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Title: ON CONVERGENCE OF INFINITE SERIES OF IMAGES

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In introducing the method of electrical images we first consider the elementary problem of finding the electric field of a single point charge and an infinite conducting plane. Then a problem involving a conducting sphere and an exterior point charge is investigated and the results are applied in the solution of two other such problems of a more involved nature. One involves a conducting sphere and conducting plane; and the other involves two conducting spheres. These latter problems both involve infinite series of image charges. The main focus of this dissertation is several different convergence proofs for these series of images.

On Convergence of Infinite Series of Images

by

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ON CONVERGENCE OF INFINITE SERIES OF IMAGES

I. INTRODUCTION

1. Preliminary Remarks

A familiar topic in many courses in electricity and magnetism is the solution of electrostatics problems by the method of electrical images. Certain of these problems require the insertion of appropriate charges at an appropriate infinite sequence of points. The resulting infinite series, representing the total charge inserted, have a particularly fascinating character. See [2], [3], [4], [6]. The question of convergence of these series is sure to be raised by the more mathematically inclined student. No proofs of convergence are given in the references cited, or in any readily available sources known to the author. In this thesis two problems are considered and a variety of convergence proofs are given.

2. The Concept of the Method of Images

We first recall that for a single point charge q in empty space the <u>electric field</u> $\vec{E}$ at a point P is the force on a unit positive charge at P. It has the direction of the radius vector drawn from q to P and magnitude q/r^2. The <u>potential</u> V at P is q/r and represents the work necessary to bring a unit positive charge from infinity to P. In a dielectric medium these quantities are reduced and become

$$\frac{q}{Kr^2} \quad \text{and} \quad \frac{q}{Kr}$$

where K is called the dielectric constant or permittivity of the medium. More generally, in the case of a continuous distribution of electricity consisting of ρ units of charge per unit volume occupying a region τ, the electric potential is

$$V = \int_{(\tau)} \frac{\rho d\tau}{r}$$

and the electric field is the negative gradient of V, i.e., $\vec{E} = -\nabla V$. It is also to be recalled that the capacitance C of a conductor with charge Q at potential V is given by

$$C = \frac{Q}{V}.$$

We turn now to the method of images and consider the electric field due to two point charges. Let an equipotential surface enclosing one of the charges be replaced by a conducting surface. Then if the charge inside is transferred to this surface, the field between the other charge and the surface remains unaltered, while that between the first charge and the surface is annhilated [4, pg. 94]. In its simplest form the method of electrical images consists of placing two point charges in such positions that one of the equipotential surfaces of the resulting field coincides with the surface of a conductor which one desires to place in the field. Then the charge on one side of the conducting surface is transferred to the surface and the field on the other side remains unaltered. Thus the field produced by a point charge and a conducting surface can be determined by the relatively simple investigation of the field of two point charges. The point charge which is transferred to the conducting surface is called the image of the other charge.

The simplest electrostatic problem solved by the method of images is that of finding the field of a point charge q and a conducting plane. Suppose q is at distance d to the right of the plane. Drop a perpendicular from q to the plane and place a charge $-q$ on this perpendicular at distance d to the left of the plane. It is immediate that the plane surface is an equipotential surface, of zero potential, of the field due to the two point charges q and $-q$. If this plane surface is a conductor and the charge $-q$ is transferred to it,the field to the right of the plane is unaltered and can be computed at any point by simply computing the field of the two point charges.

Suppose one wishes to find the field of a conducting spherical surface S and a point charge q at distance d from the center of S. See Fig. 1. Then one is required to find the magnitude and position of a point charge $-q'$ such that the spherical surface S is an equipotential of the field of the charges q and $-q'$. One readily finds [4, pg. 97] that

$$(1.1) \qquad q' = \frac{a}{d} q$$

and that $-q'$ must be placed at distance $\overline{OQ'}$ from O where

$$(1.2) \qquad \overline{OQ'} = \frac{a^2}{d} .$$

That is, q' is placed at the point Q' which is the <u>inverse</u> of Q with respect to the sphere S.

In the following chapters we extend the method to problems involving two conductors, neither of which constitutes a point charge.

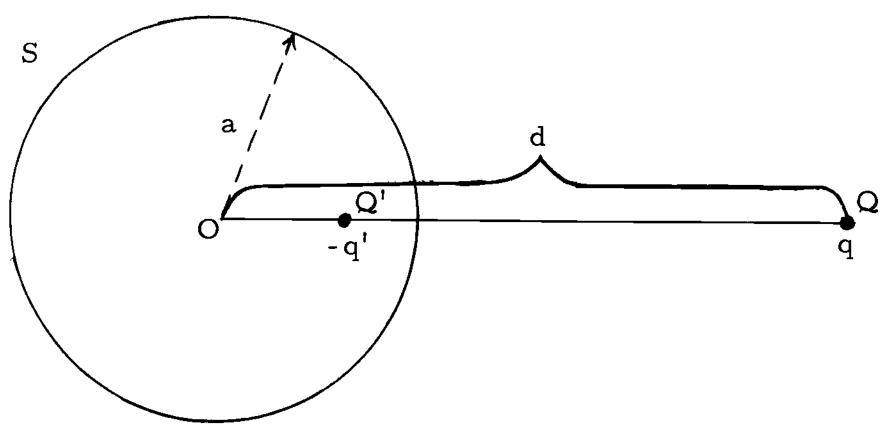

Fig. 1. Sphere, charge and image.

II. THE SPHERE-PLANE PROBLEM

1. Introduction

In finding the capacitance of a charged spherical conductor relative to an earthed conducting plane, one inserts an infinite sequence of charges within the sphere so as to make both the spherical surface and the plane equipotentials. The solution will contain the infinite series of charges, the convergence of which we will demonstrate in two different ways. We first present a cursory introduction of the sphere-plane problem, and the derivation of the series of charges, essentially as it is dealt with in the references. See [4], [6].

The first proof of convergence will involve employing a recursion relation to express the series in a form which lends itself to analysis by the ratio test.

The second, less direct method, involves showing that successive images in the sphere satisfy a difference equation which may be solved in terms of hyperbolic functions. See [1, p. 363]. The ratio test is then again invoked.

Finally, we verify the recursion relations which play an important role in both solutions.

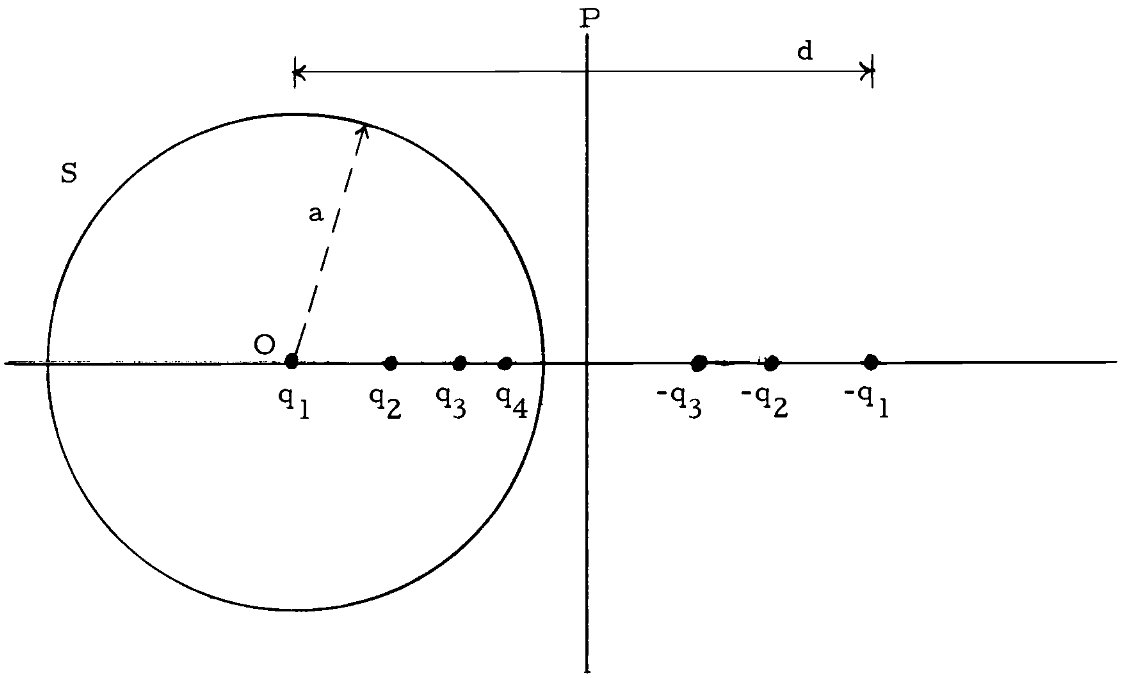

Fig. 2. Sphere, plane and images.

2. The Series and a Proof of Convergence

Using the method of images, we proceed as follows, computing the capacitance of a charged spherical conductor S of radius a at a distance $d/2$ from an infinite earthed conducting plane P. We require $d > 2a$. See Fig. 2.

Place q_1 at the center of S. Then S is equipotential but P is not. Add the image $-q_1$ of q_1 in the plane. Then P is equipotential but S is not. Now add the image q_2 of $-q_1$ in the sphere. It will be a charge $q_2 = \frac{a}{d} q_1$ at a distance $s_2 = a^2/d$ from O [(1.1), 1.2)]. Now S is equipotential and P is not. Next we add the image of q_2 in the plane, followed by the image q_3

of $-q_2$ in the sphere. Thus $q_3 = \frac{a}{s_1} q_2$ at a distance a^2/s_1 from O where s_1 is the distance of $-q_2$ from O, i.e.,

$$s_1 = d - \frac{a^2}{d} = d(1 - \frac{a^2}{d^2}) \qquad [(1.1), (1.2)].$$

So we have

$$q_3 = \frac{\frac{a^2}{d^2}}{1 - \frac{a^2}{d^2}} q_1$$

at distance

$$s_3 = d - \frac{\frac{a^2}{d}}{1 - \frac{a^2}{d^2}}$$

from O. Next insert q_4, the image of $-q_3$ in the sphere. We have

$$q_4 = \frac{\frac{a}{d}}{1 - \frac{\frac{a^2}{d^2}}{1 - \frac{a^2}{d^2}}} q_3 .$$

Proceeding in this way, we insert an infinite sequence of charges in S which makes both S and P equipotential. In S the total charge is

$$\sum_{i=1}^{\infty} q_i = q_1 \left[1+x+\frac{x^2}{1-x^2} + \frac{x^3}{(1-x^2)\left(1-\frac{x^2}{1-x^2}\right)} + \frac{x^4}{(1-x^2)\left(1-\frac{x^2}{1-x^2}\right)\left(1-\frac{x^2}{1-\frac{x^2}{1-x^2}}\right)} + \ldots \right],$$

where $x = \frac{a}{d}$.

Since q_2 is the image of $-q_1$ in the sphere, the potential of S due to this pair is zero. The same is true of the pairs q_3 and $-q_2$, q_4 and $-q_3$, etc. Therefore the potential of the sphere is

$$V = \frac{q_1}{a}$$

(taking the permittivity of the medium to be unity). Clearly the potential of the plane is zero. Therefore the capacitance of the sphere relative to the earthed plane is

$$C = \frac{Q}{V} = \frac{a\Sigma_{i=1}^{\infty} q_i}{q_1} = a\left(1+x+\frac{x^2}{1-x^2} + \ldots\right),$$

where Q here is the net charge in S.

