AN ABSTRACT OF THE THESIS OF

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Justus F. Seely

We consider the basic problem of the recovery of inter-block information. We formulate the problem in a very basic setting which consists of three pieces of information: (i) A normally distributed random variable T that is an unbiased estimator for an unknown parameter θ. (ii) An s x 1 multivariate normal random vector U whose components are independent, have zero expectation, and are possibly correlated with T. (iii) A chi-squared random variable R that is independent of T and U. In the terminology used in the recovery of inter-block information the quantities T, U, and R correspond to the intra-block estimator, the inter-block information, and the residual sum of squares respectively. When s exceeds 2, this thesis shows that T can be improved upon (as measured by the variance) by a nonlinear unbiased estimator $\hat{\delta}$ of θ. The estimator $\hat{\delta}$ depends on T, U and R. The estimator $\hat{\delta}$ belongs to the general class of nonlinear estimators described by Saleh (1987). However, we concentrate on a slightly different form than that used by Saleh. The estimator $\hat{\delta}$ is applicable under a

mixed linear model with two variance components and it can be obtained via solving a set of normal equations. An exact formula for the variance of $\hat{\delta}$ is given in terms of a one variable integral.

Efficiency comparisons between $\hat{\delta}$ and T and between $\hat{\delta}$ and the maximin-efficiency estimator of Birkes, Seely and Azzam (1981) under the random one-way model is given to illustrate the results in the thesis.

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Redacted for privacy

Chairman of Department of Statistics

Redacted for privacy

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TABLE OF CONTENTS

LIST OF TABLES

IMPROVING ON THE INTRA-BLOCK ESTIMATOR

VIA A NONLINEAR NORMAL EQUATION ESTIMATOR

I. INTRODUCTION

The problem of combining the intra- and inter-block information in incomplete block designs is widely considered in the literature. A brief presentation of the early work in the recovery of intra- and inter-block information up to the year 1983 is contained in Chapter III of Saleh (1987). In that chapter, Saleh mentioned that Yates (1939,1940) initiated this subject. Yates suggested the use of inter-block information to improve the intra-block estimator. He gave a method of combining inter- and intra-block information for balanced incomplete block designs and for Cubic Lattice designs. Since then many authors have contributed to this area of research. In order to combine the intra- and inter-block information, various conditions have been assumed on the number of treatments and/or blocks in the design. Prior to Saleh (1987), the weakest conditions were given by Khatri and Shah (1974), whose results were restricted to a proper binary equi-replicate incomplete block design where the number of blocks exceeds three. Moreover, their results were applicable only to an estimable parametric function that is a contrast of the treatment effects. Saleh (1987) generalized the idea of the intra-block estimator as well as the recovery of inter-block information to a completely general mixed model having two variance components. That is, for any linearly estimable parametric function, Saleh identified a linear estimator which is the natural generalization of the intra-block estimator and showed that this estimator can be uniformly improved upon (as measured by the variance) by a nonlinear unbiased estimator when the degrees of freedom for the primary source of variation (e.g., blocks) is three or more.

The goal of this thesis is to extend and refine the results of Saleh (1987). This is done in the context of trying to improve on the intra-block estimator by a nonlinear unbiased estimator obtained by using inter-block information. In addition, we want to be able to calculate the improved estimator via normal equations; and to be able to give an efficient way to compute the variance of the improved estimator.

Chapter II presents the canonical form of our problem and the form of a nonlinear unbiased estimator which is uniformly better than the intra-block estimator. To do this a condition similar to that given by Saleh is needed. The nonlinear improved estimator belongs to the general class of Saleh's estimators, but we concentrate on a slightly different form than that used by Saleh. Some preliminary useful facts from Saleh are included in this chapter as well as a lower bound for the efficiency of our nonlinear estimator.

Chapter III shows how the mixed linear model with two variance components can be reduced to the canonical form of Chapter II. It also shows how to calculate the estimator considered in Chapter II via normal equations.

Chapter IV introduces a formula for the variance of our estimator that requires integration in terms of one variable. This formula has allowed exact computation of efficiencies in a very efficient manner when compared with simulation as done in Saleh (1987). A computer program to compute the variance and efficiency is provided in the Appendix.

Chapter V applies the results of Chapters II and III to the random one-way model as a special case of the mixed linear model with two variance components. Some computations and numerical results for efficiency comparisons between the estimators considered in Chapter II and the intra-block estimator and the maximin-efficiency linear unbiased estimator of Birkes, Seely and Azzam (1981) are included.

Throughout the thesis, R^n is used to denote n-dimensional Euclidean space. For any matrix A, the notation A', $\underline{R}(A)$, $\underline{N}(A)$ and $\underline{r}(A)$ denotes the transpose, range, null space and rank of A respectively. The orthogonal complement of a set A is denoted by $A^{\perp}$. The notation $N_n(\mu, D)$ denotes an n-dimensional normal distribution with mean vector μ and covariance matrix D.

II. GENERAL THEORY

2.1. The canonical form.

Let T be a normally distributed random variable with mean θ and variance $\pi[(1-\gamma)h_1 + \gamma h_2]$ where $h_1 > 0$ and $h_2 \geq 0$ are known constants and where γ, θ and π are unknown parameters. We assume that γ is in the closed interval $[0,1]$, that θ is in R^1, and that $\pi > 0$. Let U be an $s \times 1$ multivariate normal random vector with mean zero and covariance matrix $\pi[(1-\gamma)I + \gamma D]$ where D is a diagonal matrix with known positive diagonal elements $d_1, \ldots, d_s$. We assume that T and U are jointly normally distributed and that the covariance vector between T and U is $\pi(1-\gamma)C$ where C is a known $s \times 1$ vector with elements $C_1, \ldots, C_s$. This assumption means that C is some vector satisfying $C'C \leq h_1$.

Let R be a random variable such that $R/\pi(1-\gamma)$ has a chi-squared distribution with q degrees of freedom. We assume that R is distributed independently of T and U.

Notice that the components of U are independently distributed among themselves, but are possibly correlated with T. If it happens that $C = 0$, then it can be shown that T is the uniformly minimum variance unbiased estimator (UMVUE) for θ. Thus, the cases we shall be interested in are the situations where C has at least one nonzero element. We will see in the next chapter that a two variance component mixed model can be reduced to the set up described above. For convenience in terminology, we will refer throughout the thesis to T, U and R as the intra-block estimator, the inter-block information, and the residual sum of squares respectively. For an additive two-way mixed model with treatments fixed and blocks random, this terminology is consistent with the usual terminology employed in the literature. Moreover, it will be clear in the next chapter from the definitions of T, U and R that these quantities can precisely be defined in terms of a general mixed two variance component model without reference to the structure imposed by the usual mixed two-way additive model.

In this chapter we study the possibility of finding a nonlinear unbiased estimator $\hat{\delta}$ for θ based on T, U, and R that strictly dominates T. That is, it has smaller variance for all $\gamma \in [0,1)$, $\theta \in R^1$ and $\pi > 0$.

2.2 The Saleh estimators.

Saleh (1987) considered the problem of dominating the intra-block estimator T for an estimable parametric function θ in a mixed linear model with two variance components. By a reduction similar to that in Chapter III of this thesis, Saleh described a vector U and a random variable R having the properties in Section 2.1.

Saleh introduced a class of nonlinear estimators of the form

(2.2.1) $$\hat{\delta} = T - \sum_{i=1}^{s} c_i \phi_i U_i .$$

Here each ϕ_i is assumed to be a measurable function of R/W_i where $W_i = \sum_j g_{ij} X_j$ with $g_{i1}, \ldots, g_{is} > 0$ and $X_1 = U_1^2, \ldots, X_s = U_s^2$, and the C_i are the components of the vector C describing the covariance structure between T and U. Saleh showed the following:

(2.2.2) a) Any such $\hat{\delta}$ is an unbiased estimator for θ provided that its expectation exists.

b) $\mathrm{Var}(\hat{\delta} \mid \pi, \gamma, \theta) = \pi \mathrm{Var}(\hat{\delta} \mid \pi = 1, \gamma, \theta = 0)$.

In the majority of her work, Saleh considered estimators where $g_{ij} = g_j$ for all i and j. In this situation one has $W_1 = \ldots = W_s$. Let W denote this common W_i. The ϕ_i functions Saleh mainly considered were of the form

(2.2.3) $$\phi_i = v_i [R/(g_0 R + W)]^{\lambda}, \quad i = 1, \ldots, s;$$

where $\lambda > 0$, $g_0 \geq 0$ and $v_1, \ldots, v_s$ are arbitrary real numbers. In this form the experimenter selects $g_0, v_1, \ldots, v_s$, and λ. In Chapter V of Saleh's thesis, she presented numerical results via simulation to study the behavior of her estimators. She used $g_1 = \ldots = g_s$ and $g_0 = 0$ or 1, and then she concentrated on how to select $v_1, \ldots, v_s$ and λ so that $\hat{\delta}$ dominates T.

2.3. The estimator $\hat{\delta}$.

We consider the same general class of nonlinear estimators described by Saleh. However, we concentrate on a slightly different form than that used by Saleh. Like Saleh, we select each ϕ_i to be a measurable function of R/W where $W = \sum_j g_j X_j$ with

$g_1, \ldots, g_s$ all positive constants. This is a special case of (2.2.1) so that $\hat{\delta}$ satisfies the properties (2.2.2).

Throughout this thesis we will consider $\hat{\delta}$ to be an estimator as in (2.2.1); but with the ϕ_i selected according to

(2.3.1) $$\phi_i = gR/[gR + d_i W], \ i = 1, \ \ldots, s;$$

where $d_1, \ldots, d_s$ represent the diagonal elements of D defined in Section 2.1 and g can be any positive real number. As can be seen by comparison with (2.2.3), one can see that our form for the ϕ_i functions is slightly different than the form of Saleh. Our form does reduce to the general form in (2.2.1). This can be seen by letting $g_{ij} = d_i g_i$ in (2.2.1).

One of our goals in the sequel is to investigate under what conditions, if any, on $g, g_1, \ldots, g_s$ is it true that

$$\mathrm{Var}(\hat{\delta} | \pi, \gamma, \theta) \le \mathrm{Var}(T | \pi, \gamma, \theta)$$

for all possible π, γ, and θ with strict inequality for some π, γ, and θ. In most of our studies we will use $g_1 = \ldots = g_s = 1$ so that only the conditions on g will be studied in some detail. Because $\hat{\delta}$ satisfies (2.2.2), we will assume, without loss of generality, that $\pi = 1$ and $\theta = 0$.

2.4. Some results of Saleh.

In this section we summarize some useful facts given in Saleh (1987). To do this, we need some notation. For each positive real number r, let G(r) denote a gamma distribution with scale parameter 1/2 and shape parameter r/2. If r is an integer, then

$G(r)$ is a chi-squared distribution with r degrees of freedom. For any real number b for which $r + 2b$ is positive, let

$$K(r,b) = 2^b\Gamma[(r+2b)/2]/\Gamma(r/2),$$

where $\Gamma(-)$ is the usual gamma function.

Lemmas 2.4.1 and 2.4.2 below are in Section 4.2 of Saleh and Lemma 2.4.3 is basically Theorem 4.3.2 of Saleh.

Lemma 2.4.1. Let $Q \sim G(r)$. Then $E(Q^b) = K(r,b)$ for all b such that $b > -r/2$.

