $-p_f^2$ which are both close to $-p^2$.

4. Summary

The parametrized relativistic quantum mechanics of Stueckelberg [13] is the only extant quantum formalism in which time is an operator, and in which the wave functions can be meaningfully normalized over space–time. The Stueckelberg formalism is therefore uniquely able to represent quantum coherence in time [19], including quantum diffraction in time in general [6] and quantum interference in time as demonstrated by Lindner et al. in particular [5, 12]. The validity of the spin–1/2 Stueckelberg equation [20] has been examined here, with the derivation of the correct leading–order value for a fundamental physical constant. The leading–order contribution of vacuum polarization to the Lamb shift [2, Ch 8] is under investigation, as is the second–order correction to the electron $g$–factor.

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Appendix A. Induced representations

A representation for the inhomogeneous Lorentz transformations acting on quantum states of a single particle of spin–1/2 cannot be unitary, irreducible and finite–dimensional. The difficulty is resolved by Wigner’s theory of induced representations [37], which is outlined by Weinberg [10, §2.5]. The application there being Quantum Field Theory, with the event $x^\mu$ a parameter, the states are indexed by the 4–momentum $p^\mu$ and irreducibility is thereby abandoned. The construction of a finite–dimensional, unitary representation is then based on the ‘little group’ of homogeneous transformations which leaves a certain timelike 4–momentum $k^\mu$ invariant. The application here is quantum mechanics, in which $x^\mu$ and $p^\mu$ are noncommuting operators. The objective is the expectation value of an operator such as $\langle x^\mu \rangle$. Hence some 4–vector other than $x^\mu$ and $p^\mu$, but commuting with both, must be chosen as the index. Horwitz et. al [22] choose a timelike 4-vector $n^\mu$ that is a constant of the dynamics. The standard vector
\( \hat{n}^\mu = (+1, 0, 0, 0) \) is phenomenologically identified with the timelike direction of the filter that prepares the states, for example a Stern–Gerlach apparatus with a magnetic field in a purely spacelike direction [21].

The dependence of both the inner product (1) and the wave equation (3) upon the 4–vector \( n^\mu \) is a superselection rule that assigns states to Hilbert spaces \( \mathcal{H}_n \) indexed by \( n^\mu \). The spaces are identical copies of \( \mathbb{C}^2 \otimes L^2(\mathbb{R}^4, d^4x) \). For any timelike \( n^\mu \) there is [10, §2.5] an homogeneous Lorentz transformation \( L(n) \) such that \( L(n)\hat{n} = n \), where \( \hat{n} \) is the standard vector \((+1, 0, 0, 0)\). For any timelike \( n \) and any proper orthochronous homogeneous Lorentz transformation \( \Lambda \), define \( D(\Lambda, n) \equiv L^{-1}(n)AL(\Lambda^{-1}n) \). It is clearly the case that \( D(\Lambda, n)\hat{n} = \hat{n} \) and so by definition \( D(\Lambda, n) \) is in the little group of \( \hat{n} \). The \( SL(2, \mathbb{C}) \) representation of \( D(\Lambda, n) \) is in \( SU(2) \), as a consequence of (6), and constitutes the induced unitary representation \( U(\Lambda, n) \) for homogeneous \( \Lambda \). That is, a \( \mathbb{C}^2 \)–valued wavefunction \( \psi \) transforms as \( \psi(x, \tau, n) = U(\Lambda, n)\hat{\psi}(\Lambda^{-1}x, \tau, \Lambda^{-1}n) \). Unitarity is defined with respect to the inner product (1). The finite–dimensional representation maps \( \mathcal{H}_{\Lambda^{-1}n} \) onto \( \mathcal{H}_n \), and thus is reducible. Choosing the basis \( \sigma^\mu = (1, \sigma) \) for \( SU(2) \) where the \( \sigma^\mu \) are the Pauli matrices, then \( \tilde{\sigma}^\mu = (1, -\sigma) \) is the dual representation. The two representations provide an extension of Pauli 2–spinors \( \tilde{\psi} \) to Dirac 4–spinors \( \hat{\psi} \) for the sake of familiarity. The details leading to the covariant formulation (1)–(5), via a natural choice for the Hamiltonian \( K \), may be found in [20].

