AN ABSTRACT OF THE THESIS OF

Kirk Eugene Lancaster for the degree of Doctor of Philosophy

in Mathematics presented on August 15, 1980

Title: ON TEMPERATE FUNDAMENTAL SOLUTIONS WITH SUPPORT IN A NONSALIENT CONE

Signature redacted for privacy.

Abstract approved: ______________________

Bent Petersen

A sufficient condition for the existence of temperate fundamental solutions of a system of constant coefficient partial differential operators with support in a cone of the form $R \times \Gamma$, where Γ is closed, convex, and salient, is proved. The approach is to solve a system of polynomial equations with weighted L^2-estimates. C^∞ dependence on the parameters is only obtained if the weight function need not change as we take derivatives in the parameters.

On Temperate Fundamental Solutions with
Support in a Nonsalient Cone

by

Kirk Eugene Lancaster

A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Commencement June 1981

APPROVED:

Signature redacted for privacy.

Professor of Mathematics
in charge of major

Signature redacted for privacy.

Chairman of Department of Mathematics

Signature redacted for privacy.

Dean of Graduate School

Date thesis is presented August 15, 1980

Typed by Suzanne Elcrat for Kirk Eugene Lancaster

ACKNOWLEDGEMENTS

I wish to express my gratitude to Professor Bent Petersen for his assistance, patience, encouragement, and, above all, generosity during my stay in Corvallis. I have enjoyed knowing Professor Petersen and learning from him.

I would like to thank all the people with whom I have conversed, especially David Finch, Harold Parks, Kennan Smith, John Leahy, Alan Coppola, and Don Solmon.

I also wish to express my gratitude to my excellent teachers, Stuart Newberger, Joel Davis, Arnold Kas, J. W. Smith, and Dave Carlson.

I wish to thank my wife, Deborah, for her patience, and my family and friends for their help and encouragement.

TABLE OF CONTENTS

ON TEMPERATE FUNDAMENTAL SOLUTIONS WITH SUPPORT IN A NONSALIENT CONE

INTRODUCTION

In this paper we prove a sufficient condition for a $p \times q$ $(p \leq q)$ system of constant coefficient partial differential operators $P(D)$ to have a temperate fundamental solution with support in $R \times \Gamma$ when Γ is a closed, convex, salient cone. A temperate fundamental solution of such a system is a $q \times p$ matrix K of temperate distributions such that $P(D)K = \delta I$. This can be considered as a partial extension of the result of Enqvist.

Ehrenpreis (1954) and Malgrange (1955) independently proved that every constant coefficient partial differential operator has a fundamental solution. Since then, conditions for existence of fundamental solutions with special properties have been studied. For an early survey, see Hörmander (1957).

We are interested in the location of the support of the fundamental solution. This question has been studied by several researchers, including Smith, Petersen (1975) and Enqvist (1976), among others.

One application of our result concerns the following overdetermined Cauchy problem:

Let λ be in the interior of the dual cone of Γ and let H be the closed half-space determined by $(0, \lambda)$, i.e. $H = R \times \{x \in R^n | \langle x,\lambda\rangle \geq 0\}$. Suppose $w \in \mathcal{D}'(R^{n+1})$ with $\text{supp}\,(P(D)^t w) \subseteq H$ and $P(D)^t w$ in a suitable convolution algebra. Does there exist $u \in \mathcal{D}'(R^{n+1})$ with $\text{supp}(u) \subseteq H$ and $P(D)^t u = P(D)^t w$?

If $P(D)$ has a temperate fundamental solution with support in $R \times \Gamma$, the answer is yes. For details, see Lancaster-Petersen (1980).

This paper can be viewed as one long proof of the result initially indicated. The method of proof is a modification of proofs by Hörmander (1969) and Petersen (1975, 1976). The paper divides into four parts.

Chapters One through Four form the first part. In Chapter One we prove that the specialization of an exact sequence of matrices of polynomials is exact off a proper variety. In Chapter Two, we show that an exact sequence of matrices of polynomials is exact over the sheaf $E0$ of germs of C^∞ functions holomorphic in some of the variables. We prove the existence of partitions of unity with polynomial growth conditions in Chapter Three. Chapter Four is the Weierstrass Preparation and Division Theorems with polynomial dependence on parameters and with estimates.

The second part, Chapter Five, consists of several lemmas in which we construct local solutions of systems of polynomial equations with smooth dependence on a parameter and with estimates. These are similar to Lemmas 1 and 1' of Petersen (1975).

In the third part, Chapter Six, we solve the Cauchy-Riemann and coboundary equations with smooth dependence on parameters and with weighted L^2 estimates. Solutions of the Cauchy-Riemann equations with smooth dependence on parameters, but without estimates, were established by B. Weinstock.

In the last part, we use the results of Chapters Five and Six and a Hilbert resolution to find global solutions of a system of polynomial equations with smooth dependence on a parameter and with estimates. We then find fundamental solutions as indicated earlier.

CHAPTER ONE

SPECIALIZATION OF HILBERT RESOLUTIONS

In this chapter, we prove a "specialization" result for exact sequences of $C[\xi,z]$-homomorphisms which will be used later. A weaker result would suffice for the purposes of this paper, but this result is aesthetically appealing. The proof is a modification of an argument from unpublished lecture notes of Aldo Andreotti.

Throughout this paper $z = (z_1, \ldots, z_n)$ will denote an n-tuple in C^n, $z' = (z_1, \ldots, z_{n-1}) \varepsilon C^{n-1}$ will denote the first $n - 1$ coordinates of z, and $\xi = (\xi_1, \ldots, \xi_m)$ will denote an m-tuple. ξ will be in C^m in this chapter (in later chapters it will be in either C^m or R^m). $C[\xi,z]$ denotes the polynomial ring $C[\xi_1, \ldots, \xi_m, z_1, \ldots, z_n]$, $C[z]$ the polynomial ring $C[z_1, \ldots, z_n]$, both over the complex field C. $C(\xi, z)$ denotes the field $C(\xi_1, \ldots, \xi_m, z_1, \ldots, z_n)$ of rational functions in $\xi_1, \ldots, \xi_m, z_1, \ldots, z_n$.

<u>Lemma 1.1</u> If P is a $p \times q$ matrix over $C[\xi, z]$ then there exists a proper (complex) algebraic variety V in C^m so that if $\xi_0 \not\varepsilon V$, $f \varepsilon C[z]^q$, and $P(\xi_0, \cdot)f = 0$, then there exists $F \varepsilon C[\xi,z]^q$ such that $PF = 0$ and $F(\xi_0, \cdot) = f$.

Proof: The proof is by induction on n. Let ρ be the rank of P over the field $C(\xi, z)$. Interchanging rows and columns, we may assume that if M is the principal $\rho \times \rho$ minor (i.e., upper left hand corner) of P, then det $M \neq 0$ and det M has largest degree in z, say μ, among all $\rho \times \rho$ minors of P. We may assume, by a unitary transformation in the z-coordinates, that

$$\det M(\xi, z) = a(\xi) z_n^{\mu} + \text{lower order terms in } z_n$$

where $a \in \mathbb{C}[\xi]$ and $a \neq 0$. (Let H be the z-homogeneous term of $\det M$ of degree μ in z and pick $\eta \in \mathbb{C}^n$ so that $H(\cdot, \eta) \neq 0$ and $|\eta| = 1$. Rotate $\mathbb{C}^n$ so $\eta \to (0, \ldots, 0, 1)$ and denote the new coordinates also by z.)

Let $P = \begin{bmatrix} R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_p \end{bmatrix} = [P_1 \ldots P_q]$ and $P' = \begin{bmatrix} R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_\rho \end{bmatrix} = [P_1' \ldots P_q']$.

Then $M = (P_1' \ldots P_\rho')$.

Let $D = \det M = \det(P_1' \ldots P_\rho')$ and $V' = \{\xi \in \mathbb{C}^n | a(\xi) = 0\}$. (In case $n = 0$, $V = \{\xi \in \mathbb{C}^n | D(\xi) = 0\}$ is the required variety.) Notice that all minors of order $\rho + 1$ in the matrix

$$\begin{bmatrix} R_i \\ R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_\rho \end{bmatrix}$$

have zero determinant, for $i = 1, \ldots, p$, since P has rank ρ. Consider the minor given by the first ρ columns and the j^{th} column

$$N = \begin{bmatrix} R_i' & P_{ij} \\ R_1' & P_{1j} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ R_\rho' & P_{\rho j} \end{bmatrix}$$

where $R'_j = (P_{j1}, \ldots, P_{j\rho})$.

Expanding by minors about the first row, we see

$$0 = \det N = \Sigma(-1)^{k+1} P_{ik} \tilde{L}_{kj} + (-1)^{\rho} P_{ij} D$$

where $\tilde{L}_{kj} = \det \begin{bmatrix} R'_1 & P_{1j} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ R'_\rho & P_{\rho j} \end{bmatrix}$ with the k^{th} column omitted.

Setting $L_{kj} = (-1)^{k+\rho+1} \tilde{L}_{kj}$ we see

$$R_i \begin{bmatrix} L_j \\ \cdot \\ \cdot \\ \cdot \\ L_{\rho j} \\ \cdot \\ \cdot \\ D \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = [P_{i1} \cdots P_{iq}] \begin{bmatrix} L_j \\ \cdot \\ \cdot \\ \cdot \\ L_{\rho j} \\ \cdot \\ \cdot \\ D \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = 0$$

where D is the j^{th} component, for $\rho + 1 \leq j \leq q$.

Thus we have $q - \rho$ vectors in ker P. Collecting these into a matrix, we get

$$C = \begin{bmatrix} & * & \\ D & & 0 \\ & \ddots & \\ 0 & & D \end{bmatrix}$$

which is a $q \times (q - \rho)$ matrix over $C[\xi, z]$ with rank $q - \rho$. Moreover, if $\xi \notin V'$ (or $\xi \notin V$ if $n = 0$), then $D(\xi, \cdot) \neq 0$ and so $C(\xi, \cdot)$ has rank $q - \rho$. By the choice of M, each L_{ij} has degree $\leq \mu$ in z.

Of course, PC = 0.

(i) Now assume $n = 0$. Suppose $\xi_0 \notin V$, $f \in C^q$, and $P(\xi_0)f = 0$.

Let $\Lambda_j = f_j/D(\xi_0)$ for $j = \rho + 1, \ldots, q$ and $\Lambda = \begin{bmatrix} \Lambda_{\rho+1} \\ \cdot \\ \cdot \\ \cdot \\ \Lambda_q \end{bmatrix}$.

Then $C(\xi_0)\Lambda = \begin{bmatrix} * \\ D(\xi_0)\ \Lambda_{\rho+1} \\ \cdot \\ \cdot \\ \cdot \\ D(\xi_0)\ \Lambda_q \end{bmatrix} = \begin{bmatrix} * \\ f_{\rho+1} \\ \cdot \\ \cdot \\ \cdot \\ f_q \end{bmatrix}$

so $f - C(\xi_0)\Lambda = \begin{bmatrix} \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_\rho \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \Phi$. Let $\theta = \begin{bmatrix} \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_\rho \end{bmatrix}$.

$P(\xi_0)\Phi = P(\xi_0)f - P(\xi_0)\ C(\xi_0)\Lambda = 0$, so

$$0 = \left[\begin{array}{c|c} M(\xi_0) & * \\ \hline * & * \end{array}\right] \begin{bmatrix} \theta \\ 0 \end{bmatrix} = \begin{bmatrix} M(\xi_0)\theta \\ * \end{bmatrix} .$$

Since $M(\xi_0)$ is nonsingular and $M(\xi_0)\theta = 0$, $\theta = 0$.

Thus $f = C(\xi_0)\Lambda$. Let $F(\xi) = C(\xi)\Lambda$.

(ii) Assume $n > 0$ and assume inductively that the lemma has been proved in dimension $n - 1$. Suppose $\xi_0 \notin V'$, $f \in C[z]^q$ and $P(\xi_0, \cdot)f = 0$. We now use the division algorithm relative to z_n to divide f_j by $D(\xi_0, \cdot)$. Recall $D(\xi, z) = a(\xi)z_n^\mu$ + lower order in z_n.

$$f_j = D(\xi_0, \cdot)\Lambda_j + \theta_j \qquad j = \rho + 1, \ldots, q.$$

where Λ_j, $\theta_j \in C[z]$ and the degree of θ_j in z_n is $< \mu$. (Note that Λ_j contains powers of $a(\xi_0)$ in its denominator.) If we set $\Lambda = \begin{bmatrix} \Lambda_{\rho+1} \\ \cdot \\ \cdot \\ \cdot \\ \Lambda_q \end{bmatrix}$ we see

$$C(\xi_0, \cdot)\Lambda = \begin{bmatrix} * \\ D(\xi_0, \cdot)\Lambda_{\rho+1} \\ \cdot \\ \cdot \\ \cdot \\ D(\xi_0, \cdot)\Lambda_q \end{bmatrix}.$$

Define $\Theta = f - C(\xi_0, \cdot)\Lambda = \begin{bmatrix} * \\ \theta_{\rho+1} \\ \cdot \\ \cdot \\ \cdot \\ \theta_q \end{bmatrix} \in C[z]^q$.

Notice $P(\xi_0, \cdot)\Theta = P(\xi_0, \cdot)f - P(\xi_0, \cdot)C(\xi_0, \cdot)\Lambda = 0$

i.e., $\Theta \in \ker P(\xi_0, \cdot)$. Considering just the first ρ rows of

$P(\xi_0, \cdot)\Theta$ we obtain

$$M(\xi_0, \cdot)\begin{bmatrix}\theta_1\\ \cdot\\ \cdot\\ \cdot\\ \theta_\rho\end{bmatrix} = -(P'_{\rho+1}(\xi_0, \cdot) \ \dots \ P'_q(\xi_0, \cdot))\begin{bmatrix}\theta_{p+1}\\ \cdot\\ \cdot\\ \cdot\\ \theta_q\end{bmatrix} = \nu.$$

By Cramer's Rule, if $1 \leq j \leq \rho$,

$$D(\xi_0, \cdot)\theta_j = \det\,(P_1'(\xi_0, \cdot) \dots \nu \dots P'(\xi_0, \cdot))$$

$$= \sum_{\ell=\rho+1}^{q} \theta_\ell C_{\ell j}(\xi_0, \cdot)$$

(with ν in the $j\underline{\text{th}}$ place)

where $C_{\ell j} = -\det\,(P_1' \dots P_\ell' \dots P_\rho')$ (P_ℓ' in $j\underline{\text{th}}$ place) which is a $\rho \times \rho$ cofactor of P and so has degree in $z \leq \mu$. Thus $D(\xi_0, \cdot)\theta_j$ $(1 \leq j \leq \rho)$ has degree $< 2\mu$ in z_n and so θ_j has degree $< \mu$ in z_n for all $j = 1, \dots, q$.

We have shown that if $\xi_0 \notin V'$ and $P(\xi_0, \cdot)f = 0$, then

$$f = C(\xi_0, \cdot)\Lambda + \Theta$$

where $\Lambda \in C[z]^{q-\rho}$, $\Theta \in C[z]^q$ and each component of Θ has degree $< \mu$ in z_n. If we write

$$P(\xi, z) = z_n^\ell\, \sigma_1(\xi, z') + \dots + \sigma_{\ell+1}(\xi, z')$$

$$\Theta(z) = z_n^{\mu-1}\phi_1(z') + \dots + \phi_\mu(z')$$

where $z' = (z_1, \dots, z_{n-1})$, σ_j is a $p\times q$ matrix over $C[\xi, z']$, and $\phi_j \in C[z]^q$, then the condition $P(\xi_0, \cdot)f = 0$ becomes $P(\xi_0, \cdot)\Theta = 0$ and this is equivalent to

$$\sigma_1(\xi_0, \cdot)\phi_1 = 0$$

$$\sigma_2(\xi_0, \cdot)\phi_1 + \sigma_1(\xi_0, \cdot)\phi_2 = 0$$

etc.

If we define $\psi = \begin{bmatrix} \sigma_1 & & \\ \vdots & \ddots & 0 \\ \sigma_{\ell+1} & & \\ & & \sigma_1 \\ 0 & & \vdots \\ & & \sigma_{\ell+1} \end{bmatrix}$

a $(\mu + \ell)p \times \mu q$ matrix over $C[\xi, z']$ and apply the induction hypothesis, we can find a proper (complex) algebraic variety $W \subseteq C^m$ so that the lemma holds for ψ. We set $V = V' \cup W$. If $\xi_0 \notin V$, then we can find $\tilde{\phi}_j \in C[\xi, z']^q (1 \leq j \leq \mu)$ so that, if we set

$$\tilde{\phi} = \begin{bmatrix} \tilde{\phi}_1 \\ \vdots \\ \tilde{\phi}_\mu \end{bmatrix}$$

then $\psi\tilde{\phi} = 0$ and $\tilde{\phi}(\xi_0, \cdot) = \phi = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_\mu \end{bmatrix}$.

