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DIFFUSION OF FLUID IN A FISSURED MEDIUM WITH MICROSTRUCTURE*

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Abstract. A system of quasilinear degenerate parabolic equations arising in the modeling of diffusion in a fissured medium is studied. There is one such equation in the local cell coordinates at each point of the medium, and these are coupled through a similar equation in the global coordinates. It is shown that the initial boundary value problems are well posed in the appropriate spaces.

Key words. porous medium, double porosity, degenerate parabolic system

AMS(MOS) subject classifications. 35K55, 35K65

1. Introduction. We shall study the Cauchy–Dirichlet problem for degenerate parabolic systems of the form

$$(1.1a)\qquad \frac{\partial}{\partial t}a(u) - \vec{\nabla}\cdot\widetilde{A}(x,\vec{\nabla}u) + \int_{\Gamma_x}\widetilde{B}(x,s,\vec{\nabla}_y U)\cdot\vec{\nu}\,ds \ni f, \qquad x\in\Omega,$$

$$(1.1b)\qquad \frac{\partial}{\partial t}b(U) - \vec{\nabla}_y\cdot\widetilde{B}(x,y,\vec{\nabla}_y U) \ni F, \qquad y\in\Omega_x,$$

$$(1.1c)\qquad \widetilde{B}(x,y,\vec{\nabla}_y U)\cdot\vec{\nu} + \mu\big(U(x,y,t)-u(x,t)\big) \ni 0, \qquad y\in\Gamma_x.$$

Here Ω is a domain in $\mathbb{R}^n$ and for each value of the macrovariable $x\in\Omega$ is specified a domain Ω_x with boundary Γ_x for the microvariable $y\in\Omega_x$. Each of a, b, μ is a maximal monotone graph. These graphs are not necessarily strictly increasing; they may be piecewise constant or multivalued. The elliptic operators in (1.1a) and (1.1b) are of p-Laplacian type, i.e., they are nonlinear in the gradient of degree $p-1>0$ and $q-1>0$, respectively, with $\frac{1}{q}+\frac{1}{n}\ge\frac{1}{p}$, so some specific degeneracy is also permitted here. Certain first-order spatial derivatives can be added to (1.1a) and (1.1b) with no difficulty, and corresponding problems with constraints, i.e., variational inequalities, can be treated similarly. A particular example important for applications is the linear constraint

$$(1.1c'),\qquad U(x,y,t)=u(x,t), \qquad y\in\Gamma_x,\quad x\in\Omega$$

which then replaces (1.1c). The system (1.1) with $\mu(s)=\frac{1}{\varepsilon}|s|^{q-2}s$ is called a *regularized microstructure model*, and (1.1a), (1.1b), (1.1c′) is the corresponding *matched microstructure model* in which (formally) $\varepsilon\to 0$. An example of such a system as a model for the flow of a fluid (liquid or gas) through a fractured medium will be given below. In such a context, (1.1a) prescribes the flow on the global scale of the fissure system and (1.1b) gives the flow on the microscale of the individual cell at a specific

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point x in the fissure system. The transfer of fluid between the cells and surrounding medium is prescribed by (1.1c) or (1.1c′). A major objective is to accurately model this fluid exchange between the cells and fissures.

Systems of the form (1.1) were developed in [21], [22], [10] in physical chemistry as models for diffusion through a medium with a prescribed microstructure. Similar systems arose in soil science [5], [14] and in reservoir models for fractured media [11], [16]. An existence-uniqueness theory for linear problems which exploits the strong parabolic structure of the system was given in [24]. Alternatively it is possible to eliminate U and obtain a single functional differential equation for u in the simpler space $L^2(\Omega)$, but the structure of the equation then obstructs the optimal parabolic type results [18]. Also see [13] for a nonlinear system with reaction–diffusion local effects.

These systems also arise from methods of homogenization. There an exact model is assumed periodic and described by a parabolic equation with periodic coefficients corresponding to the properties of the two components, the cells and fissures. The limit of this highly singular problem as the period tends to zero is the system (1.1), which is thereby justified as an approximation for the exact model. Homogenization theory provides not only a justification of the linear case of (1.1) as a model but also a means of calculating the coefficients in (1.1) in terms of those of the exact model, and a deeper analysis may describe the convergence itself [25], [17], [2], [3]. Here we study the nonlinear system directly. The task of determining the coefficients in (1.1) directly from, e.g., boundary observations, is an intriguing open problem.

The plan of this paper is as follows. In § 2 we shall give the precise description and resolution of the stationary problem in a variational formulation by monotone operators from Banach spaces to their duals. In order to achieve this we describe first the relevant Sobolev spaces, the continuous direct sums of these spaces, and the distributed trace and constant functionals which occur in the system. The operators are monotone functions or multivalued subgradients and serve as models for nonlinear elliptic equations in divergence form. We develop an abstract Green's theorem to describe the resolution of the variational form as the sum of a partial differential equation and a complementary boundary operator. Then sufficient conditions of coercivity type are given to assert the existence of generalized solutions of the variational equations. In § 3 we describe the restriction of our system to appropriate products of L^r spaces. The Hilbert space case, $r = 2$, serves not only as a convenient starting point but also leads to the generalized accretive estimates we shall need for the singular case of (1.1) in which a or b is not only nonlinear but multivalued. The stationary operator for (1.1) is shown to be m-accretive in the L^1 space, so we obtain a *generalized solution* in the sense of the nonlinear semigroup theory for general Banach spaces. As an intermediate step we shall show the special case of $a = b =$ identity is resolved as a *strong solution* in every L^r space, $1 < r < \infty$, and also in appropriate dual Sobolev spaces.

In order to motivate the system (1.1), let us consider the flow of a fluid through a fissured medium. This is assumed to be a structure of porous and permeable blocks or cells which are separated from each other by a highly developed system of fissures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume of the cell structure is much larger than that of the fissure system. There is assumed to be no direct flow between adjacent cells, since they are individually isolated by the fissures, but the dynamics of the flux exchanged between each cell and its surrounding fissures is a major aspect of the model. The distributed microstructure models that we develop here contain explicitly the local geometry of the cell matrix at each point of the fissure system, and they thereby reflect more accurately the flux exchange on the microscale of the individual cells across their intricate interface.

Let the flow region Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$. Let $\rho(x,t)$ and $p(x,t)$ be the *density* and *pressure*, respectively, at $x \in \Omega$ and $t > 0$, each being obtained by averaging over an appropriately small neighborhood of x. At each such x let there be given a cell Ω_x, a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma_x = \partial\Omega_x$. The collection of these Ω_x, $x \in \Omega$, is the distribution of blocks or cells in the structure. Within each Ω_x there is fluid of density $\tilde{\rho}(x,y,t)$ and pressure $\tilde{p}(x,y,t)$, respectively, for $y \in \Omega_x$, $t > 0$. The conservation of fluid mass in the fissure system yields the global diffusion equation

$$\text{(1.2a)} \qquad \frac{\partial}{\partial t}(\rho + a_0(p)) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\rho k_j\left(\rho\frac{\partial p}{\partial x_j}\right)\frac{\partial p}{\partial x_j}\right) + q(x,t) = f(x,t), \qquad x \in \Omega,$$

in which the total concentration $\rho + a_0(p)$ includes adsorption or capillary effects, the function k_j gives the permeability of the fissure system in the jth coordinate direction, $q(x,t)$ is the density of mass flow of fluid into the cell Ω_x at x, and f is the density of fluid sources. Similarly, we have within each cell

$$\text{(1.2b)} \qquad \frac{\partial}{\partial t}(\tilde{\rho} + b_0(\tilde{p})) = \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\left(\tilde{\rho}\tilde{k}_j\left(\tilde{\rho}\frac{\partial \tilde{p}}{\partial y_j}\right)\frac{\partial \tilde{p}}{\partial y_j}\right), \qquad y \in \Omega_x,$$

where b_0 denotes adsorption or capillary effects and the function $\tilde{k}_j$ gives the local cell permeability. Assume the flux across the cell boundary is driven by the pressure difference and is also proportional to the *average density* $\bar{\rho}$ on that pressure interval. Thus, we have the interface condition

$$\text{(1.2c)} \qquad \sum_{j=1}^{n} \tilde{\rho}\tilde{k}_j\left(\tilde{\rho}\frac{\partial \tilde{p}}{\partial y_j}\right)\frac{\partial \tilde{p}}{\partial y_j}\nu_j + \mu\big(\bar{\rho}(\tilde{p} - p)\big) \ni 0, \qquad y \in \Gamma_x,$$

where $\vec{\nu}$ is the unit outward normal on Γ_x and μ is the relation between the flux across the interface and the density-weighted pressure difference as indicated. The total mass flow into the cell is given by

$$\text{(1.2d)} \qquad q(x,t) = \int_{\Gamma_x} \tilde{\rho}\sum_{j=1}^{n} \tilde{k}_j\left(\tilde{\rho}\frac{\partial \tilde{p}}{\partial y_j}\right)\frac{\partial \tilde{p}}{\partial y_j}\nu_j\, ds.$$

In order to complete the dynamical system we need only to add a boundary condition on Γ to (1.2a) and to postulate the *state equation*

$$\text{(1.2e)} \qquad \rho = s(p)$$

for the fluid in the fissure and cell systems. Here $s(\cdot)$ is a given monotone function (or graph) determined by the fluid.

In order to place (1.2) in a more convenient form, we introduce the monotone function

$$S(w) \equiv \int_0^w s(r)\,dr$$

and the corresponding flow potentials for the fluid in the fissures and cells

$$u = S(p), \qquad U = S(\tilde{p}) .$$

In these variables with a change of notation the system (1.2) can be written in the form (1.1) together with boundary conditions on Γ for u or $\widetilde{A}(\nabla u)\cdot\nu$ and initial conditions at $t = 0$ on $a(u)$, $b(U)$. Note that the average density on the pressure interval $p, \tilde{p}$ is given by

$$\bar{\rho} = \frac{1}{p-\tilde{p}} \int_{\tilde{p}}^{p} s(r)\,dr = \frac{u-U}{p-\tilde{p}}.$$

As an alternative to (1.2c), we could require that $\tilde{p} = p$ on Γ_x and this leads to (1.1c$'$) in place of (1.1c). Finally, we note that the classical Forchheimer-type corrections to the Darcy law for fluids lead to the case $p = q = \frac{3}{2}$.

