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Title: Branching Brownian Motion with One Absorbing and One Reflecting Boundary

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In this work we will analyze branching Brownian motion on a finite interval with one absorbing and one reflecting boundary, having constant drift rate toward the absorbing boundary. Similar processes have been considered by Kesten ([12]), and more recently by Harris, Hesse, and Kyprianou ([11]). The current offering is motivated largely by the utility of such processes in modeling a biological population's response to climate change. We begin with a discussion of the beautiful theory that has been developed for such processes without boundaries, proceed through an adaptation of this theory to our finite setting with boundary conditions, and finally demonstrate a critical parameter value that answers the fundamental question of whether persistence is possible for our branching process, or if extinction is inevitable. We also include a new and simple proof of Kesten's persistence criterion for branching Brownian motion with a single absorbing boundary. The bulk of the work is done by the distinguished path (or "spine") analysis for branching processes.

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Branching Brownian Motion with One Absorbing and One Reflecting Boundary

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William F. Felder

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APPROVED:

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Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

William F. Felder, Author

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Branching Brownian Motion with One Absorbing and One Reflecting Boundary

1 Introduction

1.1 Mathematical Overview

This will be a somewhat informal overview of the main ideas and results of this paper. A more technical development will begin in Chapter 2.

We begin with one-dimensional Brownian motion, $\xi(t)$, with unit diffusion coefficient and constant drift rate $-\mu$ ($\mu > 0$). We allow this stochastic process to occupy the finite interval $[0, K]$, with 0 an absorbing boundary and K a reflecting one. In other words, we let the motion of $\xi(t)$ be governed by the infinitesimal generator

$$Lf(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) - \mu\frac{d}{dx}f(x) \quad x \in (0, K),$$

$$f(0+) = 0,$$

$$f'(K-) = 0,$$

defined for all $f \in C^2((0, K), \mathbb{R})$ which satisfy the given boundary conditions. Let $\xi(0) = x_0$ for some fixed x_0 in $(0, K]$, noting that if the process is allowed to begin at the absorbing boundary 0, nothing interesting happens. Now let $\{\mathcal{G}_t\}_{t\geq 0}$ be the natural filtration for $\xi(t)$ (i.e. $\mathcal{G}_t := \sigma(\xi(s) : s \leq t)$), and take $\mathcal{G}_\infty := \bigvee_{t\geq 0} \mathcal{G}_t$. Finally, let $\mathbb{P}$ be the distribution of ξ on $C(\mathbb{R}^+, [0, K])$. We will refer to $(\xi, \mathcal{G}_\infty, \{\mathcal{G}_t\}_{t\geq 0}, \mathbb{P})$ as ***the single-particle process***.

(Note: while $\mathcal{G}_\infty$, $\{\mathcal{G}_t\}$, and $\mathbb{P}$ each depend on both x_0 and K, these dependencies will usually be suppressed in the notation.)

Now allow the process to branch: let $\sigma_\emptyset$ be an exponential random variable (independent of ξ) with rate parameter $r > 0$, and let this represent the life-span of the initial particle. If the initial particle has not reached the absorbing boundary by time $t = \sigma_\emptyset$, it is removed and replaced by $1 + A_\emptyset$ offspring particles, where $A_\emptyset$ is distributed on the non-negative integers as $P(A_\emptyset = k) = p_k$, for $k = 0, 1, 2, \ldots$. Note that there is always at least one offspring, so that the only way for a branch to be terminated is for it to meet the absorbing boundary 0. Assume that $0 < m := \mathbb{E}A_\emptyset < \infty$, and assume further that $\mathbb{E}\left(A_\emptyset \log^+ A_\emptyset\right) < \infty$.

Each offspring particle u begins its life at the space-time location of its parent's fission event, and carries with it its own independent copies of the life-span clock (σ_u) and the offspring distribution (A_u). The spatial movement of each offspring particle u during its lifetime is governed by the operator L given above, and is independent of the movement of any other particle. The process continues in the obvious way, with each particle u (of any generation) moving according to L, undergoing fission at rate r (if not absorbed first), and giving rise to $1 + A_u$ identical offspring particles if/when fission occurs.

Let N_t be the set of particles alive at time t, and $\forall u \in N_t$ define $x_u(t)$ to be the spatial position of particle u at time t. Now we can define the branching process as

$$X(t) := \sum_{u \in N_t} \delta_{x_u(t)},$$

with δ_y the usual Dirac delta point measure at y. Note that $X(0) = \delta_{x_0}$, and that for all $t \geq 0$ we have $X(t) \in \mathcal{M}_a[0, K]$, the set of finite atomic measures on $[0, K]$. $X(t)$ is a measure-valued stochastic process. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the natural filtration for $X(t)$, let $\mathcal{F}_\infty$ be the join as before, and let P be the distribution of X on $\mathcal{M}_a[0, K]$. Now $(X, \mathcal{F}_\infty, \{\mathcal{F}_t\}_{t \geq 0}, P)$ is ***the branching process***.

(Note: as before, $\mathcal{F}$, $\{\mathcal{F}_t\}$, and P each depend on both x_0 and K, but this dependence will usually remain implicit for notational simplicity.)

Now we will state the main result, which deals with the fundamental question of extinction vs. persistence of the branching process. Let t_Ω be the time to extinction:

$$t_\Omega := \inf\{t \geq 0 : |N_t| = 0\},$$

allowing the convention that $t_\Omega = \infty$ if $|N_t| > 0 \ \forall t$. Now our main result is

Theorem 1.1.0.1. *If $\mu \geq \sqrt{2mr}$, then $P(t_\Omega < \infty) = 1$. If, however, $\mu < \sqrt{2mr}$, then there exists a minimum interval length,*

$$K_0 := \frac{\arctan\left(-\frac{\sqrt{2mr-\mu^2}}{\mu}\right) + \pi}{\sqrt{2mr-\mu^2}},$$

such that

- $K < K_0$ *implies* $P(t_\Omega < \infty) = 1$ *(i.e. extinction is inevitable),*
- $K > K_0$ *implies* $P(t_\Omega = \infty) > 0$ *(i.e. persistence is possible).*

We note here that we have not shown what happens if $\mu < \sqrt{2mr}$ and $K = K_0$.

1.2 Biological Motivation

A major factor motivating the consideration of such branching processes is their utility in modeling biological phenomena. A few examples of potential applications are given below.

1.2.1 Modeling a Population's Response to Climate Change

One major application of branching stochastic processes on finite domains is the modeling of a biological population's response to climate change. In fact, our interest in such processes first arose in response to the Zhou, Kot paper [24]. In this work, it is noted that biological populations generally have a finite "habitable range," a geographical area over which the population can survive, and outside of which extinction is assured. Generally, this habitable range is determined by both biotic factors (like predator/prey densities) and abiotic factors (like temperature, moisture, etc.). As climate change drives these aboitic parameters to new values, the habitable range of a given population will shift and move across the physical landscape. As a simple example, imagine a population that cannot tolerate extreme heat moving either poleward or to a higher altitude as its environment warms.

In [24], the population itself is modeled as a density function on $\mathbb{R}$, the finite nature of the habitable range is modeled by requiring the support of this density function to be a finite interval, and the effects of climate change are modeled by having the boundary points of this interval move to the right with constant speed. We sought to build our own model of a population responding to a changing climate by borrowing the concept of individuals occupying a finite interval with moving boundaries, while replacing the discrete-time integro-differential equation representation of individual movement and reproduction given in [24] with continuous-time branching Brownian motion. Branching Brownian motion is an ideal mathematical tool for modeling the movement and reproduction of many biological populations; single-celled algae with no motile power, for example, move according to the laws of diffusion and reproduce via binary fission. If such a population of algae were limited to certain latitudes due to, say, temperature sensitivity, then our model would provide a very biologically literal representation of how these algae would respond to changing temperatures.

So we have the movement and reproduction of a population modeled as branching Brownian motion, we have the finiteness of the population's habitable range modeled as a finite interval domain for the the branching process, and we have climate change modeled as the movement of the boundaries of that interval. It is not difficult to see (or to show) that a centered (drift-less) branching Brownian motion on an interval whose boundaries move with constant speed c to the right is equivalent to a branching Brownian motion on an interval with fixed boundaries, but with the process itself experiencing a constant rate of drift (of magnitude c) to the left.

Biologically, there are different possible outcomes when an organism comes to the edge of its habitable range, and, mathematically, these manifest as different boundary conditions for the infinitesimal generator that governs the movement of individual particles. If, for example, the border of the habitable range is lethal to the organism, this would correspond to an absorbing (i.e. "Dirchlet") boundary condition for the infinitesimal generator. But perhaps the border of the habitable range is merely impassible to the individual organism; it could be a physical boundary, or perhaps the organism is instinctively programmed to move back and away from the uninhabitable area. These biological responses to the border correspond, in the mathematical model, to a reflecting (or "Neumann") boundary condition for the infinitesimal generator. With our habitable range represented as a finite, one-dimensional interval, the border of this range consists of the two end-points of the interval. Harris, Hesse, and Kyprianou have treated the case where both end-points are taken to be absorbing ([11]); we'll be treating the case of one absorbing and one reflecting boundary.

1.2.2 Fitness Landscape Models

In the above interpretation, the finite interval on which our branching process lives is taken to represent physical space, and the positions of the particles in the interval are taken to represent the spatial positions of the individual organisms. Another view one can take is that the positions of the particles in the interval don't represent spatial position at all, but rather are a measure of the "fitness" of the corresponding biological entities in their given environment. "Movement" in this fitness landscape represents increases or decreases in fitness due to random genetic mutations (taken to be a continuous process on this time scale). If a particle were to fall to an absorbing lower boundary point on such a fitness landscape, we would say that the corresponding organism has become too unfit for its environment, and was consequently removed. A reflecting upper boundary can be viewed as representing an "optimal fitness level" for a given environment–any change from such a state would result in a reduction of fitness.

Under this new "fitness landscape" interpretation, having child particles come into being at the space-time location of their parent's fission event encodes the biological assumption of error-free reproduction. In other words, the fitness of the child at its moment of birth is exactly the same as the fitness of the parent at its moment of fission, because the two are genetically identical at that point in time. One could loosen this assumption by allowing jump discontinuities in the branching process at branching events, allowing one to model biological populations whose reproduction events are error-heavy (like viruses).

Also note that a negative drift is appropriate in this context, as the majority of random mutations are deleterious to the individual.

1.3 Organization of this Dissertation

Chapter 1 will conclude with a brief review of the relevant literature.

In Chapter 2 we'll begin a technical development of the setting for the distinguished path analysis of branching Brownian motion on $\mathbb{R}$ (subject to neither boundaries nor boundary conditions). We'll also develop and examine some important tools for that setting.

Chapter 3 will return our focus to such processes occupying a finite interval with imposed boundary conditions; here we will adapt some key results from Chapter 2 to this new, restricted setting, and will prove our main result, Theorem 1.1.0.1.

Chapter 4 offers a relatively simple proof, via distinguished path analysis, of the critical drift rate for branching Brownian motion on $[0, \infty)$ with 0 absorbing. This result was first given (under stricter assumptions than we'll impose here) in Kesten's classic paper [12].

We will conclude our work in Chapter 5 with a discussion of the main result and how it fits in with previous results in the field. We'll also offer an outline of some directions for future research.

1.4 A Brief Literature Review

One might mark the beginning of interest in branching Brownian motion to the development of the KPPF equation in the early 20th century. This important PDE was investigated simultaneously by Fisher in the context of the spread of advantageous genes ([9]), and by Kolmogorov, Petrovskii, and Piskunov in the context of a diffusive process with increasing mass, also with an eye toward biological problems ([14]). Both of these papers date to 1937. More recently we have the work of McKean ([19]) and Bramson ([4]) in the 1970's, wherein the cumulative distribution function of the right-most particle of branching

Brownian motion is considered as a solution to KPPF.

The exploration of branching Brownian motion with an absorbing boundary began with Kesten ([12]). Significant work on this topic has been done by, for instance, Berestycki, Berestycki, and Schweinsberg ([1]), and also by Harris, Hesse, and Kyprianou ([11]), to whom this author owes a significant debt.

While the history of the distinguished path analysis in general, and the so called "spine decomposition" in particular, is a complicated one, many aspects of it could be said to have been developed simultaneously by Waymire and Williams in their treatment of multiplicative cascades ([22], [23]), and by Lyons, Pemantle and Peres in their "conceptual" proof of the Kesten-Stigum criterion ([18]). A very general framework for the use of these techniques on branching diffusions is laid out quite nicely by Hardy and Harris in [10], another work to which this author is greatly indebted.

2 A Setting for Distinguished Path Analysis of Branching Brownian Motion–No Boundaries

We'll now develop a setting in which the distinguished path analysis of branching Brownian motion becomes relatively straight-forward and natural. This will include formally defining a state space for branching Brownian motion with spines, an examination of the connection between martingales for the single-particle process and certain additive martingales for the branching process, an intuitive deconstruction of the important measures involved in the analysis, and a proof of the powerful "spine decomposition." This chapter borrows *heavily* from the arXiv paper by Hardy and Harris, [10]. The treatment in [10] is more general than that presented here; we begin with Brownian motion on $\mathbb{R}$, for instance, whereas they begin with an arbitrary Markov process on an arbitrary measurable space.

In what follows, we'll be dealing with branching Brownian motion on all of $\mathbb{R}$. We'll return to branching Brownian motion on $[0, K]$ with boundary conditions in Chapter 3.

2.1 Ulam-Harris Labels, Marked Galton-Watson Trees, and Spines

The familial structure of a branching process is encoded by identifying each particle with its Ulam-Harris label. The set of such labels is the set of finite sequences of positive integers (together with the empty sequence):

$$\Omega = \{\emptyset\} \cup \bigcup_{n\in\mathbb{N}} \mathbb{N}^n,$$

for $\mathbb{N} = \{1, 2, \ldots\}$.

The idea here is that one can trace the ancestry of each particle by examining its Ulam-Harris label. The initial particle has label $\emptyset$, it's 3rd child (for example) would have label (3), the first child of *that* particle would be (31), and so forth. Another example is depicted in Figure 2.1.

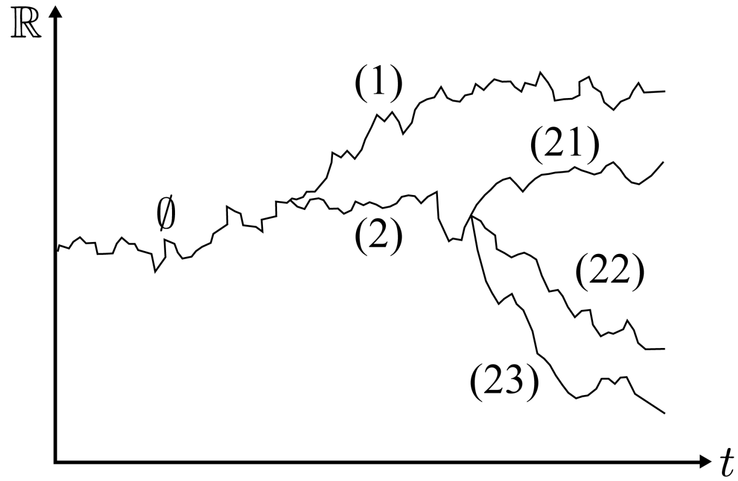

FIGURE 2.1: A demonstration of the Ulam-Harris labeling convention for a branching process.

For $u, v \in \Omega$, we let uv be the concatenation of the two sequences. The notation $v < u$ means v is an ancestor of u (i.e. $\exists w \in \Omega,\ w \neq \emptyset$, such that $vw = u$).

Certain collections of Ulam-Harris labels will make up Galton-Watson trees:

Definition 2.1.0.1. *A* ***Galton-Watson tree*** τ *is a subset of* Ω *(*$\tau \subset \Omega$*) satisfying the following:*

1. $\emptyset \in \tau$,
2. $\forall u, v \in \Omega$, $uv \in \tau \implies u \in \tau$,
3. $\forall u \in \tau,\ \exists A_u \in \{0, 1, 2, \dots\}$ *such that* $uj \in \tau \iff 1 \leq j \leq 1 + A_u$ *(for* $j \in \mathbb{N}$*).*

Informally, this means that Galton-Watson trees satisfy the following:

1. there is a single progenitor for each tree,
2. a tree contains all ancestors of every member,

3. a tree contains all children of every member, labeled consecutively (and each individual has a positive, finite number of children).

Note that in our setting, $\{A_u\}_{u\in\tau}$ will be a collection of i.i.d. random variables of the type discussed in Section 1.1.

A Galton-Watson tree encodes the familial structure of a branching process, but tells us nothing about the life-span of individuals, nor about their spatial movement during their lives. This information is carried by the more complex structure of a marked Galton-Watson tree:

Definition 2.1.0.2. *A* ***marked Galton-Watson tree*** *is a triple* $(\tau, \{\sigma_u\}_{u\in\tau}, \{x_u(t)\}_{u\in\tau})$ *such that:*

1. *τ is a Galton-Watson tree,*

2. *σ_u is a value in $\mathbb{R}^+$ that signifies the life-span of individual u,*

3. *$x_u(t)$ is an $\mathbb{R}$-valued function that gives the spatial position of u during its lifetime.*

We can here again note some particulars for our setting: $\{\sigma_u\}_{u\in\tau}$ will be a collection of i.i.d. exponential random variables with parameter r, and the movement described by $x_u(t)$ must be a fragment of a sample path of Brownian motion with drift rate $-\mu$.