Define $H(\xi, z) = z_n^{\mu-1}\tilde{\phi}_1(\xi,z') + \ldots + \tilde{\phi}_\mu(\xi,z') \in C[\xi,z]^q$ and $F = C\Lambda + H \in C[\xi, z]^q$. $PF = 0$ and $F(\xi_0, \cdot) = f$.

Theorem 1.1 If $C[\xi, z]^r \xrightarrow{Q} C[\xi, z]^q \xrightarrow{P} C[\xi, z]^p$ is exact, there exists a proper (complex) algebraic variety V in C^m so that if $\xi_0 \notin V$, then

$$C[z]^r \xrightarrow{Q(\xi_0, \cdot)} C[z]^q \xrightarrow{P(\xi_0, \cdot)} C[z]^p$$

is exact.

Proof: Clearly $P(\xi, \cdot)\, Q(\xi, \cdot) = 0$ for all $\xi \in C^m$. Let V be the variety determined in the previous lemma. Now assume $\xi_0 \notin V$, $f \in C[z]^q$, and $P(\xi_0, \cdot)f = 0$. By the previous lemma, there exists $F \in C[\xi, z]^q$ so that $PF = 0$ and $F(\xi_0, \cdot) = f$. By exactness, there exists $G \in C[\xi, z]^r$ so that $QG = F$. Now set $g = G(\xi_0, \cdot) \in C[z]^r$. Then $Q(\xi_0, \cdot)g = F(\xi_0, \cdot) = f$.

CHAPTER TWO

Exact Sequences

In this chapter we prove that if

$$C[\xi, z]^r \xrightarrow{Q} C[\xi, z]^q \xrightarrow{P} C[\xi, z]^p$$

is exact, where P and Q are matrices of polynomials, then, if we consider P and Q as maps of q-tuples and r-tuples respectively of functions C^∞ in $U \times W \subseteq R^m \times C^n$ with W convex (and U and W open) and holomorphic in W, this new sequence is also exact. We also introduce some notation that we will use throughout this paper.

We denote the coordinates in R^m by $\xi_1, \ldots, \xi_m$ and in C^n by $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$. We will use the following notation:

EE = sheaf of germs of C^∞ functions in $R^m \times C^n$

EO = sheaf of germs of C^∞ functions in $R^m \times C^n$ which are holomorphic in $z_1, \ldots, z_n$

$EE_{(p,q)}$ = sheaf of germs of C^∞ forms of type (p,q) relative to $z_1, \ldots, z_n$

AO = sheaf of germs of real-analytic functions in $R^m \times C^n$ which are holomorphic in $z_1, \ldots, z_n$

OO = sheaf of germs of holomorphic functions in C^{m+n}.

If k is a nonnegative integer, we denote by E^kE, E^kO, etc., the sheaves which are k times continuously differentiable in all variables and C^∞ (or holomorphic, etc.) in $z_1, \ldots, z_n$.

We will denote the stalk of a sheaf at 0 (for example) by subscript, e.g., EE_0, and similarly for germs. For a short discussion of sheaves, see Chapter 7 of Hörmander (1966).

$\Gamma(V, S)$ denotes the continuous sections of the sheaf S over V. Thus $\Gamma(U \times W, EE) = C^{\infty}(U \times W)$.

<u>Def</u>: Let R be a commutative ring with identity and B be a right R-module. B is <u>R-flat</u> if and only if it satisfies one of the following equivalent conditions:

(i) The functor $B \otimes_R$ is exact.

(ii) $B \otimes I \to B$ is injective, for every finitely generated left R-ideal I.

(iii) If $a_1, \ldots, a_p \in R$; $b_1, \ldots, b_p \in B$, and $\sum_{i=1}^{p} a_i b_i = 0$, then there exist $r_{ij} \in R$ and $f_j \in B$ $(i = 1, \ldots, p, j = 1, \ldots, q)$ such that $\sum_{i=1}^{p} a_i r_{ij} = 0$ $(j = 1, \ldots, q)$ and $b_i = \sum_{j=1}^{q} f_j r_{ij}$.

(i) and (ii) are equivalent by Theorems 3.53 and 3.54 of Rottman.

(iii) $\Rightarrow$ (ii) If $\alpha = \sum_j b_j \otimes a_j \in \mathrm{Ker}(B \otimes I \to B)$

i.e. $\sum_j b_j a_j = 0$

then $\alpha = \sum_j b_j \otimes a_j = \sum_j \sum_\ell f_\ell r_{j\ell} \otimes a_j = \sum_\ell f_\ell \otimes \sum_j a_j r_{j\ell} = 0$.

(i) $\Rightarrow$ (iii) Define a morphism $R^P \to R$ by $(c_1, \ldots, c_p) \to \sum_{i=1}^{p} c_i a_i$ and let K be its kernel. Then $0 \to K \to R^P \to R$ is exact. By (i), $0 \to B \otimes K \to B \otimes R^P \to B \otimes R$ is exact; thus $0 \to B \otimes K \to B^P \to B$ is exact. Notice $B \otimes K \to B^P$ is defined by $b' \otimes (c_1, \ldots, c_p) \to (b'c_1, \ldots, b'c_p)$ and $B^P \to B$ is defined by $(b_1', \ldots, b_p') \to \sum_{i=1}^{p} a_i b_i'$.

Now $(b_1, \ldots, b_p)$ is in the kernel of this last map, so there exists $B_1, \ldots, B_q \in K$ and $f_1, \ldots, f_q \in B$ so that

$$\sum_{j=1}^{q} f_j \otimes B_j \longrightarrow (b_1, \ldots, b_p).$$

If we write $B_j = (r_{1j}, \ldots, r_{pj})$, then $b_i = \sum_{j=1}^{q} r_{ij} f_j$ and $B_j \in K$ implies $\sum_{i=1}^{p} a_i r_{ij} = 0$.

Lemma 2.1 EE_0 is flat over AO_0.

Proof: Suppose $a^1, \ldots, a^p \in AO_0$, $b^1, \ldots, b^p \in EE_0$, and $\sum_{i=1}^{p} a^i b^i = 0$. We may extend $a^1, \ldots, a^p$ to holomorphic germs (by power series) and apply Oka's theorem (Theorem 6.4.1 of Hörmander (1966)). Then there exists $r^{ij} \in AO_0$ $i = 1, \ldots, p$, $j=1, \ldots, q$ such that $\sum_{i=1}^{p} a^i r^{ij} = 0$ and if $u_1, \ldots, u_p \in AO_0$ and $\sum_{i=1}^{p} a^i u_i = 0$ then there exists $v_j \in AO_0$ $j = 1, \ldots, q$ such that $u_i = \sum_{j=1}^{q} r^{ij} v_j$.

If we expand $\sum_{i=1}^{p} a^i b^i$ in formal power series about the origin, then $\sum_{i=1}^{p} a_e^i\, b_e^i = 0$ (in the formal power series ring) for e near the origin. The formal power series ring is flat over the convergent power series ring, so $b_e^i = \Sigma\, r_e^{ij}\, s_e^j$ for some formal power series s^j, $j = 1, \ldots, q$. By Malgrange's theorem (Theorem 1 of Malgrange (1960)),

$$b^i = \sum_{j=1}^{q} r^{ij}\, t_j \quad \text{for some} \quad t_j \in EE, j = 1, \ldots, q .$$

Notice that this proves EE is flat over AO, i.e., EE is flat over AO at each point of $R^m \times C^n$. We do not know if $E^k E$ is flat over AO.

Lemma 2.2 Let U be open in R^m and W open and convex in C^n. If $f \in \Gamma\,(U \times W, EE_{(p,q+1)})$ and $\bar\partial f = 0$ in $U \times W$, then there exists $u \in \Gamma\,(U \times W, EE_{(p,q)})$ such that $\bar\partial u = f$ in $U \times W$.
Recall that $\bar\partial$ denotes the Cauchy-Riemann operator in C^n only.

Proof: Let P be a matrix of polynomials over $C[x,y] = C[x_1, y_1, \ldots, x_n, y_n]$ such that the matrix of partial differential operators obtained by replacing x_j by $\frac{\partial}{\partial x_j}$ and y_j by $\frac{\partial}{\partial y_j}$ in P, which we denote $P(D(x,y))$, represents the map

$$\bar\partial\colon\ C^\infty_{(p,q)}(C^n) \longrightarrow C^\infty_{(p,q+1)}(C^n).$$

where we identify (p,q)-forms with $\binom{n}{q}\binom{n}{p}$ vectors of C^∞ functions.

Let Q be the matrix of polynomials over $C[x,y]$ such that $Q(D(x,y))$ represents the map

$$\bar\partial\colon\ C^\infty_{(p,q+1)}(C^n) \longrightarrow C^\infty_{(p,q+2)}(C^n).$$

Then

$$C[x,\ y]^A \xrightarrow{\ Q^t\ } C[x,\ y]^B \xrightarrow{\ P^t\ } C[x,\ y]^C \text{ is exact.}$$

Now consider P and Q as matrices over $C[\xi,x,y]$ (which are independent of $\xi = (\xi_1,\ldots,\xi_m)$). Then

$$C[\xi,x,y]^A \xrightarrow{\ Q^t\ } C[\xi,x,y]^B \xrightarrow{\ P^t\ } C[\xi,x,y]^C$$

is exact. Notice $Q(D(x,y))f = 0$, since $\bar\partial f = 0$.
By the Malgrange-Ehrenpreis theorem (Malgrange (1963)), there exists

$g_j \in C^\infty(U_j \times W)^A$ such that $P(D(x,y))g_j = f$ if U_j is open and convex in U. Thus $\bar\partial g_j = f$ in $U_j \times W$. Let $\{U_j\}$ be an open cover of U consisting of convex sets and x_j a partition of unity of U subordinate to U_j. Define $u = \sum_j x_j g_j \in C^\infty_{(p,q)}(U \times W)$. Then $\bar\partial u = f$ in $U \times W$.

Weinstock (1973) proves this lemma for W a domain of holomorphy and $q = p = 0$, i.e., for $(0,1)$- forms.

Corollary 2.1 Let U and W be as in Lemma 2.2 and let k be a non-negative integer. There exists an integer N (independent of k) such that:

If $f \in \Gamma\,(U \times W, E^{k+N} E^{k+N}_{(p,q+1)})$ and $\bar\partial f = 0$ in $U \times W$, then there exists $u \in \Gamma(U \times W, E^k E^k_{(p,g)})$ such that $\bar\partial u = f$ in $U \times W$.

Proof: The proof of Lemma 2.2 is still valid if we use the version of the Malgrange-Ehrenpreis theorem in Hörmander (1966).

Lemma 2.3 Suppose U is open in R^m and W is a convex, open set in C^n. Then

$$\Gamma(U \times W, EO)^r \xrightarrow{Q} \Gamma\,(U \times W, EO)^q \xrightarrow{P} \Gamma\,(U \times W, EO)^p$$

is exact if $C[\xi, z]^r \xrightarrow{Q} C[\xi, z]^q \xrightarrow{P} C[\xi, z]^p$ is exact.

Proof: Let $Q^0 = Q$, $r_0 = r$, and let $\{Q^j\}$ be a resolution of Q (which need not be finite); that is, $\rightarrow C[\xi, z]^q \xrightarrow{Q^j} \dots \xrightarrow{Q} C[\xi, z]^q \xrightarrow{P} C[\xi, z]^p$ is exact. By Lemma 7.6.3 of Hörmander (1966)

$$\longrightarrow \mathcal{O}\mathcal{O}^{r_j} \xrightarrow{Q^j} \dots \xrightarrow{Q} \mathcal{O}\mathcal{O}^q \xrightarrow{P} \mathcal{O}\mathcal{O}^p$$

is exact. Since we can identify AO_0 and $\mathcal{O}\mathcal{O}_0$, we see

$$\longrightarrow AO^{r_j} \xrightarrow{Q^j} \cdots \xrightarrow{Q} AO^q \xrightarrow{P} AO^p$$

is exact. Since EE is flat over AO,

$$\longrightarrow EE^{r_j} \xrightarrow{Q^j} \cdots \xrightarrow{Q} EE^q \xrightarrow{P} EE^p$$

is exact. By a partition of unity argument, we see that

$$\longrightarrow \Gamma(U \times W, EE)^{r_j} \xrightarrow{Q^j} \cdots \xrightarrow{Q} \Gamma(U \times W, EE)^q \xrightarrow{P}$$

is exact. This is clearly true for $EE_{(p,q)}$ also.

Suppose $f \in \Gamma(U \times W, EO)^q$ with $Pf = 0$. Then there exists

$$u \in \Gamma\ (U \times W, EE)^r$$

such that

$$Qu = f \quad \text{in} \quad U \times W.$$

Then $Q\,\bar\partial\,u = \bar\partial f = 0$ and so $\bar\partial\,u = Q\,u$, in $U \times W$ for some $u_1 \in \Gamma(U \times W, EE_{(0,1)})^{r_1}$. Now

$$Q^1 \bar\partial\, u_1 = \bar\partial\, Q^1 u_1 = \bar\partial\,\bar\partial\, u = 0,$$

so $\bar\partial\, u_1 = Q^2\, u_2$ in $U \times W$

for some $u_2 \in \Gamma(U \times W, EE_{(0,2)})^{r_2}$.

Inductively we find $u_j \in \Gamma\ (U \times W, EE_{(0,j)})^{r_j}$

with $Q^j u_j = \bar\partial\, u_{j-1}$. When $j = n$, $\bar\partial\, u_n = 0$.

By Lemma 2.2, there exists $v_{n-1} \in \Gamma\ (U \times W, EE_{(0,\ n-1)})^{r_n}$

such that $u_n = v_{n-1}$. Now

$$Q^n u_n = \bar{\partial} u_{n-1} \quad \text{and} \quad Q^n u_n = Q^n \bar{\partial} v_{n-1} = \bar{\partial} Q^n v_{n-1} ,$$

so $\bar{\partial} (u_{n-1} - Q^n v_{n-1}) = 0$.

Thus $u_{n-1} - Q^n v_{n-1} = \bar{\partial} v_{n-2}$ for some $v_{n-2} \in \Gamma(U \times W, EE_{(0,\, n-2)})^{r_{n-1}}$.
Continuing in this manner we find $v_1 \in \Gamma(U \times W, EE_{(0,1)})^{r_2}$ with

$$\bar{\partial} (u_1 - Q^2 v_1) = 0 .$$

So $u_1 - Q^2 v_1 = \bar{\partial} v_0$ for some $v_0 \in \Gamma(U \times W, EE)^{r_1}$.

Now $Q^1 u_1 = Q^1 (u_1 - Q^2 v_1) = \bar{\partial} Q^1 v$

and $Q^1 u_1 = \bar{\partial} u$.

So $\bar{\partial}(u - Q^1 v_0) = 0$ i.e., $u - Q^1 v_0 \in \Gamma(U \times W, EO)^r$.

and $Q(u - Q^1 v_0) = Q u = f$.

Corollary 2.2 If $f \in \Gamma(U \times W, EE)^r$ with $\bar{\partial} Qf = 0$, then there exists $g \in \Gamma(U \times W, EO)^r$ such that $Qg = Qf$.

Proof: Let $u = f$ in the previous proof. Then $g = u - Q^1 v_0 = f - Q^1 v_0$.

CHAPTER THREE

A PARTITION OF UNITY

We will eventually need special partitions of unity with polynomial growth and so we prove a suitable result in this chapter.

Let $\{U_j | j = 1, \ldots, \ell\}$ be a finite open cover of $R^m \times C^n$. We define

$$B(t) = \{(\xi, z) \in R^m \times C^n | \ |\xi| < t, |z| < t\},$$

$$d(\xi, z) = \sup_j \{(\mathrm{dist}\ (\xi, z), \partial U_j) | (\xi, z) \in U_j\},$$

$$D(t) = \inf \{d(\xi, z) | (\xi, z) \in B(t) - B(t - 2)\}, \text{ and}$$

$$d(t) = \min \{1, D(t)\}$$

for $t \in R$, $\xi = (\xi_1, \ldots, \xi_m) \in R^m$, and $z = (z_1, \ldots, z_n) \in C^n$. By ∂V we mean the (topological) boundary of the set V. Notice that $d(\xi, z)$ is the distance from (ξ, z) to the boundary of the set U_j which contains (ξ, z) and whose boundary is furthest from (ξ, z).

If $V \subseteq R^m \times C^n$, then $V(z)$ denotes $\{\xi \in R^m | (\xi, z) \in V\}$.

<u>Lemma 3.1</u> There exist constants $C_\alpha > 0$ for each m-multi-index α such that, for each $z \in C^n$, there exists a partition of unity $\{\psi_j\}$ of R^m subordinate to $\{U_j(z) | j = 1, \ldots, \ell\}$ so that

$$|D^\alpha \psi_j(\xi)| \leq C_\alpha (1 + |\xi| + |z|)^{|\alpha|^2} (d(t))^{-|\alpha|(2|\alpha|+m)-1}$$

if $(\xi, z) \in B(t) - \overline{B(t - 2)}$.