2. The variational formulation. We begin by stating and resolving the stationary forms of our systems. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $\Gamma = \partial\Omega$. Let $1 \le p < \infty$ and denote by $L^p(\Omega)$ the space of pth power-integrable functions on Ω, by $L^\infty(\Omega)$ the essentially bounded measurable functions, and the duality pairing by

$$(u, f)_{L(\Omega)} = \int_\Omega u(x)f(x)\,dx, \qquad u \in L^p(\Omega), \quad f \in L^{p'}(\Omega),$$

for any pair of conjugate powers, $\frac{1}{p} + \frac{1}{p'} = 1$. Let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω. $W^{m,p}(\Omega)$ is the Banach space of functions in $L^p(\Omega)$ for which each partial derivative up to order m belongs to $L^p(\Omega)$, and $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. See [1] for information on these Sobolev spaces. In addition, we shall be given for each $x \in \Omega$ a bounded domain Ω_x which lies locally on one side of its smooth boundary Γ_x. Let $1 < q < \infty$ and denote by $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ the *trace* map which assigns boundary values. Let T_x be the range of γ_x; this is a Banach space with the norm induced by γ_x from $W^{1,q}(\Omega_x)$. Since Γ_x is smooth, there is a unit outward normal $\nu_x(s)$ at each $s \in \Gamma_x$. Finally, we define $W_x^{1,q}(\Omega_x)$ to be that closed subspace consisting of those $\varphi \in W^{1,q}(\Omega_x)$ with $\gamma_x\varphi \in \mathbb{R}$, i.e., each $\gamma_x(\varphi)$ is *constant* almost everywhere on Γ_x. We shall denote by $\vec{\nabla}_y$ the gradient on $W^{1,q}(\Omega_x)$ and by $\vec{\nabla}$ the gradient on $W^{1,p}(\Omega)$.

The essential construction to be used below is an example of a *continuous direct sum* of Banach spaces. The special case that is adequate for our purposes can be described as follows. Let S be a measure space and consider the product (measure) space $Q = \Omega \times S$, where Ω has Lebesgue measure. If $U \in L^q(Q)$ then from the Fubini theorem it follows that $U(x)(z) \equiv U(x,z)$, $x \in \Omega$, $z \in S$ defines $U(x) \in L^q(S)$ at almost everywhere $x \in \Omega$, and for each $\Phi \in L^{q'}(Q)$

$$\int_\Omega \big(U(x), \Phi(x)\big)_{L(S)}\,dx \equiv \int_\Omega \left\{ \int_S U(x,z)\Phi(x,z)\,dz \right\} dx = \iint_Q U\Phi.$$

Thus $L^q(Q)$ is naturally identified with $L^q(\Omega, L^q(S))$, the Bochner qth integrable (equivalence classes of) functions from Ω to $L^q(S)$.

In order to prescribe a measurable family of cells $\{\Omega_x, x \in \Omega\}$, set $S = \mathbb{R}^n$, let $Q \subset \Omega \times \mathbb{R}^n$ be a given measurable set for which each section $\Omega_x = \{y \in \mathbb{R}^n : (x,y) \in Q\}$ is a bounded domain in $\mathbb{R}^n$. By zero-extension we identify $L^q(Q) \hookrightarrow L^q(\Omega \times \mathbb{R}^n)$ and each $L^q(\Omega_x) \hookrightarrow L^q(\mathbb{R}^n)$. Thus we obtain from above

$$L^q(Q) \cong \Big\{U \in L^q\big(\Omega, L^q(\mathbb{R}^n)\big) : U(x) \in L^q(\Omega_x) \text{ a.e. } x \in \Omega\Big\}.$$

We shall denote the duality on this Banach space by

$$(U, \Phi)_{L(Q)} = \int_\Omega \left\{ \int_{\Omega_x} U(x,y)\Phi(x,y)\, dy \right\} dx \ ,$$
$$U \in L^q(Q), \ \ \Phi \in L^{q'}(Q).$$

The *state space* for our problems will be the product $L^1(\Omega) \times L^1(Q)$.

Note that $W^{1,q}(\Omega_x)$ is continuously imbedded in $L^q(\Omega_x)$, uniformly for $x \in \Omega$. It follows that the direct sum

$$\mathcal{W}_q \equiv L^q\big(\Omega, W^{1,q}(\Omega_x)\big) \equiv \Big\{U \in L^q(Q) : U(x) \in W^{1,q}(\Omega_x) \text{ a.e. } x \in \Omega \ ,$$
$$\text{and } \int_\Omega \|U(x)\|^q_{W^{1,q}}\, dx < \infty\Big\}$$

is a Banach space. We shall use a variety of such spaces which can be constructed in this manner. Moreover, we shall assume that each Ω_x lies locally on one side of its boundary Γ_x, and Γ_x is a C^2-manifold of dimension $n-1$. We assume the trace maps $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ are *uniformly* bounded. Thus for each $U \in \mathcal{W}_q$ it follows that the *distributed trace* $\gamma(U)$ defined by $\gamma(U)(x,s) \equiv \gamma_x(U(x))(s)$, $s \in \Gamma_x$, $x \in \Omega$, belongs to $L^q(\Omega, L^q(\Gamma_x))$. The distributed trace γ maps $\mathcal{W}_q$ onto $\mathcal{T}_q \equiv L^q(\Omega, T_x) \hookrightarrow L^q(\Omega, L^q(\Gamma_x))$.

Next consider the collection $\{W^{1,q}_x(\Omega_x) : x \in \Omega\}$ of Sobolev spaces given above and denote by $\mathcal{W}_1 \equiv L^q(\Omega, W^{1,q}_x(\Omega_x))$ the corresponding direct sum. Thus for each $U \in \mathcal{W}_1$ it follows that the distributed trace $\gamma(U)$ belongs to $L^q(\Omega)$. We define $\mathcal{W}^{1,p}_0$ to be the subspace of those $U \in \mathcal{W}_1$ for which $\gamma(U) \in W^{1,p}_0(\Omega)$. Since $\gamma : \mathcal{W}_1 \to L^q(\Omega)$ is continuous, $\mathcal{W}^{1,p}_0$ is complete with the norm

$$\|U\|_{\mathcal{W}^{1,p}_0} \equiv \|U\|_{\mathcal{W}_q} + \|\gamma U\|_{W^{1,p}_0}.$$

This Banach space $W^{1,p}_0(\Omega) \times \mathcal{W}_q$ will be the *energy space* for the regularized problem (1.1) and $\mathcal{W}^{1,p}_0$ will be the energy space for the constrained problem in which (1.1c) is replaced by the Dirichlet condition (1.1c$'$). Note that $\mathcal{W}^{1,p}_0$ is identified with the closed subspace $\{[\gamma U, U] : U \in \mathcal{W}^{1,p}_0\}$ of $W^{1,p}_0(\Omega) \times \mathcal{W}_q$. Finally, we shall let $\mathcal{W}_0$ denote the kernel of γ, $\mathcal{W}_0 = \{U \in \mathcal{W}_q : \gamma U = 0 \text{ in } \mathcal{T}_q\}$.

We have defined $W^{1,q}_x(\Omega_x)$ to be the set of $w \in W^{1,q}(\Omega_x)$ for which $\gamma_x w$ is a constant multiple of $\mathbf{1}_x$, the constant function equal to one on Γ_x. Thus $W^{1,q}_x(\Omega_x)$ is the pre-image by γ_x of the subspace $\mathbb{R}\cdot\mathbf{1}_x$ of T_x. We specified the subspace $\mathcal{W}_1$ similarly as the subspace of $\mathcal{W}_q$ obtained as the pre-image by γ of the subspace $L^q(\Omega)$ of $\mathcal{T}_q$. To be precise, we denote by λ the map of $L^q(\Omega)$ into $\mathcal{T}_q$ given by $\lambda v(x) = v(x)\cdot\mathbf{1}_x$, almost everywhere $x \in \Omega$, $v \in L^q(\Omega)$; λ is an isomorphism of $L^q(\Omega)$ onto a closed subspace of $\mathcal{T}_q$. The dual map λ' taking $\mathcal{T}_q'$ into $L^{q'}(\Omega)$ is given by

$$\lambda' g(v) = g(\lambda v) = \int_\Omega g_x(\mathbf{1}_x) \cdot v(x)\,dx, \qquad g \in \mathcal{T}_q', \quad v \in L^q(\Omega),$$

so we have $\lambda' g(x) = g_x(\mathbf{1}_x)$, almost everywhere $x \in \Omega$.

Moreover, when $g_x \in L^{q'}(\Gamma_x)$ it follows that

$$g_x(\mathbf{1}_x) = \int_{\Gamma_x} g_x(y)\,dy,$$

the integral of the indicated boundary functional. Thus, for $g \in L^{q'}(\Omega, L^{q'}(\Gamma_x)) \subset \mathcal{T}_q'$, $\lambda' g \in L^{q'}(\Omega)$ is given by

$$\lambda' g(x) = \int_{\Gamma_x} g_x(y)\,dy \qquad \text{a.e. } x \in \Omega. \tag{2.1}$$

The imbedding λ of $L^q(\Omega)$ into $\mathcal{T}_q$ and its dual map λ' will play an essential role in our system below.

We consider elliptic differential operators in divergence form as realizations of monotone operators from Banach spaces to their duals. Assume we are given $\widetilde{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $1 < p < \infty$, $g_1 \in L^{p'}(\Omega)$, $g_0 \in L^1(\Omega)$, c and $c_0 > 0$

(2.2a) $\widetilde{A}(x, \vec{\xi})$ is continuous in $\vec{\xi} \in \mathbb{R}^n$ and measurable in x, and
$$|\widetilde{A}(x, \vec{\xi})| \le c|\xi|^{p-1} + g_1(x),$$

(2.2b) $\langle \widetilde{A}(x, \vec{\xi}) - \widetilde{A}(x, \vec{\eta}),\ \vec{\xi} - \vec{\eta} \rangle \ge 0,$

(2.2c) $\widetilde{A}(x, \vec{\xi}) \cdot \vec{\xi} \ge c_0 |\vec{\xi}|^p - g_0(x)$
for a.e. $x \in \Omega$ and all $\vec{\xi}, \vec{\eta} \in \mathbb{R}^n$.