Let $Q_1, \ldots, Q_m$ be independent random variables such that $Q_i \sim G(r_i)$ for $i = 1, \ldots, m$. Set

$$S = a_1Q_1 + \ldots + a_mQ_m$$

and

$$r = r_1 + \ldots + r_m,$$

where $a_1, \ldots, a_m$ are positive constants.

Lemma 2.4.2. Let $0 < \lambda < r/2$. Then $E(S^{-\lambda})$ is finite and

$$(a^*)^{-\lambda}K(r,-\lambda) \le E(S^{-\lambda}) \le (a_*)^{-\lambda}K(r,-\lambda),$$

where $a^* = \max\{a_1, \ldots, a_m\}$ and $a_* = \min\{a_1, \ldots, a_m\}$.

Lemma 2.4.3. Let $\hat{\delta}$ be defined as in (2.2.1). Assume that the variance of ϕ_iU_i exists for each $i = 1, \ldots, s$. Then $\hat{\delta}$ is an unbiased estimator for θ and

$$\begin{aligned}\mathrm{Var}(\hat{\delta}|\gamma) &= \mathrm{Var}(T|\gamma)\\ &\quad -2\textstyle\sum_i C_i^2\{(1-\gamma)E(\phi_iX_i|\gamma)/\mathrm{Var}(U_i) - (1/2)E(\phi_i^2X_i|\gamma)\}.\end{aligned}$$

2.5. Preliminary results.

In this section we gather together some facts that are needed to establish the main results in this thesis.

Lemma 2.5.1. Let $Q \sim G(r)$ and let $t, a > 0$. Then

$$E(Q^b e^{-aQt}) = K(r,b)(1+2at)^{-(r+2b)/2}$$

for all b such that $b > -r/2$.

Proof: The expectation can be expressed as

$$[2\Gamma(r/2)]^{-1}\int_o^\infty Q^b e^{-aQt}[Q/2]^{(r/2)-1}e^{-Q/2}dQ.$$

Use the change of variable $u = (2at + 1)Q$. Then it is easy to get the result.□

Let $Q_1, \ldots, Q_m$, S and r be defined as in Section 2.4. Further, let $b_1, \ldots, b_m$ be real numbers such that

$$\beta_i = (r_i + 2b_i)/2 > 0, \; i = 1, \ldots, m.$$

Set

$$P = Q_1^{b_1} \ldots Q_m^{b_m} \quad \text{and} \quad H = PS^{-\lambda},$$

where λ is a given positive real number. Further, let

$$H^* = [a_1 Q_1^* + \ldots + a_m Q_m^*]^{-\lambda}$$

where $Q_1^*, \ldots, Q_m^*$ are independent random variables satisfying

$$Q_i^* \sim G(2\beta_i) \; , \; i = 1, \ldots, m.$$

Finally, set $\beta = \beta_1 + \ldots + \beta_m$.

Lemma 2.5.2. $E(H)$ is finite if and only if $E(H^*)$ is finite, in which case

$$E(H) = K(r_1, b_1) \ldots K(r_m, b_m)\, E(H^*).$$

Proof: Apply Lemma 4.2.3 in Saleh with $h = S^{-\lambda}$.□

Lemma 2.5.3. Assume that $0 < \lambda < \beta$. Let ξ be an $m \times 1$ vector whose ith element ξ_i is such that $|\xi_i| < 1$ for $i = 1, \ldots, m$. Then the integral

$$I_\xi = \int_o^1 (1-u)^{\lambda-1} u^{\beta-\lambda-1} \Pi_{i=1}^m (1-\xi_i u)^{-\beta_i} du$$

is finite.

Proof: Let $i \in \{1, \ldots, m\}$. Because $0 < u < 1$ and $|\xi_i| < 1$, it follows that

$$0 < 1 - |\xi_i| \le 1 - |\xi_i|\, u \le 1 - \xi_i u.$$

Hence,

$$(1 - \xi_i u)^{-\beta_i} \le (1 - |\xi_i|)^{-\beta_i}.$$

It follows that

$$\begin{aligned} I_\xi &\le \Pi_{i=1}^m (1 - |\xi_i|)^{-\beta_i} \int_o^1 (1-u)^{\lambda-1} u^{\beta-\lambda-1} du \\ &= 2^\lambda \Pi_{i=1}^m (1 - |\xi_i|)^{-\beta_i} K(2\beta, -\lambda)\Gamma(\lambda). \end{aligned}$$

The last equality follows by using the beta distribution and noting that $\beta - \lambda > 0$.□

Remark 2.5.4. If we let $\xi_1 = \ldots = \xi_m = \xi_0$ in Lemma 2.5.3 and do the change of variable

$$v = u(1 - \xi_0)/(1 - \xi_0 u),$$

then we get

$$I_\xi = 2^\lambda (1-\xi_0)^{\lambda-\beta}\,\Gamma(\lambda)K(2\beta, -\lambda) .$$

Lemma 2.5.5. Let η be a fixed positive number such that $4\eta > \max\{1/a_1, \ldots, 1/a_m\}$. Set $\xi_i = 1-(2a_i\eta)^{-1}$ for $i = 1, \ldots, m$. Suppose that $0 < \lambda < \beta$. Then $E(H^*)$ is finite if and only if I_ξ is finite, in which case

$$E(H^*) = 2^{-\beta}\eta^{\lambda-\beta}[1/\Gamma(\lambda)]\,\Pi_{i=1}^m a_i^{-\beta_i}\, I_\xi .$$

Proof: Note that H^* can be expressed as

$$H^* = [\Gamma(\lambda)]^{-1}\int_0^\infty t^{\lambda-1}\exp\{-\textstyle\sum_i a_i Q_i^* t\}dt ;$$

and that

$$E(H^*) = \int_0^\infty \cdots \int_0^\infty H^* dG(Q_1^*, \ldots, Q_m^*) .$$

Use Fubini's theorem (the integrand is a non-negative measurable function) to interchange the order of integration. This leads to the expression

$$\begin{aligned} E(H^*) &= [1/\Gamma(\lambda)]\int_0^\infty t^{\lambda-1}E(\exp\{-\textstyle\sum_i a_i Q_i^* t\})dt \\ &= [1/\Gamma(\lambda)]\int_0^\infty t^{\lambda-1}\Pi_{i=1}^m E(\exp\{-a_i Q_i^* t\})dt. \end{aligned}$$

The last expression follows because of the independence of $Q_1^*, \ldots, Q_m^*$. Now apply Lemma 2.5.1 with $Q_i^* \sim G(2\beta_i)$. Then do the following change of variables:

$$u = \eta/(\eta + t) \quad \text{and} \quad \xi_i = 1-(2a_i\eta)^{-1} .$$

Since $4\eta > 1/a_i$ implies $|\xi_i| < 1$, one can apply Lemma 2.5.3 to obtain the desired result.□

In our numerical work we will numerically evaluate I_ξ. In order to make the numerical evaluation converge faster, one should consider the smallest values of η such that $4\eta > 1/a_*$. (See Morin-Wahhab (1985).)

Remark 2.5.6. Let $S = a[Q_1 + \ldots + Q_m]$. By using Lemma 2.5.5 and Remark 2.5.4 one gets

$$E(H) = a^{-\lambda} K(2\beta, -\lambda)\Pi_{i=1}^{m} K(r_i, b_i) .$$

If Q_1 is omitted from S, then we get

$$E(H) = a^{-\lambda} K(2(\beta - \beta_1), -\lambda)\Pi_{i=1}^{m} K(r_i, b_i) .$$

2.6. The main result.

In this section we will study the properties of the estimator $\hat{\delta}$ in (2.2.1) with ϕ_i defined as in (2.3.1). We suppose that $g_1, \ldots, g_s$ are given fixed constants. Our purpose is to put conditions on the choice of g so that $\hat{\delta}$ dominates T.

Recall from Section 2.1 that we have random variables R, $U_1, \ldots, U_s$ and T and parameters π, γ, and θ. Further recall (see Section 2.3) that we can assume without loss in generality that $\pi = 1$ and $\theta = 0$. Thus, our only parameter of concern is $\gamma \in [0,1]$. Recall further that $R \sim (1-\gamma)G(q)$ so that when $\gamma = 1$ we get $R = 0$ with probability one. This means that $\hat{\delta} = T$ with probability one so that

$$\mathrm{Var}(\hat{\delta} \mid \gamma = 1) = \mathrm{Var}(T \mid \gamma = 1) .$$

Thus, we need only be concerned about how the estimator $\hat{\delta}$ behaves for γ in the half-open interval $[0,1)$.

For $\gamma \in [0,1)$, let $Q_0 = R/(1-\gamma)$ and let $a_0 = g(1-\gamma)$. With these definitions observe that $gR = a_0 Q_0$ and that $Q_0 \sim G(q)$. Similarly, for each $j = 1, \ldots, s$, let $a_j = g_j[1-\gamma+\gamma d_j]$ and let $Q_j = X_j/[1-\gamma+\gamma d_j]$ so that $g_j X_j = a_j Q_j / g_j$ and $Q_j \sim G(1)$. With these definitions we can write $W = a_1 Q_1 + \ldots + a_s Q_s$ and

(2.6.1) $$\phi_i = (1-\gamma) g Q_0 / [a_0 Q_0 + d_i W], \quad i = 1, \ldots, s.$$

In these expressions we have suppressed the functional dependence of $a_0, a_1, \ldots, a_s$ on γ. However, it should always be remembered $a_0, \ldots, a_s$ are all functions of γ; and it should also be noted that a_0 is always nonnegative and that $a_1, \ldots, a_m$ are always strictly positive.

For convenience in presentation, it is helpful to define the following quantities. Set

$$d^* = \max(d_1, \ldots, d_s),$$
$$d_* = \min(d_1, \ldots, d_s),$$
$$B_1 = \max(g_1, \ldots, g_s, g_1 d_1, \ldots, g_s d_s),$$
$$B_2 = \min(g_1, \ldots, g_s, g_1 d_1, \ldots, g_s d_s),$$
$$A = \max(1, d_1, \ldots, d_s),$$

and $$B^* = \max(g, d^* B_1).$$

And when $s > 2$, set

$$M = 2d_*^2 B_2^2 s(s-2)/A(q+2)(q+s+2).$$

Recall that $d_1, \ldots, d_s$ are given quantities in our model. Also, recall that $g_1, \ldots, g_s$ have been selected by the user.

Lemma 2.6.1. For each $i = 1, \ldots, s$ we have

$$E(\phi_i Q_i \mid \gamma) \geq g(1-\gamma)q/B^*(q+s+2) ;$$

and this inequality holds for all $\gamma \in [0,1)$.

Proof: Let $\gamma \in [0,1)$ be fixed. From the definition of B^*, it can be established that

$$a_0 Q_0 + d_i \sum_{j=1}^{s} a_j Q_j \leq B^*(Q_0 + \sum_{j=1}^{s} Q_j) .$$

Using this inequality it follows that

$$E(\phi_i Q_i \mid \gamma) \geq [(1-\gamma)g/B^*] \, E(Q_0 Q_i/[Q_0 + \sum_{j=1}^{s} Q_j]) .$$

Now the expectation on the right side is equal to $q/(q+s+2)$. This follows from Remark 2.5.6 with $m = s + 1$, $a = 1$, $\lambda = 1$, $r_1 = q$, $r_2 = r_3 = \ldots = r_{s+1} = 1$, $b_1 = b_2 = 1$ and $b_3 = \ldots b_{s+1} = 0$. Because $\gamma \in [0,1)$ was selected arbitrarily, the result is established.□

Lemma 2.6.2. Assume that $s \geq 3$. Then for each $i = 1, \ldots, s$ we have

$$E(\phi_i^2 Q_i \mid \gamma) \leq (1-\gamma)^2 g^2 q(q+2)/d_*^2 B_2^2 s(s-2) ;$$

and this inequality holds for all $\gamma \in [0,1)$.

