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Fundamental Solution to the Heat Equation with a Discontinuous Diffusion Coefficient and Applications to Skew Brownian Motion and Oceanography

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Enrique Thomann

In this paper, we derive the fundamental solution to the heat equation with a discontinuous diffusion coefficient in the free space, with an absorbing boundary, and with a reflecting boundary. We use the fundamental solution with an absorbing boundary to make connections with the transition probability density of absorbed Skew Brownian Motion (SBM) and derive a formula for the first passage time of SBM. We also consider an oceanographic situation where the steepness of the continental shelf and slope at the shelfbreak are different. This creates a discontinuous coefficient in the Arrested Topographic Wave (ATW) equation, which is a form of the heat equation. Using the fundamental solution with a reflecting boundary naturally applies to the ATW and from this result, we can derive a formula the vertical velocity (the up- and downwelling) of ocean currents at the shelfbreak.

Fundamental Solution to the Heat Equation with a Discontinuous Diffusion Coefficient and Applications to Skew Brownian Motion and Oceanography

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1. INTRODUCTION

The solution to the heat equation has many applications including diffusion processes, pricing stock derivatives, and Brownian motion. It is natural to consider the case of a discontinuous diffusion coefficient because of situations involving a composite material. That is, if two media with different qualities (e.g. specific heat and thermal diffusivity) are brought into contact, then at the interface at which these two materials meet the coefficients of the equation will be discontinuous.

To solve the heat equation in other contexts, initial data and boundary conditions are needed. The interpretation of the equation determines what the necessary initial and boundary conditions are. For example, if we wish to describe the sea surface of the ocean near a coastline, we would use a reflecting boundary condition at the shoreline. This means, for a perturbation initiated somewhere on the surface of the water, if it 'hits' the coast it just 'bounces back.' Another example is if we wish to know the probability that the effects of perturbation reach a certain point (for the first time) in a given time, starting from a given position. In this case, we would use an absorbing boundary at the point of interest. We are only interested in the perturbation's effects reaching the point in question, so what it would have done afterwards is unimportant; we may as well have 'removed' the perturbation once it hits the absorbing boundary. The initial data in this oceanographic context is to prescribe the sea surface for a particular across shore cross-section.

It is these two examples that we focus on in this paper. Specifically, we consider the situation where there is an interface between media of different qualities. We consider an underwater continental shelf with a given slope and at the shelfbreak to deep ocean there is another slope. It is these differences in steepness on the shelf and that of the slope to deep ocean that cause a discontinuous diffusion coefficient in the heat equation we will use.

In Section 2, we provide the oceanographic background to derive the Arrested Topographic Wave (ATW) which is a parabolic equation that models the sea surface elevation. It is in this section that we define the notation and state the problem for which we find the fundamental solution. In Section 3, we discuss some basic probability concepts concerning Brownian motion and explain how to find the first passage time. Section 4 contains the major results which include the fundamental solution for an infinite domain, for an absorbing boundary, and for a reflecting boundary. We use these results in Section 5 to derive the first passage time for skew Brownian motion and the vertical velocity of ocean currents.

2. OCEANOGRAPHIC APPLICATION

The main motivation for this work is the application to shelfbreak upwelling currents (specifically the Malvinas Current off the coast of Argentina). Ideally, we would like to know how the steepness of the continental shelf, its slope to deep ocean, and the bottom friction coefficient effect the upwelling currents at the shelfbreak.

2.1. The Physical Problem

Assume there is a straight coastline running north-south with the ocean to the east. Let x and y be the cross-shore and alongshore coordinates, respectively. Let $x = -L$ be the location of the coastline and $x = 0$ be the location of the edge of the shelf. We make the assumptions that the jet flowing in the direction of the coastally trapped waves is steady and quasigeostrophic. Then the motion and continuity of the depth-averaged linearized equations are

$$\begin{aligned} -fv &= -g\frac{\partial \eta}{\partial x} \\ fu &= -g\frac{\partial \eta}{\partial y} - \frac{rv}{h} \\ \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} &= 0 \end{aligned} \tag{2.1}$$

where u and v are the depth mean velocities in the respective x and y directions, η is the sea surface elevation, f is the Coriolis parameter (negative in the southern hemisphere), r is the bottom friction coefficient, g is the gravitational constant, and h is the water depth. With an additional assumption that there is no alongshore variations in water depth (i.e. $\frac{\partial h}{\partial y} = 0$), eliminating u and v from (2.1), we have that

$$\frac{\partial \eta}{\partial y} = -\frac{r}{f}\left(\frac{\partial h}{\partial x}\right)^{-1}\frac{\partial^2 \eta}{\partial x^2} \tag{2.2}$$

Equation (2.2) is referred to as the arrested topographic wave equation. This can also be interpreted as the heat equation where y is the time variable and the diffusion coefficient is $D = -\frac{r}{f}\left(\frac{\partial h}{\partial x}\right)^{-1}$. Because of the different slopes on the shelf and at the shelfbreak (hereafter referred to as the 'interface'), the diffusion coefficient $D(x)$ is different on either side of interface. Let $D(x) = D_-$ be the diffusion coefficient on the shelf (when $-L < x < 0$) and $D(x) = D_+$ for $x > 0$, i.e.,

$$D(x) = \begin{cases} D_- & -L < x < 0 \\ D_+ & x > 0 \end{cases} \tag{2.3}$$

At the coast, there is no alongshore velocity ($v = 0$), so we take $\frac{\partial \eta}{\partial x} = 0$. This is called a Neumann boundary condition, or a reflecting boundary condition and in the context of the heat equation, it is analogous to an insulated boundary. The sea surface should be continuous over the shelfbreak as should the across-shore derivatives. In solving the heat equation, initial conditions are given for time, $t = 0$, but in the oceanographic context we will take initial data for $y = 0$. This corresponds to having data prescribing the sea surface elevation at a particular across shore cross-section.

Then to find the sea surface elevation η, we need to solve (2.2) under the following interface, boundary, and initial conditions

$$\left.\frac{\partial \eta}{\partial x}\right|_{x=0^+} = \left.\frac{\partial \eta}{\partial x}\right|_{x=0^-} \qquad \eta(0^+, y) = \eta(0^-, y) \tag{2.4}$$

$$\left.\frac{\partial \eta}{\partial x}\right|_{x=-L} = 0 \tag{2.5}$$

$$\eta(x, 0) = \eta_0(x) \tag{2.6}$$

This problem has been solved before (Hill 1995, and Miller et. al. 2011) using the initial data

$$\eta_0(x) = \begin{cases} 0 & -L < x < 0 \\ \eta_*(e^{-mx} - 1) & x > 0 \end{cases} \tag{2.7}$$

where η_* and m are constants. The methods that have been used to solve (2.2) combined with (2.7) in the past are very similar to the methods used in this paper. However, in previous works, the initial data was introduced at the very start of solving the problem. This paper derives a fundamental solution of a generalization of (2.2). The advantage to this approach is that we have the flexibility to apply different initial data at the end of the process, rather than at the beginning. In the end, we can find η by integrating the fundamental solution against the initial data.

2.2. Notation and Statement of the Physical Problem

Because of the various applications of the results in this paper, we will need to carefully distinguish which context we are in so as not to confuse variables (x and y in particular). We will refer to $k(x'; x, t)$ as the fundamental solution to our problem (which will be stated shortly), where x' is the location of the delta function of the initial data, x is the spatial variable, and t is the time variable.