<u>Proof</u>: We assume $z \in C^n$ is fixed and we obtain constants C_α independent of z (and of t). For $1 \leq j \leq \ell$, $t > o$, $r \in R$, define

$$K(j, t, r) = \{\xi \in R^m | (\xi, z) \in U_j \cap B(t) \text{ and } \operatorname{dist}((\xi, z), \partial U_j) > 2^{-r}\}$$

$$L(j, t, r) = K(j, t, r) - \overline{K(j, t-2, r)}.$$

Notice that if $2^{-r} < D(t)$, then $\{L(j, t, r) | j = 1, \ldots, \ell\}$ is an open cover of $B(t)(z) - \overline{B(t-2)(z)} = \{\xi \in R^m | (\xi, z) \in B(t) - \overline{B(t-2)}\}$.

Define $r(t)$ by $2^{-r(t)} = d(t)/2$. Notice $\{L(j, t, r(t)) | j=1, \ldots, \ell, t = 1, 2, 3 \ldots\}$ is a locally finite open cover of R^m with intersection number $\leq 2\ell$ consisting of relatively compact sets. Let $x(t, j)$ be the characteristic function of $L(j, t, r(t))$ and define $f \in C^\infty(R)$ by

$$f(s) = \begin{cases} \exp(1/s) & s < o \\ 0 & s \geq o \end{cases}.$$

Let $\rho(\xi) = Cf(|\xi|^2 - 1)$ with C picked so that $\int_{R^m} \rho(\xi) d\xi = 1$; ρ is a mollifier. Define

$$\rho_t(\xi) = C2^{m(r(t)+1)} f(2^{2(r(t)+1)} |\xi|^2 - 1) \text{ and } w(t,j) = x(t,j) * \rho_t.$$

Notice $\operatorname{supp}(w(t,j)) \subseteq$ closed $2^{-(r(t)+1)}$ - nbhd of $L(j,t,r(t) \subseteq U_j(z)$ and $w(t,j)(\xi) = 1$ if $\xi \in L(j,t,r(t)) - 2^{-(r(t)+1)}$nbhd of $\partial L(j,t,r(t))$.

If $\xi \in R^m$, then for some positive integer q, $(\xi,z) \in B(q) - \overline{B(q-2)}$ and $d(\xi,z) \geq d(q) \geq 2^{1-r(q)}$ and so, for some $1 \leq j \leq \ell$, $\xi \in L(j,q,r(q)) - 2^{-(r(q)+1)}$nbhd of $\partial L(j,q,r(q))$; then $w(q,j)(\xi) = 1$. Define $\theta = \sum_{j=1}^{\ell} \sum_{q=1}^{\infty} w(q,j)$. $\theta \in C^\infty(R^m)$ and $\theta \geq 1$, since

$\{\text{supp } (w(q,j)) | j = 1, \ldots, \ell, q = 1, 2, 3, \ldots\}$ is locally finite, has intersection number $\leq 3\ell$, and consists of compact sets. Define

$$\psi_j = \sum_{q=1}^{\infty} w(q,j)/\theta \qquad j = 1, \ldots, \ell.$$

$\{\psi_j | j = 1, \ldots, \ell\}$ is a partition of unity subordinate to $\{U_j(z)\}$ and it remains to estimate $|D^\alpha \psi_j(\xi)|$.

By Leibniz's formula,

$$|D^\alpha \psi(\xi)| \leq \sum_{q=1}^{\infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta} w(q,j)(\xi) D^\beta(\theta^{-1})(\xi)|$$

where $\sum\limits_{\beta \leq \alpha}$ is the sum over all m-multi-indices β such that $\beta \leq \alpha$. Note

$$D^\beta(\theta^{-1}) = \sum_\gamma (-1)^\sigma \sigma! (D\theta)^\gamma / \theta^{\sigma+1}$$

where $\sum\limits_\gamma$ is the sum over all mb-multi-indices $\gamma = (\gamma_1, \ldots, \gamma_b)$ $(b = |\beta|)$ with $\gamma_1 \leq \cdots \leq \gamma_b$, $\gamma_1 + \cdots + \gamma_b = \beta$, σ is the number of nonzero γ_j, and $(D\theta)^\gamma = D^{\gamma_1}\theta \ldots D^{\gamma_b}\theta$. Here $D^{\gamma_j} =$

$$\left(\frac{\partial}{\partial \xi_1}\right)^{\gamma_{j1}} \cdot \ldots \cdot \left(\frac{\partial}{\partial \xi_m}\right)^{\gamma_{jm}} \quad \text{and} \quad D^0\theta = 1 \ (0 = (0, \ldots, 0) \in z^m).$$

Also $|D^\beta w(t,j)(\xi)| = |x(q,j) * D^\beta \rho_t(\xi)|$

$$\leq \text{volume of } L(j,t,r(t)) \cdot \max_{|\xi| \leq t} |D^\beta \rho_t(\xi)|$$

$$\leq C 2^{m(r(t)+1)} r(t)^m (t+1)^{|\beta|} 2^{|\beta|(2r(t)+3)} \max_{\substack{0 \leq k \leq |\beta| \\ -1 \leq s \leq 0}} |f^{(k)}(s)|$$

$$\leq C'_\beta 2^{4|\beta|+2m} (|\xi| + |z| + 1)^{|\beta|} d(t)^{-2|\beta|-m} |\log_2(d(t)) - 2|^m$$

if $(\xi,z) \in B(t) - \overline{B(t-2)}$, since then $t \leq \max\{|\xi| + 2, |z| + 2\}$.

Clearly C'_β is independent of t, j, and z.

So $|D^\beta\psi_j(\xi)| \leq C''_\alpha \sum_{\beta\leq\alpha} \sum_{j=1}^{\ell} \sum_{q=1}^{\infty} |D^\beta w(q,j)(\xi)|^{|\alpha|}$ since $\theta = \sum_{j=1}^{\ell} \sum_{q=1}^{\beta} w(q,j) \geq 1$

$$\leq C_\alpha(1 + |\xi| + |z|)^{|\alpha|^2} d(q)^{-(2|\alpha|+m)|\alpha|-1}$$

(since $\{\mathrm{supp}\,(w(q,j))\}$ has intersection number $\leq 3\ell$). C_α depends on ℓ, n, m.

We will be interested in the case where $\{U_j | j=1, \ldots, \ell\}$ is a semi-algebraic open cover of $R^m \times C^n$ and we want a partition of unity with polynomial growth.

<u>Lemma 3.2</u>. Let $\{U_j | j=1, \ldots, \ell\}$ be a (real) semi-algebraic open cover of $R^m \times C^n$. Then there exist constants $C'_\alpha \geq 0$ and an integer $N \geq 0$ such that, for each $z \in C^n$, there exists a partition of unity $\{\psi_j\}$ of R^m subordinate to $\{U_j(z)\}$ such that

$$|D^\alpha\psi_j(\xi)| \leq C'_\alpha(1 + |\xi| + |z|)^{N|\alpha|^2+N}$$

for each m-multi-index α.

<u>Proof</u>: Let $\{\psi_j\}$ be the partition of unity obtained in the previous lemma; we assume $z \in C^n$ has been fixed. We consider C^n as R^{2n}. If U is (real) semi-algebraic (i.e., $U = \{(\xi, x, y) \in R^{m+2n} | P(\xi, x, y) = 0, Q(\xi, x, y) < 0\}$ with P and Q polynomials) then $\{(\mu, \xi, z) \in R^{m+2N+1} | \mu = \mathrm{dist}\,((\xi, z), \partial U)\}$ is a (real) semi-algebraic set.

Since the supremum (and so infimum) of a semi-algebraic collection of semi-algebraic sets is semi-algebraic, $\{(t, d(t)) | t > 0\}$ is semi-algebraic. Then, by the Seidenberg-Tarski theorem, the following sets are semi-algebraic: $\{(t,d(t),\mu)\,|\,t > 0,\ 0 < \mu d(t) \le 1\}$ and $M = \{(t,\mu)\,|\,t > 0,\ 0 < \mu d(t) \le 1\}$.

Let $M_t = \{\mu\,|\,(t,\mu) \in M\} = \{\mu\,|\,0 < \mu \le d(t)^{-1}\}$ and $\mu(t) = \sup_{\mu \in M_t} \mu = \sup_{0<\tau<t} d(\tau)^{-1}$. Notice $\lim_{t\downarrow 0} D(t) = d(0,0) > 0$. Thus $\lim_{\tau\downarrow 0} d(\tau)^{-1} = \max\{1, 1/d(0,0)\}$. μ is nondecreasing and finite. By Lemma 2.1 (Hörmander, 1969, p. 276), there exists a rational number $a \ge 0$ and a constant $A > 0$ so that

$$\mu(t) = A\tau^{a}(1 + o(1)) \quad \text{as} \quad t \to \infty .$$

Then there exists a constant $C > 0$ depending on μ so that

$$\mu(t) \le Ct^{a} \qquad t > 0 .$$

Let N' be an integer with $N' \ge a$, say $N' = [a] + 1$. Then

$$d(t)^{-1} \le \mu(t) \le C(1+t)^{N'} \quad \text{if} \quad t > o. \quad \text{If} \quad (\xi,z) \in B(t) - \overline{B(t-2)},$$

then $t \le \max\{|\xi| + 2, |z| + 2\} \le |\xi| + |z| + 2$, so

$$d(t)^{-1} \le C(3 + |\xi| + |z|)^{N'} \le C'(1 + |\xi| + |z|)^{N'} .$$

Using the estimates of the previous lemma, we see

$$\begin{aligned} |D^{\alpha}\psi_j(\xi)| &\le C_{\alpha}(1+|\xi|+|z|)^{|\alpha|^2}(d(t))^{-|\alpha|(2|\alpha|+m)-1} \\ &\le C'_{\alpha}(1+|\xi|+|z|)^{N|\alpha|^2+N} \end{aligned}$$

where $N = 1 + 2N' + mN'$ and $C'_{\alpha} = C_{\alpha}C'$. In the first line, $t \in (\max\{|\xi|, |z|\}, \max\{|\xi|, |z|\} + 2)$ is arbitrary.

Corollary 3.1. Lemma 3.2 is true with the estimate

$$|D^{\alpha}\psi_j(\xi)| \leq 4C'_{\alpha}(1 + |\xi|^2 + |z|^2)^{(N|\alpha|^2+N)/2} \quad .$$

Proof: Computation using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$.

CHAPTER FOUR

THE WEIERSTRASS THEOREMS

In this chapter we prove a version of the Weierstrass Preparation and Division theorems with bounds and with parameters. There are several methods of proof of the Weierstrass theorems. Here we modify the proof based on Cauchy's theorem.

If $P \in C[\xi, z]$, then we define

$$\tilde{P}(\xi)(z:r) = \sup_{|w|<r} |P(\xi,z+w)| = \sup_{|w|<1} |P(\xi,z+rw)|.$$

An <u>m-multi-index</u> is an m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers. The <u>length</u> $|\alpha|$ of an m-multi-index $(\alpha_1, \ldots, \alpha_m)$ is $\alpha_1 + \ldots + \alpha_m$. $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$ for $j=1, \ldots, m$ and $\beta < \alpha$ means $\beta \leq \alpha$ and $\beta \neq \alpha$.

We continue to use the notation of Chapters 1 and 2. Thus $\Gamma(U \times W, E0)$ denotes the set of C^∞ functions on $U \times W$ which are holomorphic in W, $\Gamma(U \times W, A0)$ denotes the set of real-analytic functions on $U \times W$ which are holomorphic on W, and $\Gamma(U \times W, E^k 0)$ denotes the set of C^k functions on $U \times W$ which are holomorphic in W.

<u>Lemma 4.1</u> Let U be a connected open subset of R^m and $C > 0$. Let $P \in C[\xi, z]$, $s \in (0, 1)$, and $r > 0$. Assume $P(\xi, \cdot) \not\equiv 0$ if $\xi \in U$. Let $z \in C^n$ and assume $\tilde{P}(\xi)(z:r) \leq C \inf_{|\tau - z_n|=rs} |P(\xi,z',\tau)|$ for each $\xi \in U$.

Let $\xi_o \in U$ and let N^+ be the number of roots of the polynomial $\tau \to P(\xi_o, z', \tau)$ which satisfy $|\tau - z_n| < rs$. Then there exists a number s', $o < s' < 1$, depending only on μ, n, and C where μ is the degree of P in z, such that the polydisc $\Delta = \{w \in C^n \mid |w_j| < s' \quad j=1,\ldots,n-1, \ |w_n| < s\}$ is contained in the unit ball $B = \{w \in C^n \mid |w| < 1\}$ and so that the following properties hold:

A. There exist unique functions P^+ and P^- so that $P = P^+P^-$ in $U \times (z + r\Delta)$ with $P^+, P^- \in \Gamma(U \times (z + r\Delta), A0)$ such that:

(i) P^+ and P^- are polynomials in w_n, $P^+(\xi,\cdot)$ and $P^-(\xi,\cdot)$ are bounded in $z + r\Delta$ and $P^-(\xi,w) \neq 0$ if $w \in z + r\,\overline{\Delta}$ and $\xi \in U$.

(ii) P^+ as a polynomial in w_n is monic, has degree N^+ in w_n, and for each $\xi \in U$ and $w' \in z' + r\Delta'$, all roots of the polynomial $\tau \to P^+(\xi, w', \tau)$ satisfy $|\tau - z_n| < rs$, where $w' = (w_1,\ldots,w_{n-1})$ and

$$\Delta' = \{w' \in C^{n-1} \mid |w_j| < s' \qquad j=1,\ldots,n-1\}.$$

(iii) There exists a constant $C_1 > 0$ depending only on μ, n, and C so that $\widetilde{P}(\xi)(z:r) \leq C_1 r^{N^+} \inf_{z+r\Delta} |P^-(\xi,\cdot)|$ and

$$\sup_{z+r\Delta} |P^-(\xi,\cdot)| \leq C_1 r^{-N^+} \widetilde{P}(\xi)(z:r).$$

B. If $f \in \Gamma(U \times (z + r\Delta), E0)$ and $f(\xi,\cdot)$ is bounded in $z + r\Delta$ for each $\xi \in U$, then $f = Pg + h$ where $g, h \in \Gamma(U \times (z+r\Delta), E0)$, h is a polynomial in w_n of degree $< N^+$ and there exists a constant $C' > 0$ depending only on μ, n, and C so that

$$\sup_{z+r\Delta} |h(\xi,\cdot)| + \widetilde{P}(\xi)(z:r) \sup_{z+r\Delta} |g(\xi,\cdot)| \leq C' \sup_{z+r\Delta} |f(\xi,\cdot)|.$$

Moreover, there exist nonnegative integers N_k depending on λ, n, and k and constants K_k depending on λ, n, m, k, C, and the top order terms of P such that

$$|D^\alpha P^+(\xi,w)| \le K_k(1+|\xi|^2+|z|^2+r^2)^{N_k}\,\tilde{P}(\xi)(z:r)^{-N^+(k+1)}$$ and the same inequality for $|D^\alpha P^-(\xi,w)r^{N^+(k+1)}|$;

$$|D^\alpha h(\xi,w)| \le K_k(1+|\xi|^2+|z|^2+r^2)^{N_k}\,\tilde{P}(\xi)(z:r)^{-N^+(k+1)} \cdot r^{-N^+(k+1)} \sum_{\beta\le\alpha} \sup_{z+r\Delta} |D^\beta f(\xi,\cdot)|$$

and the same inequality for $|D^\alpha g(\xi,w)|$

for every m-multi-index α of length $|\alpha| = k$ and for $\xi \in U$, $w \in z + r\Delta$.

C. If $f \in \Gamma(U \times (z+r\Delta), E^k 0)$ and $f(\xi,\cdot)$ is bounded in $z + r\Delta$ for each $\xi \in U$, then $f = Pg + h$ where $g, h \in \Gamma(U \times (z+r\Delta), E^k 0)$, h is a polynomial in w_n of degree $< N^+$ and there exist constants C, N, and M depending on P, k, and C so that

$$|D^\alpha h(\xi,w)| \le C(1+|\xi|^2+|z|^2+r^2)^N\,\tilde{P}(\xi)(z:r)^{-M} r^{-M} \cdot \sum_{\beta\le\alpha} \sup_{z+r\Delta} |D^\beta f(\xi,\cdot)|$$

for $\xi \in U$, $w \in z + r\Delta$ and $|\alpha| \le k$. The same estimate is true with g in place of h.

Here $D^\alpha = \left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial\xi_m}\right)^{\alpha_m}$ and λ is the total degree of P.