Proof: Let $\gamma \in [0,1)$ be fixed. From the definitions of d_* and B_2, it can be established that

$$a_0 Q_0 + d_i \sum_{j=1}^{s} a_j Q_j \geq d_* B_2 \sum_{j=1}^{s} Q_j .$$

Using this inequality it follows that

$$E(\phi_i^2 Q_i \mid \gamma) \leq [(1-\gamma)^2 g^2/d_*^2 B_2^*] E[Q_0^2 Q_i/(\sum_{j=1}^{s} Q_j)^2] .$$

Since $s \geq 3$, the expectation on the right side is equal to $q(q+2)/s(s-2)$. This follows from Remark 2.5.7 with $m = s + 1$, $a = 1$,

$\lambda = 2$, $r_1 = q$, $r_2 = \ldots = r_{s+1} = 1$, $b_1 = 2$, $b_2 = 1$ and $b_3 = \ldots = b_{s+1} = 0$. Because $\gamma \in [0,1)$ was selected arbitrarily, the result is established.□

Theorem 2.6.3 Assume that $s \geq 3$. Let $\hat{\delta}$ be defined as in (2.2.1) with ϕ_i defined as in (2.3.1). Let g be any positive real number satisfying

$$gB^* < M . \tag{2.6.2}$$

Then $\text{Var}(\hat{\delta}|\gamma) < \text{Var}(T|\gamma)$, for all $\gamma \in [0,1)$.

Proof: Let $\gamma \in [0,1)$ be fixed. Lemma 2.6.2 implies that $\text{Var}(\phi_i U_i|\gamma)$ exists. So we can use Lemma 2.4.3 to get

$$\begin{aligned}\text{Var}(\hat{\delta}|\gamma) = \text{Var}(T|\gamma) - 2\textstyle\sum_{i=1}^{s} c_i^2 \{ (1-\gamma)E(\phi_i Q_i|\gamma) \\ - (1/2)(1-\gamma+\gamma d_i)E(\phi_i^2 Q_i|\gamma)\}\end{aligned}$$

Applying Lemmas 2.6.1 and 2.6.2 gives

$$\text{Var}(\hat{\delta}|\gamma) \leq \text{Var}(T|\gamma) - 2(1-\gamma)^2 \textstyle\sum_{i=1}^{s} c_i^2 B_i(\gamma) \tag{2.6.3}$$

where

$$B_i(\gamma) = gq/B^*(q+s+2) - g^2 q(q+2)(1-\gamma+\gamma d_i)/2d_*^2 B_2^2 s(s-2).$$

Notice that

$$1 - \gamma + \gamma d_i \leq \max(1,d_i) \leq A.$$

Thus, if $B_i(\gamma) > 0$ for $i = 1, \ldots, s$, then

$$\text{Var}(\hat{\delta}|\gamma) < \text{Var}(T|\gamma) \quad , \text{ all } \gamma \in [0,1).$$

For each $i = 1, \ldots, s$, since $1 - \gamma + \gamma d_i \leq A$, we get

$$B_i(\gamma) \geq gq/B^*(q+s+2) - g^2 q/M(q+s+2).$$

By our choice of g, it can be seen that the function on the right side of the inequality is strictly positive for all $\gamma \in [0,1)$.□

Now let us study two cases for the choices of g.

Case 1. Assume that $g \le d^* B_1$. Then, $B^* = d^* B_1$. So, the inequality (2.6.2) holds true iff $g < M/d^* B_1$. In this situation we notice that as the value of q becomes greater than that of s, the inequality $M/d^* B_1 \le d^* B_1$ is always true; but as s goes to infinity we can get $M/d^* B_1 > d^* B_1$.

Case 2. Assume that $g > d^* B_1$, that is, $B^* = g$, then (2.6.2) holds true iff $g < \sqrt{M}$. This case can only be true when s becomes very large relative to q .

2.7. A lower bound for the efficiency of $\hat{\delta}$.

In this section a lower bound for the efficiency of the estimator $\hat{\delta}$ of (2.2.1) and (2.3.1), with respect to the Cramer-Rao lower bound, is derived.

If γ is known, then the minimum variance unbiased estimator of θ is given by (see Saleh)

$$\hat{\delta}_\gamma = T - (1-\gamma)\sum_i C_i U_i/(1-\gamma + \gamma d_i),$$

and its variance is given by

$$\mathrm{Var}(\hat{\delta}_\gamma | \gamma) = \mathrm{Var}(T | \gamma) - (1-\gamma)^2 \sum_i C_i^2/(1-\gamma+\gamma d_i).$$

Using Lemma 7.1 in McClellan (1984), we can conclude that $\mathrm{Var}(\hat{\delta}_\gamma | \gamma)$ is, as a function of γ, the lower bound for the variance of all unbiased estimators of θ.

Throughout this thesis, let $EF(\cdot | \gamma)$ denote the efficiency of any unbiased estimator for θ relative to the variance of $\hat{\delta}_\gamma$.

The following lemma shows that the efficiency of T is an increasing function of $\gamma \epsilon [0,1]$. This result is actually a special case of that given in Lemma 3.15 by McClellan.

<u>Lemma 2.7.1.</u> $EF(T|\gamma)$ is strictly increasing function of $\gamma \epsilon [0,1)$.

<u>Proof</u>: $EF(T|\gamma)$ is a continuous function of γ for $\gamma \epsilon [0,1)$. To show that it is increasing, it is enough to show that the derivative with respect to γ is positive for all $\gamma \epsilon [0,1)$. By differentiating $EF(T|\gamma)$ with respect to γ and doing some messy algebraic manipulations, it can be shown that

$$dEF(T|\gamma)/d\gamma = \sum_i c_i^2 [(1-\gamma)^2 h_1 d_i + (1-\gamma)(1-\gamma+2\gamma d_i)h_2]/(1-\gamma+\gamma d_i)^2 Var(T|\gamma)^2 .$$

From this expression it can be concluded that the derivative with respect to γ is positive for all $\gamma \epsilon [0,1)$.□

Notice that the efficiency of T evaluated at $\gamma = 0$ is given by

$$EF(T|\gamma = 0) = 1 - (1/h_1)\sum_{i=1}^{s} c_i^2 .$$

Since $EF(T|\gamma)$ is an increasing function we can conclude that

$$EF(T|\gamma) \geq EF(T|\gamma = 0) ;$$

and if $\hat{\delta}$ is unbiased and dominates T, then

$$EF(\hat{\delta}|\gamma) \geq EF(T|\gamma = 0) .$$

III. APPLICATION TO THE LINEAR MIXED MODEL

3.1. The model and related topics.

Consider the mixed linear model

$$Y = X\beta + Bb + e$$

where Y is an $n \times 1$ random vector, X is a known $n \times p$ matrix, β is a vector of p unknown fixed parameters, B is a known $n \times m$ matrix, and b, e are random vectors. We make the usual assumptions about b and e. In particular, $b \sim N_m(0, \sigma_b^2 I_m)$ with $\sigma_b^2 \geq 0$, $e \sim N_n(0, \sigma_e^2 I_n)$ with $\sigma_e^2 > 0$ and b and e are independent. We also assume that $\underline{r}(X,B) < n$.

Set $V = BB'$, $\pi = \sigma_e^2 + \sigma_b^2$ and $\gamma = \sigma_b^2/\pi$. Further, for each $\gamma \in [0,1]$, let $\Sigma_\gamma = (1-\gamma)I_n + \gamma V$. Then, the model assumptions along with a reparametrization of the covariance structure can be written as

$$Y \sim N_n(X\beta, \pi\Sigma_\gamma) ,$$

where $\pi > 0$ and $\gamma \in [0,1]$.

Let $\theta = \lambda'\beta$ be an estimable parametric function. For each $f \in [0,1)$ let $G_f = \Sigma_f^{-1}$ and for $f = 1$ let G_1 be any g-inverse of $XX' + V$ which has the property that $\underline{R}(G_1) = \underline{R}(X,B)$. For example, G_1 could be the Moore-Penrose inverse. For each $f \in [0,1]$ set $T_f = \lambda'\hat{\beta}_f$ where $\hat{\beta}_f$ is any random vector satisfying $X'G_fX\hat{\beta}_f = X'G_fY$. Azzam, Birkes and Seely (1986) characterized the minimal complete class $\mathscr{A}$ of all admissible linear unbiased estimators (LUE's) for θ with respect to the class of all LUE's.

They showed that the set of all linear unbiased estimators T_f as f varies in the interval [0,1] constitute the entire set $\mathscr{A}$.

Under normality, T_f is the uniformly minimum variance unbiased estimator (UMVUE) for θ with respect to the covariance structure $\pi\Sigma_f$. When $f \in [0,1)$ the matrix Σ_f is positive definite. Because of this, it can be shown that T_f is admissible within the class of all unbiased estimators for θ. The argument used to justify this last statement cannot be used for the estimator T_1 because $\Sigma_1 = V$ is not necessarily positive definite. In fact, after the next section it will be clear that T_1 is generally not admissible within the class of all unbiased estimators of θ.

This thesis is devoted to trying to find nonlinear unbiased estimators for θ that dominate T_1. To do this, it is convenient to have another characterization for T_1. In particular it can be shown that $T_1 = t'Y$ where t is the unique vector satisfying the conditions

a) $X't = \lambda$,

b) $Vt \in \underline{R}(X)$,

and c) $t \in \underline{R}(X,V)$.

See, for example, McClellan 1984.

3.2. The canonical form of Saleh.

In this section, we present a brief summary of Section 2.4 in Saleh. We have used L_1 and L_2 instead of Q and L given in Saleh.

Let $q = n - \underline{r}(X,B)$ and let L_1 be an $n \times q$ matrix satisfying the following two conditions:

a) $\underline{R}(L_1) = \underline{R}(X,B)^{\perp}$

and b) $L_1'L_1 = I_q$.

Further, let $s = \underline{r}(X,B) - \underline{r}(X)$ and let L_2 be an $n \times s$ matrix satisfying the following three conditions:

a) $\underline{R}(L_2) = \underline{R}(X,B) \cap \underline{R}(X)^{\perp}$,

b) $L_2'L_2 = I_s$

and c) $L_2'VL_2 = D$;

where D is an $s \times s$ diagonal matrix with elements $d_1, \ldots, d_s$. The matrix D is nonnegative definite. Also, it has rank s. It follows that it is a positive definite matrix and hence $d_1, \ldots, d_s$ are all positive constants.

The matrices L_1 and L_2 defined in the previous paragraph have the property that the columns of $G = (L_1, L_2)$ form an orthonormal basis for $\underline{N}(X')$. Let $Z = L_1'Y$ and $U = L_2'Y$. The following distributional properties are easily verified:

a) $Z \sim N_q(0, \pi(1-\gamma)I_q)$.

b) $U \sim N_s(0, \pi D_\gamma)$ where $D_\gamma = (1-\gamma)I_s + \gamma D$.

c) $T_1 \sim N[\theta, \pi((1-\gamma)t't + \gamma t'Vt)]$.

d) Z is independent of T_1 and U.

e) $\mathrm{Cov}(U,T_1) = \pi(1-\gamma)L_2't$.