We avoid using y as a variable in our results because of the oceanographic application; y is the along-shelf variable (but plays the role of t in our notation). The variable x in the oceanographic context is the cross shore variable (while it is a spatial variable in our notation).

With the notation of $k(x'; x, t)$, the problem we are trying to solve can be stated as

$$\frac{\partial k}{\partial t} = D(x) \frac{\partial^2 k}{\partial x^2} \tag{2.8}$$

$$\mu \frac{\partial k}{\partial x}\bigg|_{x=0^+} = (1-\mu) \frac{\partial k}{\partial x}\bigg|_{x=0^-} \qquad k\big|_{x=0+} = k\big|_{x=0^-} \tag{2.9}$$

$$k(x'; x, 0) \quad = \quad \delta_{x'}(x) \tag{2.10}$$

Now we make a few comments on the statement of this problem. By $D(x)$ we mean that $D(x) = D_-$ if $x < 0$ and similarly, $D(x) = D_+$ if $x > 0$ where $D_\pm$ are constant quantities. We are making the problem much more general than the oceanography application by using μ at the interface condition ($\mu = 1/2$ in (2.4)), which will make it easier to interpret our results in terms of transition probability densities. Note that the domain has not been specified, nor have the boundary conditions. This is because in the results section (part 4), we consider three different domains. The first is an infinite domain (with no boundaries). The second two are half-line problems; one has an absorbing boundary (which will relate to the first passage time of skew Brownian motion) and the other has a reflecting boundary (which models the oceanographic application).

As stated previously, the sea surface η can be found by integrating the fundamental solution against the initial data:

$$\eta(x,y) = \int_D k(x';x,y)\eta_0(x')dx' \tag{2.11}$$

2.3. Computing the Vertical Velocity

Let $w(z)$ be the vertical velocity (the up- or downwelling of the ocean currents at the interface) and z the coordinate that describes the depth of the water. Because of incompressible flow, we have that $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ and then using the relations in (2.1),

$$\begin{aligned} -\frac{\partial w}{\partial z} &= -\frac{g}{f}\frac{\partial^2 \eta}{\partial x \partial y} - \frac{r}{f}\frac{\partial}{\partial x}\left(\frac{v}{h}\right) + \frac{g}{f}\frac{\partial^2 \eta}{\partial x \partial y} \\ &= -\frac{r}{f}\frac{\partial}{\partial x}\left(\frac{v}{h}\right) \end{aligned}$$

Integrating over the water column, (where the depth is h), we find that

$$\begin{aligned} w(-h) &= \frac{-rh}{f}\frac{\partial}{\partial x}\left(\frac{v}{h}\right) \\ &= \frac{-rh}{f}\left(\frac{1}{h}\frac{\partial v}{\partial x} - \frac{v}{h^2}\frac{\partial h}{\partial x}\right) \\ &= \frac{rg}{f^2}\left(\frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial^2 \eta}{\partial x^2}\right) \end{aligned} \tag{2.12}$$

We will return to this computation once we have the fundamental solution explicitly.

3. PROBABILITY

A further motivation for this work is to make connections between the fundamental solution with an absorbing boundary and the first passage time of skew Brownian motion (SBM). Note, in this section only, we will use the standard probability notation of $p(t, x, y)$ and avoid using $k(x'; x, t)$ as much as possible. First, we start with a few basic concepts of stochastic processes.

3.1. Brownian Motion

3.1.1 Random Walk

There are several ways to think about the construction of Brownian motion. The simplest idea is to start from the discrete situation of a random walk and then move on to the continuous model which will be Brownian motion. To motivate why the phrase 'random walk' is used, suppose that a walker is standing at zero on a one-dimensional number line and that there is equal probability that they will walk to $+1$ or -1. In fact, at every integer, the probability that the walker will move left or right is always one half. That is, if $Z_1, Z_2, ...$ are independent random variables such that $P(Z_i = 1) = P(Z_i = -1) = 1/2$, then define a simple random walk to be $S_n = \sum_{i=1}^{n} Z_i$. Here, we assume that $S_0 = 0$ which is analogous to starting at the origin of our number line. It is easy to show that the expected value $\mathbb{E}[S_n] = 0$ and $var(S_n) = n$.

3.1.2 Brownian Motion

Brownian Motion is named after the botanist Robert Brown who observed via microscope that pollen particles moved randomly in water. He published a few papers describing this behavior, but Brownian motion wasn't treated as a mathematical concept until Einstein, Smoluchowski, and Wiener explored it (Mazo 2002). It is due to Wiener's work in stochastic processes that Brownian motion is also called a Wiener process. A Wiener process, W_t, is defined to be any process satisfying

- $W_0 = 0$
- W_t has independent increments and for $0 < s < t$, $W_{t+s} - W_t \backsim \mathcal{N}(0, s)$
- W_t is continuous in t

3.1.3 Skew Brownian Motion

To construct SBM, consider a one-dimensional Brownian motion B_t, $t \geq 0$, starting at zero. The Brownian motion reflected at zero, denoted $|B_t|$, $t \geq 0$, has countably infinitely many intervals $J_1, J_2, ...$ between consecutive pairs of zeros. Although these intervals can be enumerated, they cannot be ordered. Now consider i.i.d. Bernoulli random variables $A_1, A_2, ...$, independent of B_t, with $P(A_n = 1) = \alpha$ and $P(A_n = -1) = 1 - \alpha$, $n \geq 1$. Then define the SBM (also called α-SBM) process $B^{(\alpha)} = \{B_t^{(\alpha)} : t \geq 0\}$ by

$$B_t^{(\alpha)} = \sum_{n=1}^{\infty} \mathbf{1}_{J_n}(t) A_n |B_t|$$

If we think in terms of a skew random walk, this would mean that a walker has equal probability of moving left or right at every point *except for one point*, the interface. When the walker is at the interface instead of equal probability of either direction, the probability of going one way is α and the other $(1 - \alpha)$. Making this change at one point affects the entire walk. That is, the parameter α is important in computing the transition probability densities (see Waymire (2011)).

3.2. Transition Probability Densities

The transition probability density $p(t, x, y)$ of a stochastic process $X_t, t \geq 0$, is, roughly speaking, the probability that the process starting at x is at y at time t. It would be more exact to say that $P(X_t \in A) = \int_A p(t,x,y)dy$ for any measurable set A. The variable x is called the 'backward' variable, while y is called the 'forward' variable. It is well-known that the transition probability density function of SBM satisfies $\dfrac{\partial p}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 p}{\partial x^2}$ (where x can also be replaced by y, the forward variable). However, the interface conditions are different with respect to the backward and forward variable. Let α be the parameter of SBM, fix y, and think of $p(t,x,y)$ as a function of x, then the following conditions are satisfied at the interface $x = 0$:

$$\begin{aligned} \alpha \frac{\partial p}{\partial x}\bigg|_{x=0^+} &= (1-\alpha)\frac{\partial p}{\partial x}\bigg|_{x=0^-} \\ p\big|_{x=0^+} &= p\big|_{x=0^-} \end{aligned} \tag{3.1}$$

Compare this to fixing the backward variable, x, and looking at $p(t,x,y)$ as a function of the forward variable y:

$$\begin{aligned} (1-\alpha)p\big|_{y=0^+} &= \alpha p\big|_{y=0^-} \\ \frac{\partial p}{\partial y}\bigg|_{y=0^+} &= \frac{\partial p}{\partial y}\bigg|_{y=0^-} \end{aligned} \tag{3.2}$$

This detail is subtle, but important nonetheless. Note that the conditions in (3.1) are the same as those conditions in (2.9), so in this paper we will focus on equations in terms of the backward variable when we are in a probability context. Since the transition probability density of SBM solves (2.8) under the conditions in (3.1), the fundamental solution we find in the Results section will be the transition probability density function of SBM.