### Appendix B. Poincaré algebra

The quantities \( K^\mu, \Sigma^\mu_{\nu} \) associated with the 4–spinor representation satisfy [20] the commutation relations

\[
[K^\mu, K^\nu] = -i\Sigma^\mu_{\nu} \tag{B.1}
\]

\[
[\Sigma^\mu_{\nu}, K^\lambda] = -i\{ (g^{\mu\lambda} + n^\nu n^\lambda)K^\mu - (g^{\nu\lambda} + n^\mu n^\lambda)K^\nu \} \tag{B.2}
\]

\[
[\Sigma^\mu_{\nu}, \Sigma^\lambda_{\sigma}] = -i\{ (g^{\mu\lambda} + n^\nu n^\lambda)\Sigma^\mu_{\sigma} - (g^{\sigma\mu} + n^\nu n^\mu)\Sigma^\lambda_{\nu} - (g^{\nu\lambda} + n^\mu n^\lambda)\Sigma^\mu_{\sigma} + (g^{\sigma\nu} + n^\mu n^\nu)\Sigma^\lambda_{\mu} \} \tag{B.3}
\]

Since \( K^\mu n^\mu = n_\mu \Sigma^\mu_{\nu} = 0 \), there are only three independent \( K^\mu \) and three independent \( \Sigma^\mu_{\nu} \). From [20]: “The \( \Sigma^\mu_{\nu} \) are a covariant form of the Pauli matrices, and (B.3) is the corresponding form of \( SU(2) \) in the space–like hypersurface orthogonal to \( n^\mu \). The three independent \( K^\mu \) correspond to the non–compact part of the algebra”. Compare with e.g. [2, Ch 2 ]. The 4–momentum \( p^\mu \) commutes with \( n^\nu \) and so

\[
[p^\mu, p^\nu] = [p^\mu, K^\nu] = [p^\mu, \Sigma^\mu_{\nu}] = 0 \tag{B.4}
\]

The covariance of the Stueckelberg theory for 4–spinors follows not from the representation generated by the algebra (B.1)–(B.4), but from (5) where again \( S \) is generated in the usual way by \( \Sigma^\mu_{\nu} \equiv (i/4)[\gamma^\mu, \gamma^\nu] \).
Appendix C. Discrete symmetries

The discrete symmetries act on the wave functions as follows [20], where \( x = (t, \mathbf{x}) \) and \( n = (n^0, \mathbf{n}) \):

\[
\begin{align*}
C & : \quad \psi[x, \tau, n] \rightarrow i \gamma^2 \gamma^0 \psi^*[x, -\tau, n] \tag{C.1} \\
\mathcal{P} & : \quad \psi[x, \tau, n] \rightarrow \gamma^0 \psi[t, -\mathbf{x}, \tau, n^0, -\mathbf{n}] \tag{C.2} \\
\mathcal{T} & : \quad \psi[x, \tau, n] \rightarrow i \gamma^1 \gamma^3 \psi^*[-t, -\mathbf{x}, -\tau, -n^0, \mathbf{n}] \tag{C.3} \\
\mathcal{CPT} & : \quad \psi[x, \tau, n] \rightarrow -i \gamma^5 \psi[-t, -\mathbf{x}, -\tau, -n^0, -\mathbf{n}] \tag{C.4}
\end{align*}
\]

Appendix D. Parameter elimination

The Bohm paths of events \( x^\mu = X^\mu(y, \zeta; \tau) \) in spacetime are defined by the ordinary differential equations

\[
\frac{D}{D\tau} X^\mu(y, \zeta; \tau) = v^\mu(y, \zeta; \tau) \equiv v^\mu[X(y, \zeta; \tau), \tau] \tag{D.1}
\]
subject to

\[
X^\mu(y, \zeta; \zeta) = y^\mu \tag{D.2}
\]

where \( y^\mu \) is an event and \( \zeta \) is some fixed value of the parameter. Extending the nonrelativistic development in e.g. [9, §21–8],[38],[3] and [39], the covariant de Broglie 4–velocity \( v^\mu \) is defined here in terms of the probability current \( j^\mu \) and (suppressing the index \( n^\mu \) for clarity) the probability density \( \rho = \psi^\dagger \circ \psi \) as \( v^\mu[x, \tau] = j^\mu[x, \tau]/\rho[x, \tau] \).

With the spin–0 polar representation \( \psi = \sqrt{\rho} \exp\{iS/\hbar\} \), for arbitrary \( \hbar \) and \( c \), the de Broglie velocity is \( v^\mu[x, \tau] = (\partial^\mu S[x, \tau] - (c/e)A^\mu[x])/M \).