Then the global diffusion operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$\mathcal{A}u(v) = \int_\Omega \widetilde{A}(x, \vec{\nabla} u(x)) \vec{\nabla} v(x)\,dx, \qquad u, v \in W_0^{1,p}(\Omega).$$

Thus, each $\mathcal{A}u$ is equivalent to its restriction to $C_0^\infty(\Omega)$, the distribution

$$Au \equiv \mathcal{A}u\big|_{C_0^\infty(\Omega)} = -\vec{\nabla} \cdot \widetilde{A}(\cdot, \vec{\nabla} u),$$

which specifies the value of this nonlinear elliptic divergence operator.

In order to specify a collection of local diffusion operators, $\mathcal{B}_x : W^{1,q}(\Omega_x) \to W^{1,q}(\Omega_x)'$, assume we are given $\widetilde{B} : Q \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $1 < q < \infty$, $h_1 \in L^{q'}(Q)$, $h_0 \in L^1(Q)$, c and $c_0 > 0$

(2.3a) $\widetilde{B}(x, y, \vec{\xi})$ is continuous in $\vec{\xi} \in \mathbb{R}^n$ and measurable in $(x, y) \in Q$, and
$|\widetilde{B}(x, y, \vec{\xi})| \le c|\vec{\xi}|^{q-1} + h_1(x, y),$

(2.3b) $\langle \widetilde{B}(x, y, \vec{\xi}) - \widetilde{B}(x, y, \vec{\eta}),\ \vec{\xi} - \vec{\eta} \rangle \ge 0,$

(2.3c) $\widetilde{B}(x, y, \vec{\xi}) \cdot \vec{\xi} \ge c_0 |\vec{\xi}|^q - h_0(x, y)$
for a.e. $(x, y) \in Q$ and all $\vec{\xi}, \vec{\eta} \in \mathbb{R}^n$.

Then define for each $x \in \Omega$

$$\mathcal{B}_x w(v) = \int_{\Omega_x} \widetilde{B}\big(x, y, \vec{\nabla}_y w(y)\big)\vec{\nabla}_y v(y)\,dy\ , \qquad w, v \in W^{1,q}(\Omega_x).$$

The elliptic differential operator on Ω_x is given by the *formal part* of $\mathcal{B}_x$, the distribution

$$B_x w \equiv \mathcal{B}_x w\big|_{C_0^\infty(\Omega_x)} = -\vec{\nabla}_y \cdot \widetilde{B}(x, \cdot, \vec{\nabla}_y w)$$

in $W_0^{1,q}(\Omega_x)'$. Also, we shall denote by $\mathcal{B} : \mathcal{W}_q \to \mathcal{W}_q'$ the *distributed* operator constructed from the collection $\{\mathcal{B}_x : x \in \Omega\}$ by

$$\mathcal{B}U(x) = \mathcal{B}_x\big(U(x)\big) \qquad \text{a.e. } x \in \Omega,\ U \in \mathcal{W}_q,$$

and we note that this is equivalent to

$$\mathcal{B}U(V) \equiv \int_\Omega \mathcal{B}_x\big(U(x)\big)V(x)\,dx, \qquad U, V \in \mathcal{W}_q.$$

The coupling term in our system will be given as a monotone graph which is a subgradient operator. Thus, assume $m : \mathbb{R} \to \mathbb{R}^+$ is convex and bounded by

$$m(s) \le C(|s|^q + 1), \qquad s \in \mathbb{R}, \tag{2.4}$$

hence, continuous. Then by

$$\tilde{m}(g) \equiv \int_\Omega \int_{\Gamma_x} m\big(g(x,s)\big)\,ds\,dx, \qquad g \in L^q\big(\Omega, L^q(\Gamma_x)\big),$$

we obtain the convex, continuous $\tilde{m} : L^q(\Omega, L^q(\Gamma_x)) \to \mathbb{R}^+$. Assume $\frac{1}{q} + \frac{1}{n} \ge \frac{1}{p}$ so that $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and consider the linear continuous maps

$$\lambda : W_0^{1,p}(\Omega) \to L^q\big(\Omega, L^q(\Gamma_x)\big), \qquad \gamma : \mathcal{W}_q \to L^q\big(\Omega, L^q(\Gamma_x)\big).$$

Then the composite function

$$M[u, U] \equiv \tilde{m}(\gamma U - \lambda u), \qquad u \in W_0^{1,p}(\Omega)\ ,\ U \in \mathcal{W}_q,$$

is convex and continuous on $W_0^{1,p}(\Omega) \times \mathcal{W}_q$. The subgradients are directly computed by standard results [12]. Specifically, we have $\hat{g} \in \partial\tilde{m}(g)$ if and only if

$$\hat{g}(x,s) \in \partial m\big(g(x,s)\big) \quad \text{a.e. } s \in \Gamma_x, \quad \text{a.e. } x \in \Omega,$$

and we have $[f, F] \in \partial M[u, U]$ if and only if $f = -\lambda'(\mu)$ in $W^{-1,p'}(\Omega)$ and $F = \gamma'(\mu)$ in $\mathcal{W}_q'$ for some $\mu \in \partial\tilde{m}(\gamma U - \lambda u)$.

The following result gives sufficient conditions for the *stationary regularized problem* to be well posed.

PROPOSITION 1. *Assume $1 < p, q$, $\frac{1}{q} + \frac{1}{n} \ge \frac{1}{p}$, and define the spaces and operators λ, γ as above. Specifically, the sets $\{\Omega_x : x \in \Omega\}$ are uniformly bounded with smooth* *boundaries, and the trace maps* $\{\gamma_x\}$ *are uniformly bounded. Let the functions* $\widetilde{A}, \widetilde{B}$, *and* m *satisfy* (2.2)–(2.4), *and assume in addition that*

$$m(s) \geq c_0 |s|^q, \qquad s \in \mathbb{R}. \tag{2.5}$$

Then for each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_q'$ *there exists a solution of*

$$u \in W_0^{1,p}(\Omega) : A(u) - \lambda'(\mu) = f \quad \text{in } W^{-1,p'}(\Omega), \tag{2.6a}$$

$$U \in \mathcal{W}_q : \mathcal{B}(U) + \gamma'(\mu) = F \quad \text{in } \mathcal{W}_q', \tag{2.6b}$$

$$\mu \in L^{q'}\left(\Omega, L^{q'}(\Gamma_x)\right) : \mu \in \partial\tilde{m}(\gamma U - \lambda u). \tag{2.6c}$$

For any such solution we have

$$\int_{\Gamma_x} \mu(x,s)\, ds = \langle F(x), \mathbf{1}_x \rangle \qquad \text{a.e. } x \in \Omega, \tag{2.7}$$

where $\mathbf{1}_x$ *denotes the constant unit function in* $W^{1,q}(\Omega_x)$.

Proof. The system (2.6) is a "pseudo-monotone plus subgradient" operator equation of the form

$$\begin{aligned} &[u,U] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q : \quad \text{for all } [v,V] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q, \\ &\mathcal{A}u(v) + \mathcal{B}U(V) + \partial M[u,U]\big([v,V]\big) \ni f(v) + F(V). \end{aligned} \tag{2.6'}$$

It remains only to verify a coercivity condition, namely,

$$\frac{\mathcal{A}u(u) + \mathcal{B}U(U) + \tilde{m}(\gamma U - \lambda u)}{\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q}} \to +\infty \tag{2.8}$$

$$\text{as } \|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q} \to +\infty.$$

Choose $k = \max\{|y_n| : y \in \Omega_x,\ x \in \Omega\}$ and let $\nu_x = (\nu_x^1, \cdots, \nu_x^n)$ be the unit normal on Γ_x. For $v \in W^{1,q}(\Omega_x)$ we have by Gauss' theorem

$$\begin{aligned} \int_{\Omega_x} \left(|v|^q + y_n q |v|^{q-1} \partial_n v\right) &= \int_{\Omega_x} \partial_n \left(y_n |v(y)|^q\right) dy \\ &= \int_{\Gamma_x} \nu_x^n(s) s_n |\gamma_x v(s)|^q \, ds. \end{aligned}$$

Hölder's inequality then shows

$$\|v\|_{L^q(\Omega_x)}^q \leq k\|\gamma_x v\|_{L^q(\Gamma_x)}^q + qk\|v\|_{L^q(\Omega_x)}^{q-1} \|\partial_n v\|_{L^q(\Omega_x)},$$

and from this follows

$$\|v\|_{L^q(\Omega_x)}^q \leq 2k\|\gamma_x v\|_{L^q(\Gamma_x)}^q + (2k)^q (q-1)^{q-1} \|\partial_n v\|_{L^q(\Omega_x)}^q$$

by Young's inequality. From here we obtain

$$c_0 \|V\|_{L^q(Q)}^q \leq \|\gamma V\|_{L^q(\Omega, L^q(\Gamma_x))}^q + \|\nabla_y V\|_{L^q(Q)}^q, \qquad V \in \mathcal{W}_q. \tag{2.9}$$

Thus from the a priori estimate

$$
\begin{aligned}
&\mathcal{A}u(u) + \mathcal{B}U(U) + M(\gamma U - \lambda u) \\
(2.10) \qquad &\geq c_0 \|\vec{\nabla} u\|^p_{L^p(\Omega)} - \|g_0\|_{L^1(\Omega)} + c_0 \|\vec{\nabla}_y U\|^q_{L^q(Q)} - \|h_0\|_{L^1(Q)} \\
&\quad + c_0 \|\gamma U - \lambda u\|^q_{L^q(\Omega, L^q(\Gamma_x))}, \qquad u \in W_0^{1,p}(\Omega),\ U \in \mathcal{W}_q,
\end{aligned}
$$

the Poincaré-type inequality (2.9) and the equivalence of $\|\nabla u\|_{L^p(\Omega)}$ with the norm on $W_0^{1,p}(\Omega)$, we can obtain the coercivity condition (2.8). Specifically, if (2.8) is bounded by K, then (2.10) is bounded above by

$$
\begin{aligned}
&K\big(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} + \|\gamma U\|_{L^q(\Omega, L^q(\Gamma_x))}\big) \\
&\quad \leq K\big(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} + \|\gamma U \\
&\qquad - \lambda\varphi\|_{L^q(\Omega, L^q(\Gamma_x))} + \|\lambda u\|_{L^q(\Omega)}\big),
\end{aligned}
$$

and the last term is dominated by the first. This gives an explicit bound on each of these terms and, hence, on $\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{\mathcal{W}_q}$.