In the ATW equations and the corresponding interface conditions (2.4), the across-shore variable x must be identified with the backward variable in the probabilistic context to get the same interface conditions listed in (3.1). This forces x' to identify with y in $p(t,x,y)$. As an example to summarize the three systems of notation that have been mentioned, the sea surface η from the oceanographic application can be found by computing

$$\begin{aligned} \eta(x,y) &= \int_D k(x';x,y)\eta_0(x')dx' \\ &= \int_D p(y,x,x')\eta_0(x')dx' \end{aligned} \tag{3.3}$$

where $k(x';x,t)$ is the fundamental solution to (2.8)-(2.9) and $p(t,x,y)$ is the transition probability density of SBM. The initial data is $\eta_0(x)$ and the domain is D.

3.3. First Passage Time

The first passage time at a of a stochastic process X_t, $t \geq 0$, is the random variable $\tau = \inf\{t > 0 : X_t = a\}$. After that time τ, what the process does is unimportant, so in the context of solving the heat equation, it makes sense to use an 'absorbing boundary.' We will let $y = a$ be such an absorbing boundary.

Lemma *Given the transition probability density function of absorbed SBM $p(t,x,y)$, the first passage time at a starting at x is given by* $-\frac{1}{2}\frac{\partial p}{\partial y}\Big|_{y=a}$.

Proof. Consider the following

$$\begin{aligned} \int_0^T \frac{d}{dt}\int_{-\infty}^a p(t,x,y)dydt &= \int_0^T\int_{-\infty}^a \frac{\partial p}{\partial t}dydt \\ &= \int_0^T\int_{-\infty}^a \frac{1}{2}\frac{\partial^2 p}{\partial y^2}dydt \\ &= \frac{1}{2}\int_0^T \frac{\partial p}{\partial y}\Big|_{y=a} dt \end{aligned}$$

In the second line, we used that the transition probability density for absorbed SBM satisfies the heat equation. In the third line, we used integration by parts and the interface conditions in terms of the forward variable (3.2). Starting with the same integral, and using the fundamental theorem of calculus we find

$$\begin{aligned}\int_0^T \frac{d}{dt}\int_{-\infty}^a p(t,x,y)dydt &= \int_{-\infty}^a \left(p(T,x,y)-p(0,x,y)\right)dy \\ &= \int_{-\infty}^a p(T,x,y)dy - \int_{-\infty}^a \delta_x(y)dy \\ &= \int_{-\infty}^a p(T,x,y)dy - 1\end{aligned}$$

This calculation uses that the initial data for absorbed SBM is $p(0,x,y)=\delta_x(y)$ where δ is the Dirac delta function, and that the starting position x is less than a.

Putting these two calculations together, we have

$$1 = \int_{-\infty}^a p(T,x,y)dy - \frac{1}{2}\int_0^T \left.\frac{\partial p}{\partial y}\right|_{y=a} dt$$

The first integral is the probability that in time T, the process has not reached a. The complement of that event (the second integral) is then the probability that you have reached a by time T. Thus, in terms of $p(t,x,y)$, the first passage time density is $-\frac{1}{2}\left.\frac{\partial p}{\partial y}\right|_{y=a}$. $\square$

In the notation of the fundamental solution, recall that the probability notation y is analogous to x'. Then to find the first passage time, we will need to compute $\left.\frac{\partial k}{\partial x'}\right|_{x'=a}$. This calculation can be found in the Discussion section (part 5). Once we have this result, we can compare it to the known first passage time of SBM. The two functions will not be identical, due to the diffusion coefficient, but we can rescale our solution k and find a relationship between α and μ (the parameters for the interface condition in the probability problem and the physical problem, respectively).

3.3.1 The Relationship Between α and μ

Let $z_1 = \frac{x}{\sqrt{D(x)}}$ and $z_2 = \frac{x'}{\sqrt{D(x')}}$ so that we think of z_1 as the 'backward variable' and z_2 as the 'forward variable.' Then $p(t,z_1,z_2) = k(\sqrt{D(x')}z_2; \sqrt{D(x)}z_1, t)$. When we take the second derivative, we get the relations $\frac{\partial^2 p}{\partial z_1^2} = D(x)\frac{\partial^2 k}{\partial x^2}$ and $\frac{\partial p}{\partial t} = \frac{\partial k}{\partial t}$.

$$\frac{\partial p}{\partial t} = \frac{\partial k}{\partial t} = \frac{D(x)}{2}\frac{\partial^2 k}{\partial x^2} = \frac{1}{2}\frac{\partial^2 p}{\partial z_1^2}$$
$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial z_1^2}$$

As for the interface condition, we think of $k(x';x,t) = p(t, \frac{x}{\sqrt{D(x)}}, \frac{x'}{\sqrt{D(x')}})$ and then the derivative is $\frac{\partial k}{\partial x} = \frac{1}{\sqrt{D(x)}}\frac{\partial p}{\partial z_1}$.

$$\begin{aligned}\frac{\mu}{\sqrt{D_+}}\frac{\partial p}{\partial z_1}\bigg|_{z_1=0^+} = \mu\frac{\partial k}{\partial x}\bigg|_{x=0^+} &= (1-\mu)\frac{\partial k}{\partial x}\bigg|_{x=0^-} = \frac{(1-\mu)}{\sqrt{D_-}}\frac{\partial p}{\partial z_1}\bigg|_{z_1=0^-} \\ \frac{\mu}{\sqrt{D_+}}\frac{\partial p}{\partial z_1}\bigg|_{z_1=0^+} &= \frac{(1-\mu)}{\sqrt{D_-}}\frac{\partial p}{\partial z_1}\bigg|_{z_1=0^-}\end{aligned}$$

To write this in terms of α, we start by looking at the ratio of $\alpha/(1-\alpha)$, knowing that it should be equal to the ratio $\left(\frac{\mu}{\sqrt{D_+}}\right)/\left(\frac{(1-\mu)}{\sqrt{D_-}}\right)$.

$$\begin{aligned}\frac{\alpha}{(1-\alpha)} &= \frac{\mu}{\sqrt{D_+}}\frac{\sqrt{D_-}}{(1-\mu)} \\ \sqrt{D_+}(1-\mu)\alpha &= \sqrt{D_-}\mu - \sqrt{D_-}\mu\alpha \\ (\sqrt{D_+}(1-\mu) + \sqrt{D_-}\mu)\alpha &= \sqrt{D_-}\mu\end{aligned}$$

Therefore

$$\begin{aligned}\alpha &= \frac{\sqrt{D_-}\mu}{\sqrt{D_-}\mu + \sqrt{D_+}(1-\mu)} \\ 1-\alpha &= \frac{\sqrt{D_+}(1-\mu)}{\sqrt{D_-}\mu + \sqrt{D_+}(1-\mu)}\end{aligned}$$

With these ‘translations,’ one could switch between the notation of the results in this paper to standard probability notation for SBM.

4. RESULTS

Three major results are presented here. All of them are solutions to (2.8)-(2.9), but the difference is in the boundary conditions. The first result considers an infinite domain (no boundaries) and this result is presented to establish a method for finding the solution in the later sections. The second result has an absorbing boundary and this result is useful in computing the first passage time of SBM. The final result is with a reflecting boundary which models the oceanographic application.

