AN ABSTRACT OF THE THESIS OF

MARTIN J. STYNES for the degree of DOCTOR OF PHILOSOPHY

in MATHEMATICS presented on January 13, 1977

Title: AN ALGORITHM FOR THE NUMERICAL CALCULATION OF THE DEGREE OF A MAPPING

Abstract approved: Redacted for Privacy

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Let Φ^n be a continuous function from a connected n-dimensional polyhedron P^n to R^n. Assume Φ^n does not vanish on the boundary $b(P^n)$, so that the topological degree of Φ^n relative to the origin is defined. In this dissertation an algorithm for the computation of the degree is obtained. It utilizes certain subdivisions of $b(P^n)$ and depends on the modulus of continuity of Φ^n restricted to $b(P^n)$.

An Algorithm for the Numerical Calculation of the Degree of a Mapping

by

Martin J. Stynes

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the

degree of

Doctor of Philosophy

June 1977

APPROVED:

Redacted for Privacy

Professor of Mathematics

in charge of major

Redacted for Privacy

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Date thesis is presented January 13, 1977

Typed by Clover Redfern for Martin J. Stynes

ACKNOWLEDGMENTS

First thanks must go to my major professor, Dr. Philip M. Anselone, for his patience and most able guidance. The O.S.U. mathematics department has always been ready and willing to discuss my problems; in particular I wish to thank Professors D.H. Carlson, W.J. Firey, and F.J. Flaherty for their time and effort.

The writing of a thesis is a long and painful road, and I am proud of the unfailing encouragement I have received from my family in Ireland. The path was also made smoother by many friends in Corvallis; in particular Tom Kelley, Weldon A. Lodwick, Brian T. McCay and Dave Borer stand out in my memory. Even above these I remember Jeanne Ng with affection. They were all instrumental in making my time here a very happy one.

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AN ALGORITHM FOR THE NUMERICAL CALCULATION OF THE DEGREE OF A MAPPING

I. INTRODUCTION

1. Summary

The topological degree of a function $f: R^n \to R^n$ is an integer which when non-zero guarantees the existence of solutions x to equations of the type $f(x) = p$. The concept was first introduced by Konecker [9] in 1869. Since then it has found many applications, most notably in the fields of differential and integral equations (cf. [2, 3, 8]). However these applications generally make some assumption regarding the value of the degree because, as J. Cronin observes [2, p. 37]: "the problem of computing the degree is, to a considerable extent, unsolved except in the plane." Thus F. Stenger's paper [11], which gave an algorithm for computing the degree in R^n, represents a significant advance.

In Chapter III of this thesis a new proof is given of the basic formula used to compute the degree in that paper. It is remarkable that this formula, first given in [11], was anticipated in a simpler context as long ago as 1904 by J. Hadamard [5, pp. 452-460]. The algorithm of [11] has at least one serious drawback: it generates a sequence of numbers which eventually equal the degree, but it is in general impossible to decide when this equality has occurred. Our main result, proven in Chapters IV and V, shows that in many cases an alternative procedure is available which unambiguously computes the degree. The calculations needed to compute the degree using the basic formula of [11] are very time-consuming, involving the evaluation of many $n \times n$ determinants. In Chapter VI of this thesis a simplification is given which replaces the determinant evaluations by a "scanning" of the matrices associated with the determinants, i.e., a search for certain entries in those matrices.

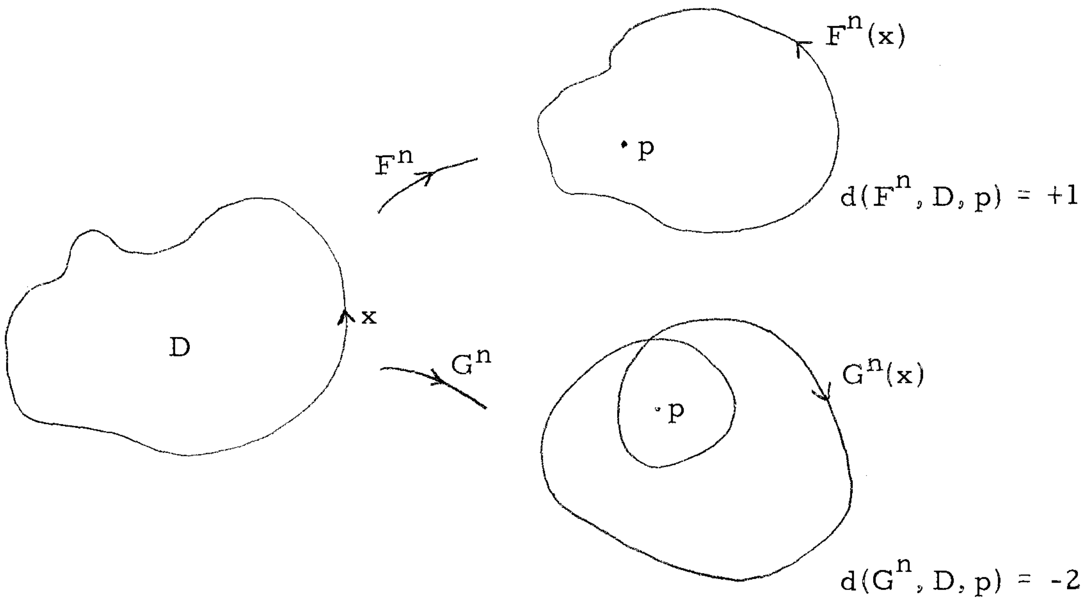
It would be of considerable interest to extend the foregoing results to the infinite dimensional situation (cf. [2, 3]).

2. Definitions of Topological Degree

We begin with an intuitive "definition" of topological degree, after which the two principal properties of the degree are listed.

Let D be a bounded region (open connected set) in R^n with $\overline{D}$ its closure. Denote the boundary of D (i.e., $\overline{D} \setminus D$) by $b(D)$. Let $F^n : \overline{D} \to R^n$ be continuous. Choose $p \in R^n$ with $p \notin F^n(b(D))$.

If x traces out $b(D)$ once "counterclockwise" then the <u>topological degree $d(R^n, D, p)$ of F^n on D relative to p</u> is an integer measuring how many times $F^n(x)$ surrounds p in a "counterclockwise" manner. In the case $n = 2$ the degree is the familiar "winding number" of complex analysis.



There are basically two important properties of $d(F^n, D, p)$:

(i) Homotopy property: if $H(t, x) : [0, 1] \times \overline{D} \to R^n$ is continuous with

$$H(t, x) \neq p \quad \forall\, t \in [0, 1], \quad \forall\, x \in b(D)$$

$$H(0, x) = F^n(x) \quad \forall\, x \in \overline{D}$$

$$H(1, x) = G^n(x) \quad \forall\, x \in \overline{D}$$

then $d(F^n, D, p) = d(G^n, D, p)$.

(ii) Existence of solutions property: if $d(F^n, D, p) \neq 0$ then $\exists$ $x \in D$ such that $F^n(x) = p$.

Next we give several strict definitions of the degree which can be shown to be equivalent. In this section we will take p in R^n to be the origin θ^n, since any other point q may be dealt with by translating F^n, i.e., by defining $d(F^n, D, q)$ to be $d(F^n - q, D, \theta^n)$.

Let D be a bounded region in R^n. Let $F^n : \bar{D} \to R^n$ be continuous with $F^n(x) \neq \theta^n \quad \forall\ x \in b(D)$.

The case $n = 1$ is treated separately: suppose $D = (a, b)$, where $-\infty < a < b < +\infty$. Set

$$d(F^1, D, \theta^1) = \frac{1}{2}\{\operatorname{sgn} F^1(b) - \operatorname{sgn} F^1(a)\},$$

where

$$\operatorname{sgn} t = \begin{cases} +1 & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t = 0, \\ -1 & \text{if} \quad t < 0. \end{cases}$$

For the rest of the section take $n > 1$.

<u>Notation.</u> If $B_i = (b_{i1}, b_{i2}, \ldots, b_{iq})$, $i = 1, 2, \ldots, q$ are q vectors, then

$$\Delta^q(B_1, \ldots, B_q) = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & & \vdots \\ b_{q1} & b_{q2} & \cdots & b_{qq} \end{vmatrix}.$$

In this array B_i is the ith row, for $1 \leq i \leq q$.

<u>1st Definition (Kronecker)</u>. (cf. [1, pp. 465-467]).

$$d(F^n, D, \theta^n) = \frac{1}{\Omega_{n-1}} \int_{X^n(U^{n-1}) \in b(D)} \frac{1}{\|F^n\|^n} \Delta^n(F^n, \frac{\partial F^n}{\partial u_1}, \ldots, \frac{\partial F^n}{\partial u_n})$$

$$\times\, du_1 \ldots du_n$$

where $\|F^n\|$ is the Euclidean norm in R^n, $\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, and $U^{n-1} = (u_1, u_2, \ldots, u_{n-1})$ is a parametrization of $b(D)$ oriented in a certain way.

Here we assume that F^n is of class C^1 on $b(D)$. If F^n is merely continuous on $b(D)$ we can approximate it by C^1 functions and show that the value of the degree is the same for all such functions close enough to F^n; finally define $d(F^n, D, \theta^n)$ to be this common value.

Note that the integral is a direct generalization of the winding number formula of complex analysis. In general it is very difficult to evaluate.

<u>2nd Definition (E. Heinz)</u>. (cf. [6]).

$$d(F^n, D, \theta^n) = \int_D \varphi(\|F^n\|) j(F) dx_1 \ldots dx_n .$$

Here we assume that $F^n \in C^1(\overline{D})$ and that the Jacobian

$$j(F) = \Delta^n\left(\frac{\partial F^n}{\partial x_1}, \ldots, \frac{\partial F^n}{\partial x_n}\right)$$

is non-zero at every point $x \in D$ such that $F^n(x) = \theta^n$. The function φ is continuous with support of $\varphi \subset [r_1, r_2]$ where $0 < r_1 < r_2 < \infty$, and r_1 and r_2 are small, depending on D and F^n, with

$$\int_{R^n} \varphi(\|x\|)dx_1 \ldots dx_n = 1.$$

If F^n is merely continuous we can again define $d(F^n, D, \theta^n)$ by approximation, using Sard's theorem.

Note that the integral is again difficult to evaluate.

<u>3rd Definition.</u> (cf. [10]). Suppose $F^n \in C^1(\bar{D})$. Assume that $\forall\ x \in D$ such that $F^n(x) = \theta^n$ we have $j(F)(x) \neq 0$.

Let N^+ (N^-) denote the number of solutions x in D of $F^n(x) = \theta^n$ such that $j(F)(x)$ is positive (negative). Both N^+ and N^- are finite since these solutions must be isolated points.

Then $d(F^n, D, \theta^n) = N^+ - N^-$.

If F^n is only continuous on D approximate once more to define $d(F^n, D, \theta^n)$.

Once again this is a difficult definition to use for calculating the degree, and in practice is not very helpful because to use it we need to know the solutions of $F^n(x) = \theta^n$; unfortunately we usually want to use the topological degree to guarantee the existence of such solutions.

Our 4th and final definition is the one which will be used for the remainder of this thesis. It might be called the "simplicial homology" definition (although no reference is ever made to homology theory!). The next chapter is a careful development of the definition.

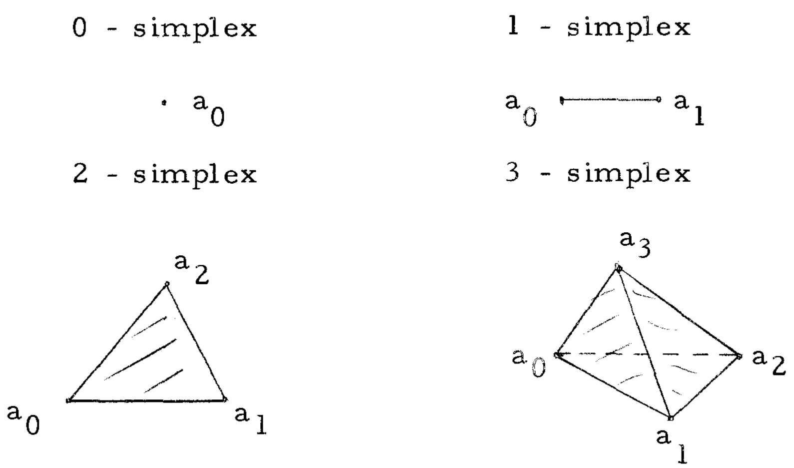
II. BACKGROUND MATERIAL

Almost all of the material presented in this chapter has been condensed from [2, Chapter 1].

Notation. In general for the remainder of this thesis superscripts will indicate dimension while subscripts are used as indices.

A set of points $\{a_0, a_1, \ldots, a_q\}$ in R^n is linearly independent if the points are not contained in any subspace of dimension $\leq q-1$.

A q-simplex S^q is the closed convex hull of $q+1$ linearly independent points $a_0, a_1, \ldots, a_q$ in R^n, and is denoted by $S^q = (a_0 a_1 \ldots a_q)$. The points $a_0, a_1, \ldots, a_q$ are called the vertices of S^q.



An r-dimensional face of S^q $(0 \leq r \leq q)$ is the closed convex hull of any $r+1$ of the points $a_0, a_1, \ldots, a_q$.

Observation 2.1. If $p \in R^n$ and p_k is the kth coordinate of p in R^n, a_{jk} the kth coordinate of a_j in R^n, $1 \leq k \leq n$, then $p \in S^q$ iff

$$p_k = \sum_{j=1}^{q} \lambda_j a_{jk} \quad \text{with all } \lambda_j \geq 0 \text{ and } \sum_{j=1}^{q} \lambda_j = 1.$$

Given a q-simplex we say that two orderings on its vertices are equivalent if one can be obtained from the other by an even permutation of symbols. This gives rise to two equivalence classes of orderings on a given q-simplex, if $q > 0$: one we call "positive", the other "negative". For $q = 0$ call the only possible ordering the positive one.

We adopt the following convention: if $x_i \in R^n$, $0 \leq i \leq n$, with $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ for each i (Cartesian coordinates), then

$$\underline{\text{orientation } (x_0 x_1 \ldots x_n)} = \operatorname{sgn} \Delta^{n+1}((1, x_0), (1, x_1), \ldots, (1, x_n))$$

$$= \operatorname{sgn} \begin{vmatrix} 1 & x_{01} & x_{02} \ldots & x_{0n} \\ 1 & x_{11} & x_{12} \ldots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} \ldots & x_{nn} \end{vmatrix} .$$

This is not zero if the x_i are linearly independent and does not change value under an even permutation of vertices.

Note: Here, given a Cartesian basis for R^n, we are essentially ordering R^n itself; this is what an "oriented R^n" will mean in the sequel. It can be shown that given a basis and orientation on R^n, the orientation with respect to another basis is the same iff the determinant of the affine mapping taking one basis to the other is positive. Consequently we see that an orientation on R^n induces an orientation on all its subspaces and their translations in a natural way.

Notation: $<x_0 x_1 \ldots x_n>$ is an oriented n-simplex (i.e., a simplex with associated orientation). We write

$<x_1 x_0 x_2 \ldots x_n> = -<x_0 x_1 x_2 \ldots x_n>$ etc.

A q-chain is a finite algebraic sum of oriented q-simplexes with integer coefficients, e.g., $2<z_0 a_1 a_2> - 3<a_3 a_1 a_4>$ is a 2-chain. In any given q-chain we assume that all possible cancellations have been made (i.e., if $c^q = \sum_j u_j S_j^q$ is a q-chain then $S_{j_1}^q \neq \pm S_{j_2}^q$ for any $j_1 \neq j_2$, and all the u_j are non-zero).