With the life-spans in hand, we can define the birth and fission times of any particle:

$$\text{(Birth time) } R_u := \sum_{v<u} \sigma_v,$$
$$\text{(Fission time) } S_u := \sum_{v\le u} \sigma_v,$$

and note that $x_u(t)$ is defined for $t \in [R_u, S_u)$. We are obeying the convention here that a particle u is removed "just before" its fission time, and is replaced by its offspring which are first present at time S_u.

We will often use the simpler notation (τ, σ, x) to refer to a particular marked Galton-Watson tree, and will use $\mathcal{T}$ to refer to the set of *all* marked Galton-Watson trees.

Notice that every particle v that comes into existence initiates a *sub-tree* from the moment of its birth onward. This sub-tree has v as its initial particle, and includes all descendents of v. We will use notation $(\tau, \sigma, x)_j^u$ to refer to the sub-tree initiated by u's jth child uj, where $1 \leq j \leq 1 + A_u$. This sub-tree comes into existence at the space-time location of u's fission event $\left(\lim_{t \to \sigma_u -} x_u(t), \sigma_u \right)$, and each such sub-tree taken relative to this root position is an independent copy of the original tree (taken relative to the initial position).

We are now ready to formally define a distinguished path:

Definition 2.1.0.3. *For a given marked Galton-Watson tree* $(\tau, \sigma, x) \in \mathcal{T}$*, a* ***Distinguished Path*** *(or* ***Spine****)* d *is a collection of nodes* ($d \subset \tau$) *that make up a unique line of descent. In other words,*

$$d := \{d_0 = \emptyset, d_1, d_2, \ldots\},$$

such that d_i *is the child of* d_{i-1} *for* $i = 1, 2, \ldots$.

We will use $node_d(t)$ to refer to the particular node in d that is alive at time t (i.e. $node_d(t) \in d \cap N_t$), and $d(t)$ to refer to the spatial position of that node (i.e., $d(t) = x_{node_d(t)}(t)$). We can also define

$$n_t := |node_d(t)|,$$

where this n_t is a "counting function," and tells us which generation the current spine node is in, or, equivalently, how many fission events have occurred along the spine by time t. Note that $|\emptyset| = 0$.

Now we can articulate our largest and most general state space:

Definition 2.1.0.4. *The set of* ***marked Galton-Watson trees with distinguished paths*** *is* $\widetilde{\mathcal{T}} = \{(\tau, \sigma, x, d) : (\tau, \sigma, x) \in \mathcal{T}, d \subset \tau$ *is a distinguished path*$\}$.

We'll need this most general state space to be able to lay out some foundational results in distinguished path analysis that will be necessary for our later work.

2.2 Four Filtrations for Marked Trees with Distinguished Paths

In this section we will define four separate filtrations on the space $\widetilde{\mathcal{T}}$ which allow us four different vantage points from which to consider and analyze a general branching process. Though all four are here defined as filtrations of $\widetilde{\mathcal{T}}$, some of them contain such limited information that they could just as well be defined to be filtrations of a simpler space (like the state space for $\xi(t)$ from Section 1.1).

2.2.1 A Filtration for the Movement of the Distinguished Path

This will be the simplest filtration:

$$\mathcal{G}_t := \sigma(d(s) : 0 \leq s \leq t),$$
$$\mathcal{G}_\infty := \bigvee_{t \geq 0} \mathcal{G}_t.$$

This filtration only knows about the spatial position along the distinguished path. It does not know how many branching events have occurred along this path, or anything about any particle not in d. It does not even know the generation number, n_t, since it does not know the fission times. It sees only a single particle moving in $\mathbb{R}$, and nothing else.

2.2.2 A Filtration for Everything About the Distinguished Path

This will be a larger filtration than $\{\mathcal{G}_t\}$, but it will still follow the distinguished path closely:

$$\widetilde{\mathcal{G}}_t := \sigma(\mathcal{G}_t, (node_s(d) : s \leq t), (A_u : u < node_d(t))),$$
$$\widetilde{\mathcal{G}}_\infty := \bigvee_{t \geq 0} \widetilde{\mathcal{G}}_t.$$

This filtration knows about movement along the spine, but it also knows which node makes up the spine at any given moment (and thus knows the times of the fission events along the spine), and it knows how many offspring were produced at each of these fission events (though it does not follow these offspring past the instant of their birth).

2.2.3 A Filtration for the Branching Process, no Distinguished Path

$\{\mathcal{F}_t\}$ will know everything about the branching process *except* which particles belong to the spine. It is defined as the join of two sigma algebras:

$$\mathcal{F}_t := \sigma((u, \sigma_u, x_u) : S_u \leq t) \bigvee \sigma((u, x_u(s)) : s \leq t, t \in [R_u, S_u))$$
$$\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t.$$

The inclusion of the first sigma algebra ensures that $\mathcal{F}_t$ will know everything about the ancestry, movement, and lifespan off all particles that undergo fission before time t. The inclusion of the second ensures $\mathcal{F}_t$ will know about the ancestry and movement *up to time t* of particles that are still alive at time t; it will not know about the life-span of those particles, or anything about their movement after time t. Note that knowledge about the number of child-particles generated by fission events that occur before time t is implicit in knowledge about the nodes present.

2.2.4 A Filtration for the Branching Process with Distinguished Path

This will be the finest filtration, and will be created by taking $\{\mathcal{F}_t\}$ and including with it information on which nodes (up to time t) make up the spine:

$$\widetilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t, (node_s(d) : s \leq t)),$$
$$\widetilde{\mathcal{F}}_\infty := \bigvee_{t \geq 0} \widetilde{\mathcal{F}}_t.$$

This filtration knows everything of interest about the process: it knows everything that can be known by time t for each particle of the branching process (as $\mathcal{F}_t$ does), and also knows which particle belongs to the distinguished path for all times up to t.

2.2.5 Filtration Summary

- $\{\mathcal{G}_t\}_{t\geq 0}$ - Knows only the spatial movement of the distinguished path.
- $\{\widetilde{\mathcal{G}}_t\}_{t\geq 0}$ - Knows movement, ancestry, and reproductive information on the distinguished path (the timing of the fission events and the number of offspring produced at each such event).
- $\{\mathcal{F}_t\}_{t\geq 0}$ - Knows everything about the branching process, nothing about the distinguished path.
- $\{\widetilde{\mathcal{F}}\}_{t\geq 0}$ - Knows everything.

It's not difficult to believe that $\widetilde{\mathcal{F}}$ contains the other three. In fact, we have the following inclusions:

$$\mathcal{G}_t \subset \widetilde{\mathcal{G}}_t \subset \widetilde{\mathcal{F}}_t,$$
$$\mathcal{F}_t \subset \widetilde{\mathcal{F}}_t.$$

These inclusions cannot be extended: $\mathcal{G}_t \not\subset \mathcal{F}_t$, and $\mathcal{F}_t \not\subset \widetilde{\mathcal{G}}_t$.

There will be other filtrations of interest, namely the natural filtrations generated by the processes restricted to a finite interval with one absorbing and one reflecting boundary. For these filtrations, knowledge about a particular line of descent will halt the moment that line reaches the absorbing boundary. This will be discussed further in Chapter 3.

2.3 Measures and Martingales

2.3.1 Extending Measures for Branching Processes to Measures for Branching Processes with Distinguished Paths

Given any probability measure P defined on $\mathcal{F}_\infty$ (such as the distribution of the branching process, discussed in Section 1.1), we can extend it to a probability measure on $\widetilde{\mathcal{F}}_\infty$ by making certain assumptions about the way the distinguished path is chosen. This approach is due to Hardy and Harris ([10]), and is a refinement of previous similar efforts, wherein the resulting measures often had time-dependent mass (as in [16]).

$\widetilde{\mathcal{T}}$ will be the underlying sample space throughout Section 2.3, and will usually not be mentioned.

We will assume from here on that a distinguished path d in a marked Galton-Watson tree (τ, σ, x) is chosen as follows. By definition the first node in d is $\emptyset$. When this initial ancestor undergoes fission, the next element of d is chosen randomly from its offspring with uniform probability. Proceed similarly; each time an element of the distinguished path undergoes fission, each child particle has an equal probability of becoming the next representative. This gives rise to the following scenario:

$$\mathrm{Prob}(u \in d) = \prod_{v<u} \frac{1}{1+A_v} \quad \forall u \in \tau. \tag{2.1}$$

Note that this does not mean that every extant particle has an equal chance of belonging to the spine; the uniformity of probability is localized to branching events along the spine. As a simple example, if we take the case of binary branching, then $\mathrm{Prob}(u \in d) = 2^{-|u|}$, and the scenario depicted in Figure 2.2 might emerge.

The above assumption and observation are key to extending a probability measure to $\widetilde{\mathcal{F}}_\infty$, as is the following theorem given in Lyons, [17]:

Theorem 2.3.1.1. *If f is a $\widetilde{\mathcal{F}}_t$-measurable function, then we can decompose it as*

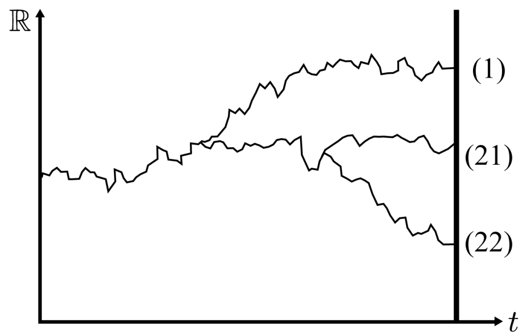

FIGURE 2.2: A depiction of the probabilities of being in the distinguished path. Note that the choice being made uniformly from among the children at each fission event does not result in an equal chance for each extant particle; in general there will have been a different number of fission events along each line at time t.

$$f = \sum_{u \in N_t} f_u 1_{\{u \in d\}}, \tag{2.2}$$

where f_u is $\mathcal{F}_t$-measurable $\forall u$, and $1_{\{.\}}$ is a standard indicator function.

We are now ready to extend a given probability measure on $\mathcal{F}_\infty$ to one on $\widetilde{\mathcal{F}}_\infty$:

Definition 2.3.1.1. *Given a probability measure P defined on $\mathcal{F}_\infty$, we can extend it to a probability measure $\widetilde{P}$ on $\widetilde{\mathcal{F}}_\infty$ as follows:*

$$\int_{\widetilde{\mathcal{T}}} f d\widetilde{P} := \int_{\widetilde{\mathcal{T}}} \sum_{u \in N_t} f_u \prod_{v<u} \frac{1}{1+A_v} dP,$$

for all $\widetilde{\mathcal{F}}_t$-measurable functions f, with the f_u as in (2.2).

To show that this new measure truly is an extension of the old, we'll need the following lemma:

Lemma 2.3.1.1.

$$\sum_{u \in N_t} \prod_{v<u} \frac{1}{1+A_v} = 1. \tag{2.3}$$

Proof. Clearly this must hold if (2.1) is true.

Let $\mathcal{M}_t := \max\{|u| : u \in N_t\}$. The proof is by strong induction on $\mathcal{M}_t$.

Base case: if $\mathcal{M}_t = 0$, then we have a single summand which is an empty product, so is equal to 1.

Induction step: assume the lemma holds whenever $\mathcal{M}_s \leq n$ for any s, and assume also that $\mathcal{M}_t = n + 1$.

Noting that $\frac{1}{1+A_\emptyset}$ is a factor of each summand in (2.3) and that N_t can be divided into the descendents of the $1 + A_\emptyset$ sub-trees that arise at time $S_\emptyset$, we may write:

$$\sum_{u \in N_t} \prod_{v<u} \frac{1}{1+A_v} = \sum_{i=1}^{1+A_\emptyset} \frac{1}{1+A_\emptyset} \underbrace{\left(\sum_{u \in N_t \cap (\tau,\sigma,x)_i^\emptyset} \prod_{\emptyset < v < u} \frac{1}{1+A_v} \right)}_{=1} = 1.$$

To see that the parenthetical term is unity, note that the sub-tree and the original tree are identically distributed up to initial position, and that the order of each node $u \in (\tau, \sigma, x)_i^\emptyset$ relative to $(\tau, \sigma, x)_i^\emptyset$ is one less than its order relative to the original tree. This, together with the induction assumption, shows the term in parentheses is equal to 1, and the proof is complete. □

Now we can show that the above language is not imprecise; $\widetilde{P}$ is indeed an extension of P. The following proof (sans lemma) is given in [10]:

Theorem 2.3.1.2. $\widetilde{P}|_{\mathcal{F}_\infty} = P$.

Proof. If f is already $\mathcal{F}_t$-measurable, then we have (via 2.3.1.1) the trivial representation,

$$f = \sum_{u \in N_t} f 1_{\{u \in d\}},$$

so that,

$$\int_{\widetilde{\mathcal{T}}} f d\widetilde{P} = \int_{\widetilde{\mathcal{T}}} f \left(\sum_{u \in N_t} \prod_{v<u} \frac{1}{1+A_v} \right) dP = \int_{\widetilde{\mathcal{T}}} f dP.$$

□

2.3.2 Single-Particle Martingales Yield Additive Branching-Process Martingales

Suppose we are given a $(\{\mathcal{G}_t\}, \widetilde{P})$-martingale, $\zeta(t)$. That is to say, $\zeta(t)$ is $\mathcal{G}_t$ measurable $\forall t \geq 0$, and is a martingale with respect to $\widetilde{P}$. Such a ζ will be referred to as a ***single-particle martingale***. As we are mainly interested in martingales that can serve as Radon-Nikodym derivatives between probability measures, assume that $\zeta(t)$ is non-negative. For the sake of notational simplicity we will also assume for the remainder of Chapter 2 that $\zeta(t)$ is normalized to have a $\widetilde{P}$-mean of 1.

Since a $\mathcal{G}_t$-measurable function is certainly $\widetilde{\mathcal{F}}_t$ measurable, 2.3.1.1 gives us a representation of the form:

$$\zeta(t) = \sum_{u \in N_t} \zeta_u(t) 1_{\{u \in d\}},$$

with $\zeta_u(t)$ $\mathcal{F}_t$-measurable. With this representation in view, we can write down what will turn out to be a martingale for the branching process–i.e. a $(\{\mathcal{F}_t\}, \widetilde{P})$-martingale:

Definition 2.3.2.1.

$$Z(t) := \sum_{u \in N_t} e^{-rmt} \zeta_u(t)$$

will be referred to as ***the branching-process martingale*** *associated with* $\zeta(t)$*.*

That the term "martingale" is not a misnomer in reference to $Z(t)$ will be established shortly. First we will collect some details on a few other martingales that will prove to be of use.

Comment 2.3.2.1. *Suppose* $\mathbb{L}^r$ *is the law of a Poisson process* n' *with rate* r *that is adapted to some filtration* $\{H_t\}_{t \geq 0}$*, and let* $\mathbb{L}^r_t := \mathbb{L}^r|_{H_t}$*. Then we have the following Radon-Nikodym derivative to change the rate parameter:*

$$\frac{d\mathbb{L}^{(1+m)r}_t}{d\mathbb{L}^r_t}(n') = e^{-mrt}(1+m)^{n'_t}.$$

Therefore $e^{-mrt}(1+m)^{n_t}$ *(with* n_t *defined as in Section 2.1) is a* $\widetilde{P}$*-martingale that will increase the rate at which fission events occur along the spine from* r *to* $(1+m)r$.

Comment 2.3.2.2. *Due to the independence of the* A_u *random variables, the following is easily seen to be a* $\widetilde{P}$*-martingale:*

$$\prod_{v<node_d(t)} \frac{1+A_v}{1+m} = \frac{1}{(1+m)^{n_t}} \prod_{v<node_d(t)} 1+A_v.$$

If this martingale is used as a Radon-Nikodym derivative to define Q' *in terms of* $\widetilde{P}$*, it will induce a size-biasing of the offspring distribution along the spine, resulting in the following scenario:*

$$Q'(A_u=k) = \begin{cases} p_k & \text{if } u \notin d \\ \frac{(1+k)p_k}{1+m} & \text{if } u \in d. \end{cases} \tag{2.4}$$

recalling that $\widetilde{P}(A_u=k)=p_k \ \forall u \in \tau$.

While the above martingale for changing the rate of a Poisson process is well known, a simple demonstration might help make clear that the second martingale discussed does indeed change the measure in the manner described:

Proof. Let $\widetilde{P}_t = \widetilde{P}|_{\widetilde{\mathcal{F}}_t}$, and $Q'_t = Q'|_{\widetilde{\mathcal{F}}_t}$. Then, by the independence of the A_v, we have $\forall u < node_d(t)$,

$$\begin{aligned} Q'_t(A_u=k) &= \int_{\widetilde{\mathcal{T}}} 1_{\{A_u=k\}} dQ' \\ &= \frac{1}{(1+m)^{n_t}} \int_{\widetilde{\mathcal{T}}} 1_{\{A_u=k\}} \prod_{v<node_d(t)} (1+A_v) d\widetilde{P} \end{aligned}$$

$$= \frac{1}{(1+m)^{n_t}} \underbrace{\left(\prod_{\substack{v<node_d(t) \\ v \neq u}} \int_{\widetilde{\mathcal{T}}} (1+A_v) d\widetilde{P} \right)}_{=(1+m)^{n_t-1}} \underbrace{\left(\int_{\widetilde{\mathcal{T}}} 1_{\{A_u=k\}} (1+A_u) d\widetilde{P} \right)}_{=(1+k)p_k}$$

$$= \frac{(1+k)p_k}{1+m}.$$

□

We may now take the above three martingales and form their product:

$$\begin{aligned} \widetilde{\zeta}(t) &:= \zeta(t) \times \left(e^{-mrt}(1+m)^{n_t}\right) \times \prod_{v<node_d(t)} \frac{1+A_v}{1+m} \\ &= \zeta(t) e^{-mrt} \prod_{v<node_d(t)} 1+A_v. \end{aligned}$$

This product of three independent martingales will itself be a martingale with respect to its natural filtration by the following lemma, taken from Cherny, [7].