We investigate the convergence of the series

$$1 + x + \frac{x^2}{1-x^2} + \dots \quad \text{for} \quad 0 < x < 1/2.$$

Note first that we may write the series as $\sum_{n=0}^{\infty} c_n$ if we define

$$(2.1) \quad \begin{cases} c_{n+1} = \dfrac{x^{n+1}}{a_n}, & n = 0, 1, 2, \dots \\ a_{n+1} = a_n b_{n+1} \\ b_{n+1} = 1 - \dfrac{x^2}{b_n} \\ a_0 = b_0 = c_0 = 1 . \end{cases}$$

The validity of this recursion relation for all n will be proved later.

First we note that all terms in the series are positive. Clearly all c_i are positive if all $a_i > 0$. All $a_i > 0$ if all $b_i > 0$. This follows simply from $a_0 = 1$, $a_{n+1} = a_n b_{n+1}$ and an induction.

Now we may prove by induction that $1/2 < b_i \leq 1$ $(i = 0, 1, 2, \dots)$, and therefore conclude that all c_i are positive: We have $b_0 = 1$. Suppose $1/2 < b_n \leq 1$. Then since $0 < x < 1/2$ we have

$$0 < \frac{x^2}{b_n} < \frac{1}{2}$$

or

$$\frac{1}{2} < 1 - \frac{x^2}{b_n} < 1$$

i. e.,

$$\frac{1}{2} < b_{n+1} < 1.$$

Now we may apply the ratio test to $\sum_{n=0}^{\infty} c_n$. We must show that

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} < 1$$

where

$$\frac{c_{n+1}}{c_n} = \frac{x^{n+1}}{a_n} \frac{a_{n-1}}{x^n} = x \frac{a_{n-1}}{a_n} = \frac{x}{b_n} .$$

We next show that $\{b_n\}$ is a monotone decreasing sequence. Clearly $b_0 - b_1 > 0$. Suppose that $b_n - b_{n+1} > 0$. Consider

$$b_{n+1} - b_{n+2} = (1 - \frac{x^2}{b_n}) - (1 - \frac{x^2}{b_{n+1}}) = \frac{x^2}{b_{n+1}} - \frac{x^2}{b_n} = \frac{x^2}{b_n b_{n+1}}(b_n - b_{n+1}).$$

This is positive according to the induction hypothesis.

Since we have shown the $\{b_n\}$ are bounded below and monotone decreasing, we can conclude $\lim_{n\to\infty} b_n$ exists, and is positive. (In fact $\lim_{n\to\infty} b_n \geq 1/2$). Also, $1/2 < b_n \leq 1 \Rightarrow 1/2 \leq \lim_{n\to\infty} b_n \leq 1$.

This, together with $c_{n+1}/c_n = x/b_n$ and $0 < x < 1/2$ implies

$$0 < \lim_{n\to\infty} \frac{c_{n+1}}{c_n} < 1$$

thereby showing the convergence of $\sum_{n=0}^{\infty} c_n$.

After showing the $\{b_n\}$ are monotone decreasing and bounded, we could follow an interesting alternate route.

Let

$$r = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \frac{c_{n+2}}{c_{n+1}}.$$

Then

$$r = \lim_{n\to\infty} \frac{c_{n+2}}{c_{n+1}} = \lim_{n\to\infty} \frac{x}{b_{n+1}} = \frac{x}{\lim_{n\to\infty}\left(1 - \frac{x^2}{b_n}\right)}$$

$$= \frac{x}{1 - x \lim_{n\to\infty}\left(\frac{x}{b_n}\right)} = \frac{x}{1 - x \lim_{n\to\infty}\left(\frac{c_{n+1}}{c_n}\right)} = \frac{1}{1-xr} .$$

Thus

$$r - xr^2 = x \quad \text{or} \quad r^2x - r + x = 0.$$

Then

$$r = \frac{1 \pm \sqrt{1-4x^2}}{2x}.$$

If the plus sign were correct then since $0 < x < 1/2$ we clearly would have $r > 1$ (for indeed, $1/2x > 1$). But if $r > 1$, then there are b_n such that

$$\frac{c_{n+1}}{c_n} = \frac{x}{b_n} > 1 \quad \text{(with } 0 < x < 1/2\text{)}.$$

Also we have $1/2 < b_n \leq 1$ so $1 \leq 1/b_n < 2$. This, with $0 < x < 1/2$ gives $x/b_n < 1$ which is a contradiction. So $r > 1$ is impossible and we reject the plus sign.

We conclude

$$r = \frac{1-\sqrt{1-4x^2}}{2x}.$$

Now showing $r < 1$ when $0 < x < 1/2$ is easy using elementary calculus: We have

$$\lim_{x \to 0^+} \left(\frac{1-\sqrt{1-4x^2}}{2x}\right) = 0 \quad \text{and} \quad \lim_{x \to 1/2^-} \left(\frac{1-\sqrt{1-4x^2}}{2x}\right) = 1.$$

Also, the derivative is seen to be positive for $0 < x < 1/2$ so the function is monotone increasing on $0 < x < 1/2$. Therefore we conclude $0 < r < 1$.

Finally we note that all the charges stay in the sphere S. The sequence of distances from O of $q_1, q_2, \ldots$ is

$$\frac{ax}{1-x^2}, \quad \frac{ax}{1-\dfrac{x^2}{1-x^2}}, \quad \frac{ax}{1-\dfrac{x^2}{1-\dfrac{x^2}{1-x^2}}}, \quad \ldots$$

or

$$\frac{ax}{b_1}, \frac{ax}{b_2}, \frac{ax}{b_3}, \ldots$$

But $1/2 \leq \lim_{n\to\infty} b_n \leq 1$ so $1 \leq \lim_{n\to\infty} 1/b_n \leq 2$, or

$a^2/d \leq \lim_{n\to\infty} ax/b_n \leq 2a^2/d$. Now

$$\frac{a}{d} < \frac{1}{2} \Rightarrow \frac{a^2}{d} < \frac{a}{2} \Rightarrow \frac{2a^2}{d} < a$$

so

$$\lim_{n\to\infty} \frac{ax}{b_n} < a.$$

3. The Series in Terms of Hyperbolic Functions

In this proof we demonstrate that successive charges in the sphere satisfy a linear finite difference equation which we shall solve in terms of hyperbolic functions. The ratio test will then be applied to the series of charges in this new form to prove convergence.

Recall we have the series $\sum_{n=0}^{\infty} c_n$. Using (2. 1) we find

(2.2) $$c_{n+1} = \frac{x^{n+1}}{a_n}$$

(2.3) $$c_{n+2} = \frac{x^{n+2}}{a_n\left(1 - \frac{x^2}{b_n}\right)}$$

(2.4) $$c_{n+3} = \frac{x^{n+3}}{a_n\left(1 - \frac{x^2}{b_n}\right)\left(1 - \frac{x^2}{1 - \frac{x^2}{b_n}}\right)}.$$

We now eliminate a_n, b_n from the above equations: From (2.2) we have

$$a_n = \frac{x^{n+1}}{c_{n+1}}.$$

Then (2.3) gives

$$c_{n+2} = \frac{x^{n+2}}{\frac{x^{n+1}}{c_{n+1}}\left(1 - \frac{x^2}{b_n}\right)} = \frac{xc_{n+1}}{1 - \frac{x^2}{b_n}}$$

or

$$1 - \frac{x^2}{b_n} = x\frac{c_{n+1}}{c_{n+2}}.$$

Now (2.4) gives

$$c_{n+3} = \frac{x^{n+3}}{\frac{x^{n+1}}{c_{n+1}}\left(\frac{xc_{n+1}}{c_{n+2}}\right)\left(1 - \frac{x^2}{x\frac{c_{n+1}}{c_{n+2}}}\right)} = \frac{xc_{n+2}}{\left(1 - x\frac{c_{n+2}}{c_{n+1}}\right)} .$$

Then

$$\frac{1}{c_{n+3}} = \frac{1 - x\frac{c_{n+2}}{c_{n+1}}}{xc_{n+2}}$$

or

$$\frac{1}{c_{n+3}} = \frac{1}{xc_{n+2}} - \frac{1}{c_{n+1}}$$

and we have the following finite difference equation with constant coefficients, which is linear in $1/c_n$:

$$\frac{1}{c_{n+3}} + \frac{1}{c_{n+1}} = \frac{1}{xc_{n+2}} . \tag{2.5}$$

We now proceed to solve the difference equation (2.5) letting $1/c_n = u^n$. Then

$$u^{n+3} + u^{n+1} = \frac{1}{x} u^{n+2}$$

or

$$u^2 + 1 = \frac{u}{x}$$

or

$$u^2 - \frac{u}{x} + 1 = 0$$

so

$$u = \frac{\frac{1}{x} \pm \sqrt{\frac{1}{x^2} - 4}}{2} = \frac{1}{2x} \pm \sqrt{\frac{1}{4x^2} - 1}\,.$$

Now since

$$\frac{1}{2x} = \frac{1}{2\frac{a}{d}} = \frac{1}{2}\,\frac{d}{a} > \frac{1}{2}(2) = 1\,,$$

we can let $1/2x = \cosh\alpha$. Then

$$u = \cosh\alpha \pm \sqrt{\cosh^2\alpha - 1} = \cosh\alpha \pm \sinh\alpha\,.$$

Therefore $u^{2n} = (u^2)^n = (\cosh\alpha + \sinh\alpha)^n$ is a solution to (2.5) as is $(\cosh\alpha - \sinh\alpha)^n$. Then the linear combination $C_1(\cosh\alpha + \sinh\alpha)^n + C_2(\cosh\alpha - \sinh\alpha)^n$ is also a solution. Simplified, this is $C_1e^{n\alpha} + C_2e^{-n\alpha}$ which is equivalent to the linear combination $A\cosh(n\alpha) + B\sinh(n\alpha)$. We now have a solution of (2.5), $\frac{1}{c_n} = A\cosh(n\alpha) + B\sinh(n\alpha)$, where the constants A, B will now be evaluated by knowing the values of the first two terms c_0, c_1 of the sequence. Since $c_0 = 1$ we find $A = 1$. Since $c_1 = x/a_0 = x$, we have

$$\frac{1}{x} = \cosh\alpha + B\sinh\alpha$$

so

$$B = \frac{1}{\sinh\alpha}\left(\frac{1}{x} - \cosh\alpha\right) = \frac{1}{\sinh\alpha}(2\cosh\alpha - \cosh\alpha).$$

Therefore $B = \dfrac{\cosh\alpha}{\sinh\alpha}$. Therefore

$$\frac{1}{c_n} = \cosh(n\alpha) + \frac{\cosh \alpha}{\sinh \alpha} \sinh(n\alpha)$$

$$= \frac{\sinh \alpha \cosh(n\alpha) + \cosh \alpha \sinh(n\alpha)}{\sinh \alpha}$$

$$= \frac{\sinh(n+1)\alpha}{\sinh \alpha} .$$

The total charge on the sphere is

$$\sum_{n=1}^{\infty} q_n = q_1 \sum_{n=0}^{\infty} c_n$$

$$= q_1 \sinh \alpha \sum_{n=0}^{\infty} [\sinh(n+1)\alpha]^{-1} = q_1 \sinh \alpha \sum_{n=1}^{\infty} \operatorname{csch}(n\alpha).$$

To show Σc_n converges, we again use the ratio test:

$$\frac{c_{n+1}}{c_n} = \frac{\sinh(n\alpha)}{\sinh(n+1)\alpha} = \frac{\sinh(n\alpha)}{\sinh(n\alpha) \cosh \alpha + \cosh(n\alpha) \sinh \alpha}$$

$$= \frac{1}{\cosh\alpha + \coth(n\alpha) \sinh \alpha} .$$

If $\alpha > 0$ we have

$$\lim_{n \to \infty} \coth(n\alpha) = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = \frac{1}{\cosh \alpha + \sinh \alpha} .$$

Clearly $\dfrac{1}{\cosh \alpha + \sinh \alpha} < 1$ since $\cosh \alpha > 1$ and $\alpha > 0$ implies $\sinh \alpha > 0$.