4.1. No Boundary

In this section, we solve (2.8)-(2.9) with an infinite domain. This introduces the method used to solve the same problem with different boundary conditions in later sections. As such, each step will be detailed in this section, but not in the later ones. We will use the Laplace transform method to solve the PDE (2.8)-(2.9).

First we note that $\widehat{\frac{\partial k}{\partial t}} = p\hat{k}(x';x,p) - k(x';x,0)$ (where the hat denotes the Laplace transform). Then (2.8)-(2.9) become after taking the Laplace transform:

$$p\hat{k}(x';x,p) - \delta_{x'}(x) = D(x)\frac{\partial^2 \hat{k}}{\partial x^2} \tag{4.1}$$

$$\mu \frac{\partial \hat{k}}{\partial x}\bigg|_{x=0^+} = (1-\mu)\frac{\partial \hat{k}}{\partial x}\bigg|_{x=0^-} \hat{k}\big|_{x=0^+} = \hat{k}\big|_{x=0^-}$$

Let's assume that $x' < 0$. Recall that $D(x) = D_\pm$ is a constant depending on the location of x, and for ease of notation, let $q_\pm = \sqrt{p/D_\pm}$. Then the solution of (4.1) is given by:

$$\hat{k}(x';x,p) = \begin{cases} Ae^{-q_+(x-x')} & x' < x < \infty \\ B_1 e^{-q_+ x} + B_2 e^{q_+ x} & 0 < x < x' \\ Ce^{q_- x} & -\infty < x < 0 \end{cases} \tag{4.2}$$

This can be easily checked to satisfy (4.1). To find the coefficients A, B_1, B_2, and C, we will need four equations. The first three come from continuity at $x = x'$, continuity at $x = 0$, and the interface condition listed in (4.1). The last equation comes from a jump of the derivative at $x = x'$.

$$A = B_1 e^{-q_+ x'} + B_2 e^{q_+ x'}$$

$$C = B_1 + B_2$$

$$\mu q_+(B_2 - B_1) = Ce^{q_-}(1-\mu)$$

$$D_+ q_+(B_2 e^{q_+ x'} - B_1 e^{-q_+ x'} + A) = 1$$

From these equations we find that

$$\begin{aligned} A &= \frac{1}{2D_+ q_+}(1 + \beta e^{-2q_+ x'}) \\ B_1 &= \frac{\beta}{2D_+ q_+} e^{-q_+ x'} \\ B_2 &= \frac{1}{2D_+ q_+} e^{-q_+ x'} \\ C &= \frac{1+\beta}{2D_+ q_+} e^{-q_+ x'} \\ \beta &= \frac{\sqrt{D_-}\mu - \sqrt{D_+}(1-\mu)}{\sqrt{D_-}\mu + \sqrt{D_+}(1-\mu)} \end{aligned} \tag{4.3}$$

Note that $|\beta| < 1$, which we will use later. Recall that the Laplace transform of $\left(\frac{D_\pm}{\pi t}\right)^{1/2} e^{-\frac{x^2}{4D_\pm t}}$ is $\frac{e^{-q_\pm x}}{q_\pm}$. Then after plugging the coefficients back into (4.2) and taking the inverse Laplace transform, we find the solution to (2.8) when $x' > 0$ is

$$k(x';x,t) = \begin{cases} \frac{1}{\sqrt{4D_+\pi t}}\left[e^{-\frac{(x-x')^2}{4D_+ t}} + \beta e^{-\frac{(x+x')^2}{4D_+ t}}\right] & x > 0 \\ \frac{1+\beta}{\sqrt{4D_+\pi t}} e^{-\frac{(\sqrt{D_-}x' + \sqrt{D_+}x)^2}{4D_- D_+ t}} & x < 0 \end{cases}$$

To find the solution when $x' < 0$, we follow the same process, but we start with:

$$\hat{k}(x';x,p) = \begin{cases} Ae^{-q_+ x} & 0 < x < \infty \\ B_1 e^{-q_-(x-x')} + B_2 e^{q_-(x-x')} & x' < x < 0 \\ Ce^{q_- x} & -\infty < x < x' \end{cases}$$

The solution in this case is

$$k(x';x,t) = \begin{cases} \frac{1-\beta}{\sqrt{4D_-\pi t}} e^{-\frac{(\sqrt{D_-}x - \sqrt{D_+}x')^2}{4D_-D_+t}} & x > 0 \\ \frac{1}{\sqrt{4D_-\pi t}} \left[e^{-\frac{(x'-x)^2}{4D_-t}} - \beta e^{-\frac{(x'+x)^2}{4t}} \right] & x < 0 \end{cases}$$

Putting everything together, we have the solution to (2.8)-(2.9) with no boundary is given by:

$$k(x';x,t) = \begin{cases} \frac{1}{\sqrt{4D_+\pi t}} \left[e^{-\frac{(x-x')^2}{4D_+t}} + \beta e^{-\frac{(x+x')^2}{4D_+t}} \right] & x' > 0, x > 0 \\ \frac{1+\beta}{\sqrt{4D_+\pi t}} e^{-\frac{(\sqrt{D_-}x' + \sqrt{D_+}x)^2}{4D_-D_+t}} & x' > 0, x < 0 \\ \frac{1-\beta}{\sqrt{4D_-\pi t}} e^{-\frac{(\sqrt{D_-}x - \sqrt{D_+}x')^2}{4D_-D_+t}} & x' < 0, x > 0 \\ \frac{1}{\sqrt{4D_-\pi t}} \left[e^{-\frac{(x'-x)^2}{4D_-t}} - \beta e^{-\frac{(x'+x)^2}{4D_-t}} \right] & x' < 0, x < 0 \end{cases} \tag{4.4}$$

Note that each of these terms is a 'scaled' Gaussian with the same variance and different means. That is, if $g(z,\sigma^2) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$, then from the above we see that the fundamental solution $k(x';x,t)$ can be expressed in terms of $\dfrac{1}{\sqrt{D(x')}} g\left(\dfrac{x'}{\sqrt{D(x')}} \pm \dfrac{x}{\sqrt{D(x)}}, 2t \right)$.

As a special case of the result (which was considered by Appuhamillage et. al. 2010), if we let $\mu = \sqrt{D_+}/(\sqrt{D_+} + \sqrt{D_-})$, then $\beta = (\sqrt{D_+} - \sqrt{D_-})/(\sqrt{D_+} + \sqrt{D_-})$ and the above results become

$$k(x';x,t) = \begin{cases} \frac{1}{\sqrt{4\pi D_+ t}}\left[e^{\frac{-(x-x')^2}{4D_+t}} + \frac{\sqrt{D_+}-\sqrt{D_-}}{\sqrt{D_+}+\sqrt{D_-}}e^{\frac{-(x+x')^2}{4D_+t}}\right] & x'>0, x>0 \\ \frac{2}{\sqrt{D_+}+\sqrt{D_-}}\frac{1}{\sqrt{4\pi t}}e^{\frac{-(\sqrt{D_+}x-\sqrt{D_-}x')^2}{4D_+D_-t}} & x'>0, x<0 \\ \frac{2}{\sqrt{D_+}+\sqrt{D_-}}\frac{1}{\sqrt{4\pi t}}e^{\frac{-(\sqrt{D_-}x-\sqrt{D_+}x')^2}{4D_-D_+t}} & x'<0, x>0 \\ \frac{1}{\sqrt{4\pi D_- t}}\left[e^{\frac{-(x-x')^2}{4D_-t}} - \frac{\sqrt{D_+}-\sqrt{D_-}}{\sqrt{D_+}+\sqrt{D_-}}e^{\frac{-(x+x')^2}{4D_-t}}\right] & x'<0, x<0 \end{cases} \tag{4.5}$$