Lemma 2.3.2.1. *Let* $\{X_t\}_{t\geq 0}$ *and* $\{Y_t\}_{t\geq 0}$ *be independent martingales with respect to their natural filtrations,* $F_t^X = \sigma(X_s : s \leq t)$ *and* $F_t^Y = \sigma(Y_s : s \leq t)$*. Then their product,* $\{X_tY_t\}_{t\geq 0}$*, is a martingale with respect to its natural filtration,* $F_t^{XY} = \sigma(X_sY_s : s \leq t)$*.*

The natural filtration for $\widetilde{\zeta}(t)$ is contained within $\widetilde{\mathcal{F}}_t$, so $\widetilde{\zeta}(t)$ is adapted to $\{\widetilde{\mathcal{F}}_t\}$, and is therefore a $(\{\widetilde{\mathcal{F}}_t\}, \widetilde{P})$-martingale. A change of measure via this martingale will result in the three alterations that have come to characterize distinguished path analysis–namely:

1. the motion along the distinguished path will be modified by ζ; for our purposes this will mean a change in the drift of the distinguished particles, as we shall see,

2. the rate of fission events along the distinguished path will be increased from r to $(1+m)r$,

3. and the distribution of offspring generated by distinguished particles (i.e. elements of d) is skewed in the manner described in (2.4).

Note that *only* the behavior of the distinguished path will be changed by such a martingale change of measure. Note also that $\widetilde{\zeta}(t)$ has a $\widetilde{P}$-mean of 1 since its three factor martingales are independent and themselves have $\widetilde{P}$-mean equal to 1.

The following theorem will allow us to see the natural connection between $\zeta(t)$ and $Z(t)$, and will also make the proof that $Z(t)$ is a branching-process martingale trivial. It is due to Hardy and Harris ([10]).

Theorem 2.3.2.1. *Both $Z(t)$ and $\zeta(t)$ are projections of $\widetilde{\zeta}(t)$ onto the appropriate filtrations:*

$$\zeta(t) = \mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{G}_t),$$
$$Z(t) = \mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t).$$

Proof. For the first claim, note that

$$\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{G}_t) = \zeta(t) \times \mathbb{E}_{\widetilde{P}}(e^{-mrt} \prod_{v<node_d(t)} 1 + A_v|\mathcal{G}_t) = \zeta(t),$$

where the final conditional expectation is 1 since the input is the product of two $\widetilde{P}$-mean-1 martingales which are independent of each other and of $\mathcal{G}_t$. For the second claim, we write the representation of $\widetilde{\zeta}(t)$ that is guaranteed by Theorem 2.3.1.1:

$$\widetilde{\zeta}(t) = \sum_{u \in N_t} \zeta_u(t) e^{-mrt} \left(\prod_{v<u} 1 + A_v \right) 1_{\{u \in d\}}.$$

Now we have

$$\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t) = \sum_{u \in N_t} \zeta_u(t) e^{-mrt} \left(\prod_{v<u} 1 + A_v \right) \mathbb{E}_{\widetilde{P}}(1_{\{u \in d\}}|\mathcal{F}_t)$$

$$= \sum_{u \in N_t} \zeta_u(t) e^{-mrt}$$

$$= Z(t),$$

since $1_{\{u \in d\}}$ is independent of $\mathcal{F}_t$, and $\mathbb{E}_{\widetilde{P}}(1_{\{u \in d\}}) = \widetilde{P}(u \in d) = \prod_{v<u} \frac{1}{1+A_v}$ $\forall u \in N_t$, as was shown in Lemma 2.3.1.1. □

Corollary 2.3.2.1. *$Z(t)$ is a $(\{\mathcal{F}_t\}, \widetilde{P})$-martingale.*

Proof. The above theorem, together with the tower property for conditional expectation, gives us, $\forall 0 \leq s < t$,

$$\begin{aligned}
\mathbb{E}_{\widetilde{P}}(Z(t)|\mathcal{F}_s) &= \mathbb{E}_{\widetilde{P}}(\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t)|\mathcal{F}_s) \\
&= \mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_s) \\
&= \mathbb{E}_{\widetilde{P}}\left(\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\widetilde{\mathcal{F}}_s)|\mathcal{F}_s\right) \\
&= \mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(s)|\mathcal{F}_s) \\
&= Z(s).
\end{aligned}$$

□

So any single-particle martingale gives rise to an associated additive branching-process martingale. Such additive martingales for branching processes are not new, and their form has always made it clear that they are connected to their associated single-particle martingale, but in this setting that connection becomes obvious and natural: they are both projections of a single, more general martingale onto the appropriate sigma algebra.

2.3.3 A Cascade of Measure Changes

Since $\widetilde{\zeta}(t)$ is a $\widetilde{P}$-mean-1, non-negative martingale, we may use it to define a new *probability* measure on $\widetilde{\mathcal{F}}_\infty$. On $\widetilde{\mathcal{F}}_t$, define

$$\frac{d\widetilde{Q}_t}{d\widetilde{P}_t} := \widetilde{\zeta}(t).$$

Having $\widetilde{P}$ defined on $\widetilde{\mathcal{F}}_\infty$ allows for measures on the sub-sigma algebras $\mathcal{G}_\infty$ and $\mathcal{F}_\infty$ as well; define

$$\mathbb{P} := \widetilde{P}|_{\mathcal{G}_\infty},$$

the distribution of the spatial motion of the distinguished path, and recall that

$$P = \widetilde{P}|_{\mathcal{F}_\infty}$$

is the distribution of the branching process with no distinguished path.

We can restrict $\widetilde{Q}$ to sub-sigma algebras similarly:

$$\mathbb{Q} := \widetilde{Q}|_{\mathcal{G}_\infty},$$
$$Q := \widetilde{Q}|_{\mathcal{F}_\infty}.$$

Alternatively, we could have obtained new measures on $\mathcal{G}_\infty$ and $\mathcal{F}_\infty$ from $\mathbb{P}$ and P via ζ and Z respectively, just as $\widetilde{Q}$ on $\widetilde{\mathcal{F}}_\infty$ was obtained from $\widetilde{P}$ via $\widetilde{\zeta}$. In fact, these different approaches agree:

Theorem 2.3.3.1.

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = \mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{G}_t) = \zeta(t),$$
$$\frac{dQ_t}{dP_t} = \mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t) = Z(t),$$

where $\mathbb{Q}_t = \mathbb{Q}|_{\mathcal{G}_t},\ P_t = P|_{\mathcal{F}_t},$ *etc.*

The theorem follows from this more general lemma:

Lemma 2.3.3.1. *Let μ and ν be probability measures defined on (Ω, S) such that ν is absolutely continuous with respect to μ, having Radon-Nikodym derivative*

$$\frac{d\nu}{d\mu} = f.$$

Then if S' is a sub-sigma algebra of S, and if $\mu' := \mu|_{S'}$ and $\nu' := \nu|_{S'}$, then ν' is absolutely continuous with respect to μ', having Radon-Nikodym derivative

$$\frac{d\nu'}{d\mu'} = \mathbb{E}_\mu(f|S').$$

Proof. Let g be an S'-measurable function, and let $A \in S'$. Then we have,

$$\begin{aligned}\int_A g d\nu' &= \int_A g d\nu \\ &= \int_A g f d\mu \\ &= \int_A \mathbb{E}_\mu(gf|S') d\mu \\ &= \int_A g\mathbb{E}_\mu(f|S') d\mu \\ &= \int_A g\mathbb{E}_\mu(f|S') d\mu',\end{aligned}$$

where the last equality holds since both factors of the integrand are S'-measurable. □

So changing the measure P via $Z(t)$ is equivalent to first changing $\widetilde{P}$ via $\widetilde{\zeta}$, then restricting attention to $\mathcal{F}_\infty$, and similarly for $\mathbb{P}$ via ζ. This means, for instance, that $Z(t)$ will induce the same 3 changes to the distinguished path that $\widetilde{\zeta}$ does (already mentioned, and to be discussed in detail in Section 2.3.4), though the process $Z(t)$ gives rise to will not know which particles make up the distinguished path.

This very lovely (if complicated), commutative situation is graphically depicted in Figure 2.3.

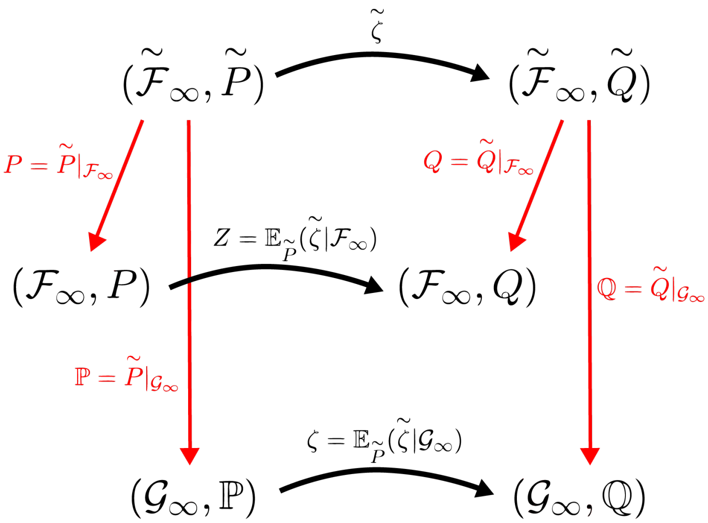

FIGURE 2.3: An illustration of the various sigma algebras, the measures defined upon them, and the relationships between them via either restriction or martingale change of measure. We have demonstrated that "moving down, then right" is equivalent to "moving right, then down."

2.3.4 Intuitive Decomposition of the Distinguished Path Change of Measure

In this section we will see that all the changes that are characteristic of distinguished path analysis have a natural and obvious origin when $\widetilde{P}$, $\widetilde{\zeta}$, and $\widetilde{Q}$ are viewed in the appropriate light.

Before breaking down $\widetilde{P}$ and $\widetilde{Q}$ to see how $\widetilde{\zeta}$ takes the one to the other, recall that $n = \{n_t\}_{t\geq 0}$ is the Poisson process that governs the number of fission events along the spine, and that $\mathbb{L}^r$ is the distribution of such a process having rate parameter r.

Now we will decompose $\widetilde{P}$. Decompositions similar to this one date back to at least the Chauvin and Rouault paper [6], though this particular decomposition is due to Hardy and Harris, and is given in [10]:

Comment 2.3.4.1. *The measure $\widetilde{P}$ on $\widetilde{\mathcal{F}}_t$ can be decomposed as follows:*

$$d\widetilde{P}(\tau,\sigma,x,d) = d\mathbb{P}(d)d\mathbb{L}^r(n) \prod_{v<node_d(t)} p_{A_v} \prod_{v<node_d(t)} \left(\frac{1}{1+A_v} \prod_{\substack{j=1 \\ vj \notin d}}^{1+A_v} dP((\tau,\sigma,x)_j^v) \right).$$

This informative decomposition can be interpreted as follows:

1. $\mathbb{P}$ governs the movement of the distinguished path d,
2. $\mathbb{L}^r$ governs the occurrence of fission events along the distinguished path,
3. at the moment of fission of $v \in d$, v is replaced with $1 + A_v$ child-particles with probability p_{A_v},
4. each of these child-particles has a $\frac{1}{1+A_v}$ chance of becoming the new representative of the distinguished path,
5. and the A_v child particles that were *not* chosen initiate sub-trees which branch off from the distinguished path, rooted at the space-time location of v's fission event, each governed by P.

Now this decomposition of $\widetilde{P}$, together with the representation of $\widetilde{\zeta}(t)$ as the product of 3 martingales,

$$\widetilde{\zeta}(t) = \zeta(t) \times \left(e^{-mrt}(1+m)^{n_t}\right) \times \prod_{v<node_d(t)} \frac{1+A_v}{1+m},$$

gives us a similar decomposition of $\widetilde{Q}$:

Comment 2.3.4.2. *On $\widetilde{\mathcal{F}}_t$ we have,*

$$\begin{aligned} d\widetilde{Q}(\tau,\sigma,x,d) &= \widetilde{\zeta}(t)d\widetilde{P}(\tau,\sigma,x,d) \\ &= (\zeta(t)d\mathbb{P}(d)) \times \left(e^{-mrt}(1+m)^{n_t}d\mathbb{L}^r(n)\right) \times \dots \end{aligned}$$

$$\cdots \times \prod_{v<node_d(t)} \frac{(1+A_v)p_{A_v}}{1+m} \prod_{v<node_d(t)} \left(\frac{1}{1+A_v} \prod_{\substack{j=1 \\ vj\notin d}}^{1+A_v} dP((\tau,\sigma,x)_j^v) \right)$$

$$= d\mathbb{Q}(d) d\mathbb{L}^{(1+m)r}(n) \times \dots$$

$$\cdots \times \prod_{v<node_d(t)} \frac{(1+A_v)p_{A_v}}{1+m} \prod_{v<node_d(t)} \left(\frac{1}{1+A_v} \prod_{\substack{j=1 \\ vj\notin d}}^{1+A_v} dP((\tau,\sigma,x)_j^v) \right).$$

This decomposition can be interpreted intuitively, much as was done for $\widetilde{P}$:

1. $\mathbb{Q}$ now governs the movement of the distinguished path (instead of $\mathbb{P}$),
2. $\mathbb{L}^{(1+m)r}$ now governs the occurrence of fission events along the distinguished path (instead of $\mathbb{L}^r$),
3. at the moment of fission of $v \in d$, v is replaced with $1 + A_v$ child particles with probability now equal to $\frac{(1+A_v)p_{A_v}}{1+m}$ (instead of p_{A_v}),
4. each of these child-particles has a $\frac{1}{1+A_v}$ chance of becoming the new representative of the distinguished path (as before),
5. and the A_v child particles that were *not* chosen initiate sub-trees which branch off from the distinguished path, rooted at the space-time location of v's fission event, each governed by P (as before).

Note again that a change of measure via $\widetilde{\zeta}$ *only* affects the behavior of the distinguished path. In fact, it induces the 3 changes typical of distinguished path analysis: the distinguished path acquires a new distribution for its movement, an accelerated rate of fission, and a skewed offspring distribution. No other particle is affected.

2.4 The Spine Decomposition

In this section we will state and prove a version of the celebrated "spine decomposition." This tool is central to many of the new or improved proofs that the techniques of distinguished path analysis have given rise to, including the main result of this paper.

First recall that $\widetilde{\mathcal{G}}_\infty$ contains all information about the distinguished path; it knows which extant particle belongs to the distinguished path at any moment, it knows their ancestry, their movement, the timing of their fission events, and the number of offspring created at each such event (though it knows nothing of what becomes of those offspring after their creation). Recall also that S_u is the fission time of any particle $u \in \tau$. With these facts in mind, we give the result. This statement and proof are adapted from [10]:

Theorem 2.4.0.1. ***(Spine Decomposition)*** *We have the following $\widetilde{Q}$ projection of the additive branching-process martingale onto $\widetilde{\mathcal{G}}_\infty$:*

$$\mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty) = \sum_{v<node_d(t)} A_v e^{-mrS_v}\zeta(S_v) + e^{-mrt}\zeta(t).$$

Proof. First, take $Z(t)$ and isolate the summand corresponding to the distinguished path:

$$Z(t) = e^{-mrt}\zeta(t) + \sum_{\substack{u\in N_t \\ u\notin d}} e^{-mrt}\zeta_u(t),$$

since $\zeta(t) = \zeta_{node_d(t)}(t)$ is the original martingale for the distinguished path.

Each particle in N_t that is not in the spine is in some sub-tree that is rooted somewhere along the spine. For all $t \geq S_u$, let

$$Z_{uj}(S_u;t) := \sum_{\substack{v\in N_t \\ v\in(\tau,\sigma,x)^u_j}} e^{-mr(t-S_u)}\zeta_v(t)$$

be the additive $\widetilde{P}$-martingale for the sub-tree branching process that has uj as its initial ancestor, and is rooted at the space-time location of u's fission event. Now we can write

$$Z(t) = e^{-mrt}\zeta(t) + \sum_{u<node_d(t)} e^{-mrS_u} \sum_{\substack{j=1 \\ uj\notin d}}^{1+A_u} Z_{uj}(S_u;t).$$

Note that since $\widetilde{\mathcal{G}}_\infty$ knows only information about the distinguished path, only the initial value of $Z_{uj}(S_u;t)$ is $\widetilde{\mathcal{G}}_\infty$-measurable; all subsequent displacements from the initial position are independent of $\widetilde{\mathcal{G}}_\infty$ by construction. This gives us

$$\mathbb{E}_{\widetilde{P}}(Z_{uj}(S_u;t)|\widetilde{\mathcal{G}}_\infty) = \zeta(S_u),$$

so that

$$\begin{aligned}
\mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty) &= e^{-mrt}\zeta(t) + \sum_{u<node_d(t)} e^{-mrS_u} \sum_{\substack{j=1 \\ uj\notin d}}^{1+A_u} \mathbb{E}_{\widetilde{Q}}(Z_{uj}(S_u;t)|\widetilde{\mathcal{G}}_\infty) \\
&= e^{-mrt}\zeta(t) + \sum_{u<node_d(t)} e^{-mrS_u} \sum_{\substack{j=1 \\ uj\notin d}}^{1+A_u} \mathbb{E}_{\widetilde{P}}(Z_{uj}(S_u;t)|\widetilde{\mathcal{G}}_\infty) \\
&= e^{-mrt}\zeta(t) + \sum_{u<node_d(t)} e^{-mrS_u} A_u\zeta(S_u).
\end{aligned}$$

Note that we have used implicitly our decomposition of $\widetilde{Q}$ from Section 2.3.4 that says that the $\widetilde{Q}$-measure of a sub-tree branching off from the spine is the same as the $\widetilde{P}$-measure of such a sub-tree, since $\widetilde{Q}$ and $\widetilde{P}$ differ only along the spine.