Finally, we apply (2.6b) to the function $V \in \mathcal{W}_q$ given by $V(x,y) = v(x)$ for some $v \in L^q(\Omega)$, and this shows

$$\mu(\gamma v) = \langle F, v \rangle$$

since $\mathcal{B}U(V) = 0$, and thus

$$\int_\Omega \lambda' \mu(x) v(x)\, dx = \mu(\lambda v) = \mu(\gamma v) = \int_\Omega \langle F(x), 1 \rangle v(x)\, dx.$$

The identity (2.7) now follows from (2.1).

For the more general case of the *degenerate stationary problem* corresponding to (1.1), we obtain the following result.

COROLLARY 1. *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ and $\Phi : \mathbb{R} \to \mathbb{R}^+$ be convex and continuous, with $\varphi(0) = \Phi(0) = 0$, and assume*

$$(2.11) \qquad \varphi(s) \leq C\big(|s|^q + 1\big), \qquad \Phi(s) \leq C\big(|s|^q + 1\big), \qquad s \in \mathbb{R}.$$

For each pair $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}'_q$, there exists a solution of

$$(2.12a) \qquad u \in W_0^{1,p}(\Omega) : a + A(u) - \lambda'(\mu) = f \quad \text{in } W^{-1,p'}(\Omega),$$

$$(2.12b) \qquad U \in \mathcal{W}_q : b + \mathcal{B}(U) + \gamma'(\mu) = F \quad \text{in } \mathcal{W}'_q,$$

$$(2.12c) \qquad \mu \in \partial \tilde{m}(\gamma U - \lambda u) \quad \text{in } L^{q'}\big(\Omega, L^{q'}(\Gamma_x)\big),$$

$$(2.12d) \qquad a \in \partial\varphi(u) \quad \text{in } L^{q'}(\Omega), \qquad b \in \partial\Phi(U) \quad \text{in } L^{q'}(Q).$$

For any such solution we have

$$(2.13) \qquad \int_{\Omega_x} b(x,y)\, dy + \int_{\Gamma_x} \mu(x,s)\, ds = \big\langle F(x), \mathbf{1}_x \big\rangle \quad \text{a.e. } x \in \Omega.$$

Proof. This follows as above but with the continuous convex function

$$\Psi[u,U] = \int_\Omega \varphi\big(u(x)\big)\,dx + \int_\Omega \int_{\Omega_x} \Phi\big(U(x,y)\big)\,dy\,dx + \tilde{m}(\gamma U - \lambda u), \qquad [u,U] \in W_0^{1,p}(\Omega) \times \mathcal{W}_q.$$

The subgradient can be computed termwise because the three terms are continuous on $L^q(\Omega)$, $L^q(Q)$, and $L^q(\Omega, L^q(\Gamma_x))$, respectively.

Remark. The lower bound (2.5) on $m(\cdot)$ may be deleted in Corollary 1 if such a lower estimate is known to hold for Φ. It is also unnecessary in the matched microstructure model; see below.

In order to prescribe the boundary condition (1.1c) explicitly, we develop an appropriate Green's formula for the operators $\mathcal{B}_x$.

Note that we can identify $L^{q'}(\Omega_x) \subset W^{-1,q'}(\Omega_x)$ since $W_0^{1,q}(\Omega_x)$ is dense in $L^q(\Omega_x)$, so it is meaningful to define

$$D_x \equiv \{w \in W^{1,q}(\Omega_x) : B_x w \in L^{q'}(\Omega_x)\}.$$

This is the domain for the *abstract Green's theorem.*

LEMMA 1. *There is a unique operator $\partial_x : D_x \to T'_x$ for which $\mathcal{B}_x w = B_x w + \gamma'_x \partial_x w$ for all $w \in D_x$. That is, we have*

$$\mathcal{B}_x w(v) = (B_x w, v)_{L(\Omega_x)} + \langle \partial_x w, \gamma_x v\rangle, \qquad v \in W^{1,q}(\Omega_x), \tag{2.14}$$

for every $w \in D_x$.

Proof. The strict morphism γ_x of $W^{1,q}(\Omega_x)$ onto T_x has a dual γ'_x which is an isomorphism of T'_x onto $W_0^{1,q}(\Omega_x)^\perp$, the annihilator in $W^{1,q}(\Omega_x)'$ of the kernel of γ_x. For each $w \in D_x$, the difference $\mathcal{B}_x w - B_x w$ is in $W_0^{1,q}(\Omega_x)^\perp$, so it is equal to $\gamma'_x(\partial_x w)$ for a unique element $\partial_x w \in T'_x$.

Remark. The identity (2.14) is a generalized decomposition of $\mathcal{B}_x$ into a partial differential operator on Ω_x and a boundary condition on Γ_x. If Γ_x is smooth, ν_x denotes the unit outward normal on Γ_x, and if $\tilde{B}(x,\cdot,\vec{\nabla}_y w) \in [W^{1,q'}(\Omega_x)]^n$, then $w \in D_x$ and from the classical Green's theorem we obtain

$$\mathcal{B}_x w(v) - (B_x w, v)_{L(\Omega_x)} = \int_{\Gamma_x} \widetilde{B}(x,s,\vec{\nabla}_y w)\vec{\nu}_x(s)\gamma v(s)\,ds, \qquad v \in W^{1,q}(\Omega_x).$$

Thus, $\partial_x w = \widetilde{B}(x,\cdot,\vec{\nabla}_y w) \cdot \vec{\nu}_x$ is the indicated normal derivative in $L^{q'}(\Gamma_x)$ when $\widetilde{B}(x,\cdot,\vec{\nabla}_y w)$ is as smooth as above, and so we can regard $\partial_x w$ in general as an extension of this nonlinear differential operator on the boundary.

The *formal part* of $\mathcal{B} : \mathcal{W}_q \to \mathcal{W}'_q$ is the operator $B : \mathcal{W}_q \to \mathcal{W}'_0$ given by the restriction $B(U) \equiv \mathcal{B}U|_{\mathcal{W}_0}$. Since $\mathcal{W}_0$ is dense in $L^q(Q)$ we can specify the domain

$$D \equiv \{U \in \mathcal{W}_q : B(U) \in L^{q'}(Q)\}$$

on which we obtain as before a distributed form of Green's theorem.

LEMMA 2. *There is a unique operator $\partial : D \to T'_q$ such that*

$$\mathcal{B}(U)(V) = (B(U), V)_{L(Q)} + \langle \partial U, \gamma V\rangle, \qquad U \in D,\ V \in \mathcal{W}_q.$$

PROPOSITION 2. *Let the Sobolev spaces and trace operators be given as above. We summarize them in the following diagrams:*

$$
\begin{array}{ccccccc}
L^q(\Omega_x) & & L^q(\Gamma_x) & L^q(Q) & & L^q(\Omega, L^q(\Gamma_x)) \\
\cup & & \cup & \cup & & \cup \\
W^{1,q}(\Omega_x) & \xrightarrow{\gamma_x} & T_x & \mathcal{W}_q & \xrightarrow{\gamma} & \mathcal{T}_q \\
\cup & & \lambda_x \uparrow & \cup & & \lambda \uparrow \\
W_x^{1,q}(\Omega_x) & \longrightarrow & \mathbb{R}\cdot \mathbf{1}_x & \mathcal{W}_1 & \xrightarrow{\gamma_1} & L^q(\Omega) \\
\cup & & \cup & \cup & & \uparrow \\
W_0^{1,q}(\Omega_x) & \longrightarrow & \{0\} & \mathcal{W}_0 & \longrightarrow & \{0\}
\end{array}
$$

in which γ_1 is the restriction of γ to $\mathcal{W}_1$. $W_0^{1,q}(\Omega_x)$, $\mathcal{W}_0$ are dense in $L^q(\Omega_x)$, $L^q(Q)$, respectively. Let operators $\mathcal{B}_x$, $x \in \Omega$, and $\mathcal{B}$ be given and define their formal parts B_x, B as above. Then construct the domains D_x, D and boundary operators ∂_x, ∂ as in Lemmas 1 and 2, respectively. It follows that for any $U \in \mathcal{W}_q$,

(a) *$BU(x) = B_x(U(x))$ in $W_0^{1,q}(\Omega_x)'$ for a.e. $x \in \Omega$, and $U \in D$ if and only if $U(x) \in D_x$ for a.e. $x \in \Omega$ and $x \mapsto B_x U(x)$ belongs to $L^{q'}(Q)$;*

(b) *for each $U \in D$,*

$$\partial U(x) = \partial_x\big(U(x)\big) \quad \textit{in } \; T_x' \textit{ for a.e. } x \in \Omega$$

and

$$\mathcal{B}U = BU + \gamma_1'(\lambda'\partial U) \quad \textit{in } \mathcal{W}_1',$$

and for each $V \in \mathcal{W}_1$ we have

$$\int_\Omega \mathcal{B}_x U(x)\big(V(x)\big)\,dx = \int_Q B_x U(x) V(x)\,dy\,dx + \int_\Omega \langle \partial_x U(x), \mathbf{1}_x\rangle (\gamma_1 V)(x)\,dx.$$

Proof. (a) For $V \in \mathcal{W}_0$ we obtain from the definitions of B, $\mathcal{B}$, and $\mathcal{B}_x$, respectively,

$$\int_\Omega BU(x)V(x)\,dx = \int_\Omega \mathcal{B}U(V)\,dx = \int_\Omega \mathcal{B}_x U(x)\big(V(x)\big)\,dx = \int_\Omega B_x U(x) V(x)\,dx,$$

and so the first equality holds since $\mathcal{W}_0' = L^{q'}(\Omega, W_0^{1,q}(\Omega_x)')$. The characterization of D is immediate now.