<u>Proof</u>: Let $Q(\xi,w) = P(\xi, z + rw)$. We can expand Q in powers of w, i.e., $Q(\xi,w) = \sum_{|\gamma|\le\mu} a_\gamma(\xi)w^\gamma$ (γ n-multi-indices). Then $P(\xi,w) = Q(\xi, (w-z)/r) = \sum_{|\gamma|\le\mu} a_\gamma(\xi)(w-z)^\gamma r^{-|\gamma|}$.

Since any two norms on the finite dimensional vector space $P(\mu,n)$ (polynomials in $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ of degree $\leq \mu$) are equivalent, there exists a constant $C_0 > 0$ depending only on μ and n so that

$$C_o^{-1}\tilde{Q}(\xi)(0:1) \leq \sum_{|\gamma|\leq\mu} |A_\gamma(\xi)| \leq C_o Q(\xi)(0:1).$$

Notice that $\tilde{Q}(\xi)(0:1) \neq 0$ if $\xi \in U$ since $P(\xi,\cdot) \neq 0$ if $\xi \in U$ and that the coefficients $a_\gamma(\xi)/\tilde{Q}(\xi)(0:1)$ of $Q(\xi,w)/\tilde{Q}(\xi)(0:1)$ vary in the compact set $[-C_o, C_o]$ depending only on μ and n. By a compactness argument, we can find $\delta > o$ depending only on μ, c and n, and s' depending only on C, μ, n, and s so that, if

$\Delta' = \{w' \in C^{n-1} | |w_j| < s' \quad j=1, \ldots, n-1\}$, $D = \{\tau \in C | |\tau| < s\}$, and $\Delta = \Delta' \times D$, then $\Delta \subseteq B$ and $|Q(\xi,w)| \geq \tilde{Q}(\xi)(0:1)/2C$, if $\xi \in U$, $w' \in \Delta'$ and $||w_n| - s| < \delta$. (Note that the hypothesis becomes $|Q(\xi, 0, w_n)| \geq \tilde{Q}(\xi)(0:1)/C$ if $|w_n| = s$.) By Rouche's theorem and the fact that $U \times \Delta'$ is connected, we see the polynomial $\tau \to Q(\xi, w', \tau)$ has exactly N^+ roots with $|\tau| < s - \delta$ if $\xi \in U$ and $w' \in \Delta'$. Denote these roots by $\tau_j(\xi, w')$ for $j=1, \ldots, N^+$.

Define $Q^+(\xi, w) = \prod_{j=1}^{N^+} (w_n - \tau_j(\xi, w'))$ for $\xi \in U$, $w' \in \Delta'$ and

$$P^+(\xi, w) = \prod_{j=1}^{N^+} (w_n - z_n - r\tau_j(\xi, (w' - z')/r)) \quad \text{for } \xi \in U,\ w' \in z' + r\Delta'.$$

$$\text{Set} \quad \theta_\ell(\xi, w') = \frac{1}{2\pi i}\int \frac{\tau^\ell \frac{\partial Q}{\partial z_n}(\xi, w', \tau)}{Q(\xi, w', \tau)}\, d\tau \qquad \text{for } \ell = 0, 1, 2, \ldots \ .$$

Since Q does not vanish on $U \times \Delta' \times \partial D$, θ_ℓ is holomorphic on an open neighborhood of $U \times \Delta'$ in C^{m+n}. By Cauchy's integral formula,

$\theta_\ell(\xi, w') = \sum_{j=1}^{N^+} (\tau_j(\xi, w'))^\ell$ and the coefficients of powers of w_n in Q^+ are elementary symmetric polynomials of $(\tau_1(\xi, w'), \ldots, \tau_{N^+}(\xi, w'))$. Since the elementary symmetric polynomials are polynomials with rational coefficients of the "power sums" $r_k(x) = \sum_{j=1}^{N^+} x_j^k$, we can find rational coefficients $b_{\ell,\alpha}$ (α are $N^+ + 1$-multi-indices) so that, with $\theta^\alpha = \theta_o^{\alpha_o} \cdot \ldots \cdot \theta_{N^+}^{\alpha_{N^+}}$,

$$Q^+(\xi, w) = \sum_\ell \sum_\alpha b_{\ell,\alpha}(\theta(\xi, w'))^\alpha w_n^\ell, \text{ for } 0 \le \ell \le N^+, |\alpha| \le N^+,$$

and so Q^+ is holomorphic in a neighborhood of $U \times \Delta' \times C$. Similarly, P^+ is holomorphic in a neighborhood of $U \times (z' + r\Delta') \times C$ and

$$P^+(\xi, w) = \sum_\ell \sum_\alpha b_{\ell,\alpha} \theta(\xi, (w' - z')/r)^\alpha (w_n - z_n)^\ell r^{N^+ - \ell} .$$

Define $P^- = P/P^+$ on $U \times (z' + r\Delta') \times C$. For each $\xi \in U$ and $w' \in z' + r\Delta'$, $\tau \to P^-(\xi, w', \tau)$ is a polynomial in τ.

By Cauchy's integral formula

$$P^-(\xi, w) = \frac{1}{2\pi i} \int_{r\partial D} \frac{P(\xi, w', \tau)}{P^+(\xi, w', \tau)(\tau - w_n)} d\tau$$

if $w_n \in z_n + rD$, so P^- is holomorphic on a neighborhood of $U \times (z + r\Delta)$.

Since $|\tau_j(\xi, w')| < s - \delta$ (by the choice of δ), we see

$|Q^+(\xi, w)| \ge (|w_n| + \delta - s)^{N^+}$ if $\xi \in U$, $w' \in \Delta'$, $|w_n| > s - \delta$, and

$|P^+(\xi, w)| \ge (|w_n - z_n| - r(s-\delta))^{N^+}$ if $\xi \in U, w' \in z' + r\Delta', |w_n - z_n| \ge r(s-\delta)$.

Then $|P^-(\xi, w)| \leq (r\delta)^{-N^+} \sup_{z+r\Delta} |P(\xi,\cdot)|$ if $|w_n - z_n| = rs$ and by the maximum principle,

$$|P^-(\xi, w)| \leq (r\delta)^{-N^+}\tilde{P}(\xi)(z:r) \quad \text{if} \quad w \in z + r\Delta,\ \xi \in U.$$

Since the roots of $Q^+(\xi, w', \tau)$ satisfy $|\tau| < s < 1$ and the coefficients of $\tau \to Q^+(\xi, w', \tau)$ are elementary symmetric polynomials in the roots, there is a constant C'' depending only on μ and n so that the coefficients of powers of τ are bounded by C''. So there exists a constant C''' depending only on μ and n so that $\sup_{|\tau|<1} |Q^+(\xi,w',\tau)| \leq C'''$. Since $P^+(\xi,w) = r^{N^+}Q^+(\xi,(w-z)/r)$ if $\xi \in U$ and $w' \in z' + r\Delta'$,

$$\sup_{|\tau-z_n|<r} |P^+(\xi, w', \tau)| \leq C'''r^{N^+}$$

i.e., $|P^+(\xi,w)| \leq C'''r^{N^+}$ if $\xi \in U$ and $w \in z + r\Delta$. Thus $|P^-(\xi,w)| \geq |P(\xi,w)|r^{-N^+}/C''' \geq \tilde{P}(\xi)(z:r)r^{-N^+}/2CC'''$ if $\xi \in U$, $w' \in z' + r\Delta'$, and $|w_n - z_n| = rs$, and so, by the maximum principle for $1/P^-$, we see

$$|P^-(\xi,w)| \geq \tilde{P}(\xi)(z:r)r^{-N^+}/2CC''' \quad \text{if} \quad \xi \in U,\ w \in z + r\Delta.$$

This proves A if we set C_1 = maximum of $2CC'''$ and δ^{-N^+}.

If $s - \delta < \rho < s$, $\xi \in U$, $w' \in z' + r\Delta'$ and $|w_n - z_n| < r\rho$, define

$$h(\xi,w) = \frac{1}{2\pi i}\int_{|\tau-z_n|=r\rho} \frac{f(\xi,w',\tau)(P^+(\xi,w',\tau) - P^+(\xi,w',w_n))}{P^+(\xi,w',\tau)(\tau-w_n)}\, d\tau$$

and $\displaystyle g(\xi,w) = \frac{1}{2\pi i P^-(\xi,w',w_n)} \int_{|\tau - z_n| = r\rho} \frac{f(\xi,w',\tau)}{P^+(\xi,w',\tau)(\tau - w_n)}\, d\tau$.

These integrals are independent of ρ since $P^+(\xi,w',\tau)$ has no roots in $|\tau - z_n| \geq r(s - \delta)$. Clearly, $g, h \in \Gamma(U \times (z + r\Delta), E0)$ and $f = Pg + h$ in $U \times (z + r\Delta)$. For each $\xi \in U$, $w' \in z' + r\Delta'$,

$$\frac{P^+(\xi,w',\tau) - P^+(\xi,w',w_n)}{\tau - w_n}$$

is a polynomial in τ (and in w_n) of degree $< N^+$, so h is a polynomial in w_n of degree $< N^+$.

As before, if $Q^+(\xi,w',\tau) = \sum_{\ell=0}^{N^+} d_\ell(\xi,w')\tau^\ell$, then $|d_\ell(\xi,w')| \leq C''$ if $\xi \in U$ and $w' \in \Delta'$ and so

$$\left| \frac{Q^+(\xi,w',\tau) - Q^+(\xi,w',w_n)}{\tau - w_n} \right| \leq C'' \sum_{\ell=1}^{N^+} \sum_{j=1}^{\ell} |\tau^{\ell-j} w_n^j| \leq C_2 \rho^{N^+-1}$$

for $|w_n| < \rho$, $|\tau| = \rho$, and $\rho \geq s - \delta$. So if $|w_n| < \rho$, $|\tau| = \rho$, $\rho > s-\delta$, and $w' \in \Delta'$, $\xi \in U$, then

$$\left| \frac{P^+(\xi,z'+rw',z_n+r\tau) - P^+(\xi,z'+rw',z_n+rw_n)}{z_n + r\tau - (z_n + rw_n)} \right|$$

$$= \left| \frac{r^{N^+}(Q^+(\xi,w',\tau) - Q^+(\xi,w',w_n))}{r(\tau - w_n)} \right| \leq C_2 (r\rho)^{N^+-1}$$

where C_2 depends on μ and C'' and so on μ and n.

By the definition of h, we see that

$$|h(\xi,w)| \leq \sup_{z+r\Delta} |f(\xi,\cdot)| C_2 (r\rho)^{N^+-1} / (|w_n - z_n| - r(s-\delta))^{N^+}$$

$$\cdot \frac{1}{2\pi} \int_{|\tau - z_n| = r\rho} |d\tau|$$

if $s - \delta \leq \rho < s$, $w' \in z' + r\Delta'$, $|w_n - z_n| < r\rho$, and $\xi \in U$. Thus

$$|h(\xi,w)| \leq C_2 \sup_{z+r\Delta} |f(\xi,\cdot) \limsup_{\rho\to s} (r\rho)^{N^+} (|w_n - z_n| - r(s-\delta))^{-N^+}$$

$$\leq C_2\, \delta^{-N^+} \sup_{z+r\Delta} |f(\xi,\cdot)|$$

$$= C_3 \sup_{z+r\Delta} |f(\xi,\cdot)|$$

where $C_3 = C_2 \delta^{-N^+}$ depends on μ, n, and C. Since $Pg = f - h$, we have

$$\sup_{z+r\Delta} |P(\xi,\cdot)||g(\xi,\cdot)| \leq (C_3 + 1) \sup_{z+r\Delta} |f(\xi,\cdot)|.$$

Also, $|P(\xi,w)| \geq \widetilde{P}(\xi)(z:r)/2C$ (by choice of δ) when $\xi \in U$, $w' \in z' + r\Delta'$, and $||w_n - z_n| - rs| < r\delta$, so, by the maximum principle, we see

$$\sup_{z+r\Delta} |g(\xi,\cdot)| \leq 2C(C_3+1)\widetilde{P}(\xi)(z:r)^{-1} \sup_{z+r\Delta} |f(\xi,\cdot)|.$$

Let $C' = C_3 + 2C(C_3 + 1)$.

Recall $P^+(\xi,w) = \sum_{\ell=0}^{N^+} \sum_{|\alpha|\leq N^+} b_{\ell,\alpha} (\theta(\xi,(w'-z')/r)^{\alpha} (w_n - z_n)^{\ell} r^{N^+ - \ell}$

and $|Q(\xi,(w'-z')/r,\tau)| \geq \widetilde{Q}(\xi)(0:1)/2C = \widetilde{P}(\xi)(z:r)/2C$ if $\xi \in U$, $w' \in z' + r\Delta'$, $w_n \in z_n + rD$, and $|\tau| = s$.

From the definition of θ_ℓ, it is clear that if γ is an m-multi-index, $|\gamma| = k$, and $D^\gamma = \left(\frac{\partial}{\partial\xi_1}\right)^{\gamma_1}\cdot\ldots\cdot\left(\frac{\partial}{\partial\xi_m}\right)^{\gamma_m}$ then

$$|D^\gamma\theta_\ell(\xi,w')| \leq (2C)^{|\gamma|+1}\widetilde{P}(\xi)(z:r)^{-|\gamma|-1}C_4 s^{\ell+1}$$

$$\cdot\ (1+|\xi|^2+|z'+rw'|^2 + |z_n|^2 + r^2s^2)^{(|\gamma|+1)(\lambda-1)/2}$$

$$\leq C_4'(\widetilde{P}(\xi)(z:r))^{-|\gamma|-1}(1+|\xi|^2+|z|^2+r^2)^{(|\gamma|+1)(\lambda-1)/2}$$

where C_4' depends on k, λ, n, C, and the top order part of P, if $\xi \varepsilon U$ and $w' \varepsilon \Delta'$. Since $\widetilde{P}(\xi)(z:r) \leq C_4''(1+|\xi|^2+|z|^2+r^2)^{\lambda/2}$, we see that there exists a nonnegative integer N' depending on λ, n, and k so that

$$|D^\gamma P^+(\xi,w)| \leq K'\widetilde{P}(\xi)(z:r)^{-N^+(k+1)}(1+|\xi|^2+|z|^2+r^2)^{N'}$$

if $|\gamma| \leq k$, $\xi \varepsilon U$, and $w \varepsilon z + r\Delta$, and where K' depends on λ, k, n, C, and the top order part of P.

Recall $P^- = P/P^+$. If $|\gamma| = k$, $\xi \varepsilon U$, $w' \varepsilon z' + r\Delta'$, and $|w_n - z_n| = rs$, then

$$|D^\gamma P^-(\xi,w)| \leq K'(r\delta)^{-N^+(k+1)}\widetilde{P}(\xi)(z:r)^{-N^+(k+1)}(1+|\xi|^2+|z|^2+r^2)^{N''}$$

and since $D^\gamma P^-(\xi,\cdot)$ is holomorphic

$$|D^\gamma P^-(\xi,\cdot)| \leq K''\widetilde{P}(\xi)(z:r)^{-N^+(k+1)}r^{-N^+(k+1)}(1+|\xi|^2+|z|^2+r^2)^{N''}$$

if $\xi \varepsilon U$, $w \varepsilon z + r\Delta$.

From the definition of g and h and the estimates on P^+ and P^-, we see that there exist constants K_1, $K_2 > 0$ and nonnegative integers

N_1, N_2 so that

$$|D^{\gamma}h(\xi,w)| \leq K_{1_k} r^{-N^+(k+1)}\widetilde{P}(\xi)(z:r)^{-N^+(k+1)}(1+|\xi|^2+r^2+|z|^2)^{N_{1_k}}$$

$$\cdot \sum_{\beta\leq\gamma} \sup_{z+r\Delta} |D^{\beta}f(\xi,\cdot)|$$

if $\xi \in U$, $w \in z + r\Delta$, and $|\gamma| = k$. Similarly for g with K_{2_k}, etc.

Now let N_k = maximum of N', N'', N_{1_k}, N_{2_k} and let K_k = maximum of K', K'', K_{1_1}, K_{2_k}.

Notice C. is essentially the same as B. as far as the proof is concerned.

CHAPTER FIVE

LOCAL EXISTENCE THEOREMS

In this chapter we prove a version of Lemma 1 of Petersen (1975) depending on one parameter. We would prefer to establish the following lemmas for any number of parameters, but the method of proof breaks down if $m > 1$. We are only interested in taking derivatives in the parameter ξ, and so we introduce the notation $D = \frac{\partial}{\partial \xi}$ for this chapter.