Now set $T = T_1$, $R = Z'Z$, $h_1 = t't$, $h_2 = t'Vt$ and $C = L_2't$. Then T, R, and U have exactly the properties assumed in Chapter II. Notice that under the assumption of this model we get that $U'U$ is the sum of squares for b adjusted for β. And if we take $g_1 = \ldots = g_s = 1$ in Section 2.3, then $W = U'U$.

3.3. Normal equations.

In this section we show how a certain class of $\hat{\delta}$ estimators can be obtained via normal equations.

Let W be defined in the usual fashion. That is, $W = g_1U_1^2 + \ldots + g_sU_s^2$ where $g_1, \ldots, g_s$ are given positive constants. Let $\hat{\gamma}$ be a function of R/W such that $0 \le \hat{\gamma} \le 1$. Set

(3.3.1) $$\phi_i = (1-\hat{\gamma})/(1-\hat{\gamma}+\hat{\gamma}d_i),\ i = 1, \ldots, s.$$

Let $\hat{\delta}$ denote the estimator of (2.2.1) using $\phi_1, \ldots, \phi_s$ as defined above. Recall that G_f is defined for all $f \in [0,1]$. Thus, $G_{\hat{\gamma}}$ is a well defined random matrix. We now show that $\hat{\delta}$ can be obtained as a solution to normal equations.

Theorem 3.3.1. Let $\hat{\gamma}$ and $\hat{\delta}$ be defined as above. If $\hat{\beta}$ satisfies the following normal equations

$$X'G_{\hat{\gamma}}X\hat{\beta} = X'G_{\hat{\gamma}}Y,$$

then $\hat{\delta} = \lambda'\hat{\beta}$.

<u>Proof</u>: Recall the defintion $\hat{\delta}_\gamma$ from Section 2.7. Let $v \in [0,1]$. From the theory of linear models, it is known that if $\gamma = v$, then $\hat{\delta}_v = \lambda' \hat{\beta}_v$ whenever $\hat{\beta}_v$ satisfies $X'G_v X\hat{\beta}_v = X'G_v Y$. Since this same argument works for any given value of $\hat{\gamma} + v$, it follows that $\hat{\delta} = \lambda' \hat{\beta}$ as claimed in the theorem.□

Let g be a positive constant. Set

$$\hat{\gamma} = W/[gR + W].$$

Then $\hat{\gamma}$ satisfies the properties previously described and our ϕ_i of (2.3.1) can be rewritten as in (3.3.1). This means our estimator can be calculated via Theorem 3.3.1. The estimators that Saleh worked with cannot be written in the form of this section. This is because her choice of ϕ_i cannot be reduced to the form in (3.3.1) for any choice of $\hat{\gamma}$.

IV. COMPUTING VAR($\hat{\delta}$).

Saleh (1987) obtained the variance of $\hat{\delta}$ via simulation. This method can be very time consuming. In this section we provide a method of computing the exact variance of $\hat{\delta}$ by means of a formula that requires integration of a single variable. We refer to this method of computing the variance of $\hat{\delta}$ as the exact method. A computer program is also provided in the Appendix.

4.1. Exact method.

Recall that the estimator $\hat{\delta}$ with which we are working is given by the formula

$$\hat{\delta} = T - \sum_i C_i \phi_i U_i \ .$$

And that the ϕ_i in this expression are defined as in equation (2.3.1).

Throughout this section and the next we assume the same notation and definitions as in Section 2.6. Also, we assume throughout that γ is some fixed value in the interval $[0,1)$.

From Section 2.6 we know for each $i = 1, \ldots, s$ that

$$\phi_i = (1-\gamma) g Q_0 / [a_0 Q_0 + d_i W] \ . \tag{4.1.1}$$

Recall that $W = \sum_{i=1}^{s} a_i Q_i$ where $a_0, \ldots, a_s$ are dependent on γ, and that $Q_0, Q_1, \ldots, Q_s$ are independent chi-squared random variables.

To calculate the variance of $\hat{\delta}$, we will use the results in Lemma 2.4.3 and Lemma 2.5.5. For this purpose it is convenient to introduce the following notation. Let

$$S_i = a_0 Q_0 + d_i W$$

$$H_{1i} = Q_0 Q_i / S_i$$

$$H_{2i} = Q_0^2 Q_i / S_i^2$$

and $\quad A_i = 1 - \gamma + \gamma d_i \,, \quad i = 1, \ldots, s \,.$

Now use the variance formula given in Lemma 2.4.3 and then consider ϕ_i defined as in equation (4.1.1). Putting these things together we can express the variance as

$$\text{(4.1.2)} \qquad \text{Var}(\hat{\delta}|\gamma) = \text{Var}(T|\gamma) - 2(1-\gamma)^2 \sum_{i=1}^{s} c_i^2 [gE(H_{1i}) - (1/2) g^2 A_i E(H_{2i})]$$

where A_i, H_{1i} and H_{2i} are defined as above. In this expression, we know how to compute the variance of T. Thus, we need only to concentrate on the expectation of H_{1i} and H_{2i}.

An evaluation of the expectation of H_{1i} and H_{2i} can be done using simulation. That is, by generating random variables of chi-squared distributions with different degrees of freedom and repeating this procedure a large number of times and then take the average. But this method is very time consuming.

Another way to evaluate the expectation of H_{1i} and H_{2i} can be done using the exact method which we now discuss. Since H_{1i} and H_{2i} are special cases of H defined in Section 2.5, we can use the results in Lemmas 2.5.2 and 2.5.5 to express the expectations of H_{1i} and H_{2i} in terms of a one variable integral.

As a general remark, we mention that the technique described in this section for computing the variance of $\hat{\delta}$ can be applied to a wide class of ϕ_i functions.

4.2. The computation formula.

From Section 2.6, recall that $a_0 = (1-\gamma)g$ and $a_j = (1-\gamma+\gamma d_j)g_j$, $j = 1, \ldots, s$. Let η be a given fixed number satisfying $\eta > 1/4a_0$ and $\eta > 1/4d_i a_j$ for $i, j = 1, \ldots, s$. Set

$$\xi_0 = 1 - (2a_0\eta)^{-1},$$

$$\xi_{ij} = 1 - (2d_i a_j \eta)^{-1},$$

and

$$M_i(\eta) = \eta^{1-\beta} a_0^{-\beta_0} \Pi_{j=1}^{s} (d_j a_j)^{-\beta_{ij}};$$

where $\beta_0 = (q+2)/2$, $\beta = (q+s+2)/2$ and

$$\beta_{ij} = \begin{cases} 3/2 \text{ if } i = j \\ & i, j = 1, \ldots, s \\ 1/2 \text{ if } i \neq j . \end{cases}$$

Then the variance of $\hat{\delta}$ is given by (4.1.2) with

(4.2.2) $E(H_{1i}) =$

$$K(q,1)M_i(\eta)\,{}_o\!\int^1 u^{\beta-2}(1-\xi_0 u)^{-\beta_0}\Pi_{j=1}^{s}(1-\xi_{ij}u)^{\beta_{ij}}du,$$

and

$$E(H_{2i}) = K(q,2)(2a_0)^{-1}M_i(\eta)\,{}_o\!\int^1 u^{\beta-2}(1-u)(1-\xi_0 u)^{-(\beta_0+1)} \times$$

$$\Pi_{j=1}^{s}(1-\xi_{ij}u)^{-\beta_{ij}}du.$$

V. THE RANDOM ONE-WAY MODEL

5.1. The model.

In the current chapter we consider the random one-way model with t different classes and n observations. For convenience in presentation, we suppose the t different classes have been separated into k groups in such a way that the ith group consists of m_i classes and that each class in the group has n_i observations. Thus, we have $t = m_1 + \ldots + m_k$ classes and $n = m_1 n_1 + \ldots + m_k n_k$ observations.

With the above description, we can express the model as

$$Y_{iju} = \mu + a_{ij} + e_{iju} ,$$

where $i = 1, \ldots, k$; $j = 1, \ldots, m_i$; and $u = 1, \ldots, n_i$.
We suppose that μ is an unknown constant and that all a_{ij} and e_{iju} are independent normal random variables with zero means and variances σ_a^2 and σ_e^2 respectively. As usual, we assume that $\sigma_e^2 > 0$ and $\sigma_a^2 \geq 0$.

Let Y_i be the vector of $m_i n_i$ observations for the m_i classes in the ith group. Assume that the observations in each Y_i are ordered lexicographically. Let $A_i = I_{m_i} \otimes 1_{n_i}$ be the Kronecker product of the $m_i \times m_i$ identity matrix and the n_i vector of ones. Set

$$Y = (Y_1', \ldots, Y_k')' \text{ and } B = \operatorname{diag}(A_1, \ldots, A_k) .$$

Then the model can be written as in Section 3.1 with $X = 1_n$, $\beta = \mu$, $\pi = \sigma_e^2 + \sigma_a^2$, $\gamma = \sigma_a^2/\pi$ and $V = BB'$.

5.2. A partial reduction.

In this section we give a partial reduction of the random one-way model to the canonical form of Section 3.2. In this section we do the straightforward part of canonical form reduction. The final reduction to the actual canonical form is given in the next section. We begin by noting that the matrix X has rank one and that its range is a subspace of $\underline{R}(B)$. From these two observations and $\underline{r}(B) = t$, we can conclude the following facts:

(5.2.1) a) $s = t - 1$.

b) $q = n - t$.

Now let us concentrate on U, T, and R as defined in Section 3.2. To do this, it is convenient to first make the nonsingular linear transformation from Y to $Z = L_1'Y$ and to the k random vectors $M_1, \ldots, M_k$ where M_i is an $m_i \times 1$ random vector whose jth component is the average of the observations in class j of group i. That is,

$$M_{ij} = (1/n_i)\sum_{u=1}^{n_i} Y_{iju}$$

for all $i = 1, \ldots, k$ and $j = 1, \ldots, m_i$. To see that this is a linear transformation it is only necessary to observe that

$$M = (M_1', \ldots, M_k')' = (B'B)^{-1}B'Y,$$

and that $\underline{R}(L_1)^{\perp} = \underline{R}(B)$.

From Section 3.2, we know that $R = Z'Z$ is the residual sum of squares from the model that treats the random effects b as fixed and that W (with $g_1 = \ldots = g_s = 1$) is the sum of squares for b adjusted for μ. Thus, R and W are in fact the within and between sum of squares from the usual one-way ANOVA table.

Now let us turn to T. From Birkes, Seely and Azzam (1981), we find that

$$(5.2.2) \qquad T = (1/t)\textstyle\sum_{ij} M_{ij} .$$

That is, T is the unweighted average of the class means.

Finally, recall that $U = L_2'Y$ where L_2 is an n x s matrix defined in Section 3.2. Since $\underline{R}(L_2) \subset \underline{R}(B)$, it follows that the components of U are linear combinations of M. Thus, to find the components of U we need to find s statistically independent linear combinations of M such that each linear combination has a zero expectation and a variance normalized so that the coefficient of $(1-\gamma)$ in the variance is one. If we can satisfy all of these conditions, the resulting vector U will be precisely the one described in Section 3.2.