4.2. Absorbing Boundary

In solving (2.8)-(2.9) with an absorbing boundary at a, we use exactly the same method as we did for the no boundary case. There are a few differences in details, but the process is the same. The first difference is that the solution is slightly more complicated because it needs to satisfy $k(x';a,t)=0$. The solution (when we assume $x'<0$), looks like:

$$\hat{k}(x';x,p) = \begin{cases} A(e^{-q_+(a-x)} - e^{q_+(a-x)}) & \text{if } 0<x<a \\ B_1 e^{-q_- x} + B_2 e^{q_- x} & \text{if } x'<x<0 \\ Ce^{q_- x} & \text{if } -\infty<x<x' \end{cases}$$

Under the same continuity and interface conditions we imposed in the no boundary case, we find that the coefficients are:

$$\begin{aligned} A &= \frac{-(1-\beta)}{2D_-q_-}\left(\frac{e^{-q_+a+q_-x'}}{1+\beta e^{-2q_+a}}\right) \\ B_1 &= \frac{e^{q_-x'}}{2D_-q_-} \\ B_2 &= \frac{e^{q_-x'}}{2D_-q_-}\left(\frac{\beta+e^{-2q_+a}}{1+\beta e^{-2q_+a}}\right) \\ C &= \frac{e^{-q_-x'}}{2D_-q_-} - \frac{e^{q_-x'}(\beta+e^{-2q_+a})}{2D_-q_-(1+\beta e^{-2q_+a})} \end{aligned}$$

Where β is given by (4.3). Attempting to take the inverse Laplace transform of terms that look like A, B_2, and C would be very difficult. Instead, we make a clever observation and re-write these coefficients as infinite series. Note that $|\beta e^{-2q_+a}| < 1$, and we can rewrite A as a geometric series.

$$A = \frac{-(1-\beta)}{2D_-q_-}e^{-q_+a+q_-x'}\sum_{k=0}^{\infty}(-\beta)^k e^{-2q_+ak}$$

Similarly, we can rewrite the other coefficients that have $1/(1+\beta e^{-2q_+a})$. Because of the linearity of the Laplace transform, we invert term-by-term and obtain the solution:

$$k(x';x,t) = \begin{cases} \frac{1-\beta}{\sqrt{4D_-\pi t}}\sum_{k=0}^{\infty}(-\beta)^k[A_k - B_{k+1}] & x'<0, x>0 \\ \frac{1}{\sqrt{4D_-\pi t}}\left[e^{-\frac{(x-x')^2}{4D_-t}} - \beta e^{-\frac{(x+x')^2}{4D_-t}} - (1-\beta^2)\sum_{k=1}^{\infty}(-\beta)^k C_k\right] & x'<0, x<0 \\ \frac{1}{\sqrt{4D_+\pi t}}\sum_{k=0}^{\infty}(-\beta)^k[D_k - E_{k+1} + \beta F_k - \beta G_{k+1}] & 0<x'<a, x>0 \\ \frac{1+\beta}{\sqrt{4D_+\pi t}}\sum_{k=0}^{\infty}(-\beta)^k[H_k - I_{k+1}] & 0<x'<a, x<0 \end{cases} \tag{4.6}$$

where

$$\begin{aligned}
A_k &= \exp\left\{-\frac{(\sqrt{D_-})(2ak+x) - \sqrt{D_+}x')^2}{4D_-D_+t}\right\} \\
B_k &= \exp\left\{-\frac{(\sqrt{D_-}(2ak-x) - \sqrt{D_+}x')^2}{4D_-D_+t}\right\} \\
C_k &= \exp\left\{-\frac{(\sqrt{D_-}2ak - \sqrt{D_+}(x+x'))^2}{4D_-D_+t}\right\} \\
D_k &= \exp\left\{-\frac{(2ak+x'-x)^2}{4D_+t}\right\} \\
E_k &= \exp\left\{-\frac{(2ak-(x+x'))^2}{4D_+t}\right\} \\
F_k &= \exp\left\{-\frac{(2ak+x'+x)^2}{4D_+t}\right\} \\
G_k &= \exp\left\{-\frac{(2ak+x-x')^2}{4D_+t}\right\} \\
H_k &= \exp\left\{-\frac{(\sqrt{D_-}(2ak+x') - \sqrt{D_+}x)^2}{4D_-D_+t}\right\} \\
I_k &= \exp\left\{-\frac{(\sqrt{D_-}(2ak-x') - \sqrt{D_+}x)^2}{4D_-D_+t}\right\}
\end{aligned}$$

Here, we notice that all of the terms are 'scaled' Gaussians (similar to the no boundary case) with the same variance ($\sigma^2 = 2t$) and different means:

$$\begin{aligned} a_k &= \frac{\sqrt{D_+}x' - \sqrt{D_-}2ak}{\sqrt{D_-D_+}} \\ b_k &= \frac{x' + 2ak}{\sqrt{D_+}} \\ c_k &= \frac{x' - 2ak}{\sqrt{D_+}} \end{aligned}$$

We also notice that when $x' < 0$, the mean of the Gaussians is $\pm a_k$ (depending on whether $x < 0$ or $x > 0$) and, similarly, when $0 < x' < a$, the mean is $\pm b_k$ or $\pm c_k$.

4.3. Reflecting Boundary

In this section, we present the solution to (2.8)-(2.9) with a reflecting boundary at $x = -L$, namely the domain is limited to $-L < x < \infty$ and we have the Neumann boundary condition $\frac{\partial k}{\partial x}\big|_{x=-L} = 0$. We solve this problem using the same method as for the absorbing boundary (including the geometric series trick for writing the coefficients). The solution then is

$$k(x';x,t) = \begin{cases} \frac{1}{\sqrt{4D_+\pi t}}\left[e^{-\frac{(x-x')^2}{4D_+t}} + \beta e^{-\frac{(x+x')^2}{4D_+t}} + (1-\beta^2)\sum_{k=1}^{\infty}(-\beta)^k\tilde{A}_k\right] & x' > 0, x > 0 \\ \frac{1+\beta}{\sqrt{4D_+\pi t}}\sum_{k=0}^{\infty}(-\beta)^k[\tilde{B}_{k+1} + \tilde{C}_k] & x' > 0, x < 0 \\ \frac{1-\beta}{\sqrt{4D_-\pi t}}\sum_{k=0}^{\infty}(-\beta)^k[\tilde{D}_k + \tilde{E}_{k+1}] & x' < 0, x > 0 \\ \frac{1}{\sqrt{4D_-\pi t}}\left[\sum_{k=0}^{\infty}(-\beta)^k[\tilde{F}_k + \tilde{H}_{k+1} + \tilde{G}_k + \tilde{I}_{k-1}] - \tilde{G}_0 - \tilde{G}_1\right] & x' < 0, x < 0 \end{cases} \tag{4.7}$$

$$\begin{aligned}
\tilde{A}_k &= \exp\left\{-\frac{(\sqrt{D_-}(x+x')+\sqrt{D_+}2Lk)^2}{4D_-D_+t}\right\} \\
\tilde{B}_k &= \exp\left\{-\frac{(\sqrt{D_-}x'+\sqrt{D_+}(x+2Lk))^2}{4D_-D_+t}\right\} \\
\tilde{C}_k &= \exp\left\{-\frac{(\sqrt{D_-}x'+\sqrt{D_+}(2Lk-x))^2}{4D_-D_+t}\right\} \\
\tilde{D}_k &= \exp\left\{-\frac{(\sqrt{D_-}x-\sqrt{D_+}(x'-2Lk))^2}{4D_-D_+t}\right\} \\
\tilde{E}_k &= \exp\left\{-\frac{(\sqrt{D_-}x+\sqrt{D_+}(x'+2Lk))^2}{4D_-D_+t}\right\} \\
\tilde{F}_k &= \exp\left\{-\frac{(2Lk+x-x')^2}{4D_-t}\right\} \\
\tilde{G}_k &= \exp\left\{-\frac{(2Lk+x'-x)^2}{4D_-t}\right\} \\
\tilde{H}_k &= \exp\left\{-\frac{(2Lk+x+x')^2}{4D_-t}\right\} \\
\tilde{I}_k &= \exp\left\{-\frac{(2Lk-(x+x'))^2}{4D_-t}\right\}
\end{aligned}$$