<u>Lemma 5.1</u> Let P be a $p \times q$ matrix over $C[\xi, w]$, let $m = 1$, and let B be the open unit ball in C^n. There exist a polynomial $a \in C[\xi]$ and $t \in (0, 1)$ such that:

If $E > 0$, then there exist constants $C_k > 0$, and nonnegative integers M_k and N_k, for $k = 0, 1, 2, \ldots$, so that if $z \in C^n$, $0 < r < E$, and $u \in \Gamma(R \times (z + rB), E0)^q$ then there exists

$v \in \Gamma(R \times (z + rtB), E0)^q$ with $Pv = aPu$ in $R \times (z + rtB)$ and

$$\sup_{z+rtB} |D^k v(\xi,\cdot)| \leq C_k(1+r^{-M_k})(1+|z|^2+|\xi|^2+r^2)^{N_k} \sum_{\ell=o}^{k} \sup_{z+rB} |D^\ell Pu(\xi,\cdot)|$$

for every $k = 0, 1, \ldots$ and $\xi \in R$.

Here C_k, M, and N depend on k and P.

<u>Proof</u>: The proof is by induction on n and p.

(i) $n = 0$, $p = 1$. $P = (P_1, \ldots, P_q)$ and we may assume $P_1 \not\equiv 0$. Let $a = P_1 \in C[\xi]$, $v_1 = Pu$, $v_j = 0$, $j = 2, \ldots, q$ and $v = (v_1, \ldots, v_q)$. Then $Pv = P_1 Pu = aPu$ and the estimates are clearly true.

(ii) Let $n \geq 0$ and $p > 1$ and assume inductively that we have proved the lemma for all $n' \leq n$ and $p' < p$.

Consider $P^1 = (P_{11}, \ldots, P_{1q})$. By hypothesis, there exists a polynomial $a' \in C[\xi]$ and there exist constants C_k', non-negative integers M_k' and N_k', $t' \in (0,1)$ and $v' \in \Gamma(R \times (z+rt'B), E0)^q$ so that $P^1 v' = a'P^1 v$ in $R \times (z+rt'B)$ and

$$|D^k v'(\xi,\cdot)| \leq C_k'(1+r^{-M_k})(1+|\xi|^2+|z|^2+r^2)^{N_k}$$

$$\cdot \sum_{\ell=o}^{k} \sup_{z+rB} |D^\ell P^1 u(\xi,\cdot)|$$

for $\xi \in R$.

Let $C[\xi,z]^r \xrightarrow{Q} C[\xi,z]^q \xrightarrow{P^1} C[\xi,z]$ be exact.

By Lemma 2.3

$$\Gamma(R \times (z+rt'B), E0)^r \xrightarrow{Q} \Gamma(R \times (z+rt'B), E0)^q$$

$$\xrightarrow{P^1} \Gamma(R \times (z+rt'B), E0)$$ is exact. Let $f \in \Gamma(R \times (z+rt'B), E0)^r$ so $Qf = a'v - v'$.

Consider the equation $PQg = PQf$, which is a system of $p - 1$ equations. By the induction hypothesis, there exist $a'' \in C[\xi]$, C_k'', M_k'', N_k'', and $t'' \in (0,1)$ and there exists $g \in \Gamma(R \times (z+rt't''B), E0)^r$ such that $PQg = a''PQf = a''P(a'u-v')$ in $R \times (z+rt't''B)$ with the estimates. Set $v = Qg + a''v'$, $t = t't''$ and $a = a'a''$.

(iii) Let $n \geq 1$ and $p = 1$ and assume inductively that we have proved the lemma for all $n' < n$ and $p' < \infty$.

$P = (P_1, \ldots, P_q)$. We may assume $P_q \not\equiv 0$ and the degree of P_q in w is μ = maximum of the degrees of P in w. We may write $P_q = bA_q$, where $b \in C[\xi]$, $A_q \in C[\xi, w]$, and $A_q(\xi, \cdot) \not\equiv 0$ for $\xi \in C$.

Let $K \subseteq B - \{0\} \subseteq C^n$ be a finite set such that its image under the evaluation map $B \to P(\mu,n)'$ is a basis for $P(\mu,n)'$, where $P(\mu,n)$ is the finite dimensional C-vector space of polynomials in $C[w]$ of degree $\leq \mu$ and $P(\mu,n)'$ is its dual. Let $\{L(e) | e \in K\} \subseteq P(\mu,n)$ be the dual basis of $P(\mu,n)$. Then

$$A_q(\xi,w) = \Sigma_{e\epsilon K} A_q(\xi,e)L(e)(w) \qquad \text{and}$$

$$\tilde{A}_q(\xi)(z:r) \leq C \max_{e\epsilon K} |A_q(\xi, z+re)|$$

where C depends only on μ and n.

Let $V(e) = \{(\xi,w) \in R \times C^n$ such that $|A_q(\xi, w + re)| > 1/2 \max_{e'\epsilon K} |A_q(\xi, w + re')|\}$. $\{V(e) | e \in K\}$ is an open cover of $R \times C^n$ and, if $(\xi,w) \in V(e)$, then $\tilde{A}_q(\xi)(w:r) \leq 2C|A_q(\xi, w + re)|$. Define $f_e(\tau) = A_q(\xi, w + re\tau) \in C[\tau]$, a polynomial in $\tau \in C$. f_e has $\sigma \leq \mu$ roots, which we call $\tau_1, \ldots, \tau_\sigma$. We divide $(1/2 - 1/8\mu, 1 + 1/8\mu)$ into $2\mu + 1$ equal intervals and let $s_\ell = 1/2 + \frac{\ell}{4\mu}$ be their midpoints. $\{|\tau_j| | j=1, \ldots, \sigma\}$ can intersect at most $2\sigma \leq 2\mu$ of these intervals. So for each $\xi \in R$, there exists $\ell \in \{0, \ldots, 2\mu\}$, such that

$$|s_\ell - |\tau_j|| > \frac{1}{8\mu} \quad \text{for each } j=1, \ldots, \sigma .$$

Define $V(e,\ell) = \{(\xi,w) \in V(e) \mid A_q(\xi, w + rse) \neq 0$ for all

$$s \in C \text{ with } |s| \in \left[\frac{1}{2} + \frac{2\ell-1}{8\mu}, \frac{1}{2} + \frac{2\ell+1}{8\mu}\right]\Bigg\} .$$

$\{V(e,\ell) \mid e \in K,\ \ell=0, \ldots, 2\mu\}$ is an open cover of $R \times C^n$ consisting of semi-algebraic sets. If $(\xi,w) \in V(e,\ell)$,

$$\tilde{A}_q(\xi)(w:r) \leq 2C(B\mu+1)^{\mu} \inf_{\substack{|\tau|=1 \\ \tau\in C}} |A_q(\xi, w+rs_\ell e\tau)| .$$

Assume now that $z \in C^n$ is fixed and $(\xi,z) \in V(e,\ell)$. We make a unitary change of coordinates and so assume that

$$\tilde{A}(\xi)(z:r) \leq 2C(8\mu+1)^{\mu} \inf_{|\tau-z_n|=rs} |A_q(\xi,z',\tau)|$$

where $s = |s_\ell e|$. Define $W = W(e,\ell) = V(e,\ell)(z)$.

We apply Lemma 4.1 to the polynomial A_q and we find $s' \in (0,1)$,

$$\Delta = \{w \in C^n \mid |w_j| < s' \quad j=1, \ldots, n-1,\ |w_n| < s\},$$

$$A^+,\ A^- \in \Gamma(W \times (z + r\Delta), A0),$$

and $g,h \in \Gamma(W \times (z + r\Delta), E0)$ such that $A_q = A^+A^-$ and $u = A_q g + h$ in $W \times (z + r\Delta)$ and we have the estimates from the Lemma. Also, h is a polynomial in w_n of degree $< N^+$ where N^+ is the number of roots of the polynomial $\tau \to A_q(\xi,z',\tau)$ in $|\tau - zn| < rs$.

By the same argument as on page 197 of Hörmander (1966), we see $P_1 f_1 + \ldots + P_{q-1} f_{q-1} + A_q f_q = h$ where $f_j \in \Gamma(W \times (z+r\Delta), E0)$

and $f_1, \ldots, f_{q-1}$ are polynomials in w_n of degree $< N^+ \leq \mu$. Thus $A_q f_q$ is a polynomial in w_n (of degree $< \mu + N^+$) and, by Lemma 7.6.9 of Hörmander (1966), $A_q^- f_q$ is a polynomial in w_n (of degree $< \mu$).

By Lemma 3.2, there exist C^∞ functions $\psi(e', \ell')$ on R ($e' \in K$, $\ell = 0, \ldots, 2\mu$) such that $\mathrm{supp}(\psi(e',\ell')) \subseteq W(e',\ell')$ and $\sum_{e' \in K} \sum_{\ell'=0}^{2\mu} \psi(e', \ell') = 1$ on R. Set $f_j' = \psi(e, \ell) A^- f_j$ and $h' = \psi(e, \ell) A^- h$. Then

$$P_1 f_1' + \ldots + P_{q-1} f_{q-1}' + A_q f_q' = h'$$

and f_j', h' are polynomials in w_n of degree $< \mu$. Also, the support of these functions is contained in $W(e, \ell) \times (z' + r\Delta') \times C$. This equation is equivalent to a system of 2μ equations in the coefficients (of powers of w_n) of the f_j' on the left and of h' and zeros on the right and in the variables (ξ, w').

By the induction hypothesis applied to this system, we can find a polynomial $a_2 \in C[\xi]$, constants $C_1 > 0$ and $t_1 \in (0,1)$, nonnegative integers M_1 and N_1 and, for $j=1, \ldots, q$, $\ell = 0, \ldots, 2\mu$, $F_{j\ell} \in \Gamma(R \times (z' + rs't, B), E0)^{2\mu}$ such that the functions $F_j(\xi, w) = \sum_{\ell=0}^{2\mu} F_{j\ell}(\xi, w') w_n^\ell$ satisfy

$$P_1 F_1 + \ldots + P_{q-1} F_{q-1} + A_q F_q = a_2 h'$$

and, if α is a nonnegative integer $\leq k$,

$$\sum_{j=1}^{q}\sum_{\ell=o}^{2\mu} \sup_{z'+rs't_1B'} |D^{\alpha}F_{j\ell}(\xi,\cdot)| \leq C_1(1+r^{-M_1})(1+|\xi|^2+|z'|^2+r^2)^N$$

$$\cdot \sum_{\ell=o}^{2\mu} \sum_{\beta\leq\alpha} \sup_{z'+rs'B'} |D^{\beta}h'_{\ell}(\xi,\cdot)|$$

where B' is the unit ball in C^{n-1} and $h'(\xi,w) = \sum_{\ell=o}^{2\mu} h_{\ell}(\xi,w')w_n^{\ell}$.

Now set $v_j(e,\ell) = a_1F_j/A^-$ $j < q$ and $v_q = a_2\psi(e,\ell)g + F_q/A^-$.

Set $a(e,\ell) = a_1a_2$. Then

$$Pv(e,\ell) = a_1(P_1F_1 + \dots + A_qF_q)/A^- + a_1a_2\psi(e,\ell)A_qg$$

$$= a(e,\ell)\psi(e,\ell)u.$$

We made a unitary change of coordinates in C^n earlier, and now we make the inverse coordinate transformation; this does not affect $a(e,\ell)$ or $\psi(e,\ell)$, which depend only on ξ.

Set $a = \Pi_{e\varepsilon K,\ \ell=o,\dots,\ 2\mu}\ a(e,\ell)$ and

$v = \sum_{e\varepsilon K} \sum_{\ell=o}^{2\mu} av(e,\ell)/a(e,\ell)$. $a \varepsilon C[\xi]$ and $v \varepsilon \Gamma(R \times (z+rtB),E0)^q$

where $t = \min\{s_\ell e,\ s'(e,\ell)t_1(e,\ell) \mid e \varepsilon K,\ \ell=o,\ \dots,\ 2\mu\}$. Then

$$Pv = \Sigma_{e,\ell}\ aPv(e,\ell)/a(e,\ell) = \Sigma_{e,\ell}\, a\psi(e,\ell)u = au.$$

We will estimate v now. Let

$A_q(\xi,w) = \sum_{\gamma} a_{\gamma}(\xi)w^{\gamma} = \sum_{\gamma} b_{\gamma}(\xi)(w - z)^{\gamma}$. Now

$b_{\gamma}(\xi) = \sum_{\omega\geq\gamma} \binom{\omega}{\gamma} z^{\omega-\gamma} a_{\omega}(\xi)$. Here α and ω are n-multi-indices of length $\leq \mu$. There exists $C_0 > 0$ depending only on μ and n so that

$$C_o^{-1}\tilde{A}_q(\xi)(z:r) \leq \sum_{\gamma} |b_{\gamma}(\xi)|^2 r^{2|\gamma|} \leq C_o\tilde{A}_q(\xi)(z:r).$$

Since $0 < r < E$, $E^{2(\mu-|\gamma|)} > r^{2(\mu-|\gamma|)}$ and so $r^{2|\gamma|} \geq E^{-2(\mu-|\gamma|)} r^{2\mu}$. If $E \geq 1$, $r^{2|\gamma|} \geq E^{-2\mu} r^{2\mu}$, and if $0 < E \leq 1$, $r^{2|\gamma|} \geq r^{2\mu}$, if $|\gamma| \leq \mu$. Thus there is a constant C_1 depending on μ, n, and E so that

$$\tilde{A}_q(\xi)(z:r) \geq C_1 r^{2\mu} \sum_{|\gamma|\leq\mu} |b_\gamma(\xi)|^2 .$$

Let $b(\xi) = \sum_{|\gamma|\leq\mu} |b_\gamma(\xi)|^2$ and

$c(\xi,w) = \sum_{|\gamma|\leq\mu} \left|\sum_{\omega\geq\gamma} \binom{\omega}{\gamma} w^{\omega-\gamma} a_\omega(\xi)\right|^2$. These are real polynomials on R and R^{2n+1}, respectively, and $C(\xi,z) = b(\xi)$. $A_q(\xi,\cdot) \not\equiv 0$ and $c(\xi,w) \neq 0$ for real ξ and $w \in C^n$. (Notice $c(\xi,w) = b_w(\xi)$ and $b_w(\xi) \neq 0$.) By a result of B. Petersen using Lemma 2.1 of Hörmander (1969), p. 276, there exists a constant $C_2 > 0$ and a rational number λ so that

$$|c(\xi,w)| \geq C_2^{-1} \min\{1, |(\xi,w)|^\lambda\} \quad \xi \in R,\ w \in C^n.$$

Thus $|c(\xi,w)|^{-1} \leq C_2 \max\{1, |(\xi,w)|^{-\lambda}\}$. In the unit ball in $R \times C^n$, c has a positive lower bound. Thus, if $H = \max\{0, -[\lambda]\}$,

$$c(\xi,w)^{-1} \leq C_3(1 + |(\xi,w)|^2)^{H/2} \leq C_3(1 + |\xi|^2 + |w|^2)^L,$$

where $L = 1 + [H/2]$. So $b(\xi)^{-1} \leq C_3(1 + |\xi|^2 + |z|^2)^L$. Notice C_3 depends on A_q. Thus

$$\tilde{A}_q(\xi)(z:r) \geq C_1 C_3^{-1} r^{2\mu}(1 + |\xi|^2 + |z|^2)^{-L} .$$

Recall that on $W(e,\ell)$, $A_q^-(\xi,w) \geq C_4 r^{-N^+} \tilde{A}_q(\xi)(z:r)$ for $w \in z + rtB$, from Lemma 4.1 (since $tB \subseteq \Delta$). The estimate of $D^k v$ now follows from the various estimates of g, F_j, A^-, etc.

Lemma 5.2 Let P be a $p \times q$ matrix over $C[\xi,w]$, $m=1$, and B be the open unit ball in C^n. There exist a polynomial $b \in C[\xi]$ and $t \in (0,1)$ such that, if $E > 0$, then there exist constants $C_k > 0$ and nonnegative integers M_k and N_k such that:

If $z \in C^n$, $0 < r < E$, and $u \in (\Gamma(R \times (z + rB), E0))^q$, then there exists $v \in \Gamma(R \times (z + rtB), E0)^q$ such that $Pv = bPu$ in $R \times (z + rtB)$ and if ϕ is any continuous real-valued function on $z + r\bar{B}$, then

$$r^{M_k} \int_{z+rtB} |D^k v(\xi,\cdot)|^2 e^{-\phi - N_k\theta}\, dV \le C_k e^a (1 + |\xi|^2)^{N_k} \cdot \sum_{\ell=0}^{k} \int_{z+rB} |D^\ell Pu(\xi,\cdot)|^2 e^{-\phi} dV$$

where $a = \sup_{z+rB} |\phi(w') - \phi(w'')|$ and $\theta(w) = \log(1 + |w|^2)$.

Proof: The proof is in Petersen (1974) and uses one of Peetre's inequalities. The proof in Petersen (1974) does not involve parameters, but the proof still works in this case.

Definition: If $t \in (0,1)$ satisfies the conclusion of this lemma, we say t is good for P.