(b) For $V \in \mathcal{W}_q$ we obtain from the definitions of $\gamma, \partial, \partial_x$, respectively, and (a)

$$\int_\Omega \partial U\big(\gamma_x V(x)\big)\,dx = \int_\Omega \partial U(\gamma V)\,dx = \int_\Omega (\mathcal{B}U - BU)(x)V(x)\,dx = \int_\Omega \partial_x\big(U(x)\big)\gamma_x V(x)\,dx.$$

Since the range of γ is $\mathcal{T}_q' = L^{q'}(\Omega, T_x')$, the first equality follows. The second is immediate from Lemma 2 since on $\mathcal{W}_1$, $\gamma = \lambda \circ \gamma_1$ and $\gamma' = \gamma_1' \lambda'$, and the third follows from the preceding remarks.

COROLLARY 2. *In the situation of Corollary* 1, $f \in L^{q'}(\Omega)$ *and* $F \in L^{q'}(Q)$ *if and only if* $Au \in L^{q'}(\Omega)$ *and* $B(U) \in L^{q'}(Q)$, *and in that case the solution satisfies almost everywhere*

$$
\begin{aligned}
&a(x) \in \partial\varphi\big(u(x)\big),\\
&a(x) + Au(x) + \int_{\Omega_x} b(x,y)\,dy = f(x) + \int_{\Omega_x} F(x,y)\,dy, \qquad x \in \Omega,\\
&u(s) = 0, \qquad s \in \Gamma,\\
&b(x,y) \in \partial\Phi\big(U(x,y)\big), \quad b(x,y) + BU(x,y) = F(x,y), \qquad y \in \Omega_x,\\
&\mu(x,s) \in \partial m\big(\gamma U(x,s) - u(x)\big), \qquad \partial_x\big(U(x)\big)(s) + \mu(x,s) = 0, \quad s \in \Gamma_x.
\end{aligned}
$$

Finally, we note that corresponding results for the *stationary matched microstructure model* are obtained directly by specializing the system (2.6′) to the space $\mathcal{W}_0^{1,p}$. This is identified with $\{[\gamma U, U] : U \in \mathcal{W}_0^{1,p}\}$ as a subspace of $W_0^{1,p}(\Omega) \times \mathcal{W}_q$, and we need only to restrict the solution $[u, U]$ and the test functions $[v, V]$, $v = \gamma V$, to this subspace to resolve the matched model. Then the coupling term M does not occur in the system; see the proof of Proposition 1, especially for the coercivity. These observations yield the following analogous results for the matched microstructure model.

PROPOSITION 1′. *Assume* $1 < p, q$, $\frac{1}{q} + \frac{1}{n} \geq \frac{1}{p}$, *and define the spaces and operators* λ, γ *as before. Let the functions* $\widetilde{A}, \widetilde{B}$, *and* m *satisfy* (2.2)–(2.4). *Then for each pair* $f \in W^{-1,p'}(\Omega)$, $F \in \mathcal{W}_1'$ *there exists a unique solution of*

$$u \in W_0^{1,p}(\Omega) : A(u) = f + \langle F, \mathbf{1}\rangle \quad \textit{in } W^{-1,p'}(\Omega), \tag{2.15a}$$

$$U \in \mathcal{W}_1 : B(U) = F \quad \textit{in } \mathcal{W}_0', \tag{2.15b}$$

$$\gamma U = \lambda u \quad \textit{in } L^q(\Omega) \subset \mathcal{T}_q. \tag{2.15c}$$

COROLLARY 1′. *Suppose* φ, Φ *are given as before and assume* (2.11). *For* f, F *as above there exists a unique solution of*

$$u \in W_0^{1,p}(\Omega) : a + \langle b, \mathbf{1}\rangle + A(u) = f + \langle F, \mathbf{1}\rangle \quad \textit{in } W^{-1,p'}(\Omega), \tag{2.16a}$$

$$U \in \mathcal{W}_1 : b + B(U) = F \quad \textit{in } \mathcal{W}_0', \tag{2.16b}$$

$$\gamma U = \lambda u \quad \textit{in } \; L^q(\Omega) \subset \mathcal{T}_q, \tag{2.16c}$$

$$a \in \partial\varphi(u) \quad \textit{in } L^{q'}(\Omega), \qquad b \in \partial\Phi(U) \quad \textit{in } L^{q'}(Q). \tag{2.16d}$$

In addition, $f \in L^{q'}(\Omega)$ and $F \in L^{q'}(Q)$ if and only if $Au \in L^{q'}(\Omega)$ and $B(U) \in L^{q'}(Q)$, and in that case the solution satisfies almost everywhere

$$\begin{aligned}
&a(x) \in \partial\varphi\big(u(x)\big),\\
&a(x) + Au(x) + \int_{\Omega_x} b(x,y)\,dy = f(x) + \int_{\Omega_x} F(x,y)\,dy\ , \qquad x \in \Omega,\\
&u(s) = 0, \qquad s \in \Gamma\ ,\\
&b(x,y) \in \partial\Phi\big(U(x,y)\big), \quad b(x,y) + BU(x,y) = F(x,y), \qquad y \in \Omega_x,\\
&U(x,s) = u(x), \qquad s \in \Gamma_x\ .
\end{aligned}$$

Remark. For the very special case of $p = q \geq 2$ and $a(u) = u$, $b(U) = U$ in the situation of Proposition 1 it follows from [7] or [20] that the Cauchy–Dirichlet problem for (1.1) is well posed in the space $L^p(0,T;W_0^{1,p}(\Omega) \times \mathcal{W}_p)$ with appropriate initial data $u(x,0)$, $U(x,y,0)$ and source functions $f(x,t)$, $F(x,y,t)$. A similar remark holds in the case of Proposition 1′ for the matched model with (1.1c′). These restrictive assumptions will be substantially relaxed in the next section.

Furthermore, variational inequalities may be resolved for problems corresponding to either the regularized or the matched microstructure model by adding the indicator function of a convex constraint set to the convex function Ψ. Thus such problems can be handled with constraints on the global variable u, the local variables U, or their difference $\lambda u - \gamma U$ on the interface.

3. The L^r-operators. Assume we are in the situation of Proposition 1. We define a relation or multi-valued operator $\mathbb{C}_2$ on the Hilbert space $L^2(\Omega) \times L^2(Q)$ as follows: $\mathbb{C}_2[u,U] \ni [f,F]$ if and only if

(3.1a) $$u \in L^2(\Omega) \cap W_0^{1,p}(\Omega) : \mathcal{A}(u) - \lambda'\mu = f \in L^2(\Omega),$$

(3.1b) $$U \in L^2(Q) \cap \mathcal{W}_q : \mathcal{B}(U) + \gamma'\mu = F \in L^2(Q)$$

$$\text{for some } \mu \in \partial\tilde{m}(\gamma U - \lambda u) \quad \text{in } L^{q'}(\Omega, L^{q'}(\Gamma_x)).$$

Thus, $\mathbb{C}_2$ is the restriction of (2.6) to $L^2(\Omega) \times L^2(Q)$. Note that $\lambda'\mu \in L^2(\Omega)$ by (2.7).

LEMMA 3. *If $\sigma : \mathbb{R} \to \mathbb{R}$ is monotone, Lipschitz, and $\sigma(0) = 0$, then for each pair*

$$\mathbb{C}_2[u_j,U_j] \ni [f_j,F_j]\ , \qquad j = 1,2,$$

there follows

$$\big(f_1 - f_2,\, \sigma(u_1 - u_2)\big)_{L^2(\Omega)} + \big(F_1 - F_2,\, \sigma(U_1 - U_2)\big)_{L^2(Q)} \geq 0.$$

Proof. Since σ is Lipschitz and $\sigma(0) = 0$, we have $\sigma(u_1 - u_2) \in W_0^{1,p}(\Omega)$ and $\sigma(U_1 - U_2) \in \mathcal{W}_q$. Also the chain rule applies to these functions, so we compute

$$\begin{aligned}
&\big\langle \mathcal{A}u_1 - \mathcal{A}u_2,\, \sigma(u_1,u_2)\big\rangle\\
&\quad = \int_\Omega \big(\widetilde{A}(x,\vec{\nabla}u_1) - \widetilde{A}(x,\nabla u_2)\big)\vec{\nabla}(u_1 - u_2)\sigma'(u_1 - u_2)\,dx,\\
&\big\langle \mathcal{B}U_1 - \mathcal{B}U_2,\, \sigma(U_1 - U_2)\big\rangle\\
&\quad = \int_\Omega \int_{\Omega_x} \big(\widetilde{B}(x,y,\vec{\nabla}_y U_1) - \widetilde{B}(x,y,\vec{\nabla}_y U_2)\big)\\
&\qquad\qquad \vec{\nabla}_y(U_1 - U_2)\sigma'(U_1 - U_2)\,dy\,dx\ .
\end{aligned}$$

Both of these are nonnegative because of (2.1b), (2.2b), and $\sigma' \geq 0$. The remaining term to check is

$$\big(-\lambda'(\mu_1 - \mu_2),\, \sigma(u_1 - u_2)\big)_{L^2(\Omega)} + \big\langle \gamma'(\mu_1 - \mu_2),\, \sigma(U_1 - U_2)\big\rangle$$
$$= \int_\Omega \int_{\Gamma_x} \big(\mu_1(x,s) - \mu_2(x,s)\big)\big(\sigma(\gamma U_1 - \gamma U_2) - \sigma(\lambda u_1 - \lambda u_2)\big)\, ds\, dx.$$

Since σ is a monotone function and ∂m is a monotone graph, this integrand is nonnegative and the result follows.

As a consequence of Lemma 3 with $\sigma(s) = s$, the operator $\mathbb{C}_2$ is monotone on the Hilbert space $L^2(\Omega) \times L^2(Q)$. Moreover, we obtain the following.