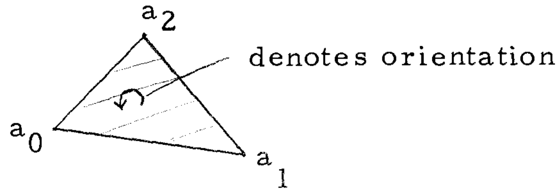
If $S^q = <a_0 a_1 \ldots a_q>$, define its boundary to be

$$b(S^q) = \sum_{i=0}^{q} (-1)^i <a_0 a_1 \ldots \hat{a}_i \ldots a_q>$$

where ^ denotes omission. Thus $b(S^q)$ is a (q-1)-chain, if $q > 0$ (we take $b(S^0) = 0$).

For example, if $S^2 = \langle a_0 a_1 a_2 \rangle$

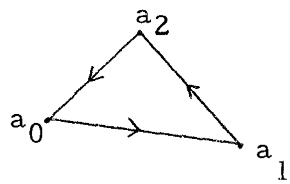
S^2:



then

$$b(S^2) = \langle a_1 a_2 \rangle - \langle a_0 a_2 \rangle + \langle a_0 a_1 \rangle$$
$$= \langle a_1 a_2 \rangle + \langle a_2 a_0 \rangle + \langle a_0 a_1 \rangle .$$

$b(S^2)$:



Given a q-chain $c^q = \sum_j u_j x_j^q$, where the u_j are integers, define $b(c^q) = \sum_j u_j b(x_j^q)$, so the boundary of a q-chain is a (q-1)-chain if $q > 0$.

<u>Remark 2.2</u>. It is easy to prove that $b(b(S^q)) = 0$ for any S^q, hence $b(b(c^q)) = 0$ for any chain c^q.

<u>Notation</u>: Let S_j^q be a collection of (possibly oriented) q-simplexes indexed by j. Then

$$\bigcup_j S_j^q = \{p\colon\ p \in S_j^q \ \text{ for some } \ j\},$$

$$\bigcap_j S_j^q = \{p\colon\ p \in S_j^q \ \text{ for all } \ j\}.$$

If $c^q = \sum_j u_j S_j^q$ is a q-chain and $u_j \neq 0 \ \ \forall\ j$, then $p \in c^q$ means $p \in S_j^q$ for some j.

If $A \subset R^n$, then $A \setminus c^q = \{x \in A\colon\ x \notin c^q\}$.

Intersection Numbers. Let S^n be an oriented n-simplex in R^n; let $p \in R^n \setminus b(S^n)$. We say that S^n and $\langle p\rangle$ are in general position.

If $p \notin S^n$ set $i(S^n, p) = 0$.

If $p \in S^n$ set $i(S^n, p) =$ orientation of S^n.

We call i the intersection number.

Let H^{n-1} be a hyperplane in an oriented R^n; let G^1 be a line in the same R^n. Suppose $H^{n-1} \cap G^1 = \{p\}$. Choose $\langle pa_1 \ldots a_{n-1}\rangle$ and $\langle pb\rangle$ positively oriented in H^{n-1} and G^1 respectively.

Then set $i(H^{n-1}, G^1) =$ orientation $\langle pa_1a_2 \ldots a_{n-1}b\rangle$. It can be checked that $\langle pa_1a_2 \ldots a_{n-1}b\rangle$ is an n-simplex and that i is independent of the choice of points $a_1, a_2, \ldots, a_{n-1}, b$.

Suppose S^{n-1} and T^1 are oriented simplexes in R^n. We say that they are in general position if $S^{n-1} \cap T^1$ is a single point or the null set. Suppose they are in general position.

If $S^{n-1} \cap T^1 = \phi$ set $i(S^{n-1}, T^1) = 0$.

If $S^{n-1} \cap T^1 \neq \phi$ then S^{n-1} and T^1 determine a hyperplane H^{n-1} and line G^1 respectively with $H^{n-1} \cap G^1$ a single point. Suppose R^n oriented so that S^{n-1} and T^1 have positive orientations. Then set $i(S^{n-1}, T^1) = i(H^{n-1}, G^1)$.

Observation 2.3. If $S^{n-1} = \langle a_1 a_2 \ldots a_n \rangle$ and $p \in S^{n-1} \setminus b(S^{n-1})$, then orientation $\langle a_1 a_2 \ldots a_n \rangle$ = orientation $\langle pa_2 \ldots a_n \rangle$. It is elementary to check this from the definition of orientation, since by Observation 2.1 we can write $p_k = \sum_{j=1}^{n} \lambda_j a_{jk}$ with each $\lambda_j > 0$ and $\sum_{j=1}^{n} \lambda_j = 1$, and a multiple of one row of a determinant can be subtracted from another row without altering the value of the determinant. Consequently if in R^n $S^{n-1} \cap T^1$ is a point p with $p \notin b(S^{n-1}) \cup b(T^1)$, where $T^1 = \langle cb \rangle$, then $i(S^{n-1}, T^1)$ = orientation $\langle pa_2 \ldots a_n b \rangle$.

Given two chains $c^p = \sum_j u_j X_j^p$, $d^q = \sum_k v_k Y_k^q$ where u_j, v_k are integers and the simplexes X_j^k, Y_k^q lie in an oriented R^n, with either $p = n-1$ and $q = 1$ or $p = n$ and $q = 0$, we say that the chains are in general position if X_j^p and Y_k^q are in general position for every pair (j, k). In this case we define

$$i(c^p, d^q) = \sum_{j,k} u_j v_k i(X_j^p, Y_k^q) .$$

Theorem 2.4 [2, Theorem 2.2]. If the chains c^n, $b(d^1)$, also $b(c^n)$, d^1 are in general position then

$$i(c^n, b(d^1)) = (-1)^n i(b(c^n), d^1) .$$

Proof. This is elementary but tedious; one considers various cases.

Order of a Point Relative to a Boundary. Suppose $z^{n-1} = b(c^n)$, some chain c^n (we say that z^{n-1} is an (n-1)-boundary). Suppose c^n and p are in general position. Then we define the order of p relative to z^{n-1} to be

$$v(z^{n-1}, p) = i(c^n, p) .$$

It is not difficult to check that v is well-defined, i.e., independent of the particular c^n satisfying $b(c^n) = z^{n-1}$.

Remark 2.5.

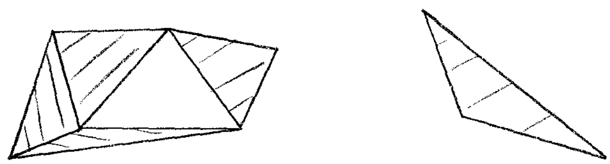
(i) If $v(z^{n-1}, p) \neq 0$ with $b(c^n) = z^{n-1}$, then $p \in c^n$

(ii) $v(az_1^{n-1} + bz_2^{n-1}, p) = av(z_1^{n-1}, p) + bv(z_2^{n-1}, p)$ for all integers a, b (provided the right-hand side is defined).

We now extend this definition by replacing z^{n-1} by a continuous image of itself.

Definition. An n-dimensional polyhedron K^n is a union of a finite number of oriented n-simplexes S_i^n, $i = 1, 2, \ldots, m$, such that for every pair S_i^n, S_j^n of these simplexes either $S_i^n \cap S_j^n$ is the empty set of $S_i^n \cap S_j^n$ is a common face, i.e., an r-simplex $(0 \leq r \leq n)$ whose vertices are vertices of both S_i^n and S_j^n. We write $K^n = \bigcup_{i=1}^{m} S_i^n$ or $K^n = \sum_{i=1}^{m} S_i^n$ depending on context.

Example.

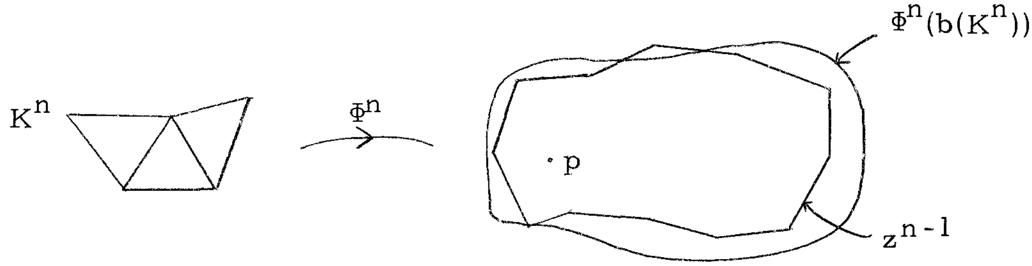


The shaded area is a 2-dimensional polyhedron.

Definition. An n-region is a connected n-dimensional polyhedron.

Definition. Let K^n be an n-region, $K^n = \sum_{i=1}^{m} S_i^n$. Let $\Phi^n : K^n \to R^n$ be continuous, and let $p \in R^n \setminus \Phi^n(b(K^n))$. Then if z^{n-1} is an (n-1)-boundary such that z^{n-1} can be deformed continuously into $\Phi^n(b(K^n))$ without passing through the point p, we define the degree of Φ^n on K^n relative to p to be

$$d(\Phi^n, K^n, p) = v(z^{n-1}, p).$$



It is shown in [2] that $d(\Phi^n, K^n, p)$ is well-defined, i.e., that for all z^{n-1} satisfying the conditions of the definition $v(z^{n-1}, p)$ has the same value. The existence of such z^{n-1} is proven by breaking up each S_i^n into a finite sum of n-simplexes, i.e., expressing each S_i^n as an n-region, then proving existence of a function $\Phi_*^n : K^n \to R^n$ which maps n-chains formed by sums of these smaller simplexes into n-chains in R^n in such a way that $\Phi_*^n(b(K^n))$ is an (n-1)-boundary z^{n-1} just as in the definition. This decomposition of K^n is called a <u>simplicial subdivision</u>, and the function Φ_*^n is called a <u>chain approximation to Φ^n on K^n with respect to p</u>. It is shown that

(i) Φ_*^n is arbitrarily close to Φ^n on K^n, and

(ii) $\Phi_*^n(b(K^n)) = b(\Phi_*^n(K^n))$.

Note that there exist arbitrarily small perturbations of Φ_*^n which will not destroy any of the properties attributed above to Φ_*^n.

<u>Observation 2.6.</u> For $n = 1$, suppose $b(K^1) = \langle x_m \rangle - \langle x_0 \rangle$. Then from the definition we can take $z^0 = \langle \Phi^1(x_m) \rangle - \langle \Phi^1(x_0) \rangle$ so $z^0 = b(\langle \Phi^1(x_0) \Phi^1(x_m) \rangle)$. By examining the various cases we see that

$$d(\Phi^1, K^1, p) = \frac{1}{2} \{\mathrm{sgn}(\Phi^1(x_m)-p) - \mathrm{sgn}(\Phi^1(x_0)-p)\} .$$

Thus, since the computation of $d(\Phi^n, K^n, p)$ is trivial for $n = 1$, this case is mentioned in the sequel only to illustrate definitions and concepts and to supply the first step in inductive definitions and proofs.

Remark 2.7.

(i) If $p \notin \Phi^n(K^n)$, then $d(\Phi^n, K^n, p) = 0$ from the definition and Remark 2.5(i);

(ii) From the definition $d(\Phi^n, K^n, p)$ is always an integer;

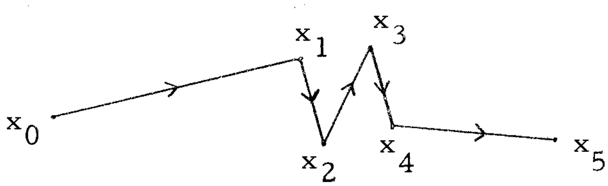
(iii) Clearly $d(\Phi^n, K^n, p)$ has the homotopy property described in Section I. 2.

III. SUFFICIENT REFINEMENTS AND COMPUTATION OF THE DEGREE

In this chapter we will use the definition of topological degree given in Chapter II to reprove the basic computational formula of [11]. This formula was originally proven using the 3rd definition of Section I. 2.

Let P^n be an n-region. We shall always assume that P^n is written as a sum of oriented n-simplexes in such a way that $b(P^n)$ coincides with the topological boundary of P^n when P^n is regarded as lying in R^n. It is also assumed that all decompositions of $b(P^n)$ preserve orientations.

Example for Case $n = 1$.



$$P^1 = \sum_{i=0}^{4} \langle x_i x_{i+1} \rangle$$

so

$$b(P^1) = \sum_{i=0}^{4} b(\langle x_i x_{i+1} \rangle) = \sum_{i=0}^{4} (\langle x_{i+1} \rangle - \langle x_i \rangle) = x_5 - x_0 .$$

In the future we shall often assume that for $n > 1$ $b(P^n)$ is written in such a way that the coefficient of each $(n-1)$-simplex is $+1$; this can be achieved by changing the order of the vertices in an oriented simplex where necessary.

Let $\Phi^n = (\varphi_1, \varphi_2, \ldots, \varphi_n) : P^n \to R^n$ be continuous with $\Phi^n(p) \neq \theta^n \quad \forall \; p \in b(P^n)$.

<u>Inductive Definition.</u> If $n = 1$, $b(P^1) = \langle x_m \rangle - \langle x_0 \rangle$ (say) is <u>sufficiently refined relative to sgn Φ^1</u> $= \operatorname{sgn} \varphi_1$ if $\varphi_1(x_0)\, \varphi_1(x_m) \neq 0$. If $n > 1$, $b(P^n)$ is <u>sufficiently refined relative to sgn Φ^n</u> if $b(P^n)$ has been subdivided so that it may be written as a union of a finite number of $(n-1)$-regions $\beta_1^{n-1}, \beta_2^{n-1}, \ldots, \beta_m^{n-1}$ in such a way that

(i) the $(n-1)$-dimensional interiors of the β_i^{n-1} are pairwise disjoint;

(ii) at least one of the functions $\varphi_1, \ldots, \varphi_n$, say φ_{r_i}, is non-zero on each region β_i^{n-1};

(iii) if $\varphi_{r_i} \neq 0$ on β_i^{n-1}, then $b(\beta_i^{n-1})$ is sufficiently refined relative to sgn $\Phi_{r_i}^{n-1}$, where $\Phi_{r_i}^{n-1} = (\varphi_1, \ldots, \hat{\varphi}_{r_i}, \ldots, \varphi_n)$ (^ signifies omission).

<u>Example 3.1.</u> In the case $n = 2$, suppose

$$b(P^2) = \sum_{j=0}^{m-1} \langle x_j x_{j+1} \rangle ,$$

where $x_m = x_0$. This is sufficiently refined relative to sgn Φ^2 if at least one of φ_1, φ_2 is non-zero on each line segment $(x_j x_{j+1})$, and neither is zero at any x_j. Here we are taking
$\beta_j^{n-1} = \langle x_j x_{j+1} \rangle$, $0 \leq j \leq m-1$.

There are some apparent differences between this definition and that given in [11]. We shall explain and justify them before proceeding further.

Let $n > 1$. Fix $i \in \{1, 2, \ldots, n\}$ and fix $\Delta = +1$ or -1. Then a $\underline{Q^n\text{-set}}$ is a connected set of points q lying in $b(P^n)$ such that $\varphi_1(q) = \ldots = \hat{\varphi}_i(q) = \ldots = \varphi_n(q) = 0$, and $\text{sgn}\, \varphi_i(q) = \Delta$. If two Q^n-sets are associated with different i's and/or different Δ's in this definition, they are said to be of <u>different types</u>.