Lemma 5.3 Let U be an open, connected subset of R^m, $z \in C^n$, $r > 0$, $s \in (0,1)$, $C > 0$, and μ and k be nonnegative integers. Suppose $P_1, \ldots, P_q \in C[\xi,w]$ all have degrees in $w = (w_1, \ldots, w_n) \le \mu$, $P_q(\xi,\cdot) \not\equiv 0$ and $\tilde{P}_q(\xi)(z:r) \le C \inf_{|\tau - z_n| = rs} |P_q(\xi, z', \tau)|$ for $\xi \in U$. Then there exists $s' \in (0,1)$ depending only on μ, n, s, and C such that $\ker(P) \subseteq \Gamma(U \times (z + r\Delta), E^k 0)^q$ is generated by those of its elements which are polynomials in w_n of degree $\le \mu$.

Here $\Delta = \{w \in C^n \mid |w_j| < s' \quad j < q, \; |w_n| < s\}$ and $P:(c_1, \ldots, c_q) \in \Gamma(U \times (z + r\Delta), E^k 0)^q \rightarrow \sum_{j=1}^{q} c_j P_j$.

Proof: This is a straightforward application of Lemma 4.1 and Lemma 7.6.9 of Hörmander (1966).

Lemma 5.4 Let B be the unit ball in C^n and U be open in R. Let

$$C[\xi,w]^r \xrightarrow{Q} C[\xi,w]^q \xrightarrow{P} C[\xi,w]^P$$

be exact with $m = 1$. Then there exists a polynomial $b \in C[\xi]$ and a constant $t \in (0,1)$ such that:

If $z \in C^n$, $r > 0$, k is a nonnegative integer and $f \in \Gamma(U \times (z + rB), E^k 0)^q$ with $Pf = 0$, then there exists $g \in \Gamma(U \times (z + rtB), E^k 0)^r$ such that $Qg = bf$ in $U \times (z + rtB)$.

Proof: The proof is by an Oka induction and the second and third cases are very similar to (ii) and (iii) of the proof of Lemma 5.1. The polynomial b comes only from the proof of the first case, which we present.

(i) $n = 0$. Let E be a $q \times q$ matrix over $C(\xi)$ so that $S = EQ$ is the unique reduced row echelon matrix of Q. Notice that $Qg' = f$ iff $Sg' = Ef$. Let μ be the rank of Q (over $C(\xi)$). Then there exists a permutation $k_1 < \ldots < k_\mu$, $\ell_1 < \ldots < \ell_s$ of $\{1, \ldots, r\}$ so that $S_{ij} = 0$ if $j < k_i$ and $S_{ik_j} = \delta_{ij}$.

Then define g' by $g'_{k_i} = \sum_{\ell=1}^{q} E_{i\ell} f_\ell$ and $0 = \sum_{\ell=1}^{q} E_{i\ell} f_\ell$ if $i > \mu$.

By Theorem 1.1, we can solve $Q(\xi)g'(\xi) = f(\xi)$ and so $S(\xi)g'(\xi) = E(\xi)f(\xi)$, for ξ off a proper variety. $S(\xi)g'(\xi)$

is a reduced row echelon matrix, so the last $q - \mu$ rows are zero, and thus the second condition in the definition of g' is satisfied. Now let b be the least common denominator of the entries of E and $g = bg'$. Now $Sg = bEf$, so $Qg = bf$.

<u>Lemma 5.5</u> Let P and B be as in Lemma 5.1. There exist a polynomial $a \in \mathbb{C}[\xi]$ and $t \in (0,1)$ such that:

If $E > 0$ and k is a nonnegative integer, then there exist $C > 0$ and nonnegative integers M and N so that if $z \in \mathbb{C}^n$, $0 < r < E$, and $U \in \Gamma(R \times (z + rB), E^k 0)^q$, then there exists $v \in \Gamma(R \times (z+rtB), E^k 0)^q$ with $Pv = aPu$ and

$$\sup_{z+rtB} |D^\ell v(\xi,\cdot)| \leq C(1+r^{-M})(1+|\xi|^2+|z|^2+r^2)^N \sum_{j=o}^{\ell} \sup_{z+rB} |D^j Pu(\xi,\cdot)|$$

for $\xi \in R$ and $\ell = 0, \ldots, k$.

<u>Proof</u>: The proof is the same as that of Lemma 5.1 except that we use Lemma 5.4 in (ii) rather than Lemma 2.3.

<u>Lemma 5.6</u> Let P be a $p \times q$ matrix over $\mathbb{C}[\xi, w]$, $m = 1$, and B be the open unit ball in $\mathbb{C}^n$. There exists a polynomial $b \in \mathbb{C}[\xi]$ and $t \in (0,1)$ such that, if $E > 0$ and k is a nonnegative integer, then there exist constants $C > 0$ and M and N nonnegative integers such that:

If $z \in \mathbb{C}^n$, $0 < r < E$, and $u \in \Gamma(R \times (z + rB), E^k 0)^q$, then there exists $v \in \Gamma(R \times (z + rtB), E^k 0)^q$ such that $Pv = bPu$ in $R \times (z + rtB)$ and if ϕ is any continuous real-valued function on $z + r\bar{B}$, then

$$r^M \int_{z+rtB} |D^\ell v(\xi,\cdot)|^2 e^{-\phi - N\theta} dV \leq Ce^a (1+|\xi|^2)^N \int_{z+rB} \sum_{j=o}^{\ell} |D^j Pu(\xi,\cdot)|^2 e^{-\phi} dV$$

for $\xi \in R$ and $\ell=0, \ldots, k$ where $\theta(w) = \log(1 + |w|^2)$ and

$$a = \sup_{z+rB} |\phi(w') - \phi(w)|.$$

<u>Proof</u>: The proof is the same as Lemma 5.2 and the proof in Petersen (1974) (without parameters) works in this case.

<u>Definition</u> If $t \in (0,1)$ satisfies the condition of Lemma 5.6, we say that t is <u>good</u> for P. For a given P, t is independent of k and the two definitions of good for P agree.

CHAPTER SIX

SOLUTIONS OF THE CAUCHY-RIEMANN AND COBOUNDARY EQUATIONS

In this chapter we solve $\bar{\partial}u(\xi,\cdot) = f(\xi, \cdot)$ with weighted L^2-estimates in the z variables and smooth dependence on ξ. The proof of this result is a modification of some arguments of L. Hörmander (1965). We then solve the coboundary equation $\delta c' = c$ with estimates and smooth dependence on parameters.

We will use the notation of Hörmander (1966, 1965) and some special notation. Suppose U is open in R^m and W is open in C^n. If p and q are nonnegative integers and ϕ is a measurable function of W, then $L^2_{(p,q)}(U, W, \phi)$ denotes the set of continuous functions from U to $L^2_{(p,q)}(W, \phi)$. $L^2_{(p,q)}(W, \phi)$ is the Hilbert space of (p, q)-forms whose coefficients are square integrable with respect to $e^{-\phi}dV$ and has the norm $||c||^2_\phi = \int_W |c|^2 e^{-\phi}dV$, where $|c|^2 = \sum_{|\alpha|=p} \sum_{|\beta|=q} |c_{\alpha,\beta}|^2$.

Let b be a nonnegative integer or $+\infty$. $L^2_{(p,q)}(U, W, \phi, b)$ denotes the subset of $L^2_{(p,q)}(U, W, \phi)$ consisting of those functions which are C^b functions as mappings from U to $L^2_{(p,q)}(W, \phi)$

$\bar{\partial}$ represents the Cauchy-Riemann operator in C^n (i.e., in the z variables only). D_j represents the partial derivative with respect to ξ_j and if α is an m-multi-index, $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_m^{\alpha_m} .$$

Let W be an open set in C^n with C^2 boundary; that is, there exists a real function $\rho \in C^2(C^n)$ such that $W = \{z \in C^n | \rho(z) < o\}$,

$\partial W = \{z \in C^n | \rho(z) = 0\}$ and $d\rho \neq 0$ on ∂W = the boundary of W. W is pseudo-convex (or Levi-convex) iff $\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0$ when $z \in \partial W$, $\sum_{j=1}^{n} \frac{\partial \rho_j}{\partial z_j}(z) w_j = 0$ and $w = (w_1, \ldots, w_n) \in C^n$. If W is open in C_n, W is pseudo-convex iff there is a continuous plurisubharmonic function u in W such that $W_c \subseteq\subseteq W$ for all $c \in R$, where $W_c = \{z \in W | u(z) < c\}$.

A function u defined in an open set $W \subseteq C^n$ is plurisubharmonic iff

(a) u is upper semicontinuous, and

(b) for z and $w \in C^n$, the function $\tau \to u(z + \tau w)$ is subharmonic in the part of C where it is defined. A C^2 function u defined in an open set $W \subseteq C^n$ is strictly plurisubharmonic iff

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k > 0$$

for $z \in W$ and $w \in C^n$, $w \neq 0$.

Lemma 6.1 Let U be open in R^m. Let W be a bounded open set in C^n with a C^2 pseudo-convex boundary, let $\phi \in C^2$ be strictly plurisubharmonic in a neighborhood of $\overline{W}$, and let $e^K \in C(W)$ be the lowest eigenvalue of the matrix $\left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \right)$.

Suppose $f \in L^2_{(p,q)}(U, W, \phi, 1)$ such that $q > 0$,

$$\bar{\partial} f(\xi, \cdot) = 0 ,$$

$$\int_W |f(\xi, \cdot)|^2 e^{-\phi - K} dV < \infty,$$

and $\int_W |D_j f(\xi, \cdot)|^2 e^{-\phi} dV < \infty$ for $j = 1, \ldots, m$ and $\xi \in U$.

Then there exists $u \in L^2_{(p,q-1)}(U, W, \phi)$ such that $\bar{\partial} u(\xi, \cdot) = f(\xi, \cdot)$ and $q \int_W |u(\xi, \cdot)|^2 e^{-\phi} dV \leq \int_W |f(\xi, \cdot)|^2 e^{-\phi - K} dV$ for all $\xi \in U$.

Proof: We will use the notation and results of Hörmander (1965). Let A be the operator on $L^2_{(p,q)}(W, \phi)$ defined by

$$Ah = \sqrt{q}\, e^{K/2} h \in L^2_{(p,q)}(W, \phi).$$

A is bounded and self-adjoint. Let S and T be as defined in Hörmander (1965, p. 99). For $h \in \mathrm{dom}(T^*) \cap \mathrm{dom}(S)$,

$$||Ah||^2_\phi \leq ||T^*h||^2_\phi + ||Sh||^2_\phi .$$

As in the proof of Theorem 1.1.4 of Hörmander (1965), we see

$$|(f(\xi, \cdot)\ g)_\phi| \leq ||h(\xi, \cdot)||_\phi \ ||T^*g||_\phi$$

for all $\xi \in U$ and $g \in \mathrm{dom}(T^*)$, where $f(\xi, \cdot) = A^*h(\xi, \cdot)$. Define $F(\xi) \in (L^2_{(p,q)}(W, \phi))'$ by $F(\xi)(k) = (f(\xi, \cdot), g)_\phi$ where $k = T^*g + k_2$, $k_2 \in R(T^*)^\perp$. $R(T^*)$ is closed since A has a positive lower bound (Hörmander, 1965). Therefore, $F(\xi)$ is well defined for $\xi \in U$ and

$$||F(\xi)|| \leq ||h(\xi)||_\phi = ||e^{-K/2} f||_\phi / \sqrt{q} .$$

Let $\{k_\ell = T^*g_\ell\}$ be an orthonormal basis of $R(T^*)$ in $L^2_{(p,q)}(W, \phi)$ and define $u(\xi) \in L^2_{(p,q-1)}(W, \phi)$ by

$$u(\xi) = \sum_\ell F(\xi)(\, k_\ell)\ k_\ell = \sum_\ell (f(\xi, \cdot), g_\ell)_\phi K_\ell$$

for $\xi \in U$. Notice

$$||u(\xi)(\cdot)||_\phi^2 = \sum_\ell |F(\xi)(k_\ell)|^2 \leq \int_W |f(\xi, \cdot)|^2 e^{-\phi-K} dV/q \quad .$$

It remains to show that $u \in L^2_{(p,q-1)}(U, W, \phi)$ where $u(\xi, \cdot) = u(\xi)(\cdot)$, since $\bar{\partial} u(\xi, \cdot) = f(\xi, \cdot)$ by construction.

Now the map $\xi \to (f(\xi, \cdot), g)_\phi$ is $C^1(U)$ if $g \in L^2_{(p,q-1)}(W, \phi)$. Also, since S is closed, $SD_j f(\xi, \cdot) = 0$ for $\xi \in U$, so by Theorem 2.2.1 of Hörmander (1965), $D_j f(\xi, \cdot) = Tw_j(\xi)(\cdot)$, where $w_j(\xi) \in L^2_{(p,q-1)}(W, \phi) \cap N(T)^\perp$ for each $\xi \in U$ and $1 \leq j \leq m$. There exists $C > 0$ depending only on T so that $||w_j||_\phi \leq C||Tw_j||_\phi$ (Theorem 1.1.1 (Hörmander, 1965)).

From Taylor's Theorem, we see

$$\begin{aligned}
&|(f(\xi, \cdot) - f(\xi^o, \cdot), g_\ell)_\phi| \\
&= \left|\int_o^1 \sum_{j=1}^m (D_j f(t\xi + (1-t)\xi^o, \cdot)(\xi_j - \xi_j^o), g_\ell)_\phi \, dt\right| \\
&\leq \left(\int_o^1 \left|\sum_{j=1}^m (w_j(t\xi + (1-t)\xi^o), k_\ell)_\phi\right|^2 dt\right)^{1/2} |\xi - \xi^o| .
\end{aligned}$$

Then $\sum_\ell |(f(\xi, \cdot) - f(\xi^o, \cdot), g_\ell)_\phi|^2$

$$\begin{aligned}
&\leq \sum_\ell \int_o^1 \left|\sum_{j=1}^m (w_j(t\xi + (1-t)\xi^o), k_\ell)_\phi\right|^2 dt |\xi - \xi^o|^2 \\
&\leq \sum_\ell \int_o^1 \sum_{j=1}^m |(w_j(t\xi + (1-t)\xi^o), k_\ell)_\phi|^2 dt |\xi - \xi^o|^2 \\
&\leq \sum_{j=1}^m \int_o^1 ||w_j(t\xi + (1-t)\xi^o)||_\phi^2 dt |\xi - \xi^o|^2 \\
&\leq C_1 |\xi - \xi^o|^2
\end{aligned}$$

where the next to last inequality is by Parsavel's relation and C_1

depends on T, f, and ξ^o (for $|\xi - \xi^o| < 1$ say). Thus

$$||u(\xi, \cdot) - u(\xi^o, \cdot)||_\phi^2 = \sum_\ell |(f(\xi, \cdot) - f(\xi^o, \cdot), g_\ell)_\phi|^2 \leq C_1|\xi - \xi^o|^2$$

implies u is continuous as a map from U to $L^2_{(p,q-1)}(W, \phi)$.

Lemma 6.2 Let U, W, ϕ, and K be as in the last lemma. Suppose b is a nonnegative integer or $+\infty$ and $f \in L^2_{(p,q)}(U, W, \phi, b+1)$ $(q > 0)$ with $\bar\partial f(\xi, \cdot) = 0$ and $\int_W |D^\alpha f(\xi, \cdot)|^2 e^{-\phi-K}\, dV < \infty$ for $\xi \in U$ and all m-multi-indices α with $|\alpha| \leq b + 1$. Then there exists $u \in L^2_{(p,q-1)}(U, W, \phi, b)$ such that $\bar\partial u = f$ and

$$q \int_W |D^\alpha u(\xi, \cdot)|^2 e^{-\phi} dV \leq \int_W |D^\alpha f(\xi, \cdot)|^2 e^{-\phi-K} dV$$

for all m-multi-indices α with $|\alpha| \leq b$.

Proof: We will prove this by induction on b. We may assume $b = 1$ and prove the lemma in this case, since the choice of u is independent of b.