If $\alpha < 0$ we have $\lim_{n \to \infty} \coth(n\alpha) = -1$. Then

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = \frac{1}{\cosh \alpha - \sinh \alpha} < 1$$

since $\cosh \alpha > 1$ and $\alpha < 0$ implies $\sinh \alpha < 0$.

Finally, we now show that the charges actually satisfy the recursion relations(2.1). To do this, we evidently must perform an induction on the distances as well as the charges.

Note that the c_{n+1} in (2.1) is actually q_{n+2}/q_1. Again let s_n be the distance from O to q_n. Then clearly $d - s_n$ is the distance from O to $-q_n$. See Fig. 2.

Now by induction we prove that for $n = 0, 1, 2, \ldots$ we have

$$\frac{q_{n+2}}{q_1} = \frac{x^{n+1}}{a_n} \quad \text{and} \quad s_{n+2} = \frac{ax}{b_n}$$

where

$$\left.\begin{aligned} a_{n+1} &= a_n b_{n+1} \\ b_{n+1} &= 1 - \frac{x^2}{b_n} \\ a_0 &= b_0 = 1 \end{aligned}\right\} \quad (n = 0, 1, 2, \ldots).$$

<u>Proof</u>: If $n = 0$ we have $q_2/q_1 = x/a_0$ or $q_2 = q_1 x$, which is true and also $s_2 = ax/b_0 = ax = a^2/d$ which is true. Now

suppose $q_{k+2}/q_1 = x^{k+1}/a_k$ and $s_{k+2} = ax/b_k$. We have [4]

$$q_{2k+3} = \frac{a}{\text{distance from O to } -q_{k+2}} q_{k+2} = \frac{a}{d-s_{k+2}} q_{k+2}$$

$$= \frac{a}{d-\frac{ax}{b_k}} \frac{x^{k+1}}{a_k} q_1 = \frac{a}{d\left(1-\frac{\frac{ax}{d}}{b_k}\right)} \frac{x^{k+1}}{a_k} q_1$$

$$= \frac{x}{1-\frac{x^2}{b_k}} \frac{x^{k+1}}{a_k} q_1 = \frac{x^{k+2}}{b_{k+1}a_k} q_1 = \frac{x^{k+2}}{a_{k+1}} q_1 .$$

Thus $q_{k+3}/q_1 = x^{k+2}/a_{k+1}$, which was to be shown.

Furthermore, [4]

$$s_{k+3} = \frac{a^2}{\text{distance from O to } -q_{k+2}} = \frac{a^2}{d-s_{k+2}}$$

$$= \frac{a^2}{d-\frac{ax}{b_k}} = \frac{a^2}{d\left(1-\frac{\frac{ax}{d}}{b_k}\right)} = \frac{ax}{1-\frac{x^2}{b_k}} = \frac{ax}{b_{k+1}}$$

which was also to be shown.

III. THE TWO-SPHERES PROBLEM

1. Introduction

In this problem, the method of images is employed to compute the capacitance between two spherical conductors. Infinite sequences of charges are inserted within each sphere, so as to put one sphere at unit potential and the other at zero potential. The solution will involve the infinite series of charges, and we will again confine our attention mainly to proving convergence of the series of charges. However, we will begin by presenting a brief introduction of the two spheres problem and the derivation of the series of charges. We will again refer the reader to the references for more complete details of this aspect of the problem.

The first proof of convergence will involve employing a recursion relation to express the series in a form which lends itself to analysis by the ratio test.

Another, but less direct, method involves showing that successive images in either sphere satisfy a difference equation which may be solved in terms of hyperbolic functions. The ratio test is then again invoked to prove the series of charges converges. This is similar to the difference equation approach employed earlier in the sphere-plane problem. See [1, p. 363].

A considerably more unwieldy approach is suggested in [2]. There may be found an outline of a novel procedure for expressing the series arising in the two spheres problem in terms of definite improper integrals. Then the proof of convergence of the series may be replaced by the task of establishing convergence of the improper integrals. We lastly make a detailed examination of this approach.

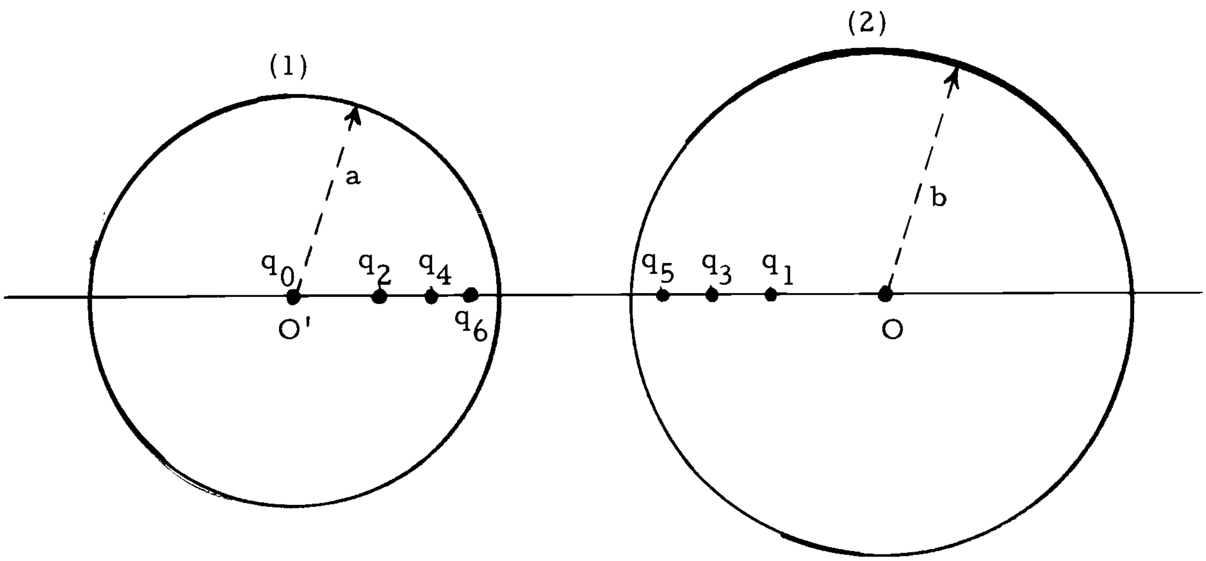

Fig. 3. Two spheres and images.

2. The Series and a Proof of Convergence

We begin by deriving the series of charges utilized in computing the self- and mutual capacitances for two charged conducting spheres, via the method of images. Except for some differences in notation, this is essentially the derivation found in [6, p. 118].

We denote the distance between the centers of the spheres by c, i.e., $c = |\overline{O'O}|$, and assume $c > a + b$ where a and b are the radii of (1) and (2) respectively. It will prove convenient to let $m = a/c$ and $n = b/c$, noting that $0 < m, n < 1$. See Fig. 3.

Put (1) at unit potential by putting a charge $q_0 = 4\pi\epsilon a$ at O', where $4\pi\epsilon$ is the permittivity of the medium. Then put (2) at zero potential by placing an image $q_1 = (-4\pi\epsilon a)\frac{b}{c} = -4\pi\epsilon an$ at a distance $d_1 = b^2/c = nb$ to the left of O. Next restore (1) to unit potential with a charge $q_2 = -q_1(\frac{a}{c-d_1}) = -q_1(\frac{a}{c-nb})$. q_2 must be placed to the right of O' a distance $d_2 = \frac{a^2}{c-d_1} = \frac{a^2}{c-nb} = \frac{ma}{1-n^2}$. Next, restore (2) to zero potential with an image q_3 where

$$q_3 = -q_2 \frac{b}{c-d_2} = -q_2 \frac{b}{c - \frac{ma}{1-n^2}} .$$

q_3 must be placed at a distance from O

$$d_3 = \frac{b^2}{c-d_2} = \frac{b^2}{c - \frac{ma}{1-n^2}} = \frac{b^2}{c\left(1 - \frac{m^2}{1-n^2}\right)} = \frac{nb}{1 - \frac{m^2}{1-n^2}} .$$

Next, q_4 returns (1) to unit potential where

$$q_4 = -q_3 \frac{a}{c-d_3} = -q_3 \frac{a}{c - \dfrac{nb}{1 - \dfrac{m^2}{1-n^2}}} .$$

q_4 lies to the right of O' a distance

$$d_4 = \frac{a^2}{c-d_3} = \frac{a^2}{c - \dfrac{nb}{1 - \dfrac{m^2}{1-n^2}}} = \frac{a^2}{c\left(1 - \dfrac{n^2}{1 - \dfrac{m^2}{1-n^2}}\right)} = \frac{ma}{1 - \dfrac{n^2}{1 - \dfrac{m^2}{1-n^2}}} .$$

q_5 restores (2) to zero potential and is a charge

$$q_5 = -q_4 \frac{b}{c-d_4} = -q_4 \frac{b}{c - \dfrac{ma}{1 - \dfrac{n^2}{1 - \dfrac{m^2}{1-n^2}}}} .$$

q_5 is located to the left of O a distance

$$d_5 = \frac{b^2}{c-d_4} = \frac{nb}{1 - \dfrac{m^2}{1 - \dfrac{n^2}{1 - \dfrac{m^2}{1-n^2}}}} .$$

Continuing this procedure, we have

$$q_6 = -q_5 \frac{a}{c-d_5} = -q_5 \cfrac{a}{c - \cfrac{nb}{1 - \cfrac{m^2}{1 - \cfrac{n^2}{1 - \cfrac{m^2}{1-n^2}}}}}$$

and q_6 lies to the right of O' a distance

$$d_6 = \frac{a^2}{c-d_5} = \cfrac{a^2}{c - \cfrac{nb}{1 - \cfrac{m^2}{1 - \cfrac{n^2}{1 - \cfrac{m^2}{1-n^2}}}}}$$

$$= \cfrac{ma}{1 - \cfrac{n^2}{1 - \cfrac{m^2}{1 - \cfrac{n^2}{1 - \cfrac{m^2}{1-n^2}}}}} .$$

The above process continues indefinitely, generating an infinite sequence of charges $\{q_i\}_{i=0}^{\infty}$ and a corresponding infinite sequence of distances $\{d_i\}_{i=0}^{\infty}$. The subsequence of charges $\{q_{2i}\}_{i=0}^{\infty}$ lies in (1); q_{2i} lying a distance d_{2i} to the right of O' for each i. Similarly, the subsequence of charges $\{q_{2i+1}\}_{i=0}^{\infty}$ lies in (2); q_{2i+1} lying a distance d_{2i+1} to the left of O for each i. After the entire sequence of charges $\{q_i\}_{i=0}^{\infty}$ has been inserted in the spheres as prescribed, both are equipotentials, (1) being at unit potential and (2) at zero potential. Therefore the capacitance of (1) relative to the earthed sphere (2) is

$$C = \frac{Q}{V} = Q = \sum_{i=0}^{\infty} q_{2i} \, .$$

We investigate the convergence of $\sum_{i=0}^{\infty} q_{2i}$.