Note: (a) From part (iii) of our definition of a sufficient refinement, if $p \in b(\beta_i^{n-1})$ for some i then $\varphi_j(p) \neq 0$ for some $j \neq r_i$. Of course $p \in b(\beta_i^{n-1})$ also implies that $\varphi_{r_i}(p) \neq 0$. Thus p cannot lie in any Q^n-set. Since the Q^n-sets are connected, this shows that each of them must lie in the interior of some β_i^{n-1}.

(b) Because $\varphi_{r_i} \neq 0$ on β_i^{n-1} and β_i^{n-1} is connected, sgn φ_{r_i} is constant and non-zero on β_i^{n-1}, and it follows that no β_i^{n-1} can contain points from Q^n-sets of different types.

Our conditions for a sufficient refinement are repeated in the definition given in [11] together with the requirements that (A) each

Q^n-set must lie in the interior of some β_i^{n-1} and (B) each β_i^{n-1} must contain at most one Q^n-set. We have show in (a) above that (A) follows from the other hypotheses, and it is evident that (B) can be weakened to the conclusion of (b) in the proof given in [11] of the computational formula for the degree. Since sufficient refinements are introduced for the purpose of deriving this formula, our definition would have sufficed in [11].

We shall prove a theorem which gives an inductive degree relation between $d(\Phi^n, P^n, \theta^n)$ and the $d(\Phi_{r_i}^{n-1}, \beta_i^{n-1}, \theta^{n-1})$, $i = 1, 2, \ldots, m$. For the proof we need two lemmas.

Lemma 3.2. Let $n > 1$. Suppose $a \neq \theta^n$ lies on one of the coordinate axes in R^n with its rth coordinate non-zero. Let S^{n-1} be an $(n-1)$-simplex with a function $F^n : S^{n-1} \to R^n$, $F^n = (f_1, f_2, \ldots, f_n)$, such that $F^n(S^{n-1})$ is an $(n-1)$-simplex. Suppose that

(i) $f_r \neq 0$ on S^{n-1} with $\operatorname{sgn} f_r |_{S^{n-1}} = \operatorname{sgn}$ (rth coordinate of a)

(ii) $i(F^n(S^{n-1}), <\theta^n a>)$ and $i(F_r^{n-1}(S^{n-1}), \theta^{n-1})$ are defined (here i is the intersection number of Chapter II and $F_r^{n-1} = (f_1, \ldots, \hat{f}_r, \ldots, f_n)$).

Then

$$i(F^n(S^{n-1}), \langle\theta^n a\rangle) = (-1)^{r+n} i(F_r^{n-1}(S^{n-1}), \theta^{n-1}) \operatorname{sgn} f_r .$$

<u>Proof</u>. If $i(F^n(S^{n-1}), \langle\theta^n a\rangle) = 0$ then $F^n(S^{n-1}) \cap \langle\theta^n a\rangle = \phi$, so $F_r^{n-1}(S^{n-1}) \cap \langle\theta^{n-1}\rangle = \phi$ also by (i) and $i(F_r^{n-1}(S^{n-1}), \theta^{n-1}) = 0$; the conclusion is verified. We may therefore assume that $F^n(S^{n-1}) \cap \langle\theta^n a\rangle = \{P\}$ say, where $P = (0, 0, \ldots, p, \ldots, 0)$ in R^n, $p \neq 0$ lying in the rth position by choice of a. Note that (i) $\operatorname{sgn} p = \operatorname{sgn} f_r$
(ii) $P \in F^n(S^{n-1}) \setminus b(F^n(S^{n-1}))$ because $i(F_r^{n-1}(S^{n-1}), \theta^{n-1})$ is defined.

Set $P' = (0, 0, \ldots, 2p, \ldots, 0)$. Suppose $F^n(S^{n-1}) = \langle y_1 \ldots y_n\rangle$. Then

$$i(F^n(S^{n-1}), \langle\theta^n a\rangle) = \text{orientation } \langle P y_2 \ldots y_n P'\rangle$$

(follows from observation 2.3)

$$= \operatorname{sgn} \Delta^{n+1}((1,P), (1, y_2), \ldots, (1, y_n), (1, P'))$$

$$= \operatorname{sgn} \begin{vmatrix} 1 & 0 & 0 & \ldots & p & \ldots & 0 \\ 1 & y_{21} & y_{22} & \ldots & y_{2r} & \ldots & y_{2n} \\ . & . & . & \ldots & . & \ldots & . \\ 1 & y_{n1} & y_{n2} & \ldots & y_{nr} & \ldots & y_{nn} \\ 1 & 0 & 0 & \ldots & 2p & \ldots & 0 \end{vmatrix},$$

where $y_m = (y_{m1}, y_{m2}, \ldots, y_{mn})$ in R^n for $1 \leq m \leq n$.

Subtract the first row from the last one and expand in terms of last row:

$$i(F^n(S^{n-1}), <\theta^n a>) = (-1)^{n+r} \text{ sgn } p \text{ sgn } \begin{vmatrix} 1 & 0 & 0 & \cdots & \hat{p} & \cdots & 0 \\ 1 & y_{21} & y_{22} & \cdots & \hat{y}_{2r} & \cdots & y_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ 1 & y_{n1} & y_{n2} & \cdots & \hat{y}_{nr} & \cdots & y_{nn} \end{vmatrix}$$

$$= (-1)^{n+r} \text{ sgn } p \text{ sgn } \begin{vmatrix} 1 & y_{11} & y_{12} & \cdots & \hat{y}_{1r} & \cdots & y_{1n} \\ 1 & y_{21} & y_{22} & \cdots & \hat{y}_{2r} & \cdots & y_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ 1 & y_{n1} & y_{n2} & \cdots & \hat{y}_{nr} & \cdots & y_{nn} \end{vmatrix}$$

(by an argument like that of observation 2.3)

$$= (-1)^{n+r} \text{ sgn } f_r . i(F_r^{n-1}(S^{n-1}), \theta^{n-1})$$

by definition; we know that $\theta^{n-1} \in F_r^{n-1}(S^{n-1})$ from our initial assumption.

<u>Lemma 3.3</u>. Let P^n be an n-region with $\Phi^n : P^n \to R^n$ continuous and $\Phi^n \neq \theta^n$ on $b(P^n)$. Suppose that $b(P^n)$ has been subdivided into a finite number of (n-1) regions β_j^{n-1}, $j \in J$, so that it is sufficiently refined relative to sgn Φ^n. Let Φ_*^n be a chain approximation to Φ^n on P^n with respect to θ^n, with $\{S_k^{n-1} : k \in K\}$ the associated simplicial subdivision of

$b(P^n) = \sum_{j \in J} \beta_j^{n-1}$. Let the point a lie on one of the coordinate axes in R^n with its rth coordinate non-zero and $\|a\| > \max_{x \in P^n} \|\Phi^n(x)\|$ (usual Euclidean norm). Then we may assume that

(i) the sufficient refinement of $b(P^n)$ relative to $\text{sgn}\, \Phi^n$ is also a sufficient refinement of $b(P^n)$ relative to $\text{sgn}\, \Phi_*^n$, with $\varphi_{r_j} \neq 0$ on β_j^{n-1} implying that $\varphi_{*r_j} \neq 0$ on β_j^{n-1} also;

(ii) $\|a\| > \max_{x \in P^n} \|\Phi_*^n(x)\|$;

(iii) for every S_k^{n-1} both $i(\Phi_*^n(S_k^{n-1}), <\theta^n a>)$ and $i(\Phi_{*r}^{n-1}(S_k^{n-1}), \theta^{n-1})$ are defined.

<u>Proof</u>. (i) We know that we may take Φ_*^n arbitrarily close to Φ^n on P^n; the proof is then easy using induction on n and the fact that Φ_*^n a chain approximation to Φ^n on P^n with respect to θ^n implies $\Phi_{*r_j}^{n-1}$ a chain approximation to $\Phi_{r_j}^{n-1}$ on β_j^{n-1} with respect to θ^{n-1}.

(ii) Since Φ_*^n may be chosen arbitrarily close to Φ^n on P^n, we may assume that (ii) holds.

(iii) If $i(\Phi_*^n(S_k^{n-1}), <\theta^n a>)$ is not defined then $\Phi_*^n(S_k^{n-1}) \cap (\theta^n a)$ is a (non-trivial) line segment, implying that $\Phi_{*r}^{n-1}(S_k^{n-1})$ is an (n-2)-simplex and θ^{n-1} lies in this (n-2)-simplex. If $i(\Phi_{*r}^{n-1}(S_k^{n-1}), \theta^{n-1})$ is not defined then $\theta^{n-1} \in b(\Phi_{*r}^{n-1}(S_k^{n-1}))$, i.e., θ^{n-1} lies in an (n-2)-chain.

Thus to ensure that both $i(\Phi_*^n(S_k^{n-1}), <\theta^n a>)$ and $i(\Phi_{*r}^{n-1}(S_k^{n-1}), \theta^{n-1})$ are defined for the given finite set $\{S_k^{n-1} : k \in K\}$, it is sufficient that θ^{n-1} does not lie in a certain finite collection of (n-2)-simplexes which are in (n-1)-dimensional space. Therefore if necessary we can perturb $<\theta^n a>$ arbitrarily slightly in a direction perpendicular to itself so that the above intersection numbers are defined for all k. This is equivalent to leaving $<\theta^n a>$ as it was originally, then perturbing Φ_*^n slightly in the opposite direction. This perturbation can be chosen so small that all the properties of Φ_*^n as a chain approximation are preserved.

<u>Theorem 3.4.</u> Let P^n, $n > 1$, be an n-region and let $\Phi^n : P^n \to R^n$ be continuous with $\Phi^n \neq \theta^n$ on $b(P^n)$ so that $d(\Phi^n, P^n, \theta^n)$ is defined. Suppose that $b(P^n)$ has been subdivided into (n-1)-regions β_j^{n-1}, $j \in J$, so that it is sufficiently refined relative to sgn Φ^n. Then if $\Phi^n = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ with $\varphi_{r_j} \neq 0$ on β_j^{n-1} for each $j \in J$,

$$d(\Phi^n, P^n, \theta^n) = \frac{1}{2n} \sum_{j \in J} (-1)^{r_j+1} d(\Phi_{r_j}^{n-1}, \beta_j^{n-1}, \theta^{n-1}) \operatorname{sgn} \varphi_{r_j} \Big|_{\beta_j^{n-1}}$$

<u>Proof</u>. Each $d(\Phi^{n-1}_{r_j}, \beta^{n-1}_j, \theta^{n-1})$ is defined since by definition $b(\beta^{n-1}_j)$ is sufficiently refined relative to $\operatorname{sgn} \Phi^{n-1}_{r_j}$, which implies that $\Phi^{n-1}_{r_j} \neq \theta^{n-1}$ on $b(\beta^{n-1}_j)$. Note that the sum above is well-defined insofar as if for some $j \in J$ both φ_s and φ_t are non-zero on β^{n-1}_j, then $d(\Phi^{n-1}_s, \beta^{n-1}_j, \theta^{n-1}) = d(\Phi^{n-1}_t, \beta^{n-1}_j, \theta^{n-1}) = 0$ because $\Phi^{n-1}_s \neq \theta^{n-1}$ and $\Phi^{n-1}_t \neq \theta^{n-1}$ on β^{n-1}_j, and so it is immaterial whether we associate φ_s or φ_t with β^{n-1}_j.

Choose a point a lying on one of the coordinate axes in R^n (with say rth coordinate non-zero) such that $\|a\| > \max_{x \in P^n} \| \Phi^n(x)\|$.

Let Φ^n_* be a chain approximation to Φ^n on P^n with respect to θ^n; by Lemma 3.3 we may assume that Φ^n_* satisfies certain conditions. Now

$$d(\Phi^n, P^n, \theta^n) = v(\Phi^n_*(b(P^n)), \theta^n) \qquad \text{(by definition)}$$

$$= v(b(\Phi^n_*(P^n)), \theta^n)$$

(property of chain approximation)

$$= i(\Phi^n_*(P^n), \theta^n) \tag{1}$$

(by definition of v)

$$= i(\Phi^n_*(P^n), \theta^n) - i(\Phi^n_*(P^n), a)$$

(by Lemma 3.3 we assume $a \notin \Phi^n_*(P^n)$)

$$= -i(\Phi^n_*(P^n), a - \theta^n) =$$

$$= -i(\Phi^n_*(P^n), b(<\theta^n a>))$$

$$= (-1)^{n+1} i(b\Phi^n_*(P^n), <\theta^n a>) \qquad \text{(Theorem 2.4)}$$

$$= (-1)^{n+1} i(\Phi^n_* b(P^n), <\theta^n a>)$$

$$= (-1)^{n+1} \sum_{j \in J} i(\Phi^n_*(\beta_j^{n-1}), <\theta^n a>)$$

where by Lemma 3.3 we assume each term in the sum is defined.

Then using the definition of intersection number

$$d(\Phi^n, P^n, \theta^n) = (-1)^{n+1} \sum_{j \in J_a} i(\Phi^n_*(\beta_j^{n-1}), <\theta^n a>) \tag{2}$$

where

$$J_a = \{j \in J : \Phi^n_*(\beta_j^{n-1}) \cap (\theta^n a) \neq \phi\}.$$

Fix $j \in J_a$. Then $r_j = r$ with $\operatorname{sgn} \varphi_{*r_j}\big|_{\beta_j^{n-1}} = \operatorname{sgn}$ (rth coordinate of a).

Let $\beta_j^{n-1} = \sum_{k \in K_j} S_k^{n-1}$ in the simplicial subdivision of P^n.

Then

$$i(\Phi^n_*(\beta_j^{n-1}), <\theta^n a>) = \sum_{k \in K_j} i(\Phi^n_*(S_k^{n-1}), <\theta^n a>) =$$

(the sum is defined by Lemma 3.3)

$$= \sum_{k \in K_j} (-1)^{r_j + n} \, i(\Phi^{n-1}_{*r_j}(S^{n-1}_k), \theta^{n-1}) \operatorname{sgn} \varphi_{*r_j}\big|_{\beta_j^{n-1}}$$

by Lemma 3.2 and (iii) of Lemma 3.3.

Thus from (2) above

$$d(\Phi^n, P^n, \theta^n) = (-1)^{n+1} \sum_{j \in J_a} \sum_{k \in K_j} (-1)^{r_j + n} \, i(\Phi^{n-1}_{*r_j}(S^{n-1}_k), \theta^{n-1}) \times \operatorname{sgn} \varphi_{*r_j}\big|_{\beta_j^{n-1}}$$

$$= \sum_{j \in J_a} (-1)^{r_j + 1} \, i(\Phi^{n-1}_{*r_j}(\beta^{n-1}_j), \theta^{n-1}) \operatorname{sgn} \varphi_{*r_j}\big|_{\beta_j^{n-1}} \quad (3)$$

Now there are $2n$ sets J_a, depending on the half of the coordinate axis on which a lies, and we get (3) for each J_a. Adding these $2n$ equations gives

$$2n \, d(\Phi^n, P^n, \theta^n) = \sum_{j \in J} (-1)^{r_j + 1} \, i(\Phi^{n-1}_{*r_j}(\beta^{n-1}_j), \theta^{n-1}) \operatorname{sgn} \varphi_{r_j}\big|_{\beta_j^{n-1}} \quad (4)$$

using (i) of Lemma 3.3 to replace $\operatorname{sgn} \varphi_{*r_j}$ by $\operatorname{sgn} \varphi_{r_j}$. Here the summation over all $j \in J$ is justified as follows: if some $j \in J$ does not lie in any J_a, this means that $\Phi^n_*(\beta^{n-1}_j) \cap (\theta^n a) = \phi$ for all possible a. Consequently $\Phi^{n-1}_{*r_j} \neq \theta^{n-1}$ on β^{n-1}_j, giving

$i(\Phi^{n-1}_{*r_j}(\beta^{n-1}_j), \theta^{n-1}) = 0$; therefore j can be included in the sum without altering its value. On the other hand it is impossible for some $j \in J$ to lie in two different sets J_a, $J_{a'}$ because this would imply that no φ_{*r}, $1 \leq r \leq n$, was non-zero on β^{n-1}_j and by (i) of Lemma 3.3 this is not so.