Let i be one of the integers 1, ..., k. Let Q_i be an $m_i - 1 \times 1$ random vector of orthonormal contrasts of $\sqrt{n_i}\, M_i$. That is, the pth component of Q_i is $U_{ip} = \sqrt{n_i}\, x_p' M_i$ where $x_p' x_p = 1$, $x_p' x_{p'} = 0$ and $1_{m_i}' x_p = 0$ for $p \neq p'$ and $p, p' = 1, \ldots, m_i - 1$. Since the components of M_i are normally and independently distributed with common mean μ and common variance $[1-\gamma+n_i\gamma]/n_i$, we get from elementary single sample statistical theory that the components of Q_i are normally and independently distributed with mean zero and variance $1 - \gamma + n_i\gamma$.

Thus, we conclude that

$$Q_i \sim N_{m_i-1}[0,(1-\gamma+n_i\gamma)I_{m_i-1}] \,.$$

We can further conclude from single sample theory that

$$Q_i'Q_i = \sum_{j=1}^{m_i} n_i(M_{ij} - \bar{Y}_i)^2 ,$$

where $\bar{Y}_i$ is the average of the class means in the ith group.

Since $M_1, \ldots, M_k$ are independent, it follows that $Q_1, \ldots, Q_k$ are independent. Now consider an arbitrary U_{ip} and an arbitrary sum $S_r = \sum_u M_{ru}$. The covariance of U_{ip} and S_r is given by

$$\mathrm{Cov}(U_{ip},S_r) = \sqrt{n_i}\, x_p'\mathrm{Cov}(M_i,M_r)1_{m_r} \,.$$

If $i \neq r$, then U_{ip} and S_r are independent because $\mathrm{Cov}(M_i,M_r) = 0$. If $i = r$, then we have the same conclusion because $\mathrm{Cov}(M_i)$ is a multiple of the identity matrix and because $x_p'1_{m_i} = 0$. Since T is a function of $S_1, \ldots, S_k$, we can conclude that all U_{ip} and T are independent. And so, $T, Q_1, \ldots, Q_k$ are mutually independent.

In the above discussion, we obtained

$$t-k = (m_1-1) + \ldots + (m_k-1)$$

of the components of U. It remains to get $k-1$ additional components. These we obtain in the next section.

5.3. LaMotte's results.

For the random one-way model LaMotte (1976) described the eigenvalues and their multiplicities for a matrix $G'BB'G$ where G is a matrix whose columns form an orthonormal basis for $\underline{N}(X')$. (Actually, LaMotte selected a particular matrix G, but the eigenvalues and their multiplicities remain the same for any choice of G. See, for example, Remark 2.1 in Olsen, Seely and Birkes (1976).) From the definitions of L_1 and L_2 in Section 3.2, it can be seen that $G = (L_1, L_2)$ is a matrix satisfying the LaMotte requirements. Because of this, it follows that the positive eigenvalues and their multiplicities given by LaMotte describe precisely the s elements of the matrix D defined in Section 3.2.

Briefly, LaMotte showed that each n_i is an eigenvalue of multiplicity $m_i - 1$. These are the eigenvalues described in the previous section that correspond to $Q_1, \ldots, Q_k$. LaMotte further showed that the remaining $k - 1$ eigenvalues are the roots to the following equation

(5.3.1) $$h(d) = \sum_{i=1}^{k} m_i n_i / (n_i - d) .$$

He showed that this equation has exactly $k - 1$ distinct roots, each of multiplicity one. For each of the $k - 1$ roots of $h(d)$, LaMotte mentioned that the $k - 1$ orthonormal contrasts among the k groups corresponding to these eigenvalues are given by

(5.3.2) $$U_\ell = [\sum_{i=1}^{k} (m_i n_i / (n_i - d_\ell)^2)]^{-1/2} \sum_{i=1}^{k} (m_i n_i \bar{Y}_i / (n_i - d_\ell)),$$

where $\ell = 1, \ldots, k - 1$.

From the previous discussion we can partition the $s \times 1$ random vector U as follows

$$U = (U_{11}, \ldots, U_{k,m_k-1}, U_1, \ldots, U_{k-1})' \quad ,$$

where the first $t-k$ components are described in Section 5.2 and the last $k-1$ components are given by (5.3.2).

In parallel with U we can partition the $s \times 1$ vector C and the $s \times s$ matrix D as

$$C = (C_{11}, \ldots, C_{k,m_k-1}, C_1, \ldots, C_{k-1})' ,$$

and $\quad D = \mathrm{diag}(d_{11}, \ldots, d_{k,m_k-1}, d_1, \ldots, d_{k-1})$.

From the previous section we know that T has zero covariance with an arbitrary U_{ij}. This means that $C_{ij} = 0$ for all i and j. Thus, the only nonzero covariances are those of T with $U_1, \ldots, U_{k-1}$ defined by (5.3.2). For $\ell = 1, \ldots, k-1$, these covariances are given by

$$\mathrm{Cov}(T, U_\ell) = (1-\gamma)C_\ell \quad ,$$

where

(5.3.3) $C_\ell = (1/t)\sum_{i=1}^{k} m_i/(n_i - d_\ell)[\sum_{i=1}^{k} m_i n_i/(n_i - d_\ell)^2]^{-1/2}$.

Here $d_1, \ldots, d_{k-1}$ are the $k-1$ roots of equation (5.3.1). Putting the information in this section, together with that in the previous section, we now have a complete summary of U, C, and D .

5.4. The estimators.

We are interested in estimating $\theta = \mu$. Birkes, Seely and Azzam (1981) showed for $f \in [0,1]$ that

$$T_f = \sum_{i=1}^{k} m_i n_i [(n_i - 1)f + 1]^{-1} \bar{Y}_i \Big/ \sum_{i=1}^{k} m_i n_i [(n_i - 1)f + 1]^{-1}.$$

Using the results in Section 3.3 and the previous equation, the normal equation estimator $\hat{\delta}_{\hat{\gamma}}$ has the explicit form

$$\hat{\delta}_{\hat{\gamma}} = \sum_{i=1}^{k} m_i n_i [(n_i - 1)\hat{\gamma} + 1]^{-1} \bar{Y}_i \Big/ \sum_{i=1}^{k} m_i n_i [(n_i - 1)\hat{\gamma} + 1]^{-1},$$

where $\hat{\gamma}$ can be any random variable in the interval [0,1]. We shall use $\hat{\gamma}$ as in (3.3.2) with R defined in the usual fashion and W selected with $g_1 = \ldots = g_s = 1$. Thus, the only variable in our choice of $\hat{\gamma}$ will be the constant g.

Birkes et al. introduced the maximin-efficiency linear unbiased estimator T^* of θ. To get the estimator T^* one proceeds as follows: for each LUE find its minimum possible efficiency for $\gamma \in [0,1]$. Then select T^* to be the LUE that maximizes these minimum efficiencies. They showed that $T^* = T_{f^*}$, where $f^* \in [0,1]$ is determined by the equation

$$n\sum_{i=1}^{k} m_i n_i [(n_i - 1)f^* + 1]^{-2} = t\sum_{i=1}^{k} m_i n_i [(n_i - 1)f^* + 1]^{-2}.$$

This equation has a unique solution and it can be solved by elementary iterative numerical procedures.

The purpose of presenting the maximin-efficiency estimator is to compute its efficiency and compare it with the efficiency of our $\hat{\delta}$ estimators.

5.5. $\hat{\delta}$ under the one-way model.

Let us suppose that the number of observations in the various classes are ordered so that $n_1 < \ldots < n_k$. Further, let d^*, d_*, B_1, B_2, B^*, A and M be defined as in Section 2.6.

Let us suppose the $k-1$ roots from (5.3.1) are ordered according to $d_1 < \ldots < d_{k-1}$. From LaMotte, we get that

$$n_i < d_i < n_{i+1}, \ i = 1, \ \ldots, \ k\text{-}1.$$

From this relationship and the previous section, we can conclude the following:

a) If $m_k > 1$, then $d^* = n_k$; otherwise $n_{k-1} < d^* < n_k$.

b) If $m_1 > 1$, then $d^* = n_1$; otherwise $n_1 < d_* < n_2$.

From the knowledge of the eigenvalues given in the previous paragraph, and using $g_1 = \ldots = g_s = 1$, we can simplify the definitions given in Section 2.6. In particular:

$$B_1 = A = d^* \ ,$$

$$B_2 = 1 \ ,$$

$$B^* = \max(g, d^{*2})$$

and $$M = 2d_*^2(t-1)(t-3)/(n-t+2)(n+1)d^*.$$

Then using Theorem 2.6.3, we can conclude that $\hat{\delta}$ dominates T if $gB^* \leq M$.

It is possible to determine some bounds on g without having to compute d_* and d^*. To see this, let

$$M1 = 2n_1^2(t-1)(t-3)/(n-t+2)(n+1)n_k.$$

Now if m_1 and m_k are both greater than 1, then $M1 = M$. Even if m_1 and m_k are equal to 1, it is true that $M1 \leq M$. To see this, simply note that $n_1 \leq d_*$ and $d^* \leq n_k$. Thus, one could always use the condition $gB^* \leq M1$, and Theorem 2.6.3 would still remain true.

5.6. Efficiency Comparisons.

In this section we describe the efficiency comparisons for our estimator $\hat{\delta}$ with the unweighted average of the class means T and the maximin-efficiency linear unbiased estimator T^*.

From Section 2.7, recall that the efficiency of any estimator, say $\hat{\theta}$, for θ is defined as

$$\text{(5.6.1)} \qquad EF(\hat{\theta}|\gamma) = Var(\hat{\delta}_\gamma|\gamma)/Var(\hat{\theta}|\gamma), \ \gamma \epsilon [0,1].$$

The quantity $Var(\hat{\delta}_\gamma|\gamma)$ is the lower bound for the variance of all unbiased estimators of θ and can be expressed as

$$Var(\hat{\delta}_\gamma|\gamma) = Var(T|\gamma) - (1-\gamma)^2\sum_{i=1}^{k-1}c_i^2/(1-\gamma+d_i\gamma).$$

For this expression, the variance of T is given by

$$Var(T|\gamma) = (1-\gamma)\sum_{i=1}^{k}(m_i/t^2n_i) + \gamma/t .$$

In our calculations, we will find the efficiency using the above expressions for the estimators T, T^* and $\hat{\delta}$ for various choices of the constant g. The variance of T has been given previously and the variance of the various $\hat{\delta}$ estimators is given by (4.1.2) and (4.2.1). Finally, the variance of T^* is given by

$$Var(T^*|\gamma) = \sum_{i=1}^{k}[m_in_i(1-\gamma+n_i\gamma)/((n_i-1)f^*+1)^2] \div [\sum_{i=1}^{k}m_in_i[(n_i-1)f^*+1]^{-1}]^2 .$$

Recall the discussion in Section 2.7 regarding the lower bound on efficiency. In that section we saw when $\hat{\delta}$ dominates T that $EF(T|\gamma=0)$ is a lower bound for the efficiency of $\hat{\delta}$. This quantity is easily computed as

$$EF(T|\gamma=0) = t^2/[n\sum_{i=1}^{k}m_i/n_i].$$

5.7. The case $k = 2$.

We consider here the special case of the random one-way model previously described when $k = 2$. For this case we have $t = m_1 + m_2$ and $n = n_1m_1 + n_2m_2$.