Let $\{k_\ell = T^* g_\ell\}$ and $u(\xi, \cdot) = \sum_\ell (f(\xi, \cdot), g_\ell)_\phi k_\ell$ be as in the previous lemma. Then

$$|(f(\xi + he_j, \cdot) - f(\xi, \cdot) - hD_j f(\xi, \cdot), g_\ell)_\phi|$$

$$= |\int_0^1 (hD_j f(\xi + the_j, \cdot) - hD_j f(\xi, \cdot), g_\ell)_\phi dt|$$

$$= |\int_0^1\int_0^1 (D_j^2 f(\xi + sthe_j, \cdot)th, g_\ell)_\phi ds dt|\ |h|$$

$$= |h|^2 |\int_0^1\int_0^1 (w_{jj}(\xi + sthe_j, \cdot), k_\ell)_\phi ds t dt|$$

$$\leq |h|^2 (\int_0^1\int_0^1 w_{jj}(\xi + sthe_j, \cdot), k_\ell)_\phi|^2 ds t^2 dt)^{\frac{1}{2}}$$

where $D_j^2 f(\xi, \cdot) = Tw_{jj}(\xi, \cdot)$, $w_{jj}(\xi, \cdot) \in N(T)^{\perp}$. Thus

$$\sum_{\ell} |(f(\xi + he_j, \cdot) - f(\xi, \cdot) - hD_j f(\xi, \cdot), g_{\ell})_{\phi}|^2$$
$$\leq \sum_{\ell} \int_0^1 \int_0^1 |(w_{jj}(\xi + sthe_j, \cdot), k_{\ell})_{\phi}|^2 ds t^2 dt \; |h|^4$$
$$\leq \int_0^1 \int_0^1 ||w_{jj}(\xi + sthe_j, \cdot)||_{\phi}^2 t^2 ds dt \; |h|^4 \leq C|h|^4$$

by Parsavel's relation and Theorem 1.1.1 of Hörmander (1965). Thus

$$\lim_{h \to 0} ||u(\xi + h) - u(\xi) - \sum_{j=1}^{m} h_j w_j(\xi, \cdot)||_{\phi}/|h| = 0$$

where $w_j(\xi, \cdot) = \sum_{\ell} (D_j f(\xi, \cdot), g_{\ell})_{\phi} k_{\ell} \in L^2_{(p,q-1)}(W, \phi)$. Then $u \in L^2_{(p,q-1)}(U, W, \phi, 1)$, $D_j u(\xi, \cdot) = \sum_{\ell} (D_j f(\xi, \cdot), g_{\ell})_{\phi} k_{\ell}$, and $\bar{\partial} u(\xi, \cdot) = f(\xi, \cdot)$. Similarly, $q \int_W |D_j u(\xi,\cdot)|^2 e^{-\phi} dV = q \int_W |w_j(\xi,\cdot)|^2 e^{-\phi} dV$

$$\leq \int_W |D_j f(\xi, \cdot)|^2 e^{-\phi - K} dV .$$

<u>Theorem 6.1</u> Let W be a pseudoconvex open set in C^n, ϕ a C^2 plurisubharmonic function on W, and $e^K \in C(W)$ be a lower bound for the plurisubharmonicity of ϕ. Let U be open in R^m, and p and q be nonnegative integers with $q > 0$.

Suppose $f \in L^2_{(p,q)}(U, W, \phi, \infty)$ with $\bar{\partial} f(\xi, \cdot) = 0$ and

$$\int_W |D^{\alpha} f(\xi, \cdot)|^2 e^{-\phi - K} dV < \infty$$

for $\xi \in U$ and all m-multi-indices α. Then there exists $u \in L^2_{(p,q-1)}(U, W, \phi, \infty)$ such that $\bar{\partial} u(\xi, \cdot) = f(\xi, \cdot)$ and

$$q \int_W |D^{\alpha} u(\xi,\cdot)|^2 e^{-\phi} dV \leq \int_W |D^{\alpha} f(\xi,\cdot)|^2 e^{-\phi - K} dV$$

for all $\xi \in U$ and all m-multi-indices α.

Proof: Notice $L^2_{(p,q-1)}(U, W, \phi, \infty) = C^\infty(U, L^2_{(p,q-1)}(W, \phi))$. Let $\sigma \in C^\infty(W)$ be a strictly pseudoconvex function on W such that $W_M = \{z \in W | \sigma(z) < M\}$ is relatively compact in W (Theorem 2.6.11 of Hörmander (1966)) and for almost all M, W_M has C^∞ pseudoconvex boundary (Hörmander, 1965). Then, by Lemma 6.2, there exists $g_\ell \in L^2_{(p,q-1)}(U, W^\ell, \phi, \infty)$ such that $\bar\partial g_\ell(\xi, \cdot) = f(\xi, \cdot)$ in $U \times W^\ell$ and $q \int_{W^\ell} |D^\alpha g_\ell(\xi, \cdot)|^2 e^{-\phi} dV \le \int_W |D^\alpha f(\xi, \cdot)|^2 e^{-\phi-K} dV$, for all $\xi \in U$, m-multi-indices α, and $\ell = 0, 1, 2, \ldots$, where $W^\ell = W_{M_\ell}$, $\{M_\ell | \ell = 0, 1, \ldots\}$ is an increasing sequence of reals, and $M_\ell \to \infty$ as $\ell \to \infty$. We extend g_ℓ by zero outside W^ℓ, so $\{g_\ell | \ell = 0, 1, 2, \ldots\} \subseteq C^\infty(U, L^2_{(p,q-1)}(W, \phi))$. Notice this sequence is bounded in $C^\infty(U, L^2_{(p,q-1)}(W, \phi))$. Let $\{\psi_j | j \ge 1\}$ be an orthonormal basis of $L^2_{(p,q-1)}(W, \phi)$. Notice that $\{(g_\ell(\cdot, \cdot\cdot), \psi_j(\cdot\cdot))_\phi | \ell = 0, 1, \ldots\}$ is bounded in $C^\infty(U)$, a Frechet-Montel space, for each $j = 1, 2, \ldots$. Then there exists a subsequence $\{g_\ell^{(1)}\}$ of $\{g_\ell\}$ such that $(g_\ell^{(1)}, \psi_1)_\phi \to a_1 \in C^\infty(U)$. Similarly, there exists a subsequence $\{g_\ell^{(s)}\}$ of $\{g_\ell^{(s-1)}\}$ such that

$$(g_\ell^{(s)}, \psi_s)_\phi \longrightarrow a_s \in C^\infty(U),$$

for $s = 2, 3, \ldots$.

Let $u(\xi, \cdot) = \sum_j a_j(\xi)\psi_j(\cdot)$. We claim that $u \in C^\infty(U, L^2_{(p,q-1)}(W,\phi))$ and $u(\xi, \cdot)$ is in the weak closure of $\{g_\ell(\xi, \cdot)\}$ for each $\xi \in U$. Set $v_\ell(\xi, \cdot) = g_\ell^{(\ell)}(\xi, \cdot)$. Now $(v_\ell(\xi, \cdot), \psi_j)_\phi \to (u(\xi, \cdot), \psi_j)_\phi$ as $\ell \to \infty$, for $j = 1, 2, \ldots$, and so if S is the subspace generated by $\{\psi_j | j = 1, 2, \ldots\}$ then $(v_\ell(\xi, \cdot), \psi)_\phi \to (u(\xi, \cdot), \psi)_\phi$ as $\ell \to \infty$ for all $\psi \in S$. Now let $\lambda \in L^2_{(p,q-1)}(W, \phi)$. Since S is dense in $L^2_{(p,q-1)}(W, \phi)$, if $\varepsilon > 0$, there exists $\psi \in S$ so that $||\lambda - \psi||_\phi < \varepsilon$. For some $\ell \geq 0$,

$$|(v_\ell(\xi, \cdot), \psi)_\phi - (u(\xi, \cdot), \Psi)_\phi| < \varepsilon,$$

where ℓ depends on ξ, of course. Then

$$\begin{aligned}
&|(v_\ell(\xi, \cdot) - u(\xi, \cdot), \lambda)_\phi| \\
&\leq |(v_\ell(\xi, \cdot) - u(\xi, \cdot), \psi)_\phi| + |(v_\ell(\xi, \cdot), \lambda - \psi)_\phi| + |(u(\xi, \cdot), \lambda - \psi)_\phi| \\
&\leq \varepsilon + C(\xi)||\eta - \psi||_\phi + C(\xi)||\lambda - \psi||_\phi \\
&\leq (1 + 2C(\xi))\varepsilon
\end{aligned}$$

where $C(\xi) = ||f(\xi, \cdot)||_{\phi+K}$. Now let $\varepsilon \to 0$. Then $v_\ell(\xi, \cdot)$ converges weakly (in $L^2_{(p,q-1)}(W, \phi)$) to $u(\xi, \cdot)$, for $\xi \in U$. Thus $\bar\partial u(\xi,\cdot) = f(\xi,\cdot)$ in $U \times W$, by standard arguments.

Now $D^\alpha g_\ell(\xi,\cdot) = \sum_j (D^\alpha g_\ell(\xi,\cdot), \psi_j)_\phi \psi_j = \sum_j D^\alpha a_{j\ell}(\xi)\psi_j$. By Parsavel's relation, $q \sum_j |D^\alpha a_{j\ell}(\xi)|^2 \leq ||D^\alpha f(\xi,\cdot)||^2_{\phi+K}$, for all $\xi \in U$ and m-multi-indices α. Thus,

$$q \sum_j |D^\alpha a_j(\xi)|^2 \leq \sup_\ell q \sum_j |D^\alpha a_{j\ell}(\xi)|^2 \leq ||D^\alpha f(\xi,\cdot)||^2_{\phi+K},$$

so "$D^\alpha u(\xi,\cdot)$" defined by $\sum_j D^\alpha a_j(\xi)\psi_j \in L^2_{(p,q-1)}(W,\phi)$, for each $\xi \in U$ and α. We claim that $D_j u(\xi, \cdot)$, defined as the limit of a difference quotient, exists and is continuous. This will show that

$u \in C^\infty(U, L^2_{(p,q-1)}(W, \phi))$ (i.e., the proof will be clear).

$$\left|\left| \frac{u(\xi+te_j,\cdot) - u(\xi,\cdot)}{t} - D_j u(\xi,\cdot)\right|\right|_\phi^2$$

$$= \left|\left|\sum_\ell \left(\frac{(a_\ell(\xi+te_j) - a_\ell(\xi))}{t} - D_j a_\ell(\xi)\right) \psi_\ell\right|\right|_\phi^2$$

$$= \left|\left|\sum_\ell (D_j a_\ell(\xi^\ell) - D_j a_\ell(\xi))\psi_\ell\right|\right|_\phi^2$$

(by the Mean-Value Theorem, where ξ^ℓ is between ξ and $\xi + te_j$)

$$= \left|\left|\sum_\ell D_j^2 a_\ell(\xi^{\ell j})(\xi_j^\ell - \xi_j)\psi_\ell\right|\right|_\phi^2$$

(where $\xi^{\ell j}$ is between ξ and ξ^ℓ)

$$\leq \sum_\ell |D_j^2 a_\ell(\xi^{\ell j})|^2 |\xi_j^\ell - \xi_j|^2$$

$$\leq |t|^2 \sum_\ell |D_j^2 a_\ell(\xi^{\ell j})|^2$$

$$\leq C_{2,X}|t|^2$$

where $C_{2,X} = \sup_{\xi \in X} \max_{|\alpha| \leq 2} ||D^\alpha f(\xi', \cdot)||_{\phi+K}^2$ and X is a compact subset of U containing a neighborhood of ξ. Thus $u \in C(U, L^2_{(p,q-1)}(W,\phi))$ and u is differentiable. A similar argument shows $u \in C^1(U, L^2_{(p,q-1)}(W,\phi))$ and $D_j u$ is differentiable. Now $||D^\alpha u(\xi,\cdot)||_\phi^2 = \sum_j |D^\alpha a_j(\xi)|^2$

$$\leq \frac{||D^\alpha f(\xi,\cdot)||_{\phi+K}^2}{q}$$

for all $\xi \in U$ and all α.

Theorem 6.2 Let W, ϕ, e^K, U, p, and q be as in Theorem 6.1. Let b be a nonnegative integer.

Suppose $f \in L^2_{(p,q)}(U, W, \phi, b+2)$ with $\bar\partial f(\xi,\cdot) = 0$ and

$$\int_W |D^\alpha f(\xi,\cdot)|^2 e^{-\phi-K}\, dV < \infty$$

for $\xi \in U$ and $|\alpha| \leq b + 2$. Then there exists $u \in L^2_{(p,q-1)}(U, W,\phi, b)$ such that $\bar\partial u(\xi,\cdot) = f(\xi,\cdot)$ and $q\int_W |D^\alpha u(\xi,\cdot)|^2 e^{-\phi} dV \leq \int_W |D^\alpha f(\xi,\cdot)|^2 e^{-\phi-K} dV$ for $\xi \in U$ and $|\alpha| \leq b$.

Proof: We modify the proof of Theorem 6.1 as follows:

We note that the inclusion of $C^{b+1}(U)$ into $C^b(U)$ is compact and the sequence $\{g_\ell\}$ is bounded in $C^{b+1}(U, L^2_{(p,q-1)}(W,\phi))$. The proof then follows.

Suppose W is a domain of holomorphy in C^n, $\{W_j | j \geq 1\}$ is an open cover of W, and U is open in R^m. If s is a nonnegative integer and b is either a nonnegative integer or $+\infty$, then $C^s(U,(W_j),Z_{(p,q)},\phi,b)$ denotes the set of all alternating s-cochains $c = (c_\alpha)$ (where α is an (s + 1)-multi-index) with $C_\alpha \in L^2_{(p,q)}(U, W_\alpha, \phi, b)$ and $\bar\partial c_\alpha(\xi,\cdot) = 0$ for $\xi \in U$. Here $W_\alpha = W_{\alpha_1} \cap \ldots \cap W_{\alpha_s}$. We define the coboundary operator (δ) in the usual manner (Petersen, 1975; Hörmander, 1965).

Lemma 6.3 Let W be a domain of holomorphy in C^n, U be open in R^m, and $(W_j | j \geq 1)$ be an open cover of W with the properties of Lemma 5 of Petersen (1976) (and A, B and a as in the lemma). Let ϕ be a C^2 strictly plurisubharmonic function in W and let e^K be a continuous lower bound for the plurisubharmonicity of ϕ with $K \leq L$ on W, for some real L. Let b be a nonnegative integer.

If s is a positive integer, then for each $c \in C^s(U, (W_j), Z_{(p,q)},\phi+K,b+2s)$ with $\delta c(\xi,\cdot) = 0$ for $\xi \in U$, there exists

$$c' \in C^{s-1}(U, (W_j), Z_{(p,q)}, \phi + 2\psi, b)$$

such that $\delta c'(\xi,\cdot) = c(\xi,\cdot)$ and

$$||D^\alpha c'(\xi,\cdot)||_{\phi+2\psi} \leq C||D^\alpha c(\xi,\cdot)||_{\phi+K}$$

for $\xi \in U$ and $|\alpha| \leq b$. Here $\psi(z) = -\log d(z) = -\log(\min\{1,\mathrm{dist}(z,\partial w)\})$ and C is a constant depending on A, B, a, L, and b.

Proof: The proof is essentially the same as the proof of Theorem 6 of Petersen (1976) with the lemmas here replacing those in Petersen (1976) as necessary. Note that the lemma is true with $b = \infty$ if we replace C by C_α's.

Lemma 6.4 Let W, U, (W_j) be as in Lemma 6.3 and let ϕ be a C^2 plurisubharmonic function in W. Suppose s is a positive integer, b is a nonnegative integer, and $q \geq 0$.

If $c \in C^s(U, (W_j), Z_{(p,q)}, \phi, b + 2s)$ such that $\delta c(\xi,\cdot) = 0$ for $\xi \in U$, then there exists

$$c' \in C^{s-1}(U, (W_j), Z_{(p,q)}, \phi + 2\psi + 2\theta, b)$$

such that $\delta c'(\xi,\cdot) = c(\xi,\cdot)$ and

$$||D^\alpha c'(\xi,\cdot)||_{\phi+2\psi+2\theta} \leq C||D^\alpha c(\xi,\cdot)||_{\phi}$$

if $\xi \in U$ and α is an m-multi-index with $|\alpha| \leq b$. Here $\theta(z) = \log(1 + |z|^2)$, $\psi(z) = -\log d(z)$, and C depends only on A, B, a, m, n, and b.

Proof: The proof is essentially that of Corollary 7 of Petersen (1976). Note that Lemma 6.4 is true with $k = \infty$ if we replace C by C_α.

CHAPTER SEVEN

GLOBAL EXISTENCE AND FUNDAMENTAL SOLUTIONS

In the chapter we prove an existence theorem with estimates similar to Theorem 1 of Petersen (1975). Using the division theorem we prove, under certain conditions, the existence of temperate fundamental solutions with support in $R \times \Gamma$, where Γ is a closed, convex, salient cone. A simple counterexample of the converse is provided.

If P is a $p \times q$ matrix over $C[\xi,z]$ we define $C^s(U,(W_j),R(P),\phi,b)$ to be the set of alternating s-cocycles $c = (c_\alpha)$ with $Pc_\alpha = 0$ in $U \times W_\alpha = U \times (W_{\alpha_0} \cap \dots \cap W_{\alpha_s})$, and

$$c_\alpha \in \Gamma(U \times W_\alpha, E^b0)^q \cap L^2(U, W_\alpha, \phi, b)^q .$$

If $f \in \Gamma(U \times W, E^b0)$ and $V \subset\subset W$, then $f \in L^2_{(0,0)}(U, V, Z_{(0,0)}, \phi, b)$ for any locally (lower) bounded measurable function ϕ on W. Conversely, if $f \in L^2_{(0,0)}(U, V, Z_{(0,0)}, \phi, b)$ for some plurisubharmonic function ϕ on W, then $f \in \Gamma(U \times V, E^b0)$. (The proof of this last comment follows from Theorem 2.2.3 of Hörmander (1966) and the fact that ϕ is upper semi-continuous).