Finally since Φ^n_* a chain approximation to Φ^n on P^n with respect to θ^n implies $\Phi^{n-1}_{*r_j}$ a chain approximation to $\Phi^{n-1}_{r_j}$ on β^{n-1}_j with respect to θ^{n-1} $j \in J$, we can use (1) to rewrite (4) to get the required result.

We are almost ready to prove our computational formula for $d(\Phi^n, P^n, \theta^n)$ using the inductive degree relation of the theorem.

Recall that if $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ in R^n, $0 \leq i \leq n$, then

$$\Delta^n(x_1, x_2, \ldots, x_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$$

and

$$\Delta^{n+1}((1, x_0), (1, x_1), \ldots, (1, x_n)) = \begin{vmatrix} 1 & x_{01} & x_{02} & \cdots & x_{0n} \\ 1 & x_{11} & x_{12} & \cdots & x_{1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} .$$

Lemma 3.5. Let $K^n = \sum_{j=1}^{\ell} \langle x_0^{(j)} x_1^{(j)} \ldots x_n^{(j)} \rangle$ be an n-region, with $b(K^n) = \sum_{j=1}^{\ell'} t_j \langle y_1^{(j)} \ldots y_n^{(j)} \rangle$, $t_j = \pm 1$. Suppose $G^n : K^n \to R^n$. Then $\sum_{j=1}^{\ell} \Delta^{n+1}((1, G(x_0^{(j)}), \ldots, (1, G(x_n^{(j)}))) = \sum_{j=1}^{\ell'} t_j \Delta^n(G(y_1^{(j)}), \ldots, G(y_n^{(j)}))$.

Proof.

$$\sum_{j=1}^{\ell} \Delta^n((1, G(x_0^{(j)})), \ldots, (1, G(x_n^{(j)})))$$

$$= \sum_{j=1}^{\ell} \sum_{i=0}^{q} (-1)^i \Delta^n(G(x_0^{(j)}), \ldots, G(x_i^{(j)})^\wedge, \ldots, G(x_n^{(j)})) \qquad (1)$$

by expanding Δ^{n+1} along its first column. Now

$$b(K^n) = \sum_{j=1}^{\ell} \sum_{i=0}^{q} (-1)^i \langle x_0^{(j)} \ldots \hat{x}_i^{(j)} \ldots x_n^{(j)} \rangle$$

by definition; also

$$b(K^n) = \sum_{j=1}^{\ell'} t_j \langle y_1^{(j)} \ldots y_n^{(j)} \rangle$$

by hypothesis, where the y's are just a relabelling of certain x's. Consequently from (1) we see that

$$\sum_{j=1}^{\ell} \Delta^n((1, G(x_0^{(j)})), \ldots, (1, G(x_n^{(j)}))) = \sum_{j=1}^{\ell'} t_j \Delta^n(G(y_1^{(j)}), \ldots, G(y_n^{(j)})) .$$

Theorem 3. 6 (computational formula for the degree). Let P^n be an n-region. Suppose $\Phi^n = (\varphi_1, \ldots, \varphi_n) : P^n \to R^n$ is continuous with $\Phi^n \neq \theta^n$ on $b(P^n)$. Suppose that $b(P^n)$ has been subdivided into (n-1)-regions β_k^{n-1}, $1 \leq k \leq m$, so that it is sufficiently refined relative to sgn Φ^n; let $b(P^n) = \sum_{j=1}^{\ell} t_j \langle y_1^{(j)} \ldots y_n^{(j)} \rangle$, $t_j = \pm 1$. Then

$$d(\Phi^n, P^n, \theta^n) = \frac{1}{2^n\, n!} \sum_{j=1}^{\ell} t_j \Delta^n(\operatorname{sgn} \Phi^n(y_1^{(j)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(j)})) ,$$

where $\operatorname{sgn} \Phi^n(y) = (\operatorname{sgn} \varphi_1(y), \ldots, \operatorname{sgn} \varphi_n(y))$.

Proof. This is the same proof as that in Section 4.3 of [11]. We use induction on n.

For $n = 1$ this clearly reduces to the formula given in observation 2. 6 for $d(\Phi^1, P^1, \theta^1)$.

Fix $n > 1$ and assume that the theorem holds in the $n-1$ case. We have

$$d(\Phi^n, P^n, \theta^n) = \frac{1}{2n} \sum_{k=1}^{m} (-1)^{r_k+1} d(\Phi_{r_k}^{n-1}, \beta_k^{n-1}, \theta^{n-1}) \operatorname{sgn} \varphi_{r_k} \Big|_{\beta_k^{n-1}} \qquad (1)$$

from Theorem 3.4.

Let $b(\beta_k^{n-1}) = \sum_{i \in I_k} \tau_i \langle z_1^{(i)} \ldots z_{n-1}^{(i)} \rangle$, $1 \leq k \leq m$, $\tau_i = \pm 1$. By definition this is sufficiently refined relative to $\operatorname{sgn} \Phi_{r_k}^{n-1}$, so by the induction hypothesis we have

$$d(\Phi_{r_k}^{n-1}, \beta_k^{n-1}, \theta^{n-1})$$

$$= \frac{1}{2^{n-1}\,(n-1)!} \sum_{i \in I_k} \tau_i \Delta^{n-1}(\operatorname{sgn} \Phi_{r_k}^{n-1}(z_1^{(i)}), \ldots, \operatorname{sgn} \Phi_{r_k}^{n-1}(z_{n-1}^{(i)}))$$

$$= \frac{1}{2^{n-1}\,(n-1)!} \sum_{j \in J_k} t_j \Delta^{n}((1, \operatorname{sgn} \Phi_{r_k}^{n-1}(y_1^{(j)})), \ldots, (1, \operatorname{sgn} \Phi_{r_k}^{n-1}(y_n^{(j)})))$$

by Lemma 3.5, where $\beta_k^{n-1} = \sum_{j \in J_k} t_j (y_1^{(j)}, \ldots, y_n^{(j)})$, $t_j = \pm 1$. Thus (1) becomes

$$d(\Phi^n, P^n, \theta^n) = \frac{1}{2^n\, n!} \sum_{k=1}^{m} \sum_{j \in J_k} (-1)^{r_k+1} t_j$$

$$\times \Delta^{n}((1, \operatorname{sgn} \Phi_{r_k}^{n-1}(y_1^{(j)})), \ldots, (1, \operatorname{sgn} \Phi_{r_k}^{n-1}(y_n^{(j)})))$$

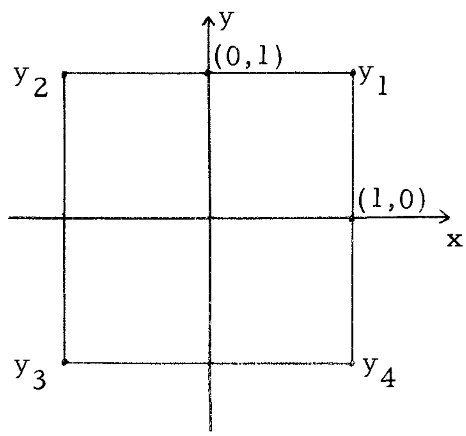
$$\times \operatorname{sgn} \varphi_{r_k} \Big|_{\beta_k^{n-1}}$$

$$= \frac{1}{2^n\, n!} \sum_{k=1}^{m} \sum_{j \in J_k} t_j \Delta^{n}(\operatorname{sgn} \Phi^n(y_1^{(j)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(j)}))$$

(moving first column of Δ^n r_{k-1} columns to the right then multiplying it by $\operatorname{sgn} \varphi_{r_k}\Big|_{\beta_k^{n-1}}$)

$$= \frac{1}{2^n\, n!} \sum_{j=1}^{\ell} t_j \Delta^n(\operatorname{sgn} \Phi^n(y_1^{(j)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(j)})).$$

<u>Example</u>. P^2 = unit square in R^2 with counterclockwise orientation. Let $\Phi^2(x, y) = (x, x^2 - 2y) = x', y')$ say.



We take $b(P^2) = \langle y_1 y_2 \rangle + \langle y_2 y_3 \rangle + \langle y_3 y_4 \rangle + \langle y_4 y_1 \rangle$. By Example 3.1 this gives a sufficient refinement of $b(P^2)$ relative to $\operatorname{sgn} \Phi^2$.

Then by Theorem 3.6

$$d(\Phi^2, P^2, \theta^2) = \frac{1}{8}\left\{\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\right\}$$

$$= \frac{1}{8}\{-2-2-2-2\}$$

$$= -1.$$

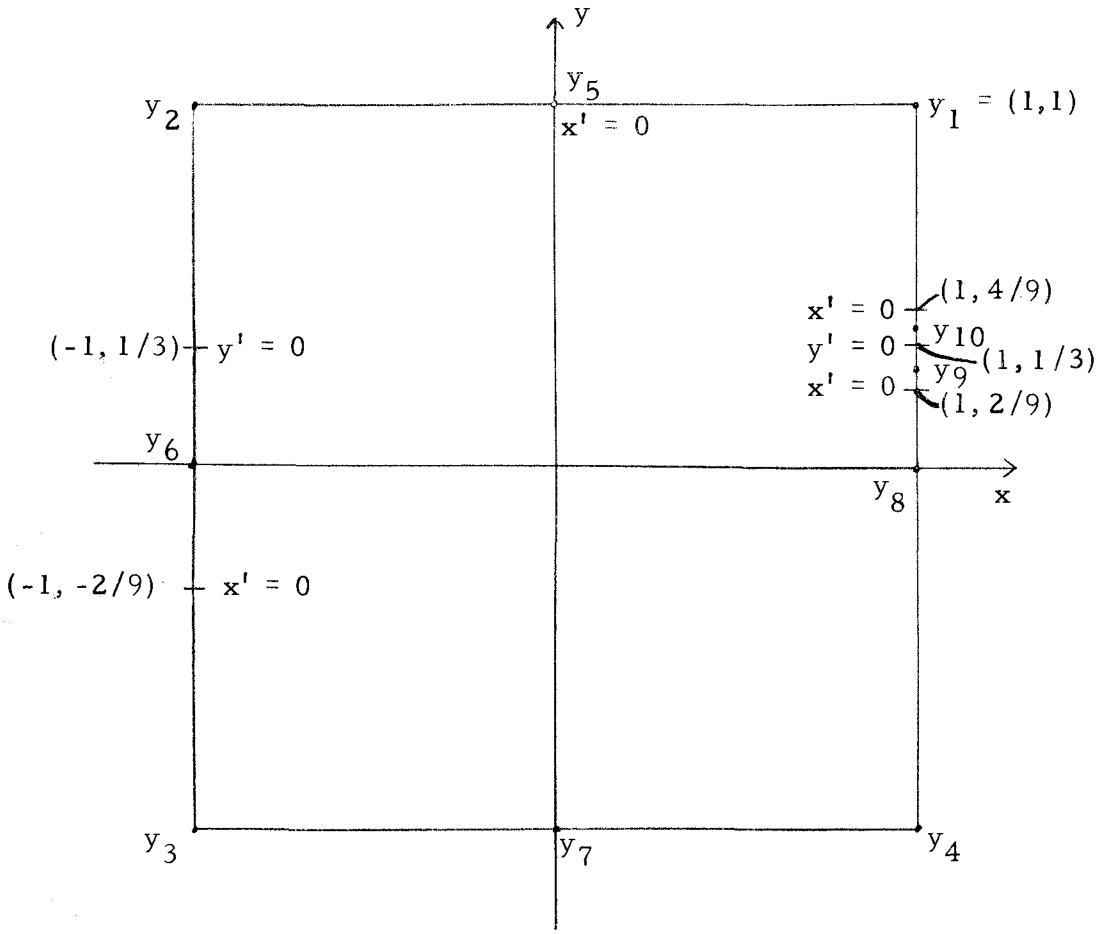
Historical note. It is a remarkable fact that J. Hadamard in [5, pp. 452-460] essentially verified Equation (3) of Theorem 3.4 for the special case $r_j = 1$. He used the Kronecker integral definition of Chapter I, simplifying the integral by ingenious manipulation. It is easy to see how to extend his work to the general case where $r_j \in \{1, 2, \ldots, n\}$. Since Equation (3) is the crucial step needed to prove the computational formula, we can say with some truth that Theorem 3.6 might well have been proven 60 years before it was.

IV. IMPARTIAL REFINEMENTS

1. Motivation

Let P^n be an n-region with $\Phi^n = (\varphi_1, \ldots, \varphi_n) : P^n \to R^n$ continuous and satisfying $\Phi^n \neq \theta^n$ on $b(P^n)$. It is shown in [11] that if we keep refining the simplexes in the original representation of $b(P^n)$ in such a way that their mesh (i.e., their largest diameter) tends to zero, then after a finite number of steps we will obtain a sufficient refinement provided that the zeroes of the φ_i satisfy certaion conditions (these conditions involve the Q^n-sets defined near the beginning of Chapter III and are similar to insisting that no φ_i can have an infinite number of isolated zeroes on $b(P^n)$). The proof of this statement relies on a contradiction argument and consequently gives no estimate regarding the number of steps needed to reach a sufficient refinement in a given situation. Moreover, as the next example illustrates, it is in general impossible to tell from the computations involved in this algorithm when a sufficient refinement has been reached. However once we have a sufficient refinement it is clear from the definition that any further refinement will not "lose" this property, so there is no question of over-refining.

Example. P^2 = unit square in R^2 with counterclockwise orientation. Let $\Phi^2(x, y) = ((y - \frac{2}{9}x)(y + \frac{5}{9}x - 1), y - \frac{1}{3}) = (x', y')$ say.



Suppose that we initially have subdivision points y_1, y_2, y_3, y_4 so that $b(P^2) = \langle y_1 y_2 \rangle + \langle y_2 y_3 \rangle + \langle y_3 y_4 \rangle + \langle y_4 y_1 \rangle$. If this is a sufficient refinement then by Theorem 3.6

$$d(\Phi^2, P^2, \theta^2) = \frac{1}{8}\left\{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\right\}$$

$$= \frac{1}{2}$$

which is impossible since the degree is always an integer. Therefore we do not have a sufficient refinement and must refine further.