Now let us consider the quantities C, U, and D as defined in Sections 5.2 and 5.3. From Section 5.3, we can conclude that C has only one nonzero component C_1. From equation (5.3.3), this can be calculated as

$$C_1 = -(n_1 - n_2)[m_1m_2/n_1n_2n]^{1/2}/t .$$

The eigenvalue associated with C_1 is, by the LaMotte equation, given by

$$d_1 = n_1n_2t/n .$$

Finally the corresponding U_1 is given by

$$U_1 = [n_1n_2m_1m_2/n]^{1/2}(\bar{Y}_1 - \bar{Y}_2) .$$

From the above results we can determine the form of the $\hat{\delta}$ estimator. In particular, using

$$\hat{\gamma} = W/[gR + W] ,$$

we can express $\hat{\delta}$ as follows:

$$\hat{\delta} = \{m_1n_1(1-\hat{\gamma}+n_2\hat{\gamma})\bar{Y}_1 + m_2n_2(1-\hat{\gamma}+n_1\hat{\gamma})\bar{Y}_2\}/[n(1-\hat{\gamma})+n_1n_2t\hat{\gamma}]$$
$$= T - C_1[gR/(gR + d_1W)]U_1 .$$

The quantities R and W are defined in the usual way as the within and between sum of squares from the usual ANOVA table.

In the remaining part of this section we give a numerical study for several examples. Associated with each example are two tables with the same identification number as the example. The first table shows the efficiencies of a set of $\hat{\delta}$ estimators corresponding to different values of g. These efficiencies are presented in the last four columns for the different values of g. The first table also gives the efficiency of the unweighted mean estimator T. The second table shows the efficiencies of a set of $\hat{\delta}$ estimators different from those given in the first table. The second table also gives the efficiency of the maximin-efficiency estimator T^*. The $\hat{\delta}$ estimators have been selected so that the ones in the first table are competitors with the estimator T; and the ones in the second table are competitors with the estimator T^*.

Before giving the examples, let us consider the condition $gB^* \le M$ in Theorem 2.6.3. It can be shown for each of the following examples that the choice of $g > d^{*2}$ is impossible (because then $gB^* > M$). Thus, we need only consider the case when $g \le (d^*)^2$ which means that $B^* = (d^*)^2$.

Example 5.7.1. For this example we take $n_1 = 1$, $n_2 = 3$, $m_1 = 3$ and $m_2 = 1$. From these values we get the following:

$n = 6$, $t = 4$, $q = 2$ and $s = 3$.

$d_* = 1$, $d^* = d_1 = 2$ and $M = 0.1071$.

$C_1 = 0.2041$, $f^* = 0.3660$ and $g \leq .0268$.□

Example 5.7.2. For this example we take $n_1 = 1$, $n_2 = 3$, $m_1 = 8$ and $m_2 = 1$. From these values we get the following:

$n = 11$, $t = 9$, $q = 2$ and $s = 8$.

$d_* = 1$, $d^* = d_1 = 2.4545$ and $M = 0.8148$.

$C_1 = 0.1094$, $f^* = 0.3660$ and $g \leq .1353$.□

Example 5.7.3. For this example we take $n_1 = 1$, $n_2 = 3$, $m_1 = 6$ and $m_2 = 2$. From these values we get the following:

$n = 12$, $t = 8$, $q = 4$ and $s = 7$.

$d_* = 1$, $d^* = 3$ and $M = 0.2991$.

$d_1 = 2$, $C_1 = 0.1443$, $f^* = 0.3660$ and $g \leq 0.0332$.□

Example 5.7.4. For this example we take $n_1 = 3$, $n_2 = 5$, $m_1 = 7$ and $m_2 = 1$. From these values we get the following:

$n = 26$, $t = 8$, $q = 18$ and $s = 7$.

$d_* = 3$, $d^* = d_1 = 4.6154$ and $M = 0.2528$.

$C_1 = 0.0335$, $f^* = 0.2052$ and $g \leq 0.0119$.□

<u>Example 5.7.5</u>. For this example we take $n_1 = 3$, $n_2 = 5$, $m_1 = 4$ and $m_2 = 4$. From these values we get the following:

$n = 32$, $t = 8$, $q = 24$ and $s = 7$.

$d_* = 3$, $d^* = 5$ and $M = 0.1468$.

$d_1 = 3.75$ $C_1 = 0.0456$ $f^* = 0.2052$ and $g \le .0059$.□

<u>Example 5.7.6</u>. For this example we take $n_1 = 3$, $n_2 = 5$, $m_1 = 1$ and $m_2 = 7$. From these value we get the following:

$n = 38$, $t = 8$, $q = 30$ and $s = 7$.

$d_* = d_1 = 3.1579$, $d^* = 5$ and $M = 0.1119$.

$C_1 = 0.0277$, $f^* = 2.052$ and $g \le .0045$.□

Table 5.7.1 Efficiencies (g values in last 4 columns)

γ	T	0.0079	0.0268	0.25	0.75
0	0.8000	0.8008	0.8027	0.8198	0.8442
.10	0.8556	0.8563	0.8578	0.8717	0.8915
.20	0.8974	0.8980	0.8991	0.9099	0.9253
.30	0.9289	0.9293	0.9301	0.9382	0.9496
.40	0.9524	0.9527	0.9533	0.9590	0.9670
.50	0.9697	0.9699	0.9703	0.9741	0.9794
.60	0.9821	0.9823	0.9825	0.9848	0.9880
.70	0.9907	0.9908	0.9909	0.9922	0.9937
.80	0.9962	0.9962	0.9963	0.9968	0.9974
.90	0.9991	0.9991	0.9991	0.9993	0.9993
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	4.00	9.00	20.0	60.0
0	0.9330	0.9060	0.9375	0.9624	0.9839
.10	0.9694	0.9399	0.9627	0.9788	0.9899
.20	0.9895	0.9612	0.9758	0.9836	0.9848
.30	0.9985	0.9743	0.9816	0.9820	0.9739
.40	0.9996	0.9822	0.9834	0.9772	0.9604
.50	0.9952	0.9871	0.9835	0.9717	0.9471
.60	0.9868	0.9901	0.9830	0.9672	0.9356
.70	0.9757	0.9925	0.9836	0.9654	0.9282
.80	0.9626	0.9948	0.9864	0.9685	0.9285
.90	0.9482	0.9976	0.9923	0.9796	0.9444
.99	0.9342	0.9999	0.9997	0.9987	0.9936

Table 5.7.2 Efficiencies (g values in last 4 columns)

γ	T	.05	.1353	1.000	4.000
0	0.8837	0.8847	0.8864	0.8996	0.9245
.10	0.9184	0.9192	0.9206	0.9312	0.9510
.20	0.9432	0.9439	0.9449	0.9532	0.9683
.30	0.9612	0.9617	0.9625	0.9687	0.9798
.40	0.9743	0.9747	0.9752	0.9797	0.9872
.50	0.9838	0.9840	0.9844	0.9874	0.9921
.60	0,9905	0.9907	0.9909	0.9927	0.9954
.70	0.9951	0.9952	0.9953	0.9963	0.9975
.80	0.9980	0.9980	0.9981	0.9985	0.9988
.90	0.9995	0.9995	0.9996	0.9997	0.9997
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	4.00	9.00	20.0	90.0
0	0.9567	0.9340	0.9440	0.9627	0.9870
.10	0.9805	0.9584	0.9659	0.9792	0.9928
.20	0.9934	0.9737	0.9789	0.9870	0.9903
.30	0.9991	0.9833	0.9866	0.9900	0.9835
.40	0.9998	0.9894	0.9909	0.9904	0.9751
.50	0.9970	0.9932	0.9934	0.9898	0.9666
.60	0,9917	0.9956	0.9949	0.9893	0.9599
.70	0.9845	0.9973	0.9962	0.9898	0.9564
.80	0.9761	0.9985	0.9975	0.9919	0.9589
.90	0.9667	0.9995	0.9990	0.9959	0.9718
.99	0.9570	0.9999	0.9999	0.9999	0.9996

Table 5.7.3 Efficiencies (g values in last 4 columns)

γ	T	0.01	0.0332	1.000	2.50
0	0.8001	0.8011	0.8031	0.8580	0.8981
.10	0.8557	0.8570	0.8581	0.9022	0.9345
.20	0.8975	0.8981	0.8994	0.9337	0.9585
.30	0.9289	0.9294	0.9303	0.9559	0.9736
.40	0.9524	0.9527	0.9534	0.9716	0.9836
.50	0.9697	0.9700	0.9704	0.9830	0.9899
.60	0,9822	0.9823	0.9826	0.9900	0.9939
.70	0.9907	0.9910	0.9910	0.9949	0.9965
.80	0.9961	0.9962	0.9963	0.9979	0.9982
.90	0.9991	0.9991	0.9991	0.9995	0.9994
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	4.00	9.00	20.0	60.0
0	0.9331	0.9207	0.9558	0.9794	0.9947
.10	0.9694	0.9520	0.9770	0.9910	0.9961
.20	0.9896	0.9708	0.9866	0.9917	0.9869
.30	0.9985	0.9819	0.9996	0.9866	0.9724
.40	0.9996	0.9883	0.9892	0.9792	0.9560
.50	0.9952	0.9918	0.9877	0.9719	0.9402
.60	0,9868	0.9940	0.9864	0.9664	0.9270
.70	0.9757	0.9955	0.9866	0.9647	0.9190
.80	0.9626	0.9971	0.9891	0.9689	0.9203
.90	0.9483	0.9988	0.9946	0.9817	0.9406
.99	0.9341	0.9999	0.9999	0.9993	0.9954

Table 5.7.4 Efficiencies (g values in last 4 columns)

γ	T	0.0093	0.0119	1.000	1.75
0	0.9716	0.9719	0.9730	0.9865	0.9907
.10	0.9861	0.9863	0.9871	0.9945	0.9968
.20	0.9926	0.9927	0.9930	0.9975	0.9987
.30	0.9959	0.9960	0.9962	0.9987	0.9993
.40	0.9978	0.9978	0.9980	0.9993	0.9996
.50	0.9988	0.9988	0.9989	0.9996	0.9997
.60	0.9994	0.9994	0.9994	0.9998	0.9998
.70	0.9997	0.9997	0.9997	0.9999	0.9999
.80	0.9999	0.9999	0.9999	0.9999	0.9999
.90	0.9999	0.9999	0.9999	0.9999	0.9999
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	4.00	9.00	16.00	25.00
0	0.9914	0.9957	0.9984	0.9993	0.9997
.10	0.9986	0.9991	0.9994	0.9989	0.9983
.20	0.9999	0.9993	0.9979	0.9963	0.9949
.30	0.9995	0.9990	0.9966	0.9940	0.9919
.40	0.9983	0.9988	0.9959	0.9926	0.9899
.50	0.9970	0.9988	0.9958	0.9923	0.9890
.60	0,9957	0.9990	0.9963	0.9927	0.9893
.70	0.9945	0.9993	0.9971	0.9940	0.9907
.80	0.9934	0.9996	0.9982	0.9960	0.9932
.90	0.9924	0.9998	0.9994	0.9983	0.9968
.99	0.9920	0.9999	0.9999	0.9993	0.9999

Table 5.7.5 Efficiencies (g values in last 4 columns)