In summary, letting $K = 4\pi\epsilon a$, the charges on (1) are

$$q_0 = K$$

$$q_2 = -q_1 \frac{m}{1-n^2}$$

$$q_4 = -q_3 \cfrac{m}{1-\cfrac{n^2}{1-\cfrac{m^2}{1-n^2}}}$$

$$q_6 = -q_5 \cfrac{m}{1-\cfrac{n^2}{1-\cfrac{m^2}{1-\cfrac{n^2}{1-\cfrac{m^2}{1-n^2}}}}}$$

$$\vdots$$

The charges on (2) are

$$q_1 = -Kn$$

$$q_3 = -q_2 \cfrac{n}{1 - \cfrac{m^2}{1 - n^2}}$$

$$q_5 = -q_4 \cfrac{n}{1 - \cfrac{m^2}{1 - \cfrac{n^2}{1 - \cfrac{m^2}{1 - n^2}}}}$$

$$q_7 = -q_6 \cfrac{n}{1 - \cfrac{m^2}{1 - \cfrac{n^2}{1 - \cfrac{m^2}{1 - \cfrac{n^2}{1 - \cfrac{m^2}{1 - n^2}}}}}}$$

$$\vdots$$

The total charge on (1) is $\sum_{k=0}^{\infty} q_{2k}$ where

$$(3.1) \quad \begin{cases} q_{2k} = -q_{2k-1} \dfrac{m}{r_{2k-2}} & (k = 1, 2, \ldots) \\ r_{2k-2} = 1 - \dfrac{n^2}{s_{2k-3}} & (k = 2, 3, \ldots) \\ s_{2k-3} = 1 - \dfrac{m^2}{r_{2k-4}} & (k = 2, 3, \ldots) \\ q_{2k-1} = -q_{2k-2} \dfrac{n}{s_{2k-3}} & (k = 1, 2, \ldots) \end{cases}$$

and $r_0 = 1-n^2$, $s_{-1} = 1$ and $q_0 = K$. The recursion relations can be verified in a manner similar to the proof by induction given for the recursion relations (2.1) of the sphere-plane problem.

In this problem, clearly $n > 0$, $m > 0$ and $n+m < 1$. Therefore $1-(m+n) = \epsilon$ where $\epsilon > 0$. Define $M = m + \frac{\epsilon}{3}$ and $N = n + \frac{\epsilon}{3}$. To employ the ratio test on Σq_{2k}, we will need the following inequalities which we proceed to prove by induction:

(3.2) $$M < r_{2k-2} \leq 1 \quad \text{and} \quad N < s_{2k-3} \leq 1 \quad \text{for all } k.$$

Note first $M = m + \frac{1}{3}(1-m-n) = \frac{1}{3} + \frac{2m-n}{3}$ and we verify $M < 1$:

$$m+n < 1 \Rightarrow m < 1-n \Rightarrow 2m < 2-2n \Rightarrow 2m < 2+n$$
$$\Rightarrow 2m-n < 2 \Rightarrow \frac{2m-n}{3} < \frac{2}{3} \Rightarrow M = \frac{1}{3} + \frac{2m-n}{3} < 1.$$

Similarly, $N < 1$.

For $k = 1$, we first show $M < r_0 \leq 1$, i.e., $M < 1-n^2 \leq 1$: Clearly, since $n < 1$, we have $n^2 < n$ or $1-n < 1-n^2$ This is $\frac{1+2(1-n)-n}{3} < 1-n^2$. Since $m < 1-n$, we have $\frac{1+2m-n}{3} < 1-n^2$, i.e., $M < 1-n^2$ (< 1). Also note that $N < s_{-1} \leq 1$ since $s_{-1} = 1$.

Now assume that for some $p > 1$, we have $M < r_{2p-2} \leq 1$ and $N < s_{2p-3} \leq 1$. We show first that $N < s_{2p-1} \leq 1$: Clearly, $M + N = (m+\frac{\epsilon}{3}) + (n+\frac{\epsilon}{3}) = m + n + \frac{2\epsilon}{3} < m + n + \epsilon = 1$. Therefore

$M < 1-N$, or $M^2/M < 1-N$, and since $m^2 < M^2$, we can conclude $m^2/M < 1-N$. Applying the induction hypothesis to the latter inequality, we obtain

$$0 \leq \frac{m^2}{r_{2p-2}} < 1-N,$$

or

$$N-1 < -\frac{m^2}{r_{2p-2}} \leq 0$$

or

$$N < 1-\frac{m^2}{r_{2p-2}} \leq 1$$

i. e.,

$$N < s_{2p-1} \leq 1.$$

We next verify $M < r_{2(p+1)-2} \leq 1$, i.e., $M < r_{2p} \leq 1$. We know $N < 1-M$ or $N^2/N < 1-M$. Since $n^2 < N^2$, it follows that $n^2/N < 1-M$. This, together with the induction hypothesis $s_{2p-3} > N$ implies $s_{2p-1} > N$. We have

$$0 \leq \frac{n^2}{s_{2p-1}} < 1-M, \quad \text{or} \quad M-1 < -\frac{n^2}{s_{2p-1}} \leq 0,$$

or

$$M < 1 - \frac{n^2}{s_{2p-1}} \leq 1, \text{ i.e., } M < r_{2p} \leq 1.$$

We next demonstrate by induction that the q_{2k}'s are positive. Note $q_0 = K > 0$. Suppose $q_{2p} > 0$. Then

$$q_{2p+2} = -q_{2p+1}\frac{m}{r_{2p}} = -(-q_{2p}\frac{n}{s_{2p-1}})\frac{m}{r_{2p}}$$

which is positive since n, m and q_{2p} are positive.

We next prove by induction that

$$(3.3) \qquad \frac{q_{2k+2}}{q_{2k}} = \frac{mn}{s_{2k-1}r_{2k}} \qquad (k = 0, 1, 2, \ldots),$$

omitting the easy verification for $k = 0$. Suppose

$$\frac{q_{2i+2}}{q_{2i}} = \frac{mn}{s_{2i-1}r_{2i}}$$

for a positive i. From (3.1) we have

$$\frac{q_{2i+4}}{q_{2i+2}} = \frac{q_{2i+3}}{q_{2i+1}}\frac{r_{2i}}{r_{2i+2}} = \frac{q_{2i+2}}{q_{2i}}\frac{s_{2i-1}}{s_{2i+1}}\frac{r_{2i}}{r_{2i+2}}.$$

Now the induction hypothesis gives

$$\frac{q_{2i+4}}{q_{2i+2}} = \frac{mn}{s_{2i-1}r_{2i}}\frac{s_{2i-1}}{s_{2i+1}}\frac{r_{2i}}{r_{2i+2}} = \frac{mn}{s_{2i+1}r_{2i+2}}$$

which establishes (3.3).

We may finally apply the ratio test to show the convergence of Σq_{2k}. We show by induction that $\{s_{2k-1}\}$ and $\{r_{2k}\}$ are nonincreasing sequences.

$$r_0 - r_2 = (1-n^2) - (1-\frac{n^2}{s_1}) = n^2(\frac{1}{s_1} - 1) \geq 0$$

since $N < s_1 \leq 1$. Also note $s_{-1} - s_1 = 1 - s_1 \geq 0$. Now suppose $r_{2k} - r_{2k+2} \geq 0$ and $s_{2k-1} - s_{2k+1} \geq 0$. We have then

$$s_{2k+1} - s_{2k+3} = (1 - \frac{m^2}{r_{2k}}) - (1 - \frac{m^2}{r_{2k+2}})$$

$$= m^2(\frac{1}{r_{2k+2}} - \frac{1}{r_{2k}}) = \frac{m^2}{r_{2k+2}r_{2k}}(r_{2k} - r_{2k+2}) \geq 0.$$

Furthermore,

$$r_{2k+2} - r_{2k+4} = (1 - \frac{n^2}{s_{2k+1}}) - (1 - \frac{n^2}{s_{2k+3}})$$

$$= n^2(\frac{1}{s_{2k+3}} - \frac{1}{s_{2k+1}})$$

$$= \frac{n^2}{s_{2k+3}s_{2k+1}}(s_{2k+1} - s_{2k+3}) \geq 0.$$

Since $\{s_{2k-1}\}$ and $\{r_{2k}\}$ are non-increasing and bounded below, they are convergent. Also, $M < r_{2k} \leq 1$, $N < s_{2k-1} \leq 1$ for $k = 1, 2, \ldots$ implies $M \leq \lim_{k\to\infty} r_{2k} \leq 1$ and $N \leq \lim_{k\to\infty} s_{2k-1} \leq 1$. Thus

$$\frac{1}{\lim\limits_{k\to\infty} r_{2k}} \leq \frac{1}{M} \quad \text{and} \quad \frac{1}{\lim\limits_{k\to\infty} s_{2k-1}} \leq \frac{1}{N},$$

so

$$\frac{1}{\lim\limits_{k\to\infty} r_{2k} \lim\limits_{k\to\infty} s_{2k-1}} \leq \frac{1}{MN}.$$

Since $m < M$ and $n < N$, we have

$$\frac{mn}{\lim\limits_{k\to\infty} r_{2k} \lim\limits_{k\to\infty} s_{2k-1}} < \frac{mn}{MN} < \frac{MN}{MN} = 1.$$

This is

$$\lim_{k\to\infty} \frac{q_{2k+1}}{q_{2k}} = \frac{mn}{\lim\limits_{k\to\infty} r_{2k} s_{2k-1}} < 1,$$

which completes this proof of convergence of Σq_{2k}.

A similar approach could surely be used to show the convergence of Σq_{2k+1}.

3. The Series in Terms of Hyperbolic Functions

Another proof of convergence of the series of charges $\sum\limits_{k=0}^{\infty} q_{2k}$ given by (2. 1) can be accomplished by expressing the series in a different form which lends itself to analysis by the ratio test. This is done by solving a linear finite difference equation which the sequence of charges $\{q_{2k}\}$ is shown to satisfy.

Also, this new form of the series, involving hyperbolic functions, is more useful for purposes of numerical computation than the original form [6, pg. 119].

From the recursion relations (3.1) we have

$$q_{2k+2} = -q_{2k+1} \frac{m}{r_{2k}}$$

or

$$q_{2k+2} = q_{2k} \frac{n}{s_{2k-1}} \frac{m}{r_{2k}}$$

and also

$$r_{2k} = 1 - \frac{n^2}{s_{2k-1}} .$$

We eliminate r_{2k} from the above equations, obtaining

$$\frac{q_{2k+2}}{q_{2k}} = \frac{mn}{s_{2k-1}\left(1 - \frac{n^2}{s_{2k-1}}\right)}$$

or

(3.4) $$\frac{q_{2k+2}}{q_{2k}} = \frac{mn}{s_{2k-1} - n^2} .$$

Replacing k by $k+1$ yields

(3.5) $$\frac{q_{2k}}{q_{2k-2}} = \frac{mn}{s_{2k-3} - n^2} .$$

We also have

$$(3.6) \qquad s_{2k-1} = 1 - \frac{m^2}{r_{2k-2}} = 1 - \frac{m^2}{1 - \frac{n^2}{s_{2k-3}}} .$$

Next, we eliminate s_{2k-1} and s_{2k-3} between (3.4), (3.5), (3.6).
From (3.4) and (3.6) we get

$$\frac{q_{2k+2}}{q_{2k}} = \frac{mn}{1 - \frac{s_{2k-3}m^2}{s_{2k-3}-n^2} - n^2}$$

and from (3.5)

$$s_{2k-3} - n^2 = \frac{mnq_{2k-2}}{q_{2k}} .$$

From these equations we obtain

$$\frac{q_{2k+2}}{q_{2k}} = \frac{mn}{1 - \frac{m^2\left(n^2 + \frac{mnq_{2k-2}}{q_{2k}}\right)}{\frac{mnq_{2k-2}}{q_{2k}}} - n^2}$$

$$= \frac{mn}{1 - m^2\left(\frac{n}{m}\frac{q_{2k}}{q_{2k-2}} + 1\right) - n^2} =$$

$$= \frac{mn}{1-(mn\frac{q_{2k}}{q_{2k-2}}+m^2)-n^2}$$

$$= \frac{mn}{1-m^2-n^2-mn\frac{q_{2k}}{q_{2k-2}}},$$

so that

$$\frac{q_{2k}}{q_{2k+2}} = \frac{1-m^2-n^2-mn\frac{q_{2k}}{q_{2k-2}}}{mn}$$

$$= \frac{1-m^2-n^2}{mn} - \frac{q_{2k}}{q_{2k-2}},$$

whence

$$\frac{1}{q_{2k+2}} = \frac{1-m^2-n^2}{mn}\frac{1}{q_{2k}} - \frac{1}{q_{2k-2}}.$$