This completes the proof.

□

We return now to our true interest, branching Brownian motion on a finite interval with one absorbing and one reflecting boundary.

3 Branching Brownian Motion with One Absorbing and One Reflecting Boundary–Extinction or Persistence?

In this section, we will present our main result for a branching Brownian motion on a finite interval with one reflecting and one absorbing boundary. This will be a division of the parameter space into a region where the extinction of the process is assured, and a region where the process will persist indefinitely with positive probability.

We'll begin by deriving and analyzing our foundational single-particle martingale.

3.1 A Martingale for Brownian Motion with One Absorbing and One Reflecting Boundary

In this section, we'll be exploring the properties of a particular martingale for the single-particle process on $[0, K]$ with 0 absorbing and K reflecting. It will turn out to be the ζ that will give rise to the branching-process martingale Z that will be so central to our final analysis.

As in Section 1.1, let $\xi(t)$ be governed by

$$Lf(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) - \mu\frac{d}{dx}f(x) \quad x \in (0, K), \tag{3.1}$$

$$f(0+) = 0, \tag{3.2}$$

$$f'(K-) = 0, \tag{3.3}$$

defined for all $f \in C^2((0, K), \mathbb{R})$ which satisfy the given boundary conditions. This is Brownian motion on the interval $[0, K]$ with drift rate $-\mu$ (we take $\mu > 0$), an absorbing boundary at 0, and a reflecting boundary at K. Let $\xi(0) = x_0 \in (0, K]$. Define $\mathcal{G}_t^{\mathcal{L}} := \sigma(\xi(s) : s \leq t)$ to be the natural filtration of this single-particle process with boundaries, where the "$\mathcal{L}$" in the notation here signifies that for this process we can associate the absorbing boundary with a "stopping line," a concept which we will discuss in Section 3.2.

Let $\mathbb{P}$ be the distribution of $\xi(t)$ when taken as a function on $\mathcal{G}_\infty^{\mathcal{L}} := \bigvee_{t\geq 0} \mathcal{G}_t^{\mathcal{L}}$.

Ignoring the boundary conditions for a moment, one can check that

$$h_1(t) = e^{\mu x} \sin(bx)$$

satisfies

$$Lh_1(x) = \left(-\frac{\mu^2}{2} - \frac{b^2}{2}\right) h_1(x).$$

The $\sin(bx)$ factor in $h_1(x)$ guarantees that (3.2) will be satisfied for any value of b; to ensure that (3.3) holds, note that

$$h_1'(x) = \mu e^{\mu x} \sin(bx) + b e^{\mu x} \cos(bx),$$

so that we require

$$h_1(K) = \mu e^{\mu K} \sin(bK) + b e^{\mu K} \cos(bK) = 0,$$

or

$$\tan(bK) = -\frac{b}{\mu}.$$

This transcendental equation has infinitely many positive solutions b, as is depicted graphically in Figure 3.1.

We will hereafter use "b" to refer to the smallest such positive solution, and note that since $\frac{\pi}{2} < bK < \pi$ (see Figure 3.1 again), we know $h_1(x) \geq 0$ on $[0, K]$ (in fact, it is positive on $(0, K]$). We have now demonstrated the following:

Theorem 3.1.0.2.

$$h_1(x) = e^{\mu x} \sin(bx)$$

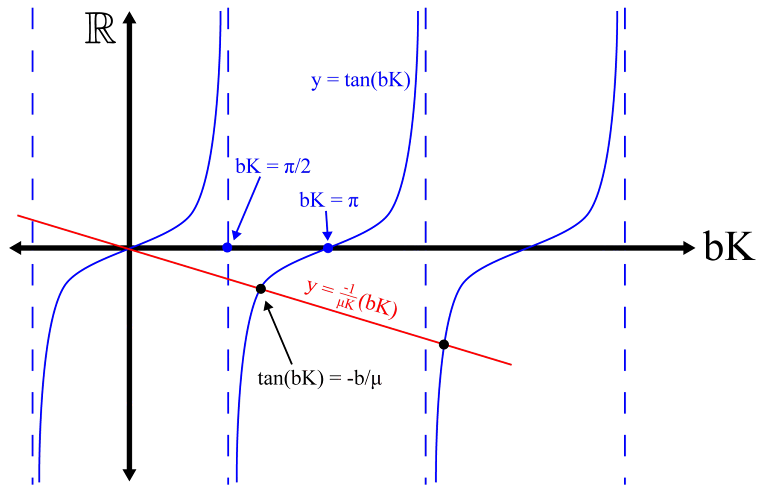

FIGURE 3.1: A "proof by picture" that $\tan(bK) = -\frac{b}{\mu}$ has a positive solution. We see that the defining equation for b in fact has infinitely many positive solutions (the points of intersection of the above graphs), and that the smallest such b satisfies $\frac{\pi}{2} < bK < \pi$. This is sufficient for the existence of a non-negative eigenfunction for (3.1)-(3.3).

is a non-negative eigenfunction for the infinitesimal generator (3.1) - (3.3), with eigenvalue

$$\lambda_1 = -\frac{\mu^2}{2} - \frac{b^2}{2}.$$

It is easy to see that

$$Lh_1(x) = \lambda_1 h_1(x)$$

$$\Longrightarrow$$

$$\left(\frac{\partial}{\partial t} + L\right) h_1(x) e^{-\lambda_1 t} = 0,$$

so that

$$h_2(x,t) := h_1(x) e^{-\lambda_1 t}$$

$$= \exp\left(\mu x - (-\frac{\mu^2}{2} - \frac{b^2}{2})t\right)\sin(bx)$$

is harmonic with respect to the space-time operator $\frac{\partial}{\partial t} + L$. This fact, together with Itô's Lemma, gives us the following:

Theorem 3.1.0.3. *Define*

$$\begin{aligned}\zeta(t) :&= h_2(\xi(t), t)\\ &= \exp\left(\mu\xi(t) - (-\frac{\mu^2}{2} - \frac{b^2}{2})t\right)\sin(b\xi(t)).\end{aligned}$$

Then $\zeta(t)$ is a non-negative $\mathbb{P}$-martingale.

We will now embark on an exploration of what kind of measure change $\frac{\zeta(t)}{\zeta(0)}$ will induce. To this end, we will state and prove a (very specific) special case of Proposition 3.4 from Chapter VIII of Revuz and Yor's work, [20]:

Lemma 3.1.0.1. *Assume $\xi(t)$ is governed by (3.1) - (3.3) under $\mathbb{P}$, and let $\{\mathcal{G}_t^{\mathcal{L}}\}$ be its natural filtration. Assume further that $h(x)$ is a positive eigenfunction for (3.1)-(3.3), with*

$$Lh(x) = \lambda h(x).$$

Then

$$M_t = \frac{h(\xi(t))}{h(\xi(0))}e^{-\lambda t}$$

is a $\mathbb{P}$-martingale, as discussed above. If we now define $\mathbb{Q}$ on $\mathcal{G}_t^{\mathcal{L}}$ via

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} := M_t,$$

then the infinitesimal generator that governs the movement of $\xi(t)$ under $\mathbb{Q}$ is

$$L' = L + \frac{h'(x)}{h(x)}\frac{d}{dx}.$$

In other words, a change in measure via such an M_t results in the same type of modification to the infinitesimal generator we would see had we used an L-harmonic function to change the measure.

Proof. We will proceed via the limit definition of the infinitesimal generator. Let $\mathbb{P}_{x_0}$, $\mathbb{Q}_{x_0}$ be conditional distributions given $\xi(0) = x_0 \in (0, K]$. For $f \in C^2((0, K))$ satisfying (3.2) and (3.3), we have

$$\begin{aligned}
L'f(x_0) &= \lim_{t\to\infty}\left(\frac{\mathbb{E}_{\mathbb{Q}_{x_0}}[f(\xi(t))] - f(x_0)}{t}\right) \\
&= \lim_{t\to\infty}\left(\frac{\mathbb{E}_{\mathbb{P}_{x_0}}\left[f(\xi(t))\frac{h(\xi(t))}{h(x_0)}e^{-\lambda t}\right] - f(x_0)}{t}\right) \\
&= \lim_{t\to\infty}\mathbb{E}_{\mathbb{P}_{x_0}}\left[\left(\frac{f(\xi(t)) - f(x_0)}{t}\right)\frac{h(\xi(t))}{h(x_0)}e^{-\lambda t}\right] \ \left(\text{Since } \mathbb{E}_{\mathbb{P}_{x_0}}M_t = 1\right) \\
&= \underbrace{\lim_{t\to\infty} e^{-\lambda t}}_{=1}\lim_{t\to\infty}\mathbb{E}_{\mathbb{P}_{x_0}}\left[\frac{f(\xi(t)) - f(x_0)}{t}\frac{h(\xi(t))}{h(x_0)}\right] \\
&= \frac{1}{h(x_0)}\lim_{t\to\infty}\mathbb{E}_{\mathbb{P}_{x_0}}\left[\frac{f(\xi(t))h(\xi(t)) - f(x_0)h(x_0)}{t} - f(x_0)\frac{h(\xi(t)) - h(x_0)}{t}\right] \\
&= \frac{1}{h(x_0)}L(fh)(x_0) - \frac{f(x_0)}{h(x_0)}Lh(x_0) \\
&= \frac{1}{h(x_0)}L(fh)(x_0) - \lambda f(x_0) \\
&= \frac{1}{h(x_0)}\left(\frac{1}{2}(fh)''(x_0) - \mu(fh)'(x_0)\right) - \lambda f(x_0) \\
&= \frac{1}{h(x_0)}\left(\frac{1}{2}(f''h + 2f'h' + fh'')(x_0) - \mu(f'h + fh')(x_0)\right) - \lambda f(x_0) \\
&= \left(\frac{1}{2}f''(x_0) - \mu f'(x_0)\right) + \frac{h'(x_0)}{h(x_0)}f'(x_0) + \underbrace{\frac{f(x_0)}{h(x_0)}\left(\frac{1}{2}h'' - \mu h'(x_0)\right)}_{+\lambda f(x_0)} - \lambda f(x_0) \\
&= Lf(x_0) + \frac{h'(x_0)}{h(x_0)}\frac{d}{dx}f(x_0),
\end{aligned}$$

and this completes the proof. □

Note that the martingale $\frac{\zeta(t)}{\zeta(0)}$ is exactly of the form given in Lemma 3.1.0.1. So we will in fact define $\mathbb{Q}$ as the lemma suggests; on $\mathcal{G}_t^{\mathcal{L}}$, let

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} := \frac{\zeta(t)}{\zeta(0)},$$

and then our lemma and simple calculus tell us the infinitesimal generator for $\xi(t)$ under $\mathbb{Q}$ will be

$$\begin{aligned} L' :&= L + \frac{h_1'(x)}{h_1(x)}\frac{d}{dx} \\ &= \frac{1}{2}\frac{d^2}{dx^2} + \frac{b}{\tan(bx)}\frac{d}{dx}, \end{aligned}$$

with boundary conditions (3.2), (3.3). Notice that the drift coefficient, $\frac{b}{\tan(bx)}$, goes to ∞ as x goes to 0. So, under $\mathbb{Q}$, as the particle $\xi(t)$ gets closer and closer to the absorbing boundary at 0, it experiences stronger and stronger drift pushing it *away* from said boundary. In fact, the following theorem will show that 0 is inaccessible to $\xi(t)$ under $\mathbb{Q}$, along with other facts about $\xi(t)$ under $\mathbb{Q}$ that will prove useful:

Theorem 3.1.0.4. *For the process $\xi(t)$ distributed as $\mathbb{Q}$, the following hold:*

1. *0 is inaccessible,*

2. *K is accessible,*

3. *and $\xi(t)$ is positive recurrent.*

Proof. For the first claim, we have the following result from Bhattacharya and Waymire's book, [3]:

$$0 \text{ is inaccessible} \iff \int_0^{x'} m(x)ds(x) = -\infty,$$

where x' is any point in $(0, K)$, and

$$m(x) := \int_{x'}^{x} \frac{2}{\sigma^2(z)} \exp\left(I(x', z)\right) dz,$$

$$ds(x) := \exp\left(-I(x', x)\right) dx,$$

and finally

$$I(x', z) := \int_{x'}^{z} \frac{2\mu(y)}{\sigma^2(y)} dy.$$

An inspection of L' shows us that $\sigma^2(x) \equiv 1$ and $\mu(x) = \frac{b}{tan(bx)}$, so that calculus tells us

$$\begin{aligned} I(x', z) &= \int_{x'}^{z} \frac{2b}{\tan(by)} dy \\ &= \ln\left(\frac{\sin(bz)}{\sin(bx')}\right)^2, \end{aligned}$$

and

$$\begin{aligned} ds(x) &= \exp\left(-\ln\left(\frac{\sin(bx)}{\sin(bx')}\right)^2\right) dx \\ &= \left(\frac{\sin(bx')}{\sin(bx)}\right)^2 dx, \end{aligned}$$

and finally

$$m(x) = 2\int_{x'}^{x} \frac{\sin^2(bz)}{\sin^2(bx')} dz.$$

Now we have

$$\int_0^{x'} m(x) ds(x) = \int_0^{x'} \left(2\int_{x'}^{x} \frac{\sin^2(bz)}{\sin^2(bx')} dz\right) \frac{\sin^2(bx')}{\sin^2(bx)} dx$$

$$= 2\int_0^{x'}\int_{x'}^{x}\frac{\sin^2(bz)}{\sin^2(bx)}dzdx$$
$$= -2\int_0^{x'}\int_{x}^{x'}\frac{\sin^2(bz)}{\sin^2(bx)}dzdx,$$

so it suffices to show

$$\int_0^{x'}\int_{x}^{x'}\frac{\sin^2(bz)}{\sin^2(bx)}dzdx = \infty.$$

Our integrand here is non-negative and measurable, so Fubini's Theorem allows us to exchange the order of integration:

$$\begin{aligned}\int_0^{x'}\int_{x}^{x'}\frac{\sin^2(bz)}{\sin^2(bx)}dzdx &= \int_0^{x'}\int_0^{z}\frac{\sin^2(bz)}{\sin^2(bx)}dxdz\\ &= \int_0^{x'}\sin^2(bz)\underbrace{\left(\int_0^{z}\frac{1}{\sin^2(bx)}dx\right)}_{=\infty}dz,\end{aligned}$$

where the inner integral is infinite since $\frac{1}{(bx)^2} \leq \frac{1}{\sin^2(bx)}$, and $\int_0^z \frac{1}{x^2}dx = \infty$. So 0 is inaccessible.

For the second claim, we again look to [3] for a necessary and sufficient condition:

$$K \text{ is accessible } \iff \int_{x'}^{K} m(x)ds(x) < \infty.$$

Similar to before, we have

$$\int_{x'}^{K} m(x)ds(x) = 2\int_{x'}^{K}\int_{x'}^{x}\frac{\sin^2(bz)}{\sin^2(bx)}dzdx.$$

For $x \in (x', K)$ with $x' > 0$, we know $\exists m$ such that $0 < m \leq \sin(bx)$. So we have

$$\begin{aligned}2\int_{x'}^{K}\int_{x'}^{x}\frac{\sin^2(bz)}{\sin^2(bx)}dzdx &\leq \frac{2}{m^2}\int_{x'}^{K}\int_{x'}^{x}\sin^2(bz)dzdx\\ &\leq \frac{2}{m^2}(K-x')^2 < \infty,\end{aligned}$$

so that K is accessible.

To prove the final claim, we first state an adapted version of Proposition 10.3 from [3]:

$$\xi(t) \text{ is positive recurrent } \iff s(0) = -\infty \text{ and } m(0) > -\infty,$$

where

$$s(x) := \int_{x'}^{x} \exp\left(-I(x', z)\right) dz.$$

We'll start by proving $s(0) = -\infty$:

$$\begin{aligned} s(0) &= \int_{x'}^{0} \frac{\sin^2(bx')}{\sin^2(bz)} dz \\ &= -\sin^2(bx') \int_{0}^{x'} \frac{1}{\sin^2(bz)} dz, \end{aligned}$$

so that it suffices to show

$$\int_{0}^{x'} \frac{1}{\sin^2(bz)} dz = \infty,$$

which, as before, holds since $\sin^2(bz) \leq (bz)^2$ and $\int_0^{x'} \frac{1}{(bz)^2} dz = \infty$.

Now we show $m(0) > -\infty$:

$$\begin{aligned} m(0) &= 2\int_{x'}^{0} \frac{\sin^2(bz)}{\sin^2(bx')} dz \\ &= -\frac{2}{\sin^2(bx')} \int_{0}^{x'} \sin^2(bz) dz \\ &\geq -\frac{2x'}{\sin^2(bx')} \\ &> -\infty. \end{aligned}$$

So, under $\mathbb{Q}$, $\xi(t)$ is positive recurrent. This completes the proof.