γ	T	.005	0.0059	1.00	1.50
0	0.9376	0.9381	0.9382	0.9979	0.9985
.10	0.9689	0.9691	0.9692	0.9988	0.9993
.20	0.9834	0.9835	0.9836	0.9993	0.9996
.30	0.9908	0.9909	0.9909	0.9995	0.9997
.40	0.9949	0.9949	0.9950	0.9997	0.9998
.50	0.9972	0.9972	0.9973	0.9998	0.9998
.60	0.9986	0.9986	0.9986	0.9999	0.9998
.70	0.9993	0.9994	0.9994	0.9999	0.9999
.80	0.9997	0.9998	0.9998	0.9999	0.9999
.90	0.9999	0.9999	0.9999	0.9999	0.9999
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	3.00	9.00	16.00	50.00
0	0.9842	0.9994	0.9998	0.9999	0.9999
.10	0.9975	0.9998	0.9998	0.9997	0.9959
.20	0.9999	0.9998	0.9992	0.9987	0.9885
.30	0.9990	0.9996	0.9984	0.9974	0.9817
.40	0.9969	0.9995	0.9975	0.9959	0.9765
.50	0.9945	0.9994	0.9967	0.9946	0.9730
.60	0,9921	0.9993	0.9963	0.9935	0.9714
.70	0.9898	0.9994	0.9964	0.9932	0.9719
.80	0.9877	0.9996	0.9971	0.9939	0.9756
.90	0.9858	0.9998	0.9986	0.9966	0.9845
.99	0.9848	0.9999	0.9997	0.9998	0.9992

Table 5.7.6 Efficiencies (g values in last 4 columns)

γ	T	0.003	0.0045	1.000	1.50
0	0.9717	0.9719	0.9720	0.9931	0.9954
.10	0.9861	0.9862	0.9863	0.9972	0.9985
.20	0.9926	0.9927	0.9928	0.9987	0.9993
.30	0.9959	0.9960	0.9960	0.9993	0.9996
.40	0.9978	0.9978	0.9978	0.9996	0.9997
.50	0.9988	0.9988	0.9988	0.9998	0.9998
.60	0,9994	0.9994	0.9994	0.9998	0.9998
.70	0.9997	0.9997	0.9997	0.9999	0.9999
.80	0.9998	0.9999	0.9999	0.9999	0.9999
.90	0.9999	0.9999	0.9999	0.9999	0.9999
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	2.5	9.00	16.00	50.00
0	0.9941	0.9975	0.9996	0.9998	0.9999
.10	0.9991	0.9994	0.9995	0.9992	0.9986
.20	0.9999	0.9996	0.9983	0.9974	0.9962
.30	0.9996	0.9995	0.9972	0.9958	0.9939
.40	0.9988	0.9994	0.9965	0.9947	0.9921
.50	0.9980	0.9994	0.9962	0.9942	0.9908
.60	0,9971	0.9995	0.9964	0.9942	0.9901
.70	0.9963	0.9996	0.9970	0.9948	0.9900
.80	0.9955	0.9998	0.9980	0.9961	0.9910
.90	0.9948	0.9999	0.9991	0.9981	0.9938
.99	0.9944	0.9999	0.9999	0.9999	0.9996

5.8. The case $k = 3$.

We consider the special case when $k = 3$ for the random one-way model previously described. For this case we have $t = m_1 + m_2 + m_3$ and $n = n_1 m_1 + n_2 m_2 + n_3 m_3$.

Now let us consider the quantities C, U and D as defined in Section 5.2 and 5.3. From Section 5.3, we know that C has only two nonzero components C_1 and C_2 . These are given by

$$C_\ell = (1/t)[\textstyle\sum_{i=1}^3 m_i/(n_i - d_\ell)][\sum_{i=1}^3 m_i n_i/(n_i - d_\ell)^2]^{-1/2} ,$$

where $\ell = 1, 2$. The eigenvalues associated with C_1 and C_2 are given by

$$d_1 = [F - (F^2 - 4n_1 n_2 n_3 tn)^{1/2}]/2n$$
$$d_2 = [F + (F^2 - 4n_1 n_2 n_3 tn)^{1/2}]/2n ,$$

where $F = m_1 n_1(n_2 + n_3) + m_2 n_2(n_1 + n_3) + m_3 n_3(n_1 + n_2)$. Finally, the corresponding U_1 and U_2 are given by

$$U_\ell = [\textstyle\sum_{i=1}^3 (m_i n_i/(n_i - d_\ell)^2)]^{-1/2} \sum_{i=1}^3 (m_i n_i \bar{Y}_i/(n_i - d_\ell)) ,$$

where $\ell = 1, 2$.

From the above results we can determine the form of the $\hat{\delta}$ estimator. In particular, using

$$\hat{\gamma} = W/[gR+W] ,$$

we can express $\hat{\delta}$ as follows:

$$\hat{\delta} = T - C_1[gR/(gR+d_1 W)]U_1 - C_2[gR/(gR+d_2 W)]U_2$$
$$= \textstyle\sum_{i=1}^3 m_i n_i[(n_i - 1)\hat{\gamma}+1]^{-1}\bar{Y}_i / \sum_{i=1}^3 m_i n_i[(n_i - 1)\hat{\gamma}+1]^{-1} .$$

The first expression for $\hat{\delta}$ is our usual form. The second expression comes from the equation in Section 5.4.

In the remaining part of this section we give a numerical study for several examples. Associated with each example are two tables with the same identification number as the example. The first table shows the efficiencies of a set of $\hat{\delta}$ estimators corresponding to different values of g. These efficiencies are presented in the last four columns for the different values of g. The first table also gives the efficiency of the unweighted mean estimator T. The second table shows the efficiencies of a set of $\hat{\delta}$ estimators different from those given in the first table. The second table also gives the efficiency of the maximin-efficiency estimator T^*. The $\hat{\delta}$ estimators have been selected so that the ones in the first table are competitors with the estimator T; and the ones in the second table are competitors with the estimator T^*.

Before giving the examples, let us consider the condition $gB^* \le M$ in Theorem 2.6.3. It can be shown for each of the following examples that the choice of $g > d^{*2}$ is impossible (because then $gB^* > M$). Thus, we need only consider the case when $g \le (d^*)^2$ which means that $B^* = (d^*)^2$.

Example 5.8.1. For this example we take $n_1 = 2$, $n_2 = 3$, $n_3 = 4$, $m_1 = 1$, $m_2 = 2$ and $m_3 = 2$. From these values we get the following:

$n = 16$, $t = 5$, $q = 11$ and $s = 4$.

$d_* = d_1 = 2.1721$, $d^* = 4$ and $M = 0.0854$.

$d_2 = 3.4529$, $C_1 = 0.0519$, $C_2 = .0384$, $f^* = 0.2544$ and $g \leq .0053$.□

Example 5.8.2. For this example we take $n_1 = 2$, $n_2 = 3$, $n_3 = 4$, $m_1 = 6$, $m_2 = 1$ and $m_3 = 1$. From these values we get the following:

$n = 19$, $t = 8$, $q = 11$ and $s = 7$.

$d_* = 2$, $d^* = d_2 = 3.6602$ and $M = 0.2942$.

$d_1 = 2.7608$, $C_1 = .0416$, $C_2 = 0.0403$, $f^* = 0.2544$ and $g \leq 0.022$.□

Table 5.8.1 Efficiencies (g values in last 4 columns)

γ	T	0.004	0.0053	0.500	0.75
0	0.9375	0.9381	0.9383	0.9671	0.9734
.10	0.9636	0.9638	0.9640	0.9819	0.9859
.20	0.9783	0.9784	0.9785	0.9896	0.9922
.30	0.9869	0.9870	0.9871	0.9939	0.9955
.40	0.9923	0.9924	0.9924	0.9964	0.9974
.50	0.9956	0.9956	0.9956	0.9980	0.9985
.60	0.9976	0.9977	0.9977	0.9989	0.9992
.70	0.9988	0.9989	0.9989	0.9995	0.9996
.80	0.9996	0.9996	0.9996	0.9998	0.9998
.90	0.9999	0.9999	0.9999	0.9999	0.9999
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	4.00	10.00	16.00	50.00
0	0.9855	0.9939	0.9982	0.9991	0.9999
.10	0.9963	0.9980	0.9992	0.9989	0.9978
.20	0.9996	0.9985	0.9971	0.9958	0.9929
.30	0.9998	0.9979	0.9947	0.9924	0.9876
.40	0.9985	0.9973	0.9926	0.9896	0.9829
.50	0.9965	0.9969	0.9914	0.9877	0.9792
.60	0,9942	0.9970	0.9911	0.9870	0.9768
.70	0.9919	0.9974	0.9917	0.9876	0.9760
.80	0.9896	0.9982	0.9937	0.9899	0.9976
.90	0.9873	0.9992	0.9968	0.9943	0.9838
.99	0.9856	0.9999	0.9998	0.9997	0.9985

Table 5.8.2 Efficiencies (g values in last 4 columns)

γ	T	0.01	0.022	1.00	1.50
0	0.9401	0.9407	0.9413	0.9687	0.9750
.10	0.9645	0.9649	0.9653	0.9835	0.9877
.20	0.9785	0.9788	0.9790	0.9909	0.9936
.30	0.9870	0.9871	0.9873	0.9949	0.9965
.40	0.9922	0.9923	0.9924	0.9972	0.9981
.50	0.9955	0.9956	0.9956	0.9984	0.9989
.60	0.9976	0.9976	0.9977	0.9992	0.9994
.70	0.9988	0.9988	0.9989	0.9996	0.9997
.80	0.9995	0.9996	0.9996	0.9998	0.9998
.90	0.9999	0.9999	0.9999	0.9999	0.9999
.99	0.9999	0.9999	0.9999	0.9999	0.9999

γ	T*	4.00	9.00	16.00	50.00
0	0.9831	0.9886	0.9955	0.9980	0.9997
.10	0.9955	0.9960	0.9989	0.9990	0.9974
.20	0.9996	0.9981	0.9978	0.9958	0.9910
.30	0.9998	0.9984	0.9957	0.9919	0.9840
.40	0.9981	0.9981	0.9940	0.9888	0.9778
.50	0.9955	0.9980	0.9930	0.9869	0.9731
.60	0.9926	0.9981	0.9930	0.9865	0.9703
.70	0.9896	0.9985	0.9940	0.9878	0.9702
.80	0.9865	0.9990	0.9958	0.9909	0.9738
.90	0.9836	0.9996	0.9983	0.9957	0.9833
.99	0.9833	0.9999	0.9999	0.9998	0.9991

5.9. Conclusion.

In this section, we draw some empirical conclusions from the numerical results obtained in the last two sections.

From the first table of each example in Sections 5.7 and 5.8 we see that T is dominated by our $\hat{\delta}$ estimators. This is true not only for $0 < g < M/d^{*2}$, but also for some values of g such that $M/d^{*2} \le g \le \alpha \le d^{*2}$ where α differs from example to example. As we have seen, $\alpha = 0.75$ in Examples 5.7.1 and 5.8.1, $\alpha = 1.5$ in Examples 5.7.5, 5.7.6 and 5.8.2 and so on.

The efficiency of the $\hat{\delta}$ estimators in the first table of each example increases as γ increases. It tends to be the same for all the $\hat{\delta}$ estimators as γ approaches one. The $\hat{\delta}$ estimators and T tends to have the same efficiency as the limit of g tends to zero. At $g = \alpha$, $\hat{\delta}$ tends to be a good estimator because it has an efficiency lower bound higher than that of T as well as the rest of the $\hat{\delta}$ estimators in the table and it tends to dominate them.

From the second table of each example in Sections 5.7 and 5.8, we notice that our $\hat{\delta}$ estimators perform very well in comparison with T^* in the range $\alpha < g \le d^{*2}$, because $\hat{\delta}$ has an efficiency lower bound greater than that of T^*. But if $g > d^{*2}$, we may get estimators with efficiency lower bound smaller than that of T^*. Further, we notice that the behavior of the efficiency of $\hat{\delta}$ is strange in the sense that it can increase, then decrease, then increase again as γ increases.