We have thus obtained a second order homogeneous linear difference equation for $1/q_{2k}$:

$$\frac{1}{q_{2k+2}} + \frac{1}{q_{2k-2}} = \frac{1-m^2-n^2}{mn}\frac{1}{q_{2k}}. \tag{3.7}$$

To solve (3.7), we presume a solution $q_{2k} = u^{-2k}$. Substituting this into (3.7) we obtain

$$u^{2k+2} + u^{2k-2} = \frac{1-m^2-n^2}{mn}u^{2k}.$$

If $u \neq 0$,

$$u^2 + u^{-2} = \frac{1-m^2-n^2}{mn},$$

whence u satisfies the quadratic in u^2

$$u^4 - \frac{1-m^2-n^2}{mn} u^2 + 1 = 0 .$$

Therefore

$$u^2 = \frac{\frac{1-m^2-n^2}{mn} \pm \sqrt{\left(\frac{1-m^2-n^2}{mn}\right)^2 - 4}}{2} . \tag{3.8}$$

Now let $\cosh \alpha = \frac{1-m^2-n^2}{2mn}$. This is possible since $\cosh^{-1} y$ exists for all $y \geq 1$, and $\frac{1-m^2-n^2}{2mn} > 1$ which we easily verify:

$$\frac{1-m^2-n^2}{2mn} = \frac{1 - \frac{a^2+b^2}{c^2}}{2\frac{ab}{c^2}} = \frac{c^2-(a^2+b^2)}{2ab} .$$

Now, $a, b, c > 0$ and $c > a+b$ imply $c^2 > (a+b)^2$, so

$$\frac{1-m^2-n^2}{2mn} = \frac{c^2-(a+b)^2}{2ab} + \frac{2ab}{2ab}$$

$$= \frac{c^2-(a+b)^2}{2ab} + 1 > 1.$$

Substituting $\cosh \alpha$ in (3.8) gives

$$u^2 = \cosh \alpha \pm \sqrt{\cosh^2 \alpha - 1}$$

$$= \cosh \alpha \pm \sinh \alpha .$$

Therefore $u^{2k} = (u^2)^k = (\cosh \alpha + \sinh \alpha)^k$ is a solution, as is $(\cosh \alpha - \sinh \alpha)^k$. Then the linear combination

$$D_1(\cosh \alpha + \sinh \alpha)^k + D_2(\cosh \alpha - \sinh \alpha)^k = D_1 e^{k\alpha} + D_2 e^{-k\alpha}$$

is also a solution. But this is equivalent to the linear combination

$$A \cosh (k\alpha) + B \sinh (k\alpha).$$

Therefore a solution of the difference equation (3.7) is

$$\frac{1}{q_{2k}} = A \cosh (k\alpha) + B \sinh (k\alpha)$$

in which A and B are to be determined from the first two q's. For $k = 0$, we have

$$\frac{1}{q_0} = A \quad \text{or} \quad A = \frac{1}{K} = \frac{1}{4\pi\epsilon a} .$$

Then

$$\frac{1}{q_2} = \frac{1}{K} \cosh \alpha + B \sinh \alpha$$

or

$$-\frac{1-n^2}{mq_1} = \frac{1}{K}\cosh\alpha + B\sinh\alpha$$

whence

$$\frac{1-n^2}{mnK} = \frac{1}{K}\cosh\alpha + B\sinh\alpha .$$

Since

$$\cosh\alpha = \frac{1-m^2-n^2}{2mn}$$

we have

$$2\cosh\alpha + \frac{m}{n} = \frac{1-m^2-n^2}{mn} + \frac{m}{n} = \frac{1-n^2}{mn},$$

so

$$\frac{1}{K}(2\cosh\alpha + \frac{m}{n}) = \frac{1}{K}\cosh\alpha + B\sinh\alpha$$

or

$$\frac{1}{K}(\cosh\alpha + \frac{m}{n}) = B\sinh\alpha .$$

With B so determined, we have

$$\frac{1}{q_{2k}} = \frac{1}{K}\cosh(k\alpha) + \frac{\cosh\alpha + \frac{m}{n}}{K\sinh\alpha}\sinh(k\alpha)$$

$$= \frac{\sinh\alpha\cosh(k\alpha) + \cosh\alpha\sinh(k\alpha) + \frac{m}{n}\sinh(k\alpha)}{K\sinh\alpha}$$

$$= \frac{\sinh(k+1)\alpha + \frac{m}{n}\sinh(k\alpha)}{K\sinh\alpha}$$

$$= \frac{b\sinh(k+1)\alpha + a\sinh(k\alpha)}{bK\sinh\alpha} .$$

From the above, we have

$$q_{2k} = 4\pi\epsilon ab \sinh \alpha[b \sinh(k+1)\alpha + a \sinh(k\alpha)]^{-1}$$

and our series of charges on (1) is

$$\sum_{k=0}^{\infty} q_{2k} = 4\pi\epsilon ab \sinh \alpha \sum_{k=0}^{\infty} [b \sinh(k+1)\alpha + a \sinh(k\alpha)]^{-1}$$

$$= 4\pi\epsilon ab \sinh\alpha \sum_{k=1}^{\infty} [b \sinh(k\alpha) + a \sinh(k-1)\alpha]^{-1} .$$

We are now in a position to prove convergence of the series in this form:

$$\sum_{k=0}^{\infty} q_{2k} = 4\pi\epsilon ab \sinh \alpha \sum_{k=1}^{\infty} \frac{1}{b \sinh(k\alpha) + a \sinh(k-1)\alpha}$$

$$= 4\pi\epsilon ab \sinh \alpha \sum_{k=1}^{\infty} c_k \quad \text{where} \quad a > 0,\ b > 0.$$

Also, let $\alpha > 0$. The case $\alpha < 0$ will be clarified later.

We employ the ratio test.

$$\frac{c_{k+1}}{c_k} = \frac{b \sinh(k\alpha) + a \sinh(k-1)\alpha}{b \sinh(k+1)\alpha + a \sinh(k\alpha)}$$

$$= \frac{b \sinh(k\alpha) + a \sinh k\alpha \cosh \alpha - a \cosh(k\alpha) \sinh \alpha}{b \sinh(k\alpha) \cosh \alpha + b \cosh(k\alpha) \sinh \alpha + a \sinh(k\alpha)}$$

$$= \frac{b + a \cosh \alpha - a \coth(k\alpha) \sinh \alpha}{b \cosh \alpha + b \coth(k\alpha) \sinh \alpha + a} .$$

Since $\lim\limits_{k \to \infty} \coth(k\alpha) = 1$ for $\alpha > 0$, we have

$$\lim_{k \to \infty} \frac{c_{k+1}}{c_k} = \frac{b + a \cosh \alpha - a \sinh \alpha}{a + b \cosh \alpha + b \sinh \alpha} .$$

We now proceed to show that this limit is less than one. The problem is divided into three cases.

<u>Case 1</u>: If $a = b$, then clearly

$$1 + \cosh \alpha - \sinh \alpha < 1 + \cosh \alpha + \sinh \alpha$$

and

$$\lim_{k \to \infty} \frac{c_{k+1}}{c_k} < 1,$$

since $\alpha > 0$ implies $\sinh \alpha > 0$.

<u>Case 2</u>: Suppose $b > a$. Then $b-a < (b-a)\cosh \alpha$ since $\alpha > 0$ implies $\cosh \alpha > 1$. Therefore,

$$b + a \cosh \alpha < a + b \cosh \alpha$$

and since $\sinh \alpha > 0$, we can conclude

$$b + a\cosh\alpha - a\sinh\alpha < a + b\cosh\alpha + b\sinh\alpha$$

and the result

$$\lim_{k\to\infty} \frac{c_{k+1}}{c_k} < 1$$

obtains.

<u>Case 3</u>: Suppose $a > b$. Since $\alpha > 0$, $1-e^{\alpha} < 0$ and $1-e^{-\alpha} > 0$, so $1-e^{\alpha}/1-e^{-\alpha} < 0$. Therefore $a/b > 1-e^{\alpha}/1-e^{-\alpha}$

$$\Rightarrow \quad a(1-e^{-\alpha}) > b(1-e^{\alpha})$$

$$\Rightarrow \quad a(1-e^{-\alpha}) + b(e^{\alpha}-1) > 0$$

$$\Rightarrow \quad (a-b) + (be^{\alpha}-ae^{-\alpha}) > 0$$

$$\Rightarrow \quad (a-b) + \frac{1}{2}[be^{\alpha}+be^{-\alpha}-ae^{\alpha}-ae^{-\alpha}+ae^{\alpha}-ae^{-\alpha}+be^{\alpha}-be^{-\alpha}] > 0$$

$$\Rightarrow \quad (a-b) + (b-a)\cosh\alpha + (a+b)\sinh\alpha > 0$$

$$\Rightarrow \quad b + a\cosh\alpha - a\sinh\alpha < a + b\cosh\alpha + b\sinh\alpha$$

and therefore

$$\lim_{k\to\infty} \frac{c_{k+1}}{c_k} < 1.$$

Finally, we clarify our earlier assumption specifying $\alpha > 0$. Recall that we first introduced α by letting

$$\cosh\alpha = \frac{1-m^2-n^2}{2mn} .$$

Of course α cannot be zero, for then $\cosh \alpha = 1$. But $m+n < 1$, so $m^2+2mn+n^2 < 1$, or $2mn < 1-m^2-n^2$. Therefore

$$\frac{1-m^2-n^2}{2mn} > 1.$$

But α can be taken to be either the positive (principal) branch of $\cosh^{-1}(\frac{1-m^2-n^2}{2mn})$ or the negative branch. If we select the negative branch, the proof holds equally well, as we shall now demonstrate.

Suppose $\alpha < 0$. Then since $\lim_{k \to \infty} \coth(k\alpha) = -1$, we have

$$\lim_{k \to \infty} \frac{c_{k+1}}{c_k} = \frac{b + a \cosh \alpha + a \sinh \alpha}{a + b \cosh \alpha - b \sinh \alpha} .$$

Again we consider three possibilities in demonstrating that this limit is less than one.

<u>Case 1</u>: If $a = b$ we must have

$$1 + \cosh \alpha + \sinh \alpha < 1 + \cosh \alpha - \sinh \alpha$$

which is true since $\sinh \alpha < 0$.