PROPOSITION 3. *The operator* $\mathbb{C}_2$ *is maximal monotone on* $L^2(\Omega) \times L^2(Q)$. *Let* $j : \mathbb{R} \to \mathbb{R}^+$ *be convex, lower-semicontinuous, and* $j(0) = 0$. *If* ∂m *is a function, then* $\mathbb{C}_2$ *is also single valued and*

$$\big(\mathbb{C}_2[u_1, U_1] - \mathbb{C}_2[u_2, U_2], [\sigma_1, \sigma_2]\big)_{L^2(\Omega)\times L^2(Q)} \geq 0 \tag{3.2}$$

for any *selections* $\sigma_1 \in \partial j(u_1 - u_2)$ *in* $L^2(\Omega)$ *and* $\sigma_2 \in \partial j(U_1 - U_2)$ *in* $L^2(Q)$.

Proof. To show $\mathbb{C}_2$ is maximal monotone it suffices to show that for any pair $[f, F] \in L^2(\Omega) \times L^2(Q)$ there is a solution of

$$u \in L^2(\Omega) \cap W_0^{1,p}(\Omega) : u + \mathcal{A}(u) - \lambda'(\mu) = f \quad \text{in } W^{-1,p'}(\Omega), \tag{3.3a}$$
$$U \in L^2(Q) \cap \mathcal{W}_q : U + \mathcal{B}(U) + \gamma'(\mu) = F \quad \text{in } \mathcal{W}_q' \tag{3.3b}$$
$$\mu \in L^{q'}\big(\Omega, L^{q'}(\Gamma_x)\big) : \mu \in \partial\tilde{m}(\gamma U - \lambda u). \tag{3.3c}$$

The existence of a (unique) solution of (3.3) follows as in Proposition 1, but by considering the pseudomonotone operator $[\mathcal{A}, \mathcal{B}]$ on the product space $L^2(\Omega) \cap W_0^{1,p}(\Omega) \times L^2(Q) \cap \mathcal{W}_q$ and the convex function, $\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|U\|^2_{L^2(Q)} + \tilde{m}(\gamma U - \lambda u)$, on that space.

To establish the estimate (3.2), we consider the lower-semicontinuous convex function

$$\tilde{\jmath}[u, U] = \int_\Omega \left(j\big(u(x)\big) + \int_{\Omega_x} j\big(U(x,y)\big)\, dy \right) dx, \qquad [u, U] \in L^2(\Omega) \times L^2(Q). \tag{3.4}$$

The subgradient of $\tilde{\jmath}$ is given on this product space by

$$\tilde{\sigma} = [\sigma_1, \sigma_2] \in \partial\tilde{\jmath}[u, U] \quad \text{if and only if}$$
$$\tilde{\sigma}[v, V] = \int_\Omega \left(\sigma_1(x) v(x) + \int_{\Omega_x} \sigma_2(x,y) V(x,y)\, dy \right) dx, \qquad [v, V] \in L^2(\Omega) \times L^2(Q),$$

where

$$\sigma_1(x) \in \partial j\big(u(x)\big), \quad \text{a.e. } x \in \Omega,$$
$$\sigma_2(x,y) \in \partial j\big(U(x,y)\big), \quad \text{a.e. } (x,y) \in Q.$$

The Yoshida approximation $\tilde{\jmath}_\varepsilon$ of $\tilde{\jmath}$ is given as in (3.4) but with j replaced by j_ε. Since the derivative of j_ε is Lipschitz, monotone, and contains the origin, it follows by

Lemma 3 that the special case of (3.2) with j_ε is true. Thus, $\mathbb{C}_2$ is $\partial\tilde{\jmath}$-monotone [8] and the desired result follows, since the single-valued $\mathbb{C}_2$ equals its minimal section.

We define the realization of (2.6) in $L^r(\Omega) \times L^r(Q)$, $1 \le r < \infty$, as follows. For $r \ge 2$, $\mathbb{C}_r$ is the restriction of $\mathbb{C}_2$ to $L^r(\Omega) \times L^r(Q)$, and for $1 \le r < 2$, $\mathbb{C}_r$ is the closure in $L^r(\Omega) \times L^r(Q)$ of $\mathbb{C}_2$.

COROLLARY 3. *The operator $\mathbb{C}_r$ is m-accretive in $L^r(\Omega) \times L^r(Q)$ for $1 \le r < \infty$.*

Proof. Let $(I + \varepsilon\mathbb{C}_2)([u_j, U_j]) \ni [f_j, F_j]$, $j = 1, 2$, and assume $[f_j, F_j] \in L^r(\Omega) \times L^r(Q)$ if $r \ge 2$. Set $j(s) = |s|^r$, $s \in \mathbb{R}$. From Proposition 4.7 of [8] it follows that

$$\|[u_1 - u_2, U_1 - U_2]\|_{L^r(\Omega)\times L^r(Q)} \le \|[f_1 - f_2, F_1 - F_2]\|_{L^r(\Omega)\times L^r(Q)}.$$

Taking $[f_2, F_2] = [0, 0]$, we see that $L^r(\Omega) \times L^r(Q)$ is invariant under $(I + \varepsilon\mathbb{C}_2)^{-1}$, and then the estimate shows this operator is a contraction on that space. We have $Rg(I + \varepsilon\mathbb{C}_r) = L^r(\Omega) \times L^r(Q)$ directly from the definition for $r \ge 2$, and for $1 \le r < 2$, $Rg(I + \varepsilon\mathbb{C}_r) \supset L^2(\Omega) \times L^2(Q)$, which is dense, so the result follows easily.

Remarks. The Cauchy–Dirichlet problem for the regularized model (1.1) is well posed in $L^r(\Omega) \times L^r(Q)$ when $a(u) = u$, $b(U) = U$, and $r > 1$. This follows from Corollary 3 and the theory of evolution equations generated by m-accretive operators in a uniformly convex Banach space. For example, from [19] we recall the following:

> If $\mathfrak{f} \in W^{1,1}(0, T; X)$ and $w_0 \in D(\mathbb{C}_r)$, where $\mathbb{C}_r$ is m-accretive on the uniformly convex Banach space X, then there exists a unique Lipschitz function $w : [0, T] \to X$ for which
>
> $$w'(t) + \mathbb{C}_r\big(w(t)\big) \ni \mathfrak{f}(t), \quad \text{a.e. } t \in (0, T),$$
> $$w(t) \in D(\mathbb{C}_r) \quad \text{for all } t \in [0, T], \quad \text{and}$$
> $$w(0) = w_0.$$

See [4] for details (Theorem III.2.3) and references. By applying this result to the operator $\mathbb{C}_r$ given in $X \equiv L^r(\Omega) \times L^r(Q)$, $1 < r < \infty$, we obtain a generalized strong solution $w(t) = [u(t), U(t)]$ of the system

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} + Au(x,t) + \int_{\Omega_x} \frac{\partial U(x,y,t)}{\partial t}\, dy \\
&\qquad = f(x,t) + \int_{\Omega_x} F(x,y,t)\, dy, \qquad x \in \Omega,\ t \in (0,T), \\
&u(s,t) = 0, \qquad s \in \Gamma, \\
&\frac{\partial U(x,y,t)}{\partial t} + BU(x,y,t) = F(x,y,t), \qquad y \in \Omega_x, \\
&\mu(x,s,t) \in \partial m\big(U(x,s,t) - u(x,t)\big), \qquad \partial_x U(x,s,t) + \mu(x,s,t) = 0, \quad s \in \Gamma_x, \\
&u(x,0) = u_0(x), \quad U(x,y,0) = U_0(x,y).
\end{aligned}$$

The restrictions on the data $\mathfrak{f}(t) = [f(t), F(t)]$ and $w_0 = [u_0, U_0]$ can be considerably relaxed in the Hilbert space case $r = 2$ [8].

By applying Proposition 1′ similarly, it follows that corresponding results for the matched model are obtained. Thus we obtain a generalized strong solution in $L^r(\Omega) \times L^r(Q)$, $1 < r < \infty$, of the system

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} + Au(x,t) + \int_{\Omega_x} \frac{\partial U(x,y,t)}{\partial t}\, dy \\
&\qquad = f(x,t) + \int_{\Omega_x} F(x,y,t)\, dt, \qquad x \in \Omega, \quad t \in (0,T), \\
&u(s,t) = 0, \qquad s \in \Gamma, \\
&\frac{\partial U(x,y,t)}{\partial t} + BU(x,y,t) = F(x,y,t), \qquad y \in \Omega_x, \\
&U(x,s,t) = u(x,t), \qquad s \in \Gamma_x, \\
&u(x,0) = u_0(x), \qquad U(x,y,0) = U_0(x,y).
\end{aligned}$$

This follows as above from the analogue of Proposition 3 and Corollary 3.

We return to consider the fully nonlinear model (1.1). The generator of this evolution system will be obtained by closing up the composition of $\mathbb{C}_2$ with the inverse of $[\partial\varphi, \partial\Phi]$ in $L^1(\Omega) \times L^1(Q)$. Thus, we begin with the following.

DEFINITION. $\mathbb{C}[a,b] \ni [f,F]$ if $\mathbb{C}_2[u,U] \ni [f,F]$ and $a \in \partial\varphi(u)$ in $L^2(\Omega)$, $b \in \partial\Phi(U)$ in $L^2(Q)$ for some pair $[u,U]$ as in (3.1).