□

This theorem shows, among other things, that ξ is (under $\mathbb{Q}$) a Brownian motion conditioned to stay within $(0, K]$ (K reflecting), similar to Knight's "taboo process," discussed in [13].

Note as a point of interest that since $\tan(bK) = -\frac{b}{\mu}$, we have drift coefficient at K equal to $\frac{b}{\tan(bK)} = -\mu$, so that our particle sees the original drift rate when it's maximally distant from the absorbing, now inaccessible boundary.

Before we prove the main theorem, we'll adapt a few results from Chapter 2.

3.2 Adapting Results for Unrestricted Branching Processes to Branching Processes with Boundaries

While the setting laid out in Chapter 2 will continue to serve as our backdrop, several critical results will need to be adapted to our current circumstance, where all particles will live and evolve in $[0, K]$ with 0 absorbing and K reflecting.

We begin by reassigning some notation. When we refer to "the (branching) process on $[0, K]$," let the boundary conditions (3.2) and (3.3) be assumed. The "unrestricted (branching) process" will refer to that (branching) process discussed in Chapter 2, where all particles are allowed to move freely on all of $\mathbb{R}$, and there are neither boundaries nor boundary conditions.

Definition 3.2.0.1. *Hereafter, we will abide by the following assignments:*

1. $\boldsymbol{\xi(t)}$ *will be the process defined in Section 3.1,*

2. $\boldsymbol{N_t}$ *will be the set of particles alive at time t for the branching process on $[0, K]$,*

3. $\boldsymbol{N'_t}$ *will be the set of particles alive at time t for the corresponding unrestricted process, and*

4. $\boldsymbol{X(t)} := \sum_{u \in N_t} \delta_{x_u(t)}$ *will be the branching process on* $[0, K]$.

Note that we are not changing the definition of n_t, r, A_u, m, or $x_u(t)$. We will also leave $\mathbb{P}$, P, and $\widetilde{P}$ as they were, but will define new sigma algebras for the various processes on $[0, K]$ that these measures can be restricted to:

Definition 3.2.0.2. *Define the following sigma algebras:*

1. $\boldsymbol{\{\mathcal{G}_t^{\mathcal{L}}\}}$, $\boldsymbol{\mathcal{G}_\infty^{\mathcal{L}}}$ *are defined as in Section 3.1,*

2. $\boldsymbol{\{\mathcal{F}_t^{\mathcal{L}}\}}$ *is the natural filtration for* $X(t)$, *and* $\boldsymbol{\mathcal{F}_\infty^{\mathcal{L}}} := \bigvee_{t \geq 0} \mathcal{F}_t^{\mathcal{L}}$ *is their join,*

3. $\boldsymbol{\widetilde{\mathcal{F}}_t^{\mathcal{L}}} := \sigma(\mathcal{F}_t^{\mathcal{L}}, (node_d(s) : s \leq t))$ *is like* $\mathcal{F}_t^{\mathcal{L}}$ *but knows which nodes are in the distinguished path, and* $\boldsymbol{\widetilde{\mathcal{F}}_\infty^{\mathcal{L}}} := \bigvee_{t \geq 0} \widetilde{\mathcal{F}}_t^{\mathcal{L}}$ *is their join.*

Notice here that $node_d(t)$ need not be in N_t, but is instead only assured to be in N_t'. It could be that the line d chosen under $\widetilde{P}$ as the distinguished path for the unrestricted process has already met the boundary by time t. To picture this, we must imagine that each time a particle meets a boundary point, a "ghost" of the particle continues on as if no boundary were present. It is for this "ghost-process" (i.e. the corresponding unrestricted process) that the distinguished path is chosen, and $\widetilde{\mathcal{F}}_\infty^{\mathcal{L}}$ carries that information.

Make note of the following important set inclusions for these sigma algebras:

- $\mathcal{G}_t^{\mathcal{L}} \subset \mathcal{G}_t$,
- $\mathcal{F}_t^{\mathcal{L}} \subset \mathcal{F}_t$,
- $\widetilde{\mathcal{F}}_t^{\mathcal{L}} \subset \widetilde{\mathcal{F}}_t$,

each of which follows because knowledge of the unrestricted process up to time t is sufficient for knowledge of the process on $[0, K]$ up to time t.

We are now in a position to clarify our use of "$\mathcal{L}$" in the notation. Stopping lines will be a major tool in our effort to bridge the gap between unrestricted processes and processes on $[0, K]$.

Definition 3.2.0.3. *Let τ be the collection of nodes for a given marked Galton-Watson tree (with or without spine). Let $\hat{\mathcal{L}} = \{\rho_u\}_{u\in\tau}$ be a collection of stopping times indexed on τ. We will define such a $\hat{\mathcal{L}}$ to be a* ***stopping line*** *for τ if $\mathcal{L} := \{u \in \tau : \rho_u < S_u\}$ has the property that $\forall u \in \mathcal{L},\ \forall v < u,\ v \notin \mathcal{L}$.*

Recall that S_u is the time of fission of particle u. The condition on $\mathcal{L}$ says that no strict ancestor of a member of $\mathcal{L}$ also belongs to $\mathcal{L}$.

This concept is developed more generally and in greater detail by Chauvin in [5]. For our purposes, we will let π_u be the first passage time of u to the absorbing boundary ($\pi_u = \inf\{t \in [R_u, S_u) : x_u(t) = 0\}$, with $\pi_u = \infty$ if no such meeting occurs), and we'll set

$$\rho_u = \begin{cases} \pi_u & \text{if } \nexists v < u \text{ s.t. } v \in \mathcal{L} \\ \infty & \text{if } \exists v < u \text{ s.t. } v \in \mathcal{L}. \end{cases}$$

With this, $\{\rho_u\}_{u\in\tau}$ will be a stopping line. Here, the nodes in $\mathcal{L}$ are the first members of their ancestral line to reach the absorbing boundary. Our definition of ρ_u says that if no ancestor of u has reached 0 (i.e., no ancestor is in $\mathcal{L}$), then u has a chance to be the first (and be included in $\mathcal{L}$), and it will be the first if and only if it reaches 0 before undergoing fission (i.e., if and only if $\pi_u < S_u$).

Knowing $\mathcal{L}$, we can define

$$\mathcal{L}_t := \{u \in \mathcal{L} : \rho_u \leq t\}$$

to be the set of nodes in $\mathcal{L}$ which entered $\mathcal{L}$ (i.e., reached 0) before time t.

With our definition of $\mathcal{L}$ in hand, we can now offer the following, more detailed description of $\mathcal{F}_t^{\mathcal{L}}$ as an aid to intuition:

Comment 3.2.0.3. *$\mathcal{F}_t^{\mathcal{L}}$ is the join of three smaller sigma algebras:*

1. $\sigma((u, \sigma_u, x_u) : S_u \leq t, \nexists v \leq u \text{ s.t. } v \in \mathcal{L})$,

2. $\sigma((u, x_u(s)) : s \leq t, t \in [R_u, S_u), \nexists v \leq u \text{ s.t. } v \in \mathcal{L})$,

3. $\sigma((u, x_u(s)) : s \leq \rho_u \leq t, u \in \mathcal{L})$.

The first sigma algebra guarantees that $\mathcal{F}_t^{\mathcal{L}}$ will know everything about particles u which undergo fission before time t, and whose ancestral line has not reached 0 by u's fission time. The second sigma algebra is similar, but applies to particles still alive at time t; if such an extant particle has an ancestral line that has not yet reached 0, $\mathcal{F}_t^{\mathcal{L}}$ will know everything that happens to it up until time t. Finally, the third sigma algebra says that for those particles that are the first of their line to reach the absorbing boundary at 0, and do so by time t, $\mathcal{F}_t^{\mathcal{L}}$ will know everything that happens to them up until the time of their absorption, ρ_u.

If we again imagine our unrestricted "ghost-process," we can say that $\mathcal{F}_t^{\mathcal{L}}$ knows nothing about any u such that $\exists v < u$ with $v \in \mathcal{L}$. That is to say, it knows nothing of nodes that occur along a line *after* the absorbing boundary has been met.

While the intuitive description of $\mathcal{F}_t^{\mathcal{L}}$ is nice, the main reason to introduce stopping lines is so that we can make use of the strong Markov branching property:

Comment 3.2.0.4. *The **strong Markov branching property** says that if for $u \in \mathcal{L}$ we define $(\tau, \sigma, x)_{\mathcal{L}}^u$ to be the subtree that begins from the moment u first enters $\mathcal{L}$, then, given $\mathcal{F}_\infty^{\mathcal{L}}$, the sub-trees $(\tau, \sigma, x)_{\mathcal{L}}^u$ taken relative to and rooted at space-time points $(x_u(\rho_u), \rho_u)$ respectively are independent copies of the original branching process.*

Actually this is a simplification of the strong Markov branching property, but it is one that will suffice for our purposes. Again, for a more general and detailed discussion, see Chauvin [5].

We will now fix the definitions of our various measure-changing martingales. (Actually $Z(t)$, as defined below, has not yet been shown to be a martingale, but this will be rectified shortly.)

Definition 3.2.0.4. *Hereafter, $\zeta(t)$ will be as in Section 3.1:*

$$\boldsymbol{\zeta(t)} := e^{\{\mu\xi(t) - (-\frac{\mu^2}{2} - \frac{b^2}{2})t\}} \sin(b\xi(t)).$$

Note that in Section 3.1, we saw how the martingale $\zeta(t)$, when used to perform a change of measure, alters the movement of $\xi(t)$–in particular, 0 becomes inaccessible.

Definition 3.2.0.5. *Hereafter, $Z(t)$ will be defined as follows:*

$$\begin{aligned}\boldsymbol{Z(t)} :&= \sum_{u \in N_t} e^{-mrt}\zeta_u(t) \\ &= \sum_{u \in N_t} e^{\{\mu x_u(t) - (mr - \frac{\mu^2}{2} - \frac{b^2}{2})t\}} \sin(b x_u(t)) \\ &= \sum_{u \in N_t} e^{\{\mu x_u(t) - \lambda t\}} \sin(b x_u(t)),\end{aligned}$$

where we've taken $\lambda := mr - \frac{\mu^2}{2} - \frac{b^2}{2}$.

Note that the sum here is only over the extant particles for the branching process on $[0, K]$. That is to say, the sum is taken over N_t, not N'_t. It is this distinction that will require us to revisit the martingale nature of $Z(t)$. When we wish to refer to the analogous additive random function for the unrestricted process (a known martingale, heretofore referred to as $Z(t)$), we will use the notation "$Z'(t)$," i.e.,

$$Z'(t) := \sum_{u \in N'_t} e^{\{\mu x_u(t) - \lambda t\}} \sin(b x_u(t)).$$

We also define, for the sub-trees $(\tau, \sigma, x)^u_{\mathcal{L}}$ with $u \in \mathcal{L}$,

$$Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t) := \sum_{\substack{v \in (\tau,\sigma,x)^u_{\mathcal{L}} \\ v \in N'_t}} e^{\{\mu x_v(t) - \lambda(t - \rho_v)\}} \sin(b x_v(t)), \text{ for } t \geq \rho_v.$$

This is a P-martingale by the Strong Markov Branching Property (since $Z'(t)$ is). Note that $Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(\rho_u) = 0 \; \forall u \in \mathcal{L}$, since $x_u(\rho_u) = 0$ for such u, implying $\sin(b x_u(\rho_u)) = \sin(b \cdot 0) = 0$.

Definition 3.2.0.6. *Hereafter, $\widetilde{\zeta}(t)$ will be defined as follows:*

$$\widetilde{\boldsymbol{\zeta}}\boldsymbol{(t)} := \zeta(t) \times \left(e^{-mrt}(1+m)^{n_t}\right) \times \prod_{v < node_d(t)} \frac{1 + A_v}{1 + m}$$

$$= \zeta(t)e^{-mrt}\prod_{v<node_d(t)} 1 + A_v.$$

This martingale is of the same type discussed in Section 2.3.4 (save that it's not yet normalized).

Before we start inducing measure changes via ζ, Z, and $\widetilde{\zeta}$, we'll need to confirm that Z is in fact a martingale, and that we have the same nice relationships between these three martingales that led to the happily commutative situation we saw in Section 2.3.3. This means verifying that $\widetilde{\zeta}(t)$, which is $\widetilde{\mathcal{F}}_t^{\mathcal{L}}$-measurable, gives rise to $Z(t)$ and $\zeta(t)$ when projected onto the appropriate sub-sigma algebras. The proof that $Z(t)$ is a P-martingale will follow as a corollary.

We'll start with the easy proof first.[1]

Theorem 3.2.0.5.

$$\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{G}_t^{\mathcal{L}}) = \zeta(t).$$

Proof. Since $\mathcal{G}_t^{\mathcal{L}} \subset \mathcal{G}_t$, and recalling from Section 2.3.3 that

$$\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{G}_t) = \zeta(t),$$

we have

$$\begin{aligned}
\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{G}_t^{\mathcal{L}}) &= \mathbb{E}_{\widetilde{P}}\left(\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{G}_t)|\mathcal{G}_t^{\mathcal{L}}\right) \\
&= \mathbb{E}_{\widetilde{P}}(\zeta(t)|\mathcal{G}_t^{\mathcal{L}}) \\
&= \zeta(t),
\end{aligned}$$

where this last line holds since $\zeta(t)$ is $\mathcal{G}_t^{\mathcal{L}}$ measurable. $\square$

This proof is made trivial by the fact that, aside from the boundary conditions, there is not much difference between ζ here and ζ in Chapter 2. There is a greater difference between Z' and Z.

[1] As Dr. John Baez once said, "Always do the easy part first, because you might get lucky and die before you get to the hard part."

Theorem 3.2.0.6.

$$\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t^{\mathcal{L}}) = Z(t).$$

Proof. We have $\mathcal{F}_t^{\mathcal{L}} \subset \mathcal{F}_t$, and we know from Section 2.3.3 that

$$\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t) = Z'(t).$$

This gives us

$$\begin{aligned}
\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t^{\mathcal{L}}) &= \mathbb{E}_{\widetilde{P}}\left(\mathbb{E}_{\widetilde{P}}(\widetilde{\zeta}(t)|\mathcal{F}_t)|\mathcal{F}_t^{\mathcal{L}}\right) \\
&= \mathbb{E}_{\widetilde{P}}(Z'(t)|\mathcal{F}_t^{\mathcal{L}}) \\
&= \mathbb{E}_P(Z'(t)|\mathcal{F}_t^{\mathcal{L}}),
\end{aligned}$$

since $Z'(t)$ is $\mathcal{F}_t$ measurable and $\widetilde{P}|_{\mathcal{F}_t} = P$.

Now, dividing N_t' up into the elements of N_t and the elements of the different subtrees $(\tau, \sigma, x)_{\mathcal{L}}^u$ for $u \in \mathcal{L}_t$, and noting that $Z(t)$ is already $\mathcal{F}_t^{\mathcal{L}}$-measurable, we have

$$\mathbb{E}_P(Z'(t)|\mathcal{F}_t^{\mathcal{L}}) = Z(t) + \sum_{u \in \mathcal{L}_t} e^{-\lambda \rho_u} \mathbb{E}_P\left(Z^{(\tau,\sigma,x)_{\mathcal{L}}^u}(t)|\mathcal{F}_t^{\mathcal{L}}\right),$$

so that it suffices to show

$$\mathbb{E}_P\left(Z^{(\tau,\sigma,x)_{\mathcal{L}}^u}(t)|\mathcal{F}_t^{\mathcal{L}}\right) = 0.$$

Because $Z^{(\tau,\sigma,x)_{\mathcal{L}}^u}(t)$ is a P-martingale whose initial value $Z^{(\tau,\sigma,x)_{\mathcal{L}}^u}(\rho_u)$ is $\mathcal{F}_t^{\mathcal{L}}$-measurable, but whose displacements after ρ_u are independent of $\mathcal{F}_t^{\mathcal{L}}$ (by the Strong Markov Branching Property), we have

$$\begin{aligned}
\mathbb{E}_P\left(Z^{(\tau,\sigma,x)_{\mathcal{L}}^u}(t)|\mathcal{F}_t^{\mathcal{L}}\right) &= Z^{(\tau,\sigma,x)_{\mathcal{L}}^u}(\rho_u) \\
&= 0.
\end{aligned}$$

This completes the proof.

□

We note that within the proof of the last theorem, it was shown that

$$\mathbb{E}_P(Z'(t)|\mathcal{F}_t^{\mathcal{L}}) = Z(t). \tag{3.4}$$

This will be useful in the following corollary:

Corollary 3.2.0.1. *$Z(t)$ is a P-martingale.*

Proof. Making free use of the tower property of conditional expectation, and bearing in mind (3.4) and the fact that $Z'(t)$ is a $\mathcal{F}_t$-measurable P-martingale, we have

$$\begin{aligned}
\mathbb{E}_P(Z(t)|\mathcal{F}_s^{\mathcal{L}}) &= \mathbb{E}_P\left(\mathbb{E}_P(Z'(t)|\mathcal{F}_t^{\mathcal{L}})|\mathcal{F}_s^{\mathcal{L}}\right) \\
&= \mathbb{E}_P(Z'(t)|\mathcal{F}_s^{\mathcal{L}}) \\
&= \mathbb{E}_P\left(\mathbb{E}_P(Z'(t)|\mathcal{F}_s)|\mathcal{F}_s^{\mathcal{L}}\right) \\
&= \mathbb{E}_P(Z'(s)|\mathcal{F}_s^{\mathcal{L}}) \\
&= Z(s),
\end{aligned}$$

which completes the proof. □

Now, in view of Lemma 2.3.3.1, we know that a change of measure via $Z(t)$ on $\mathcal{F}_t^{\mathcal{L}}$ is equivalent to first changing the measure on $\widetilde{\mathcal{F}}_t^{\mathcal{L}}$ via $\widetilde{\zeta}(t)$, and then restricting this new measure to $\mathcal{F}_t^{\mathcal{L}}$, and likewise for a change of measure via $\zeta(t)$ on $\mathcal{G}_t^{\mathcal{L}}$. Since we know what changes $\widetilde{\zeta}$ will induce (see Section 2.3.4), we know what changes Z and ζ will induce as well.