Lemma 7.1 Let $m = 1$, U be open in R, W a domain of holomorphy in C^n, and $(W_j^\ell)_{\ell=0,\dots,3n+3}^{j\geq 1}$ be a collection of open covers of W with the properties of Lemma 2 of Petersen (1975). Let $L > 0$ and ϕ be a C^2 plurisubharmonic function in W with $|\phi(z) - \phi(w)| \leq L$ if $z \in W$ and $|z - w| < \frac{1}{2} d(z)$.

Suppose P is a $p \times q$ matrix over $C[\xi, z]$ and $Q^0, \ldots, Q^\ell$ is a Hilbert resolution of P (as matrices over $C[\xi, z]$) with $t \in (0, 1)$ good for $Q^0, \ldots, Q^\ell$ and so that t satisfies the condition of Lemma 5.4. There exists a polynomial $a \in C[\xi]$ (independent of t) and a nonnegative integer $\bar{E}$ such that:

If $s \geq 1$, b is a nonnegative integer, and $c \in C^s(U, (W_j^o), R(P), \phi, b + \bar{E})$ with $\delta c(\xi, \cdot) = 0$ $(\xi \in U)$, then there exist nonnegative integers M and N, $C > 0$, and

$$c' \in C^{s-1}(U, (W_j^{3n+3}), R(P), \phi + M\psi + N\theta, b) \quad \text{with}$$

$$\delta c'(\xi, \cdot) = a(\xi)\, c(\xi, \cdot)$$

and $||D^\ell c'(\xi,\cdot)||^2_{\phi+M\psi+N\theta} \leq C(1 + |\xi|^2)^M \sum_{j=o}^{\ell} ||D^j c(\xi, \cdot)||^2_\phi$ for $\xi \in U$ and $\ell = 0, \ldots, b$. Here M and N are nonnegative integers depending on P and b, C is a constant depending on P, b, and L, and ψ and θ are as before; $\psi(z) = -\log(d(z))$ and $\theta(z) = \log(1 + |z|^2)$.

Proof: The proof is essentially the same as the proof of Theorem 3 of Petersen (1975). However, we take three times as many refinements as in Petersen (1975). One set is due to Lemma 5.4. The other is required since $f \in \Gamma(U \times W_\alpha^\ell, E^b0)$ may not be in $L^2(U, W_\alpha^\ell, \phi, b)$ but is in $L^2(U, W_\alpha^{\ell+1}, \phi, b)$.

Theorem 7.1 Let $m = 1$, P be a $p \times q$ matrix over $C[\xi, z]$, U open in R, and W a domain of holomorphy in C^n. There exists a polynomial $b \in C[\xi]$ such that the following holds:

Let b be a nonnegative integer, ϕ as in Lemma 7.1, and $u \in \Gamma(U \times W, E^{b+\bar{E}}0)^q$ (where $\bar{E}$ is from Lemma 7.1). Then there exists

$v \in \Gamma(U \times W, E^b 0)^q$ such that $Pv = bPu$ in $U \times W$ and

$$\int_W |D^\ell v(\xi, \cdot)|^2 e^{-\phi-M\psi-N\theta} dV \leq C(1 + |\xi|^2)^N \sum_{j=0}^{\ell} \int_W |D^j Pu(\xi, \cdot)|^2 e^{-\phi} dV$$

for $\xi \in U$ and $\ell = 0, \ldots, b$. Here M and N are nonnegative integers depending only on P and b and C is a constant depending on P, b, and L.

Proof: The proof is essentially the same as that of Theorem 1 of Petersen (1975).

Let Γ be a closed, convex, salient cone in R^n and let $\Gamma^+ = \{x \in R^n | \langle x, y\rangle \geq 0$ if $y \in \Gamma\}$ be its dual cone. Salient means that Γ contains no one-dimensional subspaces, which is equivalent to the interior Γ_o^+ of Γ^+ being nonempty. Suppose $m = 1$ and $P = (P_1, \ldots P_q)$ is a q-tuple of polynomials in $C[\xi, z_1, \ldots, z_n]$. If $u \in \Gamma(R \times (\Gamma_o^+ + iR^n), E0)^q$ and ϕ is a C^2 plurisubharmonic function on $W = \Gamma_o^+ + iR^n$ with $\int_W |D^\ell Pu(\xi, \cdot)|^2 e^{-\phi} dV < \infty$ for $\ell = 0, \ldots, k + \bar{E}$ and $|\phi(z) - \phi(w)| \leq L$ if $z \in W$ and $|z - w| \leq \frac{1}{2} d(z)$, then by Theorem 7.1, there exists $v \in \Gamma(R \times (\Gamma_o^+ + iR^n), E^k 0)^q$ such that $Pv = bPu$ and

$|D^\ell v(\xi, z)| \leq C(\xi)(1 + |z|^2)^{N+n+1} d(z)^{-M-2N}$ for $\xi \in R$ and $\ell = 0, \ldots, k$, where $b \in C[\xi]$, M, N, and C are from Theorem 7.1 and

$$C(\xi) = \pi^{-n} n^n 2^{2n+N+M} e^L C(1 + |\xi|^2)^N \sum_{\ell=0}^{k} ||D^\ell Pu(\xi, \cdot)||_\phi^2$$

(Petersen, 1975). Then $D^\ell v_j(\xi, \cdot)$ is the Laplace transform of a temperate distribution $T_j^\ell(\xi) \in S'(R^n)$ with support in Γ, for $\ell = 0, \ldots, k$, and we define $T_j : R \longrightarrow S'_\Gamma$ by $T_j(\xi) = T_j^o(\xi)$ $j = 1, \ldots, q$. Here the Laplace transform is as in Petersen (1972); defined formally by

$L(f)(z) = \int_{R^n} e^{-<z,x>} f(x)\, dx$. We let F denote the Fourier transform in S'. Then, if $x \in \Gamma_o^+$,

$$\frac{d}{d\xi}\,(e^{-<x,\cdot>}\,T_j(\xi),\ F(f)) = \frac{d}{d\xi}\,(\,v_j(\xi,\ x + i\cdot),\ f(\cdot))$$

$$= (\,Dv_j(\xi,\ x + i\cdot),\ f(\cdot))$$

$$= (\,e^{-<x,\cdot>}T_j^1(\xi),\ F(f))$$

if $f \in S$ and $F(f)$ has compact support. Thus, if we define $DT_j(\xi)$ as the limit of a difference quotient, $e^{-<x,\cdot>}\,DT_j(\xi) = e^{-<x,\cdot>}\,T_j^1(\xi)$ on $\mathcal{D}'(R^n)$ if $x \in \Gamma_o^+$, and so $DT_j(\xi) = T_j^1(\xi)$ as distributions. Since $T_j^1(\xi) \in S'$, $DT_j(\xi) \in S'$. Then

$D^\ell T_j(\xi) = T_j^\ell(\xi) \in S'(R^n)$, for $\ell = 0, \ldots, k$, and the map $\xi \longrightarrow (T(\xi),\ f) \in C^k(R)$ if $f \in S(R^n)$.

By the division theorem (Atiyah, 1970), there exists $G \in S'(R)$ with $GA = 1$ in $S'(R)$ if $A \in \mathbb{C}[\xi]$ $(A \neq 0)$. (Of course, this is trivial in the case of one variable.) G is continuous with respect to some norm $\|\ \|_{H,I}$ on $S(R)$, so $G \in S'_{H,I} \subseteq S'$, where $S_{H,I}$ is the Banach space determined by $\|\ \|_{H,I}$.

<u>Theorem 7.2</u> Suppose $m = 1$ and $Q = (Q_1, \ldots, Q_q)$ is a q-tuple over $\mathbb{C}[\xi, z_1, \ldots, z_n]$ which is not identically zero. Let Γ be a closed, convex, salient cone in R^n.

If $Q_j(\xi, z) = a(\xi)P_j(\xi, z)$ for $j = 1, \ldots, q$ (a and P_j polynomials) such that $P = (P_1, \ldots, P_q)$ has no common zeros in $R \times (\Gamma_o^+ + iR^n)$, then $Q(-\,iD, D)$ has a temperate fundamental solution $S = (S_1, \ldots, S_q)$

with support in $R \times \Gamma$. Here $Q_j(-iD, D)$ is the partial differential operator obtained by replacing ξ by $-i\frac{\partial}{\partial\xi}$ and z_j by $\frac{\partial}{\partial x_j}$.

<u>Proof</u>: Let $u_1', \ldots, u_q' \in \Gamma(R \times (\Gamma_0^+ + iR^n), EE)$ such that $Pu' = 1$, which we may find by a partition of unity argument. By Corollary 2.2, there exist $u_1, \ldots, u_q \in \Gamma(R \times (\Gamma_0^+ + iR^n), E0)$ with $Pu = 1$. Let $A = ab$, where b is given in Theorem 7.1 (for P), let $G \in S'(R)$ so that $GA = 1$, and pick H, I (nonnegative integers) so that $G \in S'_{H,I}$. Let $k = H$ and define S_j on $S(R^{n+1})$ by

$$(S_j, g) = (G, (T_j(\xi), F_0(g)(\xi,\cdot)))$$

where F_0 is the partial Fourier transform in the ξ-variable. Here $\xi \longrightarrow (T_j(\xi), F_0(g)(\xi, \cdot)) \in S_{H,I}$ (in fact, rapidly decreasing) and we apply G to this map. (From the estimates on $D^\ell v_j$, we see that $T_j^\ell(\xi)$ is continuous with respect to a fixed norm on $S(R^n)$, for all $\xi \in R$, $j \leq q$, and $\ell \leq k$.) Thus S_j is well-defined and a short computation shows that S_j is continuous with respect to a norm on $S(R^{n+1})$ (depending on the norms with respect to which G, T_j, and F_0 are continuous) and so $S_j \in S'(R^{n+1})$ for $j = 1, \ldots, q$. The support of S_j is contained in $R \times \Gamma$, since F_0 is the Fourier transform in ξ alone and $\operatorname{supp}(T_j(\xi)) \subseteq \Gamma$.

The Laplace transform of $T_j(\xi)$ is $v_j(\xi, \cdot)$, so the Laplace transform of $P(\xi, D)T(\xi) = \sum_{j=1}^{q} P_j(\xi,D)T_j(\xi)$ is $P(\xi,\cdot)v(\xi,\cdot) = b(\xi)$. Thus $P(\xi, D)T(\xi) = b(\xi)\delta \in S'(R^n)$. Suppose $g \in S(R^{n+1})$. Then

$$(Q(-iD, D)S, g) = \sum_{j=1}^{q} (Q_j(-iD, D)S_j, g)$$

$$= \sum_j (S_j, Q_j(-iD, D)^*g)$$

$$= \sum_j \Big(G, \ \big(T_j(\xi), \ F_0^{-1}(Q_j(-iD, D)^* g)(\xi, \cdot) \big) \Big)$$

$$= \sum_j \Big(G, \ \big(T_j(\xi), \ a(\xi) P_j(\xi, D)^* F_0^{-1}(g)(\xi, \cdot) \big) \Big)$$

$$= \Big(aG, \ \big(\sum_j P_j(\xi, D) T_j(\xi), \ F_0^{-1}(g)(\xi, \cdot) \big) \Big)$$

$$= \Big(aG, \ \big(b(\xi)\delta, \ F_0^{-1} g(\xi, \cdot) \big) \Big)$$

$$= \big(GA, \ F_0^{-1}(g)(\xi, 0) \big)$$

$$= \big(1, \ F_0^{-1}(g)(\cdot, 0) \big)$$

$$= g(0, 0)$$

since if $f \in S(R)$, then $f(0) = (\delta, f) = (F_0(\delta), F_0^{-1}(f)) = (1, F_0^{-1}(f))$. Here $Q_j(-iD, D)^*$ is the partial differential operator on R^{n+1} obtained by replacing $\frac{\partial}{\partial \xi}$ by $-\frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial x_j}$ by $-\frac{\partial}{\partial x_j}$ in $Q_j(-iD, D)$ and $P_j(\xi, D)^*$ is the partial differential operator on R^n obtained by replacing $\frac{\partial}{\partial x_j}$ by $-\frac{\partial}{\partial x_j}$ in $P(\xi, D)$.

Notice that the converse of Theorem 7.2 is not true, even if $q = 1$ (see Enqvist, 1976). If we set $\Gamma = [0, +\infty)$ and $P_1(\xi, z) = \xi$ and $P_2(\xi, z) = z + \xi - 1$, then $S = (H \otimes \delta, 0)$ is a temperate fundamental solution of $P = (P_1, P_2)$ with $\mathrm{supp}(S) = R \times \{0\}$, where H is the Heavyside function. Notice $\xi = 0$, $z = 1$ is a common zero of P.

<u>Theorem 7.3</u> Suppose P is a $p \times q$ matrix over $C[\xi, z_1, \ldots, z_n]$ (with $m = 1$) and Γ is a closed, convex, salient cone in R^n with $p \leq q$. If there is a polynomial $a \in C[\xi]$ so that $P(\xi, z) = a(\xi) R(\xi, z)$ and R has rank p in $R \times (\Gamma_0^+ + iR^n)$, then $P(-iD, D)$ has a temperate fundamental solution with support in $R \times \Gamma$.

$P(-iD, D)$ has a temperate fundamental solution means that there is a $q \times p$ matrix E over $S'(R^{n+1})$ so that $P(-iD, D)E = \delta I_p$ where I_p is the $p \times p$ identity matrix.

Proof: The proof of the sufficiency of the Lemma on page 248 of Lancaster and Petersen (1980) is still valid if we replace Γ by $R \times \Gamma$. The theorem then follows from Theorem 7.2.

Notice that we only require that there is a polynomial $a \in C[\xi]$ such that $\det M(\xi, z) = a(\xi)Q(\xi, z) \in C[\xi, z]$ for each $p \times p$ submatrix M of P and the $1 \times \binom{q}{p}$ system $(Q_j \mid |J| = p)$ has no common zeros in $R \times (\Gamma_0^+ + iR^n)$, where Q_j is a $p \times p$ submatrix of P and J is a p-multi-index (see Lancaster and Petersen 1980)).

BIBLIOGRAPHY

M. F. Atiyah, Resolution of Singularities and Division of Distributions, Communications of Pure and Applied Mathematics, XXIII (1970), pp. 145-150.

L. Ehrenpreis, Solution of Some Problems of Division, Amer. J. Math. 76 (1954), pp. 883-903.

A. Enqvist, On temperate fundamental solutions supported by a convex cone, Ark. Mat. 14 (1976), pp. 35-41.

L. Hörmander, Local and Global Properties of Fundamental Solutions, Math. Scand. 5 (1957) pp. 27-39.

L. Hörmander, L^2 Estimates and Existence Theorems for the $\bar{\partial}$ Operator, Acta Math. 113 (1965), pp. 89-152.

L. Hörmander, An Introduction to Complex Analysis in Several Variables, Van Nostrand, Princeton, N. J., 1966.

L. Hörmander, Linear Partial Differential Operators, Springer-Verlag, New York, 1969.

K. E. Lancaster and B. E. Petersen, Systems of Partial Differential Operators with Fundamental Solutions supported by a Cone, Proc. Amer. Math. Soc. 78 (1980), pp. 247-252.

S. Lojasiewicz, Ensembles Semi-Analytiques, IHES, 1965.

B. Malgrange, Existence et Approximation des Solutions des Equations aux Derivees Partielles et des Equations de Convolution, Ann. Inst. Fourier 6 (1955), pp. 271-354.

B. Malgrange, Division Des Distributions IV, Seminaire Schwartz, 1960.

B. Malgrange, Sur les Systemes Differentiels a Coefficients Constants, Coll. C. N. R. S., Paris, 1963, pp. 113-122.

B.E. Petersen, "On the Laplace Transform of a Temperate Distribution Supported by a Cone", Proc. Amer. Math. Soc. 35 (1972), pp. 123-128.

B.E. Petersen, Holomorphic Functions with Bounds, unpublished lecture notes, 1974.

B.E. Petersen, Holomorphic Functions with Growth Conditions, Trans. Amer. Math. Soc. 206 (1975), pp. 395-406.

B.E. Petersen, "Holomorphic Functions with Bounds", Complex Analysis and its Applications III, International Atomic Energy Agency, Vienna, 1976.

J. J. Rottman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

K. T. Smith, Formulas to Represent Functions by their Derivatives, Math. Ann. (1970), pp. 53-77.

B. Weinstock, Inhomogeneous Cauchy-Riemann Systems with Smooth Dependence on Parameters, Duke Math. J. 40 (1973), pp. 307-312.