The expectation of the event \( x^\mu \) for example is, as in a nonrelativistic development [40] and with obvious allowance for dimension,

\[
\langle x^\mu \rangle_n = \int x^\mu \rho[x, \tau] d^4x = \int X^\mu(y, \zeta; \tau) \rho(y, \zeta; \tau) J(y, \zeta; \tau) d^4y \tag{D.3}
\]

where \( J \) is the Jacobian determinant of the transformation \( x = X(y, \zeta; \tau) \). Recall that \( \rho \equiv \psi^\dagger \circ \psi \) depends upon \( n^\mu \). The density \( \rho J \) is conserved on Bohm paths [41, 40], that is,

\[
\frac{D}{D\tau} \rho J = 0 \tag{D.4}
\]
The ‘expectation frequency’ is the Lorentz scalar $\langle \omega \rangle_n(\zeta; \tau) \equiv \hbar^{-1}\langle K_B \rangle_n(\zeta; \tau)$ where $K_B$ is the Hamiltonian in (3), for arbitrary $\hbar$ and $c$, evaluated on Bohm paths with the substitution $\pi^\mu = Mv^\mu$. Similarly to (D.3), $\langle \omega \rangle_n(\zeta; \tau)$ reduces to

$$\langle \omega \rangle_n(\zeta; \tau) = \hbar^{-1} \int K_B(y, \zeta; \tau) \rho(y, \zeta; \zeta) d^4y$$

The labeling density $\rho(y, \zeta; \zeta)$ may for example be that of an incident beam of free Stueckelberg wave functions with a beam density independent of $\zeta$, that is, $\rho[y, \zeta] \to \rho[y]$ as $\zeta \to -\infty$. In general, inverting (D.5) yields the parameter dependence

$$\tau \equiv T(\zeta, \langle \omega \rangle_n)$$

The inversion is defined for segments of $\tau$ in which $D\langle \omega \rangle / D\tau$ is one–signed. The value of $\langle \omega \rangle$ could feasibly be estimated from measurements of $x$. The scalar event density $\rho[x, \tau]$ would then [34] be representative of all the measurements within a segment. As a first approximation, the parameter increment $d\tau$ could be assigned its value on both a Bohm and classical path in the rest frame of a free particle, that is, $d\tau = (m/M)dt$ where $m$ is the rest mass. Of course, if the ‘target mass’ $M$ is assigned the value $m$, then coordinate time intervals in the frame of the particle coincide with intervals of the Newton or universal time $\tau$ [28].

**Appendix E. Pair annihilation and the quantum potential**

Consider the spin–0 charged particle for simplicity. The polar substitution for $\psi$ into (3) leads, for arbitrary $\hbar$ and $c$, to the manifestly covariant Madelung equation

$$M \frac{D}{D\tau} v^\mu = -\partial^\mu(W + Q) + \frac{e\hbar}{c} F^{\mu\nu} v^\nu$$

on Bohm paths, where the Hamiltonian in (3) has been augmented with the additive, parameter–independent, external potential $W[x]$. See e. g. [9, §21–8],[38],[3], or [39] for non–relativistic analyses. The so–called quantum potential is $Q = (\hbar^2/2M\sqrt{\rho})\partial_\mu \partial^\mu \sqrt{\rho}$, where the partial derivatives are for fixed $\tau$.

On the other hand, the analysis of particle dynamics on classical paths satisfying the canonical equations [13] leads to (E.1) with $Q$ omitted. Such an analysis is conventionally justified by neglecting quantum mechanical dispersion $\langle x^\mu x^\nu \rangle - \langle x^\mu \rangle\langle x^\nu \rangle$ (no summation) in the Ehrenfest equations. A consequence of the omission of $Q$ is that $R = (M/2)v^\mu v_\mu + W$ is conserved on such classical paths. Neglecting the electromagnetic potentials $A^\mu(y, \zeta; \tau)$ as $\tau \to \pm \infty$ yields $v^\mu = p^\mu/M$. Neglecting also the external potential $W(y, \zeta; \tau)$ as $\tau \to \pm \infty$ yields $R = -m^2c^2/2M < 0$, where $m$ in the rest mass of the subluminal particle. It is accordingly only possible for $v^0$ to change sign on a path through an intermediate region where $W < 0$. In other words a particle can only reverse its direction in time while none of $v^j$ changes sign, the particle thereby appearing to
the observer as having been annihilated or created together with its antiparticle, if an external potential $W$ is present \cite{13} for some range of $\tau$. Yet the justified presence of the parameter–dependent quantum potential $Q = Q[x, \tau]$ in (E.1) prevents conservation of $R$ on Bohm paths, and hence allows the possible annihilation and creation of particle–antiparticle pairs even in the absence of an external potential $W$.

References

Stueckelberg wave equation