Let us now fix our definitions of $\mathbb{Q}$, Q, and $\widetilde{Q}$:

Definition 3.2.0.7. *Hereafter, $\mathbb{Q}$ will be the measure defined on $\mathcal{G}_t^{\mathcal{L}}$ by*

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} := \frac{\zeta(t)}{\zeta(0)},$$

as was done in Section 3.1.

Definition 3.2.0.8. *Hereafter, Q will be the measure defined on $\mathcal{F}_t^{\mathcal{L}}$ by*

$$\frac{dQ_t}{dP_t} := \frac{Z(t)}{Z(0)}.$$

Definition 3.2.0.9. *Hereafter, $\widetilde{Q}$ will be the measure defined on $\widetilde{\mathcal{F}}_t^{\mathcal{L}}$ by*

$$\frac{d\widetilde{Q}_t}{d\widetilde{P}_t} := \frac{\widetilde{\zeta}(t)}{\widetilde{\zeta}(0)}.$$

We have now returned to the situation depicted in Figure 2.3 (save for the explicit normalization here). As was mentioned, we have a good intuitive picture of what kinds of alterations these changes of measure affect (see Section 2.3.4). Under $\widetilde{Q}$, for instance, the following three changes have been made to the distinguished path d:

1. the motion of $u \in d$ is now governed by L' (as under $\mathbb{Q}$; see Section 3.1), so that $node_d(t) \in N_t$ $\forall t \geq 0$ (i.e., the absorbing boundary is now inaccessible to all $u \in d$),
2. the rate of occurrence of fission events along d is increased from r to $(1+m)r$, and
3. the distribution of offspring $\forall u \in d$ is now governed by $\widetilde{Q}(A_u = k) = \frac{(1+k)p_k}{1+m}$ for $k = 0, 1, 2, \ldots$.

$Z(t)$ will induce similar changes, leading to a similar picture under Q, though in this setting we can no longer ask which particles belong to d.

We emphasize that under Q we are guaranteed one immortal line.

Now we will port the final major result over from Chapter 2–the spine decomposition for the process on $[0, K]$:

Theorem 3.2.0.7. *We have a spine decomposition for $Z(t)$:*

$$\mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty) = \sum_{v < node_d(t)} A_v e^{-mrS_v}\zeta(S_v) + e^{-mrt}\zeta(t).$$

Proof. It suffices to show that

$$\mathbb{E}_{\widetilde{Q}}(Z'(t)|\widetilde{\mathcal{G}}_\infty) = \mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty),$$

since we already have a spine decomposition for $Z'(t)$. To this end, note that

$$\mathbb{E}_{\widetilde{Q}}(Z'(t)|\widetilde{\mathcal{G}}_\infty) = \mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty) + \sum_{u\in\mathcal{L}_t} \mathbb{E}_{\widetilde{Q}}(e^{-\lambda\rho_u} Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\widetilde{\mathcal{G}}_\infty),$$

so that it is now sufficient to show

$$\mathbb{E}_{\widetilde{Q}}(e^{-\lambda\rho_u} Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\widetilde{\mathcal{G}}_\infty) = 0$$

$\forall u \in \mathcal{L}_t$. On a set of $\widetilde{Q}$-measure 1, where the distinguished path is guaranteed to remain within $(0, K]$, we have $\widetilde{\mathcal{G}}_\infty \subset \mathcal{F}^{\mathcal{L}}_\infty$, since $\mathcal{F}^{\mathcal{L}}_\infty$ already knows everything that happens within the boundaries, and the distinguished path remains within those boundaries. So we can write

$$\begin{aligned}\mathbb{E}_{\widetilde{Q}}(e^{-\lambda\rho_u} Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\widetilde{\mathcal{G}}_\infty) &= \mathbb{E}_{\widetilde{Q}}\left(\mathbb{E}_{\widetilde{Q}}\left(e^{-\lambda\rho_u} Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\mathcal{F}^{\mathcal{L}}_\infty\right)|\widetilde{\mathcal{G}}_\infty\right)\\ &= \mathbb{E}_{\widetilde{Q}}\left(e^{-\lambda\rho_u}\mathbb{E}_{\widetilde{Q}}\left(Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\mathcal{F}^{\mathcal{L}}_\infty\right)|\widetilde{\mathcal{G}}_\infty\right),\end{aligned}$$

so that it now suffices to show

$$\mathbb{E}_{\widetilde{Q}}(Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\mathcal{F}^{\mathcal{L}}_\infty) = 0.$$

Because on $Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)$ we stay away from the distinguished path, which is where $\widetilde{P}$ and $\widetilde{Q}$ differ, we will have

$$\mathbb{E}_{\widetilde{Q}}(Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\mathcal{F}^{\mathcal{L}}_\infty) = \mathbb{E}_{\widetilde{P}}(Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\mathcal{F}^{\mathcal{L}}_\infty)$$

$$= Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(\rho_u)$$

$$= 0,$$

where the argument that $\mathbb{E}_{\widetilde{P}}(Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(t)|\mathcal{F}^{\mathcal{L}}_{\infty}) = Z^{(\tau,\sigma,x)^u_{\mathcal{L}}}(\rho_u)$ is given in the proof of Theorem 3.2.0.6.

This completes the proof.

□

We are now ready to move on to our main result.

3.3 Proof of the Main Result

We begin this section with a theorem on the fate of the P-martingale $\frac{Z(t)}{Z(0)}$, and we will end this section by showing that the fate of this martingale determines the fate of the branching process.

Note that since $\frac{Z(t)}{Z(0)}$ is a non-negative martingale, it has a finite, almost sure limit by Doob's martingale convergence theorem; on a set of P-measure 1, we define

$$Z(\infty) := \lim_{t\to\infty} \frac{Z(t)}{Z(0)}.$$

For convenience, we remind the reader that

$$Z(t) = \sum_{u\in N_t} e^{\{\mu x_u(t) - \lambda t\}} \sin(b x_u(t)),$$

where $\lambda = mr - \frac{\mu^2}{2} - \frac{b^2}{2}$.

Let

$$\overline{Z} := \limsup_{t\to\infty} \frac{Z(t)}{Z(0)},$$

so $Z(\infty) = \overline{Z}$ P a.s.

We'll need the following important measure-theoretic result, a proof of which can be found in, for example, Durrett's book, [8]:

Theorem 3.3.0.8.

$$\overline{Z} = \infty \quad Q \ a.s. \iff \overline{Z} = 0 \quad P \ a.s.,$$
$$\overline{Z} < \infty \quad Q \ a.s. \iff \int \overline{Z} dP = 1.$$

Now we're ready to prove the first major component of our main result. The statement here is very similar to the statement of Proposition 8 in [11], while the proof is modeled on the proof of Theorem 13 in [16].

Theorem 3.3.0.9. *For $\lambda \neq 0$, the sign of λ determines the fate of $Z(t)$; in particular*

- *if $\lambda < 0$, then $Z(\infty) = 0$ P almost surely, while*
- *if $\lambda > 0$, then $Z(\infty)$ is the $L^1(P)$ limit of $\frac{Z(t)}{Z(0)}$ (implying that $\{Z(t)\}_{t \geq 0}$ is uniformly integrable).*

Proof. The proof is divided into two cases.

Case 1; $\lambda < 0$:

Under Q, where the distinguished path is guaranteed to survive, we have

$$\begin{aligned} Z(t) &\geq e^{\{\mu d(t) - \lambda t\}} \sin(b \cdot d(t)) \\ &= e^{-\lambda t} \left(e^{\mu d(t)} \sin(b \cdot d(t)) \right). \end{aligned}$$

Because $-\lambda > 0$, to show $\overline{Z} = \infty$, it is enough to show that $\exists \epsilon > 0$ such that $\forall T \geq 0 \ \exists t > T$ with

$$e^{\mu d(t)} \sin(b \cdot d(t)) > \epsilon \quad Q \text{ almost surely.}$$

This fails if and only if $\sin(b \cdot d(t)) \to 0$ as $t \to \infty$, which occurs if and only if $d(t) \to 0$ as $t \to \infty$. But since $d(t)$ is positive recurrent under Q (see Theorem 3.1.0.4), we have $Q(d(t) \to 0) = 0$.

So for $\lambda < 0$, we have shown $\overline{Z} = \infty$ Q almost surely, which implies $\overline{Z} = 0$ P almost surely. Therefore $Z(\infty) = 0$ P almost surely, as well.

Case 2; $\lambda > 0$:

We'll begin with a few lemmas:

Lemma 3.3.0.2. *If $\mathbb{E}_P(A_\emptyset \log^+ A_\emptyset) < \infty$, then*

$$\lim_{n\to\infty} A_{d_{n-1}} e^{-cn} = 0 \quad \forall c > 0 \quad \widetilde{Q} \textit{ almost surely.}$$

Proof. Assume $\mathbb{E}_P(A_\emptyset \log^+ A_\emptyset) < \infty$.

Recall that $P(A_u = k) = p_k \ \forall u \in \tau,\ k = 0, 1, 2, \ldots,$ while

$$\widetilde{Q}(A_u = k) = \begin{cases} p_k & \text{if } u \notin d \\ \frac{(1+k)p_k}{1+m} & \text{if } u \in d. \end{cases}$$

Since $\emptyset \in d$, we can write

$$\begin{aligned} \mathbb{E}_{\widetilde{Q}}(\log^+ A_\emptyset) &= \sum_{k=2}^{\infty} (\log^+ k) \frac{(1+k)p_k}{1+m} \\ &= \frac{1}{1+m} \underbrace{\sum_{k=2}^{\infty} (\log k) p_k}_{<\infty} + \frac{1}{1+m} \underbrace{\sum_{k+2}^{\infty} (k \log k) p_k}_{<\infty}, \end{aligned}$$

where the first sum is finite since $\mathbb{E}_P(\log^+ A_\emptyset) < \mathbb{E}_P(A_\emptyset) < \infty$, and for the second sum we have assumed $\mathbb{E}_P(A_\emptyset \log^+ A_\emptyset) < \infty$.

We now know $\mathbb{E}_{\widetilde{Q}}(\log^+ A_\emptyset) < \infty$, which implies

$$\int_0^\infty \widetilde{Q}(\log^+ A_\emptyset > y) dy < \infty.$$

Letting $y = c'z$ for $c' > 0$, we get

$$\int_0^\infty \widetilde{Q}(\log^+ A_\emptyset > c'z)dz < \infty,$$

which implies (recalling that the A_{d_n} are i.i.d.),

$$\sum_{n=1}^\infty \widetilde{Q}(\log^+ A_{d_{n-1}} > c'n) < \infty,$$

so Borel-Cantelli 1 tells us

$$\widetilde{Q}(\log^+ A_{d_{n-1}} > c'n \text{ i.o.}) = 0.$$

So on a set of $\widetilde{Q}$-measure 1, we have $\frac{\log^+ A_{d_{n-1}}}{n} \leq c'$ eventually for all n. Because $c' > 0$ was arbitrary, and because the terms here are non-negative, this implies

$$\lim_{n\to\infty} \frac{\log^+ A_{d_{n-1}}}{n} = 0 \quad \widetilde{Q} \text{ almost surely.}$$

So, letting $0 < \epsilon < 1$, $\exists N_\epsilon$ such that $n > N_\epsilon$ implies $\frac{\log^+ A_{d_{n-1}}}{n} < \epsilon$, which leads to the following chain of implications for $n > N_\epsilon$:

$$\begin{aligned} \frac{\log^+ A_{d_{n-1}}}{n} < \epsilon &\implies A_{d_{n-1}} < e^{n\epsilon} \\ &\implies \frac{A_{d_{n-1}}}{e^n} < e^{-n(1-\epsilon)} \underset{n\to\infty}{\to} 0 \\ &\implies \lim_{n\to\infty} A_{d_{n-1}} e^{-n} = 0 \quad \widetilde{Q} \text{ almost surely.} \end{aligned}$$

So we've shown that

$$\lim_{n\to\infty} A_{d_{n-1}} e^{-n} = 0 \quad \widetilde{Q} \text{ almost surely.}$$

Now we'll show that, $\forall c > 0$,

$$\lim_{n\to\infty} A_{d_{n-1}} e^{-cn} = 0 \quad \widetilde{Q} \text{ almost surely.}$$

If $c \geq 1$, then $A_{d_{n-1}} e^{-n} \geq A_{d_{n-1}} e^{-cn}$, and we're done. So take $c \in (0,1)$. For any $\epsilon > 0$,

$$A_{d_{n-1}} e^{-n} < \epsilon \implies (A_{d_{n-1}})^c e^{-cn} < \epsilon^c.$$

We may assume $A_{d_{n-1}} \geq 1$ since $A_{d_{n-1}} \in \{0, 1, 2, \dots\}$ in general, and if the limit of the non-zero elements of a sequence of non-negative numbers is 0, then the limit of that sequence will be 0. So now we have

$$\begin{aligned} A_{d_{n-1}} e^{-cn} &\leq \frac{\epsilon^c}{(A_{d_{n-1}})^{c-1}} \\ &\leq \epsilon^c, \end{aligned}$$

showing

$$\lim_{n\to\infty} A_{d_{n-1}} e^{-cn} = 0 \quad \widetilde{Q} \text{ almost surely } \forall c > 0.$$

This completes the proof of this lemma.

□

Lemma 3.3.0.3. *If*

$$\lim_{n\to\infty} A_{d_{n-1}} e^{-cn} = 0 \quad \widetilde{Q} \textit{ almost surely } \forall c > 0,$$

then $\exists \delta > 0$ *such that eventually for all* n *we will have*

$$A_{d_{n-1}} e^{-cn} \leq e^{-\delta n} \quad \widetilde{Q} \textit{ almost surely.}$$

Proof. Suppose no such δ exists. Then, $\forall \delta > 0$,

$$A_{d_{n-1}}e^{-cn} > e^{-\delta n} \text{ infinitely often, } \widetilde{Q} \text{ almost surely.}$$

Fix $c > 0$ and choose δ such that $0 < \delta < c$. Now note that the above assures us that

$$A_{d_{n-1}}e^{-(c-\delta)n} > 1 \text{ infinitely often,} \widetilde{Q} \text{ almost surely,}$$

but this contradicts Lemma 3.3.0.2.