VI. SUMMARY

In this thesis, we considered an unbiased estimator T for an unknown parameter θ. Further, we considered $U_1, \ldots, U_s$ to be a set of independent unbiased estimators of zero such that T and $U_1, \ldots, U_s$ are jointly normally distributed with covariance matrix $\pi(1-\gamma)C$ where C is a known s x 1 vector with elements $C_1, \ldots, C_s$ and where π and γ are unknown parameters satisfying $\pi > 0$ and $\gamma \epsilon [0,1]$. Furthermore, we considered R to be a random variable distributed independently of T and $U_1, \ldots, U_s$; and we assumed that $R/\pi(1-\gamma)$ has a chi-squared distribution with q degrees of freedom.

When the number of the unbiased estimators of zero exceeds 2, we have shown that there is an unbiased estimator $\hat{\delta}$ for θ such that the variance of $\hat{\delta}$ is less than the variance of T for all $\gamma \epsilon [0,1)$ and for all $\pi > 0$. The estimator $\hat{\delta}$ belongs to the general class of nonlinear estimators described by Saleh. However, we concentrated on a slightly different form than that used by Saleh. We considered $\hat{\delta}$ to be defined as

$$\hat{\delta} = T - \sum_{i=1}^{s} C_i \phi_i U_i$$

where ϕ_i is a measurable function of R/W with $W = \sum_{j=1}^{s} g_j U_j^2$, and $g_1, \ldots, g_s$ positive constants. We offered a form of ϕ_i which is slightly different than the one that Saleh worked with. We showed that our $\hat{\delta}$ is applicable under the mixed linear model with two variance components and that it can be obtained via solving a set of normal equations. This result works only for our choice of the form of ϕ_i, but not for the choice used by Saleh. Next we presented an exact formula in terms of a one variable integral to evaluate the variance of $\hat{\delta}$. This formula works for a wide class of ϕ_i functions. A computer program is provided in the appendix to evaluate the variance and the efficiency of $\hat{\delta}$.

Finally, we applied our results to the random one-way model as a special case of the mixed linear model with two variance components. We established an efficiency comparison between our estimators $\hat{\delta}$ and the unweighted mean estimator T and the maximin-efficiency estimator of Birkes, Seely and Azzam (1981). We found that one can get $\hat{\delta}$ with efficiency lower bound greater than that of the maximin-efficiency estimator but not dominate it.

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APPENDIX

APPENDIX

This appendix gives a Fortran program to evaluate $\mathrm{Var}(\hat{\delta})$.

The program is structured as a main program and three subprograms, called SAMIAE, ROMB, and SARAH. All the input and output are done through the main program. After reading the input data, the main program calls the function SAMIAE to evaluate the expectation. Then, it evaluates the variance of $\hat{\delta}$ and T as well as their efficiencies, and outputs the results.

The function SAMIAE evaluates the nonintegrable part of the expectation and calls subroutine ROMB to evaluate the integrable part and returns the value to SAMIAE. SAMIAE then multiplies the value of both parts and returns the whole value of the expectation to the main program.

The subroutine ROMB (Romberg Method) carries the integration process itself. To make it a general definite integral routine, it has been written to call the function SARAH to evaluate the value of the integrable function. Note that the integrable function could be any function, and only the code in SARAH should be changed to reflect the new function.

```
$DEBUG
      INTERFACE TO SUBROUTINE DATE (N,STR)
      CHARACTER*10 STR [NEAR,REFERENCE]
      INTEGER*2 N [VALUE]
      END
      INTERFACE TO SUBROUTINE TIME(N,STR)
      CHARACTER*10 STR [NEAR,REFERENCE]
      INTEGER*2 N [VALUE]
      END
      PROGRAM MAIN
      CHARACTER*10 DSTR,TSTR
      INTEGER S,S1,Q(11)
      REAL LAMDA,LAMDAE,LASTB
      DIMENSION D(20),C(20),A(20)
      COMMON ETA,LAMDAE,S1,B(21),G(21),Q
      OPEN(1,FILE='EXPTO',STATUS='NEW')
      OPEN(2,FILE='EXPTI',STATUS='OLD')
      CALL DATE(10,DSTR)
      CALL TIME(10,TSTR)
      WRITE(1,*) 'DATE:  ',DSTR
      WRITE(1,*) 'TIME:  ',TSTR
      WRITE(*,99)
C     INPUT DATA
  100 READ(2,11) S,H1,H2,ABASE,BBASE,LAMDA,LASTB
      IF(S.EQ.0) GO TO 999
      S1=S+1
      READ(2,12)(C(K),K=1,S)
      READ(2,12)(D(K),K=1,S)
      READ(2,13)(Q(K),K=1,S1)
      WRITE(1,20)
      WRITE(1,21) S,H1,H2,ABASE,BBASE,LAMDA,LASTB
      WRITE(1,22)(C(K),K=1,S)
      WRITE(1,23)(D(K),K=1,S)
      WRITE(1,24)(Q(K),K=1,S1)
      DO 1 K=1,S
    1 A(K)=ABASE
      DO 2 K=1,S1
    2 B(K)=0.0
c     LOOP FOR G. LOOP ENDS BEFORE THE FIRST FORMAT ST.
  200 READ(2,8)GBASE
      IF(GBASE.EQ.0.0)GO TO 100
      WRITE(*,*)GBASE
      WRITE(1,10)GBASE
C     LOOP FOR GAMA. EVALUATE VARIANCE AND EFFECIENCY
      WRITE(1,14)
      DO 9 K=0,9
      GAMA=FLOAT(K)/10.
      WRITE(*,*)GAMA
      R=1.-GAMA
      VART=R*H1+GAMA*H2
      G(S1)=R*GBASE
      SUM=0.
      VM=0.0
C     NEXT LOOP (5) FOR EVALUATING EXPECTIONS E1,E2
      DO 5 I=1,S
```

```
      B(I)=BBASE
      DO 3 J=1,S
    3 G(J)=(R+GAMA*D(J))*A(J)*D(I)
C     EVALUATE ETA
      GMIN=G(1)
      DO 4 J=2,S1
      IF(G(J) .LT. GMIN) GMIN=G(J)
    4 CONTINUE
      ETA=(1./(4.*GMIN))+.000001
C     PREPARE DATA FOR E1
      LAMDAE=LAMDA
      B(S1)=LASTB
      EH1=SAMIAE()
      E1=EH1*G(S1)
C     PREPARE DATA FOR E2
      LAMDAE=LAMDA+LAMDA
      B(S1)=LASTB+LASTB
      EH2=SAMIAE()
      E2=EH2*G(S1)*G(S1)
      CC=C(I)*C(I)
      SUM=SUM+CC*(R*E1-.5*(R+GAMA*D(I))*E2)
      B(I)=0.0
C     MINIMUM VARIANCE
    5 VM=VM+CC/(R+GAMA*D(I))
      VM=VART-R*R*VM
      VARD=VART-2.*SUM
      EF1=VM/VART
      EF2=VM/VARD
      WRITE(1,16) GAMA,VART,VARD,VM,EF1,EF2
    9 CONTINUE
      GO TO 200
    8 FORMAT(F7.4)
   10 FORMAT(//' G=',F7.4)
   11 FORMAT(I2,2F6.4,4F5.2)
   12 FORMAT(20F8.4)
   13 FORMAT(40I2)
   14 FORMAT(' GAMA VARTHETA VARDELTA  VARMIN.   VM/VART    VM/VARD')
   15 FORMAT('   RUN COMPLETED')
   16 FORMAT(F4.1,3F9.5,2F11.7)
   20 FORMAT(//' S    H1      H2     ABASE  BBASE  LAMDA  LASTB')
   21 FORMAT(I2,2F8.4,4F7.2)
   22 FORMAT(' C=',10F7.4)
   23 FORMAT(' D=',10F7.4)
   24 FORMAT(' Q=',20I3)
   99 FORMAT('   PROGRAM IS RUNNING')
  999 WRITE(*,15)
      CALL DATE (10,DSTR)
      CALL TIME (10,TSTR)
      WRITE(1,*) 'DATE:  ',DSTR
      WRITE(1,*) 'TIME:  ',TSTR
      END
```

```
$DEBUG
      FUNCTION SAMIAE()
      REAL LAMDAE
      INTEGER S1,Q(21)
      COMMON ETA,LAMDAE,S1,B(21),G(21),Q
      EXTERNAL SARAH
      SUM1=0.0
      SUM2=0.0
      SUM3=1.0
      DO 5 K=1,S1
      QB=(Q(K)+2.*B(K))/2.
      Q2=Q(K)/2.
      SUM1=SUM1+QB
      SUM2=SUM2+Q2
      TERM=(1./(G(K)**QB))*(GAMMA(QB)/GAMMA(Q2))
    5 SUM3=SUM3*TERM
      SUM1=1./(ETA**(SUM1-LAMDAE))
      SUM2=1./(2.**SUM2)
      TERM=SUM1*SUM2*(SUM3/GAMMA(LAMDAE))
      CALL ROMB(0.0,1.0 ,0.001,SARAH,ANS)
      SAMIAE=TERM*ANS
      RETURN
      END
```

```
$DEBUG
      SUBROUTINE ROMB (ALIMIT,BLIMIT,EPS,SARAH,ANS)
      DIMENSION ATEMP(50)
      EQUIVALENCE (I,N)
      ATEMP(1)=0.5 *(SARAH(BLIMIT)+SARAH(ALIMIT))
      WIDTH=BLIMIT - ALIMIT
      ZL=WIDTH
      POWER=1.
      I=1
      JJ=1
  711 I=I+1
      ANS=ATEMP(1)
      TEMPL=ZL
      ZL=0.5*ZL
      POWER=0.5*POWER
      X=ALIMIT+ZL
      SUM=0.0
      DO 12 JCOUNT=1,JJ
      SUM=SUM+SARAH(X)
   12 X=X+TEMPL
      ATEMP(I)=0.5*ATEMP(I-1)+SUM*POWER
      R=1.0
      NM1=N-1
      DO 16 KOUNT=1,NM1
      KK=N-KOUNT
      R=R+R
      R=R+R
      ATEMP(KK)=ATEMP(KK+1)+(ATEMP(KK+1)-ATEMP(KK))/(R-1.)
   16 CONTINUE
      DELTA=ABS(ANS/ATEMP(1)-1.)
      IF(DELTA-EPS)40,40,30
   30 JJ=JJ+JJ
      GO TO 711
   40 ANS=ATEMP(1)*WIDTH
      RETURN
      END
```

```
$DEBUG
      FUNCTION SARAH(T)
      REAL LAMDAE
      INTEGER S1,Q(21)
      COMMON ETA,LAMDAE,S1,B(21),G(21),Q
      SUM1=0.0
      SUM2=1.0
      DO 5 K=1,S1
      QB=(Q(K)+2.*B(K))/2.
      GETA=2.*G(K)*ETA
      SUM1=SUM1+QB
      TERM=1.-(T*(GETA-1.)/GETA)
      TERM=1./(TERM**QB)
      SUM2=SUM2*TERM
    5 CONTINUE
      SUM1=T**(SUM1-LAMDAE-1.)
      SARAH=SUM1*((1.-T)**(LAMDAE-1.))*SUM2
      RETURN
      END
```