<u>Case 2</u>: Suppose $b > a$. Since $\alpha \neq 0$, $\cosh \alpha > 1$, and $b-a < (b-a)\cosh \alpha$, or $b + a \cosh \alpha < a + b \cosh \alpha$. Now, since $\sinh \alpha < 0$,

$$b + a \cosh \alpha + a \sinh \alpha < a + b \cosh \alpha - b \sinh \alpha$$

and therefore

$$\lim_{k \to \infty} \frac{c_{k+1}}{c_k} < 1 .$$

Case 3: Suppose $a < b$. Since $\alpha < 0$, $1-e^{-\alpha} < 0$ and $1-e^{\alpha} > 0$. Therefore

$$\frac{a}{b} > \frac{1-e^{-\alpha}}{1-e^{\alpha}}$$

$$\Rightarrow \quad (a-b) + (be^{-\alpha}-ae^{\alpha}) > 0$$

$$\Rightarrow \quad (a-b) + \frac{1}{2}[be^{\alpha}+be^{-\alpha}-ae^{\alpha}-ae^{-\alpha}-ae^{\alpha}+ae^{-\alpha}-be^{\alpha}+be^{-\alpha}] > 0$$

$$\Rightarrow \quad (a-b) + (b-a)\cosh \alpha - (a+b)\sinh \alpha > 0$$

$$\Rightarrow \quad b + a \cosh \alpha + a \sinh \alpha < a + b \cosh \alpha - b \sinh \alpha$$

and therefore

$$\lim_{k \to \infty} \frac{c_{k+1}}{c_k} < 1.$$

4. The Definite Integrals Approach

The convergence of Σq_{2k} will now be proved by a considerably more involved procedure whereby the series is expressed in terms of definite improper integrals.

We now write the series $\sum_{k=0}^{\infty} q_{2k}$ in a somewhat different form. Recall the solution $u^{2k} = D_1 e^{k\alpha} + D_2 e^{-k\alpha}$ of the difference equation (3. 7). Let $\gamma = e^{-\alpha}$. Supposing $\alpha > 0$, let $\gamma = e^{-\alpha}$. γ is clearly the smallest root of the quadratic equation

$$u^4 - \frac{1-m^2-n^2}{mn} u^2 + 1 = 0.$$

Since the product of the roots is unity and they are unequal (otherwise $\alpha = 0$) we have $\gamma < 1$. Therefore our solution of (3. 7) can be written

$$\frac{1}{q_{2k}} = \frac{D_1}{\gamma^k} + D_2 \gamma^k ,$$

so

$$\frac{1}{q_0} = D_1 + D_2 \quad \text{or} \quad D_1 + D_2 = \frac{1}{K}$$

and

$$\frac{1}{q_2} = \frac{D_1}{\gamma} + D_2\gamma \quad \text{or} \quad \frac{1}{q_2} = \frac{D_1 + D_2\gamma^2}{\gamma} .$$

Therefore

$$q_2 = \frac{-q_1 m}{1-n^2} = \frac{Kmn}{1-n^2} = \frac{\gamma}{D_1 + D_2\gamma^2} .$$

We now have

$$D_1 + D_2 = \frac{1}{K}$$

or

$$\frac{\gamma}{D_1+D_2\gamma^2} = \frac{Kmn}{1-n^2}$$

or

$$D_1 + D_2\gamma^2 = \frac{\gamma(1-n^2)}{Kmn}$$

or

$$(\frac{1}{K}-D_1)\gamma^2 + D_1 = \frac{\gamma(1-n^2)}{Kmn}$$

or

$$D_1(1-\gamma^2) = \frac{\gamma(1-n^2)}{Kmn} - \frac{\gamma^2}{K} .$$

But we have

$$\gamma^2 - \frac{1-m^2-n^2}{mn}\gamma + 1 = 0$$

or

$$\gamma^2 = \frac{1-m^2-n^2}{mn}\gamma - 1$$

or

$$KD_1(1-\gamma^2) = \gamma\frac{1-n^2}{mn} - \frac{1-m^2-n^2}{mn}\gamma + 1$$

or

(3.9) $$KD_1(1-\gamma^2) = \frac{m}{n}\gamma + 1.$$

It will be useful to note that the value of D_1 given by the preceding equation is

$$D_1 = \frac{1}{K(1-\xi^2)} \tag{3.10}$$

where $\xi = m + n\gamma$. We now verify this by starting with the following algebraic identity:

$$\frac{\gamma}{mn} - 2\frac{n}{m}\gamma = -\frac{n^3}{m}\gamma + \frac{m^3}{n}\gamma + \frac{m}{n}\gamma - \frac{m^3}{n}\gamma - mn\gamma - \frac{n}{m}\gamma$$
$$+ mn\gamma + \frac{n^3}{m}\gamma + \frac{\gamma}{mn} - \frac{m}{n}\gamma - \frac{n}{m}\gamma .$$

Then

$$\frac{\gamma}{mn} - 2\frac{n}{m}\gamma - 1 = m^2 - n^2 - \frac{n^3}{m}\gamma + \frac{m^3}{n}\gamma + \frac{m}{n}(1-m^2-n^2)\gamma - m^2$$
$$- \frac{n}{m}(1-m^2-n^2)\gamma + n^2 + \frac{1-m^2-n^2}{mn}\gamma - 1$$

or

$$\frac{\gamma}{mn} - \frac{n}{m}\gamma - 1 = m^2 - n^2 + \frac{n}{m}\gamma - \frac{n^3}{m}\gamma + \frac{m^3}{n}\gamma + m^2\gamma^2 - n^2\gamma^2 + \gamma^2$$

or

$$\frac{1-m^2-n^2}{mn}\gamma - 1 = -\frac{m}{n}\gamma + \frac{m^3}{n}\gamma + m^2\gamma^2 + \gamma^2 - n^2\gamma^2 - mn\gamma + m^2 \tag{3.11}$$
$$+ mn\gamma + \frac{n}{m}\gamma - \frac{n^3}{m}\gamma - n^2 .$$

Now, since

$$\xi^2 = (m+n\gamma)^2 = m^2 + 2mn\gamma + n^2\gamma^2$$

$$= m^2 + 2mn\gamma + n^2\left(\frac{1-m^2-n^2}{mn}\right)\gamma - n^2$$

$$= m^2 + 2mn\gamma + \frac{n}{m}\gamma - mn\gamma - \frac{n^3}{m}\gamma - n^2$$

$$= m^2 + mn\gamma + \frac{n}{m}\gamma - \frac{n^3}{m}\gamma - n^2 .$$

(3. 11) is equivalent to

$$\gamma^2 = -\frac{m}{n}\gamma + \frac{m}{n}\gamma\xi^2 + \xi^2$$

or

$$1 - \gamma^2 = \frac{m}{n}\gamma + 1 - \frac{m}{n}\gamma\xi^2 - \xi^2$$

or

$$\frac{1-\gamma^2}{1-\xi^2} = \frac{m}{n}\gamma + 1$$

or

$$1 - \gamma^2 = \left(\frac{m}{n}\gamma+1\right)(1-\xi^2)$$

or

$$\frac{K(1-\gamma^2)}{K(1-\xi^2)} = \frac{m}{n}\gamma + 1 ,$$

which verifies (3. 10).

Now, this equation, together with (3. 9) proves that $D_1 = \dfrac{1}{K(1-\xi^2)}$.

Now,

$$D_2 = \frac{1}{K} - D_1 = \frac{1}{K}\left[1 - \frac{1}{1-\xi^2}\right] = \frac{1}{K}\,\frac{-\xi^2}{1-\xi^2}$$

so

$$\frac{1}{q_{2k}} = \frac{-\xi^2\gamma^k}{K(1-\xi^2)} + \frac{1}{K(1-\xi^2)\gamma^k} = \frac{1-\xi^2\gamma^{2k}}{K(1-\xi^2)\gamma^k}$$

or

$$q_{2k} = \frac{K(1-\xi^2)\gamma^k}{1-\xi^2\gamma^{2k}} .$$

Thus the series $\sum_{k=0}^{\infty} q_{2k}$ becomes

$$K(1-\xi^2)\sum_{k=0}^{\infty}\frac{\gamma^k}{1-\xi^2\gamma^{2k}} .$$

We next evaluate the following definite integral, which will play a prominent role in the analysis of Σq_{2k} :

$$(3.12) \qquad \int_0^{\infty}\frac{\sin px}{e^{2\pi x}-1}\,dx \qquad (p \neq 0).$$

Note first that the integral is not actually improper at the lower limit since

$$\lim_{x\downarrow 0}\frac{\sin px}{e^{2\pi x}-1} = \lim_{x\downarrow 0}\frac{p\cos px}{2\pi e^{2\pi x}} = \frac{p}{2\pi} .$$

At the outset, we easily verify the convergence of (3. 12) by demonstrating that

$$\int_1^\infty \frac{\sin px}{e^{2\pi x}-1}\,dx$$

converges: Using the comparison test and the fact that $|\sin px| < 1$, we only need to show that

$$\int_1^\infty \frac{dx}{e^{2\pi x}-1}$$

exists. This is

$$\lim_{h\to\infty}\int_1^h \frac{e^{-2\pi x}dx}{1-e^{-2\pi x}} = \lim_{h\to\infty} \frac{1}{2\pi}\ln(1-e^{-2\pi x})\Big|_1^h$$

which clearly exists.

The integral (3. 12) will be evaluated by computing an appropriate complex contour integral: Consider

$$(3.13) \qquad f(z) = \frac{e^{ipz}}{e^{2\pi z}-1}$$

which has simple poles at $z = \pm i, \pm 2i, \ldots$. We will integrate this function around the indicated contour Ω, illustrated in Fig. 4.

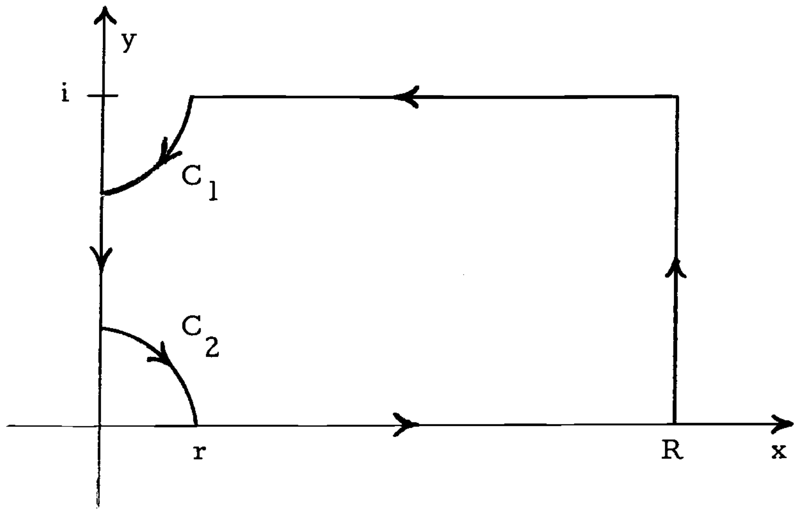

Fig. 4. The contour Ω.

r denotes the common radius of circular arcs C_1, C_2. On the circular portions of Ω, $z = x + iy$ can be represented in polar form as follows:

$$C_1:\ z - i = re^{i\theta}, \qquad -\pi/2 \leq \theta \leq 0$$

$$C_2:\ z = re^{i\theta}, \qquad 0 \leq \theta \leq \pi/2.$$

Since $f(z)$ is analytic within and on the contour, we obtain

$$\int_{\Omega} \frac{e^{ipz}}{e^{2\pi z}-1}\, dz = 0 ,$$

i.e.,

$$(3.14) \qquad \int_r^R \frac{e^{ipx}}{e^{2\pi x}-1}\, dx + \int_0^1 \frac{e^{ip(R+iy)}}{e^{2\pi(R+iy)}-1}\, idy + \int_R^r \frac{e^{ip(x+i)}}{e^{2\pi(x+i)}-1}\, dx$$

$$+ \int_0^{-\pi/2} \frac{e^{ip(i+re^{i\theta})}}{e^{2\pi(i+re^{i\theta})}-1}\, ire^{i\theta} d\theta + \int_{1-r}^{r} \frac{e^{-py}}{e^{2\pi iy}-1}\, idy \ +$$

$$+ \int_{\pi/2}^{0} \frac{e^{ipre^{i\theta}}}{e^{2\pi re^{i\theta}}-1} \, rie^{i\theta} d\theta = 0.$$

The above six integrals, which we now evaluate, will be denoted by $I_1, \ldots, I_6$ respectively. We evaluate I_2 first:

$$I_2 = \int_0^1 \frac{e^{ip(R+iy)}}{e^{2\pi(R+iy)}-1} \, idy = i \int_0^1 \frac{e^{-py} e^{ipR}}{e^{2\pi R} e^{2\pi iy}-1} \, dy \, .$$

$$|I_2| \leq \int_0^1 \frac{e^{-py}}{|e^{2\pi R} e^{2\pi iy}-1|} \, dy \, .$$

On the segment of Ω, $z = R + iy$, $(0 \leq y \leq 1)$, we have $e^{-py} \leq e^{|p|}$. Also

$$|e^{2\pi R} e^{2\pi iy}-1| \geq |e^{2\pi R} e^{2\pi iy}| - 1 = e^{2\pi R} - 1.$$

Thus

$$|I_2| \leq \frac{e^{|p|}}{e^{2\pi R}-1} ,$$

so $I_2 \to 0$ as $R \to \infty$.