Introduce four more subdivision points y_5, y_6, y_7, y_8 as indicated on the diagram so that

$b(P^2) = \langle y_1 y_5 \rangle + \langle y_5 y_2 \rangle + \ldots + \langle y_8 y_1 \rangle$. If this is a sufficient refinement then

$$d(\Phi^2, P^2, \theta^2) = \frac{1}{8}\{\begin{vmatrix} \operatorname{sgn} \varphi_1(y_1) & \operatorname{sgn} \varphi_2(y_1) \\ \operatorname{sgn} \varphi_1(y_5) & \operatorname{sgn} \varphi_2(y_5) \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} \operatorname{sgn} \varphi_1(y_8) & \operatorname{sgn} \varphi_2(y_8) \\ \operatorname{sgn} \varphi_1(y_1) & \operatorname{sgn} \varphi_2(y_1) \end{vmatrix}\}$$

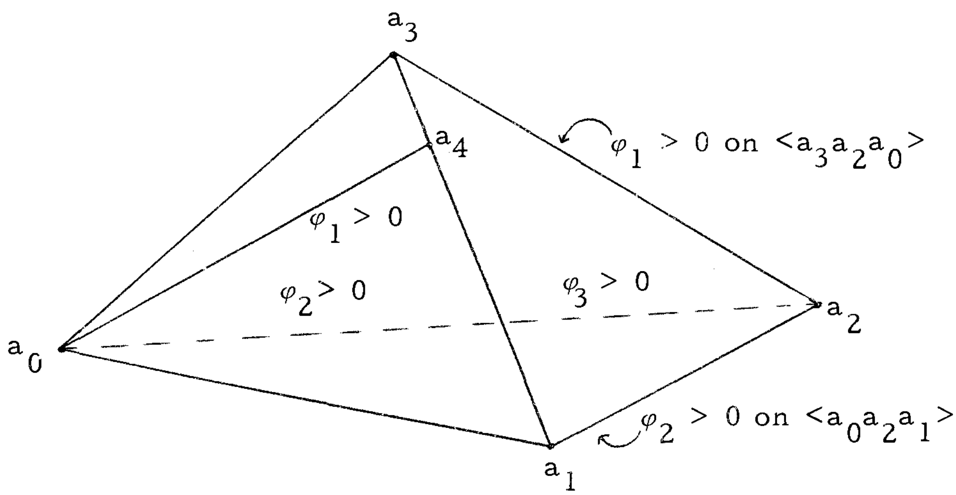
$$= 1.$$

Since we know the positions of the zeroes of x' and y' on $b(P^2)$ we see that this is not a sufficient refinement because neither x' nor y' is non-zero on the 1-simplex $\langle y_8 y_1 \rangle$. If we introduce further subdivision points and use Theorem 3.6 to compute a possible value for $d(\Phi^2, P^2, \theta^2)$ each time, we will still get the value 1 until we introduce a point such as y_9 or y_{10} on the diagram.

If $b(P^2) = \langle y_1 y_5 \rangle + \langle y_5 y_2 \rangle + \ldots + \langle y_8 y_9 \rangle + \langle y_9 y_{10} \rangle + \langle y_{10} y_1 \rangle$ we now have a sufficient refinement by Example 3.1 and consequently $d(\Phi^2, P^2, \theta^2) = 0$ using Theorem 3.6.

The point here is that if the zeroes of φ_1 and φ_2 on $b(P^2)$ were unknown, the procedure of subdividing, computing the possible degree from Theorem 3.6, further subdividing etc. might yield a sequence of possible degree values in which the term 1 occurs many successive times. We might thus be misled into believing that $d(\Phi^2, P^2, \theta^2) = 1$. In this case the calculations contained in the procedure would not indicate that we do not have a sufficient refinement because it is easy to find a function $\psi^2 : P^2 \to R^2$ for which $\operatorname{sgn} \psi^2 = \operatorname{sgn} \Phi^2$ at all previously sampled points and yet $b(P^2)$ is sufficiently refined relative to $\operatorname{sgn} \psi^2$. For example take $\psi^2(x, y) = (\frac{11}{9} x - y + 1, y - \frac{1}{3})$. Then $\operatorname{sgn} \psi^2 = \operatorname{sgn} \Phi^2$ on $b(P^2) \setminus \{(x, y) : x = 1, \frac{2}{9} < y < \frac{4}{9}\}$ with $d(\psi^2, P^2, \theta^2) = 1$, and the subdivision points $y_1, y_2, \ldots, y_8$ yield a sufficient refinement of $b(P^2)$ relative to $\operatorname{sgn} \psi^2$.

For $n > 2$ it is in general a complicated matter to check that a decomposition of $b(P^n)$ yields a sufficient refinement, because of condition (iii) of the definition. We will define what we call an "impartial refinement"; this is a stronger property than that of "sufficient refinement" but its conditions are simpler to verify. Consider the following motivating example: $P^3 = <a_0 a_1 a_2 a_3>$ (a tetrahedron) and $\Phi^3 : P^3 \to R^3$ with $\operatorname{sgn} \varphi_i$ as indicated in the diagram.



Then

$$\operatorname{sgn} \Phi^3(a_2) = \operatorname{sgn} \Phi^3(a_4) = (1, 1, 1).$$

Let

$$\operatorname{sgn} \Phi^3(a_0) = (1, 1, -1), \quad \operatorname{sgn} \Phi^3(a_1) = (-1, 1, 1), \quad \operatorname{sgn} \Phi^3(a_3) = (1, -1, 1).$$

Assume that

$$\operatorname{sgn} \varphi_i(a_j) = \operatorname{sgn} \varphi_i(a_k) = \Delta \neq 0 \Rightarrow \operatorname{sgn} \varphi_i = \Delta \quad \text{on} \quad <a_j a_k>, \quad \text{all} \quad i, j, k.$$

Now

$$b(P^3) = <a_0a_1a_4> + <a_0a_4a_3> + <a_1a_2a_3> + <a_0a_2a_1> + <a_3a_2a_0>.$$

If this decomposition into 5 2-regions yields a sufficient refinement of $b(P^3)$ relative to $\operatorname{sgn} \Phi^3$, then

$$d(\Phi^3, P^3, \theta^3) = \frac{1}{2^3\ 3!} \left\{ \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \right\}$$

$$= -\frac{1}{12}$$

which is of course impossible. Thus we do not have a sufficient refinement. The problem is that the boundary of the region $<a_1 a_2 a_3>$ is not sufficiently refined relative to $\operatorname{sgn}(\varphi_1, \varphi_2)$. For

$$b(<a_1 a_2 a_3>) = <a_1 a_2> + <a_2 a_3> + <a_3 a_1>$$

and both φ_1 and φ_2 have zeroes on $<a_3 a_1>$ since $\operatorname{sgn} \varphi_i(a_3) = -\operatorname{sgn} \varphi_i(a_1)$, $i = 1, 2$. Thus $<a_1 a_3>$ cannot be used as a 1-region in a sufficient refinement of $b(<a_1 a_2 a_3>)$.

If $<a_1 a_2 a_3>$ is divided into $<a_1 a_2 a_4> + <a_4 a_2 a_3>$ this problem is removed because now

$$b(<a_1 a_2 a_3> + <a_4 a_2 a_3>) = <a_1 a_2> + <a_2 a_3> + <a_3 a_4> + <a_4 a_1>$$

which is sufficiently refined by Example 3.1. We can now apply

Theorem 3.6 to get $d(\Phi^3, P^3, \theta^3) = 0$.

Here we obtain a sufficient refinement by making the decomposition of $b(P^3)$ more impartial insofar as $<a_1a_3>$ is no longer regarded as a single 1-simplex in the region $<a_1a_2a_3>$ while being regarded as $<a_1a_4> + <a_4a_2>$ (a sum of two 1-simplexes) in the region $(<a_0a_1a_4> + <a_0a_4a_3>)$.

2. Impartial Refinements: Definition and Properties

Let P^n be an n-region. Let $\Phi^n = (\varphi_1, \varphi_2, \ldots, \varphi_n) : P^n \to R^n$ be continuous with $\Phi^n(p) \neq \theta^n \quad \forall \ p \in b(P^n)$.

Definition. If $n = 1$, $b(P^1)$ is impartially refined relative to sgn Φ^1 if it is sufficiently refined relative to sgn Φ^1. If $n > 1$, $b(P^n)$ is impartially refined relative to sgn Φ^n if $b(P^n)$ has been subdivided so that it may be written as a union of a finite number of (n-1)-regions $\beta_1^{n-1}, \ldots, \beta_m^{n-1}$ in such a way that

(i) the (n-1)-dimensional interiors of the β_i^{n-1} are pairwise disjoint;

(ii) at least one of the functions $\varphi_1, \ldots, \varphi_n$, say φ_{r_i}, is non-zero on each region β_i^{n-1};

(iii) each region β_i^{n-1} is maximal insofar as if $\beta_j^{n-1} \cap \beta_i^{n-1} \neq \phi$ with $j \neq i$, then $r_j \neq r_i$;

(iv) if S_i^{n-1} is an (n-1)-simplex in $\beta_{i'}^{n-1}$ such that S_i^{n-1} has a face of dimension $n-2$ lying on $b(\beta_{i'}^{n-1})$, then this face is also a face of at least one (n-1)-simplex S_j^{n-1} lying in some $\beta_{j'}^{n-1}$, where $j' \neq i'$.

Before proving the fundamental theorem which states that all impartial refinements are sufficient refinements we obtain two preliminary results.

Lemma 4.1. The boundary of an n-dimensional polyhedron is an (n-1)-dimensional polyhedron.

Proof. Let K^n be an n-dimensional polyhedron. Let S^{n-1} and T^{n-1} be (n-1)-simplexes in the (n-1)-chain $b(K^n)$. We must show that $S^{n-1} \cap T^{n-1}$ is either the empty set or a common face of the simplexes.

Suppose $S^{n-1} \cap T^{n-1} \neq \phi$. Now S^{n-1} is part of the boundary of an n-simplex S^n (say) in K^n; likewise T^{n-1} comes from a T^n in K^n. Let $S^n = \langle s_0 s_1 \ldots s_n \rangle$, $T^n = \langle t_0 t_1 \ldots t_n \rangle$. Without loss of generality take

$$S^{n-1} = \langle s_1 s_2 \ldots s_n \rangle, \quad T^{n-1} = \langle t_0 t_1 \ldots t_{n-1} \rangle .$$

If $S^n = T^n$ it is easy to finish the proof, so assume not. Then because K^n is a polyhedron $S^n \cap T^n$ is a common r-dimensional face, with $0 \leq r < n$. Without loss of generality take

$$S^n \cap T^n = \left\{ p : p = \sum_{i=0}^{r} \lambda_i s_i,\ \lambda_i \geq 0,\ \sum_{i=0}^{r} \lambda_i = 1 \right\}$$

$$= \left\{ q : q = \sum_{i=0}^{r} \mu_i t_i,\ \mu_i \geq 0,\ \sum_{i=0}^{t} \mu_i = 1 \right\}$$

using observation 2.1. By linear independence and the uniqueness of the extreme points in $S^n \cap T^n$ we must have $\{s_i : 0 \leq i \leq r\} = \{t_i : 0 \leq i \leq r\}$. We may assume in fact that $s_i = t_i$ for $0 \leq i \leq r$.

Thus

$$S^{n-1} \cap (S^n \cap T^n) = \left\{ p : p = \sum_{i=1}^{r} \lambda_i s_i,\ \lambda_i \geq 0,\ \sum_{i=0}^{r} \lambda_i = 1 \right\}$$

$(\phi \neq S^{n-1} \cap T^{n-1} = S^{n-1} \cap (S^{n-1} \cap T^{n-1}) \subseteq S^{n-1} \cap (S^n \cap T^n)$

$\Rightarrow r \geq 1)$.

Likewise

$$T^{n-1} \cap (S^n \cap T^n) = \left\{ q : q = \sum_{i=0}^{r} \mu_i t_i,\ \mu_i \geq 0,\ \sum_{i=0}^{r} \mu_i = 1 \right\},$$

as $r \leq n-1$.

Recall that $s_i = t_i$ for $0 \leq i \leq r$; thus

$$S^{n-1} \cap T^{n-1} = (S^{n-1} \cap T^{n-1}) \cap (S^n \cap T^n)$$
$$= [S^{n-1} \cap (S^n \cap T^n)] \cap [T^{n-1} \cap (S^n \cap T^n)]$$
$$= \left\{ x : x = \sum_{i=1}^{r} \lambda_i s_i,\ \lambda_i \geq 0,\ \sum_{i=1}^{r} \lambda_i = 1 \right\},$$

and this is a common face of S^{n-1} and T^{n-1} as required.

Theorem 4.2. Let $n > 1$ and suppose that $b(P^n)$ is impartially refined relative to $\operatorname{sgn} \Phi^n$ with notation as in the definition. Then for any $i' \in \{1, 2, \ldots, n\}$, $b(\beta_{i'}^{n-1})$ is impartially refined relative to $\operatorname{sgn} \Phi_{r_{i'}}^{n-1}$.

Proof. Without loss of generality take $r_{i'} = 1$, $i' = 1$, and $m > 1$ (if $m = 1$ then $b(\beta_1^{n-1})$ satisfies the "impartially refined" definition vacuously).

Let $\{S_i^{n-1}\}_{i \in I}$ be the (finite) set of (n-1)-simplexes in β_1^{n-1} having an (n-2)-dimensional face lying on $b(\beta_1^{n-1})$. Let $\{S_{j(i)}^{n-1}\}_{j \in J}$ be the corresponding simplexes given by part (iv) of the definition ($i \rightarrow j(i)$ is not in general a function). For each $j \in J$ we take S_j^{n-1} as lying in $\beta_{j'}^{n-1}$, say ($j \rightarrow j'$ is a many-one map). Write S_{ij}^{n-2} for the (n-2)-simplex $S_i^{n-1} \cap S_{j(i)}^{n-1}$.

Consider

$$S_2 = \{S_{ij}^{n-2} : i \in I,\ j \in J,\ S_j^{n-1} \subsetneq \beta_{j'}^{n-1},\ r_{j'} = 2\}.$$

By Lemma 4.1 the intersection of any two distinct (n-2)-simplexes S_{ij}^{n-2} is either a common face or the empty set since all S_{ij}^{n-2} lie in $b(\beta_1^{n-1})$. We can therefore write S_2 as a disjoint union of (n-2)-regions in a unique way (by taking connected components).

Continue thus: consider, for $3 \leq k \leq n$,

$$S_k = \{S_{ij}^{n-2} : i \in I,\ j \in J,\ S_j^{n-1} \subsetneq \beta_{j'}^{n-1},\ r_{j'} = k\},$$

and write $S_k \setminus (S_2 \cup S_3 \cup \ldots \cup S_{k-1})$ as a disjoint union of (n-2)-regions.

Taking all these (n-2)-regions $\beta_1^{n-2}, \ldots, \beta_{m'}^{n-2}$ (say) gives us a decomposition of $b(\beta_1^{n-1})$ since by part (iv) of the definition all $\beta_{j'}^{n-1}$ $(j' \neq 1)$ which intersect β_1^{n-1} satisfy $r_{j'} \neq 1$. We claim that in fact this gives us an impartial refinement of $b(\beta_1^{n-1})$ relative to sgn Φ_1^{n-1}.

Clearly (i) and (iii) of the definition are satisfied.

Recall that for any i $S_{ij}^{n-2} = S_i^{n-1} \cap S_{j(i)}^{n-1}$, and consequently $\varphi_{r_{j'}} \neq 0$ on S_{ij}^{n-2}. By maximality of β_1^{n-1}, $r_{j'} \neq 1$, and so a component of Φ_1^{n-1} is non-zero on each β^{n-2}, i.e., (ii) of the definition is satisfied (this argument also takes care of the exceptional case $n = 2$).

For condition (iv) (when $n > 2$) let S_{ℓ}^{n-2} be any $(n-2)$-simplex in $\beta_{\ell'}^{n-2}$ such that an $(n-3)$-dimensional face of S_{ℓ}^{n-2} lies in $b(\beta_{\ell'}^{n-2})$. Now β_1^{n-1} is a region and $b(b(\beta_1^{n-1})) = 0$ (Remark 2.2) so any $(n-3)$-simplex in $b(\beta_{\ell'}^{n-2})$ must, since $\beta_{\ell'}^{n-2} \subseteq b(\beta_1^{n-1})$, also be an $(n-3)$-simplex in $b(\beta_{t'}^{n-2})$, some $t' \neq \ell'$. It is consequently a face of an $(n-2)$-simplex S_t^{n-2} in $\beta_{t'}^{n-2}$.