LEMMA 4. *The operator $\mathbb{C}$ is accretive on $L^1(\Omega)\times L^1(Q)$ if either ∂m is a function or if both $\partial\varphi$ and $\partial\Phi$ are functions.*

Proof. Let $\varepsilon > 0$ and suppose that $(I + \varepsilon\mathbb{C})[a_j, b_j] \ni [f_j, F_j]$ for $j = 1,2$. Thus we have $\varepsilon\mathbb{C}_2[u_j, U_j] \ni [f_j - a_j, F_j - b_j]$, $a_j \in \partial\varphi(u_j)$, $b_j \in \partial\Phi(U_j)$ as above. First we choose $\sigma(s) = \mathrm{sgn}_\delta^+(s)$, the Yoshida approximation of the maximal monotone sgn^+, apply Lemma 3 and obtain

$$\begin{aligned}
&\big(a_1 - a_2, \mathrm{sgn}_\delta^+(u_1 - u_2)\big)_{L^2(\Omega)} + \big(b_1 - b_2, \mathrm{sgn}_\delta^+(U_1 - U_2)\big)_{L^2(Q)} \\
&\qquad \le \|(f_1 - f_2)^+\|_{L^1(\Omega)} + \|(F_1 - F_2)^+\|_{L^1(Q)}.
\end{aligned}$$

If $\partial\varphi$ and $\partial\Phi$ are functions, then

$$\begin{aligned}
(a_1 - a_2)\,\mathrm{sgn}_0^+(u_1 - u_2) &= (a_1 - a_2)^+, \\
(b_1 - b_2)\,\mathrm{sgn}_0^+(U_1 - U_2) &= (b_1 - b_2)^+,
\end{aligned}$$

so letting $\delta \to 0$ gives

$$\begin{aligned}
&\|(a_1 - a_2)^+\|_{L^1(\Omega)} + \|(b_1 - b_2)^+\|_{L^1(Q)} \\
&\qquad \le \|(f_1 - f_2)^+\|_{L^1(\Omega)} + \|(F_1 - F_2)^+\|_{L^1(Q)}.
\end{aligned} \tag{3.5}$$

The same holds for negative parts, so it follows that $(I + \varepsilon\mathbb{C})^{-1}$ is an order-preserving contraction with respect to $L^1(\Omega) \times L^1(Q)$ for each $\varepsilon > 0$.

Next we suppose ∂m is a function. Choose $j(s) = s^+$, so that $\partial j = \mathrm{sgn}^+$, and then set

$$\begin{aligned}
\sigma_1(x) &= \mathrm{sgn}_0^+(u_1 - u_2 + a_1 - a_2) \in \mathrm{sgn}^+(u_1 - u_2) \cap \mathrm{sgn}^+(a_1 - a_2), \\
\sigma_2(x,y) &= \mathrm{sgn}_0^+(U_1 - U_2 + b_1 - b_2) \in \mathrm{sgn}^+(U_1 - U_2) \cap \mathrm{sgn}^+(b_1 - b_2).
\end{aligned}$$

Proposition 3 applies here to give (3.5). A similar estimate for negative parts yields the result.

Although $\mathbb{C}$ is not accretive on L^r for $1 < r$, we can obtain L^∞ estimates when the graphs $\partial\varphi$, $\partial\Phi$ are not too dissimilar.

COROLLARY 4. *If* $(I + \varepsilon\mathbb{C})[a, b] \ni [f, F]$ *with* $\varepsilon > 0$, *then*

$$\|a^+\|_{L^\infty(\Omega)} \le \max\big(a_0(k), \|f^+\|_{L^\infty(\Omega)}\big),$$
$$\|b^+\|_{L^\infty(Q)} \le \max\big(b_0(k), \|F^+\|_{L^\infty(Q)}\big), \tag{3.6}$$

where $k \equiv \max(a_0^{-1}(\|f^+\|_{L^\infty}), b_0^{-1}(\|F^+\|))$.

Remarks. Here a_0 is the *minimal section* $(\partial\varphi)_0$, a_0^{-1} is the minimal section of $(\partial\varphi)^{-1}$, and b_0, b_0^{-1} are defined similarly from $\partial\Phi$. Specifically, we obtain an explicit a priori bound on $\|a^+\|_{L^\infty(\Omega)}$ and $\|b^+\|_{L^\infty(Q)}$ when $\|f^+\|_{L^\infty(\Omega)} inRg(\partial\varphi)$ and $\|F^+\|_{L^\infty(Q)} \in Rg(\partial\Phi)$. By similar estimates for negative parts, we obtain explicit estimates on $\|a\|_{L^\infty(\Omega)}$ and $\|b\|_{L^\infty(Q)}$ for *any* pair $f \in L^\infty(\Omega)$, $F \in L^\infty(\Omega)$ if $Rg(\partial\varphi) = \mathbb{R}$ and $Rg(\partial\Phi) = \mathbb{R}$ or (trivially) if both $Rg(\partial\varphi)$ and $Rg(\partial\Phi)$ are bounded in $\mathbb{R}$. Finally, we note that in the special case $\varphi = \Phi$, we obtain

$$\max\big(\|a^+\|_{L^\infty(\Omega)}, \|b^+\|_{L^\infty(Q)}\big) \le \max\big(\|f^+\|_{L^\infty(\Omega)}, \|F^+\|_{L^\infty(Q)}\big).$$

Proof. By the choice of $k \ge 0$ we have

$$\partial\varphi(k) \ni \ell_1 \ge \|f^+\|_{L^\infty}, \qquad \partial\Phi(k) \ni \ell_2 \ge \|F^+\|_{L^\infty}$$

for some pair ℓ_1, ℓ_2. Subtract these from the operator equation, multiply by either

$$\operatorname{sgn}_\delta^+(u - k), \qquad \operatorname{sgn}_\delta^+(U - k)$$

or by

$$\operatorname{sgn}_0^+(a - \ell_1 + u - k), \qquad \operatorname{sgn}_0^+(b - \ell_2 + U - k)\ ,$$

depending on whether $\partial\varphi$ and $\partial\Phi$ are functions or ∂m is a function, respectively. Apply Lemma 3 and let $\delta \to 0$ or apply Proposition 3, respectively, to obtain

$$\begin{aligned}&\|(a - \ell_1)^+\|_{L^1(\Omega)} + \|(b - \ell_2)^+\|_{L^1(Q)}\\ &\quad \le \|(f^+ - \ell_1)^+\|_{L^1(\Omega)} + \|(F^+ - \ell_2)^+\|_{L^1(Q)}.\end{aligned}$$

The right side is zero, so the result follows.

PROPOSITION 4 (Moser). *Let* $(u, U) \in W_0^{1,p}(\Omega) \times \mathcal{W}_q$ *be a solution to*

$$\begin{aligned}\mathcal{A}(u) - \lambda'\mu \ni f \quad &\text{in } W^{-1,p'}(\Omega),\\ \mathcal{B}(U) + \gamma'\mu \ni F \quad &\text{in } \mathcal{W}_q',\\ \mu \in \partial m(\gamma U &- \lambda u).\end{aligned}$$

(a) *If* $(f, F) \in L^{\tau'}(\Omega) \times L^{\tau'}(Q)$ *with* $\tau' > \frac{n}{p}$, *and*

$$\widetilde{A}(x, \vec{\xi}) \cdot \vec{\xi} \ge c_0|\xi|^p - g_0(x) \tag{2.2c$'$}$$

where $g_0 \in L^{\tau'}(\Omega)$, *then* $u \in L^\infty(\Omega)$.

(b) *If, additionally,* $F \in L^\infty[\Omega; L^{t'}(\Omega_x)]$ *with* $t' > \frac{n}{q}$,

$$\widetilde{B}(x,y,\vec{\xi}) \cdot \vec{\xi} \geq c_0|\xi|^q - h_0(x,y), \tag{2.3c$'$}$$

where $h_0 \in L^\infty[\Omega; L^{t'}(\Omega_x)]$, *and* m *satisfies the growth condition* (2.5) *and* $m(0) = 0$, *then* $U \in L^\infty(Q)$.

Proof. (a) Estimate (2.7) of Proposition 1 shows that $\lambda'\mu \in L^{r'}(\Omega)$, so that

$$\mathcal{A}(u) = f - \lambda'\mu = \tilde{f} \in L^{r'}(\Omega).$$

Lemma 3 of [23] can now be used to conclude $u \in L^\infty(\Omega)$.

(2) Define $\widetilde{U} = U - u\mathbf{1}$. Since $\mathcal{B}(\widetilde{U}) = B(U)$, it follows that

$$\mathcal{B}(\widetilde{U}) + \gamma'\mu = F \text{ in } \mathcal{W}_q', \qquad \mu \in \partial m(\gamma\widetilde{U}),$$

and for almost every $x \in \Omega$, and every $V \in \mathcal{W}_q$

$$\int_{\Omega_x} \widetilde{B}(x,\cdot,\nabla_y\widetilde{U}(x)) \cdot \nabla_y V(x) + \int_{\Gamma_x} \mu(x)\gamma V(x) = \int_{\Omega_x} F(x,\cdot)V(x), \tag{$*$}$$

with $\mu(x) \in \partial m(\gamma\widetilde{U}(x))$. We will now use Moser iteration with $(*)$ to conclude $\|\widetilde{U}(x)\|_{L^\infty(\Omega_x)} \leq C$, where C is to be chosen independently of $x \in \Omega$.

If $\widetilde{U}(x) \in L^r(\Omega_x)$ ($r = q$ suffices for the first iterate), define $s = 1 + (r - t/tq)$ $(\frac{1}{t} + \frac{1}{t'} = 1)$. Let $H \in C^1(\mathbb{R})$ satisfy $H(s) = |s|^s$ if $|s| \leq s_0$, H affine for $|s| > s_0$, and define $G(s) = \int_0^s |H'(\xi)|^q \, d\xi$. Since H has linear growth, it follows that $G(\widetilde{U}) \in \mathcal{W}_q$. Substituting $G(\widetilde{U})$ for V in $(*)$ gives

$$\int_{\Omega_x} \widetilde{B}(x,\cdot,\nabla_y\widetilde{U}(x)) \cdot \nabla_y\widetilde{U}(x)G'(\widetilde{U}(x)) + \int_{\Gamma_x} \mu(x)\gamma G(\widetilde{U}(x)) = \int_{\Omega_x} F(x,\cdot)G(\widetilde{U}(x)).$$

The first term of the formula above is bounded below using (2.3c$'$). To estimate the second term, use

(i) $\mu\widetilde{U} \geq m(\widetilde{U})$ (as $m(0) = 0$), and

(ii) $\operatorname{sgn}(\widetilde{U}) = \operatorname{sgn}(G(\widetilde{U}))$ (so that $G(\widetilde{U})/\widetilde{U} \geq 0$ when $\widetilde{U} \neq 0$)

to get

$$\begin{aligned} \mu G(\widetilde{U}) &= \mu\widetilde{U}\ G(\widetilde{U})/\widetilde{U} \geq m(\widetilde{U})\ G(\widetilde{U})/\widetilde{U} \\ &\geq c_0|\widetilde{U}|^q\ G(\widetilde{U})/\widetilde{U} = c_0|\widetilde{U}|^{q-1}|G(\widetilde{U})|, \end{aligned}$$

$$c_0\int_{\Omega_x} |\nabla_y\widetilde{U}|^q G'(\widetilde{U}) + c_0\int_{\Gamma_x} |\widetilde{U}|^{q-1}|G(\widetilde{U})| \leq \int_{\Omega_x} FG(\widetilde{U}) + h_0G'(\widetilde{U}).$$