Now consider I_4 and its limit as $r \to 0$. We can interchange the limit and integral operations by invoking a corollary of the following familiar theorem.

<u>Theorem</u>: If $f(x,t)$ is continuous for $a \leq t \leq b$, and $A \leq x \leq B$, then

$$F(x) = \int_a^b f(x,t)dt$$

is a continuous function of x on $A \leq x \leq B$. We apply the following corollary of the above theorem to I_4.

<u>Corollary:</u> Under the conditions of the above theorem we have

$$\lim_{x \to x_0} \int_a^b f(x,t)dt = \int_a^b \lim_{x \to x_0} f(x,t)dt = \int_a^b f(x_0,t)dt$$

if $A \leq x_0 \leq B$.

Therefore,

$$\lim_{r \to 0} I_4 = \lim_{r \to 0} e^{-p} \int_0^{-\pi/2} \frac{re^{ipre^{i\theta}} ie^{i\theta}}{e^{2\pi re^{i\theta}} - 1} d\theta$$

$$= e^{-p} \int_0^{-\pi/2} ie^{i\theta} \lim_{r \to 0} \left(\frac{re^{ipre^{i\theta}}}{e^{2\pi re^{i\theta}} - 1} \right) d\theta$$

$$= e^{-p} \int_0^{-\pi/2} ie^{i\theta} \lim_{r \to 0} \left(\frac{ripe^{i\theta} e^{ipre^{i\theta}} + e^{ipre^{i\theta}}}{2\pi e^{i\theta} e^{2\pi re^{i\theta}}} \right) d\theta$$

(using L'Hopital's Rule)

$$= e^{-p} \int_0^{-\pi/2} ie^{i\theta} \frac{1}{2\pi e^{i\theta}} d\theta = \frac{e^{-p} i}{2\pi} \left(-\frac{\pi}{2}\right) = -\frac{i}{4} e^{-p}.$$

I_6 lends itself to similar analysis:

$$\lim_{r \to 0} I_6 = \int_{\pi/2}^{0} ie^{i\theta} \lim_{r \to 0} \left(\frac{re^{ipre^{i\theta}}}{e^{2\pi re^{i\theta}}-1} \right) d\theta$$

$$= \int_{\pi/2}^{0} ie^{i\theta} \lim_{r \to 0} \left(\frac{ripe^{i\theta} e^{ipre^{i\theta}} + e^{ipre^{i\theta}}}{2\pi e^{i\theta} e^{2\pi re^{i\theta}}} \right) d\theta$$

(using L'Hopital's Rule)

$$= \int_{\pi/2}^{0} ie^{i\theta} \frac{1}{2\pi e^{i\theta}} d\theta = -\frac{i}{4}.$$

Note that the sum of I_1 and I_5 is

$$\int_r^R \frac{e^{ipx}}{e^{2\pi x}-1} dx - e^{-p} \int_r^R \frac{e^{ipx}}{e^{2\pi x}-1} dx = (1-e^{-p}) \int_r^R \frac{e^{ipx}}{e^{2\pi x}-1} dx .$$

Finally, taking limits as $r \to 0$, $R \to \infty$, in (3.14) yields

$$(1-e^{-p}) \int_0^\infty \frac{e^{ipx}}{e^{2\pi x}-1} dx - \frac{i}{4} e^{-p} - \frac{i}{4} + \lim_{r \to 0} I_5 = 0.$$

Taking the imaginary part in the above equation we obtain

$$(1-e^{-p}) \int_0^\infty \frac{\sin px}{e^{2\pi x}-1} dx - \frac{1}{4}(e^{-p}+1) + \mathrm{Im}(\lim_{r \to 0} I_5) = 0$$

or

$$(1-e^{-p}) \int_0^\infty \frac{\sin px}{e^{2\pi x}-1} dx = \frac{1}{4}(e^{-p}+1) - \lim_{r \to 0} (\mathrm{Im} I_5)$$

or

$$\int_0^\infty \frac{\sin px}{e^{2\pi x}-1}\,dx = \frac{1}{4}\,\frac{e^p+1}{e^p-1} - \frac{\lim\limits_{r\to 0}\,(\mathrm{Im} I_5)}{1-e^{-p}}\,.$$

Also, I_5 is $\displaystyle\int_{1-r}^{r} \frac{ie^{-py}}{\cos 2\pi y - 1 + i\sin 2\pi y}\,dy$, and we readily find that its imaginary part is

$$\int_{1-r}^{r} \frac{e^{-py}(\cos 2\pi y-1)}{(\cos 2\pi y-1)^2+\sin^2 2\pi y}\,dy$$

$$= \int_{1-r}^{r} \frac{e^{-py}(\cos 2\pi y-1)}{1-2\cos 2\pi y+1}\,dy = -\frac{1}{2}\int_{1-r}^{r} e^{-py}\,dy$$

$$= \frac{1}{2p}\,[e^{-pr}-e^{-p(1-r)}]\,.$$

Therefore $\lim\limits_{r\to 0}\,(\mathrm{Im} I_5) = \frac{1}{2p}\,(1-e^{-p})$, and we finally have

$$\int_0^\infty \frac{\sin px}{e^{2\pi x}-1}\,dx = \frac{1}{4}\,\frac{e^p+1}{e^p-1} - \frac{1}{2p}\,. \tag{3.15}$$

We now return to the series

$$\sum_{k=0}^{\infty} q_{2k} = K(1-\xi^2)\sum_{k=0}^{\infty}\frac{\gamma^k}{1-\xi^2\gamma^{2k}} \qquad (\xi = m+n\gamma).$$

Ignoring the factor $K(1-\xi^2)$, we proceed to express the series in terms of definite integrals. In (3.15) set $p = \log(\xi^2\gamma^{2k})$ and obtain

$$\int_0^\infty \frac{\sin[(\log \xi^2\gamma^{2k})t]}{e^{2\pi t}-1}\,dt = \frac{1}{4}\,\frac{\xi^2\gamma^{2k}+1}{\xi^2\gamma^{2k}-1} - \frac{1}{2\log(\xi^2\gamma^{2k})}$$

or

$$\frac{1}{2}\,\frac{\xi^2\gamma^{2k}+1}{1-\xi^2\gamma^{2k}} = -\frac{1}{\log(\xi^2\gamma^{2k})} - 2\int_0^\infty \frac{\sin[(\log \xi^2\gamma^{2k})t]}{e^{2\pi t}-1}\,dt\,.$$

Now multiply each term by γ^k and also note that

$$\frac{1+\xi^2\gamma^{2k}}{2(1-\xi^2\gamma^{2k})} = \frac{2-(1-\xi^2\gamma^{2k})}{2(1-\xi^2\gamma^{2k})} = \frac{1}{1-\xi^2\gamma^{2k}} - \frac{1}{2}\,.$$

We get

$$\frac{\gamma^k}{1-\xi^2\gamma^{2k}} = \frac{\gamma^k}{2} - \frac{\gamma^k}{\log(\xi^2\gamma^{2k})} - 2\int_0^\infty \frac{\gamma^k\sin[(\log \xi^2\gamma^{2k})t]}{e^{2\pi t}-1}\,dt\,.$$

Sum on k and note that since $0 < \gamma < 1$ we have $\sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}$.
Then

$$(3.16)\qquad \sum_{k=0}^{\infty} \frac{\gamma^k}{1-\xi^2\gamma^{2k}} = \frac{1}{2(1-\gamma)} - \sum_{k=0}^{\infty} \frac{\gamma^k}{\log(\xi^2\gamma^{2k})}$$

$$- 2\sum_{k=0}^{\infty}\int_0^\infty \frac{\gamma^k\sin[(\log \xi^2\gamma^{2k})t]}{e^{2\pi t}-1}\,dt\,.$$

Consider the last term in the above equation. Ignore for the present the justification for the interchange of summation and integration. Interchanging, we have

$$\int_0^\infty \sum_{k=0}^{\infty} \frac{\gamma^k \sin[(\log \xi^2 \gamma^{2k})t]}{e^{2\pi t}-1}\, dt \,.$$

Factoring $\frac{1}{e^{2\pi t}-1}$ outside the summation, we re-express the series as follows:

$$\sum_{k=0}^{\infty} \gamma^k \sin[(\log \xi^2 \gamma^{2k})t] = \sum_{k=0}^{\infty} \gamma^k \frac{e^{i(\log \xi^2 \gamma^{2k})t} - e^{-i(\log \xi^2 \gamma^{2k})t}}{2i}$$

$$= \sum_{k=0}^{\infty} \frac{\gamma^k}{2i} [(\xi^2 \gamma^{2k})^{it} - (\xi^2 \gamma^{2k})^{-it}]$$

or

$$(3.17) \qquad \frac{\xi^{2it}}{2i} \sum_{k=0}^{\infty} \gamma^{(1+2it)k} - \frac{\xi^{-2it}}{2i} \sum_{k=0}^{\infty} \gamma^{(1-2it)k} \,.$$

Therefore the series becomes

$$\frac{\xi^{2it}}{2i} \frac{1}{1-\gamma^{1+2it}} - \frac{\xi^{-2it}}{2i} \frac{1}{1-\gamma^{1-2it}}$$

$$= \frac{1}{2i} \frac{\xi^{2it}(1-\gamma^{1-2it}) - \xi^{-2it}(1-\gamma^{1+2it})}{(1-\gamma^{1+2it})(1-\gamma^{1-2it})} =$$

$$= \frac{1}{2i} \frac{\xi^{2it}-\xi^{-2it}-\gamma\xi^{2it}\gamma^{-2it}+\gamma\xi^{-2it}\gamma^{2it}}{1-\gamma(\gamma^{2it}+\gamma^{-2it})+\gamma^2}$$

$$= \frac{1}{2i} \frac{e^{2it\log\xi}-e^{-2it\log\xi}-\gamma[(\frac{\xi}{\gamma})^{2it}-(\frac{\xi}{\gamma})^{-2it}]}{1-\gamma(\gamma^{2it}+\gamma^{-2it})+\gamma^2}$$

$$= \frac{1}{2i} \frac{e^{2it\ln\xi}-e^{-2it\ln\xi}-\gamma[e^{2it\log(\xi/\gamma)}-e^{-2it\log(\xi/\gamma)}]}{1-2\gamma\left(\frac{e^{2it\log\gamma}+e^{-2it\log\gamma}}{2}\right)+\gamma^2}$$

$$= \frac{\sin(2t\log\xi)-\gamma\sin(2t\log\frac{\xi}{\gamma})}{1-2\gamma\cos(2t\log\gamma)+\gamma^2} .$$

We now proceed to justify the interchange of integration and summation

(3. 18)

$$\sum_{k=0}^{\infty}\int_0^{\infty} \frac{\gamma^k \sin[(\log \xi^2\gamma^{2k})t]}{e^{2\pi t}-1}\, dt = \int_0^{\infty}\sum_{k=0}^{\infty} \frac{\gamma^k \sin[(\log \xi^2\gamma^{2k})t]}{e^{2\pi t}-1}\, dt$$

performed earlier. We first rewrite the integral over $[0,\infty)$ as a sum of integrals over the intervals $[0,1]$ and $[1,\infty)$, denoting these latter integrals simply by $\int_0^1$ and $\int_1^\infty$ respectively. After showing we can interchange limits in the case of $\int_0^1$, we will invoke the Lebesgue Dominated Convergence Theorem to justify the interchange in the case of $\int_1^\infty$.