This completes the proof of this lemma. □

Now we begin on the proof of this case, where $\lambda > 0$. The first step will be to show

$$\limsup_{t\to\infty} \mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty) < \infty \quad \widetilde{Q} \text{ almost surely.} \tag{3.5}$$

To that end, consider the spine decomposition of $Z(t)$:

$$\begin{aligned}\mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty) &= \sum_{i=1}^{n_t} A_{d_{i-1}}e^{\{\mu\cdot d(S_{d_{i-1}})-\lambda S_{d_{i-1}}\}}\sin(b\cdot d(S_{d_{i-1}})) + e^{\{\mu\cdot d(t)-\lambda t\}}\sin(b\cdot d(t)) \\ &\leq e^{\mu K}\left(\sum_{i=1}^{n_t} A_{d_{i-1}}e^{-\lambda S_{d_{i-1}}} + \underbrace{e^{-\lambda t}}_{\to 0}\right).\end{aligned}$$

So to show (3.5), it is enough to show

$$\limsup_{t\to\infty} \sum_{i=1}^{n_t} A_{d_{i-1}}e^{-\lambda S_{d_{i-1}}} < \infty \quad \widetilde{Q} \text{ almost surely.} \tag{3.6}$$

Recall that since $S_{d_{i-1}}$ is the fission time of particle d_{i-1} (an element of the distinguished path), we have

$$S_{d_{i-1}} = \sigma_{d_0} + \cdots + \sigma_{d_{i-1}},$$

where the lifespans $\{\sigma_{d_k}\}$ form a collection of i.i.d. exponential random variables with common $\widetilde{Q}$-mean $\mathbb{E}_{\widetilde{Q}}(\sigma_{d_k}) = \frac{1}{(1+m)r}$. So the strong law of large numbers gives us

$$\frac{S_{d_{i-1}}}{i} = \frac{\sigma_{d_0} + \cdots + \sigma_{d_{i-1}}}{i} \underset{i\to\infty}{\to} \frac{1}{(1+m)r} \quad \widetilde{Q} \text{ almost surely.}$$

Fix ϵ such that $0 < \epsilon < \frac{1}{(1+m)r}$. Now we know $\exists N_\epsilon$ such that $i > N_\epsilon \implies$

$$\frac{S_{d_{i-1}}}{i} > \frac{1}{(1+m)r} - \epsilon.$$

so that we now have (for all t large enough to have $n_t \geq N_\epsilon + 1$)

$$\begin{aligned}
\sum_{i=1}^{n_t} A_{d_{i-1}} e^{-\lambda S_{d_{i-1}}} &< \underbrace{\sum_{i=1}^{N_\epsilon} A_{d_{i-1}} e^{-\lambda S_{d_{i-1}}}}_{:=C_1} + \sum_{i=N_\epsilon+1}^{n_t} A_{d_{i-1}} e^{-\lambda\left(\frac{1}{(1+m)r}-\epsilon\right)i} \\
&\leq C_1 + \sum_{i=1}^{\infty} A_{d_{i-1}} e^{-\lambda\left(\frac{1}{(1+m)r}-\epsilon\right)i} \\
&= C_1 + \sum_{i=1}^{\infty} A_{d_{i-1}} e^{-ci},
\end{aligned}$$

where $c := \lambda\left(\frac{1}{(1+m)r} - \epsilon\right) > 0$. Our long-ago assumption that $\mathbb{E}_P(A_\emptyset \log^+ A_\emptyset) < \infty$, together with Lemma 3.3.0.2 and Lemma 3.3.0.3, tell us that $\exists\delta > 0$ for which $\exists N_\delta$ such that

$$A_{d_{i-1}} e^{-ci} \leq e^{-\delta i} \quad \forall i > N_\delta.$$

So that we can now write

$$\sum_{i=1}^{\infty} A_{d_{i-1}} e^{-ci} \leq \underbrace{\sum_{i=1}^{N_\delta} A_{d_{i-1}} e^{-ci}}_{\text{finite } \widetilde{Q} \text{ almost surely}} + \underbrace{\sum_{i=N_\delta+1}^{\infty} e^{-\delta i}}_{\text{a finite geometric series}}.$$

We have now verified that (3.6) holds, which gives us (3.5). Now, using Fatou's Lemma for conditional expectation, we have

$$\infty > \limsup_{t\to\infty} \mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty)$$

$$\geq \liminf_{t\to\infty} \mathbb{E}_{\widetilde{Q}}(Z(t)|\widetilde{\mathcal{G}}_\infty)$$
$$\geq \mathbb{E}_{\widetilde{Q}}(\liminf_{t\to\infty} Z(t)|\widetilde{\mathcal{G}}_\infty) \quad \widetilde{Q} \text{ almost surely.}$$

The Doob-Blackwell Theorem (see, for instance, Theorem 2.8 in Bhattacharya and Waymire's [2]) tells us that, since $\mathbb{R}$ is a Polish space, we can choose a regular version of the above conditional distribution, so that $\widetilde{Q}(\cdot|\widetilde{\mathcal{G}}_\infty)$ will be a probability measure, and we'll have

$$\mathbb{E}_{\widetilde{Q}}(\liminf_{t\to\infty} Z(t)|\widetilde{\mathcal{G}}_\infty) < \infty \ \widetilde{Q} \text{ almost surely}$$

$$\implies \widetilde{Q}(\liminf_{t\to\infty} Z(t) < \infty|\widetilde{\mathcal{G}}_\infty) = 1. \tag{3.7}$$

Apply $\mathbb{E}_{\widetilde{Q}}(\cdot)$ to both sides of (3.7) to get

$$\widetilde{Q}(\liminf_{t\to\infty} Z(t) < \infty) = 1,$$

which, since $Z(t)$ is $\mathcal{F}_t$-measurable, implies

$$Q(\liminf_{t\to\infty} Z(t) < \infty) = 1. \tag{3.8}$$

Now, since $\frac{dQ_t}{dP_t} = Z(t)$, we know that $\frac{1}{Z(t)}$ is a non-negative Q-martingale, so has a non-negative, finite limit almost surely by Doob's martingale convergence theorem; i.e., on a set of Q-measure 1 we have

$$0 \leq \lim_{t\to\infty} \frac{1}{Z(t)} < \infty.$$

Because we have equality of the two events

$$\left[\lim_{t\to\infty} \frac{1}{Z(t)} = 0\right] = \left[\liminf_{t\to\infty} Z(t) = \infty\right],$$

(3.8) tells us

$$0 < \lim_{t\to\infty} \frac{1}{Z(t)} \quad Q \text{ almost surely,}$$

so that

$$\limsup_{t\to\infty} Z(t) < \infty \quad Q \text{ almost surely.}$$

Now Theorem 3.3.0.8 gives us

$$\int \overline{Z} dP = 1,$$

so that

$$\int Z(\infty) dP = 1$$

also.

Now a final lemma will complete the proof. This lemma is borrowed from Kusolitsch, [15]. It says that almost sure pointwise convergence and convergence in mean is enough for L^1 convergence.

Lemma 3.3.0.4. *If a sequence of L^1 integrable functions f_n converges a.e. to an L^1 integrable function f and $\lim_n \|f_n\|_1 = \|f\|_1$ holds true, then $\lim_n \|f_n - f\|_1 = 0$.*

Proof. Consider the functions

$$f_n^* = \begin{cases} f_n, & |f_n| \le |f|, \\ |f| sgn(f_n) & |f_n| > |f|, \end{cases}$$

which are dominated by the L^1 integrable $|f|$ and converge to f a.e. So the functions $|f_n^* - f|$ are dominated by $2|f|$ and vanish a.e., and the dominated convergence theorem yields

$$\lim_n \int |f_n^*| = \int |f|$$

and also

$$\lim_n \int |f_n^* - f| = 0.$$

By definition f_n^* always has the same sign as f_n and $|f_n^*| \leq |f_n|$. So one gets $|f_n - f_n^*| = |f_n| - |f_n^*|$, and

$$\int |f_n - f_n^*| = \int |f_n| - \int |f_n^*|.$$

Since both integrals on the right hand side converge to $\int |f| < \infty$, this yields the conclusion.

□

This completes the proof of Case 2, and of the theorem.

□

The next theorem will show us that the fate of the martingale $Z(t)$ is tied to the fate of the process $X(t)$, so that the sign of λ decides the question of extinction or persistence. Both the statement and proof are adapted from Proposition 9 of [11].

Theorem 3.3.0.10. *For $\lambda \neq 0$, the two events $[Z(\infty) = 0]$ and $[t_\Omega < \infty]$ agree P almost surely.*

Proof. If $t_\Omega < \infty$, then $|N_t| = 0$ for all t large enough, and because $Z(t)$ will be an empty sum for all such t, certainly we have $Z(\infty) = 0$ in this case. So

$$[t_\Omega < \infty] \subset [Z(\infty) = 0].$$

We will complete the proof by showing

$$P\left([Z(\infty) = 0] \cap [t_\Omega = \infty]\right) = 0.$$

Assume $\lambda < 0$. In this case, Theorem 3.3.0.9 tells us that $Z(\infty) = 0$ P almost surely. Assume now that $t_\Omega = \infty$, and we will demonstrate a contradiction on a set of P-measure 1.

Because $t_\Omega = \infty$, we know $|N_t| \not\to 0$ as $t \to \infty$. So to have $Z(\infty) = 0$, we must have the summands

$$\exp\{\mu x_u(t) - \lambda t\} \sin(b x_u(t))$$

going to zero $\forall u \in N_t$. But with $-\lambda > 0$, this can only happen if $x_u(t) \to 0$ for all such u. In particular, for all $\epsilon > 0$ the branching process must eventually enter $[0, \epsilon]$ and survive there indefinitely, with 0 absorbing and ϵ reflecting.

Let $P^{[0,\epsilon]}$ be the distribution of such a branching process on $[0, \epsilon]$ beginning from a single particle at $x_0 \in (0, \epsilon]$. It is enough to show that, for ϵ small enough, such a process will become extinct $P^{[0,\epsilon]}$ almost surely.

Let $N_t^{[0,\epsilon]}$ be the set of particles alive at time t under $P^{[0,\epsilon]}$, and let

$$X^{[0,\epsilon]}(t) := \sum_{u \in N_t^{[0,\epsilon]}} \delta_{x_u(t)} \quad t \geq 0$$

signify the branching process itself.

For the analogous process on $[-\delta, \epsilon + \delta]$ with $\delta > 0$, we'll define $P^{[-\delta,\epsilon+\delta]}$, $N_t^{[-\delta,\epsilon+\delta]}$, and $X^{[-\delta,\epsilon+\delta]}(t)$ similarly.

Under $P^{[-\delta,\epsilon+\delta]}$, we can define

$$N_t^{[-\delta,\epsilon+\delta]}|_{(0,\epsilon]} := \{u \in N_t^{[-\delta,\epsilon+\delta]} : \forall s \leq t, \forall v \in N_s^{[-\delta,\epsilon+\delta]} \text{ s.t. } v \leq u, \text{ we have } x_v(s) \in (0, \epsilon]\},$$

the set of particles in $N_t^{[-\delta,\epsilon+\delta]}$ whose ancestral lines have not left $(0, \epsilon]$ by time t. This allows us to define

$$X^{[-\delta,\epsilon+\delta]}(t)|_{(0,\epsilon]} := \sum_{u \in N_t^{[-\delta,\epsilon+\delta]}|_{(0,\epsilon]}} \delta_{x_u(t)} \quad t \geq 0.$$

It is clear that $\{X^{[-\delta,\epsilon+\delta]}(t)|_{(0,\epsilon]}\}_{t\geq 0}$ under $P^{[-\delta,\epsilon+\delta]}$ has the same law as $\{X^{[0,\epsilon]}(t)\}_{t\geq 0}$ under $P^{[0,\epsilon]}$.

Suppose for a moment that $P^{[0,K']}$ is the distribution of one of these processes on $[0, K']$, and recall that the parameter $\lambda = \lambda(K')$ in the definition of Z is defined as $\lambda(K') = mr - \frac{\mu^2}{2} - \frac{b^2(K')}{2}$ where $b(K')$ satisfies $\frac{\pi}{2K'} < b(K') < \frac{\pi}{K'}$. So, by letting $K' \to 0$, we can make $b(K')$ large, and thereby make $\lambda(K')$ negative.

So we can choose δ and ϵ small enough so that $\lambda(\epsilon + 2\delta) < 0$, and under $P^{[-\delta,\epsilon+\delta]}$ we can define (by spatially shifting our process to the right by δ)

$$Z^{[-\delta,\epsilon+\delta]}(t) := \sum_{u \in N_t^{[-\delta,\epsilon+\delta]}} e^{\{\mu(x_u(t)+\delta)-\lambda(\epsilon+2\delta)t\}} \sin(b(\epsilon + 2\delta)(x_u(t) + \delta)),$$

and this will be the type of martingale treated in Theorem 3.3.0.9.

Now consider the contribution to this martingale from the elements of $N_t^{[-\delta,\epsilon+\delta]}|_{(0,\epsilon]}$: the set $N_t^{[-\delta,\epsilon+\delta]}|_{(0,\epsilon]}$ is non-empty by our assumption that the $P^{[0,\epsilon]}$-branching diffusion persists, and after our spatial shift to the right by δ, the terms $x_u(t) + \delta$ for such u will occupy $(\delta, \epsilon + \delta]$. So we can bound the associated summands from below: $\exists c > 0$ such that $\forall v \in N_t^{[-\delta,\epsilon+\delta]}|_{(0,\epsilon]}$, we have

$$c \leq e^{\{\mu(x_v(t)+\delta)\}} \sin(b(\epsilon + 2\delta)(x_v(t) + \delta)).$$

Now we have

$$\begin{aligned} Z^{[-\delta,\epsilon+\delta]}(t) &\geq c \left| N_t^{[-\delta,\epsilon+\delta]}|_{(0,\epsilon]} \right| e^{-\lambda(\epsilon+2\delta)t} \\ &\geq ce^{-\lambda(\epsilon+2\delta)t}, \end{aligned}$$

since $\left|N_t^{[-\delta,\epsilon+\delta]}|_{(0,\epsilon]}\right| \geq 1$. Because $-\lambda(\epsilon+2\delta) > 0$, we see that $Z^{[-\delta,\epsilon+\delta]}(\infty) = \infty$ $P^{[-\delta,\epsilon+\delta]}$ almost surely, but this contradicts Doob's martingale convergence theorem since $Z^{[-\delta,\epsilon+\delta]}(t)$ is a positive martingale.

So we've seen that for $\lambda < 0$ the martingale limit $Z(\infty)$ cannot be 0 on survival.

Now assume $\lambda > 0$. Assume also, expecting contradiction, that the event

$$[Z(\infty) = 0] \cap [t_\Omega = \infty] \tag{3.9}$$

has positive probability. Let P_x be the distribution of our branching process on $[0, K]$ with initial value at $x \in (0, K]$, and define

$$z(x) := P_x(Z(\infty) = 0).$$

Now define $M_\infty := 1_{\{Z(\infty)=0\}}$, and set

$$\begin{aligned} M_t :&= \mathbb{E}_{P_x}(M_\infty | \mathcal{F}_t^{\mathcal{L}}) \\ &= \prod_{u \in N_t} z(x_u(t)), \end{aligned}$$

where the second equality follows from the strong Markov branching property. Now $\{M_t\}_{t \geq 0}$

- is a martingale by the tower property of conditional expectation,
- is uniformly integrable since it's uniformly bounded by 1, and
- has a.s. limit equal to M_∞.

On event (3.9) we clearly have $M_\infty = 1$ P_x almost surely, which requires (since $z(x) < 1$ $\forall x \in (0, K]$ by Theorem 3.3.0.9, recalling that $\lambda > 0$ here) that $x_u(t) \to 0$ as $t \to \infty$. But as we have already seen in the first half of this proof, we cannot have the process persisting on $[0, \epsilon]$ indefinitely for all $\epsilon > 0$. We have reached a contradiction.

So for $\lambda > 0$, the martingale limit cannot be zero on condition of survival. This completes the proof.

□

We are now just a stone's throw from the full demonstration of our main result. Consider λ, the critical parameter whose sign determines extinction or persistence:

$$\lambda = mr - \frac{\mu^2}{2} - \frac{b^2}{2}.$$

Because $\frac{b^2}{2}$ is positive, we can see that if we have

$$mr - \frac{\mu^2}{2} \leq 0,$$

i.e.,

$$\mu \geq \sqrt{2mr},$$

this will guarantee $\lambda < 0$, and extinction.

If, however, $\mu < \sqrt{2mr}$, then the magnitude of b will decide the sign of λ and the fate of the process. Under the assumption that $\mu < \sqrt{2mr}$, we have

$$b > \sqrt{2mr - \mu^2} \implies \text{extinction is inevitable,} \tag{3.10}$$

while

$$b < \sqrt{2mr - \mu^2} \implies \text{persistence is possible.} \tag{3.11}$$

So our critical value for b is $b_0 := \sqrt{2mr - \mu^2}$; we can find the associated critical interval length K_0 through the relation

$$\tan(b_0 K_0) = -\frac{b_0}{\mu},$$

or

$$\tan(\sqrt{2mr-\mu^2}K_0) = -\frac{\sqrt{2mr-\mu^2}}{\mu}.$$

Using the arctan function (and adding π to arrive on the correct branch of arctan; see Figure 3.1), we have

$$K_0 = \frac{\arctan\left(-\frac{\sqrt{2mr-\mu^2}}{\mu}\right)+\pi}{\sqrt{2mr-\mu^2}}.$$

All that remains is to show that b decreases monotonically to 0 as $K \to \infty$. A simple, calculus-based lemma will suffice:

Lemma 3.3.0.5. *The value $b = b(K)$ decreases monotonically to 0 as $K \to \infty$.*

Proof. Clearly $b \to 0$ as $K \to \infty$, since $\frac{\pi}{2K} < b < \frac{\pi}{K}$.

To show monotonicity, we begin with the defining equation for b,

$$\tan(bK) = -\frac{b}{K},$$

and differentiate both sides with respect to K:

$$\begin{aligned}
\frac{d}{dK}\tan(bK) &= \frac{d}{dK}\left(-\frac{b}{\mu}\right) \\
\sec^2(bK)\left(\frac{db}{dK}K + b\right) &= -\frac{1}{\mu}\frac{db}{dK} \\
\left(K\sec^2(bK)+\frac{1}{\mu}\right)\frac{db}{dK} &= -b\sec^2(bK) \\
\frac{db}{dK} &= -\frac{b\sec^2(bK)}{K\sec^2(bK)+\frac{1}{\mu}} < 0.
\end{aligned}$$

This completes the proof of the lemma.

□

We have now shown our main result, Theorem 1.1.0.1.

4 A Proof of Kesten's Critical Drift Speed

We'll now use the techniques developed and deployed to handle branching Brownian motion on $[0, K]$ with one absorbing and one reflecting boundary to give a relatively simple, direct proof of Kesten's critical drift speed for branching Brownian motion on $[0, \infty)$, first stated in [12]. Before now, this author has been unable to find a direct proof in the literature. Kesten himself says, in [12], "So far we have only an ugly and complicated proof...and we shall therefore not prove Theorem 1.1 here."

Note that while Kesten assumes a finite second factorial moment ($\mathbb{E}(A_\emptyset(A_\emptyset - 1)) < \infty$ in our notation), our assumption of a Kesten-Stigum condition ($\mathbb{E}(A_\emptyset \log^+ A_\emptyset) < \infty$) is weaker.

We'll need to reclaim and reset some notation. Let $X(t)$ be branching Brownian motion on $[0, \infty)$ initiated from a single particle at $x_0 \in (0, \infty)$, where the movement of each particle is governed by

$$Lf(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) - \mu\frac{d}{dx}f(x) \quad x \in (0, \infty), \tag{4.1}$$

$$f(0+) = 0, \tag{4.2}$$

with $\mu > 0$. Note the absorption at 0. Let P be the distribution of X, and let A_u, m, r, σ_u, S_u, N_t, $x_u(t)$, $d(t)$, n_t, and t_Ω be as before. Also let the various sigma algebras and measures be defined for this process, and let them be denoted as before.