The first term may be written as $|\nabla_y H(\widetilde{U})|^q$ which, using the Sobolev embedding theorem, is bounded below by

$$c(\varepsilon)\|H(\widetilde{U})\|^q_{L^{\frac{nq}{n-q}}(\Omega_x)} - \varepsilon\int_{\Gamma_x} |H(\widetilde{U})|^q,$$

where $\varepsilon > 0$ can be chosen arbitrarily small (see (2.9)). The right-hand side is bounded using Hölder's inequality.

$$c(\varepsilon)\|H(\widetilde{U})\|^q_{L^{nq/(n-q)}(\Omega_x)} + \int_{\Gamma_x} |\widetilde{U}|^{q-1}|G(\widetilde{U})| - \varepsilon|H(\widetilde{U})|^q$$

$$\leq \frac{1}{c_0}\left(\|F\|_{L^{t'}(\Omega_x)}\|G(\widetilde{U})\|_{L^t(\Omega_x)} + \|h_0\|_{L^{t'}(\Omega_x)}\|G'(\widetilde{U})\|_{L^t(\Omega_x)}\right)$$

when $s_0 \to \infty$, $H(\widetilde{U}) \to |\widetilde{U}|^s$, and $|\widetilde{U}|^{q-1}|G(\widetilde{U})| \to \eta(r)|\widetilde{U}|^{sq}$, where $\eta(r) = \frac{t}{r}s^q = \frac{t}{r}(1 + (r - t/qt))^q$. If ε is chosen as $\varepsilon = \min_{q\leq r<\infty} \eta(r)$, it follows that

$$\|\widetilde{U}\|_{L^{sq(n/n-q)}(\Omega_x)} \leq (cs)^{1/s} \max\left[1, \|\widetilde{U}\|_{L^r(\Omega_x)}\right].$$

The result now follows by iteration of the above estimate.

THEOREM 1. *Assume the hypotheses of Proposition* 1, *Corollary* 1, *Lemma* 4, *and Proposition* 4. *Also, assume that $Rg(\partial\varphi)$ and $Rg(\partial\Phi)$ are both bounded or that both are equal to* $\mathbb{R}$. *Then $\overline{\mathbb{C}}$, the closure of $\mathbb{C}$ in $L^1(\Omega) \times L^1(Q)$, is m-accretive.*

Proof. Let $f \in L^\infty(\Omega)$ and $F \in L^\infty(Q)$. Corollary 1 asserts there is a solution of (2.12). If the graphs $\partial\varphi$ and $\partial\Phi$ have bounded range, then $a \in L^\infty(\Omega)$, $b \in L^\infty(Q)$, and it follows from Proposition 4 that $u \in L^2(\Omega)$ and $U \in L^2(Q)$. This shows $\mathbb{C}_2[u, U] \ni [a - f, b - F]$, so $(I + \mathbb{C})([a, b]) \ni [f, F]$. Thus, $Rg(I + \overline{\mathbb{C}})$ is dense in and, hence, equal to $L^1(\Omega) \times L^1(Q)$.

If the ranges of $\partial\varphi$ and $\partial\Phi$ equal $\mathbb{R}$, then by Corollary 4 any solution satisfies

$$\|a\|_{L^\infty(\Omega)} \leq K, \qquad \|b\|_{L^\infty(Q)} \leq K,$$

where K depends on f and F. Replace $\partial\varphi, \partial\Phi$ by the appropriately truncated $\partial\varphi_K, \partial\Phi_K$.

The solution with these truncated graphs, then, is a solution of the equation with the original graphs, so we are done.

COROLLARY 5. *Under the hypotheses of Theorem* 1, *problem* (1.1) *has a unique generalized solution $(a, b) \in C[0, T; L^1(\Omega) \times L^1(Q)]$, provided the data satisfy $(f, F) \in L^1[0, T; L^1(\Omega) \times L^1(Q)]$, and $(a(0), b(0)) \in \overline{D(\mathbb{C})}$.*

This follows from the Crandall–Liggett theorem [9], which is proved by showing that the step functions (a^N, b^N), constructed from solutions to the differencing scheme

$$(a^n, b^n) - (a^{n-1}, b^{n-1}) + \tau\mathbb{C}(a^n, b^n) \ni \tau(f^n, F^n) \tag{3.7}$$

$(\tau = \frac{T}{N})$, converge uniformly when the operator $\mathbb{C}$ is m-accretive. Benilan [6] proves that these generalized solutions are unique.

All of our results hold for the matched microstructure model problem. Specifically, Lemma 4 and Corollary 4 are obtained from Proposition 3, and Proposition 4 is actually simpler for the matched problem. The analogues of Theorem 1 and Corollary 5 show that the matched problem (1.1a), (1.1b), (1.1c′) has a unique generalized solution $(a, b) \in C[0, T; L^1(\Omega) \times L^1(Q)]$.

The next theorem shows that if the data is further restricted, the generalized solutions will satisfy the partial differential equation (1.1). The following notation is used:

$$\begin{aligned}
&L^r(T) = L^r\left[0, T; L^r(\Omega) \times L^r(Q)\right], \qquad 1 \leq r \leq \infty,\\
&V = W_0^{1,p}(\Omega) \times \mathcal{W}_q,\\
&\mathcal{V}(T) = L^p\left[0, T; W_0^{1,p}(\Omega)\right] \times L^q[0, T; \mathcal{W}_q],\\
&\widehat{\mathcal{V}}(T) = W^{1,p'}\left[0, T; W^{-1,p'}(\Omega)\right] \times W^{1,q'}[0, T; \mathcal{W}_q'].
\end{aligned}$$

THEOREM 2. *Assume the hypotheses of Theorem* 1 *hold and in addition that* $(f, F) \in L^1(T) \cap \mathcal{V}(T)'$ *and* $(a(0), b(0)) \in \overline{D(\mathbb{C})} \cap V'$. *Then the generalized solutions of Corollary* 5 *satisfy*

$$(3.8a) \qquad (a, b) \in \widehat{\mathcal{V}}(T), \qquad (u, U) \in \mathcal{V}(T),$$

$$(3.8b) \qquad \frac{\partial}{\partial t}(a, b) + \big(\mathcal{A}(u) - \lambda'\mu,\ \mathcal{B}(U) + \gamma'\mu\big) = (f, F) \text{ in } \mathcal{V}(T)',$$

$$(3.8c) \qquad (a, b) \in \big(\partial\phi(u), \partial\Phi(U)\big), \qquad \mu \in \partial\tilde{m}(\lambda u - \gamma U).$$

Proof. The results of Grange and Mignot [15] show that the step functions (a^N, b^N) and (u^N, U^N) generated from the differencing scheme (3.7) converge weakly in $\widehat{\mathcal{V}}(T)$ and $\mathcal{V}(T)$, respectively. Moreover, equation (3.8) will be satisfied in the limit, provided the weak limits (a, b) and (u, U) satisfy $(a, b) \in (\partial\phi(u), \partial\Phi(U))$. To establish this inclusion, let $(v, V) \in \mathcal{V}(T)$ and $(\tilde{a}, \tilde{b}) \in (\partial\phi(v), \partial\Phi(V))$. The growth conditions on ϕ and Φ guarantee that (a^N, b^N) and $(\tilde{a}, \tilde{b}) \in \mathcal{V}(T)'$ are functions, so it is possible to define $(a^N - \tilde{a},\ b^N - \tilde{b})_s$ to be the pair of functions truncated above and below by $\pm s$ $(s > 0)$. This pair of functions is bounded in $L^\infty(T)$ and converges in $L^1(T)$ to $(a - \tilde{a},\ b - \tilde{b})_s$, and so converges in $L^r(T)$ for $1 \le r < \infty$. If $r \ge \max(p', q')$, it follows that $L^r(T) \subset \mathcal{V}(T)'$, so the sequence $(a^N - \tilde{a},\ b^N - \tilde{b})_s$ converges strongly in $\mathcal{V}(T)'$. The monotonicity of $\partial\phi$ and $\partial\Phi$ imply

$$0 \le \Big\langle (a^N - \tilde{a},\ b^N - \tilde{b})_s,\ (u^N - v,\ U^n - V) \Big\rangle.$$

Passing to the limit as $N \to \infty$ and then letting $s \to \infty$ yields

$$0 \le \Big\langle (a - \tilde{a},\ b - \tilde{b})\ ,\ (u - v,\ U - V) \Big\rangle, \qquad (\tilde{a}, \tilde{b}) \in \big(\partial\phi(v), \partial\Phi(V)\big).$$

Since $(\partial\phi(\cdot), \partial\Phi(\cdot))$ is maximally monotone, it follows that $(a, b) \in (\partial\phi(u), \partial\Phi(U))$.

Finally, we note that the corresponding solution of the matched problem satisfies

$$(3.8\mathrm{a}') \qquad (a, b) \in \widetilde{\mathcal{V}}(T), \qquad (\gamma U, U) \in \mathcal{V}(T),$$

$$(3.8\mathrm{b}') \qquad \frac{\partial}{\partial t}(a, b) + \big(\mathcal{A}(\gamma U), \mathcal{B}(U)\big) = (f, F) \ \text{ in } \mathcal{V}_0(T)',$$

$$(3.8\mathrm{c}') \qquad (a, b) \in \big(\partial\varphi(\gamma U), \partial\Phi(U)\big), \qquad U \in \mathcal{W}_0,$$

where the space $\mathcal{V}_0(T)$ is given by

$$\mathcal{V}_0(T) \equiv \Big\{U \in L^q[0, T; \mathcal{W}_1] : \gamma(U) \in L^p\big[0, T; W_0^{1,p}(\Omega)\big]\Big\}$$

with the appropriate norm for which $(\gamma(U), U) \in \mathcal{V}(T)$ for each $U \in \mathcal{V}_0(T)$.

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