Interchange of limits in the case of $\int_0^1$, i.e.,

$$\sum_{k=0}^{\infty} \int_0^1 = \int_0^1 \sum_{k=0}^{\infty}$$

is justified since

$$(3.19) \qquad \sum_{k=0}^{\infty} \frac{\gamma^k \sin[(\log \xi^2 \gamma^{2k})t]}{e^{2\pi t}-1}$$

is a uniformly convergent series on $0 \leq t \leq 1$, as we will now show with the aid of the following lemma.

Lemma: On $0 \leq t \leq 1$, suppose

(1) $F(t)$ and $G(t)$ are continuous.

(2) $|F(t)| \leq 1$, $0 \leq |G(t)| \leq M$.

(3) $G(t)$ is monotone and $G(0) = 0$.

(4) $\lim_{t \downarrow 0} \frac{F(t)}{G(t)} = L$.

Then on $0 \leq t \leq 1$, $\frac{F(t)}{G(t)}$ is bounded, i.e., $\left|\frac{F(t)}{G(t)}\right| \leq M_2 < \infty$ (defining $\frac{F(0)}{G(0)} = L$).

Proof: Since $\lim_{t \downarrow 0} \frac{F}{G}$ exists, $\frac{F}{G}$ is bounded on some interval $0 \leq t \leq t_1$ where $0 < t_1 < 1$. Then on the interval $t_1 \leq t \leq 1$, $\frac{F}{G}$ is a quotient of continuous functions with G nonvanishing, and so is a continuous function on this closed interval, and is hence bounded on this interval.

Now returning to the series (3.19), let $F(t) = \sin[(\log \xi^2 \gamma^{2k})t]$ and $G(t) = e^{2\pi t} - 1$. Noting that

$$\lim_{t \downarrow 0} \frac{\sin[(\log \xi^2 \gamma^{2k})t]}{e^{2\pi t}-1} = \frac{\log(\xi^2 \gamma^{2k})}{2\pi}$$

we see the above lemma applies. Thus

$$\left| \frac{\gamma^k \sin[(\log \xi^2 \gamma^{2k})t]}{e^{2\pi t}-1} \right| < \gamma^k M \qquad \text{on} \quad 0 \leq t \leq 1.$$

Since $\sum_{k=0}^{\infty} M\gamma^k$ converges, we know the series (3.19) converges uniformly on $0 \leq t \leq 1$ by the Weierstrass M-Test.

We now examine the interchange of limits in the case of $\int_1^{\infty}$. Recall Lebesgue's Dominated Convergence Theorem [5, pg. 246].

Let $f_k(t)$ be the kth partial sum of (3.19). Returning to the expression of the series in (3.17) (and restoring the factor $\frac{1}{e^{2\pi t}-1}$) we see that

$$f_k(t) = \frac{1}{e^{2\pi t}-1} \left[\frac{\xi^{2it}}{2i} \frac{1-\gamma^{k(1+2it)}}{1-\gamma^{1+2it}} - \frac{\xi^{-2it}}{2i} \frac{1-\gamma^{k(1-2it)}}{1-\gamma^{1-2it}} \right].$$

(Recall, the kth partial sum of the geometric series $\sum_{k=0}^{\infty} x^k$ is

$\frac{1-x^k}{1-x}$.) Since $|1-\gamma^{1\pm 2it}| \geq 1 - |\gamma^{1\pm 2it}| = 1 - \gamma$, the triangle inequality gives

$$|f_k(t)| \leq \frac{1}{e^{2\pi t}-1} \left[\frac{1}{2}\frac{1+\gamma^k}{1-\gamma} + \frac{1}{2}\frac{1+\gamma^k}{1-\gamma}\right] = \frac{1}{e^{2\pi t}-1}\frac{1+\gamma^k}{1-\gamma} < \frac{1}{e^{2\pi t}-1}\frac{1+1}{1-\gamma},$$

since $0 < \gamma < 1$. This shows the sequence of partial sums of (3.19) is dominated by $g(t) = \frac{2}{1-\gamma}\frac{1}{e^{2\pi t}-1}$. Also, we have already noted that $g(t)$ is integrable, i.e.,

$$\int_1^\infty \frac{2}{1-\gamma}\frac{dt}{e^{2\pi t}-1}$$

converges. (Note that we had to rewrite the original integral $\int_0^\infty$ as $\int_0^1 + \int_1^\infty$ since $\int_0^\infty \frac{dt}{e^{2\pi t}-1}$ diverges at the lower limit.)

Obviously we wish to exclude the extended case of the Lebesgue Dominated Convergence Theorem, so we now demonstrate that

$$\lim_{k\to\infty} \int_0^\infty f_k(t)dt$$

and

$$\int_0^\infty f(t)dt = \int_0^\infty \sum_{k=0}^\infty \frac{\gamma^k \sin[(\log \xi^2\gamma^{2k})t]}{e^{2\pi t}-1} dt$$

$$= \int_0^\infty \frac{\sin(2t \log \xi) - \gamma \sin(2t \log \frac{\xi}{\gamma})}{1-2\gamma \cos(2t \log \gamma)+\gamma^2} dt$$

are both finite. Since the integrand has a finite limit as $t \to 0^+$, we see the integral is not really improper at the lower limit. Hence it suffices to consider $\int_1^\infty$. Recall, for $1 \leq t < \infty$, we showed that the $f_k(t)$ were dominated by

$$g(t) = \frac{2}{1-\gamma} \frac{1}{e^{2\pi t}-1} .$$

Thus their limit, the sum (3.19), is also dominated by $g(t)$. Therefore, since $\int_1^\infty g(t)dt$ is finite, so also is $\int_1^\infty f(t)dt$.

In summary, we have justified the interchange of limits (3.18) and have also shown that the limits involved are all finite.

To complete the proof of the convergence of Σq_{2k}, it only remains to prove the convergence of

$$\sum_{k=0}^{\infty} \frac{\gamma^k}{\log(\xi^2 \gamma^{2k})}$$

in (3.16). We now attend to this problem. Consider the following integral I:

$$I = \int_0^\infty \frac{\xi^{2t} dt}{1-\gamma^{2t+1}} \qquad (0 < \xi < 1,\ 0 < \gamma < 1).$$

Now

$$\frac{1}{1-\gamma^{2t+1}} = \sum_{k=0}^{\infty} \gamma^{(2t+1)k} ,$$

so

$$I = \int_0^{\infty} \xi^{2t} \sum_{k=0}^{\infty} \gamma^{(2t+1)k} dt$$

$$= \int_0^{\infty} \sum_{k=0}^{\infty} \gamma^k (\xi^2 \gamma^{2k})^t dt$$

$$= \sum_{k=0}^{\infty} \gamma^k \int_0^{\infty} (\xi^2 \gamma^{2k})^t dt$$

$$= \sum_{k=0}^{\infty} \gamma^k \left. \frac{(\xi^2 \gamma^{2k})^t}{\log(\xi^2 \gamma^{2k})} \right|_0^{\infty}$$

$$= - \sum_{k=0}^{\infty} \frac{\gamma^k}{\log(\xi^2 \gamma^{2k})} .$$

We now justify these formal procedures by showing the integral I exists and justifying the interchange of summation and integration.

We first show that $\xi < 1$. Recall in (3. 10) we defined ξ by $\xi = m + n\gamma$. Also recall $\gamma = e^{-\alpha}$ where $\cosh \alpha = \frac{1-m^2-n^2}{2mn}$. Thus

$$\xi = m + n(\cosh \alpha - \sinh \alpha)$$

$$= m + n\left(\frac{1-m^2-n^2}{2mn} - \sqrt{\left(\frac{1-m^2-n^2}{2mn} \right)^2 - 1} \right).$$

Hence we must show that the above expression is less than one, or equivalently that

$$(3.20) \qquad \frac{(1-m-n)(1+n-m)}{2m} < n\sqrt{\left(\frac{1-m^2-n^2}{2mn}\right)^2-1} .$$

Obviously $(m-1)^2 > 0$ from which follows $-4m^3 + 8m^2 - 4m < 0$. To both sides add $1 - 2m^2 + m^4$ to obtain $(m-1)^4 + 4mn^2 < (1-m^2)^2$. Next add $n^4 - 2m^2n^2 - 2n^2$ and we have $[(1-m)^2-n^2]^2 < 1 - 2m^2 - 2n^2 - 2m^2n^2 + m^4 + n^4$. This is $(1-m-n)^2(1+n-m)^2 < (1-m^2-n^2)^2 - 4m^2n^2$ or

$$\left[\frac{(1-m-n)(1+n-m)}{2m}\right]^2 < n^2\left[\left(\frac{1-m^2-n^2}{2mn}\right)^2-1\right] .$$

Observe that $1-m-n$ and $1+n-m$ are positive and extract square roots to obtain the desired inequality (3.20). Therefore $\xi < 1$.

Turning our attention to the convergence of

$$I = \int_0^\infty \frac{\xi^{2t}dt}{1-\gamma^{2t+1}} ,$$

note that γ is fixed and $0 < \gamma < 1$, so $\frac{1}{1-\gamma^{2t+1}}$ is bounded, say by M, for $t \geq 0$. Clearly $\int_0^\infty M\xi^{2t}dt$ converges since $\xi < 1$, so by comparison, I converges.

To justify the interchange of summation and integration in the computations involving I, we invoke the Lebesgue Monotone Convergence Theorem [5, pg. 243]. Clearly the sequence of partial sums of $\sum_{k=0}^{\infty} \gamma^{(2t+1)k}$ is monotone increasing since $\gamma > 0$. Therefore the sequence of partial sums of

$$\xi^{2t} \sum_{k=0}^{\infty} \gamma^{(2t+1)k} = \sum_{k=0}^{\infty} \gamma^{k}(\xi^2\gamma^{2k})^t$$

is monotone increasing also. Since I, the integral of the infinite series of positive terms $\sum_{k=0}^{\infty} \gamma^{k}(\xi^2\gamma^{2k})^t$ converges, by comparison, the sequence of integrals of partial sums of the above series must be bounded. Therefore the Lebesgue Monotone Convergence Theorem shows the interchange of summation and integration to be valid.

In summary, we have justified writing the series $\sum_{k=0}^{\infty} q_{2k}$ (deleting the factor $K(1-\xi^2)$) in the interesting form

$$\frac{1}{2(1-\gamma)} + \int_0^\infty \frac{\xi^{2t}}{1-\gamma^{2t+1}}\,dt - 2\int_0^\infty \frac{\sin(2t \log \xi) - \gamma \sin(2t \log \frac{\xi}{\gamma})}{(e^{2\pi t}-1)[(1-2\gamma\cos(2t\log\gamma)+\gamma^2]}\,dt.$$

Since we have demonstrated the convergence of these improper integrals, we have established again the convergence of

$$\sum_{k=0}^{\infty} q_{2k} .$$

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