We will show the following:

Theorem 4.0.0.11. *The following criterion distinguishes assured extinction from possible persistence when $\mathbb{E}\left(A_\emptyset \log^+ A_\emptyset\right) < \infty$:*

- *If $\mu \geq \sqrt{2mr}$, then $P(t_\Omega < \infty) = 1$.*
- *If $\mu < \sqrt{2mr}$, then $P(t_\Omega = \infty) > 0$.*

Note: as before, we'll actually show more than just $P(t_\Omega = \infty) > 0$ for the case where $\mu < \sqrt{2mr}$; we'll actually show the uniform integrability of our key branching-process martingale, $Z(t)$. But we prefer the simplicity of this statement.

Proof. Begin by noticing that $h(x) = xe^{\mu x}$ satisfies

$$Lh(x) = \left(-\frac{\mu^2}{2}\right) h(x),$$

and

$$h(0) = 0.$$

So we have a non-negative eigenfunction for (4.1) and (4.2). As before, we can create a single-particle martingale from h: if $\xi(t)$ gives the position of a single particle governed by (4.1) and (4.2), and $\mathbb{P}$ is its distribution, then

$$\zeta(t) := \xi(t)e^{\left(\mu\xi(t) - \left(-\frac{\mu^2}{2}\right)t\right)}$$

is a $\mathbb{P}$-martingale. Lemma 3.1.0.1 tells us what the infinitesimal generator for ξ will be after a change of measure via $\frac{\zeta(t)}{\zeta(0)}$:

$$\begin{aligned} L' &= L + \frac{h'(x)}{h(x)}\frac{d}{dx} \\ &= \frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}, \end{aligned}$$

so that ξ will be a Bessel-3 process after the measure change.

The single-particle martingale gives rise to an additive branching-process martingale in the usual way:

$$Z(t) := \sum_{u \in N_t} e^{-mrt} x_u(t) e^{\left(\mu x_u(t) - \left(-\frac{\mu^2}{2}\right)t\right)}$$

$$= \sum_{u \in N_t} x_u(t) e^{\left(\mu x_u(t) - \left(mr - \frac{\mu^2}{2}\right)t\right)}$$

$$= \sum_{u \in N_t} x_u(t) e^{(\mu x_u(t) - \lambda t)},$$

where we've taken $\lambda := mr - \frac{\mu^2}{2}$. Let $Z(\infty) = \lim_{t\to\infty} Z(t) \geq 0$, which exists and is finite P almost surely by Doob's martingale convergence theorem.

Now define Q on $\mathcal{F}_t^{\mathcal{L}}$ via

$$\frac{dQ_t}{dP_t} := \frac{Z(t)}{Z(0)}$$

(where $P_t = P|_{\mathcal{F}_t^{\mathcal{L}}}$, etc.), and Section 2.3.4 tells us what kind of behavior to expect from X under Q. Along a randomly chosen distinguished path d

- the motion will be governed by L' (instead of by L),
- the rate of fission will be increased from r to $(1+m)r$,
- the offspring distribution will be size-biased to $Q(A_u = k) = \frac{(1+k)p_k}{1+m}$ for $u \in d$.

No particle not of this line will be affected.

It is a simple matter, using the techniques employed in the proof of Theorem 3.1.0.4, to show that, for $d(t)$ under Q, the boundary point 0 is inaccessible.

Because $d(t)$ is a Bessel-3 process under Q, Theorem 3.2 of [21] tells us that $\forall \epsilon_1 > 0\ \exists T_1 > 0$ such that $t > T_1 \implies$

$$t^{\frac{1}{2}-\epsilon_1} < d(t) < t^{\frac{1}{2}+\epsilon_1} \quad Q \text{ almost surely.}$$

In particular, this says $d(t) \to \infty$ as $t \to \infty$.

Consider the case where $\lambda \leq 0$ (i.e. where $\mu \geq \sqrt{2mr}$). Because the distinguished path is guaranteed to persist under Q, and because the summands are all non-negative, we have

$$Z(t) = \sum_{u \in N_t} x_u(t) e^{(\mu x_u(t) - \lambda t)}$$
$$\geq d(t) e^{(\mu d(t) - \lambda t)}.$$

Now since $e^{-\lambda t} \geq 1$ and $d(t) \to \infty$ Q almost surely, we see that

$$Z(\infty) = \infty \quad Q \text{ almost surely,}$$

so that Theorem 3.3.0.8 tells us

$$Z(\infty) = 0 \quad P \text{ almost surely.}$$

So we have shown

$$\mu \geq \sqrt{2mr} \implies Z(\infty) = 0 \quad P \text{ almost surely.}$$

Now consider the case where $\lambda > 0$ (i.e. where $\mu < \sqrt{2mr}$). Using the spine decomposition, we have

$$\mathbb{E}_{\widetilde{Q}}\left(Z(t) | \widetilde{\mathcal{G}}_\infty\right) = \sum_{i=1}^{n_t} A_{d_{i-1}} d(S_{d_{i-1}}) e^{\left(\mu d(S_{d_{i-1}}) - \lambda S_{d_{i-1}}\right)} + d(t) e^{(\mu d(t) - \lambda t)}.$$

Let $0 < \epsilon_1 < \frac{1}{2}$ and T_1 be as before, and note that since $t^{\frac{1}{2}+\epsilon_1 - 1} \to 0$ as $t \to \infty$, we may pick $0 < \epsilon_2 < \frac{\lambda}{\mu}$ and be assured that $\exists T_2 > 0$ such that $t > T_2 \implies t^{\frac{1}{2}+\epsilon_1-1} < \epsilon_2$. Now let $t > \max\{T_1, T_2\}$, and we have

$$\mu d(t) - \lambda t < \mu t^{\frac{1}{2}+\epsilon_1} - \lambda t$$
$$= -\left(\lambda - \mu t^{\frac{1}{2}+\epsilon_1 - 1}\right) t$$
$$< -\left(\lambda - \mu \epsilon_2\right) t,$$

$\widetilde{Q}$ almost surely. Note that $\lambda - \mu\epsilon_2 > 0$. Now we can assert that

$$d(t)e^{(\mu d(t) - \lambda t)} < t^{\frac{1}{2}+\epsilon_1} e^{-(\lambda - \mu\epsilon_2)t} \to 0 \quad \widetilde{Q} \text{ almost surely}$$

as $t \to \infty$. This shows us that the question of whether

$$\mathbb{E}_{\widetilde{Q}}\left(Z(t)|\widetilde{\mathcal{G}}_\infty\right)$$

is infinite or finite is decided by the question of whether

$$\sum_{i=1}^{n_t} A_{d_{i-1}} d(S_{d_{i-1}}) e^{\left(\mu d(S_{d_{i-1}}) - \lambda S_{d_{i-1}}\right)}$$

is infinite or finite.

Recalling that $S_{d_{i-1}} = \sigma_{d_0} + \cdots + \sigma_{d_{i-1}}$, the strong law of large numbers assures us that

$$\frac{S_{d_{i-1}}}{i} = \frac{\sigma_{d_0} + \cdots + \sigma_{d_{i-1}}}{i} \to \frac{1}{(1+m)r} \quad \widetilde{Q} \text{ almost surely}$$

as $i \to \infty$. Pick $0 < \epsilon_3 < \frac{1}{(1+m)r}$, and let N_3 be such that $i > N_3 \implies$

$$0 < \frac{1}{(1+m)r} - \epsilon_3 < \frac{S_{d_{i-1}}}{i} < \frac{1}{(1+m)r} + \epsilon_3.$$

Now let t be large enough to have $n_t > N_3$, and we'll have

$$\sum_{i=1}^{n_t} A_{d_{i-1}} d(S_{d_{i-1}}) e^{\left(\mu d(S_{d_{i-1}}) - \lambda S_{d_{i-1}}\right)} \leq \sum_{i=1}^{N_3} A_{d_{i-1}} d(S_{d_{i-1}}) e^{\left(\mu d(S_{d_{i-1}}) - \lambda S_{d_{i-1}}\right)} + \ldots$$
$$\cdots + \sum_{i=N_3+1}^{n_t} A_{d_{i-1}} d(S_{d_{i-1}}) e^{\left(\mu d(S_{d_{i-1}}) - \lambda S_{d_{i-1}}\right)}.$$

The first sum on the right hand side is certainly finite $\widetilde{Q}$ almost surely, it being a finite sum of almost surely finite terms. For the second sum, we'll be taking $t > \max\{T_1, T_2\}$ again, and we'll have:

$$\sum_{i=N_3+1}^{n_t} A_{d_{i-1}} d(S_{d_{i-1}}) e^{\left(\mu d(S_{d_{i-1}})-\lambda S_{d_{i-1}}\right)} \leq \sum_{i=N_3+1}^{n_t} A_{d_{i-1}} \left(S_{d_{i-1}}\right)^{\frac{1}{2}+\epsilon_1} e^{-(\lambda-\mu\epsilon_2)S_{d_{i-1}}}$$
$$\leq \sum_{i=N_3+1}^{n_t} A_{d_{i-1}} \left(\left(\frac{1}{(1+m)r}+\epsilon_3\right) i\right)^{\frac{1}{2}+\epsilon_1} e^{-(\lambda-\mu\epsilon_2)\left(\frac{1}{(1+m)r}-\epsilon_3\right)i}$$
$$\leq \sum_{i=1}^{\infty} A_{d_{i-1}} \left(\left(\frac{1}{(1+m)r}+\epsilon_3\right) i\right)^{\frac{1}{2}+\epsilon_1} e^{-(\lambda-\mu\epsilon_2)\left(\frac{1}{(1+m)r}-\epsilon_3\right)i}.$$

Now, taking $c_1 := \frac{1}{(1+m)r} + \epsilon_3$ and $c_2 := (\lambda - \mu\epsilon_2)\left(\frac{1}{(1+m)r} - \epsilon_3\right) > 0$, and making use of Lemmas 3.3.0.2 and 3.3.0.3 to find a N_4 such that $n > N_4 \implies A_{d_{n-1}} e^{-c_2 n} \leq e^{-\delta n}$ $\widetilde{Q}$ almost surely for some $\delta > 0$, we see

$$\sum_{i=1}^{\infty} A_{d_{i-1}} (c_1 i)^{\frac{1}{2}+\epsilon_1} e^{-c_2 i} = \sum_{i=1}^{N_4} A_{d_{i-1}} (c_1 i)^{\frac{1}{2}+\epsilon_1} e^{-c_2 i} + \sum_{i=N_4+1}^{\infty} A_{d_{i-1}} (c_1 i)^{\frac{1}{2}+\epsilon_1} e^{-c_2 i}$$
$$\leq \sum_{i=1}^{N_4} A_{d_{i-1}} (c_1 i)^{\frac{1}{2}+\epsilon_1} e^{-c_2 i} + \sum_{i=N_4+1}^{\infty} (c_1 i)^{\frac{1}{2}+\epsilon_1} e^{-\delta i}.$$

Again, the finite sum is certainly finite $\widetilde{Q}$ almost surely. The infinite sum is also almost surely finite, since

$$\int_1^{\infty} x^{\frac{1}{2}+\epsilon_1} e^{-\delta x} dx \leq \int_1^{\infty} x e^{-\delta x} dx < \infty$$

by the integration by parts formula.

Now we've seen that $\lambda > 0 \implies$

$$\limsup_{t\to\infty} \mathbb{E}_{\widetilde{Q}} \left(Z(t) | \widetilde{\mathcal{G}}_{\infty} \right) < \infty \quad \widetilde{Q} \text{ almost surely.}$$

The path from here to the statement

$$\int Z(\infty) dP = 1$$

is the same as it was for the analogous claim in the proof of Theorem 3.3.0.9.

So we have shown

$$\mu < \sqrt{2mr} \implies \int Z(\infty)dP = 1.$$

We now have

- $\mu \geq \sqrt{2mr} \implies Z(\infty) = 0$ P almost surely, and
- $\mu < \sqrt{2mr} \implies \int Z(\infty)dP = 1.$

All that is needed now is an analog to Theorem 3.3.0.10; that is to say, the equivalence of the events $[Z(\infty) = 0]$ and $[t_\Omega < \infty]$. As the proof is identical to that already given (save when $\lambda = 0$, which is handled the same as the $\lambda < 0$ case), we omit it.

This completes the proof.

□

5 Conclusion

5.1 Discussion of the Main Result

First, let's informally consider $\sqrt{2mr}$ as a critical parameter value for μ. Recall that m is the mean number of offspring per fission event (minus one), and r is the average rate of occurrence of such events, so that their product, mr, is something like the "aggregate growth rate" for the process. Recall also that μ is the magnitude of the drift for our process. So if drift toward the absorbing boundary outweighs the process's ability to replenish lost individuals, we see certain extinction.

As we discussed in Chapter 4, Kesten found this same $\sqrt{2mr}$ critical parameter value for μ in his classic 1969 paper, [12]. In his formulation, a branching Brownian motion lives and evolves on the positive half-line, with constant drift toward the single absorbing boundary at 0. There, particles are allowed to wander arbitrarily far away from the absorbing boundary, but even so it was found that $\mu \geq \sqrt{2mr}$ was enough to guarantee extinction. No surprise then that the same condition will guarantee extinction in our treatment, where no particle can ever move further than distance K from the deadly boundary.

Now consider the case where $\mu < \sqrt{2mr}$. Here the "aggregate growth rate" outweighs the drift toward absorption, and persistence is possible. But only if the interval $[0, K]$ is large enough. In the Harris, Hesse, and Kyprianou paper [11], a very similar result is obtained for a branching Brownian motion on $[0, K]$ with *both* boundary points taken to be absorbing. The only difference between their result and the one given here being that their minimum interval length,

$$K_0' = \frac{\pi}{\sqrt{2mr - \mu^2}},$$

is larger than our minimum interval length, K_0:

$$\frac{K_0'}{2} < K_0 = \frac{\arctan\left(-\frac{\sqrt{2mr-\mu^2}}{\mu}\right) + \pi}{\sqrt{2mr-\mu^2}} < K_0',$$

since the arctan expression here returns a value strictly between $-\frac{\pi}{2}$ and 0.

So from the results in [12], [11], and here, we now have the intuitively satisfying situation that if a given branching process with constant negative drift is able to survive on $[0, K]$ when both boundary points are taken to be absorbing, then it's also able to survive in the less-lethal environment of $[0, K]$ when K is taken to be reflecting instead. And if this process is able to survive in $[0, K]$ with 0 absorbing and K reflecting, then it's certainly also able to survive in the even more benign environment of $[0, \infty)$ with 0 absorbing.

So we find that our main result is nestled nicely between two already existing results in the way one would expect. It has also been seen (here and in [11]) that all three results are attainable via distinguished path analysis.

We note again here that the results of Section 3.3 and Chapter 4 give much more than $P(t_\Omega = \infty) > 0$ when $\lambda > 0$; in fact they give the uniform integrability of $\{Z(t)\}_{t\geq 0}$. The main results of those sections are stated as they are for the sake of simplicity, and because we have posed the fundamental question as one of persistence vs. extinction.

We also point out that there is a gap in our understanding; we do not know what happens for branching Brownian motion on $[0, K]$ with 0 absorbing and K reflecting when $\lambda = 0$. This corresponds to the case of $\mu < \sqrt{2mr}$ with $K = K_0$. In the analogous statement in [11] for branching Brownian motion on $[0, K]$ with both boundaries taken to be absorbing, it is claimed without proof that $\lambda = 0$ implies $Z(\infty) = 0$ P almost surely, so that the process becomes extinct in finite time almost surely. This author has not been able to show the same for the process considered here.

5.2 Directions for Future Research

First and foremost, we'd like to show what happens when $\lambda = 0$. In this case we have

$$Z(t) = \sum_{u \in N_t} e^{\mu x_u(t)} \sin(b x_u(t)),$$

and trying to show either $\limsup_{t\to\infty} Z(t) = \infty$ or $\limsup_{t\to\infty} Z(t) < \infty$ Q almost surely (in the spirit of the proof of Theorem 3.3.0.9) is complicated by the fact that $Z(t)$ neither grows nor decays exponentially with time with $\lambda = 0$.

Another area that is ripe for future research is the consideration of different combinations of boundary conditions at 0 and at K. Harris, Hesse, and Kyprianou have treated the case of Dirchlet/Dirchlet conditions in [11], as we've noted, and we've begun to address the Dirchlet/Neumann case here. This author has all other cases where combinations are made from Dirchlet, Neumann, and Robin boundary conditions listed in his notes, and has investigated when non-negative eigenfunctions exist for the infinitesimal generators in those cases. The results follow:

- Dirchlet/Robin - non-negative eigenfunctions exist, under certain conditions,
- Neumann/Neumann - eigenfunctions exist, but are not non-negative (and of course the question of extinction vs. persistence is not interesting in this case),
- Neumann/Robin - non-negative eigenfunctions exist, under certain conditions, and
- Robin/Robin - non-negative eigenfunctions exist, under certain conditions.

Since the existence of a non-negative eigenfunction for the infinitesimal generator played such a central role in our analysis, the above abundance of such eigenfunctions bodes well for future researchers using techniques similar to those deployed here.

A Robin boundary condition analog to Kesten's problem would also be of interest. One would expect branching Brownian motion on the half-line to survive more easily with a Robin boundary than with a Dirchlet one.

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