The principal result is the following.

Theorem 4.3. If $b(P^n)$ is impartially refined relative to $\operatorname{sgn} \Phi^n$, then it is also sufficiently refined relative to $\operatorname{sgn} \Phi^n$.

Proof. Use induction on n.

For $n = 1$ the assertion is trivial. Fix $n > 1$ and assume that the theorem holds for any P^{n-1} and Φ^{n-1}.

Suppose that $b(P^n)$ is impartially refined relative to $\operatorname{sgn} \Phi^n$ using the regions $\beta_1^{n-1}, \ldots, \beta_m^{n-1}$. We only have to show that (iii) of the "sufficiently refined" definition holds, the rest being automatically satisfied.

For any $i \in \{1, 2, \ldots, m\}$ $\varphi_{r_i} \neq 0$ on β_i^{n-1} and by Theorem 4.2 $b(\beta_i^{n-1})$ is impartially refined relative to $\operatorname{sgn} \Phi_{r_i}^{n-1} \Rightarrow b(\beta_i^{n-1})$ is sufficiently refined relative to $\operatorname{sgn} \Phi_{r_i}^{n-1}$ by the inductive hypothesis, and we are done.

V. ALGORITHM FOR COMPUTATION OF THE DEGREE USING IMPARTIAL REFINEMENTS

1. Introduction and Description of Algorithm

Let P^n be an n-region. Let $\Phi^n = (\varphi_1, \ldots, \varphi_n) : P^n \to R^n$ with $\Phi^n \neq \theta^n$ on $b(P^n)$. We now describe an algorithm for computing $d(\Phi^n, P^n, \theta^n)$, assuming that we can majorize the modulus of continuity of $\Phi^n|_{b(P^n)}$ by a known function $\Omega(\delta)$ which is $O(\delta)$. This happens for example if we know that $\Phi^n|_{b(P^n)}$ satisfies a Lipschitz condition of order $\alpha > 0$.

The algorithm constructs an impartial refinement of $b(P^n)$ then terminates. By Theorem 4.3 this impartial refinement is also a sufficient refinement. We can thus compute $d(\Phi^n, P^n, \theta^n)$ with assurance using the formula of Theorem 3.6.

Take $n > 1$ as the computation of $d(\Phi^1, P^1, \theta^1)$ is trivial.

Let $\omega_i(\cdot)$ be the modulus of continuity of $\varphi_i|_{b(P^n)}$, $i = 1, 2, \ldots, n$. Suppose that $\omega_i(\delta) \leq \Omega_i(\delta)$ for $1 \leq i \leq n$, where the Ω_i are known functions and $\Omega_i(\delta)$ is $O(\delta)$ for each i.

For any i and any $x \in b(P^n)$ with $\varphi_i(x) \neq 0$, choose $\delta_i > 0$ such that $\Omega_i(\delta_i) < |\varphi_i(x)|$. Let $\|\cdot\|$ denote the Euclidean norm in R^n. Then for any $y \in b(P^n)$ such that $\|x-y\| \leq \delta_i$, we have

$$\begin{aligned}|\varphi_i(y)| &\geq |\varphi_i(x)| - |\varphi_i(x)-\varphi_i(y)| \\ &\geq |\varphi_i(x)| - \omega_i(\delta_i) \\ &\geq |\varphi_i(x)| - \Omega_i(\delta_i) \\ &> 0.\end{aligned}$$

Let $p \in b(P^n)$. Choose $\delta(p) = \max\limits_{1 \leq i \leq n} \delta_i$, where the δ_i are chosen as large as possible such that $\Omega_i(\delta_i) < |\varphi_i(p)|$ (if $\varphi_i(p) = 0$ take $\delta_i = 0$). For at least one i

$$|\varphi_i(p)| \geq m = \min_{x \in b(P^n)} \| \Phi^n(x) \| > 0$$

so $\delta(p) \geq \min\limits_{1 \leq i \leq n} \delta_i'$ where each $\delta_i' > 0$ is chosen as large as possible such that $\Omega_i(\delta_i') < m$. Thus $\delta(p) \geq c$ say, where c is positive and independent of p.

From the preceding paragraph we see that if $y \in b(P^n)$ satisfies $\|p-y\| \leq \delta(p)$, then $|\varphi_i(y)| > 0$ for some i independent of y.

Thus, given any point $p \in b(P^n)$, we can surround it by a ball B of radius at least c (a fixed positive constant) such that some component of Φ^n is non-zero on $B \cap b(P^n)$.

Definition. A simplex is acceptable if at least one component of Φ^n is known to be non-zero thereon.

Remark 5.1. Any subdivision of an acceptable simplex yields acceptable simplexes.

Where necessary our algorithm will subdivide simplexes in $b(P^n)$ until all simplexes in $b(P^n)$ are acceptable. Also, the original representation of $b(P^n)$ is a polyhedron by Lemma 4.1 and the algorithm will preserve this property.

Definition. An edge of a simplex is a one-dimensional face of that simplex.

Let $S_1^{n-1}, \ldots, S_\ell^{n-1}$ be a list of the (oriented) simplexes in $b(P^n)$ with $b(P^n) = \sum_{j=1}^{\ell} S_j^{n-1}$. At first all these simplexes are unacceptable. Let $p_1, \ldots, p_m$ be a list of the vertices of the S_j^{n-1}.

Algorithm.

1. Set $i = 1$ and go to step 3.
2. If $i = m$, terminate. Otherwise replace i by $i+1$ and continue.
3. Compute $\delta(p_i)$ as described above.
4. Set $j = 1$ and go to step 6.
5. If $j = \ell$ go to step 2. Otherwise replace j by $j+1$ and continue.

6. If S_j^{n-1} is acceptable or if p_i is not a vertex of S_j^{n-1} to to step 5. Otherwise continue.

7. Without loss of generality assume that $S_j^{n-1} = \langle p_i y_2 \cdots y_n \rangle$. Set $k = 2$ and go to step 9.

8. If $k = n$ list S_j^{n-1} as acceptable and to to step 5. Otherwise replace k by $k+1$ and continue.

9. Compute ℓ_k, the length of the edge $\langle p_i y_k \rangle$.

   (a) If $\ell_k \leq \delta(p_i)$, go to step 8.

   (b) If $\ell_k > \delta(p_i)$, let p_{m+1} be the point lying on $\langle p_i y_k \rangle$ at a distance $\min\{\delta(p_i), \frac{1}{2}\ell_k\}$ from p_i. In step 2 replace m by $m+1$. Replace the oriented simplex S_j^{n-1} by the two oriented simplexes

$$S_j^{n-1} = \langle p_i y_2 \cdots y_{k-1} p_{m+1} y_{k+1} \cdots y_n \rangle$$

and

$$S_{\ell+1}^{n-1} = \langle p_{m+1} y_2 \cdots y_n \rangle .$$

In step 5 replace ℓ by $\ell+1$. In exactly the same way replace every other oriented simplex having $\langle p_i y_k \rangle$ as an edge by two new oriented simplexes whose "sum" is the original oriented simplex, increasing ℓ in step 5 to $\ell+1$ each time a new simplex is created. Got to step 8.

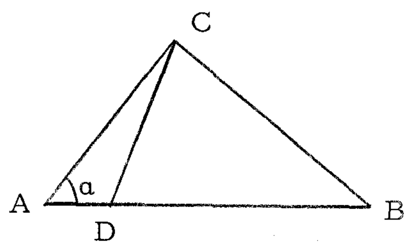
Summary of Algorithm. Consider the vertices $p_1, p_2, \ldots$ in turn. For each p_i consider in turn those unacceptable simplexes having p_i as a vertex. For each such simplex S_j^{n-1} consider the edges emanating from p_i. Subdivide those edges whose length exceeds $\delta(p_i)$ so that the new length of each edge of S_j^{n-1} having p_i as an endpoint is at most $\delta(p_i)$. Then S_j^{n-1} must be acceptable. Here we also subdivide all other simplexes sharing the above edges with S_j^{n-1} in order to preserve the polyhedral property of $b(P^n)$.

2. Proof of Convergence of Algorithm

Observation 5.2. By virtue of step 6, it is sufficient to prove that after a finite number of iterations all simplexes are acceptable.

For this proof we use the following two lemmas.

Lemma 5.3.



Let c and k be positive constants. Write 'AB' for 'length of AB' etc. Given a triangle ABC as shown with diameter less than or equal to k, $AB > c$, and D lying on AB subject to the constraints:

(a) if $AB \leq 2c$, then $AD = \frac{1}{2} AB$;

(b) if $AB > 2c$, then $c \leq AD \leq \frac{1}{2} AB$.

Then

$$CD^2 \leq k^2 - \frac{c^2}{4} .$$

<u>Proof</u>. We break up the proof into three cases:

(i) $AB \leq 2c$ and $AB \geq BC$

(ii) $AB \leq 2c$ and $AB < BC$

(iii) $AB > 2c$.

<u>Case (i)</u>. Now $BC^2 = AC^2 + AB^2 - 2AC\ AB \cos \alpha$ so $BC^2 - AB^2 = AC^2 - 2AC\ AB \cos \alpha$ and

$$\begin{aligned} BC \leq AB &\Leftrightarrow AC^2 - 2AC\ AB \cos \alpha \leq 0 \\ &\Leftrightarrow AC - 2\ AB \cos \alpha \leq 0 \\ &\Leftrightarrow \cos \alpha \geq \frac{AC}{2\ AB} . \end{aligned}$$

Since $AB \leq 2c$, we have by hypothesis $AD = \frac{1}{2} AB$. Thus

$$\begin{aligned} CD^2 &= AC^2 + AD^2 - 2AC\ AD \cos \alpha \\ &= AC^2 + \frac{1}{4} AB^2 - AC\ AB \cos \alpha \qquad (1) \end{aligned}$$

From above $AB \geq BC \Rightarrow \cos \alpha \geq \frac{AC}{2\ AB}$ so from (1)

$$CD^2 \leq AC^2 + \frac{1}{4} AB^2 - AC \quad AB \frac{AC}{2\ AB}$$

$$= \frac{1}{2} AC^2 + \frac{1}{4} AB^2$$

$$\leq \frac{3}{4} k^2 .$$

Case (ii). If $AB \geq AC$, then by symmetry we can obtain the same result as in Case (i). Therefore take $AB < AC$. By symmetry we can without loss of generality take $BC \geq AC$. Then

$$BC^2 = AC^2 + AB^2 - 2AC \quad AB \cos \alpha \tag{2}$$

$$BC^2 - AC^2 = AB^2 - 2AC \quad AB \cos \alpha$$

so

$$AB^2 - 2AC \quad AB \cos \alpha \geq 0$$

$$AB - 2AC \quad \cos \alpha \geq 0$$

$$\cos \alpha \leq \frac{AB}{2\ AC} \tag{3}$$

Combining (2) with (1) gives

$$CD^2 = BC^2 - \frac{3}{4} AB^2 + AC \quad AB \cos \alpha$$

$$\leq BC^2 - \frac{3}{4} AB^2 + AC \quad AB \frac{AB}{2\ AC}$$

$$= BC^2 - \frac{1}{4} AB^2$$

$$\leq k^2 - \frac{c^2}{4} .$$

<u>Case (iii)</u>. Now $c \leq AC \leq \frac{1}{2} AB$.

$$CD^2 = AC^2 + AD^2 - 2AC\ AD \cos \alpha \tag{4}$$

For the given triangle CD depends only on AD. Differentiating,

$$\frac{d(CD^2)}{d(AD)} = 2\,AD - 2\,AC \cos \alpha$$

$$= 0 \quad \text{for} \quad \cos \alpha = \frac{AD}{AC}, \quad \text{i.e.,} \quad CD \perp AB;$$

this gives a minimum value for CD. Thus CD achieves its absolute maximum value wither at $AD = c$ or at $AD = \frac{1}{2} AB$. If this maximum is at $AD = \frac{1}{2} AB$ we are done by Cases (i) and (ii) (note that in these cases the fact that $AB \leq 2c$ was used only to deduce that $AD = \frac{1}{2} AB$). We therefore suppose that the absolute maximum occurs when $AD = c$, i.e., (from (4)) that

$$AC^2 + c^2 - 2\,AC\,c \cos \alpha \geq AC^2 + \frac{1}{4} AB^2 - AC\ AB \cos \alpha$$

($AD = c$ on left-hand side, $AD = \frac{1}{2} AB$ on right-hand side) so

$$c^2 - \frac{1}{4} AB^2 \geq (AC \cos \alpha)(2c - AB)$$

$$(c + \frac{1}{2} AB)(c - \frac{1}{2} AB) \geq (2AC \cos \alpha)(c - \frac{1}{2} AB)$$

$$\Rightarrow \qquad c + \frac{1}{2} AB \leq 2AC \cos \alpha, \quad \text{since} \quad c - \frac{1}{2} AB < 0$$

i. e.,

$$\frac{c + \frac{1}{2} AB}{2\,AC} \leq \cos \alpha .$$

Now with $AD = c$

$$CD^2 = AC^2 + c^2 - 2AC\, c \cos \alpha$$

$$\leq AC^2 + c^2 - 2AC\, c \,\frac{c + \frac{1}{2} AB}{2\, AC}$$

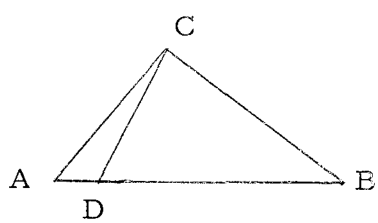
$$= AC^2 - \frac{1}{2} c\, AB$$

$$\leq k^2 - \frac{1}{2} c^2 .$$

Finally, combining the results of the three cases yields the conclusion of the lemma (use the fact that $k \geq AB > c$).

Definition. When dealing with a vertex p_i in the algorithm (i.e. running through steps 3-9), a new edge is an edge formed by subdivision which is not part of any edge previously present.

Example.



Suppose that when dealing with A we divide the edge AB at D, then join D to C. Now CD is a new edge; AD and DB are not.

Observation 5.4. When dealing with a fixed p_i the algorithm does not increase the number of simplexes having p_i as a vertex. Consequently, after considering all S_j^{n-1} of which p_i is a vertex, step 6 will always return us to step 5 and so we eventually get returned to step 2. We have now "surrounded" p_i by acceptable simplexes, i.e., every simplex of which p_i is a vertex is now acceptable, by virtue of steps 7, 8, and 9. This property will be retained for the remainder of the algorithm (recall Remark 5.1).

Notation. Let S_0 denote the largest diameter of the original simplexes $S_1^{n-1}, \ldots, S_\ell^{n-1}$. Let S_r, $r \geq 1$, denote the largest diameter of the unacceptable simplexes present after we have surrounded $p_1, p_2, \ldots, p_r$ by acceptable simplexes (we take $S_r = 0$ iff all simplexes are acceptable).

Lemma 5.5. When dealing with any vertex p_i in the algorithm, any new edge constructed in an unacceptable simplex lies in a triangle such that the edge and triangle satisfy the properties ascribed to CD and ABC respectively in Lemma 5.3, with $k = S_{i-1}$ and c that given in Section V. 1.