We note that this result is similar to our result for the absorbing boundary case. In fact, replacing L with a and switching D_- for D_+, the terms $\tilde{F}_k, \tilde{G}_k, \tilde{H}_k, \tilde{I}_k$ look exactly the same. The other terms are different by signs, but this makes sense since there are different boundary conditions (recall that the absorbing boundary used a Dirichlet condition whereas the reflecting boundary had a Neumann condition). As before, all the terms are 'scaled' Gaussians with variance $2t$ and the following different means:

$$\begin{aligned}
\tilde{a}_k &= \frac{\sqrt{D_-}x'+\sqrt{D_+}2Lk}{\sqrt{D_-D_+}} \\
\tilde{b}_k &= \frac{x'+2Lk}{\sqrt{D_-}} \\
\tilde{c}_k &= \frac{x'-2Lk}{\sqrt{D_-}}
\end{aligned}$$

5. DISCUSSION

In this part, we will use our fundamental solution for the two applications mentioned in the introduction. For the oceanographic application, using our result with a reflecting boundary is most relevant since it has the same boundary condition as that in (2.5). For the probabilistic application, using the result with an absorbing boundary makes sense since we are computing the first passage time.

5.1. First Passage Time of Skew Brownian Motion

The first passage time of SBM can be found by computing the derivative of $k(x';x,t)$ (as given in (4.6)) with respect to x' and evaluating that derivative at $x' = a$ (see the justification for this after the lemma in section 3). Because $a > 0$ is fixed, it is impossible to evaluate at $x' = a$ if $x' < 0$. Therefore, we only consider the cases when $x' > 0$.

Referring to our result with an absorbing boundary, the first passage time of SBM may be obtained from (4.6) by computing $-\frac{1}{2}\frac{\partial k}{\partial x'}\Big|_{x'=a}$
for $0 < x' < a, x < 0$

$$-\frac{1}{2}\frac{\partial k}{\partial x'}\Big|_{x'=a} = \frac{(1+\beta)}{4\sqrt{\pi D_-}(D_+t)^{3/2}}\sum_{k=0}^{\infty}(-\beta)^k(\sqrt{D_-}(2k+1)a - \sqrt{D_+}x)e^{-\frac{(\sqrt{D_-}(2k+1)a-\sqrt{D_+}x)^2}{4D_-D_+t}}$$

Comparing our results with those found by Appuhamillage and Sheldon (2011), we see that this is the same formula they obtained (using a completely different method). In making this comparison, we must be very careful with the different notations. Their x is our x and their y is our a. In the Probability section, we noted that there was a relationship between μ and α and with our definition of β in (4.3), the relationship can be summarized by $2\alpha = 1 + \beta$. In the following case, it appears that we have obtained a simpler formula than that presented by Appuhamillage and Sheldon, but further work should be done to compare this more complicated case.

for $0 < x' < a, x > 0$,

$$
\begin{aligned}
-\frac{1}{2}\frac{\partial k}{\partial x'}\bigg|_{x'=a} &= \frac{1}{4\sqrt{\pi}(D_+t)^{3/2}}\sum_{k=0}^{\infty}(-\beta)^k((2k+1)a-x)e^{-\frac{((2k+1)a-x)^2}{4D_+t}} \\
&\quad +\frac{1}{4\sqrt{\pi}(D_+t)^{3/2}}\sum_{k=0}^{\infty}(-\beta)^k\beta((2k+1)a+x)e^{-\frac{((2k+1)a+x)^2}{4D_+t}}
\end{aligned}
$$

5.2. Vertical Velocity of Ocean Currents at the Interface

Returning to (2.13), we are now able to compute the vertical velocity, $w(x, y, z)$, using the relations given in (2.11).

$$
\begin{aligned}
w(x,y,-h) &= \frac{rg}{f^2}\left(\frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial}{\partial x}\int_{-L}^{\infty}k(x';x,y)\eta_0(x')dx' - \frac{\partial^2}{\partial x^2}\int_{-L}^{\infty}k(x';x,y)\eta_0(x')dx'\right) \\
&= \frac{rg}{f^2}\int_{-L}^{\infty}\left(\frac{1}{h}\frac{\partial h}{\partial x}\frac{\partial k}{\partial x} - \frac{D(x')}{D(x)}\frac{\partial^2 k}{\partial x'^2}\right)\eta_0(x')dx' \\
&= -\frac{rg}{f^2}\frac{1}{D(x)}\int_{-L}^{\infty}\left(\frac{1}{h}\frac{r}{f}\frac{\partial k}{\partial x} + D(x')\frac{\partial^2 k}{\partial x'^2}\right)\eta_0(x')dx'
\end{aligned}
$$

At this point, we can use whatever initial data is desired. As an example, let $\eta_0(x)$ be that given in (2.7). For ease of notation, let $h_x = \dfrac{\partial h}{\partial x}$.

$$
\begin{aligned}
w(x,y,-h) &= \frac{rg}{f^2}\int_{0}^{\infty}\left(\frac{h_x}{h}\frac{\partial k}{\partial x} - \frac{D(x')}{D(x)}\frac{\partial^2 k}{\partial x'^2}\right)\eta_*(e^{-mx'}-1)dx' \\
&= \frac{rg}{f^2}\int_{0}^{\infty}\frac{h_x}{h}\frac{\partial k}{\partial x}\eta_*(e^{-mx'}-1)dx' - \frac{rg}{f^2}\int_{0}^{\infty}\frac{D_+}{D(x)}\frac{\partial^2 k}{\partial x'^2}\eta_*(e^{-mx'}-1)dx' \\
&= \frac{-rg\eta_*}{f^2}\frac{h_x}{h}\int_{0}^{\infty}\frac{\partial k}{\partial x}dx' - \frac{rg\eta_*}{f^2}\int_{0}^{\infty}\left(\frac{D_+}{D(x)}\frac{\partial k}{\partial x'} - \frac{h_x}{h}\frac{\partial k}{\partial x}\right)e^{-mx'}dx'
\end{aligned}
$$