Proof. If $(p_i p_{i'})$ is an edge which we divide at p_j $(p_j \neq p_i, p_{i'})$ then the new edges resulting from this division are of the form $(p_k p_j)$ where $k \neq i, i'$ (see step 9(b)). Any such edge will lie in the triangle $(p_i p_{i'} p_k)$ which is a face of an unacceptable simplex by hypothesis. Here we have a correspondence $p_i \to A$, $p_{i'} \to B$, $p_k \to C$, $p_j \to D$ to Lemma 5.3.

We must have length $(p_i p_{i'}) > c$ as otherwise this edge would not have been divided (see Section V.1 and step 9 of the algorithm). The diameter of the triangle $(p_i p_{i'} p_k)$ is at most S_{i-1} by definition of S_{i-1}. Finally, the restrictions on the length of $(p_i p_j)$ follow from step 9(b) of the algorithm and the fact that $\delta(p_i) \geq c$.

<u>Theorem 5.6.</u> After a finite number of iterations of the algorithm all simplexes are acceptable.

<u>Proof</u>. Clearly $0 \leq S_{r+1} \leq S_r$ for $r = 0, 1, 2, \ldots$. We show that for some R, $S_R \leq c$; from this point onwards step 9(a) of the algorithm will always return us to step 8. As a result all simplexes will eventually be acceptable (recall Observation 5.4).

Consider the original vertices $p_1, p_2, \ldots, p_m$. After we have dealt with these vertices, any part of an original edge which now lies in an unacceptable simplex has either resulted from a bisection or from a subdivision which removed a length of at least c. Thus its length is at most $\max\{\frac{1}{2} S_0, S_0 - c\}$. Regarding other edges of unacceptable simplexes (i.e., those new edges constructed in the algorithm), by Lemma 5.5 such edges arise precisely as CD does in Lemma 5.3, and it follows that the length of any such edge is at most $(S_0^2 - \frac{1}{4}c^2)^{1/2}$. Combining these results gives

$$S_m^2 \leq \max\{\frac{1}{4}S_0^2, (S_0 - c)^2, S_0^2 - \frac{1}{4}c^2\} .$$

We now have a list $p_1, \ldots, p_m, p_{m+1}, \ldots, p_{m'}$ say of vertices. Apply the same argument to $p_{m+1}, \ldots, p_{m'}$ to get

$$S_{m'}^2 \leq \max\{\frac{1}{4}S_m^2, (S_m - c)^2, S_m^2 - \frac{1}{4}c^2\} .$$

Continue thus. It is clear that the sequence $S_0, S_m, S_{m'}, \ldots$ must eventually be less than or equal to c. By the opening remarks the proof is complete.

<u>Corollary 5.7</u>. After a finite number of iterations the algorithm terminates.

<u>Proof</u>. Use Observation 5.2.

<u>Observation 5.8.</u> Since $b(b(P^n)) = 0$ for the original representation of $b(P^n)$, every (n-2)-simplex lying in $b(S_1^{n-1})$ for some S_1^{n-1} in $b(P^n)$ must also lie in $b(S_2^{n-1})$ for at least one other (n-1)-simplex S_2^{n-1} in $b(P^n)$. If in the algorithm this (n-2)-simplex is subdivided, it is subdivided in all (n-1)-simplexes containing it and from step 9(b) we see that new (n-2)-simplexes appear in exactly as many boundaries of (n-1)-simplexes as their

ancestors did, and with exactly the same orientation. Thus $b(b_1(P^n)) = 0$, where $b_1(P^n)$ is the new representation of $b(P^n)$ obtained by means of the algorithm.

Computation of the Degree. Since P^n is an n-region, by Lemma 4.1 the original representation of $b(P^n)$ is a polyhedron. The subdivisions of the algorithm are such that it preserves this property of $b(P^n)$. We will thus obtain a polyhedral decomposition of $b(P^n)$ into acceptable simplexes. This gives an impartial refinement of $b(P^n)$ relative to $\operatorname{sgn} \Phi^n$: take maximal connected collections of simplexes to be (n-1)-regions, avoiding overlapping, and use the fact that $b(b_1(P^n)) = 0$ (Observation 5.8) to check part (iv) of the definition. The degree $d(\Phi^n, P^n, \theta^n)$ can then be computed using the formula of Theorem 3.6.

VI. SIMPLIFICATION OF THE COMPUTATIONS

In this chapter a procedure is given which replaces the computation of the $n \times n$ determinants in Theorem 3.6 by a "scanning" of the matrices associated with these determinants, that is a counting of ± 1's in certain positions within the matrices.

As usual let P^n be an n-region with $\Phi^n = (\varphi_1, \ldots, \varphi_n) : P^n \rightarrow R^n$ continuous. The next lemma and its two corollaries will be used later when we need to choose simplexes on which certain components of Φ^n have certain signs.

Lemma 6.1. Suppose that $b(P^n)$ is impartially refined relative to $\operatorname{sgn} \Phi^n$ in such a way that it is a polyhedron. Then if any coordinate function φ_i has the same sign Δ $(\Delta = \pm 1)$ at all vertices of any simplex S_j^{n-1} in $b(P^n)$, we may assume that $\operatorname{sgn} \varphi_i = \Delta$ on all of S_j^{n-1} without altering the value of $d(\Phi^n, P^n, \theta^n)$.

Proof. Suppose that we do not have $\operatorname{sgn} \varphi_i = \Delta$ on all of S_j^{n-1}. Now S_j^{n-1} lies in some region $\beta_{j'}^{n-1}$ and so $\varphi_{r_{j'}} \neq 0$ on S_j^{n-1}, where of course $r_{j'} \neq i$.

By continuity $\varphi_{r_{j'}} \neq 0$ on some open neighborhood M of S_j^{n-1}. Take M so small that it does not contain any vertex (of any simplex in the impartial refinement of $b(P^n)$) at which $\operatorname{sgn} \varphi_i = 0$

or $-\Delta$ (this can be done because of the assumed polyhedral property of the impartial refinement). Choose another open set N such that $S_j^{n-1} \subset N \subset \bar{N} \subset M$. Now homotopically deform φ_i where necessary so that inside N it has constant sign Δ, while outside M it is unchanged.

Since $\varphi_{r_{j'}} \neq 0$ on M, $\Phi^n \neq \theta^n$ during the homotopy and by [2, Theorem 6.4, p. 31] $d(\Phi^n, P^n, \theta^n)$ is unaffected. Note that the same impartial refinement can be retained since the sign of φ_i is unchanged at all points considered in the definition.

<u>Corollary 6.2.</u> Under the same hypotheses as those of Lemma 6.1, we may assume that S_j^{n-1} lies in a region $\beta_{j'}^{n-1}$ of the impartial refinement such that $r_{j'} = i$.

<u>Proof.</u> Since the refinement $b_1(P^n)$ of $b(P^n)$ is polyhedral and refinements are always assumed to preserve orientations, $b(b_1(P^n)) = 0$ (see Observation 5.8).

Form $\beta_{j'}^{n-1}$ as the maximal connected union containing S_j^{n-1} of those $(n-1)$-simplexes in $b_1(P^n)$ on which $\operatorname{sgn} \varphi_i = \Delta$. Then after redefining some other $(n-1)$-regions to eliminate overlapping if necessary, $b_1(P^n)$ is still an impartial refinement because it is a polyhedron and because $b(b_1(P^n)) = 0$.

<u>Notation</u>. If $B_i = (b_{i1}, b_{i2}, \ldots, b_{iq})$, $i = 1, 2, \ldots, q$ are q

vectors, then

$$M^q(B_1, \dots, B_q) = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \cdot & \cdot & \dots & \cdot \\ b_{q1} & b_{q2} & \dots & b_{qq} \end{pmatrix}$$

(q x q matrix).

Of course $\det(M^q(B_1, \dots, B_q)) = \Delta^q(B_1, \dots, B_q)$ in the notation of Section I. 2.

Let $b(P^n) = \sum_{k=1}^{\ell} t_k \langle y_1^{(k)} \dots y_n^{(k)} \rangle$ be a sufficient refinement relative to $\operatorname{sgn} \Phi^n$, where $t_k = \pm 1$. Its associated matrices are

$$t_k M^n(\operatorname{sgn} \Phi^n(y_1^{(k)}), \dots, \operatorname{sgn} \Phi^n(y_n^{(k)})), \quad k = 1, 2, \dots, \ell \tag{1}$$

Let $\{\beta_j^{n-1} : j \in J_{\Delta_n}\}$ be the set of regions β_j^{n-1} in the given sufficient refinement for which the associated coordinate function is φ_{r_n} with $\operatorname{sgn} \varphi_{r_n} \big|_{\beta_j^{n-1}} = \Delta_n$; here $r_n \in \{1, 2, \dots, n\}$ and $\Delta_n \in \{-1, 1\}$.

Take $\beta_j^{n-1} = \sum_{k \in K_j} \langle y_1^{(k)} \dots y_n^{(k)} \rangle$ for each $j \in J_{\Delta_n}$ so

$$b(\beta_j^{n-1}) = \sum_{k \in K_j} \sum_{i=1}^{n} (-1)^{i+1} \langle y_1^{(k)} \dots \hat{y}_i^{(k)} \dots y_n^{(k)} \rangle$$

with associated matrices

$$(-1)^{i+1}M^{n-1}(\operatorname{sgn} \Phi_{r_n}^{n-1}(y_1^{(k)}), \ldots, \widehat{\operatorname{sgn}} \Phi_{r_n}^{n-1}(y_i^{(k)}), \ldots, \operatorname{sgn} \Phi_{r_n}^{n-1}(y_n^{(k)}))$$

$$i = 1, 2, \ldots, n, \; k \in K_j, \; j \in J_{\Delta_n} \qquad (2)$$

Corollary 6.3. Suppose that the above sufficient refinement is actually a polyhedral impartial refinement. Then the matrices M^{n-1} of (2) can be obtained as follows: choose from (1) those M^n whose r_nth column consists entirely of Δ_n's, then delete the ith row and r_nth column from these matrices, where i ranges over the values $1, 2, \ldots, n$.

Proof. Consider the matrices

$$t_k M^n(\operatorname{sgn} \Phi^n(y_1^{(k)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(k)})), \quad \langle y_1^{(k)} \ldots y_n^{(k)} \rangle \subseteq \beta_j^{n-1},$$

$$j \in J_{\Delta_n} \qquad (3)$$

Each matrix in (3) is a matrix in (1) whose r_nth column consists entirely of Δ_n's; by Corollary 6.2 we may assume that all such matrices in (1) are actually matrices in (3). Thus choosing from (1) those matrices whose r_nth column consists entirely of Δ_n's is a means of listing the matrices in (3). Now each matrix in (2) is obtained by deleting the ith row and r_nth column from a matrix in (3). This completes the proof.

Using the above notation, consider the matrices

$$t_k M^n(\operatorname{sgn} \Phi^n(y_1^{(k)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(k)})), \quad k = 1, 2, \ldots, \ell \qquad (1)$$

Recall that their associated determinants Δ^n were used to compute $d(\Phi^n, P^n, \theta^n)$ in Theorem 3.6. In the procedure below a signature is assigned to certain of these matrices, then the signatures are added; it is shown later that this yields $d(\Phi^n, P^n, \theta^n)$.

Procedure. Let $\{r_i : 1 \leq i \leq n\} \subseteq \{1, 2, \ldots, n\}$ with $r_i \leq i$ $\forall$ i. Let $\{\Delta_i : 1 \leq i \leq n\} \subseteq \{-1, 1\}$.

Choose from (1) all $t_k M^n$ whose r_nth column consists entirely of Δ_n's. Assign a temporary signature $t_k \Delta_n (-1)^{r_n+1}$ to each such matrix; assign the signature zero to every other M^n.

Delete the r_nth column from each chosen matrix to form an $n \times (n-1)$ array. In the r_{n-1}th column of this array pick all combinations of $n-1$ rows having $n-1$ Δ_{n-1}'s as entries, if any such combination exists (if not, discard the matrix, i.e., assign it the signature zero) (if the matrix is not discarded there will be either 1 or n such combinations).

If there is one such combination, suppose that the q_{n-1}th row is the unique row with entry 0 or $-\Delta_{n-1}$. Delete this row to give an $(n-1) \times (n-1)$ matrix and assign a temporary signature $t_k \Delta_n \Delta_{n-1} (-1)^{r_n + 1 + r_{n-1} + q_{n-1}}$ to the matrix.

If there are n such combinations (i.e., r_{n-1}th column contains n Δ_{n-1}'s) then deleting each row in turn gives $n-1$ Δ_{n-1}'s in the r_{n-1}th column. Do this, obtaining n $(n-1) \times (n-1)$ matrices with associated temporary signatures $t_k \Delta_n \Delta_{n-1} (-1)^{r_n + 1 + r_{n-1} + q_{n-1}}$, where the q_{n-1}th row was the one deleted (so q_{n-1} runs through the values $1, 2, \ldots, n$).

Now deal with each $(n-1) \times (n-1)$ matrix just as each original $n \times n$ matrix was dealt with after assigning the first temporary signature, replacing n by $n-1$ throughout. Continue reducing until left with 1×1 matrices. At this stage the sum of the temporary signatures of those 1×1 matrices whose ancestor was a particular M^n is taken as the signature of that M^n. Finally add all the signatures of the M^n.

We now recall some equations which will be used to prove that the procedure computes $d(\Phi^n, P^n, \theta^n)$.

Again suppose that $b(P^n) = \sum_{k=1}^{\ell} t_k \langle y_1^{(k)} \ldots y_n^{(k)} \rangle$ is sufficiently refined relative to $\operatorname{sgn} \Phi^n$, where $t_k = \pm 1$. Recall Equation (3) of Theorem 3.4:

$$d(\Phi^n, P^n, \theta^n) = \sum_{j \in J_a} (-1)^{r_j+1} i(\Phi^{n-1}_{*r_j}(\beta^{n-1}_j), \theta^{n-1}) \operatorname{sgn} \varphi_{*r_j}\Big|_{\beta^{n-1}_j} .$$

Using (1) of that theorem and Lemma 3.3 this can be written as

$$d(\Phi^n, P^n, \theta^n) = \sum_{j \in J_a} (-1)^{r_j+1} d(\Phi^{n-1}_{r_j}, \beta^{n-1}_j, \theta^{n-1}) \operatorname{sgn} \varphi_{r_j}\Big|_{\beta^{n-1}_j} .$$

Changing the notation a little we have

$$d(\Phi^n, P^n, \ ^n) = \sum_{j \in J_{\Delta_n}} (-1)^{r_n+1} d(\Phi^{n-1}_{r_n}, \beta^{n-1}_j, \theta^{n-1}) \Delta_n$$

$$= (-1)^{r_n+1} \Delta_n \sum_{j \in J_{\Delta_n}} d(\Phi^{n-1}_{r_n}, \beta^{n-1}_j, \theta^n) \qquad (1)$$

where the summation is over all regions β^{n-1}_j in $b(P^n)$ whose associated coordinate function is φ_{r_n} with $\operatorname{sgn} \varphi_{r_n}\Big|_{\beta^{n-1}_j} = \Delta_n$; here $r_n \in \{1, 2, \ldots, n\}$ and $\Delta_n \in \{-1, 1\}$ are arbitrary but fixed.