On the second line, we used integration by parts on the second integral, and rearranged terms on the third line so that all the terms involving $e^{-mx'}$ are contained within one integral. We cannot proceed any further with these calculations unless we make an assumption about the location of x relative to the interface. From (2.9), we know that $\frac{\partial k}{\partial x}$ is discontinuous at the interface. Also recall that h_x is the slope of the sea floor, which is different on either side of the interface. For these reasons, we consider the $\lim_{x\to 0^-} w(-h)$ and $\lim_{x\to 0^+} w(-h)$ separately. If we assume $x > 0$, for example, we use integration by parts, completing the square, and geometric series sum to get

$$\begin{aligned} w(x,y,-h) &= \frac{rg\eta_*}{2f^2\sqrt{D_+\pi y}}\left[\beta(1+\beta)m + \frac{h_x}{h}(1-\beta)\left(1-(1+\beta)\sum_{k=1}^{\infty}(-\beta)^{k-1}e^{-\frac{(Lk)^2}{D_-y}}\right)\right] \\ & -\frac{rg\eta_*}{2f^2}\left(m(1+\beta)+\frac{h_x}{h}(1-\beta)\right)me^{D_+ym^2}\operatorname{erfc}(m\sqrt{D_+y})) \\ & +\frac{rg\eta_*}{2f^2}(1-\beta^2)\left(-m+\frac{h_x}{h}\right)\sum_{k=1}^{\infty}(-\beta)^{k-1}me^{D_+ym^2+\sqrt{\frac{D_+}{D_-}}2Lkm} \\ & \times\operatorname{erfc}\left(\frac{\sqrt{D_-D_+}m+Lk}{\sqrt{D_-y}}\right) \end{aligned} \tag{5.1}$$

where the complementary error function $\operatorname{erfc}(x)$ is defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-\psi^2}d\psi$$

For $x < 0$, we get a very similar result, except that some of the coefficients with β are changed (i.e. some terms will be $(1+\beta)$ instead of $(1-\beta)$). All the exponential and error functions are the same however. This being a complicated formula, we will focus on the effects of only three parameters that appear in the formula: the bottom friction coefficient r, the interface parameter μ, and the initial data parameter m. The effects of r and μ are hidden in (5.1) because $D_\pm$ depends on r and β depends on μ. The initial data parameter m is easy to see in (5.1), but a prediction of how it effects the vertical velocity is not as obvious.

Using MATLAB, we can find numerical approximations and graphs of $w(-h)$ for various values of the three parameters we are considering. For these computations, only the first three terms of the infinite series in (5.1) were used. The constants were assumed to be

$$
\begin{aligned}
r &= 0.001ms^{-1}\\
g &= 9.8ms^{-2}\\
\eta_* &= 0.1m\\
f &= 0.0001s^{-1}\\
D_+ &= 0.3m\\
D_- &= 0.02m\\
m &= 0.00005m^{-1}\\
h_x^- &= 0.002\\
h_x^+ &= 0.03\\
h &= 150m\\
L &= 50km
\end{aligned}
$$

In all the graphs, we see that the vertical velocity decreases as the distance from the origin increases in the along-shore direction. Since we assumed $x > 0$ in creating this graph, we are only looking at the interface approached from the right.

We see that for both of the parameters r and μ, there is a monotonicity effect. That is, as r increases, the vertical velocity decreases. However, if μ increases, the vertical velocity also increases.

In Figure 5.3 we see a more complicated effect. The vertical velocity increases as m increases, but we also see on this graph that there is a switch from upwelling to downwelling for larger values of m. For $1/40,000 < m < 1/10,000$, we see that there is upwelling for at least $200km$ in the along-shore direction. When m is increased to $1/7,500$, at about $120km$ out the upwelling changes to downwelling (the vertical velocity becomes negative). Then the effect of the initial data parameter m has more to do with the extent of downwelling in the along-shore direction.

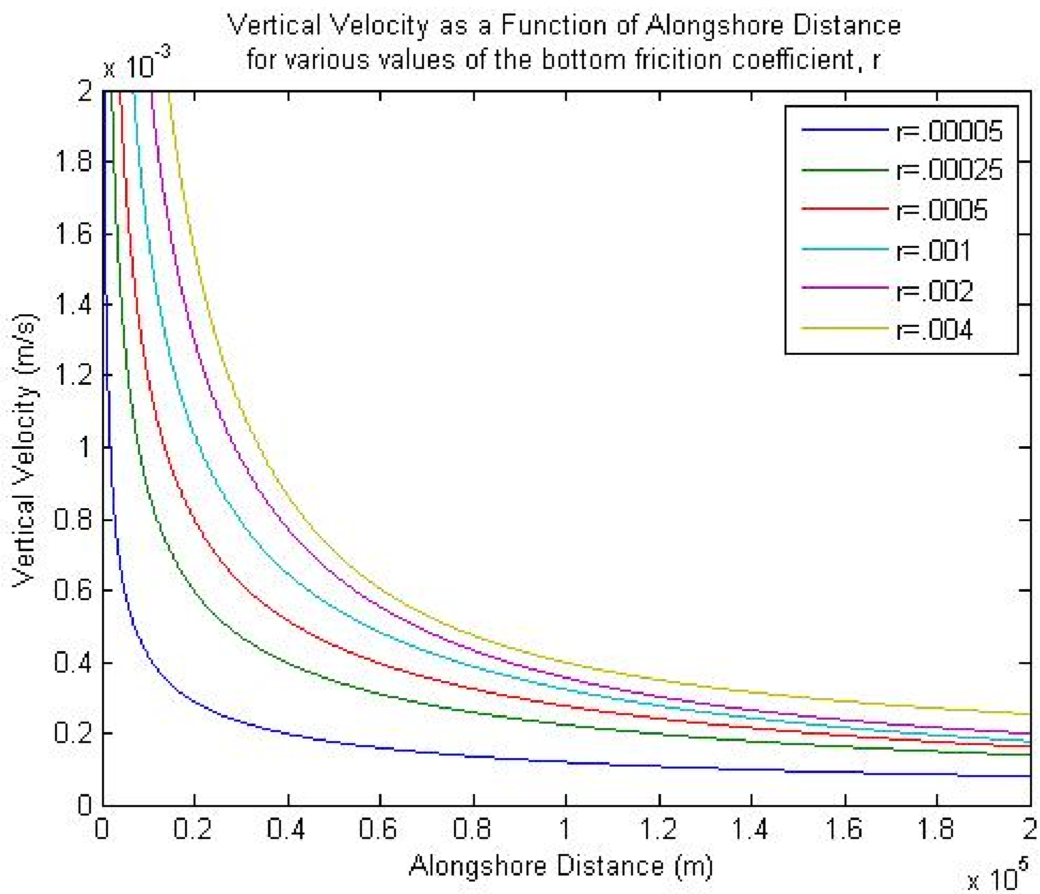

FIGURE 5.1: Vertical velocity as a function of alongshore coordinate with various values of bottom friction coefficient, r