For each $j \in J_{\Delta_n}$ take

$$\beta^{n-1}_j = \sum_{k \in K_j} \langle y^{(k)}_1 \ldots y^{(k)}_n \rangle$$

so

$$b(\beta_j^{n-1}) = \sum_{k \in K_j} \sum_{i=1}^{n} (-1)^{i+1} \langle y_1^{(k)} \dots \hat{y}_i^{(k)} \dots y_n^{(k)} \rangle .$$

Consider the associated matrices

$$(-1)^{i+1} M^{n-1}(\operatorname{sgn} \Phi_{r_n}^{n-1}(y_1^{(k)}), \dots, \widehat{\operatorname{sgn} \Phi_{r_n}^{n-1}(y_i^{(k)})}, \dots, \operatorname{sgn} \Phi_{r_n}^{n-1}(y_n^{(k)})),$$

$$i = 1, 2, \dots, n, \; k \in K_j, \; j \in J_{\Delta_n} \tag{2}$$

<u>Observation 6.4.</u> Consider the special case $n = 1$. Let $P^1 = \sum_{i=0}^{m-1} \langle x_i x_{i+1} \rangle$ so that $b(P^1) = \langle x_m \rangle - \langle x_0 \rangle$, with associated 1×1 matrices $\Phi^1(x_m)$, $-\Phi^1(x_0)$.

Of necessity $r_1 = 1$. As usual $\Delta_1 \in \{-1, 1\}$. If $\Phi^1(x_m) = \Delta_1$ it is assigned the signature $\Delta_1 (-1)^{r_1+1} = \Delta_1$; if not, it is assigned the signature zero. If $\Phi^1(x_0) = \Delta_1$ it is assigned the signature $(-1)\Delta_1 (-1)^{r_1+1} = -\Delta_1$; if not, it is assigned the signature zero. Finally the signatures are added.

<u>Theorem 6.5.</u> Let P^n be an n-region with $\Phi^n : P^n \to R^n$ continuous. Suppose $b(P^n)$ has a polyhedral impartial refinement relative to $\operatorname{sgn} \Phi^n$ so that it is sufficiently refined with notation as above. Then the procedure computes $d(\Phi^n, P^n, \theta^n)$.

Proof. Use induction on n. By inspection the theorem holds for the case $n = 1$ (see Observations 6.4 and 2.6). Now fix $n > 1$ and assume that the theorem is true for the $n-1$ case. Then the procedure deals with the matrices

$$t_k M^n(\operatorname{sgn} \Phi^n(y_1^{(k)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(k)})), \quad k = 1, 2, \ldots, \ell \tag{3}$$

By Theorem 4.2 and Lemma 4.1 we are justified in applying the inductive hypothesis to calculate $\sum_{j \in J_{\Delta_n}} d(\Phi_{r_n}^{n-1}, \beta_j^{n-1}, \theta^{n-1})$. Thus, letting j vary over J_{Δ_n}, choose from (2) all those matrices $(-1)^{i+1} M^{n-1}$ whose r_{n-1}th column consists entirely of Δ_{n-1}'s and assign a temporary signature

$$(-1)^{i+1} \Delta_{n-1} (-1)^{r_{n-1}+1} = \Delta_{n-1} (-1)^{r_{n-1}+i}$$

to each. Apply the procedure from this point to compute the signature of each $(-1)^{i+1} M^{n-1}$, then add these signatures.

To now calculate $d(\Phi^n, P^n, \theta^n)$, use (1): multiply the signature of each $(-1)^{i+1} M^{n-1}$ by $(-1)^{r_n+1} \Delta_n$ and then add these values. By inspection, using Corollary 6.3, this method for computing $d(\Phi^n, P^n, \theta^n)$ is seen to coincide with the procedural method applied to the matrices in (3).

Corollary 6.6. Any $n \times n$ matrix $(n > 1)$ having two columns each of which consists entirely of $+1$'s or of -1's will be assigned the signature zero.

Proof. Suppose that the rth column consists of Δ_r's, the sth column of Δ_s's, where $\{\Delta_r, \Delta_s\} \subseteq \{-1, 1\}$ and $r \neq s$.

Take P^n to be a n-simplex with a continuous function $\Phi^n : P^n \to R^n$ such that

$$\text{sgn } \varphi_s = \Delta_s \quad \text{on all of } P^n$$

and

$$\text{sgn } \varphi_r = \begin{cases} \Delta_r & \text{on } n \text{ vertices of } P^n \\ -\Delta_r & \text{on the remaining vertex.} \end{cases}$$

Then $b(P^n)$ is a polyhedron and is impartially refined relative to $\text{sgn } \Phi^n$, taking all of $b(P^n)$ as an $(n-1)$-region on which $\varphi_s \neq 0$. We may therefore apply Theorem 6.5 with $r_n = r$ and $\Delta_n = \Delta_r$.

In the procedure only one matrix will be chosen because for only one simplex in $b(P^n)$ does $\text{sgn } \varphi_r = \Delta_r$ at all vertices. This matrix has an rth column of Δ_r's and an sth column of Δ_s's; the other entries (if any) are arbitrary. Its signature must be $\pm d(\Phi^n, P^n, \theta^n)$ by Theorem 6.5; however this is zero by Remark 2.7(i) because $\varphi_s \neq 0$ on $P^n \Rightarrow \Phi^n \neq \theta$ on P^n.

Theorem 6.7. Let P^n be an n-region with $\Phi^n : P^n \to R^n$ continuous. Suppose $b(P^n)$ is sufficiently refined relative to sgn Φ^n. Then the procedure computes $d(\Phi^n, P^n, \theta^n)$.

Proof. The proof is almost identical to that of Theorem 6.5. No alteration is needed for the $n = 1$ case. For $n > 1$ the same basic argument holds, but some changes are needed in the results quoted from elsewhere.

Instead of quoting Theorem 4.2 and Lemma 4.1 we appeal to part (iii) of the definition of a sufficient refinement.

We cannot prove a version of Corollary 6.3 for arbitrary sufficient refinements. It guaranteed that any matrix M^n whose r_nth column consisted of Δ_n's corresponded to a simplex in some β_j^{n-1}, $j \in J_{\Delta_n}$ (notation of Theorem 6.5), and so should be chosen at the beginning of the procedure. However any such matrix corresponding to a simplex in a β_k^{n-1} where $k \notin J_{\Delta_n}$ will then have two columns each consisting of $+1$'s or of -1's because of the sufficient refinement, and by Corollary 6.6 may be included among the chosen matrices because it will be assigned the signature zero in any case. This observation should be used in place of Corollary 6.3 in the proof.

Corollary 6.6 indicates that there is superfluous computation involved in the original procedure. We now give a modified procedure designed to circumvent this. This modified procedure will also explicitly discard matrices having a column consisting entirely of zeroes since such matrices are clearly discarded at some stage.

Modified Procedure. Suppose $b(P^n)$ is sufficiently refined relative to $\operatorname{sgn} \Phi^n$ with $b(P^n) = \sum_{k=1}^{\ell} t_k \langle y_1^{(k)} \ldots y_n^{(k)} \rangle$, $t_k = \pm 1$. Consider the associated matrices

$$t_k M^n(\operatorname{sgn} \Phi^n(y_1^{(k)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(k)})), \quad k = 1, 2, \ldots, \ell \tag{1}$$

A signature is assigned to certain of these matrices, then the signatures are added; by what has gone before this will give us $d(\Phi^n, P^n, \theta^n)$.

Let $\{r_i : 1 \leq i \leq n\} \subseteq \{1, 2, \ldots, n\}$ with $r_i \leq i \quad \forall\, i$. Let $\{\Delta_i : 1 \leq i \leq n\} \subseteq \{-1, 1\}$.

Choose from (1) all matrices $t_k M^n$ whose r_nth column consists entirely of Δ_n's. Assign a temporary signature $t_k \Delta_n (-1)^{r_n + 1}$ to each such matrix.

If any other column of the matrix is constant (i.e., all its entries have the same value) assign the signature zero to the matrix, i.e., discard M^n. Otherwise delete the r_nth column to form an $n \times (n-1)$ array. In the r_{n-1}th column of this array pick the combination of $n-1$ rows having $n-1$ Δ_{n-1}'s as entries, if such a combination exists (if not, discard M^n) (there will be at most one such combination otherwise M^n would have been discarded already). Suppose the q_{n-1}th row is the unique row with entry 0 or $-\Delta_{n-1}$. Delete this row to give an $(n-1) \times (n-1)$ matrix and assign a temporary signature $t_k \Delta_n \Delta_{n-1} (-1)^{r_n+1+r_{n-1}+q_{n-1}}$ to $t_k M^n$.

Deal with this $(n-1) \times (n-1)$ matrix just as we dealt with the $n \times n$ situation after choosing the matrices M^n and assigning the first temporary signature, replacing n by $n-1$ thoughout. Continue reducing until left with a 1×1 matrix. The temporary signature at this stage is the signature of $t_k M^n$, i.e.,

$$\text{signature}\,(t_k M^n) = t_k \Delta_n \Delta_{n-1} \cdots \Delta_1 (-1)^{r_n+1+r_{n-1}+q_{n-1}+\cdots+r_1+q_1}.$$

Finally add the signatures of the chosen M^n's.

Example. We recompute $d(\Phi^3, P^3, \theta^3)$ for the tetrahedron example of Section IV.1. This is done for two different sets $\{(r_1, \Delta_1), (r_2, \Delta_2), (r_3, \Delta_3)\}$.

In the notation of Section IV. 1, a sufficient refinement of $b(P^3)$ is given by

$$b(P^3) = \langle a_0 a_1 a_4 \rangle + \langle a_0 a_4 a_3 \rangle + \langle a_1 a_2 a_4 \rangle + \langle a_4 a_2 a_3 \rangle$$
$$+ \langle a_0 a_2 a_1 \rangle + \langle a_3 a_2 a_0 \rangle .$$

The associated matrices are

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Case (i). $r_3 = 1$, $\Delta_3 = -1$, $r_2, r_1, \Delta_2, \Delta_1$ arbitrary. Begin by choosing those matrices whose first column consists entirely of -1's. Since there are none, all matrices are discarded and $d(\Phi^3, P^3, \theta^3) = 0$.

Case (ii). $r_3 = r_2 = r_1 = 1$; $\Delta_3 = \Delta_2 = 1$, $\Delta_1 = -1$. Choose all matrices whose first column consists entirely of +1's; assign a temporary signature $(+1)(-1)^{1+1} = +1$ to each, writing it in front of the matrix:

$$(+1)\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & \cdot 1 & 1 \end{pmatrix}, \quad (+1)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad (+1)\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Discard the second matrix because its third column is also constant.

Delete the first column from the remaining two matrices:

$$(+1)\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (+1)\begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Choose the arrays whose first column contains two $+1$'s; assign the temporary signature $(+1)(+1)(-1)^{1+i}$ to each where the ith row is the one containing the -1; delete the ith row:

$$(+1)\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (+1)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Delete the first column from each matrix:

$$(+1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad (+1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Choose the arrays whose first column contains one -1; assign the temporary signature $(+1)(-1)(-1)^{1+i}$ to each where the ith row is the one containing the $+1$; delete the ith row. Since this gives 1×1 matrices the temporary signature becomes the signature. Thus the signatures here are $+1$, -1.

Finally $d(\Phi^3, P^3, \theta^3)$ = sum of signatures

$= 1 - 1$

$= 0.$

Remark 6.8. In the modified procedure it is clear that the signature of any matrix is -1, 0, or $+1$.

Corollary 6.9 (to Theorem 6.7). Let P^n be an n-region, $n > 1$, with $\Phi^n : P^n \rightarrow R^n$ continuous. Suppose that $b(P^n)$ is sufficiently refined relative to sgn Φ^2. Let $m = |d(\Phi^n, P^n, \theta^n)|$. Then $b(P^n)$ is subdivided into at least $m\, n\, 2^{n-1}$ simplexes.

Proof. By Theorem 6.7 the modified procedure gives $d(\Phi^n, P^n, \theta^n)$ for any choice of $\{r_i : 1 \leq i \leq n\} \subseteq \{1, 2, \ldots, n\}$ satisfying $r_i \leq i \ \forall\, i$, and $\{\Delta_i : 1 \leq i \leq n\} \subseteq \{-1, 1\}$.

There are $2n$ possibilities for the (ordered) pair (r_n, Δ_n). Now any matrix chosen for two distinct pairs will have signature zero by the modified procedure (two columns will be constant). Consequently we can assume that there is no overlap in choice; count the minimum number of matrices needed to give $|d(\Phi^n, P^n, \theta^n)| = m$ for a fixed pair (r_n, Δ_n) then multiply this number by $2n$.

Fix (r_n, Δ_n). Consider the matrices M^n chosen for this pair. Delete the r_nth column from each M^n to leave an $n \times n-1$ array. Choose r_{n-1} satisfying $1 \leq r_{n-1} \leq n-1$. Then the matrices chosen at this stage for the pair $(r_{n-1}, 1)$ must not overlap with those chosen for the pair $(r_{n-1}, -1)$ if $n > 2$ because overlapping implies the existence of an $n \times (n-1)$ array with a column containing $(n-1)$ $+1$'s and $(n-1)$ -1's. Thus the minimum number of matrices needed for the pair (r_n, Δ_n) is at least twice the minimum number needed for the pair $(r_{n-1}, 1)$.

We can apply the last argument repeatedly going from the ith to (i-1)th stage for $i = n, n-1, \ldots, 4, 3$. Each stage yields a factor of two, so with the original factor of $2n$ this gives a factor $(2n)(2)^{n-2} = n\,2^{n-1}$.

When $i = 2$ is reached all that can be said is that by Remark 6.8 at least m matrices are needed. This gives the final lower bound of $m\,n\,2^{n-1}$.

Corollary 6.10. Let P^n be an n-region, $n > 1$, with $\Phi^n : P^n \to R^n$ continuous. Suppose that $b(P^n)$ is sufficiently refined relative to sgn Φ^2. If $d(\Phi^n, P^n, \theta^n) \neq 0$, then $b(P^n)$ is subdivided into at least $n\,2^{n-1}$ simplexes.

Proof. Immediate from Corollary 6.9.

Example. Consider once more the tetrahedron example of Section IV.1. Here $n = 3$ so $n\,2^{n-1} = 12$. However we have a sufficient refinement of $b(P^3)$ consisting of 6 2-simplexes. By Corollary 6.10 $d(\Phi^3, P^3, \theta^3) = 0$.

Remark. The result of Corollary 6.9 can be improved for $n \geq 6$ using the Theorem 3.6 formula

$$d(\Phi^n, P^n, \theta^n) = \frac{1}{2^n\, n!} \sum_{k=1}^{\ell} t_k\, \Delta^n(\operatorname{sgn} \Phi^n(y_1^{(k)}), \ldots, \operatorname{sgn} \Phi^n(y_n^{(k)})).$$

Hadamard's determinant theorem [4] tells us that for any determinant Δ^n in this sum, $|\Delta^n| \leq n^{n/2}$. Consequently under the hypotheses of Corollary 6. 9 the above sum must contain at least $m\, 2^n\, n!/n^{n/2}$ terms, and for $n \geq 6$ it is easy to check that this is greater than $m\, n\, 2^{n-1}$. We conjecture that in fact the lower bound can be increased to $2m\, n!$. Then Corollary 6. 10 would have the lower bound $2\, n!$. If so, this is certainly the best possible estimate: let P^n be the cube of side 2 in R^n with vertices all of whose coordinates are ± 1. We give P^n the standard "counterclockwise" orientation. If $\Phi^n(x) = x$ for all x in P^n, then it is easy to show that $d(\Phi^n, P^n, \theta^n) = 1$. Now [7] gives a simplicial decomposition of P^n which, as can be checked, readily yields a sufficient refinement of $b(P^n)$ relative to $\operatorname{sgn} \Phi^n$, and this sufficient refinement contains $2\, n!$ $(n-1)$-simplexes.

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