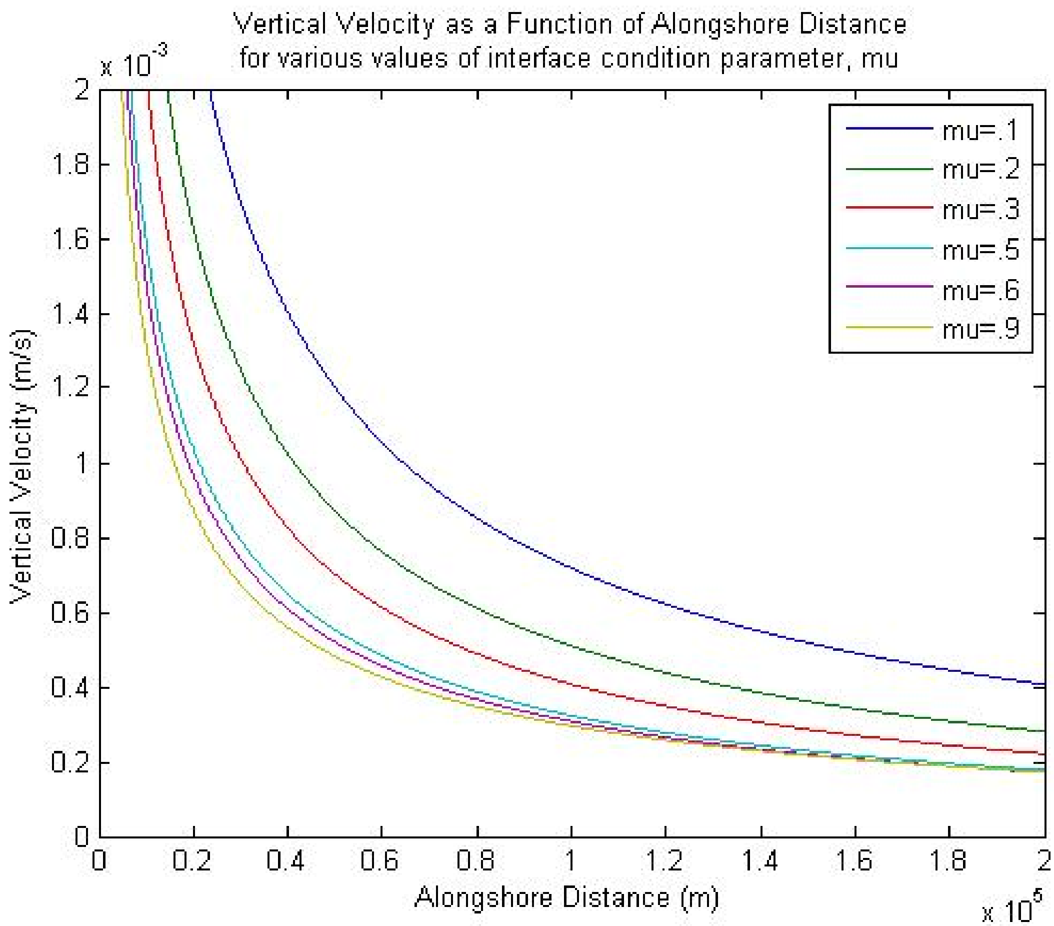

FIGURE 5.2: Vertical velocity as a function of alongshore coordinate with various values of interface parameter μ

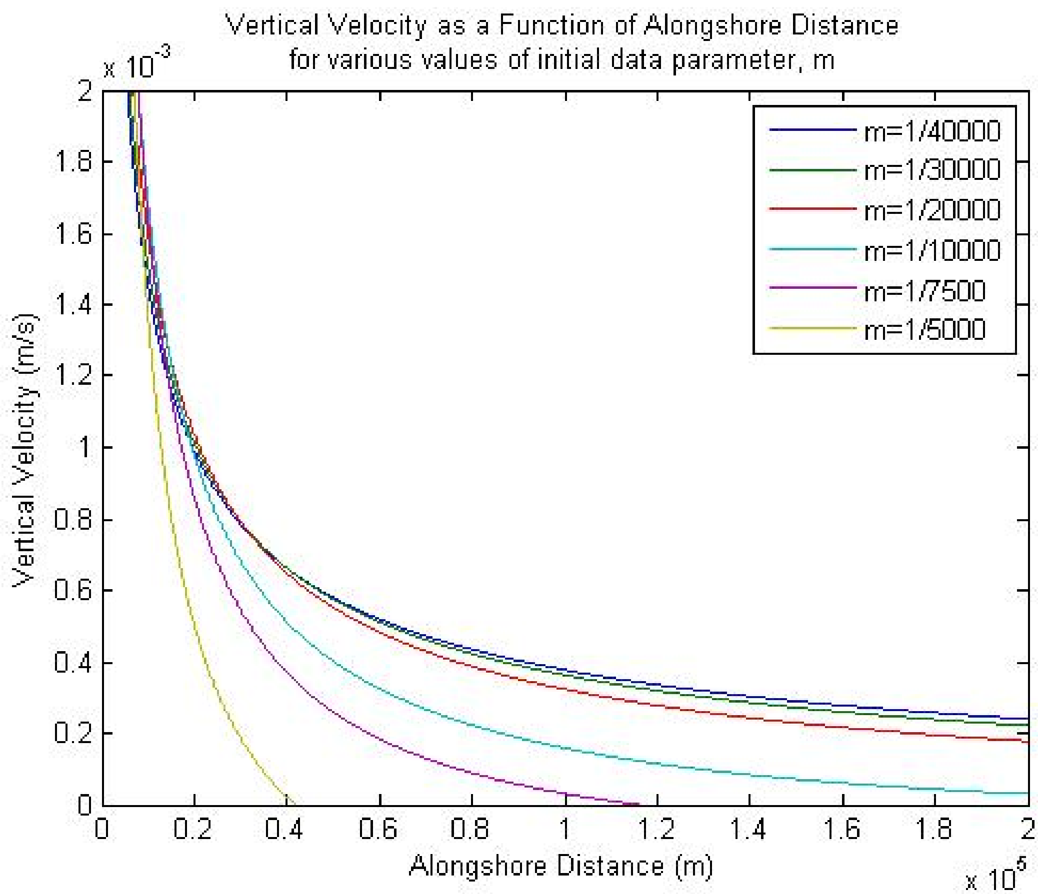

FIGURE 5.3: Vertical velocity as a function of alongshore coordinate with various values of initial data parameter, m

6. CONCLUSION

In this paper, we have presented the fundamental solution of the heat equation with a discontinuous diffusion coefficient and used this result in two applications.

One of these applications was also the motivation for solving this problem. In order to understand why there is such an abundance of aquatic life along the shelfbreak off the coast of Argentina, it is important to understand how the ocean currents behave. The assumptions that oceanographers use about the sea surface elevation lead to a derivation of an equation that is the heat equation with a diffusion coefficient that is dependent on the bottom friction coefficient and the steepness of the shelf and shelfbreak slope. By finding the fundamental solution, the sea surface elevation can be calculated given any initial data. Knowing the sea surface elevation is key to calculating the vertical velocity at the shelfbreak, and it is exactly the vertical velocity that will offer insight about the up- and downwelling currents.

In this work, we considered very specific initial data that has been used in previous publications. Using the fundamental solution, we presented an explicit formula for the vertical velocity at the shelfbreak. To see the effect that three parameters (bottom friction coefficient, interface and initial) have on the vertical velocity, we enlisted the assistance of MATLAB to plot a few curves where these parameters are changed slightly. Although this was not the focus of this paper, further investigation in how these parameters affect the vertical velocity would be interesting. The results here could be used for such an investigation.

The second application of these results is to Skew Brownian Motion. We have shown that the fundamental solution we have derived is the transition probability density of SBM (up to a translation of constants) and from that we can calculate the first passage time. When finding the first passage time, care was taken to ensure that we were working with respect to the correct variable. The problem we solved to find the fundamental solution required the interface conditions that are associated with the backward variable of SBM's transition probability density. With these interface conditions, we can make connections between the oceanographic application and SBM.

If a perturbation starts on one side of the interface, what is the probability that it will reach a given position on the opposite side of the interface in a given time? This is the type of question that the first passage time result can answer. From further analysis of the first passage time formula, we can determine what the difference is between moving from shelf to deep ocean and deep ocean to shelf. This difference, of course, is because of the different diffusion coefficients on either side of the interface.

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